ON THE FIBRED SPACES OF SEIFERT

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FOR $h = 0$, $r = 0, 1, 2$ the Seifert fibred spaces (**6**)

$$(Oo;\ h\,|\,b;\ \alpha_1, \beta_1;\ \ldots, \alpha_r, \beta_r)$$

are lens spaces (or $S^1 \times S^2$). It is known that the topological classification (**1**) of the lens spaces differs from their homotopy classification. Surprisingly, the 'wickedness' of the lens spaces is not shared by their 'big brothers' for $h > 0$, $r = 0, 1$. As I shall show below, the topological classification of these spaces follows from homology for $r = 0$, and, for $r = 1$, from their fundamental groups *via* the elementary ideals (**3**). However, this is not true for $r = 2$, and this case seems to require new methods.

The lens spaces can be classified topologically by considering the complementary spaces of simple closed curves in these spaces. R. H. Fox has suggested that other 3-dimensional manifolds might be classified by examining the complementary spaces of *pairs* of simple closed curves. The results of this paper show that this is, in fact, the case.

I shall prove that the topological classification coincides with the (unoriented) fibre classification for those spaces

$$(Oo;\ h\,|\,b;\ \alpha_1, \beta_1;\ \alpha_2, \beta_2)$$

for which

(i) $(\alpha_1, \alpha_2) = 1$,

(ii) the torsion number $p = |b\alpha_1\alpha_2 + \beta_2\alpha_1 + \beta_1\alpha_2|$ is greater than $1 + \max\{\alpha_1, \alpha_2\}$.

Condition (ii) necessarily holds if $b \neq -1$. If these conditions are not satisfied we may encounter the algebraic difficulties found in the Poincaré spaces, and our method fails. I shall, however, derive some invariants for the case $(\alpha_1, \alpha_2) > 1$.

The homology group H_1 distinguishes the lens spaces from the spaces $(Oo;\ h)$ for which $h > 0$, $r = 0, 1, 2$. Hence it will be assumed throughout this paper that $h > 0$.

1. The invariant Δ^{**}

Throughout this paper, the same symbol will often be used for a geometrical object and a closely associated element of some group. The symbol H, for instance, is used for a fibre of a fibred space M and also for an associated generator of $\pi_1(M)$: there are other cases of the same kind and it is hoped that the convention will assist rather than confuse the reader. By a *knot* we mean a simple closed curve which is polygonal in some triangulation of M.

Let M be a compact, orientable 3-dimensional manifold such that the Betti group $B_1(M)$ is non-trivial. Let γ_j $(j = 1, 2)$ be elements of $H_1(M)$ and let k_j be disjoint knots carrying γ_j. By Theorem 3.1 of (**1**), the torsion group $T_1(M-k_j)$ depends only upon γ_j. Suppose that γ_j are such that $T_1(M-k_j)$ are trivial. The Alexander polynomial $\Delta(M-k_1-k_2)$ is an element of the integral group ring $JB_1(M-k_1-k_2)$. It is defined up to a multiplicative unit $\pm u$, where $u \in B_1(M-k_1-k_2)$. Denote by

$$**\colon\ JB_1(M-k_1-k_2) \to JH_1(M)$$

the homomorphism induced by the injection $M-k_1-k_2 \to M$. This is well-defined since we have the composition

$$M-k_1-k_2 \to M-k_j \to M$$

and $H_1(M-k_j)$ is assumed torsion-free.

THEOREM 1.1. *The image under* $**$ *of* $\Delta(M-k_1-k_2)$, *defined up to a multiplicative unit* $\pm v$, *where* $v \in H_1(M)$, *depends only upon* γ_1 *and* γ_2.

Proof. Let k_1' be another knot carrying γ_1 and disjoint from k_2; k_1' may also be assumed disjoint from k_1 since a polygonal knot carrying γ_1 and disjoint from k_1, k_1', and k_2 is easily constructed.

Case I: $\gamma_1 \in T_1(M)$. Let m, n be knots disjoint from k_1, k_1', k_2 and each other, and having the properties

(i) the cycles carried by m and by n are of infinite order in $H_1(M)$;

(ii) the cycles carried by k_1 and k_1' are homologous and of infinite order in $H_1(M-m-n)$. Such a pair m, n does exist, as is shown in the proof of Theorem 3.2 (Case II b) of (**1**). Consider the commutative diagram

$$\begin{tikzcd}
 & B_1(M-k_1-k_2-m-n) \arrow[rr] \arrow[dr, "i"] & & B_1(M-k_1-k_2) \arrow[dr] & \\
B_1(M-k_1-k_1'-k_2-m-n) \arrow[ur] \arrow[dr] & & \bar{H}_1(M-m-n) \arrow[rr] & & H_1(M), \\
 & B_1(M-k_1'-k_2-m-n) \arrow[rr] \arrow[ur, "i'"] & & B_1(M-k_1'-k_2) \arrow[ur] &
\end{tikzcd}$$

where all homomorphisms are induced by injections, and $\bar{H}_1(M-m-n)$

is the quotient group of $H_1(M-m-n)$ by the subgroup generated by the images

$$iT_1(M-k_1-k_2-m-n) \quad \text{and} \quad i'T_1(M-k_1'-k_2-m-n).$$

In the remainder of this proof we omit symbols for homomorphisms induced by injection.

By Theorem 2.11 of (**1**), we have

$$\Delta(M-k_1-k_1'-k_2-m-n) = \delta(k_1')\,\Delta(M-k_1-k_2-m-n)$$

in $JB_1(M-k_1-k_2-m-n)$, where $\delta(k_1') = 1-k_1'$, and

$$\Delta(M-k_1-k_1'-k_2-m-n) = \delta(k_1)\,\Delta(M-k_1'-k_2-m-n)$$

in $JB_1(M-k_1'-k_2-m-n)$; hence

$$\delta(k_1')\,\Delta(M-k_1-k_2-m-n) = \delta(k_1)\,\Delta(M-k_1'-k_2-m-n)$$

in $J\bar{H}_1(M-m-n)$, and, since $\delta(k_1) = \delta(k_1')$ is not a zero divisor of $J\bar{H}_1(M-m-n)$,

$$\Delta(M-k_1-k_2-m-n) = \Delta(M-k_1'-k_2-m-n)$$

in $J\bar{H}_1(M-m-n)$. Again, by two successive applications of the theorem,

$$\Delta(M-k_1-k_2-m-n) = \delta(m)\delta(n)\,\Delta(M-k_1-k_2)$$

in $JB_1(M-k_1-k_2)$, and

$$\Delta(M-k_1'-k_2-m-n) = \delta(m)\delta(n)\,\Delta(M-k_1'-k_2)$$

in $JB_1(M-k_1'-k_2)$. Hence

$$\delta(m)\delta(n)\,\Delta(M-k_1-k_2) = \delta(m)\delta(n)\,\Delta(M-k_1'-k_2)$$

in $JH_1(M)$, and, since $\delta(m)$, $\delta(n)$ are not zero divisors of $JH_1(M)$,

$$\Delta(M-k_1-k_2) = \Delta(M-k_1'-k_2)$$

in $JH_1(M)$.

Case II: $\gamma_1 \notin T_1(M)$. Then

$$\Delta(M-k_1-k_1'-k_2) = \delta(k_1')\,\Delta(M-k_1-k_2)$$

in $JB_1(M-k_1-k_2)$, and

$$\Delta(M-k_1-k_1'-k_2) = \delta(k_1)\,\Delta(M-k_1'-k_2)$$

in $JB_1(M-k_1-k_2)$. Hence

$$\delta(k_1')\,\Delta(M-k_1-k_2) = \delta(k_1)\,\Delta(M-k_1'-k_2)$$

in $JH_1(M)$, and, since $\delta(k_1) = \delta(k_1') = 1-\gamma_1$ is not a zero divisor of $JH_1(M)$,

$$\Delta(M-k_1-k_2) = \Delta(M-k_1'-k_2)$$

in $JH_1(M)$.

We define $$\Delta^{**}(\gamma_1, \gamma_2) = **(\Delta(M-k_1-k_2)).$$

2. No singular fibres

Let $M = (Oo;\, h\,|\,p)$. Using the notation of [(6) § 10], we let $C_i = A_{\frac{1}{2}(i+1)}$ for i odd and $C_i = B_{\frac{1}{2}i}$ for i even $(i = 1, 2, \ldots, 2h)$. Then $\pi_1(M)$ is presented by

$$(C_i, H, Q_0 \mid [C_i, H], [Q_0, H], Q_0 F^{-1}, Q_0 H^p),$$

where $F = \prod_{i=1}^{h} [C_{2i-1}, C_{2i}]$. After Tietze transformations, this becomes

$$(C_i, H \mid [C_i, H], FH^p).$$

Omitting commutators we have

$$H_1(M) = (C_i, H \mid H^p).$$

Thus $p_1(M) = h$, $T_1(M) = Z_p$, and so h and p are topological invariants. The Alexandrian is

$$\begin{array}{c}
\begin{array}{ccccc} C_1 & C_2 & \cdots & C_{2h} & H \end{array} \\
\left\| \begin{array}{ccccc}
 & & & & C_1-1 \\
 & & & & \vdots \\
 & & & & C_{2h}-1 \\
1-C_2 & C_1-1 & \cdots & C_{2h-1}-1 & p
\end{array} \right\|
\end{array}$$

(in this and other matrices, blank spaces will indicate zeros). Let $\mathfrak{J}$ denote the fundamental ideal of $JB_1(M)$. Then the non-trivial elementary ideals are $\mathfrak{E}_{2h} = (\mathfrak{J}, p)$ and $\mathfrak{E}_{2h-1} = \mathfrak{J}^2$. Notice that, if $p = 0$, it follows from Lemma 1.2 of (1) that $\mathfrak{E}_1 = 0$ and hence the Alexander polynomial $\Delta_1 = 0$.

3. One singular fibre

Let $M = (Oo;\, h\,|\,b;\, \alpha, \beta)$. Then $\pi_1(M)$ is presented by

$$\begin{aligned}
&(C_i, H, Q_0, Q_1 \mid [C_i, H], [Q_0, H], [Q_1, H], Q_0 H^b, Q_1^\alpha H^\beta, Q_0 Q_1 F^{-1}) \\
&\qquad \to (C_i, H, Q_1, R \mid [C_i, R^\alpha], FR^{b\alpha+\beta}, HR^{-\alpha}, Q_1 R^\beta) \\
&\qquad \to (C_i, R \mid [C_i, R^\alpha], FR^{b\alpha+\beta})
\end{aligned}$$

by Tietze transformations. Omitting commutators we write

$$H_1(M) = (C_i, R \mid R^p),$$

and therefore h and $p = |b\alpha+\beta|$ are invariants, and $p \neq 0$ because $\alpha > \beta > 0$. The Alexandrian of $\pi_1(M)$ is

$$\begin{array}{c}
\begin{array}{ccccc} C_1 & C_2 & \ldots & C_{2h} & R \end{array} \\
\left\| \begin{array}{ccccc}
 & & & & \alpha(C_1-1) \\
 & & & & \vdots \\
 & & & & \alpha(C_{2h}-1) \\
1-C_2 & C_1-1 & \ldots & C_{2h-1}-1 & b\alpha+\beta
\end{array} \right\|.
\end{array}$$

Let $\mathfrak{J}$ denote the fundamental ideal of $JB_1(M)$. The only non-trivial elementary ideals are $\mathfrak{E}_{2h-1} = \alpha.\mathfrak{J}^2$ and $\mathfrak{E}_{2h} = (\mathfrak{J}, p)$. Therefore α is an invariant. Furthermore, $\mathfrak{E}_{2h-1}$ distinguishes the spaces $(Oo; h \mid b; \alpha, \beta)$ from the spaces $(Oo; h \mid b)$ since α is assumed to be greater than 1.

Conversely, suppose h, p, and α given, $(\alpha, p) = 1$, and let b and β be such that $p = |b\alpha+\beta|$. The *class space* $\bar{F}_0$ of the spaces $(Oo; h \mid b; \alpha, \beta)$ is the topological product of a circle and a perforated surface of genus h. Let H and Q_0 be a naturally chosen longitude and meridian, respectively, on the torus $\dot{\bar{F}}_0$ bounding $\bar{F}_0$. Let V be an anchor ring and M_1 its meridian. Then it is clear by the discussion of [(6) 183] that $(Oo; h \mid b; \alpha, \beta)$ is obtained by matching $\dot{\bar{F}}_0$ with $\dot{V}$ so that $\alpha Q_0 + pH \to M_1$. This proves the theorem:

THEOREM 3.1. *The numbers* h, α, $|b\alpha+\beta|$ *are a complete set of topological invariants for the (unoriented) spaces* $(Oo; h \mid b; \alpha, \beta)$.

4. Two singular fibres

Let $M = (Oo; h \mid b; \alpha_1, \beta_1; \alpha_2, \beta_2)$. Then $\pi_1(M)$ is presented by

$$\begin{aligned}
&(C_i, H, Q_0, Q_1, Q_2 \mid [C_i, H], [Q_0, H], [Q_1, H], [Q_2, H], Q_0 Q_1 Q_2 F^{-1}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad Q_0 H^b, Q_1^{\alpha_1} H^{\beta_1}, Q_2^{\alpha_2} H^{\beta_2}) \\
&\quad \to (C_i, H, Q_1, Q_2, R_1, R_2 \mid [C_i, H], Q_1 R_1^{\beta_1}, Q_2 R_2^{\beta_2}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad H R_1^{-\alpha_1}, H R_2^{-\alpha_2}, R_1^{\beta_1} H^b F R_2^{\beta_2}) \\
&\quad \to (C_i, H, R_1, R_2 \mid [C_i, H], H R_1^{-\alpha_1}, H R_2^{-\alpha_2}, R_1^{\beta_1} H^b F R_2^{\beta_2})
\end{aligned}$$

and

$$\begin{aligned}
H_1(M) &= (C_i, H, R_1, R_2 \mid H R_1^{-\alpha_1}, H R_2^{-\alpha_2}, R_1^{\beta_1} H^b R_2^{\beta_2}) \\
&\qquad\qquad \to (C_i, R_1 \mid R_1^p) = (C_i, R_2 \mid R_2^p).
\end{aligned}$$

Hence h and $p = |b\alpha_1\alpha_2 + \beta_2\alpha_1 + \beta_1\alpha_2|$ are invariants.

Note that the centre of $\pi_1(M)$ is generated by the element H. For H is certainly in the centre, and the quotient group $\pi_1(M)/(H)$ has the presentation
$$(C_i, R_1, R_2 \mid R_1^{\alpha_1}, R_2^{\alpha_2}, R_2^{\beta_2} R_1^{\beta_1} F),$$

which is the free product of the free group $X = (C_i)$ and the group $(R_1, R_2 \mid R_1^{\alpha_1}, R_2^{\alpha_2})$, with amalgamated infinite cyclic subgroups (F) and $(R_1^{-\beta_1} R_2^{-\beta_2})$ respectively. It is well known that a non-trivial amalgamated free product (i.e. the amalgamated subgroup is a proper subgroup of each factor) has no centre if at least one of its factors has no centre [see e.g. (4)]. Since X has no centre, $\pi_1(M)/(H)$ has no centre.

For future reference, we prove that, if $(\alpha_1, \alpha_2) = 1$, then H is of infinite order in $\pi_1(M)$: for in that case, by adjoining the relator $[R_1, R_2]$, we obtain the group

$$G = (C_i, R \mid [C_i, R^{\alpha_1 \alpha_2}], R^p F),$$

where $R_1 = R^{\alpha_2}$, $R_2 = R^{\alpha_1}$, and this in turn maps homomorphically onto

$$(C_i, R \mid [C_i, R], R^p F, [C_i, [C_j, C_k]], i, j, k = 1, \ldots, 2h),$$

which is the quotient group of the direct sum $(X/X_3) \circ (R)$ by the normal subgroup generated by the element $F \circ R^p$. The elements of this normal subgroup are of the form $F^n \circ R^{pn}$ since F is in the centre of X/X_3. Now F is of infinite order in X_2/X_3 [see (**2**)]. Hence F is certainly of infinite order in X/X_3; therefore none of the elements $F^n \circ R^{pn}$ can be a non-trivial power of R. Hence R is of infinite order in G and so is H, which maps onto $R^{\alpha_1 \alpha_2}$.

We shall now prove that $\alpha = (\alpha_1, \alpha_2)$ is an invariant. First consider the case $p \neq 0$. Then H, R_1, and R_2 represent the identity of $B_1(M)$. The Alexandrian of $\pi_1(M)$ reads

$$\begin{array}{ccccccc} C_1 & \cdot\;\cdot\;\cdot & C_{2h} & R_1 & R_2 & H \end{array}$$
$$\left\| \begin{array}{cccccc} & & & & & C_1 - 1 \\ & & & & & \cdot \\ & & & & & \cdot \\ & & & & & \cdot \\ & & & & & C_{2h} - 1 \\ 1 - C_2 & \cdot\;\cdot\;\cdot & C_{2h-1} - 1 & \beta_1 & \beta_2 & b \\ & & & \alpha_1 & 0 & -1 \\ & & & 0 & \alpha_2 & -1 \end{array} \right\|.$$

The non-trivial elementary ideals are

$$\mathfrak{E}_{2h-1} = \alpha_1 \alpha_2 \mathfrak{F}^2 \quad \text{and} \quad \mathfrak{E}_{2h} = (\alpha \mathfrak{F}, p).$$

Hence the residue $\alpha \pmod p$ is an invariant. Since α divides p, α itself is an invariant.

Returning to the case $p = 0$, the conditions

$$\alpha_1 > \beta_1 > 0, \quad \alpha_2 > \beta_2 > 0, \quad (\alpha_1, \beta_1) = (\alpha_2, \beta_2) = 1$$

require that $\quad \alpha_1 = \alpha_2 = \alpha, \quad b = -1, \quad \alpha = \beta_1 + \beta_2.$

The last presentation of $\pi_1(M)$ becomes

$$(C_i, H, R_1, R_2 \mid [C_i, H], HR_1^{-\alpha}, HR_2^{-\alpha}, H^{-1}FR_2^{\beta_2}R_1^{\beta_1})$$
$$\to (C_i, R_1, R_2 \mid [C_i, R_2^{\alpha}], R_1^{\alpha}R_2^{-\alpha}, FR_2^{\beta_2}R_1^{-\beta_1})$$

and
$$R_1 = R_2 = R \quad \text{in} \quad H_1(M).$$

Denoting by $\phi_n(x)$ the polynomial $(1-x^n)/(1-x)$, we can then write the Alexandrian as

$$\begin{array}{c} \begin{array}{ccccccc} C_1 & \cdot & \cdot & \cdot & C_{2h} & R_1 & R_2 \end{array} \\ \left\| \begin{array}{ccccccc} 1-R^{\alpha} & & & & & & (C_1-1)\phi_\alpha(R) \\ & \cdot & & & & & \cdot \\ & & \cdot & & & & \cdot \\ & & & \cdot & & & \cdot \\ & & & & 1-R^{\alpha} & & (C_{2h}-1)\phi_\alpha(R) \\ & & & & & \phi_\alpha(R) & -\phi_\alpha(R) \\ 1-C_2 & \cdot & \cdot & \cdot & C_{2h-1}-1 & \phi_{\beta_1}(R) & -\phi_{\beta_2}(R) \end{array} \right\|. \end{array}$$

The order ideal $\mathfrak{E}_0$ is zero since $H_1(M)$ is infinite. By Lemma 1.2 of (**1**), $(R-1)\mathfrak{E}_1 = \mathfrak{J}\mathfrak{E}_0'$, where $\mathfrak{E}_0'$ is the ideal generated by the $(2h+1)$th order determinants of the first $2h+1$ columns. Thus

$$\mathfrak{E}_0' = (1-R^{\alpha})^{2h}(\phi_\alpha(R), \phi_{\beta_1}(R)),$$
$$(R-1)\mathfrak{E}_1 = (1-R^{\alpha})^{2h}(\phi_\alpha(R), \phi_{\beta_1}(R))\mathfrak{J}.$$

Hence the Alexander polynomial Δ_1 satisfies

$$(R-1)\Delta_1 = (1-R^{\alpha})^{2h}.$$

This proves that α is also an invariant in the case $p = 0$.

We now assume that $\alpha = 1$, and that $p \neq 1$. In that case

$$(\alpha_1, p) = (\alpha_1, \alpha_2) = 1,$$

and $T_1(M)$ is generated by $H = R_1^{\alpha_1}$. We proceed to calculate the invariant Δ^{**} for the pair H^{n_j}, $(n_j, p) = 1$ $(j = 1, 2)$. Let T_j $(j = 1, 2)$ be disjoint tubular neighbourhoods of non-singular fibres. Let k_j be a torus knot of type $\{n_j, 1\}$ in T_j. Then $\pi_1(T_j - k_j)$ is presented by

$$(H, y_j, z_j \mid [y_j, H], H^{n_j}y_j z_j^{-n_j}),$$

$\pi_1(M - T_1 - T_2)$ by

$$(C_i, H, Q_{0j}, Q_1, Q_2 \mid [C_i, H], [Q_{0j}, H], [Q_1, H], [Q_2, H],$$
$$Q_{01}Q_{02}Q_1Q_2F^{-1}, Q_1^{\alpha_1}H^{\beta_1}, Q_2^{\alpha_2}H^{\beta_2})$$
$$\to (C_i, H, Q_{0j}, R_1, R_2 \mid [C_i, H], [Q_{0j}, H], Q_{01}Q_{02}R_1^{-\beta_1}R_2^{-\beta_2}F^{-1},$$
$$H^{-1}R_1^{\alpha_1}, H^{-1}R_2^{\alpha_2}),$$

and $\pi_1(M - k_1 - k_2)$ by generators

$$C_i,\ H,\ Q_{0j},\ R_1,\ R_2,\ y_j,\ z_j$$

and relators

$$[C_i, H],\ [Q_{0j}, H],\ [y_j, H],\ H^{-1}R_1^{\alpha_1},\ H^{-1}R_2^{\alpha_2},\ H^{n_j}y_j z_j^{-n_j},$$
$$y_j^{-1}Q_{0j}H^{b(j-1)},\ Q_{01}\,Q_{02}\,R_1^{-\beta_1}R_2^{-\beta_2}F^{-1},$$

and thus, after Tietze transformations, by

$$(C_i, H, R_1, R_2, z_j \mid [C_i, H], [z_1^{n_1}, H], H^{-1}R_1^{\alpha_1}, H^{-1}R_2^{\alpha_2},$$
$$z_1^{n_1}z_2^{n_2}R_1^{-\beta_1}R_2^{-\beta_2}H^{-n_1-n_2-b}F^{-1}).$$

Omitting commutators, we write $H_1(M-k_1)$, say, as

$$(C_i, H, R_1, R_2, z_1 \mid z_1^{-n_1}R_2^{\beta_2}R_1^{\beta_1}H^{b+n_1},\ R_1^{\alpha_1}H^{-1},\ R_2^{\alpha_2}H^{-1}),$$

and, if $(n_j, p) = 1$, one can calculate that $H_1(M-k_j)$ is free abelian. The Alexandrian of $\pi_1(M-k_1-k_2)$ can be written

$$\begin{array}{c}
\begin{matrix} C_1 & \cdot & \cdot & \cdot & C_{2h} & R_1 & R_2 & z_1 & z_2 & H \end{matrix}\\
\left\| \begin{matrix}
1-H & & & & & & & & & \cdot \\
 & \cdot & & & & & & & & \cdot \\
 & & \cdot & & & & & & & \cdot \\
 & & & \cdot & & & & & & \cdot \\
 & & & & 1-H & & & & & \cdot \\
 & & & & & \phi_{\alpha_1}(R_1) & & & & \cdot \\
 & & & & & & \phi_{\alpha_2}(R_2) & & & \cdot \\
 & & & & & & & (1-H)\phi_{n_1}(z_1) & & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \phi_{n_2}(z_2) & \cdot
\end{matrix} \right\|,
\end{array}$$

where the remaining non-zero entries are confined to the last row and last column, and are irrelevant to our calculation. By Lemma 1.1 of (**1**),

$$\Delta(M-k_1-k_2) = eu\,(1-H)^{2h}\,\phi_{\alpha_1}(R_1)\,\phi_{\alpha_2}(R_2)\,\phi_{n_1}(z_1)\,\phi_{n_2}(z_2),$$

where $e = \pm 1$ and $u \in B_1(M-k_1-k_2)$, and

$$\Delta^{**}(H^{n_1}, H^{n_2}) = eu^*\,(1-H)^{2h}\,\phi_{\alpha_1}(H^{\bar{\alpha}_1})\,\phi_{\alpha_2}(H^{\bar{\alpha}_2})\,\phi_{n_1}(H)\,\phi_{n_2}(H),$$

where $\alpha_j\,\bar{\alpha}_j \equiv 1 \pmod p$.

Now $H^{\pm 1}$ are *distinguished generators* of $T_1(M)$ since they are the images of the generators of the centre of $\pi_1(M)$. Therefore it is topologically meaningful to ask: 'what are the pairs $\{n_1, n_2\}$ such that $\Delta^{**}(H^{n_1}, H^{n_2}) = (1-H)^{2h}$?' Clearly the unit u^* must be of the form H^ν. The condition is

$$eH^\nu\,(1-H)^{2h}\,\phi_{\alpha_1}(H^{\bar{\alpha}_1})\,\phi_{\alpha_2}(H^{\bar{\alpha}_2})\,\phi_{n_1}(H)\,\phi_{n_2}(H) = (1-H)^{2h} \pmod{H^p-1},$$

or, when we multiply by units of $JH_1(M)$,

$$eH^\nu\,(1-H)^{2h}\,\phi_{n_1}(H)\,\phi_{n_2}(H) = (1-H)^{2h}\,\phi_{\bar{\alpha}_1}(H)\,\phi_{\bar{\alpha}_2}(H) \pmod{H^p-1}.$$

This is true only if

$$eH^\nu\,\phi_{n_1}(H)\,\phi_{n_2}(H) = \phi_{\bar{\alpha}_1}(H)\,\phi_{\bar{\alpha}_2}(H) \quad (\text{mod } (H^p-1)/(H-1));$$

multiplying by $(1-H)^2$, we have

$$eH^{\nu}(H^{n_1}-1)(H^{n_2}-1) = (H^{\bar{\alpha}_1}-1)(H^{\bar{\alpha}_2}-1) \pmod{H^p-1}.$$

First suppose that $e = -1$. Expanding, we have

$$H^{\bar{\alpha}_1}+H^{\bar{\alpha}_2}+H^{n_1+\nu}+H^{n_2+\nu} = H^{\nu}+H^{n_1+n_2+\nu}+H^{\bar{\alpha}_1+\bar{\alpha}_2}+1,$$

and the exponents on opposite sides must cancel in pairs $(\operatorname{mod} p)$. The only possibility is $n_1+\nu \equiv 0$, say, and then

$$H^{\bar{\alpha}_1}+H^{\bar{\alpha}_2}+H^{n_2-n_1} = H^{-n_1}+H^{n_2}+H^{\bar{\alpha}_1+\bar{\alpha}_2}.$$

Therefore $n_1 \equiv -\bar{\alpha}_1$, say, and $n_2 \equiv \bar{\alpha}_2$.

Suppose that $e = 1$. Then

$$H^{n_1+n_2+\nu}+H^{\nu}+H^{\bar{\alpha}_1}+H^{\bar{\alpha}_2} = H^{\bar{\alpha}_1+\bar{\alpha}_2}+H^{n_1+\nu}+H^{n_2+\nu}+1.$$

Either $n_1+n_2+\nu \equiv 0$ or $\nu \equiv 0$. In the former case,

$$H^{-n_1-n_2}+H^{\bar{\alpha}_1}+H^{\bar{\alpha}_2} = H^{\bar{\alpha}_1+\bar{\alpha}_2}+H^{-n_1}+H^{-n_2},$$

and so $n_1 \equiv -\bar{\alpha}_1$, say, and $n_2 \equiv -\bar{\alpha}_2$. If $\nu \equiv 0$,

$$H^{n_1+n_2}+H^{\bar{\alpha}_1}+H^{\bar{\alpha}_2} = H^{\bar{\alpha}_1+\bar{\alpha}_2}+H^{n_1}+H^{n_2},$$

and so $n_1 \equiv \bar{\alpha}_1$, say, and $n_2 \equiv \bar{\alpha}_2$.

Therefore the pairs of homology classes satisfying the condition are $\{H^{\pm\bar{\alpha}_1}, H^{\pm\bar{\alpha}_2}\}$.

THEOREM 4.1. *The residue classes of* $\pm\alpha_1$, $\pm\alpha_2$ $(\operatorname{mod} p)$ *are invariants of those spaces* $(Oo; h \mid b; \alpha_1, \beta_1; \alpha_2, \beta_2)$ *for which* $p \neq \alpha$.

Proof. We have proved the theorem for the spaces satisfying the additional condition $\alpha = 1$. Suppose that $\alpha > 1$. Let

$$\alpha_1 = \alpha_1' \alpha, \qquad \alpha_2 = \alpha_2' \alpha.$$

Consider all possible subgroups $B_1^k(M) \subset H_1(M)$ such that $H_1(M)$ is the direct sum $B_1^k(M) \circ T_1(M)$. Let $\psi \colon \pi_1(M) \to H_1(M)$ be the abelianizing homomorphism and let $T_1^{\alpha}(M) = (R_1^{\alpha})$ be the subgroup of index α of $T_1(M)$. Consider the subgroup

$$G^k = \psi^{-1}(B_1^k(M) \circ T_1^{\alpha}(M)),$$

of index α in $\pi_1(M)$. The corresponding covering space $U^k(M)$ is a fibred space [(6) § 9] of the form

$$(Oo; h' \mid b'; \alpha_1', \beta_1'; \alpha_2', \beta_2'),$$

because $H \in G^k$. Thus to each M considered there is a topologically defined class of covering spaces $U^k(M)$ for which the multiplicities α_1', α_2' are relatively prime. The torsion number p' of $U^k(M)$ is a multiple of p/α. For H, which generates $T_1(U^k(M))$, maps into $H = R_1^{\alpha_1}$, which is of order p/α in $T_1(M)$. Since the residue classes of α_1' and α_2' $(\operatorname{mod} p')$

are invariants, their classes $(\bmod\, p/\alpha)$ are invariants. Hence the classes of α_1 and $\alpha_2 \,(\bmod\, p)$ are invariants.

We add the remark that $p = \alpha$ is possible only if $b = -1$.

THEOREM 4.2. *The Betti number h, the torsion number*

$$p = |b\alpha_1\alpha_2+\beta_2\alpha_1+\beta_1\alpha_2|,$$

and the unordered pair $\{\alpha_1, \alpha_2\}$ are a complete set of topological invariants for the (unoriented) spaces $(Oo; h \mid b; \alpha_1, \beta_1; \alpha_2, \beta_2)$ satisfying $(\alpha_1, \alpha_2) = 1$ and $p > 1+\max\{\alpha_1, \alpha_2\}$.

Proof. The numbers α_1, α_2 are then determined by the residue classes of $\pm\alpha_1$, $\pm\alpha_2$. For a given triple $\{p, \alpha_1, \alpha_2\}$ with $(\alpha_1, \alpha_2) = 1$, the equation $p = |b\alpha_1\alpha_2+\beta_2\alpha_1+\beta_1\alpha_2|$ has at most two solutions $\{b, \beta_1, \beta_2\}$ for which $\alpha_1 > \beta_1 > 0$, $\alpha_2 > \beta_2 > 0$. The fibred spaces corresponding to these two solutions differ only in orientation [(6) 184].

The spaces $(Oo; h \mid b; \alpha_1, \beta_1; \alpha_2, \beta_2)$ are distinguished from the spaces $(Oo; h \mid p)$ by the elementary ideal $\mathfrak{E}_{2h-1}$ if $p \neq 0$, and by the Alexander polynomial Δ_1 if $p = 0$. Can a space $M_2 = (Oo; h \mid b; \alpha_1, \beta_1; \alpha_2, \beta_2)$ be homeomorphic to a space of the form $M_1 = (Oo; h \mid b'; \alpha', \beta')$? (The referee has pointed out that $(Oo; h \mid b; 3, 1; 4, 1)$ and $(Oo; h \mid b; 12, 7)$, for example, have isomorphic fundamental groups.) We can supply a partial answer as follows. If we compare the values of $\mathfrak{E}_{2h-1}$, homeomorphism would require $\alpha' = \alpha_1\alpha_2$. If we use $\mathfrak{E}_{2h}$, $(\alpha\mathfrak{F}, p) = (\mathfrak{F}, p)$, and hence $\alpha \equiv 1 \,(\bmod\, p)$; therefore $\alpha = 1$. Also $p \neq 0$ since $\alpha' > \beta' > 0$. If $p \neq 1$, then, the calculation of the invariant Δ^{**} proceeds as before if we regard M_1 as a space with $r = 2$ and $\alpha_2 = 1$, $\beta_2 = 0$. The previous argument showing that H was a distinguished generator of $T_1(M_2)$ breaks down in the case of M_1. However, we know there is some distinguished generator (call it $K = H^q$, $(q, p) = 1$), and that

$$\Delta^{**}(K^{\bar{\alpha}_1}, K^{\bar{\alpha}_2}) = (1-K)^{2h}.$$

Substituting $K = H^q$, and using the above expression for Δ^{**} in terms of H, we have, after simplifying, the condition

$$\pm u\,(1-H^q)^{2h} \equiv (1-H)^{2h} \,(\bmod\, H^p-1) \quad (u \in H_1(M)).$$

Clearly u is of the form H^ν. Mapping H into θ, a primitive pth root of unity, we have

$$\phi_q^{2h}(\theta) = \pm\theta^\nu,$$

which is impossible unless $q \equiv 1 \,(\bmod\, p)$, for otherwise $|\phi_q(\theta)| > 1$. Hence H is indeed a distinguished generator. Thus Theorem 4.1 also applies in our broader sense, i.e. allowing $\alpha_2 = 1$, $\beta_2 = 0$. Therefore

THEOREM 4.3. *If $p \neq 1$, the space* $(Oo;\, h \mid b;\, \alpha_1, \beta_1;\, \alpha_2, \beta_2)$ $(\alpha_i > \beta_i > 0)$ *is not homeomorphic to any space* $(Oo;\, h \mid b';\, \alpha', \beta')$.

5. Open questions

How does the homotopy classification of the Seifert spaces compare with the results found here and, more generally, with the fibre classification?

Since the fundamental groups of the Seifert spaces Oo are non-abelian for $h > 0$, the Reidemeister torsion is inapplicable. Can the theory of simple homotopy types (**7**) yield the information derived here (assuming that the homotopy classification could not)?

The examples of non-fibreable 3-manifolds given by Seifert (**6**) are topological sums of lens spaces, the latter being fibreable. Can every 3-manifold be decomposed into a sum of fibreable spaces? Conversely, does there exist a non-trivial (i.e. not including S^3) sum of fibred spaces which can be fibred? An affirmative answer to the former question, and a negative answer to the latter, together with Milnor's results (**5**) would bring the 3-manifold classification problem significantly closer to solution (*modulo* the Poincaré conjecture), provided that the results of the present paper could be generalized to all fibred spaces.

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