Cobordism theories

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1. Introduction

Since 1954 when Thom introduced cobordism theory, much attention has been given to cobordism theories corresponding to subgroups of the orthogonal group O. One of the main results in each of these cases (except the identity subgroup) is that a manifold is cobordant to zero if and only if certain characteristic numbers vanish. Also, one has been able to describe quite completely the cobordism ring associated to the given subgroup (e.g., see [11] and [12] for the groups O and SO).

In this paper we shall study cobordism theories which contain the ordinary differentiable cobordism (corresponding to the groups O and SO). The particular examples to keep in mind are the piecewise linear cobordism theories. The techniques we will employ will be of a completely homotopy theoretic nature to study the Thom complex and the classifying space associated to a given theory. Thus they will apply to more general situations where such Thom complexes and classifying spaces exist, but where they are not known to have interpretations in a cobordism theory.

In particular, we will assume given a classifying space BG (for some stable fibre space theory) and a Thom spectrum MG associated with it (see § 2 for details). We assume a map $BG \times BG \to BG$ (*Whitney sum*) inducing $MG \wedge MG \to MG$, and a map $BO \to BG$ with $MO \to MG$ induced by it, commuting with *Whitney sum*. Define $\mathfrak{N}_*^G = \pi_*(MG)$, the homotopy groups of the spectrum. If $G = \mathrm{O}$ or PL (i.e., BPL is the classifying space for piecewise linear microbundles), then $\mathfrak{N}_*^G$ is a cobordism ring (see [11] and [16]). The case where $BG = BF$, the classifying space for stable spherical fibre spaces, is a case of interest where no such interpretation is known.

Our first main theorem is that $H^*(MG; Z_2)$ is a free left module over the mod 2 Steenrod algebra $\mathbf{A}$, and that the Hurewicz homomorphism $h: \pi_*MG \to H_*(MG; Z_2)$ is a monomorphism. It follows that if $[M] \in \mathfrak{N}_*^{\mathrm{PL}}$, $[M] = 0$ if and only if all characteristic numbers associated to elements in $H^*(B\mathrm{PL}; Z_2)$ vanish.

Next we show that there is a Hopf algebra $C(G)$ over Z_2 such that $H^*(BG; Z_2) \approx H^*(B\mathrm{O}; Z_2) \otimes C(G)$, as Hopf algebras over the Steenrod algebra,

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and that $\mathfrak{N}_*^G \approx \mathfrak{N}_*^O \otimes C(G)^*$, as algebras. Thus information about $H^*(BG; Z_2)$ gives information abut $\mathfrak{N}_*^G$. In the oriented case, i.e., for BSG the classifying space of the associated oriented bundle theory, we prove that $H^*(BSG; Z_2) \approx H^*(B\mathrm{SO}; Z_2) \otimes C(G)$ as a Hopf algebra, and that if $x \in \pi_*(MSG)$, then x is of odd order if and only if $h_r(x) = 0$ for all r, for $h_r: \pi_*(MSG) \to H_*(MSG; Z_{2^r})$. This yields a corresponding statement in oriented piecewise linear cobordism, that if $[M] \in \Omega_*^{\mathrm{PL}}$, $[M]$ is of odd order if and only if all its characteristic numbers associated to $H^*(BSG; Z_{2^r})$ vanish, for all r.

Next, we calculate the ideal of relations among characteristic classes in $H^*(B\mathrm{PL}; Z_2)$ for $p.l.$ n-manifolds; the answer is analogous to that of Brown and Peterson [5] for the orthogonal case.

Finally, we make some remarks about $C(\mathrm{PL})$. Note, that except for the latter calculations, each theorem stated for $G = \mathrm{PL}$, would hold whenever a suitable t-regularity theorem could be proved for G, analogous to those of [11] and [16].

We state the theorems precisely in §2, and prove them in the later sections.

2. Statements of results

We shall always assume the following. We have a sequence of spaces

$$(*) \qquad BG(1) \longrightarrow BG(2) \longrightarrow \cdots \longrightarrow BG(n) \longrightarrow \cdots$$

and maps $B\mathrm{O}(n) \xrightarrow{g_n} BG(n) \xrightarrow{h_n} BF(n)$ such that $h_n g_n = j_n$, where $BF(n)$ is the classifying space for fibre spaces with fibre the homotopy type of S^{n-1} and j_n is the classifying map of the canonical S^{n-1} bundle over $B\mathrm{O}(n)$. We will denote by ξ_n the classifying fibre space over $BF(n)$, and let $\eta_n = h_n^*(\xi_n)$, $\gamma_n = j_n^*(\xi_n)$, so that $g_n^*(\eta_n) = \gamma_n$. We will assume the existence of an operation of *Whitney sum* for G-bundles, i.e., there is a map $\mu: BG(n) \times BG(m) \to BG(n + m)$ which restricted to either factor is the map coming from our sequence $(*)$, and such that the diagrams

$$\begin{tikzcd}
B\mathrm{O}(n) \times B\mathrm{O}(m) \arrow[r] \arrow[d] & BG(n) \times BG(m) \arrow[r] \arrow[d] & BF(n) \times BF(m) \arrow[d] \\
B\mathrm{O}(n + m) \arrow[r] & BG(n + m) \arrow[r] & BF(n + m)
\end{tikzcd}$$

$$\begin{tikzcd}
 & BG(n+1) \times BG(m) \arrow[dr] & & \\
BG(n) \times BG(m) \arrow[ur] \arrow[r] \arrow[dr] & BG(n + m) \arrow[r] & BG(n + m + 1) \\
 & BG(n) \times BG(m + 1) \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}
BG(n) \times BG(m) \arrow[rr] \arrow[dr] & & BG(m) \times BG(n) \arrow[dl] \\
 & BG(n + m) &
\end{tikzcd}$$

commute up to homotopy.

Under these assumptions, we may define the Thom complex of η_n, $MG(n)$ to be the mapping cone of the projection map of η_n, so that the Thom isomorphism holds[1]:

$$\Phi^G\colon H^i(BG(n)) \xrightarrow{\approx} H^{i+n}(MG(n)) ,$$

and there are natural maps of the suspension $\Sigma MG(n) \to MG(n+1)$ and $MG(n) \wedge MG(m) \to MG(n+m)$. Thus $MG(n)$ is a ring spectrum (see [15]), denoted by MG. Then define $\pi_k(MG) = \operatorname{dir\,lim}_{n\to\infty} \pi_{k+n}(MG(n))$ and $H_k(MG) = \operatorname{dir\,lim}_{n\to\infty} H_{k+n}(MG(n))$. If we define $BG = \lim BG(n)$ with the weak topology, then we have the Thom isomorphism $\Phi^G\colon H^k(BG) \to H^k(MG)$, and we have a commutative diagram:

$$\begin{array}{ccccc}
\pi_*(MG) \otimes \pi_*(MG) & \longrightarrow & \pi_*(MG \wedge MG) & \longrightarrow & \pi_*(MG) \\
\downarrow & & \downarrow & & \downarrow \\
H_*(MG) \otimes H_*(MG) & \xrightarrow{\approx} & H_*(MG \wedge MG) & \longrightarrow & H_*(MG) \\
{\scriptstyle \Phi^* \otimes \Phi^*}\downarrow{\scriptstyle \approx} & & {\scriptstyle \Phi^*}\downarrow{\scriptstyle \approx} & & {\scriptstyle \Phi^*}\downarrow{\scriptstyle \approx} \\
H_*(BG) \otimes H_*(BG) & \xrightarrow{\approx} & H_*(BG \times BG) & \longrightarrow & H_*(BG)
\end{array}$$

where $MG \wedge MG$ denotes the spectrum defined by $MG(n) \wedge MG(m)$. We shall assume that $H_k(BG; Z)$ is finitely generated for each k, so that we have the commutative diagram

$$\begin{array}{ccccc}
H^*(MG) & \longrightarrow & H^*(MG \wedge MG) & \xrightarrow{\approx} & H^*(MG) \otimes H^*(MG) \\
{\scriptstyle \approx}\uparrow{\scriptstyle \Phi^G} & & {\scriptstyle \approx}\uparrow{\scriptstyle \Phi^G} & & {\scriptstyle \approx}\uparrow{\scriptstyle \Phi^G \otimes \Phi^G} \\
H^*(BG) & \xrightarrow{\psi_G} & H^*(BG \times BG) & \xrightarrow{\approx} & H^*(BG) \otimes H^*(BG)
\end{array}$$

so that $H^*(MG)$ and $H^*(BG)$ are isomorphic commutative coalgebras. The map $BO \xrightarrow{g} BG$ induced by g_n induces $MO \xrightarrow{m} MG$, and $g^*\colon H^*(BG) \to H^*(BO)$ and $m^*\colon H^*(MG) \to H^*(MO)$ are maps of coalgebras over the mod 2 Steenrod algebra $\mathbf{A}$ (g^* actually being a map of Hopf algebras over $\mathbf{A}$). Similarly for $h\colon BG \to BF$ and $n\colon MG \to MF$. Note that Φ^G is *not* a map of modules over $\mathbf{A}$.

Define the Stiefel-Whitney classes in H^*BG (after Thom) by the formula

$$W^G = (\Phi^G)^{-1}(\mathrm{Sq}\, \Phi^G(1)) = 1 + W_1^G + W_2^G + \cdots .$$

Since $\psi_F W^F = W^F \otimes W^F \in H^*(BF) \otimes H^*(BF)$, it follows that

$$\psi_G W^G = W^G \otimes W^G \in H^*(BG) \otimes H^*(BG) .$$

We define the G-bordism groups of K by $\mathfrak{N}_*^G(K) = \pi_*(K \wedge MG)$, where $K \wedge MG$

[1] Unless otherwise stated, all coefficient groups will be assumed to be Z_2.

is the spectrum $K \wedge MG(n)$ ($\pi_*(K \wedge MG) = H_*(K; MG)$ according to G. W. Whitehead [15]). Hence $\mathfrak{N}^G_*(\text{point}) = \pi_*(MG)$. We denote $\mathfrak{N}^G_*$ by $\mathfrak{N}_*$.

In particular, we have

THEOREM 2.1. *$G = \mathrm{PL}$ satisfies all the above hypotheses, and in addition $\mathfrak{N}^{\mathrm{PL}}_*$ is isomorphic to the unoriented piecewise linear cobordism ring.*

This is proved in Williamson [16] and Hirsch [6], [7].

Now we proceed to state our theorems about any BG satisfying the above assumptions (keeping in mind the examples $G = \mathrm{PL}$ or F).

THEOREM 2.2. *$H^*(MG)$ is a free left $\mathbf{A}$-module.*

Then, as in [11], we get

COROLLARY 2.3. *The Hurewicz homomorphism $\pi_*(MG) \to H_*(MG)$ is a monomorphism, and thus any element in $\pi_*(MG)$ is detected by an element in $H^*(MG)$.*

Note that $\pi_*(MG)$ is a Z_2-module, which follows from the fact that there is a map of the Eilenberg-MacLane spectrum $K(Z_2)$ into MG and MG is a ring spectrum. This approach also leads to a direct proof of (2.2) and (2.3) (see [15]). If $\pi_*(MG)$ represents an unoriented cobordism group, then obviously $2[M] = 0$, as in [11].

COROLLARY 2.4. *Let $[M] \in \mathfrak{N}^{\mathrm{PL}}_*$. Then $[M] = 0$ if and only if all characteristic numbers defined by elements of $H^*(B\mathrm{PL})$ vanish.*

Unfortunately, $H^*(B\mathrm{PL})$ is not known. The following two theorems shed some light on the structure of $H^*(BG)$.

THEOREM 2.5. *There is a Hopf algebra over the Steenrod algebra $C(G)$, such that $H^*(BG) \approx H^*(B\mathrm{O}) \otimes C(G)$ as Hopf algebras over the Steenrod algebra.*

$H^*(B\mathrm{O})$ is a free right $\mathbf{A}$-module (see [5]). We define a right $\mathbf{A}$-module structure on $H^*(B\mathrm{O}) \otimes C(G)$ by the formula $(b \otimes c)a = \sum (b)a'_i \otimes \chi(a''_i)(c)$ where $\psi(a) = \sum a'_i \otimes a''_i$ is the diagonal map in $\mathbf{A}$, and χ is the canonical anti-automorphism of the Steenrod algebra (see [10]).

THEOREM 2.6. *The isomorphism of Theorem 2.5 is an isomorphism of right $\mathbf{A}$-modules and $H^*(B\mathrm{O}) \otimes C(G)$ is a free right $\mathbf{A}$-module on generators $\mu_i \otimes c_j$, where $\{\mu_i\}$ is a right $\mathbf{A}$-base for $H^*(B\mathrm{O})$, and $\{c_j\}$ is a vector space basis for $C(G)$.*

$C(G)$ also gives us the structure of $\mathfrak{N}^G_*$ in the following theorem.

THEOREM 2.7. *$\mathfrak{N}^G_* \approx \mathfrak{N}_* \otimes C(G)^*$ as an algebra.*

COROLLARY 2.8. *If $[M], [N] \in \mathfrak{N}^{\mathrm{PL}}_*$, M not cobordant to a C^∞-manifold,*

$[N] \in \mathfrak{N}_* \subset \mathfrak{N}_*^{\mathrm{PL}}$, $[N] \neq 0$, *then* $M \times N$ *is not cobordant to a* C^∞*-manifold.*

We now turn to the orientable case. Let $BSG(n)$ denote the two-fold covering space of $BG(n)$ defined by $W_1(\gamma_n) \in H^1(BG(n))$. Then $BSG(n)$ satisfies the same hypotheses as $BG(n)$ with $\mathrm{O}(n)$ replaced by $\mathrm{SO}(n)$. Let $'\mathbf{A}_p \subset \mathbf{A}_p$, the mod p Steenrod algebra, be the Hopf subalgebra generated by all P^I, where p is an odd prime.

THEOREM 2.9. *$H^*(\mathbf{M}SG)$ is a direct*[2] *sum of copies of* $\mathbf{A}$ *and* $\mathbf{A}/\mathbf{A}(\mathrm{Sq}^1)$. *$H^*(\mathbf{M}SG; Z_p)$ is a free $'\mathbf{A}_p$-module for p odd.*

COROLLARY 2.10. *Let $[M] \in \Omega_*^G = \pi_*(MSG)$. Then $[M]$ is of odd order if and only if all characteristic numbers defined by elements of $H^*(BSG; Z_{2^r})$ vanish.*

THEOREM 2.11. *$H^*(BSG) \approx H^*(B\mathrm{SO}) \otimes C(G)$ as a Hopf algebra over the Steenrod algebra (operating on both the left and right). There is a Hopf algebra over the Steenrod algebra, $C_3(G)$, such that*

$$H^*(BSG; Z_3) \approx H^*(BS\mathrm{O}; Z_3) \otimes C_3(G)$$

as a Hopf algebra over the mod 3 *Steenrod algebra.*

THEOREM 2.12. *The additive structure of $\Omega_*^g/$(elements of odd order) is determined by $C(G)$ as a left module over* $\mathbf{A}$ *and the Bockstein spectral sequence of $H^*(BSG)$.*

Let M^n be an n-dimensional, closed, PL-manifold. Let $\tau_M \colon M \to B\mathrm{PL}$ be the classifying map. Define $I_n(\mathrm{PL}, 2) \subset H^*(B\mathrm{PL})$ by $I_n(\mathrm{PL}, 2) = \bigcap_M \operatorname{Ker} \tau_M^*$, where the intersection is taken over all such n-dimensional, closed, PL-manifolds. The following is a generalization of the main theorem in [5].

THEOREM 2.13. $I_n(\mathrm{PL}, 2) = \sum_{2i > n-j} H^j(B\mathrm{PL})\mathrm{Sq}^i$.

COROLLARY 2.14. (a) *$I_n(\mathrm{PL}, 2)^q = 0$ if $2q \leqq n$.*

(b) *$I_n(\mathrm{PL}, 2)^n$ is generated by $[\mathrm{Sq}^i + v_i u](H^{n-i}(B\mathrm{PL}))$ for $i = 1, \cdots, n$, where $v_i = (1)\mathrm{Sq}^i \in H^i(B\mathrm{PL})$.*

THEOREM 2.15. (a) $C^i(\mathrm{PL}) = 0$, $1 \leqq i \leqq 7$.

(b) $C^8(\mathrm{PL}) = Z_2$, $C^9(\mathrm{PL}) = Z_2 \oplus Z_2$, $C^{10}(\mathrm{PL}) \neq 0$.

(c) $C^i(\mathrm{PL}) \neq 0$, $i \geqq 24$.

The proofs of Theorems 2.2–2.8 will be given in § 3, of 2.9–2.12 in § 4, and of 2.13–2.14 in § 5. Theorem 2.15 (a) and (b) is known (see [14]), and (c) is very technical and complicated. A proof can be constructed using the techniques of [2], [3], and [4], but we omit the details.

[2] $H^*(\mathbf{M}SG)$ is a simple module over $\mathbf{A}$ in the notation of Wall [10].

3. The unoriented case

We first state Theorem 4.4 of Milnor and Moore [10]. Let A be a connected Hopf algebra over a field F. Let M be a coalgebra over F and a left module over A such that the diagonal map is a map of A-modules.

THEOREM 3.1. *Assume M is connected with $1 \in M^0$. Define $\nu\colon A \to M$ by $\nu(a) = a(1)$, and assume ν is a monomorphism. Then M is a free left A-module.*

Using this theorem, we now prove Theorem 2.2. Let $A = \mathbf{A}$ and $\nu\colon A \to H^*(\mathbf{M}G)$ be as above. Then $m^*\nu\colon \mathbf{A} \to H^*(\mathbf{MO})$ is a monomorphism by a theorem of Thom, and hence ν is a monomorphism. Theorem 2.2 follows.

The proofs of Corollaries 2.3 and 2.4 are similar to the classical case (see [9]).

LEMMA 3.2. *There is a map $f\colon H^*(BO) \to H^*(BG)$ such that $g^*f = \mathrm{id}$, f is a map of Hopf algebras, and f is a map of left and right A-modules.*

PROOF. Define $f(W_i) = W_i^G$. This defines a map of Hopf algebras, since $\psi_G W = W(\eta) \otimes W(\eta)$, and $H^*(BO) = Z_2[W_1, \cdots]$. Then $g^*f =$ identity as $m^*\Phi^G = \Phi^0 g^*$, and W^G is defined analogously to W. f is a map of left $\mathbf{A}$-modules by a theorem of Hsiang [8] as he shows W^G satisfies the Wu relations. The right $\mathbf{A}$-module structure in $H^*(BG)$ is defined by

$$b(a) = (\Phi^G)^{-1}\big(\chi(a)\Phi^G(b)\big)\ ,$$

and satisfies $(b)a = \sum (1)a_i' \cdot \chi(a_i'')(b)$, where $(1)a_i'$ is a polynomial in W_j^G (see [5]). Since f preserves W's, the left $\mathbf{A}$-module structure, and products, f preserves the right $\mathbf{A}$-module structure.

We now proceed to prove Theorem 2.5. The basic tool is Theorem 4.7 of Milnor and Moore [10]. Define $C(G)$ to be the kernel of γ which is the following composite:

$$\begin{aligned} H^*(BG) &\xrightarrow{\psi} H^*(BG) \otimes H^*(BG) \\ &\xrightarrow{g^*\otimes 1} H^*(BO) \otimes H^*(BG) \xrightarrow{\eta\otimes 1} \bar{H}^*(BO) \otimes H^*(BG)\ . \end{aligned}$$

In the notation of Milnor and Moore, $C(G) = Z_2 \square_{H^*(BO)} H^*(BG)$. Let $i\colon C(G) \to H^*(BG)$ be the inclusion and $\cdot\colon H^*(BG) \otimes H^*(BG) \to H^*(BG)$ be the multiplication. Then Theorem 4.7 of [10] shows that

$$\theta = \cdot(f \otimes i)\colon H^*(BO) \otimes C(G) \longrightarrow H^*(BG)$$

is an isomorphism of left $H^*(BO)$-comodules and right $C(G)$-modules, where $C(G)$ is an algebra, and i a map of algebras. Since $H^*(BG)$ is commutative as an algebra, $\cdot$ is a map of algebras. Hence θ is a map of algebras. To show

θ is a map of coalgebras, we need to show that $C(G)$ is closed under the diagonal map in $H^*(BG)$. Now $C(G)\otimes H^*(BG)$ is the kernel of

$$\gamma\otimes 1\colon H^*(BG)\otimes H^*(BG)\longrightarrow \bar{H}^*(BO)\otimes H^*(BG)\otimes H^*(BG)\,.$$

The following diagram is commutative as $H^*(BG)$ is co-associative:

$$\begin{array}{ccc} H^*(BG) & \xrightarrow{\ \psi\ } & H^*(BG)\otimes H^*(BG) \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\gamma\otimes 1} \\ H^*(BG)\otimes H^*(BG) & \xrightarrow{\ \eta g^*\otimes\psi\ } & \bar{H}^*(BO)\otimes H^*(BG)\otimes H^*(BG)\,. \end{array}$$

Hence $(\gamma\otimes 1)\psi(C(G))=0$, and $\psi C(G)\subset C(G)\otimes H^*(BG)$. Since ψ is commutative, $\psi C(G)\subset C(G)\otimes C(G)$. Finally, $\cdot$, f, and i are maps of left $\mathbf{A}$-modules and hence so is θ.

We now wish to show θ is a map of right $\mathbf{A}$-modules, where the right $\mathbf{A}$-structure in $H^*(BO)\otimes C(G)$ is given before Theorem 2.6. Let $b\in H^*(BO)$ and $c\in C$. Then

$$\begin{aligned}\theta\big((b\otimes c)a\big) &= \theta\big(\textstyle\sum (b)a_i'\otimes\chi(a_i'')(c)\big)=\sum f\big((b)a_i'\big)\cdot\chi(a_i'')(c)\\ &=\textstyle\sum \big(f(b)\big)a_i'\cdot\chi(a_i'')(c)=\big(f(b)\cdot c\big)a=\big(\theta(b\otimes c)\big)a\,.\end{aligned}$$

Let $\{\mu_i\}$ be a right $\mathbf{A}$-basis for $H^*(BO)$, and $\{c_j\}$ a Z_2-basis for $C(G)$. By counting, it is clear that $H^*(BO)\otimes C(G)$ and the free right $\mathbf{A}$-module generated by $\{\mu_i\otimes c_j\}$ have the same number of elements in each dimension. Hence we must show that $\{\mu_i\otimes c_j\}$ generate $H^*(BO)\otimes C(G)$ as a right $\mathbf{A}$-module. We show $(\mu_i)a\otimes c_j$ is in this submodule by induction on r, the dimension of a.

$$\begin{aligned}(\mu_i)a\otimes c_j &= (\mu_i\otimes c_j)a+\textstyle\sum_{\dim a_k'<r}(\mu_i)a_k'\otimes\chi(a_k'')(c_j)\\ &=(\mu_i\otimes c_j)a+\textstyle\sum_{\dim a_k'<r}(\mu_i)a_k'\otimes\sum c_{j,k}\,,\end{aligned}$$

which belongs to the above right $\mathbf{A}$-module by induction. This proves Theorem 2.6, and, of course, gives another proof of Theorem 2.2.

We now prove Theorem 2.7. Let $\eta^G\colon H^*(BG)\to(\mathfrak{N}^G)^*$ be the evaluation map on elements of $\pi_*(MG)$, i.e., $H^*(BG)\to H^*(MG)\to(\pi_*(MG))^*$, $\eta^G=h^*\Phi^G$, where h is the Hurewicz homomorphism. By our results, η^G is an epimorphism with Ker $\eta^G=H^*(BG)\bar{\mathbf{A}}$, where $\bar{\mathbf{A}}$ denotes the positive dimensional elements of $\mathbf{A}$. Also, η^G is a map of coalgebras. We first study the case $G=O$.

Let $S\subset H^*(BO)$ be the vector space generated by $s_\omega(\bar{W})$ for all partitions ω having no numbers of the form 2^j-1 (see [11]). The following lemma may be of independent interest.

LEMMA 3.3. *$\bar{\eta}=\eta^O\,|\,S\colon S\to\mathfrak{N}^*$ is an isomorphism of coalgebras.*

PROOF. Note $\psi(s_\omega)=\sum_{\omega'\omega''=\omega}s_{\omega'}\otimes s_{\omega''}$ (see [11]), hence S is a subcoalgebra of $H^*(BO)$. η^0 is a map of coalgebras, hence so is $\bar{\eta}$, and it is classical that $\bar{\eta}$ is an isomorphism of vector spaces.

Define $\theta\colon \mathfrak{N}^* \otimes C(G) \to \mathfrak{N}^{G*}$ by $\theta = \eta^G(\bar{\eta}^{-1} \otimes \mathrm{id})$. η^G, $\bar{\eta}^{-1}$, and id are maps of coalgebras, thus θ is. By Theorem 2.6, θ is an isomorphism of vector spaces. Hence $\theta^*\colon \mathfrak{N}^G_* \to \mathfrak{N}_* \otimes C(G)^*$ is an isomorphism of algebras.

Corollary 2.8 is immediate from Theorem 2.7.

4. The oriented case

We first prove Theorem 2.11. By the same proof as for Theorem 2.5, we have $H^*(BSG) \approx H^*(BSO) \otimes C'$ where C' is the kernel of the composition

$$\begin{aligned} H^*(BSG) &\xrightarrow{\psi} H^*(BSG) \otimes H^*(BSG) \\ &\longrightarrow H^*(BSO) \otimes H^*(BSG) \longrightarrow \bar{H}^*(BSO) \otimes H^*(BSG)\ . \end{aligned}$$

Let $\pi_G\colon BSG \to BG$. π_G^* is a map of Hopf algebras, hence $\psi_G^*\colon H^*(BG) \to H^*(BSG)$ is an epimorphism with kernel, the ideal generated by W_1^G (see [1]). Let $f^S\colon H^*(BSO) \to H^*(BSG)$ and note that $f^S\pi_0^* = \pi_G^* f\colon H^*(BO) \to H^*(BSG)$. Hence, the following two diagrams are commutative:

$$\begin{array}{ccccccc} C(G) & \longrightarrow & H^*(BG) & \xrightarrow{\psi} & H^*(BG) \otimes H^*(BG) & \longrightarrow & \bar{H}^*(BO) \otimes H^*(BG) \\ \downarrow{\scriptstyle \pi_G^* \mid C(G)} & & \downarrow{\scriptstyle \pi_G^*} & & \downarrow & & \downarrow \\ C' & \longrightarrow & H^*(BSG) & \xrightarrow{\psi} & H^*(BSG) \otimes H^*(BSG) & \longrightarrow & \bar{H}^*(BSO) \otimes H^*(BSG) \end{array}$$

$$\begin{array}{ccc} H^*(BO) \otimes C(G) & \xrightarrow{\theta} & H^*(BG) \\ \downarrow{\scriptstyle \pi_0^* \otimes \pi_G^* \mid C(G)} & & \downarrow{\scriptstyle \pi_G^*} \\ H^*(BSO) \otimes C' & \xrightarrow{\theta^S} & H^*(BSG)\ . \end{array}$$

Since $\mathrm{Ker}\, \pi_G^* = W_1^G \cdot H^*(BG)$ and $W_1^G \cdot H^*(BG) \cap C(G) = 0$, $\pi_G^* \mid C(G) = g$ is a monomorphism. Consider

$$\begin{aligned} H^*(BO) \otimes C(G) &\xrightarrow{\pi_0^* \otimes g} H^*(BSO) \otimes C' \\ &\longrightarrow H^*(BSO) \otimes C'/\bar{H}^*(BSO) \otimes C' \approx C'\ . \end{aligned}$$

This is an epimorphism. $\bar{H}^*(BO) \otimes C(G)$ is in the kernel, hence

$$C(G) \approx H^*(BO) \otimes C(G)/\bar{H}^*(BO) \otimes C(G) \longrightarrow C'$$

is an epimorphism, or g is an isomorphism.[3] The second part of Theorem 2.11 is proved in a way similar to the proof of Theorem 2.5 as $H^*(BSG; Z_3) \to H^*(BSO; Z_3)$ is an epimorphism, and there is a splitting map.[4]

[3] A similar theorem holds for $H^*(B\,\mathrm{Spin}\,G)$, where $B\,\mathrm{Spin}\,G$ is obtained from BSG by killing W_2^G.

[4] $H^*(BSG; Z_p) \to H^*(BSO; Z_p)$ is not necessarily an epimorphism if $p > 3$, so our methods fail in that case.

The second part of Theorem 2.9 follows from Lemma 3.1 just as Theorem 2.2 does. To prove the first part of Theorem 2.9, we define, following Wall [13], a simple right **A**-module to be an **A**-module which is a direct sum of a free **A**-module and copies of $\mathbf{A}/\mathrm{Sq}^1\mathbf{A}$. Wall [12] proves that $H^*(B\mathrm{SO})$ is a simple right **A**-module. Let $\{\mu_i\}$ be a basis for the free part, and $\{\nu_j\}$ the generators of copies of $\mathbf{A}/\mathrm{Sq}^1\mathbf{A}$. Write

$$C(G) = C_1 \oplus C_2 \oplus \mathrm{Im}\,\mathrm{Sq}^1 ,$$

where $\mathrm{Sq}^1(C_2) = 0$, and $\mathrm{Sq}^1\colon C_1 \to \mathrm{Im}\,\mathrm{Sq}^1$ is an isomorphism. Let $\{c_1^k\}$ and $\{c_2^l\}$ be Z_2-bases for C_1 and C_2 respectively, and $\{c^m\}$ a basis for $C(G)$. Then one can show that $H^*(B\mathrm{SO}) \otimes C(G)$ is a direct sum of a free right **A**-module on $\{\mu_i \otimes c^m\} \cup \{\nu_j \otimes c_1^k\}$ and copies of $\mathbf{A}/\mathrm{Sq}^1\mathbf{A}$ generated by $\{\nu_j \otimes c_2^l\}$. The proof is similar to that of Theorem 2.6. Rather than give the details, we give an alternate proof based on Wall [13]. He shows that a right **A**-module M is simple if and only if there is a free module F and an exact sequence

$$0 \longrightarrow M \xrightarrow{\;f\;} F \xrightarrow{\;g\;} M \longrightarrow 0 ,$$

where f and g have degrees 1 and 0 respectively. Let $0 \to H^*(B\mathrm{SO}) \to F \to H^*(B\mathrm{SO}) \to 0$ be such a sequence. Consider the exact sequence

$$0 \longrightarrow H^*(B\mathrm{SO}) \otimes C(G) \longrightarrow F \otimes C(G) \longrightarrow H^*(B\mathrm{SO}) \otimes C(G) \longrightarrow 0 .$$

By the proof of Theorem 2.6, $F \otimes C(G)$ is a free right **A**-module. Hence $H^*(BSG) \approx H^*(B\mathrm{SO}) \otimes C(G)$ is a simple right **A**-module, and $H^*(\mathbf{M}SG)$ is a simple left **A**-module.

We now consider Corollary 2.10. We prove the following lemma, which seems to be of independent interest, from which Corollary 2.10 follows.

LEMMA 4.1. *Let* **M** *be a spectrum such that* $H^*(\mathbf{M})$ *is a simple left* **A**-*module. Then there is a map* $h\colon \mathbf{M} \to \prod_i \mathbf{K}(\pi, n_i)$, *where* $\pi = Z_{2^r}$ *or* Z, *and* $h^*\colon H^*(\prod_i \mathbf{K}(\pi, n_i)) \to H^*(\mathbf{M})$ *is an isomorphism.*

PROOF. Replace **M**, if necessary, by an Ω-spectrum [15]. Consider the mod 2 Adams spectral sequence for **M**. Each generator of a copy of **A** in $H^{m_i}(\mathbf{M})$ gives an element $u_i \in E_2^{0,m_i}$. Each generator of a copy of $\mathbf{A}/\mathbf{A}(\mathrm{Sq}^1)$ in $H^{n_i}(\mathbf{M})$ gives elements $h_0^s v_i \in E_2^{s,n_i+s}$, $s \geqq 0$. These are all the non-trivial elements in $E_2^{s,t}$. Since $h_0 u_i = 0$, the only non-trivial d^r's are, after a change of basis, $d_{r_i}(h_0^s v_{2i}) = h_0^{s+r_i} v_{2i-1}$, $i = 1, \cdots, k$, where $n_{2i-1} = n_{2i} - 1$. Thus the only non-trivial elements in E_∞ are $u_i \in E_\infty^{0,m_i}$, $h_0^s v_i$ for $i > 2k$, and $h_0^s v_{2i-1}$ for $i = 1, \cdots, k$, and $s = 0, \cdots, r_i - 1$. Hence $\pi_*(M)/$(elements of odd order) has a Z_2 for each u_i, a Z for each v_i, $i > 2k$, and a $Z_{2^{r_i}}$ for each v_{2i-1}, $i = 1, \cdots, k$. Note that in the stable range, $H^*(K(Z, n)) \approx \mathbf{A}/\mathbf{A}(\mathrm{Sq}^1)$ as a left **A**-module, and $H^*(K(Z_{2^r}, n))$, $r > 1$, is $\mathbf{A}/\mathbf{A}(\mathrm{Sq}^1) \oplus \mathbf{A}/\mathbf{A}(\mathrm{Sq}^1)$, the two generators being ι

and $\delta_r^*(\iota)$.

We now show, by induction on dimension, that the two-torsion Postnikov invariants of $\mathbf{M}$ are all zero. Assume true in dimensions $< n$. Then $H^*(\mathbf{M}^{(n-1)})$ is a simple $\mathbf{A}$-module on generators of dimension $< n$. Since $H^*(K(Z_{2^r}, m); Z)$ and $H^*(K(Z, m); Z)$ have no elements of order 4 in stable dimensions $\geqq m + 2$, $k^{n+1} \in H^{n+1}(M^{(n-1)}; \pi_n(M)/(\text{elements of odd order}))$ is of order two or is zero.

The following statement is easy to prove, and we omit the proof. Let δ_r be the Bockstein associated with the sequence $0 \to Z_2 \to Z_{2^{r+1}} \to Z_{2^r} \to 0$. If $H^{n+2}(X; Z)$ and $H^{n+1}(X; Z)$ have no elements of order 4 or of infinite order, and $k \in H^{n+1}(X; Z_{2^r})$, $k \neq 0$, then k reduced mod $2 \neq 0$ or $\delta_r(k) \neq 0$. In our case, if $k^{n+1} \neq 0$, then a mod 2 relation is introduced into $H^{n+1}(\mathbf{M}^{(n)})$ or $H^{n+2}(\mathbf{M}^{(n)})$ on the $\mathbf{A}$-module generated by elements of dimension $< n$. This contradicts the hypothesis that $H^*(\mathbf{M})$ is simple. Hence $k^{n+1} = 0$, and the lemma is proved.

Corollary 2.10 now follows easily, as does Theorem 2.12.

5. Relations among PL-characteristic classes

The proof of Theorem 2.13 is essentially identical to the proof of [5, Th. 3.5]. We need the following facts to use this proof. First, that $H^*(B\mathrm{PL})$ is a free right A-module; this is Theorem 2.2. Second, that $\mathfrak{N}_*^{\mathrm{PL}}(K)$, defined geometrically as bordism classes of maps of PL-manifolds into K, is isomorphic to $H_*(K; M\mathrm{PL})$; this follows from [16]. The proof is now formally the same as that in [5]. The proof of Corollary 2.14 is also the same as that of [5, Cor. 3.6].

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