MANIFOLDS AND HOMOTOPY THEORY

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The history of classification theorems for manifolds really began with the classification theorem for 2-dimensional manifolds (the case of dimensions < 2 being elementary). In this case one can describe simple algebraic invariants (such as H_1, or the Euler characteristic and the orientability) whose values completely characterize the homeomorphism type of a compact 2-manifold.
As one increases the dimension it becomes clear that the complexity of the invariants required for even a classification of the homotopy type must increase
For example the fundamental group $\pi_1(M)$ for manifolds of dimension $m \geq 4$ can be an arbitrary finitely presented group, and in general the $([\frac{m}{2}] - 1)$-skeleton can be the homotopy type of an arbitrary $[\frac{m}{2}] - 1$ dimensional finite complex.
Hence it is natural to try to describe classification theorems in terms of the number of different structures of given type on a space given with less structure.

The history of such theorems begins with [Milnor 1956] where it was shown that there were ≥ 7 different differentiable structures (up to diffeomorphism) on the topological or piecewise linear (PL) S^7. Subsequently the theory of classification of smooth structures on a given PL manifold was developed extensively and serves as a model for such a theory. We give a brief description (see [Lashof-Rothenberg] for the finished version of the theory).

Let M^m be a PL manifold, and consider smooth (differentiable) structures on M compatible with the PL structure (so that the smooth functions are smooth on each simplex). Two such smoothings α_0, α_1 of M are called concordant if there is a smooth structure on $M \times [0, 1]$ which is α_i on $M \times i$, $i = 0, 1$.
(We ignore technical details, which would make it more desirable to take $M \times R$ instead of $M \times [0, 1]$, etc.).

A PL manifold M has a PL tangent bundle, which is induced by a map $f : M \to B_{PL}$, B_{PL} being the classifying space for stable PL bundle theory. There is a natural map of B_0, the classifying space for linear bundle theory, into B_{PL}, which can be considered a fibration $\pi : B_0 \to B_{PL}$, with fibre $PL/_0$.

The following theorem is due to Hirsch and Mazur, and Lashof and Rothenberg:

Theorem. Let M be a PL manifold. Concordance classes of smoothings on M (compatible with the PL structure) are in 1 - 1 correspondence with homotopy classes of cross-sections of the bundle $f^*(\pi)$ over M, induced by f from $\pi : B_0 \to B_{PL}$.

The fibration π is homotopically like a principal bundle, so that if $f^*(\pi)$ has a cross-section, (i.e. if M has some smooth structure) then it is trivial, equivalent to $M \times PL/_0$ and we get:

Theorem. Let M be a smooth manifold. Concordance classes of smooth structures on M (compatible with its PL structure) are in 1 - 1 correspondence with homotopy classes of maps into $PL/_0$, i.e. $[M, PL/_0]$.

The key to this theory is the Cains-Hirsch theorem [Hirsch] which essentially states that smooth structures $(M \times R)_\beta$ on $M \times R$ are of the form $M_\alpha \times R$ for a smooth structure α on M, (unique up to concordance). It is the precision of this theorem which leads to the satisfactory simplicity of the smoothing theory.

One would like to construct analogous theories starting from even less information than the PL type of a manifold, for example the homotopy type. This has been successful to a large extent, and the resulting theory and its differences from the above situation illustrates some deep underlying problems of homotopy theory, and algebra.

The underlying structure that we will start from will be an analog in homotopy theory of a closed oriented manifold, namely a space satisfying Poincaré duality. (We will assume that all spaces considered are CW complexes having finitely generated homology in each dimension).

Definition. An (oriented) Poincaré complex is a CW complex X and a class $[X] \in H_m(X)$ such that $[X] \cap : H^q(X) \to H_{m-q}(X)$ is an isomorphism for all q. An (oriented) Poincaré pair is a CW pair (X, Y) and a class $[X] \in H_m(X, Y)$ such that $[X] \cap : H^q(X) \to H_{m-q}(X, Y)$ is an isomorphism for all q. m is defined to be the dimension of X.

It is easy to prove that if (X, Y) is Poincaré pair then Y is a Poincaré complex and that the definition of Poincaré pair is symmetric, i.e. $[X] \cap : H^q(X) \to H_{m-q}(X, Y)$ is an isomorphism for all q if and only if $[X] \cap : H^q(X, Y) \to H_{m-q}(X)$ is an isomorphism for all q, (see [Browder, S]).

While a suitable definition of tangent bundle or fibre space is not known for Poincaré complexes, a stable version has been defined by [Spivak]. Its most natural definition is in terms of the analog of the normal bundle of a manifold in a high dimensional euclidean space.

Definition. The Spivak normal fibre space of a Poincaré complex of dimension m is an oriented $(k - 1)$-spherical fibration ξ over X, $k > m + 1$, and a class $\alpha \in \pi_{m+k}(T(\xi))$ such that

$$h(\alpha) \cap U = [X] .$$

Here if ξ is the fibration $p : E_0 \to X$, then $T(\xi) = X \cup_p cE_0$ (the "Thom complex of ξ"), $U \in H^k(T(\xi))$ is the Thom class, $h : \pi_*(\ ) \to H_*(\ )$ is the Hurewicz map.

The cap product is defined

$$\cap : H_{m+k}(T(\xi)) \otimes H^k(T(\xi)) \to H_m(X)$$

by interpreting the terms in various relative groups, (i.e. $H_*(T(\xi)) \cong H_*(E, E_0)$ where $E \to X$ is a fibre space with contractible fibre containing $E_0 \to X$ analogous to the disk bundle associated to a sphere bundle).

This is the analog of the normal bundle ν^k for a smooth manifold $M^m \subset S^{m+k}$. For recall that there is a neighborhood of M diffeomorphic to the total space E of ν (called a tubular neighborhood) and a map

$$S^{m+k} \to S^{m+k}/_{S^{m+k} - \mathrm{int}(E)} = E/_{E_0} = T(\nu)$$

defined by pinching the exterior of the tubular neighborhood E of M to a point, ($E_0 = \partial E$ is the sphere bundle of ν). This is called the natural collapsing map for the manifold M, normal bundle ν.

Theorem. For a Poincaré complex X, for $k > m + 1$, there exists a Spivak normal fibre space (ξ^k, α) and if (ξ'^k, α') is another in the same dimension $k > m + 1$, there exists a fibre homotopy equivalence $b : \xi \to \xi'$ such that $T(b)_*(\alpha) = \alpha'$, $T(b) : T(\xi) \to T(\xi')$ being the induced map of Thom complexes. Further such a b is unique up to fibre homotopy.

This is an improvement of [Wall, P]'s version of [Spivak]. Note that we require no hypotheses on π_1 or on homology with local coefficients, etc. It will be proved in [Browder, N].

One may now attempt to develop the analog of the smoothing theory of PL manifolds in this situation by studying the classifying map of the Spivak normal fibre space and the problem of lifting it to a linear (or PL or Top) bundle. If we let B_G be the classifying space for stable spherical fibrations then we have natural maps

$$B_O \to B_G , \; B_{PL} \to B_G , \; B_{Top} \to B_G ,$$

and we consider the diagram

$$\begin{tikzcd} & B_H \arrow[d, "p"] \\ X \arrow[ur, dashed] \arrow[r, "\xi"'] & B_G \end{tikzcd} \qquad (H = 0, \ PL \ \text{or} \ Top).$$

Now let us discuss only the case of $H = 0$, and the question of finding and classifying smooth manifolds of the homotopy type of X, with the understanding that for the other cases PL and Top a similar theory can be developed utilizing smoothing theory (see [Browder-Hirsch]) and the recent work of Kirby and Siebenmann in the topological case.

Suppose then that there is a lifting of the Spivak normal fibre space ξ to a linear bundle η, i.e. there is a fibre homotopy equivalence $b : \xi \to \eta$. Then the induced map of Thom complexes, $T(b) : T(\xi) \to T(\eta)$, sends $\alpha \in \pi_{m+k}(T(\xi))$ into $\beta = T(b)_*\alpha \in \pi_{m+k}(T(\eta))$ and by naturallity, $h(\beta) \cap U_\eta = [X] \in H_m(X)$.
If $g : S^{m+k} \to T(\eta)$ represents β, we may use the Thom transversality theorem to get a representative (call it again g) such that $g^{-1}(X) = N^m$ is a closed manifold $N^m \subset S^{m+k}$ with normal bundle ν^k, $g|N : N \to X$ is a map of degree 1 and $g|E(\nu)$ is a linear bundle map of a tubular neighborhood $E(\nu)$ of N into $E(\eta)$.
One asks the questions:

<u>Questions</u>: (a) Is g homotopic to a (transversal) g' such that $N' = g'^{-1}(X)$ is homotopy equivalent to X ?

(b) If so, how many different such N' are there, up to diffeomorphism?

An affirmative answer to (a) and a unique solution in (b) would be a very good analog of the Cains-Hirsch theorem, and would lead to a very exact analogy with smoothing theory.
However, there is a non-trivial obstruction to solving (a) and there is non-uniqueness in (b) which leads to a different theory with many interesting ramifications.

<u>Definition</u>. Let X be an m-dimensional Poincaré complex, η a linear k-plane bundle over X. A <u>normal map</u> into (X, η) is a pair of maps (f, b), where $f : N^m \to X$, N^m a smooth closed m manifold, f degree 1, $b : \nu \to \eta$ a linear bundle map over f, where ν is the normal bundle of $N^m \subset S^{m+k}$, $k >> m$.

(An analogous definition can be made for normal maps of manifolds with boundary into Poincaré pairs.).

<u>Definition</u>. A <u>normal cobordism</u> of two normal maps $(f_0, b_0), (f_1, b_1)$, $f_i : N_i \to X$, $b_i : \nu_i \to \eta$, is a pair (F, B) where $F : U \to X \times [0, 1]$,

$\partial U = M_0 \cup M_1$, $F|M_i = (f_i, i) : M_i \to X \times i$ $i = 0, 1$, $B : \omega \to \nu \times [0, 1]$ covering F where ω is the normal bundle of $U \subset S^{m+k} \times [0, 1]$ and $B|\nu_i = b_i$, etc.

(An analogous definition can be made for normal cobordisms of Poincaré pairs. Such a cobordism is called rel ∂ if the cobordism of the boundary is a product).

The Thom transversallity theorem implies (as above) that normal cobordism classes of normal maps into (X, η) correspond to homotopy classes of maps $\beta : S^{m+k} \to T(\eta)$, $\beta \in \pi_{m+k}(T(\eta))$, such that $h(\beta) \cap U_\eta = [X]$.

Thus questions (a) and (b) above can be translated into problems about normal maps and cobordisms. In case $m \geq 5$ and $\pi_1 X = 0$ the situation is very well understood. We give a brief account following [Browder, S] .

Fundamental Theorem of Surgery for $\pi_1 = 0$.

Let (f, b) be a normal map, $f : (M, \partial M) \to (X, Y)$, (X, Y) a Poincaré pair of dimension $m \geq 5$, $\pi_1 X = 0$, $(f|\partial M)_* : H_*(\partial M) \to H_*(Y)$ an isomorphism. Then (f, b) is normally cobordant rel ∂ to a homotopy equivalence (into X) if and only if an obstruction $\sigma(f, b) = 0$,

where $\sigma(f, b) \in P_m$, $P_m = \begin{cases} 0 & m \text{ odd} \\ Z & m = 4k \\ Z_2 & m = 4k + 2 \end{cases}$.

This theorem was first proved by [Kervaire and Milnor] when $X = S^m$ or D^m , and in this generality is in the work of Novikov and the author. This specific form is from [Browder, S] .

We note that if X is a Poincaré complex, i.e. $Y = \emptyset$, then normal cobordism rel ∂ = normal cobordism, and the condition on $f|\partial M$ is empty.

The obstruction σ has a very simple definition for $m = 4k$ as follows:

If $[X] \in H_m(X, Y)$, $m = 4k$, define a symmetric bilinear pairing

(1) $\quad (\ , \ ) : H^{2k}(X, Y) \otimes H^{2k}(X, Y) \to Z$ by $(x, y) = (x \cup y)[X]$.

Tensoring with the rational numbers Q we get a symmetric bilinear form on a finite dimensional Q vector space $H^{2k}(X, Y; Q)$ and we define $I(X)$ = signature of the form. Then define $\sigma(f, b) = \frac{1}{8}(I(M) - I(X))$.
(It is a theorem that this is an integer). Note that its value does not depend on the bundle map b , (nor even on the map f).

The definition of $\sigma(f, b) \in P_{4k+2}$ is considerably more subtile, (the Kervaire invariant), and depends on the choice of b as well as f . We omit it, (see [Browder, S, Chapter III]) .

The invariant σ has many good properties, convenient for calculations:

(P1). (Cobordism property). If (f, b), $f: (M, \partial M) \to (X, Y)$ is a normal map, then $\sigma(f|\partial M, b|\partial M) = 0$.

(P2). (Additivity property). If a normal map (f, b), $f: (M, \partial M) \to (X, Y)$ is the "union" of two normal maps (f_i, b_i) , $f_i : (M_i, \partial M_i) \to (X_i, Y_i)$, so that $M = M_1 \cup M_2$, $X = X_1 \cup X_2$ along "submanifolds" of the boundary, and if $\sigma(f, b)$, $\sigma(f_i, b_i)$ $i = 1, 2$ are defined (hypothesis as in Fundamental Theorem) then $\sigma(f, b) = \sigma(f_1, b_1) + \sigma(f_2, b_2)$.

(P3) (Hirzebruch formula). If $m = 4k$, $Y = \emptyset$, then $\sigma(f, b) = \frac{1}{8}(L_k(p_1(\eta^1), \ldots)[X] - I(X))$, where L_k is the Hirzebruch polynomial.

(P4) (Product formula). If N is closed, $1 : N \to N$ the identity, $\sigma(f \times 1, b \times 1) = I(N)\sigma(f, b)$, where $f \times 1 : (M, \partial M) \times N \to (X, Y) \times N$.

The last property is a well known property of index and Sullivan's formula for the Kervaire invariant (see [Browder, S, (III §5)] , [Rourke, Sullivan]) .

Another indispensible tool is the theorem of [Kervaire, Milnor] :

<u>Plumbing Theorem</u>. For each value $x \in P_m$, there is a normal map (g, c) , $g : (V, \partial V) \to (D^m, S^{m-1})$ satisfying the hypothesis of the Fundamental Theorem, with $\sigma(g, c) = x$. If $m > 4$, $g|\partial V : \partial V \to S^{m-1}$ is a homotopy equivalence.

From the above results one may read off many powerful theorems:

<u>Homotopy Type of Smooth Manifolds</u>. Let X be a 1-connected Poincaré complex of dimension $m \geq 5$. Let η^k be a linear k-plane bundle over X , $\alpha \in \pi_{m+k}(T(\eta))$ such that $h(\alpha) \cap U_\eta = [X]$, (i.e. (η, α) is a lift of the Spivak normal fibre space).

(1) If m is odd or (2) if $m = 4k$ and $L_k(p_1(\xi^{-1}), \ldots, p_R(\xi^{-1}))[X] = I(X)$, then X is the homotopy type of a smooth manifold with η as its stable normal bundle, α its natural collapsing map.

This is due to the author and S.P. Novikov. It follows immediately from the Fundamental Theorem, and (P3) , noting that our hypothesis yield a normal map into X, η, and that in these cases there is no obstruction to getting a normal cobordism to a homotopy equivalence.

One can write down a simpler theorem in the PL case [Browder-Hirsch] , which extends to the topological case easily using [Kirby, Siebenmann] :

<u>Homotopy Type of PL (Top) Manifolds</u>. Let X be a Poincaré complex of dimension $m \geq 5$, $\pi_1 X = 0$. Then X is of the homotopy type of a PL (Top) manifold if and only if the Spivak normal fibre space is fibre homotopy equivalent to a PL (Top) bundle.

We note that it is easy to avoid the obstruction in the PL and Top cases at the cost of weakening the conclusion slightly.

The answer to our question (b) requires for clarity the following definition first introduced by [Sullivan, T] :

Definition. Let $(X, \partial X)$ be a Poincaré pair with ∂X possibly empty. Define the set of homotopy structures on X, $\mathcal{S}^0(X)$ $(\mathcal{S}^{PL}(X), \mathcal{S}^{Top}(X))$ to be the set of concordance classes of pairs (M, h) where M is a smooth (PL, Top) manifold and $h : (M, \partial M) \to (X, \partial X)$ is a homotopy equivalence. Two such (M_i, h_i) $i = 0, 1$, are called concordant if there is a cobordism U, $\partial U = M_0 \cup M_1 \cup V$, $\partial V = \partial M_0 \cup \partial M_1$ and a homotopy equivalence $H : (U, V) \to (X \times [0, 1], \partial X \times [0, 1])$ with $H|M_i = h_i : M_i \to X \times i$, $i = 0, 1$.

We note that such a concordance U is an h-cobordism so that if $\pi_1 X = 0$, $\partial X = \emptyset$, $m = \text{dimension } X \geq 5$, it follows from [Smale]'s h-cobordism theorem that $h_1^{-1} h_0$ is homotopic to a diffeomorphism (PL equivalence, homeomorphism) $d : M_0 \to M_1$.

If (M, h) represents an element in $\mathcal{S}(X)$, define a normal map (h, b) by taking $\eta = (h^{-1})^*(\nu_M)$ over X, and b some bundle map over h. Then b is well defined up to a bundle automorphism of η.

Definition. Let (X, Y) be a Poincaré pair (Y possibly empty). Then $N^0(X)$ $(N^{PL}(X), N^{Top}(X))$ is defined to be equivalence classes of normal maps (f, b) into X, η, for all linear (PL, Top) bundles η over X. Two such (f_i, b_i), $f_i : (M_i, \partial M_i) \to (X, Y)$, $i = 0, 1$, are equivalent if (f_0, b_0) is normally cobordant to (f_1, ab_1), where $a : \eta \to \eta$ is a linear (PL, Top) bundle automorphism of η (over $1 : X \to X$).

Then we have the result of [Sullivan, T] (See also [Browder, S, (II.4.4.)]):

Theorem. If X is smooth then $N^0(X) \cong [X, G/_0]$ where $[\ ,\ ]$ denotes homotopy classes of maps and $G/_0$ is the fibre of the map of classifying spaces, $B_0 \to B_G$, (similarly for PL and Top).

In other words, $N^0(X)$ is in 1-1 correspondence with homotopy classes of lifts of the Spivak normal fibre space ξ of X from B_G to B_0, and given one such (from the smooth structure) the others correspond to maps into the fibre. Thus it is $N(X)$ which corresponds exactly to the lifts of ξ and there is a map $n : \mathcal{S}(X) \to N(X)$ (defined above), but it is not an isomorphism, which is where the analogy with the smoothing theory of PL manifolds breaks down.

If dimension $X = m$, let us define an action of P_{m+1} on $\mathcal{S}^0(X)$ in the following way: Let $x \in P_{m+1}$, and let (g, c) be the normal map of the Plumbing Theorem, $g : (V, \partial V) \to (D^{m+1}, S^m)$ such that $\sigma(g, c) = x$, $g|\partial V : \partial V \to S^m$ a homotopy equivalence. If (M, H) represents an element of $\mathcal{S}^0(X)$, consider the "connected sum along the boundary $M \times 1$" of $h \times 1 : M \times [0, 1] \to X \times [0, 1]$ and $g : V \to D^{m+1}$, so that

$$(h \times 1) \amalg g : (M \times [0, 1]) \underset{M \times 1}{\amalg} V \to (X \times [0, 1]) \underset{X \times 1}{\amalg} D^{m+1}$$

where $\amalg$ means identify a disk $D^m \subset M \times 1$ with a disk $D^m \subset \partial V$ similarly on the right. If we arrange the maps h and g so that the identified disks are sent to the disks in X and S^m which are identified on the right, then the union defines a new map and the right hand end $(M \times 1) \# V$ is mapped by a homotopy equivalence h' into $X \times 1$. We define $x(M, h) = (M', h')$, where $M' = (M \times 1) \# \partial V$.

It is not hard to see that $(h \times 1) \amalg g$ is covered by a bundle map defined by some map over h and c, so that $n(x(M, h)) = n(M, h)$, In particular, if $\pi_1 X = 0$ and $m \geqq 4$, it follows from Additivity (P2) that the obstruction to finding a normal cobordism of $(h \times 1) \amalg g$ to a concordance (of (M, h) and (M',h') is exactly the element $x \in P_{m+1}$ which we started out with, and it then follows from Additivity (P2) and the Fundamental Theorem that the operation of P_{m+1} on $\mathcal{S}^O(X)$ is a well defined operation of the group on the set, denoted by $\omega : P_{m+1} \dashrightarrow \mathcal{S}(X)$, $(\omega : P_{m+1} \times \mathcal{S}^O(X) \to \mathcal{S}^O(X))$. We note that the operation is trivial on $\mathcal{S}^{PL}(X)$ and $\mathcal{S}^{Top}(X)$.

Exact Sequence of Surgery $(\pi_1 = 0)$. Let X be a Poincaré complex of dimension $m \geqq 5$, $\pi_1 X = 0$. Then we have an exact sequence of sets

$$P_{m+1} \overset{\omega}{\dashrightarrow} \mathcal{S}^H(X) \xrightarrow{n} N^H(X) \xrightarrow{\sigma} P_m$$

for $H = 0$, PL or Top, and $n(\alpha) = n(\beta)$, $\alpha, \beta \in \mathcal{S}(X)$ if and only if $\alpha = \omega(x, \beta)$ for some $x \in P_{m+1}$.

This exact sequence was first developed by [Kervaire, Milnor] in case $X = S^m$, and was generalized to this situation by [Sullivan, H]. It contains within it the uniqueness theorem of [Novikov]. Sullivan actually showed that the sequence extends to a long exact sequence of abelian groups on the left if X is an H-manifold, where the next stage involves $\mathcal{S}(X \times [0, 1]$ rel $X \times 0 \cup X \times 1)$ = homotopy structures on $X \times [0, 1]$ which are the identity on $X \times 0 \cup X \times 1$, and $N^H((X \times [0, 1])/\partial) = [\Sigma X, G/_H]$, where H=O, PL or Top.

In case we have (X, Y) a Poincaré pair and $Y \neq \emptyset$ we have an exact analogy with the smoothing theory of PL manifolds, namely:

Theorem. Let (X, Y) be a Poincaré pair of dimension $m \geq 6$, X, Y 1-connected, $Y \neq \emptyset$. Then elements of $\mathcal{S}^H(X)$ $(H = 0$, PL or Top$)$ are in 1 - 1 correspondence with homotopy classes of cross-sections of $f^*(p_H)$, where $f : X \to B_G$ is the classifying map of the Spivak normal fibre space and $p_H : B_H \to B_G$ is the natural map made into a fibration, $H = 0$, PL or Top . Given one smooth (PL or Top) structure on X , one gets an isomorphism $\mathcal{S}^H(X) \cong [X, G/_H]$. $H = 0$ (PL or Top).

This is a combination of [Wall, G] (see also [Golo]) in the smooth case, together with the work of [Spivak] and [Sullivan, T] above, and [Browder, Hirsch] , [Kirby, Siebenmann] to extend to the PL and Top cases.

To prove this theorem in the smooth case, we note first that the hypothesis gives us a normal map (f, b) , $f : (M, \partial M) \to (X, Y)$. By the cobordism property (P1) , $\sigma(f|\partial M, b|\partial M) = 0$, so by the Fundamental Theorem , $(f|\partial M , b|\partial M)$ is normally cobordant to a homotopy equivalence. Gluing this cobordism to (f, b) along ∂M we obtain a normal cobordism of (f, b) to (f', b') such that $f' : (M', \partial M') \to (X, Y)$, $f'|\partial M'$ is a homotopy equivalence.

Let (g, c) , $g : (V, \partial V) \to (D^m, S^{m-1})$ be a normal map such that $g|\partial V$ is a homotopy equivalence and $\sigma(g, c) = -\sigma(f', b')$. Taking connected sum along the boundary of (f', b') and (g, c) we get

$$(f' \perp\!\!\perp g, b' \perp\!\!\perp c) , \quad f' \perp\!\!\perp g : (M' \perp\!\!\perp V , \partial) \to (X, Y) ,$$
$$f' \perp\!\!\perp g|\partial(M' \perp\!\!\perp V) : \partial M' \# \partial V \to Y$$

is still a homotopy equivalence, and by Additivity (P2) , $\sigma(f' \perp\!\!\perp g, b' \perp\!\!\perp c) = 0$. Hence by the Fundamental Theorem, $(f' \perp\!\!\perp g, b' \perp\!\!\perp c)$ is normally cobordant <u>rel ∂</u> to a homotopy equivalence. This proves the existence part of the theorem, the uniqueness part following by a similar argument, (compare with the proof of the Exact Sequence) .

Besides these basic theorems, there have been many fruitful applications of these methods to other problems in topology, (e.g. see [Browder, T] for a discussion of some applications to the theory of transformation groups).

There are still many unsolved problems in the simply connected case, in particular, about the behaviour of the surgery obstruction σ in case the dimension is $4k + 2$, (i.e., the Kervaire invariant case). We note the absence of these dimensions in the Homotopy Type of Smooth Manifolds Theorem. It has so far proved very difficult to describe the homotopy conditions on a $(4k + 2)$-dimensional Poincaré complex to make it homotopy equivalent to a smooth manifold, though a few partial results are known (c.f. [Browder, K]) .
In general, the map $\sigma : N^O(M) \to P_{4k+2}$, remains mysterious, in the case where M is smooth. In the cases $H = PL$ or Top , $N^H(X) \to P_*$ is onto for any X . [Sullivan, G] has given a cohomological formula for σ (see also [Rourke, Sullivan]).

By Sullivan's Theorem on normal maps, $N^H(M) = [M, G/_H]$ if M is an H-manifold, $H = 0$, PL or Top ,and one would like to analyze $G/_H$.

This remains very difficult in the case $H = 0$, but in the other cases there has been spectacular success. First we note that one can prove that $\pi_*(G/_{PL}) = P_*$, using the Plumbing Theorem for dimensions ≥ 5, and special arguments in lower dimensions. If V is a graded module, denote by $K(V) = \Pi K(V_j, j)$, the product of Eilenberg-MacLane spaces with homotopy groups V_j in dimension j. Let Z_0 = ring of rational numbers with odd denominators. We note without proof that $G/_H$ is a homotopy commutative loop space, so that $[X, G/_H]$ is an abelian group.

<u>Theorem.</u> [Sullivan, T and G]. At the prime 2, $G/_{PL} \cong Y \times K(P'_*)$, where $P'_i = P_i$, $i > 4$, $P'_i = 0$, $i \leq 4$, Y is a fibre space over $K(Z_2, 2)$ induced by the map $\phi : K(Z_2, 2) \to K(Z, 5)$, such that $\phi^*(\iota_5) = \delta Sq^2 \iota_2$, ($\delta$ is the integral Bockstein). In other words

$$[X, G/_{PL}] \otimes Z_0 \cong [X, Y] \otimes Z_0 + [X, K(P'_*)] \otimes Z_0 \text{, (isomorphism as sets).}$$

<u>Theorem.</u> [Sullivan, N, G and A].

$$[X, G/_{PL}] \otimes Z[\tfrac{1}{2}] = KO(X) \otimes Z[\tfrac{1}{2}].$$

In other words, $G/_{PL} \cong B_0$ at odd primes.

<u>Theorem.</u> [Kirby, Siebenmann]. $Top/_{PL} \cong K(Z_2, 3)$, $G/_{Top} \cong K(P_*)$ at the prime 2, $G/_{Top} \cong G/_{PL} \cong B_0$ at odd primes.

We will outline the proof of Sullivan's theorem at the prime 2. Using the composition

$$[M, G/_{PL}] \cong N^{PL}(M) \xrightarrow{\sigma} P_m ,$$

the cobordism property (P1) defines a homomorphism of the bordism group $\Omega_*(G/_{PL}) \to P_*$.
Using the Plumbing Theorem, it is easy to see that the composition

$$\pi_i(G/_{PL}) \to \Omega_i(G/_{PL}) \to P_i$$

is an isomorphism for $i \neq 4$, and one can show that $\pi_4(G/_{PL}) \to P_4$ is multiplication by 2. But at the prime 2, $\Omega_*(X) \cong H_*(X, \Omega_*)$, (see [Conner, Floyd, D]), so that one gets a map of $G/_{PL} \to K(P'_*)$ which induces an isomorphism on π_i, $i > 4$. One constructs another map $G/_{PL} \to Y$ by a special argument, and the resulting map $G/_{PL} \to Y \times K(P'_*)$ is a homotopy equivalence at the prime 2.

The argument for the odd prime theorem is similar in spirit but much more difficult to carry out. One uses the homomorphism $\sigma : \Omega_*(G/_{PL}) \to P_*$, considered as a Ω_* module map (operating by multiplication by the Index on P_*) together with analogous maps $\sigma_n : \Omega_*(G/_{PL} ; Z_n) \to P_* \otimes Z_n$, n odd.

By a generalization of [Conner, Floyd, K], one gets that
$\Omega_*(X) \otimes_{\Omega_*} Z[\frac{1}{2}] \cong KO_*(X) \otimes Z[\frac{1}{2}]$, (using the "Index" Ω_* -module structure of $Z[\frac{1}{2}]$) .
Hence σ and σ_n induce $\sigma' : KO_*(G/_{PL}) \to Z[\frac{1}{2}]$, $\sigma'_n : KO_*(G/_{PL} ; Z_n) \to Z_n$.
From some universal coefficient theormes in KO theory, one deduces that a set of compatible homomorphisms

$$KO_*(X) \to Z[\tfrac{1}{2}] \quad \text{and} \quad KO_*(X; Z_n) \to Z_n$$

(as above) are induced by an element of $KO^*(X) \otimes Z[\frac{1}{2}]$ (using the Kronecker pairing of cohomology and homology).
This yields a map $G/_{PL} \to B_0$ (at odd primes) which induces the isomorphism of the theorem.

We say nothing about the [Kirby, Siebenmann] theory, except to remark that it uses the PL classification of homotopy tori of [Hsiang, Shaneson] and [Wall, T] , which emerges as one of the end products of surgery theory in the non-simply connected case. Thus all the results of doing surgery on topological manifolds are based first on the development of the whole surgery theory for smooth and PL manifolds, together with some elaborate calculations in the theory.

The theory developed by [Sullivan, N] has also produced the remarkable characterization of PL-bundles at odd primes.

Theorem. The theory of PL-bundles at odd primes is equivalent to the theory of spherical fibre spaces with KO'-theory orientation $(KO'(X) = K'(X) \otimes Z[\frac{1}{2}])$.

The question of a characterization of PL-bundles at the prime 2 is still a mystery, seemingly related to difficult questions about the Kervaire invariant, and the "Kervaire manifolds" of Kervaire invariant 1 .

Now we shall describe the theory of surgery when $\pi_1 \neq 0$.

One may first discuss the question of the appropriate definition of Poincaré complex or pair when $\pi_1 \neq 0$ and there are several possibilities.

Suppose that C is a free chain complex over $Z[\pi]$, the group ring of a group π , (acting on the left) $\Delta : C \to C \otimes C$ is a diagonal such that $\Delta(gc) = (g \otimes g)\Delta(c)$ for $g \in \pi$, $c \in C$, i.e. we have an equivariant map $\Delta : C \to C \otimes C$ with the diagonal action of π on $C \otimes C$. This is the case for C = chains of $\tilde{X}$, $\tilde{X}$ = universal cover of X , $\pi = \pi_1(X)$.
Then Δ defines $\Delta_0 : C/_\pi \to C \otimes_\pi C$, where π acts on the right on C by $cg = g^{-1}c$, $g \in \pi$, $c \in C$.
If $x \in \mathrm{Hom}_\pi(C, M)$, some π-module M , then

$$x \otimes_\pi 1 : C \otimes_\pi C \to M \otimes_\pi C ,$$

is a chain map and we have defined a chain map

$$\cap : C/_{\pi} \underset{Z}{\otimes} \mathrm{Hom}_{\pi} (C, M) \to M \underset{\pi}{\otimes} C ,$$

by $z \cap x = (x \underset{\pi}{\otimes} 1)\Delta_0(z)$.

Passing to homology we get

$$\cap : H_m(C/_{\pi}) \otimes H^i(C; M) \to H_{m-i}(C; M) ,$$

where $H^i(C; M)$ = ith homology of the cochain complex $\mathrm{Hom}_{\pi} (C, M)$, and $H_j(C; M)$ is the j-th homology of the chain complex $M \underset{\pi}{\otimes} C$.

One possible definition of Poincaré complex in the non-simply connected case might be the following:

Definition. A CW complex X satisfies Poincaré duality with local coefficients if there is a class $[X] \in H_m(X)$ such that $[X] \cap : H^i(X; M) \to H_{m-i}(X; M)$ is an isomorphism for all i , all π-modules M , $\pi = \pi_1(X)$. Here $H^i(X; M) = \mathrm{Hom}_{\pi} (C_*(\tilde{X}), M))$, $H_i(X; M) = H_i(M \underset{\pi}{\otimes} C_*(\tilde{X}))$.

Closed oriented manifolds satisfy Poincaré duality of this type, but they also have stronger duality properties.

We note in passing that one could have included the case of non-orientable manifolds by taking twisted integer coefficients for the fundamental class $[X]$, and defining cap product in this context. This gives greater generallity as in [Wall, S] but at the cost of increasing complication, so we will restrict ourselves to the oriented case.

It follows from the existence of the Spivak normal fibre space and [Wall, F II, Theorem 8] that if X satisfies Poincaré duality with local coefficients and if $\pi_1 X$ is finitely presented, then it is dominated by a finite complex, [Browder, N] , though it may not be the homotopy type of a finite complex, [Wall, P. Theorem 1.5] . If π_1 were not finitely presented on the other hand, it would be impossible to find a manifold with the same fundamental group, which leeds us to:

Definition. A (oriented) Poincaré complex is a CW complex X with $\pi_1 X$ finitely presented, which satisfies Poincaré duality with local coefficients.

We will call X a finite Poincaré complex if in addition X has the homotopy type of a finite complex.

If X is a finite complex, then the chains and cochains of the universal cover $\tilde{X}$ have natural $\pi = \pi_1 X$ bases, as free π-modules.
If $x \in C_m(X)$ is a chain representing $[X] \in H_m(X)$ and if $x \cap : C^*(\tilde{X}) \to C_*(\tilde{X})$ is a simple chain homotopy equivalence, then we call X a simple Poincaré complex. This is the definition used by [Wall, S] .

Poincaré pairs of these different types may be defined analogously, and smooth (PL) manifolds are Poincaré complexes in each sense.

One may attempt to develop a theory of surgery in the context of (i.e. for normal mappings into) Poincaré complexes of these three types. This has been done in [Wall, S] for simple Poincaré complexes and the other cases can be dealt with similarly. We now describe Wall's theory.

Let X be a Poincaré complex, and define the set of homotopy structures $\mathcal{S}^H(X)$ as in the simply connected case. If X is a simple Poincaré complex, define the set of simple homotopy structure $\mathcal{S}^H_S(X)$ analogously to $\mathcal{S}^H(X)$ by requiring the homotopy equivalences and h-cobordisms in the definition to be simple.

<u>Theorem</u>. [Wall, S] . There exists a functor from finitely presented groups into graded abelian groups, $L_n(\ )$ and if X is a simple Poincaré complex of dimension $n \geq 5$ with $\pi_1(X) = \pi$, there is an exact sequence of sets

$$L_{n+1}(\pi) \to \mathcal{S}^H_S(X) \xrightarrow{\eta} N^H(X) \xrightarrow{\sigma} L_n(X)$$

where $H = 0$, PL or Top.
Further $L_{n+1}(\pi)$ acts on $\mathcal{S}^H_S(X)$ so that $\eta(x) = \eta(y)$ if and only if $x = ty$ for some $t \in L_{n+1}(\pi)$. (Note that $\mathcal{S}^H_S(X)$, $N^H(X)$ may be empty.)

There are analogous sequences for pairs with a functor $L_n(f)$ for group homomorphisms $f : \pi \to \pi'$, and similar theories for Poincaré complexes and finite Poincaré complexes without simplicity, but we shall concentrate our attention on this case.

Partly generalizing the Product Formula for the obstructions in the simply connected case we have the

<u>Periodicity Theorem</u>. [Wall, S] . There is a natural isomorphism $\phi : L_n(\pi) \to L_{n+4}(\pi)$ for all n such that the diagram

$$\begin{tikzcd}
L_{n+1}(\pi) \arrow[r] \arrow[d, "\phi"] & \mathcal{S}^H_S(X) \arrow[r, "\eta"] \arrow[d] & N^H(X) \arrow[r, "\sigma"] \arrow[d] & L_n(\pi) \arrow[d, "\phi"] \\
L_{n+5}(\pi) \arrow[r] & \mathcal{S}^H_S(X \times \mathbb{C}P^2) \arrow[r, "\eta"] & N^H(X \times \mathbb{C}P^2) \arrow[r, "\sigma"] & L_{n+4}(\pi)
\end{tikzcd}$$

commutes, where the map on $\mathcal{S}^H_S$ and N^H are induced by multiplying a normal map or simple homotopy equivalence by the identity map of $\mathbb{C}P^2$ to itself.

The calculation of these groups $L_n(\pi)$ seems to be a difficult problem, and not much is yet known about it. It can be approached algebraicly using the definition of [Wall, S] . For example the group $L_{2k}(\pi)$ is defined to be a Grothendieck

group of quadratic forms of a certain type on free finitely generated $Z\pi$-modules, with some extra structure. For $\pi = Z_2$ this was calculated in [Wall, N] , and for $\pi = Z_p$ more recently by Wall and Petrie. In [Petrie, AS] , it is shown how to use the Atiyah-Singer theory to help calculate these groups, and the relations between various surgery problems.
In [Wall, N] , $L_{2k+1}(Z_2)$ is also calculated, and [R, Lee, C] has shown that $L_3(Z_p) = 0$ for p an odd prime, by algebraic arguments.

In this paper, I would like to discuss geometric approaches to the calculation of $L_n(\pi)$ and to ways of calculating the map σ .

Let us begin by considering $\pi = Z$. This case was essentially done in [Browder, Z] before the theory of [Wall, S] had been developed. The analysis in that case is based on the observation that if W^{m+1} is a connected manifold with $\pi_1 W = Z$ and if $m \geq 4$, then there is a 1 - connected U^{m+1} with $\partial U = M \times 0 \cup M \times 1$, M_i 1 - connected, so that W is diffeomorphic to $U \cup (M \times [0, 1])$. Then one can analyze a normal map into W by analyzing the induced normal map over $M \times \frac{1}{2}$ and over U , i.e. two 1-connected problems.
More explicitly, let (f, b), $f : W' \to W$ be a normal map, and make f transverse regular to $M \times \frac{1}{2} \subset W$, so that $f^{-1}(M \times [0, 1]) = N \times [0, 1] \subset W'$, $f^{-1}(M \times t) = N \times t$, and let $V = W' - N \times (0, 1)$.
Then the restrictions of f and b define normal maps $f_0 : N \to M$, $f_1 : V \to U$, where ∂f_1 is two copies of f_0 .
A normal cobordism of f_0 induces normal cobordisms of f and f_1 . Then if one can make f_0 a homotopy equivalence $N \to M$, and take the resulting map $V \to U$ (which is now a homotopy equivalence on $\partial V = N \times 0 \cup N \times 1$) and find a normal cobordism rel ∂ of it to a homotopy equivalence into U then the result will be a normal cobordism of f to a homotopy equivalence.

Similarly if one has a normal cobordism of two normal maps (f_i, b_i) $i = 0, 1$ which are homotopy equivalences $f_i : W_i' \to W$, then one could use the same technique as above to try to do surgery on the cobordism to make it an h-cobordism, provided that for each i , on $f_i^{-1}(M) = N_i$ and $f_i^{-1}(U) = V_i$ the restrictions of f_i are homotopy equivalences. Such a situation may be achieved when W is a fibre bundle over S^1 with fibre M , using the fibering theorem of [Browder-Levine] , and in general using [Farrell-Hsiang] .

These arguments lead to a calculation of $L_n(Z)$ by using the theorem of [Wall, S] which describes the action of $L_{n+1}(\pi)$ on $\mathcal{S}^H_S(X)$:

<u>Theorem</u>. Let M^m be a manifold of dimension $m \geq 5$ with $\pi_1 M = \pi$. For each element $x \in L_{m+1}(\pi)$ there is a normal cobordism W^{m+1} of the identity map $M \to M$ to a simple homotopy equivalence $M' \to M$ such that the obstruction to finding

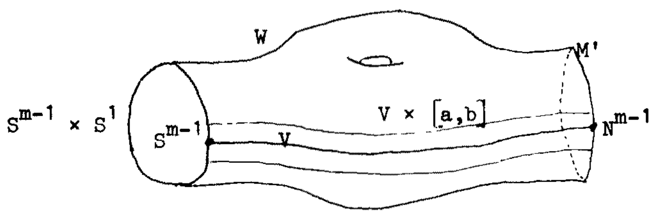
a normal cobordism rel ∂ of W to a simple homotopy equivalence is x . Further, two such normal cobordisms are normally cobordant rel ∂ .

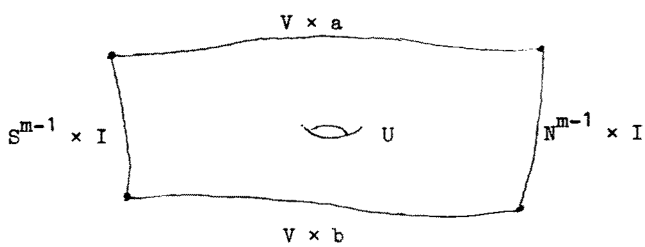
In other words, $L_{m+1}(\pi)$ can be considered as the set of normal cobordism classes rel ∂ of normal cobordisms of the identity of M to a simple homotopy equivalence for any m-manifold M with $\pi_1 M = \pi$.

Then to calculate $L_{m+1}(Z)$ following [Shaneson] we take $M = S^{m-1} \times S^1$ and study normal cobordisms of M to a homotopy equivalent M' . Using [Browder, Levine] M' is a fibre bundle over S^1 with fibre N^{m-1}, a homotopy sphere. If $F : W^{m+1} \to S^{m-1} \times S^1 \times [0, 1]$ is the normal cobordism between M and M' , we can cut W along $F^{-1}(S^{m-1} \times [0, 1]) = V^m$ and obtain $W = U \cup V \times [a, b]$, where

$$\partial U = V \times a \cup V \times b \cup S^{m-1} \times I \cup N^{m-1} \times I ,$$

$$\partial V = S^{m-1} \cup N^{m-1} \quad , \text{(see diagram)}.$$





Then the problem of doing surgery on W to make it an h-cobordism is reduced to the rpoblem of first making V into an h-cobordism by surgery leaving $S^{m-1} \cup N^{m-1} = \partial V$ fixed and then doing surgery on the resulting U rel ∂U to make it homotopy equivalent to $S^{m-1} \times I \times [0, 1]$. On deduces

<u>Theorem</u>. (Shaneson and Wall).

$$L_m(Z) = L_m(0) + L_{m-1}(0) .$$

One can use a similar argument employing [Farrell] to prove the more general theorem of [Shaneson] (which has also be proved in [Wall, S]) .

<u>Theorem.</u> There is a split exact sequence

$$0 \to L_m(G) \to L_m(G \times Z) \to L^h_{m-1}(G) \to 0$$

where L^h_{m-1} is the surgery obstruction group for homotopy equivalences, L_m the obstruction group for simple homotopy equivalences.

A similar program was initiated by R. Lee to show $L_m(G_1 * G_2) = L_m(G_1) + L_m(G_2)$ where $G_1 * G_2$ is the free product. He proposed to prove a codimension 1 embedding theorem to show that in the PL category if $W = M_1 \# M_2$ (connected sum) and $W \to W'$ is a simple homotopy equivalence, then $W' = M_1' \# M_2'$, where $M_i \to M_i'$ is a simple homotopy equivalence, generalizing a simply connected theorem of [Browder, E] .
This would replace the [Farrell] and [Browder-Levine] theorems in the argument. His program unfortunately was only partially successful, in that he succeeded in proving his codimension 1 theorem only in half the dimensions necessary.
His program was carried out and greatly generalized by Cappell, who has proved a very general codimension 1 embedding theorem for a submanifold $N^{m-1} \subset W^m$ such that $\pi_1 N$ <u>injects</u> into $\pi_1 W$ and a technical condition called "$\pi_1 N$ two sided in $\pi_1 W$" . With Cappell's theorem it is possible to extend the calculations of the surgery obstruction groups to many free products with amalgamation, some of which were carried out by Quinn.

These results suggest that the groups $L_m(\pi)$ may be closely related to the structure of submanifolds of a manifold M with $\pi_1 M = \pi$.
Applying the theorem of [Shaneson] inductively to calculate $L_m(A)$ where A is free abelian, for example, shows how if we take $M = T^n = S^1 \times \ldots \times S^1$ n-times, it is exactly the lattice of subtoruses $T^i \subset T^n$ which are related to the structure of $L_m(A)$ (compare [Hsiang-Shaneson]) . Similarly [Lopez de Medrano] has shown how one can calculate $L_m(Z_2)$ using the structure of codimension 1 manifolds of P^m , as studied in [Browder-Livesay] . It seems reasonable that there is a similar (though necessarily more complicated) method to calculate $L_m(Z_p)$, or perhaps $L_m(G)$ for finite G , explointing the structure of codimension 1 and 2 submanifolds of M (where $\pi_1 M = G$) , which reflects the structure of the 2-skeleton of $K(G, 1)$, (compare [Browder, F]) .

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