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Surgery and the Theory of Differentiable Transformation Groups

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Recent progress in differential topology has developed many powerful techniques for the study of differentiable manifolds, for classification of manifolds, embeddings, diffeomorphisms, etc. These methods have been applied to the study of differentiable transformation groups with some notable successes. I will try to outline in this paper some results of the theory of surgery (or spherical modification) which have given many good results about manifolds, and show how these results may be applied to study actions of groups, in particular actions of S^1 and S^3.

For the most part, the results of Chapters I and II are either well known, or in print in various places, or both, and include the work of many persons. Many of the results of Chapter III on semifree actions of S^1 are new however.

Chapter I is an exposition of the theory of surgery on 1-connected manifolds as developed by J. Milnor, M. Kervaire, S. P. Novikov, the author, and others, and gives some general applications in differential topology. The point of view follows that of [7] in general.

Chapter II discusses free actions of S^1 and S^3 on homotopy spheres, ways of constructing them, and questions about invariant and characteristic spheres. The results in this area are due for the most part to the Hsiang brothers, Montgomery and Yang, and M. Rothenberg (unpublished in the latter case).

In Chapter III, we discuss semifree actions of S^1 on homotopy spheres, that is, actions which are free outside the fixed point set F, under the additional assumption that F is a homotopy sphere. The approach here is new and yields many new results as well as new proofs of known results. In particular, we construct infinitely many inequivalent semifree S^1 actions on homotopy spheres with fixed point set the standard sphere.

We have omitted many things. In Chapter I we have not discussed Sullivan's point of view on the surgery problem [47], or surgery on non-simply connected manifolds, such as the theory of [51]. In Chap-

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ters II and III we have emphasized the construction of examples, rather than classification. In Chapter III we have discussed only S^1 actions, although a similar theory may be constructed for S^3 actions. Also the results may be extended to construct infinitely many actions with S^1 as fixed point set, by extending some of the results of Chapter I in a simple way. Throughout, we have not discussed $\mathbb{Z}_2$-actions, though many results are known.

Many questions remain open which seem amenable to attack in this direction:

1. What are the homotopy spheres which are being operated on in our constructions, and what is the knot type of the fixed point set?

2. In Chapter III, in the cases where our theorem does not construct an infinite number of semifree S^1 actions, are there in reality only a finite number? It seems likely when the dimension of the fixed point set F^q (a homotopy sphere) is even.

3. How can one study the problem of invariant spheres for non-free actions, in particular, for semifree actions?

This work on transformation groups has benefited greatly from contact with many people. In particular my collaboration with G. R. LIVESAY began my interest in this direction, and it was further stimulated and educated by conversations and lectures by W.-C. HSIANG, W.-Y. HSIANG, D. MONTGOMERY, C. T. YANG, G. BREDON, C. GIFFEN, and others.

I. The Surgery Problem and the Fundamental Results

All manifolds we shall deal with will be compact oriented with boundary, equipped with a differential structure, and diffeomorphisms will preserve orientation. In fact all the theorems of this section are valid in the piecewise linear (p. l.) category also (see [15]), using p. l. microbundles instead of linear bundles (see [32]) for stable normal bundles, and using p. l. "block bundles" (see [43, 39]), in place of normal bundles of embeddings. However we shall restrict our attention for the most part to the differentiable case.

In this chapter we shall describe the results of the theory of surgery for simply connected manifolds. The account here follows that of [7] closely, and may be considered a globalization of the paper of KERVAIRE-MILNOR [29] due to S. P. NOVIKOV ([40, 41]) and the author [8].

1. Poincaré Pairs

A pair (X,Y) is called an *m-dimensional Poincaré pair over R*, (R a ring) if there is an element $[X]\in H_m(X,Y;R)$ such that

$[X]\cap: H^q(X;R)\to H_{m-q}(X,Y;R)$ is an isomorphism for all q. If $R=\mathbb{Z}$, the ring of integers, we call (X,Y) a Poincaré pair, and if $Y=\phi$ we call X a Poincaré complex over R. (Recall that $x\cap y$ is defined on the chain level by the formula $x\cap y=\Sigma_i(y(x_i'))x_i$, where $y\in C^*$, $x\in C_*$, and $\Delta x=\Sigma_i x_i\otimes x_i'$, $\Delta: C_*\to C_*\otimes C_*$ being a chain approximation to the diagonal.) The element $[X]\in H_m(X,Y;R)$ is called the *fundamental class* of X, or the *orientation class* if $R=\mathbb{Z}$.

We recall some simple properties of Poincaré pairs developed in [7]:

(1.1) The diagram below is commutative (up to a sign) and all the vertical arrows are isomorphisms.

$$\begin{array}{ccccccccc}
\cdots\longrightarrow & H^{q-1}(Y;R) & \xrightarrow{\delta} & H^q(X,Y;R) & \xrightarrow{j^*} & H^q(X;R) & \xrightarrow{i^*} & H^q(Y;R) & \longrightarrow\cdots \\
 & \downarrow{\scriptstyle(\partial[X])\cap} & & \downarrow{\scriptstyle[X]\cap} & & \downarrow{\scriptstyle[X]\cap} & & \downarrow{\scriptstyle(\partial[X])\cap} & \\
\cdots\longrightarrow & H_{m-q}(Y;R) & \xrightarrow{i_*} & H_{m-q}(X;R) & \xrightarrow{j_*} & H_{m-q}(X,Y;R) & \xrightarrow{\partial} & H_{m-q-1}(Y;R) & \longrightarrow\cdots
\end{array}$$

(In particular Y is a Poincaré complex over R of dimension $m-1$ with fundamental class $[Y]=\partial[X]$.)

Now we shall describe how to "add" Poincaré pairs over a Poincaré pair contained in the boundary. One must have a compatibility relation among the fundamental classes.

Let $X=X_1\cup X_2$, $X_0=X_1\cap X_2$, $Y_i=Y\cap X_i$. Let $x\in H_m(X,Y;R)$, and let $x_i\in H_m(X_i,X_0\cup Y_i;R)$ be the image under the composite

$$H_m(X,Y;R)\longrightarrow H_m(X,X_{i+1}\cup Y;R)\xleftarrow{\approx} H_m(X_i,X_0\cup Y_i;R),\ i=1,2,$$

$$(i+1=1 \text{ for } i=2),\ x_0\in H_{m-1}(X_0,Y_0;R),\ x_0=\partial_1 x_1=-\partial_2 x_2,$$

$\partial_i: H_m(X_i,X_0\cup Y_i;R)\to H_{m-1}(X_0,Y_0;R)$ being the boundary operator in the triple $(X_i,X_0\cup Y_i,Y_i)$, composed with the inverse of the excision $H_{m-1}(X_0,Y_0;R)\xrightarrow{\approx} H_{m-1}(X_0\cup Y_i,Y_i;R)$.

(1.2) (Addition property) Two of the following three statements imply the third:

a) (X,Y) is a Poincaré pair over R with fundamental class x.

b) $(X_i,X_0\cup Y_i)$ is a Poincaré pair over R with fundamental class x_i, $i=1$ and 2.

c) (X_0,Y_0) is a Poincaré pair over R with fundamental class x_0.

The theorem (1.2) may be used in various directions for various purposes: for example, in the form (a), (c) imply (b) it is an important algebraic step in the proof of [9, Theorem 1.1]. In this account we shall use it to define sums of Poincaré complexes, i. e., we use (b), (c) imply (a).

If (X,Y) and (X',Y') are Poincaré pairs (over R), a map $f:(X,Y)\to(X',Y')$ is said to be *of degree* 1 if $\bar{f}_*[X]=[X']$, where $\bar{f}_*$ denotes the induced homology map $\bar{f}_*: H_m(X,Y;R)\to H_m(X',Y';R)$. If f is a map of degree 1, then we may define inverses for $\bar{f}_*: H_i(X, Y; R) \to H_i(X', Y'; R)$ and $f_*: H_i(X; R) \to H_i(X'; R)$ and the analogous cohomology maps as follows:

Consider the diagram

$$
\text{(1.3)} \qquad
\begin{array}{ccc}
H^{m-i}(X;R) & \xleftarrow{f^*} & H^{m-i}(X';R) \\
{\scriptstyle [X]\cap}\downarrow{\scriptstyle \cong} & & {\scriptstyle [X']\cap}\downarrow{\scriptstyle \cong} \\
H_i(X,Y;R) & \xrightarrow{\bar{f}_*} & H_i(X',Y';R)
\end{array}
$$

It follows easily from the properties of $\cap$-products that $\bar{f}_*([X]\cap f^*(x')) = [X']\cap x'$, for $x' \in H^{m-i}(X';R)$. If we define $\bar{\alpha}_*: H_i(X',Y';R) \to H_i(X,Y;R)$ by $\bar{\alpha}_*([X']\cap x') = [X]\cap f^*(x')$, then $\bar{f}_*\bar{\alpha}_* = 1$. A similar definition defines $\alpha_*: H_i(X';R) \to H_i(X;R)$ such that $f_*\alpha_* = 1$ and similar splitting maps in cohomology, $\bar{\alpha}^*$, α^*, as follows: $\alpha^*: H^{m-i}(X;R) \to H^{m-i}(X';R)$ is defined by $[X']\cap \alpha^*(x) = \bar{f}_*([X]\cap x)$ for $x \in H^{m-i}(X;R)$, and $\bar{\alpha}^*: H^{m-i}(X,Y;R) \to H^{m-i}(X',Y';R)$ is defined similarly, so that $\alpha^* f^* = 1$ on $H^*(X';R)$, $\bar{\alpha}^*\bar{f}^* = 1$ on $H^*(X',Y';R)$.

Clearly the same definitions work for Y to provide $\beta_*: H_*(Y';R) \to H_*(Y;R)$ and $\beta^*: H^*(Y;R) \to H^*(Y';R)$ and we define:

$$
\text{(1.4)} \qquad
\begin{aligned}
K_q(X;R) &= (\ker f_*)_q \subset H_q(X;R), \\
K^q(X;R) &= (\ker \alpha^*)^q \subset H^q(X;R), \\
K_q(X,Y;R) &= (\ker \bar{f}_*)_q \subset H_q(X,Y;R), \\
K^q(X,Y;R) &= (\ker \bar{\alpha}^*)^q \subset H^q(X,Y;R), \\
K_q(Y;R) &= (\ker (f|Y)_*)_q \subset H_q(Y;R), \\
K^q(Y;R) &= (\ker \beta^*)^q \subset H^q(Y;R).
\end{aligned}
$$

One may then prove the following properties of the K^* and K_*:

(1.5) There is a commutative (up to sign) diagram with exact rows and split columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\longrightarrow & & K^{q-1}(Y;R) & \longrightarrow & K^q(X,Y;R) & \longrightarrow & K^q(X;R) & \longrightarrow & K^q(Y;R) & \longrightarrow \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\longrightarrow & & H^{q-1}(Y;R) & \longrightarrow & H^q(X,Y;R) & \longrightarrow & H^q(X;R) & \longrightarrow & H^q(Y;R) & \longrightarrow \\
 & & \uparrow{\scriptstyle (f|Y)^*} & & \uparrow{\scriptstyle \bar{f}^*} & & \uparrow{\scriptstyle f^*} & & \uparrow & \\
\longrightarrow & & H^{q-1}(Y';R) & \longrightarrow & H^q(X',Y';R) & \longrightarrow & H^q(X';R) & \longrightarrow & H^q(Y';R) & \longrightarrow \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
 & & 0 & & 0 & & 0 & & 0 &
\end{array}
$$

A similar diagram exists in homology with similar properties.

(1.6) The functors K^* and K_* satisfy Poincaré duality, that is $[X]\cap$ carries $K^*(X,Y;R)$ into $K_*(X;R)$ etc. and $[X]\cap$ induces

$$
\begin{aligned}
[X]\cap &: K^q(X,Y;R) \longrightarrow K_{m-q}(X;R), \\
[X]\cap &: K^q(X;R) \longrightarrow K_{m-q}(X,Y;R), \\
(\partial[X])\cap &: K^{q-1}(Y;R) \longrightarrow K_{m-q}(Y;R)
\end{aligned}
$$

which are isomorphisms for all q.

(1.7) K^* and K_* satisfy the formulas of Universal Coefficients. In particular, if $f:(X,Y) \to (X',Y')$ is a map of degree 1 of Poincaré pairs (over $\mathbb{Z}$) then

$$K^q(X,Y;R) = \mathrm{Hom}(K_q(X,Y;\mathbb{Z}),R) + \mathrm{Ext}(K_{q-1}(X,Y;\mathbb{Z}),R)$$

and

$$K_q(X,Y;R) = K_q(X,Y;Z) \otimes R + \mathrm{Tor}(K_{q-1}(X,Y;\mathbb{Z}),R),$$

and similar formulas for $K^*(X;R)$, $K^*(Y;R)$, etc. If R is a field and f is a map of degree 1 of Poincaré pairs (over R) then

$$K^q(X,Y;R) = \mathrm{Hom}(K_q(X,Y;R),R)$$

and similar formulas for $K^*(X;R)$, $K^*(Y;R)$.

Taking $R = \mathbb{Q}$, the rational numbers, suppose f is a map of degree 1 over $\mathbb{Q}$ and suppose that $(f|Y)^*: H^*(Y';\mathbb{Q}) \to H^*(Y;\mathbb{Q})$ is an isomorphism. Then from (1.5) it follows that $K^*(X,Y;\mathbb{Q}) \cong K^*(X;\mathbb{Q})$. Then from (1.8) and the identity relating cup and cap products: $z([X]\cap y) = (y \cup z)([X])$, $z \in H^*(X;R)$, $y \in H^*(X,Y;R)$, we may deduce that the pairing:

$$
\text{(1.8)} \qquad
\begin{aligned}
&K^i(X,Y;R) \otimes K^{m-i}(X,Y;R) \to R, \\
&(x,y) = (x \cup y)[X],
\end{aligned}
$$

is for $R = \mathbb{Q}$, a non-singular pairing, and if $i = m - i$ is even, it is symmetric.

Thus suppose $f:(X,Y) \to (X',Y')$ is a map of degree 1 of Poincaré pairs over $\mathbb{Q}$, and that $(f|Y)^*: H^*(Y';\mathbb{Q}) \to H^*(Y;\mathbb{Q})$ is an isomorphism, and that $m = 4q$. Then we may define the index of f

$$\text{(1.9)} \qquad I(f) = \text{signature of } (\ ,\ ) \text{ on } K^{2q}(X,Y;\mathbb{Q}).$$

Now in the splitting

$$H^*(X,Y;\mathbb{Q}) = \text{image } \bar{f}^* + K^*(X,Y;\mathbb{Q})$$

it is easy to prove that the two factors are orthogonal under the pairing $(\ ,\ )$ and the pairing on image $\bar{f}^*$ is isomorphic to that on $H^*(X',Y';\mathbb{Q})$. It follows that if we define $I(X) = $ signature of $(\ ,\ )$ on $H^{2q}(X,Y;\mathbb{Q})$,

$$\text{(1.10)} \qquad I(f) = I(X) - I(X').$$

Now suppose $m=4k+1$, so that dimension $Y=4k$ and suppose $f:(X,Y)\to(X',Y')$ is a map of degree 1 of Poincaré pairs over $\mathbb{Q}$. Then $I(f|Y)$ is defined. On the other hand in $K^{2k}(Y;\mathbb{Q})$ the image of $K^{2k}(X;\mathbb{Q})$ is a subspace of half the rank, by Poincaré duality (1.6) and (1.7), and annihilates itself under the pairing. It follows that

(1.11) $$I(f|Y)=0.$$

Now suppose the pairs (X,Y) and (X',Y') are the "sums" of pairs, $X=X_1\cup X_2$, $X_1\cap X_2=X_0$, $Y_i=Y\cap X_i$, (X,Y) and $(X_i,X_0\cup Y_i)$ are Poincaré pairs over R, $i=1,2$ with compatible orientations (see (1.2)), and similarly for (X',Y'). Suppose $f:(X,Y)\to(X',Y')$ is a map of degree 1 over R and that $f(X_i)\subset X_i'$. It follows easily that $f_i=f|X_i$: $(X_i,X_0\cup Y_i)\to(X_i',X_0'\cup Y_i')$, $i=1,2$, and $f_0=f|X_0,Y_0)\to(X_0',Y_0')$ are maps of degree 1 over R.

Suppose that $(f_i|Y_i\cup X_0)^*$ $i=1,2$ are isomorphisms and that f_0^* is an isomorphism (all with $\mathbb{Q}$ coefficients). Then it is not hard to see that $K^{2k}(X,Y;\mathbb{Q})$ splits as the direct sum of $K^{2k}(X_1,X_0\cup Y_1;\mathbb{Q})$ and $K^{2k}(X_2,X_0\cup Y_2;\mathbb{Q})$ and we get if $\dim(X,Y)=4k$:

(1.12) (Addition property of index) $I(f)=I(f_1)+I(f_2)$.

We will want to define an analogous invariant in $\mathbb{Z}_2$ in dimensions $4k+2$, using $\mathbb{Z}_2$ cohomology, but its definition requires much more structure that the definition of index, and we shall not give its explicit definition here, (see [7, 29 and 14]).

2. Normal Maps and Cobordisms

Let (X,Y) be a pair of spaces, ξ^k a linear k-plane bundle over X. Let $(M^m,\partial M^m)$ be an m-dimensional compact differential manifold with boundary, ν^k its normal bundle for a smooth embedding

$$(M,\partial M)\subset(D^{m+k},S^{m+k-1}),\quad k>m+1.$$

(In such dimensions it follows that embeddings exist and are unique up to isotopy, so that ν^k is also unique.)

A normal map of $((M,\partial M),\nu)$ into $((X,Y),\xi)$ will consist of two maps (f,b), where $f:(M,\partial M)\to(X,Y)$ is a continuous map, and $b:\nu\to\xi$ is a linear bundle map lying over f.

A cobordism of a map $f:(M,\partial M)\to(X,Y)$ is a cobordism W of $(M,\partial M)$, i.e., a manifold W^{m+1} with $\partial W=M\cup V\cup M'$, where $\partial V=\partial M\cup\partial M'$, together with a map $F:(W,V)\to(X,Y)$ such that $F|M=f$. If ω^k is the normal bundle of $(W,V)\subset(D^{m+k}\times I,\ S^{m+k-1}\times I)$ (where $(M,\partial M)\subset(D^{m+k}\times 0,\ S^{m+k-1}\times 0)$, $(M',\partial M')\subset(D^{m+k}\times 1,\ S^{m+k-1}\times 1)$), and if (f,b) is a normal map, $b:\nu\to\xi$ over f, then a normal cobordism of (f,b) will be a cobordism (W,F) of f together with a linear bundle map $B:\omega\to\xi$ lying over F such that $B|(\omega|M)=B|\nu=b$.

We shall say that the cobordism is rel Y if $V=(\partial M\times I)$, and $F|V=f|\partial M$, $B|(\omega|V)=b|(\nu|\partial M)$. (We note that if $Y=\emptyset$, this is automatic.)

Now there is a natural collapsing map $C:(D^{m+k},S^{m+k-1})\to(T(\nu),T(\nu|\partial M))$ where $T(\nu)$ is the Thom complex of ν, $T(\nu)=E(\nu)/E_0(\nu)$, where $E(\nu)$ is the unit disk bundle, $E_0(\nu)$ is the unit sphere bundle associated to ν. The linear map $b:\nu\to\xi$ induces a map of Thom complexes $T(b):(T(\nu),T(\nu|\partial M))\to(T(\xi),T(\xi|Y))$, and $T(b)_*:\pi_{m+k}(T(\nu),T(\nu|\partial M))\to\pi_{m+k}(T(\xi),T(\xi|Y))$. Then the element $T(b)_*(\{C\})\in\pi_{m+k}(T(\xi),T(\xi|Y))$ will be called the *Thom invariant* of the normal map (f,b). Then as a standard application of the Thom transversality theorem (see [49, 1, 7]) we have

(2.1) Theorem. *The Thom invariant establishes a one to one correspondence between normal cobordism classes of maps of m-manifolds into* (X,Y), ξ, *and elements in* $\pi_{m+k}(T(\xi),T(\xi|Y))$.

This theorem allows one to apply the methods of homotopy theory to the study of normal cobordism of normal maps, to calculate upper bounds to the number of such classes etc.

Now suppose (X,Y) is a Poincaré pair of dimension m and that (f,b) is a normal map $f:(M^m,\partial M^m)\to(X,Y)$, $b:\nu\to\xi$ such that f is of degree 1. Suppose that ν and ξ are oriented and that b preserves the orientation. First we recall

(2.2) Thom Isomorphism Theorem. *If ξ is an oriented k-plane bundle over X, then there is a class $U\in H^k(T(\xi))$ such that U restricted to each fibre is the orientation class, and such that*

$$\begin{aligned}\cup U&:H^q(X)\longrightarrow H^{q+k}(T(\xi))\\ \cup U&:H^q(X,Y)\longrightarrow H^{q+k}(T(\xi),T(\xi|Y))\\ \cap U&:H_p(T(\xi))\longrightarrow H_{p-k}(X)\\ \cap U&:H_p(T(\xi),T(\xi|Y))\to H_{p-k}(X,Y)\end{aligned}$$

are isomorphisms.

(See [49, 33, 27].)

Here the cup and cap products are defined using the isomorphisms

$$H_*(T(\xi))\cong H_*(E(\xi),E_0(\xi)),$$
$$H_*(T(\xi),T(\xi|Y))\cong H_*(E(\xi),E_0(\xi)\cup E(\xi|Y)),$$

(and the corresponding isomorphisms in cohomology) and the appropriate products in the pair, $(E(\xi), E_0(\xi))$, etc.

It follows easily that if $b: v \to \xi$ preserves orientation, then the appropriate diagrams involving the Thom isomorphisms commute, for example:

$$\begin{array}{ccc} H_p(T(v), T(v|\partial M)) & \xrightarrow{T(b)_*} & H_p(T(\xi), T(\xi|Y)) \\ \cap U_v \downarrow & & \cap U_\xi \downarrow \\ H_{p-k}(M, \partial M) & \xrightarrow{f_*} & H_{p-k}(X, Y) \end{array}$$

Hence if f_* is of degree 1, then $T(b)_*$ is of "degree 1" also. In particular, since if g is a generator of $H_{m+k}(D^{n+k}, S^{m+k-1})$, $C_*(g) = x \in H_{m+k}(T(v), T(v|\partial M))$, such that $x \cap U_v = [M]$, it follows that the Thom invariant $\alpha \in \pi_{m+k}(T(\xi), T(\xi|Y))$ of the normal map (f, b) of degree 1 satisfies

(2.3) $$h(\alpha) \cap U_\xi = [X],$$

where $h: \pi_* \to H_*$ is the Hurewicz homomorphism.

Using (2.1) and (2.3) we get:

(2.4) *Normal cobordism classes of normal maps of degree* 1 *of m-manifolds into* (X, Y), ξ^k *are in one-to-one correspondence with elements* $\alpha \in \pi_{m+k}(T(\xi), T(\xi|Y))$ *such that* $h(\alpha) \cap U_\xi = [X]$, *the correspondence being given by the Thom invariant.*

In particular two such elements differ by an element in kernel h so that we get:

Normal cobordism classes of normal maps of degree 1 *of m manifolds into* (X, Y), ξ *are in one-to-one correspondence with the elements of kernel h,*

$$h: \pi_{m+k}(T(\xi), T(\xi|Y)) \to H_{m+k}(T(\xi), T(\xi|Y)).$$

In the stable range, it is a well known result of homotopy theory that kernel h is finite. Thus there are only a finite number of normal cobordism classes of normal maps of degree 1.

Now if (f, b) is a normal map of degree 1, such that $(f|\partial M)_*: H_*(\partial M) \to H_*(Y)$ is an isomorphism, $m = 4k$, then we have:

(2.5) $I(f)$ *is divisible by* 8.

This follows from the fact that $I(f)$ is the index of the intersection form on $K_{2k}(M; \mathbb{Z})$/torsion, and this form has determinant 1 and is even (i. e. $(x,x) \in 2\mathbb{Z}$ for all $x \in K_{2k}(M; \mathbb{Z})$/torsion). But a unimodular symmetric matrix over $\mathbb{Z}$ with even diagonal entries has determinant divisible by 8, (see [34]). That the form is even may be deduced from

the fact that a normal map sends Wu classes to Wu classes, or in several other ways (c. f. [7]).

Then besides $I/8$ in dimension $4k$, for a normal map (f, b), $f: (M^m, \partial M) \to (X, Y)$, $m = 4k+2$, such that $(f|\partial M)_*: H_*(\partial M; \mathbb{Z}_2) \to H_*(Y; \mathbb{Z}_2)$ is an isomorphism, we may define an invariant called the *Kervaire invariant*, or *Arf invariant* (it is given as the Arf invariant of a quadratic form on $(\ker f_*)_{2k+1}$ over $\mathbb{Z}_2$). Then we have the basic theorem:

(2.6) Invariant Theorem. *Let* (f, b), $f: (M, \partial M) \to (X, Y)$, $b: v \to \xi$ *be a normal map of degree* 1, $f_*: H_*(\partial M) \to H_*(Y)$ *an isomorphism,* (X, Y) *a Poincaré pair of dimension m, etc. Then there is an invariant* σ *defined with the following properties:*

(i) $\sigma \in \mathbb{Z}$, *if* $m = 4k$, $\sigma = I(f)/8$,
$\sigma \in \mathbb{Z}_2$, *if* $m = 4k+2$, $\sigma =$ *Kervaire invariant,*
$\sigma = 0$, *if* m *is odd.*

(ii) *If* (f, b) *is normally cobordant* rel Y *to* (f', b') *such that* $f'_*: H_*(M') \to H_*(X)$ *is an isomorphism, then* $\sigma = 0$.

(iii) $\sigma(f|\partial M, b|(v|\partial M)) = 0$.

(iv) σ *is additive, i.e. if* $M = M_1 \cup M_2$, $M_0^{m-1} = M_1 \cap M_2$, $X = X_1 \cup X_2$, $X_0 = X_1 \cap X_2$, *etc., as in* (1.2), $(M, \partial M)$ *and* (X, Y) *are the sum of two manifolds and two Poincaré pairs, respectively,* (f, b) *is a normal map of degree* 1, $f(M_i) \subset X_i$, *so that* $f_i = f|M_i$, $b_i = b|(v|M_i)$, *etc.,* (f_i, b_i) $i = 1, 2$ *are normal maps, and* σ *is defined for each. Then*

$$\sigma(f) = \sigma(f_1) + \sigma(f_2).$$

(v) *If* $Y = \phi$ *and* $m = 4k$, *then* $\sigma(f, b) = \frac{1}{8}$ (Index M − Index X).

In the case $m = 4k$, most of these properties follow easily from the results of § 1. The definition and properties of the Kervaire invariant $(m = 4k+2)$ are much more difficult to deduce, and we refer to [7] for the complete account in the form given here.

It turns out that in the simply connected situation when dimension $m \geq 5$, the invariant σ is the only obstruction to finding a normal cobordism rel Y to a homotopy equivalence. Explicitly we have:

(2.7) Fundamental Theorem of Surgery. *Let* (X, Y) *be a Poincaré pair of dimension* $m \geq 5$, X 1*-connected,* ξ^k *a linear k-plane bundle over* X, *and let* (f, b) *be a normal map of degree* 1, $f: (M, \partial M) \to (X, Y)$, $b: v \to \xi$, *such that* $f_*: H_*(\partial M) \to H_*(Y)$ *is an isomorphism. Then* (f, b) *is normally cobordant* rel Y *to* (f', b') *with* $f': M' \to X$ *a homotopy equivalence if and only if* $\sigma(f, b) = 0$. *In any case* (f, b) *is normally cobordant to* (f', b') *with* f' $[m/2]$*-connected, where* $[a] =$ *greatest integer* $\leq a$.

This theorem was first proved by KERVAIRE and MILNOR [29] in the case where X was a disk or a sphere, and subsequently generalized by S. P. NOVIKOV [40, 41] and the author [8]. A complete account in the present form is given in [7]. Extensions of these results for the non-simply connected case have been given by WALL [51, 52].

The following theorem of KERVAIRE-MILNOR was to appear in the second part of [29], which unfortunately has not been published to date. A proof is given in [7].

(2.8) Plumbing Theorem. *For $(X,Y)=(D^m,S^{m-1})$, ξ^k=trivial bundle, $m\geq 5$, all values are realized for $\sigma(f,b)$, for (f,b) normal maps of degree 1, $f:(M^m,\partial M)\to(D^m,S^{m-1})$ with $f:\partial M\to S^{m-1}$ a homotopy equivalence.*

(2.9) Remark. For $m=6$, 14, 30 all values are realized for $\sigma(f,b)$, $f: M^m\to S^m$.

The proof is well known for $m=6$, 14, and for $m=30$ it is proved in [14].

One may now deduce some powerful corollaries such as the following due to S. P. NOVIKOV [41] and the author [8].

(2.10) Homotopy Type of Manifolds. *Let X be a 1-connected Poincaré complex of dimension $m\geq 5$, ξ a linear k-plane bundle over X, $k>m+1$, $\alpha\in\pi_{m+k}(T(\xi))$ such that $h(\alpha)\cap U_\xi=[X]\in H_m(X)$. If either*

(i) *m is odd,*

(ii) *$m=4k$ and condition H below holds*

or

(iii) *$m=6$, 14 or 30,*

then there is a homotopy equivalence f of a smooth m-manifold M^m with X, $f: M\to X$ such that $f^*(\xi)$ is the normal bundle of $M^m\subset S^{m+k}$. In cases (i) *and* (ii) *above, α represents the normal cobordism class of the homotopy equivalence f.*

The condition (H) is essentially that the Hirzebruch Index Theorem [24] hold. Namely, let ξ^{-1} be the bundle inverse of ξ, p_i be the Pontryajin class, and L_k be the Hirzebruch polynomial [24].

(H) $$\text{Index}\, X = L_k(p_1(\xi^{-1}),\ldots,p_k(\xi^{-1}))\ [X]\ .$$

Then (2.10) is an immediate consequence of (2.4), (2.7), in case $m=4k$ of (2.6) (v) and the Hirzebruch index theorem, and in case $m=6$, 14 or 30, of (2.9) and (2.6) (iv).

If (f_i,b_i), $f_i:(M_i^m,\partial M_i)\to(X_i,Y_i)$ are normal maps of degree 1, $i=1,2$, then we may define the sum of the two along a cell in the two boundaries, (see (1.2)). We choose a component V_i of Y_i, $i=1,2$ and let $V_i=V_i^0\cup e_i^{m-1}$, with attaching map $\alpha_i: S^{m-2}\to V_i^0$, e_i^{m-1} being the top dimensional cell of V_i, so that $H_{m-1}(V_i^0)=0$. Then the sum $X_1\coprod X_2$ along V_1 and V_2 is defined $X_1\coprod X_2=X_1\cup(e^{m-1}\times I)\cup X_2$, with

$$e^{m-1}\times 0 \text{ identified with } e_1^{m-1}\subset V_1,$$
$$e^{m-1}\times 1 \text{ identified with } e_2^{m-1}\subset V_2,$$

and the sum $Y_1\# Y_2$ along V_1 and V_2 is defined by $Y_1^0\cup(S^{m-2}\times I)\cup Y_2^0$ with $Y_i^0=V_i^0\cup$ other components of Y_i, $i=1,2$ and

$$S^{m-2}\times 0 \text{ identified with } \alpha_1(S^{m-2})\subset V_1^0,$$
$$S^{m-2}\times 1 \text{ identified with } \alpha_2(S^{m-2})\subset V_2^0.$$

Then $(X_1\coprod X_2, Y_1\# Y_2)$ is the Poincaré pair which is the sum (see (1.2)).

The connected sum $M_1\coprod M_2$ along components of ∂M_1 and ∂M_2 is defined similarly using differentiably embedded cells in ∂M_1 and ∂M_2, $\partial M_1\#\partial M_2$ defined similarly. There is a canonical way to put a differential structure on the sum known as "straightening the angle" (see [19, Chapter I, § 3]).

If we choose the cells in ∂M_i so that f_i maps them into those chosen for Y_i, we may, if necessary changing f_i by a homotopy, and covering by a homotopy of b_i, get an induced map

$$f_1\coprod f_2:(M_1\coprod M_2,\partial M_1\#\partial M_2)\to(X_1\coprod X_2, Y_1\# Y_2),$$

of degree 1, and arrange a map $b_1\coprod b_2$ covering $f_1\coprod f_2$.

It follows from results of [18, 42] that the sum of manifolds depends only on the components and orientations of the cells chosen, and a similar fact is true for Poincaré pairs. On the other hand, the maps defined may depend on the choice of homotopies if Y_i is not simply connected.

(2.11) Theorem. *Let (X,Y) be an m-dimensional Poincaré pair with X 1-connected, $Y\neq\phi$, $m\geq 5$, and let (f,b), $f:(W^m,\partial W^m)\to(X,Y)$ be a normal map of degree 1 such that $(f|\partial W)_*: H_*(\partial W)\to H_*(Y)$ is an isomorphism. Then there is a normal map of degree 1, (g,c), $g:(U^m,\partial U^m)\to(D^m,S^{m-1})$ with $g|\partial U$ a homotopy equivalence, such that $(f,b)\coprod(g,c)$ is normally cobordant* rel Y *to a homotopy equivalence. In particular, (f,b) is normally cobordant to a homotopy equivalence.*

Proof. Let $\sigma=\sigma(f,b)$ be the obstruction to surgery rel Y in (2.6). Let $g:(U^{m+1},\partial U^{m+1})\to(D^{m+1},S^m)$ (g,c) a normal map, with $\sigma(g,c)=-\sigma$, and $g|\partial U:\partial U\to S^m$ a homotopy equivalence, which exists by the Plumbing Theorem (2.8). Take the sum of the two normal maps (f,b) and (g,c) along a cell in ∂W and another in ∂U. Then by (2.6) (iv), $\sigma(f\coprod g, b\coprod c)=\sigma-\sigma=0$, so by the Fundamental Theorem (2.7), $(f\coprod g, b\coprod c)$ is normally cobordant rel Y to a homotopy equivalence.

Recall that a cobordism W, $\partial W = M \cup M'$ is called an h-cobordism if the inclusions $M \subset W$ and $M' \subset W$ are homotopy equivalences. It is a result of SMALE [44] that if $\partial M = \phi$, $\dim W \geqslant 6$, and W is 1-connected then W is diffeomorphic to $M \times I$ and $M' \times I$, and in particular, M is diffeomorphic to M'.

As a consequence of (2.11) we have a classification theorem due to S. P. NOVIKOV [40, 41],

(2.12) Classification of Manifolds. *Let* (f_i, b_i) *be normal maps,* $f_i: M_i \to X$, $i=0,1$, *such that* f_i *are homotopy equivalences,* X *a 1-connected Poincaré complex of dimension* $m \geqslant 4$. *Suppose* (f_0, b_0) *and* (f_1, b_1) *are normally cobordant. Then* M_0 *is h-cobordant to* $M_1 \# \Sigma$, *where* Σ *is a homotopy m-sphere which bounds a parallelizable manifold. In particular if m is even* >4, *then* M_0 *is diffeomorphic to* M_1.

KERVAIRE and MILNOR [29] have shown that the group of h-cobordism classes of homotopy m-spheres which bound parallelizable manifolds $m \geqslant 4$ is a cyclic group of finite order, 0 if m is even, or $m=5,13$, order at most 2 if $m=4k+1$, and for $m=4k+3$ they calculated its order, up to a factor of 2 in some cases. In [17], it was shown the order is 2 for $m=8k+1$, and in [14], it is shown that the order is 2 for $m=4k+1$ except possibly for $m=2^q-3$, and for $m=29$ the group is zero. This information together with (2.12) gives some upper bounds for the number of closed manifolds in a given normal cobordism class.

Proof of (2.12). If (F,B) $F: W \to X$ is the normal cobordism, consider F as a map

$$F:(W, M_0 \cup M_1) \to (X \times I, X \times 0 \cup X \times 1),$$

$F(M_i) \subset X \times i$, $i=0,1$. Then (F,B) satisfies the hypotheses of (2.11), so $(F,B) \coprod (g,c)$ (along M_1) is normally cobordant $\operatorname{rel} X \times 0 \cup X \times 1$) to a homotopy equivalences, i. e. an h-cobordism between M_0 and $M_1 \# \partial U$, $(\partial U = \Sigma)$. Since $g:(U, \partial U) \to (D^{m+1}, S^m)$ and D^{m+1} is contractible, it follows that the stable normal bundle of U is trivial, and since U has non-empty boundary, it follows that U is parallelizable.

As another application of (2.11) we have the following theorem of WALL [53] which extends (2.10) and (2.12) to the case of bounded manifolds.

(2.13) Theorem. *Let* (X,Y) *be an m-dimensional Poincaré pair,* $Y \neq \phi, m \geqslant 6$, X *and* Y *1-connected,* ξ *a k-plane bundle over* X, $k > m+1$, *and* $\alpha \in \pi_{m+k}(T(\xi), T(\xi|Y))$ *such that* $h(\alpha) \cap U_\xi = [X] \in H_m(X,Y)$. *Then there is a normal map* (f,b) *in the class determined by* $\alpha, f:(M, \partial M) \to (X,Y)$, *such that* f *is a homotopy equivalence, and such an* (f,b) *(and hence* $(M, \partial M)$*) is unique up to diffeomorphism.*

Proof. Consider a representative (h,a) of the normal cobordism class of $\alpha, h:(N, \partial N) \to (X,Y)$. Then by (2.6) (iii) and (2.7), $(h|\partial N, a|\partial N)$ is normally cobordant to a homotopy equivalence. This normal cobordism extends in an obvious way to a normal cobordism of (h,a) to (h',a') such that $h':(N', \partial N') \to (X,Y)$, $h'|\partial N': \partial N' \to Y$ is a homotopy equivalence. Then (2.11) implies (h',a') is normal cobordant rel Y to a homotopy equivalence.

To prove uniqueness we consider (f_i, b_i), $f_i:(M_i, \partial M_i) \to (X,Y)$ homotopy equivalences, $i=0,1$, $F:(W^{m+1}, V^m) \to (X,Y)$, (F, B) a normal cobordism between (f_0,b_0) and (f_1,b_1). As in the proof of (2.12), we consider F as a map $F:(W, V, M_0, M_1) \to (X \times I, Y \times I, X \times 0, X \times 1)$ (we recall that $\partial W = M_0 \cup V \cup M_1$, $\partial V = \partial M_0 \cup \partial M_1$. Then $\sigma(F|\partial W) = \sigma(F|V) + \sigma(F|M_0) + \sigma(F|M_1)$ by (2.6) (iv), $\sigma(F|\partial W) = 0$ by (2.6) (iii), and $\sigma(F|M_i) = 0$, $i=0,1$, since $F|M_i = f_i$ is a homotopy equivalence. Hence $\sigma(F|V) = 0$ so there is by (2.7) a normal cobordism of $F|V$ rel $Y \times 0 \cup Y \times 1$ to a homotopy equivalence. Hence we get a new normal cobordism (F',B') between (f_0,b_0) and (f_1,b_1) which is an h-cobordism between ∂M_0 and ∂M_1 in $Y \times I$. Then as in (2.11) we may add a normal map (g,c), $g:(U^{m+1}, \partial U^{m+1}) \to (D^{m+1}, S^m)$ to (F',B') so that the result is normally cobordant to a homotopy equivalence, i.e. h-cobordism. But if (g,c) is added to (F',B') along the part of the boundary of W' between ∂M_0 and ∂M_1, i.e. away from M_0 and M_1, then $(F' \coprod g, B' \coprod c)$ is still a normal cobordism between (f_0,b_0) and (f_1,b_1). Hence we arrive at an h-cobordism between them, and applying the h-cobordism theorem of SMALE [44] twice, first to the boundary, and then to the interior, we arrive at the result.

Now we give an example of an embedding theorem that can be proved with these methods. The theorem is similar to a theorem of the author in [10] and the proof is identical.

Suppose the total space of a q-plane bundle α^q over a space A is contained as an open set in a space X, and let ξ^k be a k-plane bundle over X. If (f,b), $f: M^m \to X$, $b: \nu \to \xi$ is a normal map, then by making f transverse regular to $A \subset X$, we may suppose $f^{-1}(A) = N^n$, N^n has normal bundle η^q in M^m, $m-n=q$, and $f|E(\eta)$ is a linear bundle map, $c: \eta \to \alpha$, where $E(\eta)$ is the total space of the vector bundle η which is a tubular neighborhood of N in M. Then $(f|N, c+b|(\nu|N))$ is a normal map into A, with bundle $\alpha + (\xi|A)$, (since the normal bundle of N in D^{m+k} is the sum of its normal bundle in M and the restriction of M's normal bundle in D^{m+k}, i.e. $\eta + (\nu|N)$.)

Suppose X, A Poincaré complexes are, and the collapsing map $X \to T(\alpha)$ is of degree 1, and suppose M a closed manifold and f is of degree 1. Then $(f|N, c+b|(\nu|N))$ is a normal map of degree 1, and hence $\sigma(f|N, c+b|(\nu|N))$ is defined. We will set $\sigma(f|N, c+b|(\nu|N)) = \sigma_A(f,b)$

to emphasize that its value does not depend on the particular way in which f was made t-regular on A (see (2.6) (iii)).

Now suppose $f: M^m \to M'^m$ is a homotopy equivalence of smooth manifolds. Then if we take over M' the bundle ξ^k such that $f^*(\xi)=\nu$, the normal bundle of M in D^{m+k}, $k>m+1$, then choosing a bundle map $b: \nu \to \xi$, (f,b) is a normal map. If N'^n is a smooth closed submanifold of M'^m, then as above $\sigma_{N'}(f,b)$ is defined, and it can be shown to be independent of the choice of b, above, so we may write in this case $\sigma_{N'}(f,b)=\sigma_{N'}(f)$.

(2.14) Theorem. *Let $f: M^m \to M'^m$ be a homotopy equivalence of closed smooth 1-connected manifolds, $N'^n \subset M'^m$ a 1-connected smooth submanifold, α' its normal bundle, such that $M'-N'$ is 1-connected, $n \geqslant 5$. If $\sigma_{N'}(f)=0$, then (f,b), for any choice of b, is normally cobordant to (f',b'), $f': M \to M'$ a homotopy equivalence, such that, if $f'^{-1}(N')=N$, $f':(M,N,M-N) \to (M',N',M'-N')$ is a homotopy equivalence on each term. In particular, M has a submanifold homotopy equivalent to N', with complement homotopy equivalent to $M'-N'$ and normal bundle induced from α'.*

This theorem is a special case of a very general theorem about submanifolds and "supermanifolds" which has many applications, to existence and isotopy of embeddings, to study of manifolds with free or free abelian fundamental group (c.f. [11, 10, 9], and a general treatment will be given in a later paper.

Proof. If $\sigma_{N'}(f,b)=0$, then $f^{-1}(N')=N''$ is normally cobordant to an N homotopy equivalent to N'. Let (G,B), $G:(U^{n+1}, N'' \cup N) \to (N' \times I, N' \times 0 \cup N' \times 1)$ be the normal cobordism, $B: \omega \to \alpha' + (\xi | N')$, ω the normal bundle of $U \subset S^{m+k} \times I$. Then $B^{-1}(\alpha')=\alpha_0$ is a q-plane bundle over U, $\alpha_0 | N''=\alpha''$, the normal bundle of $N'' \subset M$. Then $M \times I \cup E(\alpha_0)$ with $E(\alpha'') \times 1$ identified with $E(\alpha_0 | N'')$, defines a cobordism W of M, and f on $M \times I$ together with $B | \alpha_0$ defines a map $F: W \to M'$ with $F^{-1}(N')=U$, (see figure). Then b over $M \times I$ and B

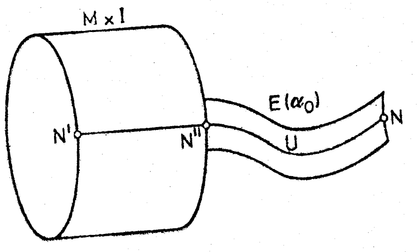

Fig.

restricted to the orthogonal complement of α_0 defines a map of the normal bundle of W into ξ, to get a normal cobordism of (f,b) with (f_1,b_1). Now $f_1: M_1 \to M'$ is no longer a homotopy equivalence, but $f_1^{-1}(N')=N$, $f_1 | N: N \to N'$ is a homotopy equivalence, $f_1^*(\alpha')=\alpha=$ normal bundle of $N \subset M_1$, so $f_1 | \partial E(\alpha): \partial E(\alpha) \to \partial E(\alpha')$ is also a homotopy equivalence. Hence (f_1,b_1) is the sum of two maps

$$f_1 | E(\alpha): (E(\alpha), \partial E(\alpha)) \to (E(\alpha'), \partial E(\alpha'))$$

and

$$f_1 | (M_1 - E(\alpha)), (M_1 - E(\alpha), \partial E(\alpha)) \to (M' - E(\alpha'), \partial E(\alpha'))$$

along $\partial E(\alpha)$. Hence

$$\sigma(f_1)=\sigma(f_1 | E(\alpha)) + \sigma(f_1 | M_1 - E(\alpha))$$

by (2.6) (iv), and since $f_1 | E(\alpha)$ is a homotopy equivalence, $\sigma(f_1 | E(\alpha))=0$ and

$$\sigma(f_1)=\sigma(f_1 | M_1 - E(\alpha)).$$

Now f_1 is normally cobordant (by the construction) to f, and f is a homotopy equivalence, so $\sigma(f_1)=\sigma(f)=0$, by (2.6) (iii), and hence $\sigma(f_1 | M_1 - E(\alpha))=0$. Now $M' - E(\alpha')$ is homotopy equivalent to $M'-N'$ which is 1-connected by hypothesis, so from (2.7) it follows that $f_1 | M_1 - E(\alpha)$ is normally cobordant rel $\partial E(\alpha')$ to a homotopy equivalence. This gives a normal cobordism of f_1 with $f_2: M_2 \to M'$, $f_2=f_1$ on $E(\alpha)$ and $f_2 | M_2 - E(\alpha)$ is a homotopy equivalence with $M' - E(\alpha')$. It follows from the Mayer-Vietoris theorem that $f_{2*}: H_*(M_2) \to H_*(M')$ is an isomorphism, and hence a homotopy equivalence, by the theorem of J. H. C. Whitehead, and $f_2:(M_2, N, M_2 - N) \to (M', N', M' - N')$ is a homotopy equivalence on each term. Now (f_2,b_2) is normally cobordant to (f,b), so by Novikov's theorem (2.12), (f,b) is diffeomorphic to $(f_2,b_2) \# (g,c)$, where $g: \Sigma^m \to S^m$ is a homotopy equivalent. But if Σ is added to $M_2 - N$, i.e. away from N, then $M=M_2 \# \Sigma$ still contains N with normal bundle α, complement homotopy equivalent to $M'-N'$, etc. This completes the proof of (2.14).

Next we use the methods of surgery to construct exotic diffeomorphisms of simply connected manifolds. This method was developed in [13] and we review it in the special case of diffeomorphisms homotopic to the identity.

Let W^{m+1} be a manifold, $M^m \subset W^{m+1}$ a smooth submanifold, M 1-connected, and let $h: W^{m+1} \to M^m \times S^1$ be a homotopy equivalence with the property that $h^{-1}(M \times s)=M \subset W$, some $s \in S^1$; and $h'=h | M: M \to M \times s$ is given by $h'(x)=(x,s)$, $x \in M$. Let $m \geqslant 5$.

(2.15) *There is a diffeomorphism $f: M \to M$ such that f is homotopic to the identity and W is diffeomorphic to the* mapping torus M_f *of* f, $M_f = M \times I$ *modulo the identification* $(x,0) = (f(x), 1)$, $x \in M$.

If we "cut W along M", i.e. remove a neighborhood $M \times I$ of M in W, we obtain a manifold $\bar{W}$, $\partial \bar{W} = M \times 0 \cup M \times 1$, and f restricts to a map

$$\bar{f}: (\bar{W}, M \times 0 \cup M \times 1) \to (M \times I, M \times 0 \cup M \times 1)$$

with $\bar{f}|M \times i = \text{identity}$, $i = 0, 1$, (thinking of $M \times S^1 = (M \times I) \cup (M \times I)$ and assuming $f|M \times I = \text{identity}$, $M \times I \subset W$). Since M is 1-connected and f is a homotopy equivalence, it is easy to deduce that $\bar{W}$ is 1-connected and $\bar{f}$ is a homotopy equivalence, and hence $\bar{W}$ is an h-cobordism between $M \times 0$ and $M \times 1$. Since $m \geqslant 5$, SMALE's h-cobordism theorem implies that there is a diffeomorphism $F: \bar{W} \to M \times I$ such that $F|M \times 0 = \text{identity}$. Define $f = F|M \times 1: M \to M$, which is a diffeomorphism. Then $W = \bar{W} \cup (M \times I)$ with $M \times i \subset \bar{W}$ identified by the identity with $M \times i \subset M \times I$, $i = 0, 1$. Then $G = F \cup \text{identity}$: $W = \bar{W} \cup (M \times I) \to M_f = (M \times I) \cup (M \times I)$ (with $M \times 0$ identified with $M \times 0$ by the identity, $M \times 1$ identified with $M \times 1$ by f) defines a diffeomorphism of W with M_f = the mapping torus of f.

Now $hG^{-1}: M_f \to M \times S^1$ is a homotopy equivalence and $hG^{-1}|M = \text{identity}: M \to M \times s$, $s \in S^1$. "Cutting along M" again we get a map $H: M \times I \to M \times I$, $H|M \times 0 = \text{identity}$, $H|M \times 1 = hG^{-1}|M \times 1 = (h|M \times 1) f^{-1} = f^{-1}$ since $h|M \times 1 = \text{identity}$, $M \times 1 \subset W$. Hence H defines a homotopy between f^{-1} and the identity, and hence f is homotopic to the identity, which proves (2.15).

We note that the analog of (2.15) is true with a similar proof if we take another bundle over S^1 instead of $M \times S^1$. Also similar results hold for bounded M (see [13]).

Now we investigate the problem of finding such manifolds W as in (2.15), using the methods of surgery.

Let ξ^k be a linear k-plane bundle over $M^m \times S^1$ where M^m is a smooth 1-connected manifold of dimension $m \geqslant 5$, $k > m+2$. Let ν^{k+1} be the normal bundle of M^m in S^{m+k+1} and let $b: (\xi^k|M \times s) + \varepsilon^1 \to \nu^{k+1}$ be a linear bundle equivalence, $s \in S^1$. Let $\eta: T(\xi) \to T(\xi|M \times s) + \varepsilon^1)$ be the natural map which collapses $T(\xi|M \times I)$ to a point, where $M \times I \subset M \times S^1$ is disjoint from the fibre $M \times s$. Let $c \in \pi_{m+k+1}(T(\nu))$ be the homotopy class of the natural collapsing map.

(2.16) Theorem. *Let $\alpha \in \pi_{m+k+1}(T(\xi))$ be such that $T(b)_* \eta_*(\alpha) = c \in \pi_{m+k+1}(T(\nu))$, and let (f,b) be the normal map $f: W \to M \times S^1$ corresponding to α. Then (f,b) is normally cobordant to (f',b'), $f': W' \to M \times S^1$, $W' = M_g$, $g: M \to M$ a diffeomorphism, $f'|M = \text{identity}$, if and only if $\sigma(f,b) = 0$.*

This theorem is essentially a special case of a theorem about manifolds with $\pi_1 = \mathbb{Z}$ proved in [11]. The proof is similar to that of (2.14).

Proof. Since $T(b)_* \eta_*(\alpha) = c$, it follows by a transversality argument that $f|f^{-1}(M \times s)$ is normally cobordant to the identity $M \to M \times s$. By the same argument as in the proof of (2.14) we may find a normal cobordism of f to $f': W \to M \times S^1$, $M \times I \subset W$ and $f'|M \times I: M \times I \to M \times I \subset M \times S^1$ is the identity. Let $U = W - \text{int}\, M \times I$ so that $f'' = f'|U: U \to M \times I$ (the other half of $M \times S^1 = (M \times I) \cup (M \times I)$). Then f'' is a normal map, $f''|\partial U$ is the identity, so $\sigma(f'', b'')$ is defined, and by (2.6) (iv), $\sigma(f'', b'') = \sigma(f', b')$, while $\sigma(f', b') = \sigma(f, b)$ since they are normally cobordant. Hence if $\sigma(f,b) = 0$, then $\sigma(f'', b'') = 0$, and since $M \times I$ is 1-connected, dimension $\geqslant 5$, it follows from (2.7) that (f'', b'') is normally cobordant rel $M \times 0 \cup M \times 1$ to a homotopy equivalence, $f''': (U', \partial U') \to (M \times I, M \times \{0,1\})$, $\partial U' = M \times 0 \cup M \times 1$ and $f'''|M \times 0 \cup M \times 1 = \text{identity}$. Then the union $U' \cup M \times I = W'$, and $f''' \cup \text{id}: W' \to M \times S^1$ is a homotopy equivalence satisfying the hypotheses of (2.15) and the result follows.

Let $\varphi: W^{m+1} \to M^m \times S^1$ be a homotopy equivalence such that $M \subset W$, $\varphi|M = \text{identity}: M \to M \times s$, and let ξ^k be the normal bundle of W in S^{m+d+1}, $k > m+2$. Then it follows that $\xi|M + \varepsilon^1 = \nu^{d+1}$ is the normal bundle of M in S^{m+k+1}, since $M \subset M \times I \subset W$, and it follows from the theorem of HIRSCH [23] and ATIYAH [2] that there is a fibre homotopy equivalence $\beta: \xi \to \nu \times \varepsilon^0$ covering φ such that $\beta|(\xi|M)$ is a linear equivalence.

Thus in looking for bundles ξ to use in applying (2.16), we may restrict our attention to bundles ξ over $M \times S^1$ such that there is a fibre homotopy equivalence $\beta: \xi \to \nu \times \varepsilon^0$ which restricted to $\xi|M$ is a linear equivalence. Now $M \times S^1/M \times s$ is homeomorphic to $\Sigma(M_+)$, the reduced suspension of M_+, where $M_+ = M \cup (\text{point})$. Then the virtual bundle $\xi - (\nu \times \varepsilon^0)|M \times s$ is trivial (as a linear bundle), so $\xi - (\nu \times \varepsilon^0) = h^*(\gamma)$ where $h: M \times S^1 \to \Sigma(M_+)$ is the collapsing map, $\gamma \in KO(\Sigma(M_+))$. Since $M \times s \subset M \times S^1$ is a retract it follows that

$$0 \to KO(\Sigma(M_+)) \to KO(M \times S^1) \to KO(M \times s) \to 0$$

so γ is unique, and it follows similarly that γ is fibre homotopically trivial. We denote by $L(X) \subset KO(X)$ the subgroup of fibre homotopically trivial bundles.

Conversely if $\gamma \in L(\Sigma(M_+))$, then one may assume the fibre homotopy trivialization of γ is linear at the base point. Then this induces a fibre homotopy equivalence $\beta: h^*(\gamma^q) \to \varepsilon^q$, such that $\beta|(h^*(\gamma)|M\times s)$ is linear. Hence $\alpha = \mathrm{id} + \beta: \nu\times\varepsilon^0 + h^*(\gamma) \to (\nu\times\varepsilon^0)+\varepsilon^q = (\nu^k+\varepsilon^q)\times\varepsilon^0$ is a fibre homotopy trivialization, linear over $M\times s$, and $\nu^k+\varepsilon^q$ is the normal bundle of M in S^{m+k+q}. Let $c\in\pi_{m+k+q+1}(T((\nu^k+\varepsilon^q)\times\varepsilon^0))$ be the collapsing map for $M\times S^1$, where $(\nu^k+\varepsilon^q)\times\varepsilon^0$ is the normal bundle of $M\times S^1$ in $S^{m+k+q+1}$. Then $\xi=\nu\times\varepsilon^0+h^*(\gamma)$ and $T(\alpha^{-1})_*(c)$ satisfy the hypotheses of (2.16).

(2.17) Lemma. *The Pontryagin class defines a linear map* $P: KO(\Sigma X)\to H^{4*}(\Sigma X)$, *i.e.* $P(x+y)=P(x)+P(y)-1$, *where* $x,y\in KO(\Sigma X)$, $P=1+p_1+p_2+\cdots$ *is the total Pontryagin class,* $p_i\in H^{4i}$.

Proof. In general $P(\alpha+\beta)=P(\alpha)P(\beta)$ (see [33, 27]), but ΣX being a suspension, products of positive dimensional classes are zero, and the result follows.

If $I=(i_1,i_2,\ldots,i_n)$, define the Pontryagin number of a bundle δ over a manifold N, by

$$P(I,\delta)[N] = \langle p_{i_1}(\delta)p_{i_2}(\delta)\ldots p_{i_n}(\delta), [N]\rangle.$$

(2.18) Lemma. *The map* $R: KO(\Sigma(M_+))\to\mathbb{Q}$ *given by* $R(\gamma)=\Sigma_k\lambda_k P(I_k,(\nu\times\varepsilon^0)+h^*(\gamma))[M\times S^1]$, $\lambda_k\in\mathbb{Q}$ *is a linear map.*

Proof. Since all products of positive dimensional elements of $H^*(\Sigma(M_+))$ are zero, and $h^*(P(\gamma))=P(h^*(\gamma))$, it follows that for any polynomial $G(x_1,\ldots,x_s)$, $G(a_1,a_2,\ldots,p_1(h^*(\gamma)),\ldots,p_t(h^*(\gamma)))$ is linear in the elements $p_i(h^*(\gamma))$, and the coefficients depend only on the a_i's and the coefficients of G. Since $P(\nu\times\varepsilon^0+h^*(\gamma))$ is a polynomial, $P(I_k,(\nu\times\varepsilon^0)+h^*(\gamma))$ is a polynomial, and the result follows.

(2.19) Theorem. *Let* M^m *be a 1-connected manifold,* $m\geq 5$, *m even or* $m=5$, *13 or 29. Then for each element* $\gamma\in L(\Sigma(M_+))$ *there is a diffeomorphism* $f: M\to M$ *such that* f *is homotopic to the identity, and the bundle over* $M\times S^1$ *corresponding to the normal bundle of the mapping torus* M_f *is* $\nu\times\varepsilon^0+h^*(\gamma)$, $h: M\times S^1\to\Sigma(M_+)$, ν *the stable normal bundle of* M. *If* $m\equiv 3 \bmod 4$, *this is true for a submodule* L' *of* $L(\Sigma(M_+))$, *where* L' = *kernel of the linear map* $L(\Sigma(M_+))\to\mathbb{Q}$ *induced by the Hirzebruch L-genus.*

This follows immediately from (2.16), the discussion following and (2.18).

Now let us consider the question of whether the diffeomorphisms corresponding to different elements of $L(\Sigma(M_+))$ are actually different, up to isotopy or pseudo-isotopy. If $\mathscr{D}(M)$ = group of pseudo-isotopy classes of orientation preserving diffeomorphisms $f: M\to M$, and let $\mathscr{D}_0(M)$ = subgroup of f which are homotopic to the identity. One would like to define a map $\mathscr{D}_0(M)\to L(\Sigma(M_+))$ inverse to the construction above but there is a difficulty created by the fact that there may be several different ways of defining a homotopy of $f\in\mathscr{D}_0(M)$ to the identity.

Let $f: M\to M$ be a map, $h_i: M\times I\to M\times I$, $i=1,2$, two homotopies of f with the identity, i.e. $h_i(x,0)=(x,0)$, $h_i(x,1)=(f(x),1)$, $i=1,2$. Define $\hat{h}_i: M\times S^1\to M_f$ by $\hat{h}_i(x,t)=(h_i(x,t),t)$, which respects the identifications of $M\times I$ to $M\times S^1$ and M_f. Thus $\hat{h}_i$ has the properties:

(i) $\hat{h}_i$ is a homotopy equivalence,

(ii) $\hat{h}_i|M\times s$ = inclusion of the fibre $M\subset M_f$,

(iii) $\bar{p}\hat{h}_i=p$, where $\bar{p}: M_f\to S^1$, $p: M\times S^1\to S^1$

are the natural projections induced by $(x,t)\to(t)$.

Let us consider the induced homotopy equivalence $g=\hat{h}_2^{-1}\hat{h}_1$: $M\times S^1\to M\times S^1$. Let $\sim$ mean "homotopic to".

(2.20) Lemma. *Let* $g: M\times S^1\to M\times S^1$ *be the homotopy equivalence induced by two different homotopies of a map* f *to the identity (as above). Then*

a) $g|M\times s\sim$ *inclusion,*

b) $pg\sim p$, *where* $p: M\times S^1\to S^1$ *is projection.*

The proof follows immediately from the properties (ii) and (iii) of the $\hat{h}_i$, $i=1,2$.

Let $\mathscr{T}(\omega)$ = group of homotopy classes of bundle maps $b:\omega\to\omega$, ω some stable bundle over M, b covering $\rho(b): M\to M$, the induced map of base spaces, which is assumed a homotopy equivalence. Then $\rho:\mathscr{T}(\omega)\to\mathscr{H}(M)$ = the group of homotopy classes of homotopy equivalences, is a homomorphism of groups. There is an easy argument that the groups $\mathscr{T}(\omega)$, $\mathscr{T}(\omega^{-1})$ and $\mathscr{T}(\varepsilon^q)$ are all isomorphic, ε^q the trivial bundle, provided they are all stable (i.e. fibre dimension $\gg \dim M$).

Let $\mathscr{T}_0(\omega)=\ker\rho$, and define $\mathscr{T}_\#(\omega)$ = group of homotopy classes of bundle maps $c:\omega\to\omega$ which lie over the identity of M (as do the homotopies). Then there is a natural homomorphism $\zeta:\mathscr{T}_\#(\omega)\to\mathscr{T}_0(\omega)$, which is clearly onto by the bundle covering homotopy theorem, but may not in general be $1-1$.

If $b:\omega\to\omega$ is a bundle map covering the map $f: M\to M$ of base spaces, then we define a bundle $\bar{\omega}$ over M_f by identifying $\omega\times 0$ and $\omega\times 1$ in $\omega\times I$ over $M\times I$, using the bundle map b. Then $\bar{\omega}|M\times s=\omega$. Suppose f = identity, so that $\bar{\omega}$ is a bundle over $M\times S^1$ and $\bar{\omega}|M\times s = \omega\times\varepsilon^0$. Then $(\bar{\omega}-\omega\times\varepsilon^0)|M\times s=0$ in $KO(M\times s)$ so that $\bar{\omega}-\omega\times\varepsilon^0$

maps in rational cohomology. For the special case of K-theory we refer to [3] or [4].

Now we recall that for any space X

(2.26) Lemma. *$L(X) \subset KO(X)$ is a submodule of maximal rank.*

Proof. $L(X)$ is defined to be the kernel $J: KO(X) \to J(X)$ (see [2]). But $J(X)$ is finite [2], so $L(X)$ is of maximal rank.

Combining (2.24), (2.25) and (2.26) we get the result:

(2.27) Corollary. *With hypotheses of* (2.24) *if m is even or $m=5$, 13, or 29, $A_0(\mathscr{D}(M))$ has rank* $=\operatorname{rank} H^{4*}(\Sigma(M_+))$, *while* $\operatorname{rank} A_0(\mathscr{D}_0(M)) = \operatorname{rank} H^{4*}(\Sigma(M_+))-1$ *if* $m=4n-1$.

Without special assumptions on M we may define invariants using Pontryagin numbers.

Let $f_i: M_i \to M_i$ be diffeomorphisms, $i=1,2$. We shall say that f_1 is cobordant to f_2 if there is a W, cobordism between M_1 and M_2, and a diffeomorphism $F: W \to W$ such that $F|M_i = f_i$. In the usual way one may speak of oriented cobordism (of orientation preserving diffeomorphisms of oriented manifolds) etc., just as in the usual theory of cobordism. Clearly, if f_1 is cobordant to f_2, then M_{f_1} is cobordant to M_{f_2} (W_F being the cobordism). As usual the Pontryagin numbers depend only on the cobordism class and we get:

(2.28) Theorem. *The mapping torus construction defines a map $0: \mathscr{D}(M^m) \to \Omega_{m+1}$ such that*

(i) *$0(f)$ depends only on the cobordism class of f,*
(ii) *$0(f^n) = n\,0(f)$,*
(iii) *If $f_1, f_2 \in \mathscr{D}_0(M)$, $0(f_1 f_2) = 0(f_1) + 0(f_2)$.*

Proof. (i) is obvious. (ii) follows from the fact that M_{f^n} is the n-fold cover of M_f. (iii) follows from (2.18), (2.22) and the fact that such a g is of degree 1.

II. Free Actions of S^1 and S^3 on Homotopy Spheres

The first strong application of the modern methods of differential topology to transformation groups was the paper of W. C. and W. Y. HSIANG [26] on S^1 and S^3 actions on S^{11}, followed soon by the paper of MONTGOMERY and YANG [35], on S^1 actions on homotopy seven

spheres. In the former, the special results of EELLS and KUIPER [20] on 8-manifolds were used, while in the latter, special properties of dimension 7 were used to translate the classification of free S^1 actions into a question about knotted S^3's in S^6, and then the results of HAEFLIGER [22] were applied.

In order to generalize the theorems of the HSIANGS and MONTGOMERY and YANG to higher dimensions it is necessary to use the techniques of surgery, using the results of Chapter I. This was first done by W. C. HSIANG [25], as well as in unpublished work of M. ROTHENBERG. More recently SULLIVAN [47, 48] obtained some interesting examples, and some very comprehensive piecewise linear results, but we will not discuss them here.

In § 3, we shall discuss the construction of free S^1 and S^3 actions, studying the orbit space with the results of Chapter I, and in § 4 we study the question of finding invariant embedded spheres.

3. Construction of Free S^1 and S^3 Actions

Let us denote by (M^m, φ, G) a differentiable action of the Lie group G on the smooth manifold M, i. e. $\varphi: G \times M \to M$ such that, if $m \in M$, $x, y \in G$,

(i) $\varphi(x, \varphi(y,m)) = \varphi(xy, m)$,
(ii) $\varphi(e,m) = m$, $e =$ identity of G,
(iii) φ is a C^∞ map.

The action φ is called *free* if for any $m \in M$

$$\varphi(g,m) = m \text{ implies } g = e.$$

We recall that if $M^m/\varphi = N$ is the orbit space, $N = \{Gm, m \in M\}$, then for a free G-action, $M^m \to N$ is a principal G-bundle, N is a differentiable n-manifold, $m-n =$ dimension of G.

Now if M^m is a homotopy m-sphere, then M is $(m-1)$-connected and thus by [45, (19.4)], the principal G-bundle $M \to N$ is m-universal, and N is the homotopy type of the classifying space of G, up to dimension m. If $G = S^i$, $i = 0,1,3$, a classifying space for G is the projective space over the reals, complexes, or quaternions, respectively, and the fibrations of spheres, $S^n \to \mathrm{RP}^n$, $S^{2n+1} \to \mathrm{CP}^n$, $S^{4n+3} \to \mathrm{HP}^n$, are k-universal for $k = n$, $2n+1$, $4n+3$ respectively, for the different groups.

Let (Σ^m, φ, S^1) be a free action of the circle group S^1 on a homotopy m-sphere Σ^m. By the Lefschetz fixed point theorem, since S^1 is connected and the action is free (so without fixed points), it follows that m must be odd, $m = 2n+1$. Then the orbit space $N = \Sigma/\varphi$ is of dimension $2n$,

$=h^*(\gamma)$, $\gamma\in KO(\Sigma(M_+))$ where $\Sigma(M_+)=M\times S^1/M\times s$, and γ is unique, since $M\times s$ is a retract of $M\times S^1$.

(2.21) Proposition. *The correspondence $b\to\gamma$ above, depends only on the class of $b\in\mathcal{T}_\#(\omega)$, and defines a homomorphism $\beta:\mathcal{T}_\#(\omega)\to KO(\Sigma(M_+))$.*

Proof. To check that the definition depends only on the class in $\mathcal{T}_\#(\omega)$ is routine.

To show that β is a homomorphism, we use the fact that for a stable bundle ω, $\mathcal{T}_\#(\omega)\cong\mathcal{T}_\#(\omega+\varepsilon^q)$, any q. If $b_1,b_2\in\mathcal{T}_\#(\omega)$, $\beta b_i=\bar{\omega}_i-\omega\times\varepsilon^0$, then consider $1+1+b_2$ and $b_1+1+1\in\mathcal{T}_\#(\omega+\omega^{-1}+\omega)$. Clearly b_1 and b_1+1+1 have the same image in $KO(\Sigma(M_+))$, similarly for b_2. Also $b_1b_2\in\mathcal{T}_\#(\omega)$ corresponds to $(b_1+1+1)(1+1+b_2)=b_1+1+b_2\in\mathcal{T}_\#(\omega+\omega^{-1}+\omega)$. But the bundle corresponding to b_1+1+b_2 over $M\times S^1$ is clearly $\bar{\omega}+\omega^{-1}\times\varepsilon^0+\bar{\omega}_2$. Then since $\omega^{-1}=-\omega$ in KO, $\beta(b_1+1+b_2)=\bar{\omega}_1+\omega^{-1}\times\varepsilon^0+\bar{\omega}_2-\omega\times\varepsilon^0=\bar{\omega}_1-\omega\times\varepsilon^0+\bar{\omega}_2-\omega\times\varepsilon^0$ $=\beta(b_1)+\beta(b_2)$, which proves (2.21).

(2.22) Lemma. *Let $x_0,x_1\in\mathcal{T}_\#(\omega)$ such that $\zeta(x_0)=\zeta(x_1)\in\mathcal{T}_0(\omega)$, and let $\omega_0,\omega_1\in KO(M\times S^1)$ be the elements corresponding to x_0 and x_1. Then there is a homotopy equivalence $g:M\times S^1\to M\times S^1$ such that* (a) *$g|M\times s\sim$inclusion,* (b) *$pg\sim p$, where $p:M\times S^1\to S^1$ is projection, and such that $g^*\omega_1=\omega_0$, and if $\bar{g}:\Sigma(M_+)\to\Sigma(M_+)$ is the homotopy equivalence induced by g, $\bar{g}^*(\beta(x_1))=\beta(x_0)$.*

The lemma follows from the bundle covering homotopy theorem and (2.20).

Proof. Let $b_i:\omega\to\omega$ be representatives of $x_i\in\mathcal{T}_\#(\omega)$, $i=0,1$, so that b_i lies over $1:M\to M$. Then $\zeta(x_0)=\zeta(x_1)$ implies there is a bundle map $B:\omega\times I\to\omega\times I$ covering $h:M\times I\to M\times I$ such that $B|\omega\times i=b_i$, so that $h|M\times i=$identity, $i=0,1$. Taking the bundle map $B'=B(b_0^{-1}\times 1)$: $\omega\times I\to\omega\times I$ over h, we find $B'|\omega\times 0=$identity, $B'|\omega\times 1=b_1b_0^{-1}$, and thus defines a map of the identified bundles $\omega_0\to\omega_1$, where $\omega_i=\omega\times I$ with $(v,0)$ identified with $(b_i(v),1)$, $v\in E(\omega)$. For $B'(v,0)=(v,0)$ and $B'(b_0(v),1)=(b_1b_0^{-1}b_0(v),1)=(b_1(v),1)$, so preserves identification. Hence the map g induced by h, $g:M\times S^1\to M\times S^1$ has properties (a) and (b) and $g^*(\omega_1)=\omega_0$.

With strong hypotheses on M we may define some strong invariants.

Condition K. *A manifold satisfies condition K if for any homotopy equivalence $g:M\times S^1\to M\times S^1$ such that* a) *$g|M\times s\sim$inclusion and* b) *$pg\sim p$, we have $g^*:H^{4*}(M\times S^1;\mathbb{Q})\to H^{4*}(M\times S^1;\mathbb{Q})$ is the identity.*

(2.23) Lemma. *If M satisfies condition* (K) *above then the formula $A(y)=P(\beta x)$, where $\zeta x=y$, defines a linear map $\mathcal{T}_0(\omega)\to H^{4*}(\Sigma(M_+);\mathbb{Q})$.*

Proof. A is defined by the digram

$$\begin{array}{ccccc}\mathcal{T}_\#(\omega) & \xrightarrow{\beta} & KO(\Sigma(M_+)) & \xrightarrow{P} & H^{4*}(\Sigma(M_+);\mathbb{Q})\\ \zeta\downarrow & & & \nearrow A & \\ \mathcal{T}_0(\omega) & & & & \end{array}$$

Since ζ is a homomorphism, β is a homomorphism by (2.21) and P is linear by (2.17), and ζ is onto, the result will follow if A is well defined. By (2.22), if $\zeta(x_1)=\zeta(x_2)$ then $\beta(x_2)=\bar{g}^*\beta(x_1)$, and hence $P(\beta(x_2))$ $=P(\bar{g}^*\beta(x_1))=\bar{g}^*P(\beta(x_1))=P(\beta(x_1))$, since $g^*=$identity implies that $\bar{g}^*=$identity since $h^*:H^*(\Sigma(M_+))\to H^*(M\times S^1)$ is a monomorphism. This proves (2.23).

We may now sum up with the following theorem which we will apply in § 6.

(2.24) Theorem. *Let M^m be a closed 1-connected manifold of dimension $m\geq 5$, which satisfies condition* (K) *above. Then there is a homomorphism $A_0:\mathcal{D}_0(M)\to H^{4*}(\Sigma(M_+);\mathbb{Q})$ such that*

a) *If m is even or $m=5,13,29$, then A_0 is onto the image $P(L(\Sigma(M_+)))$.*

b) *If $m=4n-1$, $A_0(\mathcal{D}_0(M))\subset P(L(\Sigma(M_+)))$ is a submodule of rank one less.*

Proof. The differential $f\to df$ defines a homomorphism $d:\mathcal{D}_0(M)\to\mathcal{T}_0(\tau)$, $\tau=$tangent bundle of M. Then A of (2.23) composed with d defines $A_0=Ad$. But $d(f)$ is easily seen to induce the tangent bundle of M_f in $S^{m+k+q+1}$. Then (2.19), (2.23) complete the proof.

We note that if $m=4n-1$ the Hirzebruch polynomial is not zero on $L(\Sigma(M_+))$, because it contains p_n with a non-zero coefficient, so that the rank is always reduced by 1 in applying (2.19), to satisfy the index condition, (H).

(2.25) Lemma. *$KO(X)$ modulo torsion is isomorphic to $\sum_r H^{4r}(X)$ modulo torsion, and the Pontryagin classes are a complete set of invariants, modulo torsion, (X finite complex).*

This is a standard fact, a special case of a very general theorem that maps into H-spaces are determined modulo torsion by their induced and the principal bundle $\Sigma^m \to N^{2n}$ is classified by a map $f: N^{2n} \to \mathrm{CP}^n$, since $S^{2n+1} \to \mathrm{CP}^n$ is $(2n+1)$-universal. Since for the m-classifying space B of a group G, $\pi_i(B) \cong \pi_{i-1}(G)$, $i \leq m-1$, [45, (19.9)], it follows that $f_*: \pi_i(N) \to \pi_i(\mathrm{CP}^n)$ is an isomorphism for $i \leq 2n$, and it follows that since N and CP^n are $2n$-dimensional that N is homotopy equivalent to CP^n. Similar arguments in the case of S^0 and S^3 may be made, so that we get:

(3.1) Proposition. *Let (Σ^m, φ, S^i), $i=0,1$ or 3 be a free action of a sphere group on a homotopy m-sphere Σ^m. Then the orbit space $N=\Sigma/\varphi$ is the homotopy type of projective space, real, complex or quaternionic, respectively, for S^0, S^1 or S^3.*

On the other hand if two actions $(\Sigma_1^m, \varphi_1, S^i)$ and $(\Sigma_2^m, \varphi_2, S^i)$ are equivalent, i.e. there is an equivariant diffeomorphism $f: \Sigma_1^m \to \Sigma_2^m$, then f induces a diffeomorphism of the orbit spaces $\bar{f}: N_1 \to N_2$. On the other hand a diffeomorphism $\bar{f}: N_1 \to N_2$ induces a diffeomorphism of the principal bundles $f: \Sigma_1 \to \Sigma_2$ which is a G-bundle map, i.e. an equivalence of the actions. It follows that:

(3.2) Proposition. *Equivalence classes of actions (Σ^m, φ, S^i) are in $1-1$ correspondence with diffeomorphism classes of manifolds homotopy equivalent to projective space, over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ respectively, for $i=0$, 1 or 3.*

This problem is exactly of the type which may be studied using the surgery techniques of Chapter I, in the case of S^1 and S^3.

We recall the theorem of Hirsch [23] and Atiyah [2]:

(3.3) Theorem. *Let M_1 and M_2 be two closed m-manifolds embedded in S^{m+k} with normal bundles ν_1 and ν_2 respectively, $k > m+1$. If $f: M_1 \to M_2$ is a homotopy equivalence, then there is fibre homotopy equivalence $b: \nu_1 \to \nu_2$ lying over f.*

Thus the stable normal bundle of a manifold homotopy equivalent to projective space has its stable normal bundle fibre homotopy equivalent to that of projective space.

Let ξ^k be a linear k-plane bundle over $X = \mathrm{CP}^n$ (or QP^n) which is fibre homotopy equivalent to the stable normal bundle ν^k of X in S^{m+k}, $m = \dim X = 2n$ for CP^n ($m=4n$ for QP^n). Then it follows that the fibre homotopy equivalence $f: \nu \to \xi$ induces a homotopy equivalence of the Thom complexes $T(f): T(\nu) \to T(\xi)$. Also $T(f)_*$ commutes with the Thom isomorphism since $T(f)^*(U_\xi) = U_\nu$, so it follows that if $c \in \pi_{2n+k}$ $(T(\nu))$ (or $\pi_{4k+k}(T(\nu))$) is the class of the natural collapsing map (see § 2), then $h(c) \cap U_\nu = [X]$ and it follows that $h(T(f)_*(c)) \cap U_\xi = [X]$.

Then if $X = \mathrm{CP}^n$ and $n = 3$, 7 or 15 we may immediately apply (2.10) to get a smooth manifold M^{2n}, homotopy equivalent to CP^n with normal bundle ξ^k in S^{2n+k}.

A simple way to construct such bundles ξ^k over CP^n, $n=3$, 7 or 15, is the following: Let η^k be a bundle over S^{2n-2}, $2n-2=4q$, let $g: \mathrm{CP}^{n-1} \to S^{2n-2}$ be of degree 1 and suppose that $p_q(\eta) \neq 0$ and η is fibre homotopically trivial. Since $n-1$ is divisible by 2, if $x \in H^2(\mathrm{CP}^n; \mathbb{Z}_2)$, then $Sq^2(x^{n-1}) = 0$, by the Cartan formula, and therefore, since x^{n-1} generates $H^{2n-2}(\mathrm{CP}^n; \mathbb{Z}_2)$, the map $g: \mathrm{CP}^{n-1} \to S^{2n-2}$ extends to $f: \mathrm{CP}^n \to S^{2n-2}$ (see [46]) and $f^*(\eta)$ is a fibre homotopically trivial bundle with q-th Pontryagin class $p_q(f^*(\eta)) \neq 0$. We note that such bundles η over S^{4q} exist since $\pi_{4q}(B_{s_0}) = \mathbb{Z}$ and the number of fibre homotopy classes of spherical fibre spaces is finite, being the order of $\pi_{4q+k}(S^k)$, (k large), (see [2]).

Then if we set $\xi = \nu + f^*(\eta)$, since $f^*(\eta)$ is fibre homotopically trivial, ξ is fibre homotopy equivalent to $\nu + \varepsilon^\ell$, ε^ℓ = the trivial bundle, which is again the normal bundle of CP^n in $S^{2n+k+\ell}$. Hence we may apply (2.10) to get a manifold M^{2n} homotopy equivalent to CP^n but with $p_q(M)$ different from $p_q(\mathrm{CP}^n)$, ($n=3$, 7 or 15, $q=1$, 3 or 7). The different possible η's each give different M^{2n} with different p_q, so that we get a different action of S^1 on S^{2n+1} for each of these, and we have proved:

(3.3) Theorem. *One can construct infinitely many different free S^1 actions on homotopy $(2n+1)$-spheres for $n=3$, 7 or 15, distinguished by the Pontryagin class p_q of the orbit space, $q=1$, 3 or 7, with p_i for $i<q$, being the same as for CP^n.*

In dimension 6 we can get a slightly finer result to reconstruct all the Montgomery-Yang examples [35]. Namely if $\alpha \in \pi_4(B_{s_0})$ is a generator, then the smallest multiple of α which is fibre homotopy trivial is 24α. However, if $g: \mathrm{CP}^2 \to S^4$ is of degree 1, it can be shown that $g^*(12\alpha)$ is fibre homotopy trivial. Then the extension of $g^*(12\alpha)$ to CP^3 is still fibre homotopy trivial since $\pi_{5+k}(S^k) = 0$ (see [50]). Using all multiples of this bundle yields the Montgomery-Yang examples, which are all of the actions of S^1 on homotopy 7-spheres [35].

For S^1 actions on homotopy $(4n+1)$-spheres and S^3 actions on homotopy $(4n+3)$-spheres, one may proceed in a similar way, trying to find bundles ξ over CP^{2n} or QP^n which are fibre homotopy equivalent to the normal bundle ν, and with the additional restriction that condition (H) holds, i.e. that the Hirzebruch index formula is true for ξ in place of ν. This was shown by W. C. HSIANG [25], and we refer to that paper for the explicit construction of these bundles, by which he proves:

(3.4) Theorem. *There are infinitely many free S^1 (or S^3) actions on homotopy $(2n+1)$-spheres ($(4n+3)$-spheres) $n>3$ ($n>1$) which are inequivalent and distinguished by the rational Pontryagin classes of the quotient space.*

We note that the exotic S^1 actions on $(4n+3)$-spheres are constructed as the restrictions of the exotic S^3 actions (see [25]).

4. Characteristic and Invariant Spheres of Free S^1 and S^3 Actions on Homotopy Spheres

Let (Σ^m, φ, S^i), $i=0$, 1 or 3 be a free S^i action on the homotopy sphere Σ^m. As we saw in § 3, the quotient space $\Sigma^m/\varphi = N^n$, $m-n=i$, is a differentiable manifold, homotopy equivalent to the corresponding projective space P^k (real, complex or quaternionic according to whether $i=0$, 1 or 3). Let $f: N \to P^k$ be the homotopy equivalence, and suppose f is transverse regular on $P^\ell \subset P^k$, so that $Q = f^{-1}(P^\ell)$ is a smooth submanifold of N with normal bundle induced from that of $P^\ell \subset P^k$. We call Q a characteristic submanifold of N and the induced S^i principal bundle $\tilde{Q} \subset \Sigma^m$, a characteristic submanifold of Σ^m. Note that the codimensions of Q and $\tilde{Q}$ are the same, and are a multiple of $i+1$. We consider the question: *When does Σ^m have a characteristic homotopy sphere of a given codimension, i.e., is f homotopic to f' such that $\tilde{Q}$ is a homotopy sphere (or equivalently Q is homotopy equivalent to P^ℓ)?* This question for $i=0$ and codimension 1 was studied by G. R. LIVESAY and the author [16], and an obstruction was defined. Though the results were similar, the methods go outside the scope of the present context, and we will restrict ourselves here to $i=1$ or 3, i.e., free actions of S^1 and S^3 (see [31 and 37]).

A smooth submanifold $X \subset \Sigma^m$ is called *invariant* if $\varphi(G \times X) \subset X \subset \Sigma^m$.

Now the smooth submanifolds V of the orbit space N, yield invariant submanifolds $\tilde{V}$ of Σ^m, where $\tilde{V}$ is the induced principal S^i bundle over V, and conversely invariant submanifolds of Σ^m under the action φ, yield smooth submanifolds of N. We may also ask if (Σ^m, φ, S^i) has invariant homotopy spheres of various dimensions, which is the same thing as asking if the mapping $P^\ell \longrightarrow P^k \xrightarrow{f^{-1}} N$ is homotopic to an embedding of a homotopy P^ℓ into N. This may in general be much easier than finding a characteristic sphere of this dimension; for example the freedom

in choosing the normal bundle should make the problem considerably easier. Thus if $k > 2\ell$, the map $P^\ell \to N$ is homotopic to an embedding, and thus every free action has all the standard actions of less than half the dimension embedded in it as *invariant* spheres. This will not be true for characteristic spheres, as we shall see later.

(4.1) Proposition. *Let (Σ^m, φ, S^i) be a free S^i action, $i=1$ or 3, and let $V \subset \Sigma^m$ be an invariant manifold of codimension $i+1$, corresponding to $\tilde{V} \subset N^n$, such that $i_*: H_{n-i-1}(V) \xrightarrow{\cong} H_{n-i-1}(N)$. Then with appropriate choice of orientation, the normal bundle of V in N has Euler class $\chi = i^*(\alpha) \in H^{i+1}(V)$, where $\alpha \in H^{i+1}(N)$ is a generator which is the Euler class of the canonical bundle over N, the linear bundle associated with $\Sigma^m \to N$, ($i: V \to N$, the inclusion).*

Proof. Recall that if $j: V \to T(\nu)$ is the inclusion of the zero-cross-section in the Thom complex, then $\chi = j^*(U)$, where $U \in H^*(T(\nu))$ is the Thom class (see [33, 27]). Now j factors, $V \xrightarrow{i} N \xrightarrow{\eta} T(\nu)$, $j = \eta i$, where η is the natural collapsing map. Hence $\chi = i^*(\eta^* U)$.

Consider the inclusion of pairs, where E is a tubular neighborhood of V in N:

$$j_0: (E, \partial E) \to (N, N_0)$$

where $N_0 = N - \operatorname{int} E$, so j_0 is an excision, and consider the inclusion $k: N \to (N, N_0)$. Then $j_0^*: H(N, N_0) \to H(E, \partial E)$ is an isomorphism, and $\eta^* = k^* j_0^{*-1}$. Also $k^*: H^n(N, N_0) \to H^n(N)$ is an isomorphism. Now we have the commutative diagrams with cup products

a)
$$\begin{array}{ccc} H^q(N, N_0) \otimes H^{n-q}(N) & \longrightarrow & H^n(N, N_0) \\ \downarrow{\scriptstyle j_0^* \otimes i^*} & & \downarrow{\scriptstyle j_0^*}\,\cong \\ H^q(E, \partial E) \otimes H^{n-q}(E) & \longrightarrow & H^n(E, \partial E) \end{array}$$

b)
$$\begin{array}{ccc} H^q(N, N_0) \otimes H^{n-q}(N) & \longrightarrow & H^n(N, N_0) \\ \downarrow{\scriptstyle k^* \otimes 1} & & \downarrow{\scriptstyle k^*}\,\cong \\ H^q(N) \otimes H^{n-q}(N) & \longrightarrow & H^n(N) \end{array}$$

Now $i^*: H^s(N) \to H^s(E)$ is an isomorphism for $s = n-(i+1)$, since i_* is. Hence, if $j^*(U') = U \in H^{i+1}(E, \partial E)$, $U \cup: H^{n-q}(E) \to H^n(E, \partial E)$ is an isomorphism by the Thom isomorphism, so that $U' \cup: H^{n-q}(N) \to H^n(N, N_0)$ is also an isomorphism, using diagram (a). Using (b) we get $k^*(U') \cup: H^{n-q}(N) \to H^n(N)$ is an isomorphism. But if $\alpha \in H^{i+1}(N)$

is the generator, $\alpha \cup$ is an isomorphism, so $k^*(U')=\pm\alpha$ and it follows that $\chi=i^*\eta^*(U)=i^*k^*(j_0^{*-1}U)=i^*k^*(U')=\pm i^*\alpha$, and with appropriate choice of orientation, the proposition is proved.

(4.2) Proposition. *Let (Σ^m,φ,S^1) be a free S^1 action, $\tilde{V}\subset\Sigma^m$ a codimension 2 invariant manifold such that $i_*:H_{n-2}(V)\to H_{n-2}(N)$ is an isomorphism (with notation as in (4.1)). Then $\tilde{V}$ is characteristic.*

Proof. First we note that an oriented 2-plane bundle is determined by its Euler class. If we take the map of V into CP^k $(m=2k+1, n=2k)$ coming from the inclusion of V into N followed by the homotopy equivalence N to CP^k, this map factors through CP^{k-1} since V is $2k-2$ dimensional. Then from (4.1) and the remark above it follows that this map $s:V\to\mathrm{CP}^{k-1}$ induces the normal bundle of V in N from that of CP^{k-1} in CP^k. Now the complement of CP^{k-1} (or a tubular neighborhood of it) in CP^k is a $2k$-cell D^{2k}, so that the bundle map of normal bundles extends to the complement of V in N to D^{2k}. The union of these maps induces a map $t:N^{2k}\to\mathrm{CP}^k$, which is still a homotopy equivalence, since the map induces an isomorphism on H^2, as $t^*=s^*$ on H^2, and $t^{-1}(\mathrm{CP}^{k-1})=V$. Hence V is characteristic.

The analogous theorem is true for S^0 actions, but seems less likely for S^3 actions. For 4-plane bundles are not characterized by the Euler class, and in fact submanifolds of codimension 4 correspond to maps $N\to\mathrm{MSO}(4)$, rather than into HP^∞, the quaternionic projective space. In fact it is false in the piecewise linear case as we shall see later.

Now we shall consider the question of finding characteristic spheres, using (2.14). We consider the homotopy equivalence of N with the projective space P^k over C or H, for S^1 or S^3 actions. Utilizing the notation of (2.14), we consider the homotopy equivalence $f:N\to P^k$, let ξ be a bundle over P^k such that $f^*(\xi)=\nu$ is the normal bundle of N in a high dimensional Euclidean space, and we consider bundle maps $b:\nu\to\xi$ covering f.

(4.3) Theorem. *If $\sigma_{P^{k-\ell}}(f)=0$ then N has a characteristic homotopy $(i+1)(k-\ell)+i$ sphere, provided $(i+1)(k-\ell)>4$.*

This is a direct application of (2.14). It was first proved by Rothenberg in unpublished work.

We recall that for dimension of $P^{k-\ell}=4q$, the definition of $\sigma_{P^{k-\ell}}(f,\mathrm{b})$ is simply the difference of two indices, namely index $P^{k-\ell}-$index $V=1-$index V, where V is the characteristic submanifold. Now index V may be calculated using the Hirzebruch Index Theorem [24].

The normal bundle of V in Euclidean space $\mathbb{R}^Q$ is the sum of the normal bundle of N in $\mathbb{R}^Q$ restricted to V, plus $\ell\eta$ where η is the canonical

bundle ρ over N restricted to V. Hence the stable tangent bundle of V is

$$\tau_V=(\nu|\zeta)^{-1}+(\ell\eta)^{-1}=\tau_N|V+\ell\eta^{-1}.$$

Suppose $(i+1)(k-\ell)=4q=\dim V$. Then

$$\begin{aligned}\sigma(V)&=\langle L_q(p_1(\tau_N|V+\ell\eta^{-1}),\ldots),[V]\rangle\\&=\langle i^*L_q(p_1(\tau_N+\ell\rho^{-1}),\ldots),[V]\rangle\\&=\langle L_q(p_1(\tau_N+\ell\rho^{-1}),\ldots),\chi_{k-\ell}\rangle\end{aligned}$$

where $i_*[V]=\chi_{k-\ell}$, $f_*(\chi_{k-\ell})=\text{image}[P^{k-\ell}]$ in $H_*(P^k)$. Hence we get

(4.4) Corollary. *Let $(\Sigma^{2k+1},\varphi,S^1)$ (or $(\Sigma^{4k+1},\varphi,S^3)$) be a free differentiable action. Then the action has a characteristic homotopy $(4q+1)$-sphere ($(4q+3)$-sphere) $q>1$ if and only if $\langle L_q(p_1(\tau_N+\ell\rho^{-1}),\ldots),\chi\rangle=1$ where ρ is the canonical bundle over N, χ is the generator of $H_{4q}(N)$, $\ell=k-2q$ (or $\ell=k-q$).*

Applying (4.4) to the actions on $4q+3$ homotopy spheres constructed in (3.3), we find that since $p_i(N)=p_i(\mathrm{CP}^k)$ for $i<q$ and $p_q(N)\neq p_q(\mathrm{CP}^k)$, $(2k=4q+2)$, it follows easily that none of these actions have characteristic homotopy $(4q+1)$-spheres, while they all have them of dimension $(4i+1)$, $i<q$.

We now construct similar examples to those of (3.3), as follows. Recall that the ring of complex vector bundles, $K(P^k)\cong H^*(P^k)$, $P^k=\mathrm{CP}^k$ or HP^k, and the Chern character $(\mathrm{ch}:K(P^k)\to H^*(P^k;\mathbb{Q}))$ is a monomorphism (see [3]). It follows that we can find complex bundles θ over P^k with $c_1=0$ and $c_2\neq 0$. It follows from the formula relating Pontryagin and Chern classes that $p_1(\theta)\neq 0$, considering θ as a real bundle (see [33, 27]). Since $J(P^k)$ is finite, some multiple $n\theta=\theta'$ is fibre homotopically trivial (see [2]). Setting $\xi=\nu+n\theta'$, and suppose $k=7$ or 15, we may proceed as in the proof of (3.3) to prove:

(4.5) Theorem. *There are infinitely many different actions of S^1 on homotopy 15 or 31 spheres, distinguished by p_1 of the orbit space, so that none of them has a characteristic homotopy 5-sphere.*

This then demonstrates that in high codimensions invariant and characteristic manifolds are quite different.

We end with a few remarks about the piecewise linear (p.ℓ.) situation. Considering the analogous questions about p.ℓ. submanifolds of the orbit space N, we can show:

(4.6) Remark. *Any S^3 action on S^{4k+3} has an invariant p.ℓ. S^{4k-1}, i.e. the orbit space N^{4k} has HP^{k-1} embedded in it.*

Proof. It is easy to see that $N-$(point) is homotopy equivalent to HP^{k-1}. Now HAEFLIGER (see [22]) and CASSEN and SULLIVAN have shown that:

If $f: N^n \to M^m$ is a homotopy equivalence of p.ℓ. manifolds, $m-n \geqslant 3$, N 1-connected, closed, $m \geqslant 6$, then f is homotopic to a p.ℓ. embedding.

Thus HP^{k-1} embeds p.ℓ. in N^{4k}, so S^{4k+3} has the standard action on S^{4k-1} embedded in it piecewise linearly

On the other hand, a characteristic submanifold has a linear normal bundle, so a p.ℓ. characteristic submanifold of a smooth orbit space N can be smoothed, using the smoothing theory of HIRSH-MAZUR, LASHOF-ROTHENBERG (see [30]). Hence the problem for characteristic submanifolds is the same, p.ℓ. or smooth.

We remark that MONTGOMERY-YANG [36] have shown that characteristic spheres of codimension 2 of S^1 actions are unknotted.

III. Semi-free S^1 Actions

An action (M^m, φ, G) is called *semi-free* if it is free off of the fixed point set, i.e. there are two types of orbits, fixed points and G. We shall study the situation where (Σ^m, φ, S^1) is a semi-free differentiable action of S^1 on a homotopy sphere Σ^m, and the fixed point set F^q is a homotopy sphere. For a general discussion of semi-free S^1 actions we refer to [6].

We shall show how to use the results of surgery to construct many such exotic actions, generalizing the constructions of MONTGOMERY-YANG [35] in dimension 6, and in other dimensions [36].

In § 5 we describe the reduction of the problem to conventional problems in differential topology. In § 6 we use these results to construct actions with exotic spheres as fixed point sets, and then show how to get infinitely many exotic actions with the standard sphere as fixed point set. The most powerful theorem proved is (6.22).

5. Constructions of Semi-free Actions of S^1

Let (Σ^m, φ, S^1) be a semi-free action, with fixed point set $F^q \subset \Sigma^m$, F^q a homotopy q-sphere (i.e. S^1 acts freely outside F). Then S^1 acts freely and linearly on the normal space to F at each point of F (see [38] or [19, page 58]). Considering $S^1 = \{z \in C, |z| = 1\}$, this action defines a complex structure on the normal space, since it defines multiplication by $i \in S^1$ with the right properties. The action of S^1 is then linear with

respect to the structure of a complex vector space, and it follows that the action of S^1 is exactly the action of the complex numbers of unit modulus in a complex vector space. Hence the normal bundle of F has a complex structure, and the action of S^1 on it, is just that induced by the complex structure*. In particular the codimension $m-q=2k$.

Let η be the complex bundle over F defined by the action.

(5.1) Theorem. *If (Σ^m, φ, S^1) and $(\Sigma^m, \varphi', S^1)$ are equivalent, then F is diffeomorphic to F' and η is equivalent to η'.*

Proof. That F is diffeomorphic to F' is evident, and clearly the equivalence $f: \Sigma \to \Sigma$ defines a complex map of the normal bundles of F and F' so that they are equivalent as complex bundles.

Numerous examples in which F is an exotic sphere have been constructed (see [36] and [6]), and we shall construct some below.

Now let E be an equivariant tubular neighborhood of F in Σ (see [19, page 57]), and let S^{2k-1} be the boundary of a fibre of E. Now it has been shown** by MONTGOMERY-YANG [36] that if $k=1$, then $\Sigma^m = S^m$, $F = S^{m-2}$ embedded as usual, and the action is linear. Therefore we restrict ourselves to $k>1$. It follows from a homology argument that if $k>1$, then $S^{2k-1} \subset \Sigma - F$ is a homotopy equivalence. Now let $N = \Sigma - E_0$, where E_0 is the interior of an equivariant tubular neighborhood of F, with $\bar{E}_0 \subset \operatorname{int} E$. Then S^1 acts freely on N, and on $S^{2k-1} \subset N$, and S^{2k-1} is homotopy equivalent to N. It follows from the exact homotopy sequence of the fibre maps, using the diagram

$$\begin{array}{ccc} S^{2k-1} & \longrightarrow & N \\ \downarrow & & \downarrow \\ S^{2k-1}/S^1 & \longrightarrow & N/S^1 \end{array}$$

that $S^{2k-1}/S^1 \to N/S^1$ is a homotopy equivalence. Set $\bar{N} = N/S^1$. Since the action of S^1 on S^{2k-1} is linear, $S^{2k-1}/S^1 = \mathrm{CP}^{k-1}$, and since S^{2k-1} is the fibre of E over F it follows that its normal bundle is equivariantly trivial, so that we get an embedding $\mathrm{CP}^{k-1} \times D^{q+1} \subset \bar{N}^{m-1}$, and it is a homotopy equivalence. It follows similarly that the region between $\partial \bar{N}$ and $\mathrm{CP}^{k-1} \times S^q$ is an h-cobordism, so if $m>6$, by the h-cobordism theorem of SMALE if $q>1$ (see [44]) or its generalization, the s-cobordism

* This observation was made to me by G. BREDON, who also conjectured (5.5) below.

** (Added in proof) This result first appears in WU-YI HSIANG. On the unknottedness of the fixed point set of differentiable circle group actions on spheres – PA Smith conjecture, Bull. AMS 70 (1964) 678—680.

theorem [28] if $q=1$, it is diffeomorphic to the product $CP^{k-1}\times S^q\times I$, and hence $\bar{N}$ is diffeomorphic to $CP^{k-1}\times D^{q+1}$, and $N\to\bar{N}$ is equivalent to $h\times 1: S^{2k-1}\times D^{q+1}\to CP^{k-1}\times D^{q+1}$, where $h: S^{2k-1}\to CP^{k-1}$ is the Hopf map, i.e. the principal bundle $N\to\bar{N}$ is induced by the map $\bar{N}\to CP^{k-1}$.

Hence we have shown

(5.2) Theorem. *Let (Σ^m,φ,S^1) be a semi-free action on a homotopy sphere Σ^m, with fixed point set F a homotopy q-sphere, $m-q=2k$, $k\geqslant 1$, $m\geqslant 6$. If N is the complement of an open tubular neighborhood of F in Σ^m, then N is equivariantly diffeomorphic to $S^{2k-1}\times D^{q+1}$, with the standard action on S^{2k-1}, trivial action on D^{q+1}. In particular the projective space bundle P associated to η, which is $\partial N/S^1$, is diffeomorphic to $CP^{k-1}\times S^q$.*

Now we study the bundle η.

(5.3) Theorem. *The bundle of projective spaces $\pi: P\to F$ with fibre CP^{k-1} associated with $E\to F$ is fibre homotopically trivial, i.e., there is a map $t: P\to CP^{k-1}\times F$ such that t is a homotopy equivalence and commutes with the projections, $p_2 t=\pi$, ($p_2: CP^{k-1}\times F\to F$ is projection on the second factor). Further, $t^*(\alpha\times\varepsilon^0)=\alpha'$ the canonical line bundle over P, where α is the canonical line bundle over CP^{k-1}.*

Proof. Clearly $P=\partial\bar{N}$ and by (5.2) there is a diffeomorphism $h: P\to CP^{k-1}\times S^q$ such that $h^*(\alpha\times\varepsilon^0)=\alpha'$, the canonical line bundle over P, where α is the canonical line bundle over CP^{k-1}. Let $t: P\to CP^{k-1}\times F$ be defined by $t(z)=(p_1 h(z),\pi(z))$. Clearly $p_2 t=\pi$, so t preserves fibres.

Now $p_1^*(\alpha)=\alpha\times\varepsilon^0$, and hence $t^*(\alpha\times\varepsilon^0)=t^* p_1^*(\alpha)=(p_1 t)^*\alpha=(p_1 h)^*(\alpha)=h^* p_1^*(\alpha)=h^*(\alpha\times\varepsilon^0)=\alpha'$, by (5.2), so the last condition of the theorem is satisfied for t. On the other hand, if $i: CP^{k-1}\to P$ is the inclusion of a fibre, then $i^*(\alpha')=\alpha$. It follows easily that if t' is the restriction of t to the fibres $t': CP^{k-1}\to CP^{k-1}$, then $t'^*(\alpha)=\alpha$. Hence t' is a homotopy equivalence on the fibres and hence t is a homotopy equivalence, which completes the proof.

(5.4) Question. *Are there non-trivial complex bundles whose projective space bundles are fibre homotopy trivial by a trivialization t sending the canonical line bundles into each other, as in (5.3)? Such bundles are stably trivial (see (5.5)).*

As another application of (5.2) we get:

(5.5) Theorem. *The Chern classes of η are 0 so that η is stably trivial (as a complex bundle).*

Proof. We recall the definition of the Chern classes after GROTHENDIECK (see [4]):

Let $P\to F$ be the CP^{k-1} bundle associated to η, and let $x\in H^2(P)$ be the first Chern class of the canonical line bundle over P. Then the powers of x, i.e., $1,x,x^2,\ldots,x^{k-1}$ are a basis for $H^*(P)$ over $H^*(F)$ and hence we have a relation $x^n+\sum_{i=1}^{n} c_i x^{n-i}=0$. Then $c_i=c_i(\eta)$ are the Chern classes.

Now by (5.2), P is diffeomorphic to $CP^{k-1}\times S^q$, and the canonical bundle over P is induced from that over CP^{k-1}. Hence if $h: P\to CP^{k-1}\times S^q$, $x=h^*(y)$, $y\in H^2(CP^{k-1})\subset H^2(CP^{k-1}\times S^q)$, and $x^k=h^*(y^k)=0$. Hence $c_i(\eta)=0$ for $i>0$. But a stable bundle over a homotopy sphere is determined by its Chern classes, so η is stably trivial.

(5.6) Corollary. *If $q\leqslant 2k$, then η is trivial.*

Proof. In these dimensions, η is already stable, hence trivial by (5.5).

It would be interesting to know if there are examples of semi-free S^1 actions (Σ^m,φ,S^1) with η non-trivial. In fact we may characterize the bundles η which occur by the following theorem, which describes how to construct semi-free S^1 actions.

Let F^q be a homotopy q-sphere, η a complex k-plane bundle over F. Let $\pi: P\to F$ be the associated CP^{k-1} bundle to η, and suppose $h: P\to CP^{k-1}\times S^q$ is an orientation preserving diffeomorphism such that $h^*(y)=x$, where $y=p_1^*(c_1)$, $p_1: CP^{k-1}\times S^q\to CP^{k-1}$, c_1 is the Chern class of the canonical bundle over CP^{k-1} and x is the first Chern class of the canonical line bundle over P.

(5.7) Theorem. *There is a semi-free S^1 action (Σ^m,φ,S^1) with fixed point set F embedded in Σ^m with (complex) normal bundle η, and such that the orbit space is $C_\pi\bigcup_h CP^{k-1}\times D^{q+1}$, where C_π is the mapping cylinder of π, and $\bigcup_h$ means we identify $P\subset C_\pi$ with $CP^{k-1}\times S^q\subset CP^{k-1}\times D^{q+1}$ via the diffeomorphism h. Every semi-free action of S^1 on a homotopy sphere of dimension >6 with fixed point set a homotopy sphere is given this way.**

* One can remove the condition that dimension >6 if one substitutes h-cobordism for diffeomorphism in the statement.

(5.8) Theorem. *Let $(\Sigma_i^m, \varphi_i, S^1)$ be semi-free S^1 actions constructed as in (5.7), $i=0,1$, with F_i, η_i, h_i, $i=0,1$. Then $(\Sigma_0^m, \varphi_0, S^1)$ is equivalent to $(\Sigma_1^m, \varphi_1, S^1)$ if and only if there are*

(i) *A diffeomorphism $f: F_0 \to F_1$;*

(ii) *A complex bundle equivalence $b: \eta_0 \to \eta_1$ covering the diffeomorphism f;*

(iii) *A diffeomorphism $g: \mathrm{CP}^{k-1} \times D^{q+1} \to \mathrm{CP}^{k-1} \times D^{q+1}$ such that the diagram below commutes:*

$$\begin{array}{ccc} P_0 & \xrightarrow{P(b)} & P_1 \\ {\scriptstyle h_0}\downarrow & & \downarrow{\scriptstyle h_1} \\ \mathrm{CP}^{k-1} \times S^q & \xrightarrow{\partial g} & \mathrm{CP}^{k-1} \times S^q \end{array}$$

where P_i is the associated CP^{k-1} bundle to η_i, $P(b)$ is the map induced by b, and ∂g is the restriction of g.

These two theorems give us the necessary tools to attempt to construct and distinguish semi-free S^1 actions, using the surgery results of Chapter I.

Proof of (5.7). Consider the semi-free S^1 action on the total space of η, $E(\eta)$, defined by the complex structure, and the free S^1 action on $S^{2k-1} \times D^{q+1}$ defined by the linear free action on S^{2k-1}. Then the diffeomorphism $h: P \to \mathrm{CP}^{k-1} \times S^q$ induces an equivariant diffeomorphism $h': E_0(\eta) \to S^{2k-1} \times S^q$, where $E_0(\eta) = \partial E(\eta)$ is the associated sphere bundle. Then $M = E(\eta) \bigcup_{h'} (S^{k-1} \times D^{q+1})$ is then a manifold with a semi-free S^1 action, fixed point set F with normal bundle η. It remains to show that M is a homotopy sphere.

Since we have assumed $k>1$ throughout this section, it follows that $\pi_1(E_0(\eta)) \cong \pi_1(F)$, $S^{2k-1} \times D^{q+1}$ is 1-connected and $\pi_0(E_0(\eta)) \cong \pi_0(F)$. Therefore, by Van Kampen's theorem, M is 1-connected.

Consider the commutative diagram

$$\begin{array}{ccccc} E_0(\eta) & \xrightarrow{h'} & S^{2k-1} \times S^q & & \\ {\scriptstyle \pi}\downarrow & & \downarrow & \searrow{\scriptstyle p_2} & \\ P & \xrightarrow{h} & \mathrm{CP}^{2k-1} \times S^q & \xrightarrow{p_2} & S^q \end{array}$$

Let $a \in H_{2k-1}(E_0(\eta))$ be the image of the generator of the homology of the fibre $H_{2k-1}(S^{2k-1})$. Then $\pi_*(a)=0$, so $(p_2 h \pi)_*(a)=0$, and hence $(p_2 h')_*(a)=0$. Hence $(h')_*(a) = \lambda(g \otimes 1) \in H_{2k-1}(S^{2k-1}) \otimes H_0(S^q)$ $\subseteq H_{2k-1}(S^{2k-1} \times S^q)$, $\lambda \in \mathbb{Z}$, $g \in H_{2k-1}(S^{2k-1})$, a generator. Since h_1 is a diffeomorphism, $\lambda = \pm 1$. If $j: S^{2k-1} \times S^q \to S^{2k-1} \times D^{q+1}$ is inclusion we deduce:

(5.9) $$j_* h'_*(a) = \pm g.$$

Now consider the Mayer-Viettoris sequence for M:

(5.10) $$\cdots \longrightarrow H_{s+1}(M) \longrightarrow H_s(E_0(\eta)) \xrightarrow{i_* + \rho_*} H_s(E(\eta)) + H_s(S^{2k-1} \times D^{q+1}) \longrightarrow H_s(M) \longrightarrow \cdots$$

where $\rho = jh'$. Then $(i_* + \rho_*)a = i_*(a) + \rho_*(a) = 0 \pm g$. Assume the sign is $+$. Let $b \in H_q(E_0(\eta))$ be another generator for the homology of $E_0(\eta)$, such that $\pi_*(b)$ is a generator of $H_q(F)$, so that a and b are a basis for $H_*(E_0(\eta))$ in dimensions between 0 and $2k+q-1$. Let $h'_*(b) = \mu g \otimes 1 + \gamma 1 \otimes c$, c a generator of $H_q(S^q) \subset H_q(S^{2k-1} \times S^q)$. Choose as a new generator $b' = b - \mu a$. Then $(h')_*(b') = (h')_*(b - \mu a) = \mu g \otimes 1 + \gamma 1 \otimes c - \mu g \otimes 1 = \gamma 1 \otimes c$. Since b' is a generator, $h_{1*}(b')$ is a generator and $\gamma = \pm 1$. Now $\pi_*(b') = \pi_*(b) - \pi_*(\mu a) = \pi_*(b)$. Hence $i_* + \rho_*$ is an isomorphism between $H_s(E_0(\eta))$ and $H_s(E(\eta)) + H_s(S^{2k-1} \times D^{q+1})$ for $0 < s < 2k+q-1$, and it follows from (5.10) that M is a homology sphere, hence a homotopy sphere. This completes the construction.

On the other hand (5.2) shows that all such semi-free actions arise from this construction. Thus (5.7) is proved.

Proof of (5.8). It follows from condition (iii) of (5.7) that the diagram commutes:

$$\begin{array}{ccc} E_0(\eta_0) & \xrightarrow{b_0} & E_0(\eta_1) \\ {\scriptstyle h'_0}\downarrow & & \downarrow{\scriptstyle h'_1} \\ S^{2k-1} \times S^q & \xrightarrow{\partial g'} & S^{2k-1} \times S^q \end{array}$$

where $'$ denotes the map induced on principal bundles by the maps of base spaces, $\partial g'$ is the restriction of g'. Thus the maps $b: E(\eta_0) \to E(\eta_1)$ and $g': S^{2k-1} \times D^{q-1} \to S^{2k-1} \times D^{q+1}$ respect the identifications by h'_0 and h'_1 and are equivariant, so define an equivariant diffeomorphism of Σ_0^m to Σ_1^m. Hence conditions (i), (ii) and (iii) yield an equivalence of actions.

Now suppose $\psi: \Sigma_0 \to \Sigma_1$ is an equivalence of actions, i.e. an equivariant diffeomorphism. As in (5.1), $f = \psi | F_0 : F_0 \to F_1$ is a diffeomorphism and ψ restricted to an equivariant tubular neighborhood N_0 of F_0 induced a complex bundle equivalence $b: \eta_0 \to \eta_1$ covering f. Let $N_1 = \psi(N_0)$, and $V_i = \Sigma_i - \mathrm{int}\, N_i$. Then since $\psi(N_0) = N_1$ and ψ is a diffeomorphism, $\psi(V_0) = V_1$, and $\psi | V_0 = g' : V_0 \to V_1$ is an equivariant diffeomorphism, and $\partial V_i = E_0(\eta_i)$, $g' | \partial V_0 = b_0 = b | E_0(\eta_0)$, and let $\bar{g}: V_0/S^1 \to V_1/S^1$ be the induced diffeomorphism. By (5.2) there are diffeomorphisms $k_i: V_i/S^1 \to CP^{k-1} \times D^q$, so we define $g: CP^{k-1} \times D^q \to CP^{k-1} \times D^q$ by $g = k_1 \bar{g} k_0^{-1}$. Since $g'|\partial V_0 = b_0$, it follows that $\bar{g}|(\partial V_0/S^1) = \bar{g}|P_0 = P(b)$, and it follows that, since $h_i = k_i|P_i$, $(P_i = \partial V_i/S^1)$, the diagram of (iii) commutes, which proves (5.8).

6. Applying Surgery to Construct Semi-free Actions

In this paragraph we apply results of surgery together with (5.7) and (5.8) to construct semi-free actions.

By (5.7), the problem is to find over a homotopy sphere F^q a complex k-plane bundle η and a diffeomorphism $h: P(\eta) \to CP^{k-1} \times S^q$, where $P(\eta)$ is the associated bundle with fibre CP^{k-1}, satisfying the condition on Chern classes. If $q \neq 2$, then any diffeomorphism h can be made to have this property by composing h if necessary with $\alpha \times \varepsilon: CP^{k-1} \times S^q \to CP^{k-1} \times S^q$ where $\alpha^*(c_1) = -c_1 \in H^2(CP^{k-1})$ and ε = identity if k is even, ε is orientation reversing if k is odd.

As I do not know of any non-trivial bundles η with the above property, I will always take η to be the trivial complex k-plane bundle. We will say in this case F is untwisted. We shall first consider the problem of constructing actions with exotic spheres as fixed point set, and we get some results extending those of BREDON [5] and MONTGOMERY-YANG [36].

(6.1) Theorem. *Let F^{2n-1} be a homotopy sphere which bounds a parallelizable manifold, $n \geq 2$ ($F^{2n-1} \in bP^{2n}$ in the notation of [29]). Then for each even $k > 1$, there is a semi-free action of S^1 on a homotopy sphere $\Sigma^{2n-1+2k}$ with F as untwisted fixed point set.*

If M^m is a closed manifold, let $I(M) = \{\Sigma \in \Theta^m$ such that $M \# \Sigma$ is diffeomorphic to $M\}$. $I(M)$ is a subgroup and is called the *inertia group* of M. Let $I_0(M) = I(M) \cap bP^{m+1}$. We recall from [29] that bP^{4s} is a finite cyclic group, and we denote its order by m_s.

(6.2) Theorem. *Suppose $q = 4t-1$ and $k > 1$, k odd, and let ℓ = order of $I_0(CP^{k-1} \times S^q)$. Then an element $F \in bP^{4t}$, $t > 1$, occurs as an untwisted fixed point set of a semi-free action of S^1 on a homotopy sphere Σ^{4s+1}, where dimension $4s+1 = 4t-1+2k$ if and only if $F \in (m_s/\ell) bP^{4t}$.*

(6.3) Theorem. *Suppose $q = 4n+1$, k odd, $k > 1$, and $F \neq 0$ in bP^{4n+2}. Then F occurs as an untwisted fixed point set of a semi-free S^1 action on a homotopy sphere Σ of dimension $4n+1+2k = 4s+3$ if and only if $I_0(CP^{k-1} \times S^q) = bP^{4s+2}$.*

In case $bP^{4s+2} = 0$ for example when $s = 1$, 3 or 7 (see [14]) then the condition is of course satisfied for any F.

The proofs of these three theorems are very similar, utilizing surgery techniques to create a diffeomorphism of $CP^{k-1} \times F$ with $CP^{k-1} \times S^q$.

Let $F = \partial W^{2n}$, W parallelizable. We may consider $W_0 = (W - \operatorname{int} D^{2n})$ as a parallelizable cobordism between F and S^{2n-1}, and thus we may define a normal map $G: (W_0, F \cup S^{2n-1}) \to (S^{2n-1} \times I, S^{2n-1} \times 0 \cup S^{2n-1} \times 1)$ with $G|S^{2n-1}$ = identity. By (2.7), we may assume W_0 to be $(n-1)$-connected. Multiplying by CP^{k-1} we get a normal map

$$1 \times G: \; CP^{k-1} \times (W_0, F \cup S^{2n-1}) \to CP^{k-1} \times (S^{2n-1} \times I, S^{2n-1} \times 0 \cup S^{2n-1} \times 1)$$

with $1 \times G | CP^{k-1} \times S^{2n-1}$ = identity. The remainder of the proofs of the three theorems is computing the obstruction σ for this normal cobordism, and using this to determine if $CP^{k-1} \times F$ is diffeomorphic to $CP^{k-1} \times S^{2n-1}$. The three theorems correspond to the three cases:

(6.1) k is even,

(6.2) k is odd, n is even,

(6.3) k is odd, n is odd.

(6.4) Lemma. $\ker(1 \times G)_* = H_*(CP^{k-1}) \otimes \ker G_*$.

Proof. By the Künneth formula, since $H_*(CP^{k-1})$ is torsion free $H_*(CP^{k-1} \times (W_0, F \cup S^{2n-1})) \cong H_*(CP^{k-1}) \otimes H_*(W_0, F \cup S^{2n-1})$, and $(1 \times G)_* = 1 \otimes G_*$. Thus the lemma follows.

Now $\dim W_0 = 2n$, $\dim CP^{k-1} = 2k-2$ and W_0 is $(n-1)$-connected, so that $\ker(1 \times G)_* = H_*(CP^{k-1}) \otimes H_n(W_0)$. If k is even, then $k-1$ is odd, and thus $H_{k-1}(CP^{k-1}) = 0$. Hence $(\ker(1 \times G)_*)_{n+k-1} = 0$ and hence $\sigma(1 \times G) = 0$ and by (2.7), the Fundamental Theorem, $1 \times G$ is normally cobordant $\operatorname{rel} CP^{k-1} \times S^{2n-1} \times \{0,1\}$ to a homotopy equivalence, i.e. an h-cobordism between $CP^{k-1} \times F$ and $CP^{k-1} \times S^{2n-1}$. Now $\dim(CP^{k-1} \times S^{2n-1}) = 2k-2+2n-1 = 2n+2k-3$ and $n \geq 2$, $k > 1$ so that $n+k \geq 4$ so that $2n+2k-3 \geq 5$. Hence Smale's h-cobordism theorem applies, and $CP^{k-1} \times F$ is diffeomorphic to $CP^{k-1} \times S^{2n-1}$. Applying (5.7), it follows that there is a semi-free action of S^1 on some homotopy sphere Σ^m with F as untwisted fixed point set, $m = 2n-1+2k$. This proves (6.1).

In case $k = 2q+1$ then $(\ker(1 \times G)_*)_{n+2q} = H_{2q}(CP^{2q}) \otimes H_n(W_0)$ and it follows easily, in the notation of § 1, that $K^{n+2q}(CP^{k-1} \times W_0) = H^{2q}(CP^{2q}) \otimes H^n(W_0)$. If $x \in H^2(CP^{2q})$ is a generator, then x^q generates $H^{2q}(CP^{2q})$ and x^{2q} generates $H^{4q}(CP^{2q})$. Hence

$$(x^q \otimes a, x^q \otimes b) = ((x^q \otimes a) \cup (x^q \otimes b))[CP^{2q} \times W_0]$$
$$= ((x^{2q})[CP^{2q}])(ab)(W_0] = (ab)[W_0] = (a, b).$$

Hence the pairing on $K^{n+2q}(\mathrm{CP}^{2q}\times W_0)$ is isomorphic to the pairing on $K^n(W_0)$, so it follows if n is even that $I(1\times G)=I(G)$. It follows as in the proof of (2.12) NOVIKOV's Classification Theorem and (2.11), that $\mathrm{CP}^{2q}\times F$ is diffeomorphic to $(\mathrm{CP}^{2q}\times S^{2n-1})\# L$ where $L\in \mathrm{bP}^{2n}$, and $L=\partial U$, $U\in P^{2n}$, index U = index W.

Hence by (5.7) $\mathrm{CP}^{2q}\times S^{2n-1}$ is diffeomorphic to $(\mathrm{CP}^{2q}\times S^{2n-1})\# L=\mathrm{CP}^{2q}\times F$, i.e. $L\in I_0(\mathrm{CP}^{2q}\times S^{2n-1})$, if and only if F is the untwisted fixed point set of a semi-free action of S^1 on a homotopy $2q+2n+1$ sphere, where if $F=mg_n$, g_s a generator of bP^{2s}, s even, then $L=mg_{n+2q}$. It follows that F is the untwisted fixed point set if and only if m is divisible by r, where rg_{n+2q} generates

$$I_0(\mathrm{CP}^{2q}\times S^{2n-1})\subset \mathrm{bP}^{4q+2n},$$

and

$$r=\text{order } \mathrm{bP}^{4q+2n}/\text{order } I_0(\mathrm{CP}^{2q}\times S^{2n-1}),$$

which proves (6.2).

The proof of (6.3) is similar where one shows that $\sigma(1\times G)=\sigma(G)$ using properties of σ in dimension $\equiv 2 \pmod 4$ which we have not discussed here (such as the definition). We refer to [14] or [7] for the necessary techniques. In particular, SULLIVAN has proved the following formula (in unpublished work): Let $f:(M,\partial M)\to(X,Y)$, (f,b) a normal map of degree 1, $(f|\partial M)_*: H_*(\partial M)\to H_*(Y)$ an isomorphism as in § 2, $m=\dim M=4k+2$, N a smooth manifold $n=\dim N=4\ell$. Then $1\times f: N\times(M,\partial M)\to N\times(X,Y)$ and $1\times b$ give a normal map and

$$\sigma(1\times f)=\chi(N)\sigma(f).$$

One cannot say much about the homotopy sphere on which S^1 acts from this construction. However if one takes the "equivariant connected sum" of Σ with itself ℓ-times, i.e. connected sum along a cell which intersects the fixed point set F and is invariant, we get a semi-free S^1 action on $\Sigma\#\Sigma\#\cdots\#\Sigma=\ell\Sigma$ with $\ell F=F\# F\#\cdots\# F$ as fixed point set, (see [6; § 3]). Hence if $\ell\Sigma=S^m$ we arrive at an action on S^m with ℓF as fixed point set. For example, order $\theta_{11}=992$, order $\theta_7=28$ (see [29]), so starting from any $F\in\theta_7=\mathrm{bP}^8$, using (6.1), and the above we may obtain $992F$ as the fixed point set of a semi-free action on S^{11}, and $992\theta_7=4\theta_7$ so we get the result of MONTGOMERY-YANG [36] (see [6]):

(6.5) There are semi-free S^1 actions on S^{11} with every element of $4\theta_7$ as untwisted fixed point set.

In applying (6.2), unfortunately not much is known about calculating $I_0(\mathrm{CP}^{2q}\times S^{2n-1})$, (c.f. [12]). However m_s is certainly a multiple of m_s/ℓ so we may use the same numbers used above in deriving (6.5) to show that there are semi-free actions of S^1 on homotopy 13-spheres

with every element of $4\theta_7$ as untwisted fixed point set. But since $\theta_{13}=\mathbb{Z}_3$, we may use the connected sum method above to show that

(6.6) There are semi-free S^1 actions on S^{13} with every element of $4\theta_7$ as untwisted fixed point set.

Now we shall apply the methods of § 2 to construct exotic semi-free S^1 actions on homotopy spheres with fixed point set the standard sphere. We will construct infinitely many inequivalent such actions using (2.19), (2.27), etc. To apply these results we must have a manifold M which satisfies condition (K) in § 2:

Condition (K). If $g: M\times S^1\to M\times S^1$ is a homotopy equivalence such that

(i) $g|M\times s\sim$ inclusion,

(ii) $pg\sim p, p: M\times S^1\to S^1$, then $g^*: H^{4*}(M\times S^1;\mathbb{Q})\to H^{4*}(M\times S^1;\mathbb{Q})$ is the identity.

(6.7) Proposition. *$M=\mathrm{CP}^{k-1}\times S^q$, q odd satisfies condition (K).*

Proof. Let $g:\mathrm{CP}^{k-1}\times S^q\times S^1\to\mathrm{CP}^{k-1}\times S^q\times S^1$ be a homotopy equivalence satisfying (i) and (ii), above. Then as a map into the product

$$\begin{array}{ccc} M & \xrightarrow{\Delta_2} & M\times M\times M \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle (p_1 g, p_2 g, p_3 g)} \\ & & \mathrm{CP}^{k-1}\times S^q\times S^1 \end{array}$$

commutes, where $\Delta_2(m)=(m,m,m)$, $m\in M$. Hence $g^*(x)=(g^*p_1^*x)\cup(g^*p_2^*x)\cup(g^*p_3^*x)$ for $x\in H^*(\mathrm{CP}^{k-1}\times S^q\times S^1)$. Now by (ii) $pg\sim p$, and $p=p_3:\mathrm{CP}^{k-1}\times S^q\times S^1\to S^1$, so $g^*p_3^*=p_3^*$. If $q>1$, then $j^*: H^2(\mathrm{CP}^{k-1}\times S^q\times S^1)\to H^2(\mathrm{CP}^{k-1})$ is an isomorphism, $j:\mathrm{CP}^{k-1}\to\mathrm{CP}^{k-1}\times S^q\times S^1$ inclusion. By (i) above, $j^*g^*=j^*$, so that $j^*g^*p_1^*=j^*p_1^*=$ identity on $H^2(\mathrm{CP}^{k-1})$. Hence $g^*p_1^*=p_1^*$ on $H^2(\mathrm{CP}^{k-1})$, and since $H^2(\mathrm{CP}^{k-1})$ generates $H^*(\mathrm{CP}^{k-1})$ under $\cup$-product, $g^*p_1^*=p_1^*$ on $H^*(\mathrm{CP}^{k-1})$.

Using condition (i), if $z\in H^q(S^q)$, $x\in H^2(\mathrm{CP}^{k-1})$, $y\in H^1(S^1)$ are the generators, then $p_2^*z=1\otimes z\otimes 1$ and $g^*p_2^*(z)=1\otimes z\otimes 1+\lambda x^k\otimes 1\otimes y$, for some $\lambda\in\mathbb{Z}$, $q=2k+1$, since $1\otimes z\otimes 1$ and $x^k\otimes 1\otimes y$ are a basis for $H^q(\mathrm{CP}^{k-1}\times S^q\times S^1)$. A basis of $H^{4i}(\mathrm{CP}^{k-1}\times S^q\times S^1)$ is given by $x^{2i}\otimes 1\otimes 1$, $x^\ell\otimes z\otimes y$, where $2\ell+q+1=4i$. Hence $g^*(x^{2i}\otimes 1\otimes 1)=x^{2i}\otimes 1\otimes 1$, and $g^*(x^\ell\otimes z\otimes y)=(x^\ell\otimes 1\otimes 1)(1\otimes z\otimes 1+\lambda x^k\otimes 1\otimes y)(1\otimes 1\otimes y)=x^\ell\otimes z\otimes y+(x^{\ell+k}\otimes 1\otimes y^2)=x^\ell\otimes z\otimes y$, since $y^2=0$. Hence $g^*=$ identity on H^{4i}, and condition (K) is satisfied, if $q>1$.

If $q=1$, then $g^* p_1^*(x)=x\otimes 1\otimes 1+\lambda(1\otimes g\otimes y)$. Since $x^k=0$, $g^* p_1^*(x^k)=0$, since $g^2=y^2=0$,

$$g^* p_1^*(x^k)=(g^* p_1^*(x))^k=(x\otimes 1\otimes 1+\lambda(1\otimes g\otimes y))^k$$

$$=x^k\otimes 1\otimes 1+k\lambda(x^{k-1}\otimes g\otimes y)=k\lambda(x^{k-1}\otimes g\otimes y)=0.$$

Hence $\lambda=0$, and $g^* p_1^*=p_1^*$. The rest of the proof proceeds as before, and therefore condition (K) is satisfied for any odd q.

(6.8) Lemma. rank $H^{4*}(\Sigma((CP^{k-1}\times S^q)_+))=[(k-1)/2]+1$ *if q is odd and $q+2k\equiv 1 \bmod 4$. Also* rank $H^{4*}(\Sigma((CP^{k-1}\times S^q)_+))=[k/2]$ *if q is odd and $q+2k\equiv -1 \bmod 4$.*

(6.9) Corollary. *If $q>1$, q odd, $k>1$, and $m=q+2k-2=4n-1$ then $A_0(\mathscr{D}_0(CP^{k-1}\times S^q))$ has rank $[(k-1)/2]$. If $m=5$, 13 or 29, then $A_0(\mathscr{D}_0(CP^{k-1}\times S^q))$ has rank $[k/2]$.*

This follows from (6.8), (6.7) and (2.27).

Now each element of $\mathscr{D}(CP^{k-1}\times S^q)$, gives rise to a semi-free S^1 action on a homotopy $(q+2k)$-sphere with S^q as untwisted fixed point set, and we shall use the homomorphism $A_0: \mathscr{D}_0(CP^{k-1}\times S^q)\to H^{4*}(\Sigma(CP^{k-1}\times S^q)_+)$ and (5.8) to distinguish different actions.

By (5.8), if $f_i: CP^{k-1}\times S^q\to CP^{k-1}\times S^q$, $i=1,2$ are diffeomorphisms, then the induced S^1 actions are equivalent if and only if there are diffeomorphisms $d: S^q\to S^q$, $g: CP^{k-1}\times D^{q+1}\to CP^{k-1}\times D^{q+1}$ and a complex bundle map $b: C^k\times S^q\to C^k\times S^q$ covering d such that the diagram commutes:

$$\text{(U)}\qquad \begin{array}{ccc} CP^{k-1}\times S^q & \xrightarrow{P(b)} & CP^{k-1}\times S^q \\ \big\downarrow{\scriptstyle f_1} & & \big\downarrow{\scriptstyle f_2} \\ CP^{k-1}\times S^q & \xrightarrow{\partial g} & CP^{k-1}\times S^q \end{array}$$

Therefore we shall study such diffeomorphisms $P(b)$ and g.

(6.10) Lemma. *There is a map $\alpha: S^q\to U(k)$ such that $P(b)=(1\times d)(f_\alpha)$ where $f_\alpha: CP^{k-1}\times S^q\to CP^{k-1}\times S^q$ is the diffeomorphism $f_\alpha(x,y)=(x\cdot\alpha(y),y)$, where $U(k)$ acts on CP^{k-1} on the right.*

Proof. Any complex bundle map covering the identity is f_α for some $\alpha: S^q\to U(k)$. But $(1\times d)^{-1}P(b)$ is a complex bundle map covering the identity and the result follows.

(6.11) Lemma. *The correspondence which sends $\alpha: S^q\to U(k)$ into the diffeomorphism $f_\alpha: CP^{k-1}\times S^q\to CP^{k-1}\times S^q$ (defined above) defines a homomorphism of groups*

$$\varphi: \pi_q(U(k))\to\mathscr{D}(CP^{k-1}\times S^q).$$

Proof. Let $\alpha,\beta: S^q\to U(k)$ and let $h: S^q\times I\to U(k)$ be a homotopy of α to β. Then $H: CP^{k-1}\times S^q\times I\to CP^{k-1}\times S^q$, $H(x,y,t)=(x\cdot h(y,t),y)$ is a diffeomorphism for each t, and hence an isotopy. Hence the isotopy class of f_α depends only on the homotopy class of α.

Let $S^q=D_1^q\cup D_2^q$, $S^q-\operatorname{int} D_i^q\subset\operatorname{int} D_j$, $(i,j)=(1,2)$ and $(2,1)$. Since $\alpha|D_1$ is homotopic to a constant and $U(k)$ is connnected, using the homotopy extension theorem, α is homotopic to α' such that $\alpha'|D_1=e$, the identity of $U(k)$. Similarly, β is homotopic to β', $\beta'|D_2=e$. Hence $f_{\alpha'}|CP^{k-1}\times D_1=$identity, and $f_{\beta'}|CP^{k-1}\times D_2=$identity. If we take $\alpha'\beta': S^q\to U(k)$, $\alpha'\beta'(y)=\alpha'(y)\beta'(y)$, then $\alpha'\beta'|D_1=\beta'$, $\alpha'\beta'|D_2=\alpha'$, so that $f_{\alpha'\beta'}=f_{\beta'}$ on $CP^{k-1}\times D_1^q$ and $f_{\alpha'\beta'}=f_{\alpha'}$ on $CP^{k-1}\times D_2^q$. Also $f_{\alpha'}f_{\beta'}=f_{\beta'}$ on $CP^{k-1}\times D_1$ and $=f_{\alpha'}$ on $CP^{k-1}\times D_2$. Hence $f_{\alpha'\beta'}=f_{\alpha'}f_{\beta'}$. But $\alpha'\beta'$ is homotopic to $\alpha'+\beta'$ by a well known argument. Hence $f_{\alpha'\beta'}$ is isotopic to $f_{\alpha'+\beta'}$. Since $\alpha\sim\alpha'$, $\beta\sim\beta'$, then $\alpha'+\beta'\sim\alpha+\beta$, and $f_{\alpha'+\beta'}$ is isotopic to $f_{\alpha+\beta}$, $f_{\alpha'}$ is isotopic to f_α, $f_{\beta'}$ is isotopic to f_β. Hence $f_{\alpha+\beta}$ is isotopic to $f_\alpha f_\beta$. Hence $\varphi: \pi_q(U(k))\to\mathscr{D}(CP^{k-1}\times S^q)$ is a homomorphism.

(6.12) Proposition. *Let $f: CP^{k-1}\times S^q\to CP^{k-1}\times S^q$ be a map such that $f^*(x\otimes 1)=x\otimes 1$, $x\in H^2(CP^{k-1})$, and suppose f is the restriction to the boundary of a map $F: CP^{k-1}\times D^{q+1}\to CP^{k-1}\times D^{q+1}$. Then F is homotopic to the identity.*

Proof. $CP^{k-1}\times D^{q+1}$ is homotopy equivalent to CP^{k-1} and $j: CP^{k-1}\to CP^\infty$ is a homotopy equivalence up to dimension $2k-1$, the first non-zero homotopy group $\pi_i(CP^{k-1})$ for $i>2$ being $\pi_{2k-1}(CP^{k-1})$. The dimension of CP^{k-1} is $2k-2$, so the homology class of F is determined by $F^*|H^2(CP^{k-1})$, which is determined in turn by f^*, so the result follows.

(6.13) Corollary. *Notation as in (6.12), the mapping torus W_F of F is homotopy equivalent to the product $CP^{k-1}\times S^1$ $(W=CP^{k-1}\times D^{q+1})$.*

(6.14) Proposition. *Let $F: CP^{k-1}\times D^{q+1}\to CP^{k-1}\times D^{q+1}$ be a diffeomorphism homotopic to the identity, so $W_F\cong CP^{k-1}\times S^1$. Then $P(W_F)=p_1^*(P(CP^{k-1}))$, $p_1=$projection of the first factor.*

Proof. Let $h: W_F \to \Sigma(W_+)$, $W = \mathrm{CP}^{k-1} \times D^q$, be the map collapsing $W \times s$ to a point, so that $\tau(W_F) - p_1^*(\tau(\mathrm{CP}^{k-1}) \times \varepsilon^0) \in \operatorname{im} h^*$. But $H^{4*}(\Sigma(\mathrm{CP}_+^{k-1})) = 0$, and the result follows.

(6.15) Corollary. *Suppose* $f \in \mathscr{D}_0(\mathrm{CP}^{k-1} \times S^q)$ *and* $f = F|\mathrm{CP}^{k-1} \times S^q$, *F a diffeomorphism of* $\mathrm{CP}^{k-1} \times D^{q+1}$. *Then* $A_0(f) = 0$.

Proof. The mapping torus $M_f = \partial W_F$, $M = \mathrm{CP}^{k-1} \times S^q$, $W = \mathrm{CP}^{k-1} \times D^{q+1}$. Hence $\tau(M_f) = \tau(W_F)|M_f$, and $P(\tau(M_f)) = i^* P(\tau(W_F))$ $i^* p_1^*(P(\mathrm{CP}^{k-1})) = P(\tau(\mathrm{CP}^{k-1} \times S^q) \times \varepsilon^0)$, as stable bundles, i. e. in $KO(\mathrm{CP}^{k-1} \times S^q \times S^1)$. Hence $P(\nu(M_f)) = P(\nu(\mathrm{CP}^{k-1} \times S^q) \times \varepsilon^0)$ and it follows from the definition of A_0 (see (2.24)), that $A_0(f) = 0$.

(6.16) Lemma. *Suppose in diagram* (*U*), $f_i \in \mathscr{D}_0(\mathrm{CP}^{k-1} \times S^q)$, $i = 1, 2$. *Then the maps* $P(b)$ *and* ∂g *are homotopic to the identity. Further* $1 \times d$ *and* f_α *are homotopic to the identity.*

Proof. Since $P(b)$ comes from a complex bundle map $P(b)^*(x) = x$ where $x \in H^2(\mathrm{CP}^{k-1} \times S^q)$ is the Chern class of the canonical complex line bundle over $\mathrm{CP}^{k-1} \times S^q$, and a generator. Since f_1 and $f_2 \in \mathscr{D}_0(\mathrm{CP}^{k-1} \times S^q)$, it follows that $\partial g = f_2(P(b)) f_1^{-1} \sim P(b)$, so that $(\partial g)^*(x) = x$. By (6.12), it follows that $g: \mathrm{CP}^{k-1} \times D^{q+1} \to \mathrm{CP}^{k-1} \times D^{q+1}$ is homotopic to the identity. It follows that if $i: \mathrm{CP}^{k-1} \times S^q \to \mathrm{CP}^{k-1} \times D^{q+1}$ is inclusion, then $i(\partial g) = g i \sim i$. If p_1 is projection on the first factor, then $p_1 i = p_1$, so $p_1(\partial g) = p_1 i(\partial g) = p_1 g i \sim p_1 i = p_1$.

Now since $P(b)$ is orientation preserving and $P(b)^*(x) = x$, it follows that $P(b)^*(y) = y$, y a generator of $H^q(\mathrm{CP}^{k-1} \times S^q) \cong H^q(S^q)$, in order to preserve orientation. Now since $P(b) = (1 \times d)(f_\alpha)$ by (6.10), $\alpha: S^q \to U(k)$, it follows that $d \sim 1$, and hence $p_2 P(b) \sim p_2$. Therefore we have $P(b) \sim \partial g$, $p_1(\partial g) \sim p_1$, $p_2(P(b)) \sim p_2$, so $p_2(\partial g) \sim p_2$. But $\partial g = (p_1 \partial g, p_2 \partial g) \sim (p_1, p_2) =$ identity, so $\partial g \sim$ identity and hence $P(b) \sim$ identity. We have shown $1 \times d \sim$ identity, so $f_\alpha = (1 \times d)^{-1} P(b) \sim$ identity, which completes the proof of (6.16).

(6.17) Lemma. $A_0(1 \times d) = 0$, $d \in \mathscr{D}_0(S^q)$, $1 \times d \in \mathscr{D}_0(\mathrm{CP}^{k-1} \times S^q)$.

Proof. If $t = 1 \times d$, then the mapping torus of t is $\mathrm{CP}^{k-1} \times (S_d^q)$. Now the mapping torus $(S_d^q) \cong S^q \times S^1$, so the normal bundle ν_d of $(S^q)_d$ is fibre homotopy equivalent to the trivial bundle, by the Atiyah-Hirsch Theorem [2], and is trivial along S^q, and along S^1 (since $(S^q)_d$ is orientable). Hence $\nu_d = h^*(\gamma)$, $\gamma \in KO(S^q \times S^1, S^q \vee S^1) \cong KO(S^q \wedge S^1)$ $\cong KO(S^{q+1})$. Then since the index $(S^q)_d = 0$, it follows from the Index Theorem of HIRZEBRUCH [24], that $P(\gamma) = 0$ and (6.17) follows.

(6.18) Proposition. *Let* $\alpha \in \pi_q(U(k))$, $q > 1$. *If* $f_\alpha: \mathrm{CP}^{k-1} \times S^q \to \mathrm{CP}^{k-1} \times S^q$ *is homotopic to the identity, then* α *is of finite order.*

Proof. (c. f. (5.5)). The element α is the characteristic map (see [45]) of a complex vector bundle η^k over S^{q+1}, and if α is of infinite order, it follows that the Chern class $C(\eta) \neq 1$, (see [3]), so $c_n(\eta) \neq 0$, where $q + 1 = 2n$. Recall the relation [4], that if $x \in H^2(P(\eta))$ is the Chern class of the canonical line bundle over $P(\eta) =$ the projective space bundle associated to η, then $\sum_{i=0}^{k} c_{k-i}(\eta) x^i = 0$, so that $x^k + c_n(\eta) x^{k-n} = 0$. Now $P(\eta) = \mathrm{CP}^{k-1} \times D^{q+1} \cup \mathrm{CP}^{k-1} \times D^{q+1}$ with boundaries identified by f_α. If f_α is homotopic to the identity, then it follows that there is a homotopy equivalence $\psi: P(\eta) \to \mathrm{CP}^{k-1} \times S^{q+1}$, and since $q > 1$ $\psi^*(x') = \pm x$, $x' \in H^2(\mathrm{CP}^{k-1} \times S^{q+1})$ a generator. Now $x'^k = 0$, so $\psi^*(x'^k) = \pm x^k = 0$, so $c_n(\eta) = 0$. Hence if f_α is homotopic to the identity, then $\alpha \in \pi_q(U(k))$ is of finite order.

(6.19) Corollary. $A_0(\pi_q(U(k)) \cap \mathscr{D}_0(\mathrm{CP}^{k-1} \times S^q)) = 0$ *for* $q > 1$.

(6.20) Corollary. *The natural map of* $\pi_q(U(k))$ *into homotopy equivalences of* $\mathrm{CP}^{k-1} \times S^q$ *has kernel a torsion group for* $q > 1$.

(6.21) Theorem. *Let* $f_1, f_2 \in \mathscr{D}_0(\mathrm{CP}^{k-1} \times S^q)$, $q > 1$, *and suppose that* f_1 *and* f_2 *define equivalent* S^1 *actions, constructed using* (5.7). *Then* $A_0(f_1) = A_0(f_2)$.

Proof. By (5.8) we have that there are b and g such that diagram (*U*) above is commutative, where $P(b) = (1 \times d)(f_\alpha)$, $\alpha: S^q \to U(k)$, $g: \mathrm{CP}^{k-1} \times D^{q+1} \to \mathrm{CP}^{k-1} \times D^{q+1}$, a diffeomorphism, etc., that is $f_2(1 \times d) f_\alpha f_1^{-1} = \partial g$. By (6.16), all these maps are homotopic to the identity, so that we may apply the homomorphism $A_0: \mathscr{D}_0(\mathrm{CP}^{k-1} \times S^q) \to H^{4*}(\Sigma(\mathrm{CP}^{k-1} \times S^q)_+)$. Hence $A_0(f_2) + A_0(1 \times d) + A_0(f_\alpha) - A_0(f_1)$ $= A_0(\partial g)$. By (6.15), $A_0(\partial g) = 0$, by (6.17), $A_0(1 \times d) = 0$, and by (6.19), $A_0(f_\alpha) = 0$, so $A_0(f_2) = A_0(f_1)$.

(6.22) Corollary. *Let q be odd,* $q > 1$. *If* (a) $q + 2k \equiv 1 \bmod 4$ *and* $k > 2$, *or* (b) *if* $q + 2k = 7$, 15 *or* 31 *and* $k > 1$, *then there are an infinite number of distinct semi-free* S^1 *actions on homotopy* $(q+2k)$*-spheres with* S^q *as untwisted fixed point set.*

The case $q = 3$, $k = 2$ is a theorem of MONTGOMERY and YANG [35].

Proof. By (6.9), $A_0(\mathscr{D}_0(\mathrm{CP}^{k-1} \times S^q)) \neq 0$ under the given conditions on q, k, and $q + 2k$, and therefore there are an infinite number of elements of $\mathscr{D}_0(\mathrm{CP}^{k-1} \times S^q)$ with different images under A_0. Then the result follows from (6.21).

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PART I

Differentiable Transformation Groups

Exotic Actions on Spheres

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In this review article I attempt to give a compendium of examples of exotic differentiable actions of compact Lie groups on homotopy spheres. It was written mainly for the benefit of the participants in the Tulane Conference on Compact Transformation Groups, May 8–June 2, 1967.

The basic arrangement of the examples is done through the linear representations that they resemble. Thus Section 1 treats examples resembling the standard representation of $O(n)$ plus a trivial representation. The examples in Section 2 resemble twice the standard representation of $O(n)$ and some of the many varied examples arising from these are given in Section 3. Section 4 treats examples resembling twice the standard representation plus a trivial 2-dimensional representation. In Section 5 we discuss analogues of the product of several standard representations plus a trivial representation. In Section 6, we consider semifree actions of $S^1=U(1)$ and $S^3=Sp(1)$. These are actions which are free outside the fixed point set and hence resemble some multiple of the standard representation plus a trivial one. The case of free actions of S^1 and S^3 is discussed in Section 7 and free involutions and cyclic actions are discussed in Sections 8 and 9.

Some actions cannot be said to resemble any linear representation, and we discuss some of these in Section 10.

We confine ourselves almost exclusively to the differentiable case, although some comments are made about other cases, and differentiability assumptions will be taken for granted throughout the article unless there is specific mention to the contrary.

Although we are mainly concerned here with various properties of examples, we also discuss a few theorems of a positive nature since these are sometimes of basic importance for adequate discussion of the examples.

There are many examples discussed here, due to myself as well as to others, which are quite recent and have not yet been published. There are also a few examples given which have more of a "folklore" nature.

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