Complete Intersections and the Kervaire Invariant

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The topology of non-singular complete intersections (i.e. a non-singular variety V of complex dimension n defined by k polynomials in $\mathbb{C}P^{n+k}$) has received considerable attention in recent years. It was observed by Thom in the early 1950's that the degrees of the k polynomials determine the diffeomorphism type. From the Lefschetz Theorem, it follows that for a complete intersection $V^n \subset \mathbb{C}P^{n+k}$, the pair $(\mathbb{C}P^{n+k}, V^n)$ is n-connected, so that the inclusion $i : V \longrightarrow \mathbb{C}P^{n+k}$ induces $i_* : \pi_i(V) \longrightarrow \pi_i(\mathbb{C}P^{n+k})$ which is an isomorphism for $i < n$, and an epimorphism for $i = n$. Thus, in some sense, the topology of V is "concentrated in the middle dimension."

In particular $i_* : H_i(V) \longrightarrow H_i(\mathbb{C}P^{n+k})$ is injective for $i \neq n$, and $(\text{image } i_*)_{2n} = d(H_{2n}(\mathbb{C}P^{n+k}))$ where $d = d_1 \cdot \ldots \cdot d_k$ is the total degree of V, d_i = degree of the i-th polynomial P_i, $P_1, \ldots, P_k$ define V, so that (image i_*) is then completely determined by Poincaré duality.

If s is the largest integer less than $n/2$, we can embed $\mathbb{C}P^s \subset V$, and V will be a bundle neighborhood U of $\mathbb{C}P^s$ with handles $D^r \times D^q$ attached (along $S^{r-1} \times D^q$) with $r \geq n$, $r + q = 2n$. In fact, V can be described as $W \cup U'$ where $W = U \cup$ (n-handles) and U' is another copy of U, and union is along the boundaries. The diffeomorphism type of V is then determined by the attaching maps of the n-handles, and the "gluing" map of $\partial(U')$ to ∂W.

The attaching maps of the n-handles will be closely connected to the middle

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dimensional intersection form. When n is even, close analysis of this form (as in [Kulkarni-Wood]) leads to interesting results on the topology of V (see also [Wood], [Libgober], [Libgober-Wood]).

When n is odd, the intersection form is skew symmetric and the analysis of the middle dimensional handles relies on more subtle homological information. A basis for the middle dimensional homology $H_n(V)$, when $n > 1$, n odd, can be represented, using Whitney's and Haefliger's embedding theorems, by embedded spheres $S^n_{1i}, S^n_{2i} \subset V$, with $S_{ij} \cap S_{ik} = \emptyset$, any j, k, $S_{1j} \cap S_{2k} = \emptyset$ for $j \neq k$ and $S_{1i} \cap S_{2i}$ = one point, every i, so that $\{S_{ij}\}$ represent a symplectic base for $H_n(V)$. If each S_{ij} could be chosen to have trivial normal bundle, then a neighborhood of $S_{1i} \cup S_{2i}$ would be diffeomorphic to $(S^n \times S^n - (2n\text{-disk}))$ and it would follow that

$$V = (U \cup U') \;\#\; \mathop{\#}_{q} S^n \times S^n ,$$

the connected sum of $(U \cup U')$ with q copies of $S^n \times S^n$, where $U \cup U'$ is the "twisted double" of U, i.e. two copies of the disk bundle U over $\mathbb{C}P^s$ $(n = 2s + 1)$, glued by a diffeomorphism of the boundary.

The question of finding a basis for $H_n(V)$ represented by embedded spheres with trivial normal bundle, $(S^n \times D^n \subset V)$, can be studied by the methods of the Kervaire invariant arising in surgery theory. This involves defining a quadratic form $\psi : H_n(V) \longrightarrow \mathbb{Z}/2$, such that $\psi(x) = 0$ if and only if $x \in H_n(V)$ is represented by a $S^n \times D^n \subset V$. We give conditions in terms of the degrees of the defining polynomials of V for such a quadratic form to be defined, and show that <u>when</u> ψ <u>cannot be defined</u>, that any $x \in H_n(V)$ can be represented by $S^n \times D^n$.

When ψ can be defined one can find the sought for basis if and only if the Arf invariant of ψ (called the Kervaire invariant) is zero. We give a formula for computing it in these cases.

Our specific results are as follows:

Let $V \subset \mathbb{C}P^{n+k}$ be a non-singular complete intersection of complex dimension $n = 2s + 1$, defined by k-polynomials of degree $d_1, \ldots, d_k$, and let $d = d_1 \ldots d_k$ (= the degree of V).

Theorem A. Suppose exactly ℓ of the degrees $d_1, \ldots, d_k$ are even (so $k - \ell$ are odd). If the binomial coefficient $\binom{s+\ell}{s+1}$ is odd, and $n \neq 1, 3$ or 7, then there exists a homologically trivial $S^n \subset V$ with non-trivial (stably trivial) normal bundle, and every element $x \in H_n(V)$ can be represented by $S^n \times D^n \subset V$. If $n = 1, 3$ or 7 every embedded $S^n \subset V$ has trivial normal bundle.

This was originally proved by [Morita, H] and [Wood, H] for hypersurfaces and [Wood, CI] for complete intersections.

Theorem B. Notation as in A, if $\binom{s+\ell}{s+1}$ is even, then a quadratic form is defined $\psi : H_n(V) \longrightarrow \mathbb{Z}/2$ such that $x \in H_n(V)$ $(n \neq 1, 3$ or $7)$ is represented by $S^n \times D^n \subset V$ if and only if $\psi(x) = 0$.

Theorem C. With hypothesis as in B, $H_n(V)$ has a symplectic basis represented by embeddings $S^n \times D^n \subset V$, mutually intersecting exactly as in the intersection matrix if and only if the Arf invariant of ψ (the Kervaire invariant) $K(V) = 0$, $(K(V) \in \mathbb{Z}/2)$.

(i) If all $d_1, \ldots, d_k$ are odd,

$$K(V) = \begin{cases} 0 & \text{if } d \equiv \pm 1 \pmod 8 \\ 1 & \text{if } d \equiv \pm 3 \pmod 8 \end{cases}$$

(ii) If some d_i's are even, $K(V) = 1$ if and only if $\ell = 2$, $4 | s$

and $8 \nmid d$.

In C , note that the condition $\binom{s+\ell}{s+1} \equiv 0 \pmod 2$ of Theorem B , imposes a condition on the number of d_i which may be even. For example $\ell \neq 1$, and for $\ell = 2$, $\binom{s+2}{s+1} = s+2$ is even if and only if s is even, ($n = 2s+1$) .

As well as Theorem A , [Morita, H] proved B and C for hypersurfaces $(k = 1)$. This case was also done in [Wood, H] . The author first proved C(i) at that time, and [Wood, CI] gave another proof. [Ochanine] first proved C(ii) in the case where V is a $8q + 2$ dimensional Spin manifold, (so $s = 2q$) , which is the only case of even degree when K(V) might equal 1 .

In §1 we discuss the definition of the quadratic form in a general context and prove that when the form is not defined (for framed $M^{2n} \times \mathbb{R}^k \subset W$) there is an embedded sphere $S^n \subset M^{2n}$ $(n \neq 1, 3, 7)$ which is homologically trivial (mod 2) in M , with non-trivial (stably trivial) normal bundle. In §2, we consider complete intersections and prove A , B and C . To prove C we invoke a theorem relating the Kervaire invariant of V^n and its hyperplane section V_0^{n-1} which will be proved elsewhere.

§1. Quadratic forms.

In [Browder, K], a definition of the quadratic form arising in surgery or in framed manifolds was given using functional Steenrod squares. We give here a geometrical version of this definition, and then study when it can be defined for complete intersections, and its meaning.

First note:

(1.1) Proposition. For any $x \in H_n(M^{2n}; \mathbb{Z}/2)$, one can find an embedded $N^n \subset M^{2n}$, with $i_*[N] = x$.

The proof of this is standard as in Thom's proof of representability of homology by maps of manifolds, but using the additional fact that for the canonical n-plane bundle γ^n, the first non-trivial k-invariant occurs in dimension $2n+1$, so that there is no obstruction to finding a map $f : M^{2n} \longrightarrow T(\gamma^n)$ such that $[M] \cap f^*(U) = x$, ($[M] \cap$ is Poincaré duality, $U \in H^n(T(\gamma^n); \mathbb{Z}/2)$ is the Thom class). Similarly we get:

(1.2) Proposition. If $M^{2n} \times \mathbb{R}^q \subset W^{2n+q}$, W connected, and $y \in H_{n+1}(W, M; \mathbb{Z}/2)$, we can find $N \subset M$ representing ∂y as in (1.1), with $N = \partial V$, $V \subset W \times [0, 1)$, V connected, with $[V]$ representing y, $[V] \in H_{n+1}(V, N; \mathbb{Z}/2)$ the fundamental class. Further V meets $W \times 0$ transversally in $N \subset M$.

Now the normal bundle of N in $W \times 0$ has a q-frame given by the product $M \times \mathbb{R}^q$ restricted to N. The obstructions to extending this frame to a normal q-frame on $V \subset W \times [0, 1)$ lie in $H^{i+1}(V, N; \pi_i(V_{n+q,q}))$ where $V_{n+q,q}$ is the space of orthogonal q-fames in $\mathbb{R}^{n+q}$, so $V_{n+q,q} = O(n+q)/O(n)$, and is $(n-1)$-connected. Hence all these obstructions are zero except the last, $\alpha \in H^{n+1}(V, N; \pi_n(V_{n+q,q})) \cong \mathbb{Z}/2$. Evaluating α on $[V]$ we get an element in $\mathbb{Z}/2$, and we would like to define $\psi(x) = \alpha[V]$, (for $x = \partial y$, $y \in H_n(W, M; \mathbb{Z}/2)$) but we have made a number of choices in this process which depend on more than the homology class x, namely the choice of N and the choice of V, with $\partial V = N$.

From the theory of the Stiefel-Whitney class (see [Steenrod]) the first obstruction to finding a q-frame in an $(n+q)$ plane bundle is the Stiefel-Whitney class W_{n+1}, which becomes the ordinary Stiefel-Whitney class w_{n+1} when reduced mod 2. In the relative situation we are discussing, this is in fact the relative Stiefel-Whitney class in the sense of [Kervaire]. Thus it is homologically defined provided that this relative class does not depend on the chocie of V. This will be true provided that any closed manifold $X^{n+1} \subset W^{n+q} \times [0,1)$ admits a normal q-frame, that is, its normal Stiefel-Whitney class $\overline{w}_{n+1}(X) = 0$, so that adding X to V will not change the relative class of V.

If ξ^{n+q} is the normal bundle of X in $W \times [0, 1)$, the normal class $\overline{w}_{n+1}(X)$ is given by the formula

$$\overline{w}_{n+1} \cup U = Sq^{n+1} U \quad , \quad U \in H^{n+q} (T(\xi) \; ; \; \mathbb{Z}/2)$$

the Thom class. The natural collapsing map $c : \Sigma W = (W \times [0,1]) / \text{boundary} \longrightarrow T(\xi)$ has degree 1 (mod 2), and it follows that:

(1.3) The following are equivalent:

(a) $\overline{w}_{n+1}(X) = 0$ for all $X^{n+1} \subset W \times [0, 1)$

(b) $Sq^{n+1} : H^{n+q-1} (W / \partial W \; ; \; \mathbb{Z}/2) \longrightarrow H^{2n+q} (W / \partial W \; ; \; \mathbb{Z}/2)$ is zero

(c) $v_{n+1}(W) = 0$ (v_{n+1} = the Wu class).

Thus we get the condition:

(1.4) The obstruction to extending a q-frame over N to V described above defines a quadratic form $\psi : K \longrightarrow \mathbb{Z}/2$ where $K = \text{kernel } H_n(M \; ; \; \mathbb{Z}/2) \longrightarrow H_n(W \; ; \; \mathbb{Z}/2)$ if and only if $v_{n+1}(W) = 0$.

It is not difficult to translate this relative Stiefel-Whitney class definition into the functional Sq^{n+1} definition of [Browder, K], which shows it defines a quadratic form.

One may prove (1.4) directly as follows:

Since we have shown that the definition is evaluation of a relative Stiefel-Whitney class it follows that the definition depends only on homology class. To show it quadratic, we first prove it in the special case of $S^n \times S^n \times \mathbb{R}^q \subset S^{2n+q} = \partial D^{2n+q+1}$, which may be done directly. It is clear that the function is additive on two non-intersecting manifolds $N_1^n , N_2^n \subset M^{2n}$, (which then bound non-intersecting $V_1^{n+1} , V_2^{n+1} \subset W \times [0 , 1)$.

If N_1 , N_2 have even intersection number, then if $n > 1$, we may find a bordism of N_1 to N_1' in M, disjoint from N_2', (simply the first few lines of the Whitney process produces the cobordism from each pair of intersection points). Take two intersection points $a , b \in N_1 \cap N_2$ and draw an arc on N_2 joining them. If N_2 were not connected we could first make a bordism of N_2 to a connected submanifold, if M were connected. If M were not connected, we would first take connected sum of its components, without changing the quadratic forms.

A neighborhood of this arc would be of the form $D^1 \times D^{n-1} \times D^n$ where $D^1 \times D^{n-1} \times 0$ is a neighborhood in N_2. Then $D^1 \times 0 \times D^n$ defines a handle which when added to N_1, produces a bordism of N_1 to N_1' which has 2 less intersection points with N_2.

This shows that $\psi(x_1 + x_2) = \psi(x_1) + \psi(x_2)$ whenever $x_1 \cdot x_2$ is even. If $x_1 \cdot x_2$ is odd, let $g_1 , g_2 \in H_n(S^n \times S^n)$ be the generators corresponding to the factors, so that $\psi(g_1) = \psi(g_2) = 0$, $\psi(g_1 + g_2) = 1$. Then $(x_1 + g_1) \cdot (x_2 + g_2) = x_1 \cdot x_2 + g_1 \cdot g_2$ is even, so that

(a) $$\psi((x_1 + g_1) + (x_2 + g_2)) = \psi(x_1 + g_1) + \psi(x_1 + g_2) = \psi(x_1) + \psi(x_2) .$$

But $(x_1 + x_2) \cdot (g_1 + g_2) = 0$ so

(b) $$\psi((x_1 + x_2) + (g_1 + g_2)) = \psi(x_1 + x_2) + \psi(g_1 + g_2) = \psi(x_1 + x_2) + 1$$

Equating (a) and (b) we get

$$\psi(x_1 + x_2) + 1 = \psi(x_1) + \psi(x_2)$$

and $(x_1 \cdot x_2) \equiv 1 \mod 2$, which completes the proof that ψ is quadratic. $\square$

(1.5) Theorem. Suppose $M^{2n} \times \mathbb{R}^q \subset W^{2n+q}$, $n \neq 0, 1, 3$ or 7, W is 1-connected, (W,M) n-connected, and suppose $v_{n+1}(W) \neq 0$. Then there exists an embedded $S^n \subset M^{2n}$ and $U^{n+1} \subset M^{2n} \times \mathbb{R}^{q+1}$ with $\partial U = S^n$ such that the normal bundle ξ to S^n in M^{2n} is non-trivial, but $\xi + \varepsilon^1$ is trivial. Hence S^n is homologically trivial (mod 2) with non-trivial normal bundle.

Proof: Since $v_{n+1}(W) \neq 0$ we can find an embedding of a closed manifold $j : X^{n+1} \subset W$ whose normal bundle ξ does not admit a q-frame in $W \times [0,1)$. For $v_{n+1}(W) \neq 0$ means there is an $x \in H^{n+q-1}(W/\partial W ; \mathbb{Z}/2)$ such that $(Sq^{n+1} x)[W] \neq 0$, and hence $q \geq 2$. By Thom's theorem, since $n + q - 1 > n$, there is a map $r : (W, \partial W) \longrightarrow (T(\gamma_{n+q-1}), \infty)$ such that $r^* U = x$ (U the Thom class in $H^{n+q-1}(T(\gamma_{n+q-1}) ; \mathbb{Z}/2)$, and the transverse inverse image of the 0-section will be our manifold X^{n+1}, which we may assume connected, by choosing a component with the above property.

Let $X_0 = X - (\text{int } D^{n+1})$ so that $\partial X_0 = S^n$. Since X was connected, X_0 has the homotopy type of a n dimensional complex. Since (W, M) is n-connected, it follows that there is a map $f : X_0 \longrightarrow M$ such that

$$\begin{array}{ccc} X_0 & \xrightarrow{f} & M \\ \cap & & \cap\, i \\ X & \longrightarrow & W \end{array}$$

commutes up to homotopy.

Let g be the composite $S^n = \partial X_0 \subset X_0 \xrightarrow{f} M$, so that $g_*[S^n] = 0$. Since M is 1-connected, the Whitney process will produce a homotopy of g to an embedding (again called g), and we wish to show the normal bundle ζ^n to this embedded sphere $g(S^n)$ is non-trivial.

Using the Whitney general position embedding theorem, we may deform

$X_0 \xrightarrow{f} M \times \mathbb{R}^q \times (-1, 0]$ to an embedding $g_0 : X_0 \longrightarrow M \times \mathbb{R}^q \times (-1, 0]$ extending g which meets $M \times \mathbb{R}^q \times 0$ transversally in $g_0(\partial X_0) = g(S^n) \subset M \subset M \times \mathbb{R}^q \times 0$.

On the other side, the embedding $g : S^n \subset M$ extends to an embedding (using general position) $\bar{g} : D^{n+1} \subset W \times [0, 1)$ meeting $W \times 0$ transversally in $g(S^n) = \bar{g}(\partial D^{n+1}) \subset M \times 0 \subset M \times \mathbb{R}^k \times 0 \subset W \times 0$. The two embeddings $g_0 : X_0 \subset M \times \mathbb{R}^q \times (-1, 0]$, $\bar{g} : D^{n+1} \subset W \times [0, 1)$ together define an embedding $g_1 : X_0 \cup D^{n+1} = X \longrightarrow W \times (-1, 1)$ which is isotopic (by general position) to our original embedding $j : X \subset W \subset W \times (-1, 1)$.

The product structure $M \times \mathbb{R}^q \subset W \times 0$ defines a q-frame in the normal bundle ζ_1^{n+q} of $g(S^n) \subset M \times \mathbb{R}^q \subset W \times 0$ (so that $\zeta^n + \varepsilon^q = \zeta_1^{n+q}$) and let $\mathcal{O} \in \pi_{n+1}(V_{n+q,q})$ be the obstruction to extending this k-frame over the normal bundle of $\bar{g}(D^{n+1}) \subset W \times [0, 1)$.

(1.6) <u>Lemma</u>. If $n \neq 1, 3$ or 7, the obstruction $\mathcal{O} = 0$ if and only if ζ^n is trivial.

Proofs of (1.6) can be found in [Wall] or [Browder, S ; (IV 4.2)].

We assume ζ^n is trivial and produce a contradiction. In that case we can find a framed handle $D^{n+1} \times D^n \times \mathbb{R}^q \subset W \times [0, 1)$ (using (1.6)) with $D^{n+1} \times 0 \times 0 = \bar{g}(D^{n+1})$ and $S^n \times D^n \times 0$ a neighborhood of $g(S^n) \subset M \subset M \times \mathbb{R}^q \times 0 \subset W \times [0, 1)$. Let V be the cobordism of M defined by $V = M \times [-1, 0] \cup (D^{n+1} \times D^n)$, so that $V \times \mathbb{R}^q \subset W \times [-1, 1]$, and $g_1(X) \subset (\text{int } V) \times \mathbb{R}^q$.

Hence we have a factorization of the collapsing map $Y = W \times [-1, 1] / \partial(W \times [-1, 1]) \xrightarrow{a} \Sigma^q V/\partial V \xrightarrow{b} T(\xi + \varepsilon^1)$ so that $(ba)^*(U) = \Sigma x \in H^{n+q}(Y ; \mathbb{Z}/2)$ and $(Sq^{n+1}(x))[W] = (Sq^{n+1}(\Sigma x))[Y] \neq 0$, $U \in H^{n+q}(T(\xi) ; \mathbb{Z}/2)$ the Thom class. It follows that $(Sq^{n+1}(b^*U))(\Sigma^q[V]) \neq 0$, so that $Sq^{n+1}(\Sigma^{-q}(b^*U))[V] \neq 0$, where $\Sigma^{-q}(b^*U) \in H^n(V/\partial V ; \mathbb{Z}/2)$, which leads to sought after contradiction since Sq^{n+1} annihilates cohomology of dimension n . This completes the proof of (1.5). □

On the other hand we have:

(1.7) <u>Proposition</u>. If $v_{n+1}(W) = 0$, (W, M) n-connected, $n \neq 1, 3, 7$, n odd , and $\varphi : S^n \subset M^{2n}$ with φ nullhomotopic in W . Then the normal bundle of $\varphi(S^n)$ is trivial if and only $\psi(\varphi_*[S^n]) = 0$ where ψ is the quadratic form of (1.4) .

(1.7) follows easily from (1.6) and the definition of ψ .

§2. Complete intersections, their normal bundles and the quadratic form.

In this paragraph, we apply the results of §1 to the case of non-singular complete intersections $V^n \subset \mathbb{C}P^{n+\ell}$, give the conditions for the quadratic form of §1 to be defined, and calculate the Kervaire invariant when it is defined.

Recall that a submanifold $V \subset \mathbb{C}P^{n+k}$ is a non-singular complete intersection if V is the locus of zeros of k homogeneous polynomials $P_1, \dots, P_k$ where $\dim V = 2n$ (real dimension) and $\operatorname{codim} V = 2k$. The degrees d_i of P_i completely determine V up to diffeomorphism. Thus any question we ask in differential topology about V must have an answer in the form of a formula involving only n, and $d_1, \dots, d_k$, and we will use the notation $V = V^n(d_1, \dots, d_k)$.

From the topological point of view it is convenient to view V as a transversal inverse image, to have its normal bundle in $\mathbb{C}P^{n+k}$ evident. We may assume $P_i(1, 0, \dots, 0) \neq 0$ for all i. Define maps $\bar{P}_i : \mathbb{C}P^{n+k} \longrightarrow \mathbb{C}P^{n+k}$ by $\bar{P}_i(z_0, \dots, z_{n+k}) = (P_i(z_0, \dots, z_{n+k}), z_1^{d_i}, \dots, z_{n+k}^{d_i})$, so $\bar{P}_i^{-1}(z_0 = 0)$ is a hypersurface of degree d_i.

Define $\bar{P} : \mathbb{C}P^{n+k} \longrightarrow \prod_{i=1}^{k} \mathbb{C}P^{n+k}$ by $\bar{P} = \Pi \bar{P}_i$. Then $V = \bar{P}^{-1}\left(\bigcap_{i=1}^{k} (z_0^{(i)} = 0)\right)$, where $z_0^{(i)}$ is the 0-th coordinate in the i-th copy of $\mathbb{C}P^{n+k}$.

Small perturbation of the coefficients (if necessary) will make $\bar{P}$ transversal and V will be a non-singular manifold and we get:

(2.1) Proposition. The non-singular complete intersection V^n defined by the vanishing of $P_1, \dots, P_k$ on $\mathbb{C}P^{n+k}$ represents the homology class $dx_n \in H_{2n}(\mathbb{C}P^{n+k})$, and the normal bundle ξ of $V \subset \mathbb{C}P^{n+k}$ has a natural bundle map into the bundle $(\alpha^{d_1} + \alpha^{d_2} + \dots + \alpha^{d_k})$ over $\mathbb{C}P^{n+k}$ where x_n

is the generator dual to $y^k \in H^{2k}(\mathbb{C}P^{n+k})$, $y = c_1(\alpha)$, α the canonical $\mathbb{C}$ bundle over $\mathbb{C}P^{n+k}$, $d = d_1 \ldots d_k$ the total degree of V .

We may transform this situation into the situation of §1 by embedding $\mathbb{C}P^{n+k} \subset E$ = the total space of a bundle γ which is stably inverse to $(\alpha^{d_1} + \ldots + \alpha^{d_k})$. Then the normal bundle of $V \subset E$ has a bundle map into $(\alpha^{d_1} + \ldots + \alpha^{d_k}) + \gamma$ which has a natural trivialization. Hence

(2.2) Proposition. The complete intersection $V^n \subset \mathbb{C}P^{n+k}$ has a natural framing in E , $V \times \mathbb{R}^q \subset E$, where $E = E(\gamma)$, γ a representative of $-(\alpha^{d_1} + \ldots + \alpha^{d_k}) \in K(\mathbb{C}P^{n+k})$.

Note that the framing is determined by the structure of V as a complete intersection, and the polynomials $P_1, \ldots, P_k$, not simply by the differentiable structure of V .

To apply §1 , we need to calculate $v_{n+1}(E(\gamma))$, to see if the quadratic form is well defined. Note that if n is even, $v_{n+1} = 0$ since it lies in a zero group.

(2.3) Theorem. $v_{n+1}(E) \neq 0$ if and only if $\binom{s+\ell}{s+1} \not\equiv 0 \pmod 2$, where ℓ = the number of even integers among the degrees $d_1, \ldots, d_k$, $n = 2s + 1$.

Proof: The Stiefel-Whitney class

$$W(E) = W(\tau_{\mathbb{C}P^{n+k}} + \gamma) = W((n+k+1)\alpha - (\alpha^{d_1} + \ldots + \alpha^{d_k}))$$

$$= \frac{(1+x)^{n+k+1}}{\prod_{i=1}^{k}(1+d_i x)} = \frac{(1+x)^{n+k+1}}{(1+x)^{k-\ell}} = (1+x)^{n+\ell+1} .$$

Hence $W(E) = W(\tau_{\mathbb{C}P^{n+\ell}})$ in dimensions where both cohomologies agree, and hence $v_{n+1}(E) = v_{n+1}(\mathbb{C}P^{n+\ell})$. But $v_{n+1}(\mathbb{C}P^{n+\ell}) \neq 0$ if and only if Sq^{n+1} : $H^{n+2\ell-1}(\mathbb{C}P^{n+\ell}; \mathbb{Z}/2) \longrightarrow H^{2n+2\ell}(\mathbb{C}P^{n+\ell}; \mathbb{Z}/2)$ is non-zero. The group $H^{n+2\ell-1}(\mathbb{C}P^{n+\ell}; \mathbb{Z}/2)$ is generated by $x^{s+\ell}$, where x generates $H^2(\mathbb{C}P^{n+\ell}; \mathbb{Z}/2)$ (since $n = 2s + 1$) , so $Sq^{n+1}(x^{s+\ell}) = Sq^{2s+2}(x^{s+\ell})$

$= \binom{s+\ell}{s+1} x^{n+\ell}$, which completes the proof.$\square$

In E , x^{s+1} is represented by $X = \mathbb{C}P^{s+1}$, so X_0 in the proof of (1.5) may be taken oriented, and we get:

(2.4) <u>Corollary</u>. (Morita, Wood) If $n = 2s + 1$, V non-singular in $\mathbb{C}P^{n+k}$ defined by $P_1, \ldots, P_k$ of degree $d_1, \ldots, d_k$, and if $\binom{s+\ell}{s+1}$ is odd , ℓ = number of even degrees among the d_i's , then there is an embedded $S^n \subset V$ which is homologically trivial, and has a non-trivial normal bundle, (provided $n \neq 1$, 3 or 7).

To calculate the Kervaire invariant (the Arf invariant of ψ) in the other cases (where ψ is well defined) we use the following theorem. The proof will be given in another paper, and it follows from a combination of an additivity theorem for the Kervaire invariant (analogous to Novikov's theorem on index) and the product formula for the Kervaire invariant.

(2.5) <u>Theorem</u>. Let $V^n \subset \mathbb{C}P^{n+k}$ be a non-singular complete intersection, and let $V_0^{n-1} \subset \mathbb{C}P^{n+k-1}$ be a non-singular hyperplane section. If the quadratic forms are defined for both V^n and V_0^{n-1}, then their Kervaire invariants are equal, $K(V) = K(V_0)$.

Note that the definition of $K(V_0)$ may have some extra subtlety as we will see in the calculation.

We can immediately derive the formula for $K(V)$ when $d = d_1 \ldots d_k$ is odd. In that case (2.3) implies that the quadratic forms are defined for all the iterated hyperplane sections $V_0^{n-1} \supset V_1^{n-2} \ldots \supset V_{n-1}^0$ so that (2.5) implies $K(V^n) = K(V_{n-1}^0)$, and we are left with the problem of computing $K(V^0)$ for the zero dimensional complete intersection of degree d , that is, for d similarly oriented points.

This calculation is a special case of that of [Browder, FPK] and is actually equivalent to it using a product formula. We do it explicitly as follows.

Suppose $V^0(d)$ = d disjoint points, embedded in W . A <u>framing</u> of

$V^0(d)$ is simply an orientation on a neighborhood of each point and the condition $v_1(W) = 0$ means that W is orientable. Suppose W is connected, d is odd, and the orientations at all the points are the same.

(2.6) Proposition. $K(V^0(d)) = \begin{cases} 0 & \text{if} \quad d \equiv \pm 1 \mod 8 \\ 1 & \text{if} \quad d \equiv \pm 3 \mod 8 . \end{cases}$

Proof: Since W^m is connected we may assume that $V^0(d) \subset D^m \subset W$ and that the symmetric group Σ_d acts on $V^0(d)$, preserving the framed embedding, so that

$$\psi(\sigma x) = \psi(x) \quad \text{for} \quad \sigma \in \Sigma_d ,$$

$$x \in K_0 = \ker H_0 (V^0(d) ; \mathbb{Z}/2) \longrightarrow H_0 (D^m ; \mathbb{Z}/2) .$$

Now K_0 has a basis $\{x_1 + x_0 , x_2 + x_0 , \dots , x_{2s} + x_0\}$, where the d points are $x_0 , \dots , x_{2s}$, $d = 2s + 1$. Since Σ_d acts transitively on this basis,

(2.7) $\psi(x_i + x_0) = \psi(x_j + x_0)$ for all i , j .

Further the intersection product

(2.8) $(x_i + x_0) \cdot (x_j + x_0) = \begin{cases} 0 & i = j \\ 1 & i \neq j \end{cases}$.

Define a module A_s with quadratic form ψ by letting $a_1 , a_2 , \dots , a_{2s}$ be a basis for A_s , $\psi(a_i) = 0$ for all i , and

$$(a_i , a_j) = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases}$$

(the opposite of an orthonormal basis). It is easy to check that the bilinear form $(\, , \,)$ is non-singular on A_s and is the associated bilinear form to ψ .

Similarly, define B_s , φ by the basis $b_1 , \dots , b_{2s}$, $\varphi(b_i) = 1$, all i , and

$$(b_i , b_j) = \begin{cases} 0 & i = j \\ 1 & i \neq j . \end{cases}$$

(2.9) Lemma. $$A_s \cong A_1 + B_{s-1}$$
$$B_s \cong B_1 + A_{s-1}$$

(as modules with quadratic forms).

Proof: Define a new basis for A_s by $a_i' = a_i + a_1 + a_2$, and let $A_1 \subset A_s$ be generated by a_1 , a_2 , $B'_{s-1} \subset A_s$ be generated by $a_3' , \ldots , a_{2s}'$. Then $A_1 \perp B'_{s-1}$, and it is easy to check that $B'_{s-1} \cong B_{s-1}$.

Similary, define a new basis for B_s by $b_i' = b_i + b_1 + b_2$, let $B_1 \subset B_s$ be generated by b_1 , b_2 , $A'_{s-1} \subset B_s$ generated by $b_3' , \ldots , b_s'$. Then $B_s \cong B_1 + A'_{s-1}$ as orthogonal direct sum, and $A'_{s-1} \cong A_{s-1}$.
Since $\operatorname{Arf}(A_1) = 0$, $\operatorname{Arf}(B_1) = 1$, we get :

$$\operatorname{Arf}(A_s) = \operatorname{Arf}(B_{s-1})$$
$$\operatorname{Arf}(B_s) = 1 + \operatorname{Arf}(A_{s-1})$$

so that

$$\operatorname{Arf}(A_s) = 1 + \operatorname{Arf}(A_{s-2})$$
$$\operatorname{Arf}(B_s) = 1 + \operatorname{Arf}(B_{s-2}) .$$

Hence:

(2.10) Proposition. $$\operatorname{Arf}(B_s) = \begin{cases} 1 & \text{if } s \equiv 1 \text{ or } 2 \mod 4 \\ 0 & \text{if } s \equiv 3 \text{ or } 4 \mod 4 . \end{cases}$$

But in (2.6) , $K_0 \cong B_s$ if $d = 2s + 1$ which completes the proof of (2.6) and the calculation of $K(V^n (d_1 , \ldots , d_k))$ when $d = d_1 \ldots d_k$ is odd , i.e. Theorem C(i) .

The case of even degree d is more difficult, since the quadratic form may not be defined for the iterated hyperplane sections. However, it is always defined for the first hyperplane section $V_0^{n-1} \subset V^n$, (since the appropriate Wu class v_n lies in a zero group when n is odd). To make the calculation in V_0^{n-1} we need:

(2.11) Proposition. Let $M^{2m} \times \mathbb{R}^k \subset W$, $v_{m+1}(W) = 0$, m even, and M oriented. If $x \in H_m(M ; \mathbb{Z})$, $i_*(x_2) = 0$, $i : M \longrightarrow W$ inclusion, $x_2 \in H_m(M ; \mathbb{Z}/2)$ the reduction of x mod 2 , then $\psi(x_2) \equiv \frac{x \cdot x}{2}$ mod 2 .

We sketch a proof, (compare [Morita, P] , [Brown]) .

First note that if $i_*(x_2) = 0$, then $v_m(M)(x_2) = i^*(v_m(W))(x_2) = v_m(W)(i_*x_2) = 0$, so $x \cdot x \equiv x_2 \cdot x_2$ mod 2 , and $x_2 \cdot x_2 = (y \cup y)[M] = (v_m(M) \cup y)[M] = v_m(M)([M] \cap y) = v_m(M)(x_2) = 0$ (where $[M] \cap y = x_2$) . Hence $x \cdot x$ is even, so $\varphi(x) = \frac{x \cdot x}{2}$ mod 2 is a well defined quadratic form.

(2.12) Lemma. Let ψ be our usual quadratic form $\psi : K \longrightarrow \mathbb{Z}/2$ ($K = \ker H_m(M^{2m} ; \mathbb{Z}/2) \longrightarrow H_m(W ; \mathbb{Z}/2)$) as in §1 , and $\varphi : K \longrightarrow \mathbb{Z}/2$ another quadratic form defined in these circumstances such that $\psi(x) = 0$ implies $\varphi(x) = 0$. Then $\psi = \varphi$.

The proof is similar to that of [Browder, S ; (IV. 4.7)] . For the condition implies that on the diagonal $\Delta \in H_m(S^m \times S^m ; \mathbb{Z}/2)$ (for $S^m \times S^m \subset S^{2m+1}$) , $\varphi(\Delta) = 1 = \psi(\Delta)$. Then by adding $S^m \times S^m$ to M^{2m} and adding Δ to an arbitrary $x \in K$ if necessary we get $\varphi(x) = \psi(x)$ (compare the proof of (1.4)) .

Thus to prove (2.11) it suffices to show $\psi(x) = 0$ implies $\varphi(x) = 0$.

(2.13) Lemma. If k is large, $\psi(x) = 0$ implies there exists a framed bordism $U^{2m+1} \times \mathbb{R}^k \subset W \times [0 , 1]$, $\partial(U \times \mathbb{R}^k) = M \times \mathbb{R}^k \times 0 \cup M' \times \mathbb{R}^k \times 1 \subset W \times \{0 , 1\}$, and $V^{m+1} \subset U$, $\partial V = N^m \subset M$ with $[N^m] = x_2 \in H_m(M ; \mathbb{Z}/2)$.

Proof: As in §1 , we can find $N^m \subset M^{2m}$ representing $x \in H_m(M ; \mathbb{Z}/2)$ and $V^{m+1} \subset W \times [0 , 1)$ with $\partial V = N \subset M \times 0$. Then $\psi(x) = 0$ implies that the normal bundle to U admits a k-frame extending that of N (coming from the framing of M in W). The complement of this frame is a D^m bundle over

V which meets M^{2m} in the normal disk bundle of N^m in M, and adding this disk bundle to $M \times [0, \varepsilon]$ clearly defines a framed cobordism of the type required except for the condition $M' \subset W \times 1$, but this may be achieved by an isotopy if k is large.

Now to show $\varphi(x) = 0$ we note that if $[M] \cap \bar{x} = x$, since $x_2 = \partial y$, $y \in H_{m+1}(U, M \cup M'; \mathbb{Z}/2)$, by Poincaré duality, $\bar{x}_2 = i^*\bar{y}$, $\bar{y} \in H^m(U; \mathbb{Z}/2)$, $i'^*\bar{y} = 0$, $i : M \longrightarrow U$, $i' : M' \longrightarrow U$ the inclusions.

Now $x \cdot x = \bar{x}^2[M] \equiv \mathcal{P}(\bar{x}_2)[M] \pmod 4$ where $\mathcal{P} : H^m(M; \mathbb{Z}/2) \longrightarrow H^{2m}(M; \mathbb{Z}/4)$ is the Pontryagin square (see [Morita, P]). Now $i_* : H_{2m}(M; \mathbb{Z}/2) \cong H_{2m}(U; \mathbb{Z}/2)$ so that $x_2 \cdot x_2 \equiv \mathcal{P}(\bar{x}_2)[M] = \mathcal{P}(i^*\bar{y})[M] = (i^*\mathcal{P}(\bar{y}))[M] = \mathcal{P}(\bar{y})(i_*[M]) \equiv 0 \bmod 2$ and hence $\mathcal{P}(\bar{y}) \in j_* H^{2m}(U; \mathbb{Z}/2)$, where $0 \longrightarrow \mathbb{Z}/2 \xrightarrow{j} \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0$.

Since $i'^*(\bar{y}) = 0$, it follows that

$$x \cdot x \equiv \mathcal{P}(i^*\bar{y})[M] = (\mathcal{P}(i^*\bar{y}) + \mathcal{P}(i'^*\bar{y}))([M] - [M'])$$

$$= (\bar{i}^*(\mathcal{P}(\bar{y}))[\partial U] = \mathcal{P}(\bar{y})(\bar{i}_*[\partial U])$$

(all mod 4), $\bar{i} = i \cup i' : M \cup M' = \partial U \longrightarrow U$.

Since $\bar{i}_*[\partial U] \equiv 0 \bmod 2$ it follows that $(\bar{i}_*[\partial U])_4 \in j_*H_{2m}(U; \mathbb{Z}/4)$ $(j : \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4)$. But $\mathcal{P}(\bar{y}) \in j_* H^{2m}(U; \mathbb{Z}/2)$ and $j_* H^{2m}$ and j_*H_{2m} are paired to zero (we get a factor of 2 from each j_* which multiply to become 0 mod 4)

Hence $x \cdot x \equiv 0 \bmod 4$ so $\varphi(x) = 0$, which complete the proof of (2.11) $\square$.

We now proceed to the calculation of $K(V^n(d_1, \dots, d_k))$ for $d = d_1 \cdots d_k$ even. Recall ℓ = number of even d_i's and $\binom{s+\ell}{s+1} \equiv 0 \bmod 2$ (to have ψ defined), where $n = 2s + 1$. By (2.5), $K(V^n(d_1, \dots, d_k)) = K(V_0^{n-1}(d_1, \dots, d_k))$ but we must make this statement more precise.

The quadratic form ψ is defined on $L = \ker(H_{n-1}(V_0; \mathbb{Z}/2) \longrightarrow H_{n-1}(\mathbb{C}P^{n+k-1}; \mathbb{Z}/2))$ and since $n - 1$ is even and d is even, the associated bilinear form is singular on L. Thus, for the Arf invariant of ψ on L to be defined it is necessary that if $r \in L$ and $(r, x) = 0$ for all $x \in L$, then $\psi(r) = 0$ (see [Browder, FPK]), but this is implicitly included in (2.5).

We will now study the middle dimensional intersection form on V_0^{n-1} and using coefficients in $\mathbb{Z}_{(2)}$ (i.e. introducing all odd denominators) we will put it in a form in which the Arf invariant of φ can be easily computed.

Since $(\mathbb{C}P^{n+k-1}, V_0)$ is $(n-1)$-connected, $i_* : H_{n-1}(V_0) \longrightarrow H_{n-1}(\mathbb{C}P^{n+k-1}) \cong \mathbb{Z}$ is onto, and therefore splits. Hence $i^* : H^{n-1}(\mathbb{C}P^{n+k-1}) \longrightarrow H^{n-1}(V_0)$ also splits and we let $h = i^*(g)$, where g generates $H^{n-1}(\mathbb{C}P^{n+k-1})$.

(2.14) <u>Lemma</u>. The Poincaré dual of L = the annihilator of h under $(\ ,\ )$.

<u>Proof</u>. $g(i_* x) = i^*(g)(x) = h(x) = h([V_0] \cap \bar{x}) = (h \cup \bar{x})[V_0]) = (h, \bar{x})$. □

Now let $\beta \in H^{n-1}(V_0)$ be such that $(h, \beta) = 1$, (which is possible since h is indivisible.) Now $(h, h) = (i^* g \cup i^* g)[V_0] = (g \cup g)\, i_*[V_0] = (g \cup g)(d[\mathbb{C}P^{n-1}]) = d$. Hence the quadratic form on A = the submodule generated by h, β, has the matrix

$$\begin{pmatrix} d & 1 \\ 1 & a \end{pmatrix}$$

and hence has <u>odd</u> determinant $ad - 1$, since d is even.

Hence, over $\mathbb{Z}_{(2)}$, we can find a complementary summand B to A so that the matrix for $H^{n-1}(V_0; \mathbb{Z}_{(2)}) = A + B$ becomes

$$\begin{pmatrix} \begin{pmatrix} d & 1 \\ 1 & a \end{pmatrix} & 0 \\ 0 & T \end{pmatrix}$$

Since $B \perp h$, and B is the largest submodule of (annihilator (h)) on which the bilinear form is non-singular (mod 2) , it follows that $\mathrm{Arf}\ \varphi = K(V_0)$ is the Arf invariant of the quadratic form $\frac{x \cdot x}{2}$ (mod 2) on B , and T is the matrix for this intersection form.

(2.15) Proposition. The Arf invariant

$$\mathrm{Arf}\ (B) = \begin{cases} 0 & \text{if } \det T \equiv \pm 1 \mod 8 \\ 1 & \text{if } \det T \equiv \pm 3 \mod 8 . \end{cases}$$

See for example [Hirzebruch-Mayer ; (9.3)] . We sketch the proof here.

Over $\mathbb{Z}_{(2)}$, a matrix T with even diagonal entries and odd determinant may be put in form of the sum of 2×2 blocks

$$STS^t = \begin{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & b_1 \end{pmatrix} & & 0 \\ & \begin{pmatrix} a_2 & 1 \\ 1 & b_2 \end{pmatrix} & \\ 0 & & \begin{pmatrix} a_t & 1 \\ 1 & b_t \end{pmatrix} \end{pmatrix}$$

For given a generator g of B , since $\det T$ is odd, there is $g' \in B$ such that (g , g') is odd so that over $\mathbb{Z}_{(2)}$ g and g' generate a submodule whose matrix may be made into $\begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}$ and we may then split off this module (over $\mathbb{Z}_{(2)}$) and proceed by induction.

For $\begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}$ (a , b even) clearly the Arf invariant is 1 if and only if both $\frac{a}{2}$ and $\frac{b}{2}$ are odd. Then $ab - 1 \equiv \pm 3 \mod 8$ (i.e. $ab - 1$ is not a quadratic residue mod 8) . The result then follows, adding Arf invariants and multiplying determinants of the 2×2 blocks.

Since the bilinear form is unimodular on all of $H^{n-1}(V_0)$, it follows that $\det T = (ad - 1)^{-1}$. The condition $\binom{s + \ell}{s + 1} \equiv 0 \mod 2$ (for defining ψ) implies $\ell \geq 2$ so that $4 \mid d$, (ℓ = number of even d_i's) .

Hence $-(ad-1)^{-1} = 1 + ad + a^2d^2 + \dots \equiv 1 + ad \pmod 8$. Hence, by (2.15), Arf B $= 0$ if $8 \mid d$ or if a is even. It remains to calculate $a = (\beta . \beta)$.

But $(\beta , \beta) = (v_{n-1}(V_0) , \beta) \bmod 2$ so a is odd if and only if $v_{n-1}(V_0)$ is nonzero and equal to $h \pmod 2$. But $v_{n-1}(V_0) = i^* (v_{n-1}(E))$ where $E = E(-(\alpha^{d_1} + \dots + \alpha^{d_k}))$, the total space of this stable bundle over $\mathbb{C}P^{n+k-1}$, so $a = (\beta , \beta)$ is odd if and only if $v_{n-1}(E) \neq 0$.

As in (2.3) we get that $v_{n-1}(E) \neq 0$ if and only if $v_{n-1}(\mathbb{C}P^{n-1+\ell}) \neq 0$. The latter happens if and only if $\binom{s+\ell}{s} \not\equiv 0 \bmod 2$ which completes the proof of Theorem C. □

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