# A REMARK CONCERNING IMMERSIONS OF $S^n$ IN $\mathbf{R}^{2n}$

*By* EDGAR H. BROWN, JR.†

[Received 6 October 1972]

IN this note we give a curious fact about immersions of $S^n$ in $\mathbf{R}^{2n}$. Let

$$S^n = \{x \in \mathbf{R}^{n+1}\colon |x| = 1\}.$$

Let $P^n = S^n/\sim$ where $x \sim -x$, and let $p\colon S^n \to P^n$ be the projection, $p(x) = (x, -x) = \{x\}$. Suppose $h\colon P^n \to \mathbf{R}^{2n}$ is a smooth immersion. Then $hp\colon S^n \to \mathbf{R}^{2n}$ is a smooth immersion. Deform $hp$ by a regular homotopy into an immersion $k$ such that $k(S^n)$ is in general position, that is, $k(S^n)$ cuts itself transversally and all intersections are double points. $k(S^n)$ has an intersection number $i(k)$, in the integers if $n$ is even, and in the integers modulo two if $n$ is odd. $i(k)$ is obtained by counting the intersection points ($y \in \mathbf{R}^{2n}$ such that $h(x) = h(x') = y$, $x \neq x'$) with signs when $n$ is even and modulo two when $n$ is odd. $i(k)$ is an invariant of regular homotopy, and in fact $k$ is regularly homotopic to an embedding if and only if $i(k) = 0$.

*Remark.* If $\nu$ is the normal bundle of $h(P^n)$ in $\mathbf{R}^{2n}$, $p^*\nu$ is stably trivial and hence is determined by its Euler number when $n$ is even and is trivial or isomorphic to the tangent bundle of $S^n$ when $n$ is odd. Furthermore, when $n$ is even, $2i(k)$ is the Euler number, and when $n$ is odd, and $n \neq 1, 3, 7$, $i(k)$ is 0 or 1 according as $p^*\nu$ is trivial or not.

THEOREM 1. $i(k) \not\equiv 0 \bmod 2$ *if and only if* $n = 2^j - 1$.

Theorem 1 and Adams's solution of the vector field problem on spheres in (**1**) yield James's non-immersion results in (**2**), namely:

THEOREM 2. *If* $v(n)$ *is the maximum number of linearly independent tangent vector fields on* $S^n$ *and if* $n = 2^i - 1$, *then* $P^n$ *does not immerse in* $\mathbf{R}^{2n-v(n)-1}$.

Theorem 1 suggests why James's techniques have not been extended to other dimensions.

*Proof of Theorem* 1. Let $\zeta$ be the canonical line bundle of $P^n$ and let $\nu$ be the normal bundle of $h(P^n)$ in $\mathbf{R}^{2n}$. $0^l$ denotes the trivial $l$-plane bundle.

† This work was supported by NSF Grant 28938.

We may assume that $h(P^n)$ is in general position. Also we may assume that $k$ is obtained from $hp$ by moving a short distance along the fibres of $\nu$, and hence that $k$ has the form

$$k(x) = h(\{x\})+v(x),$$

where $v(x) \in \mathbf{R}^{2n}$ is normal to $h(P^n)$ at $h(\{x\})$ and $k(x)$ lies in a tubular neighbourhood of $h(P^n)$. The self-intersections of $k(S^n)$ will arise from points $\{x\} \in P^n$ such that $v(x) = v(-x)$, and from pairs of points $\{x\}$, $\{y\} \in P^n$ such that $h(\{x\}) = (\{y\})$. We may assume that these two phenomena are well separated. A self-intersection of $h(P^n)$ gives four intersections of $h(S^n)$ and hence may be ignored. Thus $i(k)$ is the number of points $\{x\} \in P^n$ such that $v(x) = v(-x)$, modulo two.

Note
$$\nu \otimes \zeta = \{(x, v): x \in S^n,\ v \in \nu,\ p(x) = p_\nu(v)\}/\sim,$$
where $(x, v) \sim (-x, -v)$. Let $s\colon P^n \to \nu \otimes \zeta$ be the section defined by

$$s(\{x\}) = (x, v(x)-v(-x)).$$

One easily checks that $s$ cuts the zero section transversally and that $s(\{x\}) = 0$ if and only if $v(x) = v(-x)$. Hence $i(k)$ equals the number of times $s$ crosses the zero section, modulo two. Hence,

$$i(k) = w_n(\nu \otimes \zeta)(P^n),$$

where $w_n$ is the $n$th Stiefel-Whitney class.

Since
$$\nu+(n+1)\zeta = 0^{2n+1},$$
$$\nu \otimes \zeta+0^{n+1} = (2n+1)\zeta.$$

Therefore
$$w_n(\nu \otimes \zeta)(P^n) = \binom{2n+1}{n} \not\equiv 0 \bmod 2$$
if and only if $n = 2^i-1$.

*Proof of Theorem* 2. Suppose that $n = 2^i-1$ and that $P^n$ immerses in $\mathbf{R}^{2n-v(n)-1}$. Let $\nu'$ be the normal bundle of this immersion. Then $\nu = \nu'+0^{v(n)+1}$ is the normal bundle of an immersion of $P^n$ in $\mathbf{R}^{2n}$ and hence by Theorem 1 and the remark preceding it, $p^*\nu$, which is equivalent to the tangent bundle of $S^n$, has $v(n)+1$ linearly independent sections, contradicting the definition of $v(n)$.

## REFERENCES

1. J. F. Adams, 'Vector fields on spheres', *Ann. of Math.* 75 (1962) 603–32.
2. I. M. James, 'On the immersion problem for real projective spaces', *Bull. Amer. Math. Soc.* 69 (1963) 231–8.