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TWISTED TENSOR PRODUCTS, I

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The objective of this paper, and a subsequent one, is the study of the singular homology structure of fiber spaces. Suppose $p: X \to B$ is a fibering with fiber $F = p^{-1}(b_0)$. For any space X, $S(X)$ will denote the singular chains on X. A fairly complete theory is known for the case when $X = B \times F$. There is first a topological theorem, the Eilenberg-Zilber theorem [3], which says that $S(B) \otimes S(F)$ is chain equivalent to $S(B \times F)$ when it is given the differentiation

$$\partial(T \otimes S) = (\partial T) \otimes S + (-1)^q T \otimes \partial S, \qquad q = \dim T .$$

Then there is an algebraic result, the Künneth formula, which deals with the homology groups of the tensor product of two chain complexes. The objective of this paper is to generalize the Eilenberg-Zilber theorem to fiber spaces. In a subsequent paper we will study the algebraic construction resulting from this generalization.

The Eilenberg-Zilber theorem may be generalized as follows: Let α be a loop in B based at b_0 and let $x \in F$. Lift α to a path $\tilde{\alpha}$ ending at x and let αx denote the initial point of $\tilde{\alpha}$. In Section 1 we show that if $p: X \to B$ is a reasonable fibering (e.g., B paracompact and X locally a product space) $\tilde{\alpha}$ can be chosen for each loop α so that αx defines a continuous action of $\Omega(B)$ on F ($\Omega(B)$ denotes the space of loops on B.) In Section 4 we define a cochain $\Phi \in C^*(B; S(\Omega(B)))$ which assigns to each q-chain of B a q-1-chain of $\Omega(B)$ and satisfies the identity:

$$\partial \Phi_q = \Phi_{q-1}\partial - \sum_{k=1}^{q-1} (-1)^k \Phi_k \smile \Phi_{q-k} \tag{1}$$

where $\Phi = \sum \Phi_q$, $\Phi_q \in C^q(B; S(\Omega(B))$ and the cup product is formed using the Pontrjagin multiplication in $S(\Omega(B))$. (We use Moore paths so as to have an associative multiplication.) A cochain similar to Φ was introduced by J. F. Adams in his paper on the cobar construction [1]. Using Φ we define a differentiation ∂_Φ on $S(B) \otimes S(F)$ by:

$$\partial_\Phi(T \otimes S) = (\partial T) \otimes S + (-1)^q (T \otimes \partial S + (T \otimes S) \smile \Phi), \qquad q = \dim T .$$

Note that the cap product makes sense since $T \otimes S \in C_*(B; S(F))$, $\Phi \in C^*(B; S(\Omega(B)))$ and $S(\Omega(B))$ acts on $S(F)$. Equation (1) above implies that $\partial_\Phi^2 = 0$. We call $S(B) \otimes S(F)$, with ∂_Φ as differentiation, the twisted tensor product of $S(B)$ and $S(F)$ with respect to Φ and denote it by $S(B)_\Phi \otimes S(F)$. Our main result is that $S(B)_\Phi \otimes S(F)$ and $S(X)$ are chain equivalent. The technique used to prove this theorem and the existence of Φ is similar to one used by Eilenberg and Zilber; namely, by means of acyclic models. As a result, the theory is functorial.

In Section 2 we give some algebraic preliminaries and in Section 3 we describe the above construction in purely algebraic terms. In Section 4 we state the main theorems of this paper and in Sections 5 and 6 we prove them. Sections 7, 8, 9, and 10 contain some applications.

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1. Fiber spaces

In this paper we adopt Moore's definition of path space. Let R^+ denote the non-negative real numbers and I_r the interval $0 \leqq t \leqq r$, $r \in R^+$. The space of paths $P(B)$ in a topological space B is defined by:

$$P(B) = \{(\alpha, r) \mid \alpha : I_r \to B,\ r \in R^+\}$$

Let $h : P(B) \to B^I \times R^+ (I = I_1)$ be given by $h(\alpha, r) = (\alpha', r)$ where $\alpha'(t) = \alpha(tr)$, $0 \leqq t \leqq 1$. $P(B)$ is topologized by requiring that h be a homeomorphism. Paths (α, r) and (β, s), such that $\alpha(r) = \beta(0)$, are multiplied as follows: $(\alpha, r)(\beta, s) = (\gamma, r+s)$ where

$$\begin{aligned} \gamma(t) &= \alpha(t) & 0 \leqq t \leqq r \\ &= \beta(t-r) & r \leqq t \leqq r+s \end{aligned}$$

$(e_b, 0)$, $b \in B$, will denote the path defined by $e_b(0) = b$. Then $(e_{\alpha(0)}, 0)$ and $(e_{\alpha(r)}, 0)$ are respectively a left and a right identity for (α, r).

Let $b_0 \in B$. $E(B)$ and $\Omega(B)$ will denote respectively, the subspace of $P(B)$ consisting of all paths ending at b_0 and the subspace of all paths beginning and ending at b_0. The multiplication in $P(B)$ defines an associative multiplication with a unit in $\Omega(B)$ and defines the action of $\Omega(B)$ on the right of $E(B)$. We will assume that in each space B a fixed point b_0 has been chosen as base point for $E(B)$ and $\Omega(B)$, that maps take base points into base points and that when p: $X \to B$ is a fibering, "the fiber" is $p^{-1}(b_0)$.

Hereafter we will denote (α, r) simply by α and $\alpha(r)$ by $\varepsilon\alpha$.

Suppose $p : X \to B$ is a map. Let $U_p \subset P(B) \times X$ be given by

$$U_p = \{(\alpha, x) \mid \varepsilon\alpha = p(x)\}$$

A *lifting function* for the map p is a map λ: $U_p \to X$ such that $p\lambda(\alpha, x) =$

$\alpha(0)$. This is substantially the notion of lifting function given by Hurewicz in [4]. According to Hurewicz a lifting function is a map $\Lambda : U_p \to P(X)$ such that $p\Lambda(\alpha, x) = \alpha$ and $\varepsilon\Lambda(\alpha, x) = x$. Note that if Λ exists, then λ defined by $\lambda(\alpha, x) = \Lambda(\alpha, x)(0)$ is a lifting function in the sense defined above. A lifting function λ is *transitive* if

(1.1) $\lambda(e_b, x) = x, x \in X, b = p(x)$,
(1.2) $\lambda(\alpha\beta, x) = \lambda(\alpha, \lambda(\beta, x))$ when $\alpha\beta$ is defined and $(\beta, x) \in U_p$.

λ is *weakly transitive* if (1.1) holds for $b = b_0$ and (1.2) holds when $\varepsilon\alpha = \beta(0) = b_0$. A (transitive, weakly transitive) fiber space is a quadruple (X, B, p, λ) where $p : X \to B$ and λ is a (transitive, weakly transitive) lifting function for p. It is easily shown that the existence of a lifting function is equivalent to the covering homotopy property for all spaces.

If (X, B, p, λ) is a weakly transitive fiber space, then $\Omega(B)$ acts on the left of $F = p^{-1}(b_0)$ by $\alpha x = \lambda(\alpha, x)$, $\alpha \in \Omega(B)$ and $x \in F$. By (1.1), $e_b x = x$ and by (1.2), $(\alpha\beta)x = \alpha(\beta x)$.

REMARK. If (X, B, p, λ) is weakly transitive, the action of $\Omega(B)$ on F determines X to within weak homotopy type. Let X' be the identification space formed from $E(B) \times F$ by identifying $(\alpha, \beta x)$ with $(\alpha\beta, x)$ for $\alpha \in E(B)$, $\beta \in \Omega(B)$ and $x \in F$. Let $p' : X' \to B$ by $p'\{\alpha, x\} = \alpha(0)$. Then it is easily shown that λ induces a map $h : X' \to X$ such that $p' = ph$ and such that $h_* : \pi_q(X') \approx \pi_q(X)$, $q \geqq 0$.

REMARK. If $p : X \to B$ satisfies the covering homotopy theorem for all spaces, it is homotopic to a transitive fiber space in the following sense. Let $p' : U_p \to B$ be given by $p'(\alpha, x) = \alpha(0)$ and let $\lambda' : U_{p'} \to U_p$ by $\lambda'(\alpha, (\beta, x)) = (\alpha\beta, x)$. Then (U_p, B, p', λ') is a transitive fiber space and there are maps $\nu : U_p \to X$ and $\mu : X \to U_p$ such that $p\nu = p'$, $p'\mu = p$ and such that $\nu\mu$ and $\mu\nu$ are homotopic to the identity maps via homotopies which move points along fibers.

A *map* of one fiber space (X, B, p, λ) into another (X', B', p', λ') is a pair (f, g) where $f : X \to X'$ and $g : B \to B'$ such that $p'f = gp$ and $f\lambda(\alpha, x) = \lambda'(g\alpha, f(x))$ for $\alpha \in \Omega(B)$ and $x \in F$. Note that if $x \in F$ and $\alpha \in \Omega(B)$, then $f(\alpha x) = (g\alpha)f(x)$; that is, the actions of $\Omega(B)$ on F and $\Omega(B')$ on F' commute with f and g.

Suppose $p' : X' \to B'$ is the fibering induced from (X, B, p, λ) by the map $g : B' \to B$, that is,

$$X' = \{(b', x) \in B' \times X \mid g(b') = p(x)\}$$
$$p'(b', x) = b' .$$

Let $f : X' \to X$ be given by $f(b', x) = x$.

(1.3) p' admits a lifting function λ' such that (f, g) is a map of

(X', B', p', λ') into (X, B, p, λ). λ' can be chosen to be transitive (weakly transitive) if λ is transitive (weakly transitive).

PROOF. Let $\lambda'(\alpha, (b', x)) = (\alpha(0), \lambda(g\alpha, x))$.

We conclude this section with some theorems concerning the existence of weakly transitive lifting functions.

(1.4) THEOREM. *If X is locally a product space over B, $p : X \to B$ is the projection and B is paracompact, then p admits a weakly transitive lifting function.*

(1.5) THEOREM. *If (X, p, B, F, G) is a Steenrod fiber bundle and B is paracompact, then p admits a weakly transitive lifting function λ and a product preserving map $k: \Omega(B) \to G$ such that $\lambda(\alpha, x) = k(\alpha)x$ for $x \in F$ and $\alpha \in \Omega(B)$.*

Since (1.4) and (1.5) can be proved by the same technique, we give only the proof of (1.5).

PROOF of (1.5). Let $(\bar{X}, \bar{p}, B, G, G)$ be the associated principal bundle of (X, p, B, F, G) and let $P: \bar{X} \times F \to X$ be the principal map (see [6]). Let $\{U_\gamma\}$ be a set of coordinate neighborhoods covering B and $\varphi_\gamma : U \times G \to \bar{X}$ coordinate maps. Let $b_0 \in U_{\gamma_0}$. We identify G with $p^{-1}(b_0)$ and F with $p^{-1}(b_0)$ by identifying $g \in G$ with $\varphi_{\gamma_0}(b_0, g)$ and $x \in F$ with $P(e, x)$ where e is the identity of G. Note $P(g, x) = gx$ for $g \in G$ and $x \in F$. We first show that (1.5) is implied by:

(1.6) p admits a weakly transitive lifting function $\bar{\lambda}$ such that $\bar{\lambda}(\alpha, xg) = \bar{\lambda}(\alpha, x)g$ for $(\alpha, x) \in U_{\bar{p}}$ and $g \in G$. (Recall G acts on the right of $\bar{X}$.)

Let $\lambda(\alpha, x) = P(\bar{\lambda}(\alpha, y), z)$ where $(\alpha, x) \in U_p$, $y \in \bar{X}$, $z \in F$ and $P(y, z) = x$. By (1.6), $\lambda(\alpha x)$ is independent of the choice of y and z. Let $k: \Omega(B) \to G$ by $k(\alpha) = \bar{\lambda}(\alpha, e)$. Then,

$$\begin{aligned} k(\alpha\beta) &= \bar{\lambda}(\alpha\beta, e) \\ &= \bar{\lambda}(\alpha, \bar{\lambda}(\beta, e)) \\ &= \bar{\lambda}(\alpha, e\bar{\lambda}(\beta, e)) \\ &= \bar{\lambda}(\alpha, e)\bar{\lambda}(\beta, e) \\ &= k(\alpha)k(\beta) \end{aligned}$$

If $\alpha \in \Omega(B)$ and $x \in F$, then $\lambda(\alpha, x) = P(\bar{\lambda}(\alpha, e), x) = P(k(\alpha), x) = k(\alpha)x$.

PROOF OF (1.6). Let λ_γ be the lifting function for $\bar{p} \mid \bar{p}^{-1}(U_\gamma) : \bar{p}^{-1}(U_\gamma) \to U_\gamma$ defined by $\lambda_\gamma(\alpha, x) = \varphi_\gamma(\alpha(0), g)$ where $\alpha \in P(U_\gamma)$, $x \in \bar{p}^{-1}(U_\gamma)$, $\varepsilon\alpha = \bar{p}(x)$ and $\varphi_\gamma(\varepsilon\alpha, g) = x$. λ_γ is clearly transitive and $\lambda_\gamma(\alpha, xg) = \lambda_\gamma(\alpha, x)g$, $g \in G$.

(1.7) $\bar{p}$ admits a lifting function λ' such that for each path $\alpha : I_r \to B$ there is a partition $0 = t_0 < t_1 < \cdots < t_n = r$ satisfying:

(i) $\alpha([t_{i-1}, t_i]) \subset U_{\gamma_i}$

(ii) Let $\alpha_i \in P(U_{\gamma_i})$ by $\alpha_i(t) = \alpha(t + t_{i-1})$, $0 \leqq t \leqq t_i - t_{i-1}$.

Then,

$$\lambda'(\alpha x) = \lambda_{\gamma_1}(\alpha_1, \cdots, \lambda_{\gamma_{n-1}}(\alpha_{n-1}, \lambda_{\gamma_n}(\alpha_n, x)) \cdots) .$$

Hurewicz has substantially proved (1.7) in [4], so we give only an outline of the proof. Since B is paracompact, we may assume that $\{U_\gamma\}$ is a locally finite cover and that for each γ there is a real valued function u_γ on B which is positive on U_γ and zero elsewhere. Let $W_{\gamma_1\gamma_2} \cdots {}_{\gamma_m} \subset P(B)$ consist of all $\alpha : I_r \to B$, $r \geqq 0$, such that:

$$\alpha\left(\left[\frac{(i-1)r}{m}, \frac{ir}{m}\right]\right) \subset U_{\gamma_i}, \qquad i = 1, 2, \cdots, m .$$

For each $\alpha \in W_{\gamma_1\gamma_2 \cdots \gamma_m}$ and $x \in X$ such that $p(x) = \varepsilon\alpha$ define $\lambda_{\gamma_1\gamma_2 \cdots \gamma_m}(\alpha, x) \in X$ as in ((1.7) (ii)), taking $t_i = ir/m$. The sets $W_{\gamma_1\gamma_2} \cdots {}_{\gamma_m}$ cover $P(B)$ and have a locally finite subcover $\{W_\omega\}$. Using the functions u_γ a real valued function v_ω can be defined on $P(B)$ which is positive on W_3 and zero elsewhere. Well order the elements of $\{W_\omega\}$. λ' is now defined by,

$$\lambda'(\alpha, x) = \lambda_{\omega_1}(\alpha_1, \cdots, \lambda_{\omega_{n-1}}(\alpha_{n-1}, \lambda_{\omega_n}(\alpha_n, x)) \cdots)$$

where $W_{\omega_1} < W_{\omega_2} < \cdots < W_{\omega_n}$ are all the sets containing α and

$$\alpha_i(t) = \alpha(t + s_{i-1}) \qquad 0 \leqq t \leqq s_i - s_{i-1}$$

$$s_i = r\sum_{j=1}^{i} v_{\omega_j}(\alpha) \Big/ \sum_{j=1}^{n} v_{\omega_j}(\alpha) .$$

Finally we define $\bar{\lambda}$. Since B is paracompact and therefore normal, we may assume that there is an open set $V \subset V_{\gamma_0}$ such that $b_0 \in V$ and $V \cap U_\gamma = 0$ for $\gamma \neq \gamma_0$. Choose an open set $V' \subset B$ such that $b_0 \in V' \subset \bar{V}' \subset V$ and let $W = B - \bar{V}'$. Then $B = W \cup V$, $V \subset U_{\gamma_0}$ and $V \cap U_\gamma = 0$ for $\gamma \neq \gamma_0$. $\alpha : I_r \to B$ and $x \in X$ such that $\bar{p}(x) = \varepsilon\alpha$. Choose a partition $0 = t_0 < t_1 < \cdots < t_n = r$ such that $\alpha([t_{i-1}, t_i]) \subset V$ or W and $\alpha(t_i) \in V \cap W$. Define $\bar{\lambda}(\alpha, x)$, as in ((1.7) (ii)), using λ' when $\alpha_i \in P(W)$ and λ_{γ_0} when $\alpha_i \in P(V)$. Note that ((1.7) (ii)) implies that when $\alpha_i \in P(W)$ is lifted by λ', any portion of α_i meeting V must be lifted by λ_{γ_0}. But λ_{γ_0} is transitive. Hence $\bar{\lambda}(\alpha, x)$ is independent of the choice of partition for I_r. By the same argument, if α and $\beta \in P(B)$ and $b_0 = \varepsilon\alpha = \beta(0)$, then $\bar{\lambda}(\alpha\beta, x) = \bar{\lambda}(\alpha, \bar{\lambda}(\beta, x)$, that is, $\bar{\lambda}$ is weakly transitive.

2. Algebraic preliminaries

We adopt Cartan's definitions of DGA algebra, module, etc. Let Λ be a commutative ring with a unit 1. A DGA Λ-*module* is a Λ-module M with a differentiation $\partial: M \to M$, a grading by submodules M_q, $q \geqq 0$, and an augmentation $a: M \to \Lambda$ satisfying the following conditions:

(2.1) $M = \sum M_q$ (direct sum)

(2.2) ∂ is a Λ-linear map such that $\partial\partial = 0$, $\partial M_q \subset M_{q-1}$ and $\partial M_0 = 0$.

(2.3) a is a Λ-linear epimorphism such that $a\partial = 0$ and $a(M_q) = 0$ for $q > 0$.

Let M and M' be DGA Λ-modules. A *map* $f: M \to M'$ is a Λ-homomorphism preserving all the DGA structures. The *tensor product* of M and M' is the DGA Λ-module $M \otimes M'$ ($M \otimes_\Lambda M'$) where,

$$
\begin{aligned}
(M \otimes M')_q &= \textstyle\sum_{r+s=q} M_r \otimes M'_s \\
\partial(m \otimes m') &= (\partial m) \otimes m' + (-1)^q m \otimes \partial m', \qquad m \in M_q,\ m' \in M' \\
a(m \otimes m') &= a(m)a(m')
\end{aligned}
$$

A DGA *algebra* is an associative Λ-algebra A with a unit 1 which is also a DGA Λ-module such that $a(1) = 1$ and such that the product map $A \otimes A \to A$ is a DGA map. Let A be a DGA algebra. A DGA A-*module* is a DGA Λ-module M which is also a left A-module such that the product map $A \otimes M \to M$ is a DGA map. A DGA *coalgebra* is a DGA Λ-module K together with a DGA map $\nabla: K \to K \otimes K$ and an element $1 \in K_0$ satisfying the following conditions:

$(\nabla \otimes \mathrm{id})\nabla = (\mathrm{id} \otimes \nabla)\nabla$ where id is the identity map.

$\nabla(1) = 1 \otimes 1$, and

$\nabla(k) = 1 \otimes k + k \otimes 1 + k'$ for $k \in K$, $k' \in K \otimes K$ and $a(k) = a(k') = 0$.

$a(1) = 1$.

Let K be a DGA coalgebra with coproduct ∇, let G, N, and H be Λ-modules and let $\mu: G \otimes N \to H$ be a homomorphism. Let $C^p(K; G) = \mathrm{Hom}(K_p, G)$, $C^*(K; G) = \sum C^p(K; G)$ and define $\delta: C^p(K; G) \to C^{p+1}(K; G)$ by $\delta U = U\partial$. Let $U \in C^*(K; G)$, $V \in C^*(K; N)$ and $c \in K \otimes N$. We define the cup product $U \smile V \in C^*(K; H)$ and the cap product $c \frown U \in K \otimes H$ as follows:

$$
\begin{aligned}
U \smile V &= \mu(U \otimes V)\nabla \\
c \frown U &= (i_1 \otimes \mu)(i_1 \otimes U \otimes i_2)(\nabla \otimes i_2)(c)
\end{aligned}
$$

where $i_1: K \to K$ and $i_2: N \to N$ are the identity maps. It is easily shown that:

$$
\begin{aligned}
\delta(U \smile V) &= (\delta U) \smile V + (-1)^p U \frown \delta V, && U \in C^p(K; G) \\
\partial(c \frown U) &= (\partial c) \frown U + (-1)^{q-p} c \frown \delta U, && U \in C^p(K; G),\ c \in K_p \otimes N \\
U \smile (V \smile W) &= (U \smile V) \smile W \\
(c \frown U) \frown W &= c \frown (U \smile W)
\end{aligned}
$$

Let X be a topological space and let $S(X)$ be its total singular complex. For the remainder of this section we take Λ to be the integers. $S(X)$ is a DGA Z-module ($Z =$ integers) under the usual differentiation, grading and augmentation. If T is a singular q-simplex, $\partial_i T$, $0 \leqq i \leqq q$, will denote the i^{th} face of T. Let $\nabla : S(X) \to S(X) \otimes S(X)$ be given by

$$\nabla(T) = \sum_{k=0}^{q} (\partial_{k+1})^{q-k} T \otimes (\partial_0)^k T$$

Choose a zero simplex in $S(X)$ and denote it by 1. It is well known $S(X)$ is a DGA coalgebra with ∇ as coproduct and 1 as unit. Suppose Y is a topological space with an associative multiplication $\mu : Y \times Y \to Y$ and unit e. Let $\eta : S(Y) \otimes S(Y) \to S(Y \times Y)$ be the Eilenberg-Zilber map. Let 1 denote the 0-simplex in $S(Y)$ whose image is e. Then $S(Y)$ is a DGA algebra under the multiplication $\mu_{\#}\eta : S(Y) \otimes S(Y) \to S(Y)$ and unit 1. Suppose $\nu : Y \times X \to X$ defines an action of Y on X, that is, $\nu(e, x) = x$ and $\nu(yy', x) = \nu(y, \nu(y', x))$, $x \in X$, y, $y' \in Y$. Then $S(X)$ is a DGA $S(Y)$-module under the action of $S(Y)$ on $S(X)$ defined by $\nu_{\#}\eta : S(Y) \otimes S(X) \to S(X)$.

3. Twisted tensor products

Let K be a DGA coalgebra and let A be a DGA algebra. A *twisting cochain* is a cochain $\varphi = \sum \varphi_q \in C^*(K ; A)$ such that

(3.1) $\varphi_q \in C^q(K ; A)$, $\varphi_0 = 0$, $\varphi_q(K_q) \subset A_{q-1}$

(3.2) $a\varphi_1 = 0$ and $\partial\varphi_q = \varphi_{q-1}\partial - \sum_{k=1}^{q-1} (-1)^k \varphi_k \smile \varphi_{q-k}$

where the cup product is formed by using the multiplication in A.

Let L be a DGA A-module. The *twisted tensor product* of K and L with respect to a twisting cochain $\varphi \in C^*(K ; A)$ is the DGA Λ-module $K_\varphi \otimes L$ defined as follows : With respect to grading and augmentation, $K_\varphi \otimes L = K \otimes L$. The differentiation ∂_φ on $K_\varphi \otimes L$ is given by :

(3.3) $\partial_\varphi(k \otimes l) = (\partial k) \otimes l + (-1)_q(k \otimes \partial l + (k \otimes l) \frown \varphi)$ where $k \in K_q$, $l \in L$ and the cap product is formed by utilizing the pairing $A \otimes L \to L$ defined by the A-module structure on L. It follows from (3.2) by a straightforward calculation that $\partial_\varphi \partial_\varphi = 0$ and $a\partial_\varphi = 0$. By (3.1) ∂_φ lowers dimension by one.

We define an increasing filtration on $K_\varphi \otimes L$ as follows :

$$A_p = \sum_{q=0}^{p} K_q \otimes L$$

Clearly $K_\varphi \otimes L = \bigcup A_p$ and $\partial_\varphi(A_p) \subset A_p$. Let $E^{p,q}$ denote the resulting spectral sequence in the sense of Serre [5]. Then (3.4) yields immediately :

(3.5) Suppose K is Λ free. Then $E_1^{p,q} = K_p \otimes H_q(L)$ and if $H_0(A) = \Lambda$, then $E_2^{p,q} = H_p(K; H_q(L))$.

Suppose K and K' are DGA coalgebras, A and A' are DGA algebras, L and L' are respectively, DGA A- and A'-modules, $\varphi \in C^*(K; A)$ $\varphi' \in C^*(K'; A')$ are twisting cochains and $f: K \to K'$, $g: L \to L'$ and $h: A \to A'$ are maps such that $g(al) = h(a)g(l)$, $a \in A$, $l \in L$, and $\varphi' f = h\varphi$. Then $f \otimes g$: $K_\varphi \otimes L \to K'_{\varphi'} \otimes L'$ is a map.

(3.6) Suppose K and K' are Λ free and $H_0(A) = H_0(A') = \Lambda$. If any two of f, g and $f \otimes g$ induce isomorphisms in homology, then the third does.[1]

PROOF. (3.6) follows from (3.5) and Moore's theorem [2, Ch. 3].

(3.7) Suppose $K = K'$, f is the identity and K is Λ free. If g induces an isomorphism in homology, then so does $f \otimes g$.

PROOF. (3.7) follows from (3.5) by well known spectral sequence arguments.

REMARK. Considering A as a DGA A-module, we may form $K_\varphi \otimes A$. Then $(A, K, K_\varphi \otimes A)$ is a construction in the sense of Cartan [2].

4. Main theorems

Let B be a pathwise connected space. $S(B)$ will hereafter denote the singular chains generated by singular simplexes taking the vertices of the standard simplex into b_0. We identify I with the standard 1-simplex Δ_1 by the map $h: I \to \Delta_1$ where $h(t) = (1-t)d_0 + td_1$, d_0 and d_1 the first and second vertices of Δ_1. Then a 1-simplex in $S(B)$ is also a point in $\Omega(B)$. We also identify any singular 0-simplex T with its image $T(d_0)$. If $p: X \to B$ is a fibering, $S(X)$ will denote the chains generated by singular simplexes taking the vertices of the standard simplex into $F = p^{-1}(b_0)$. $S(F)$ and $S(\Omega(B))$ will denote the total singular complexes of F and $\Omega(B)$.

(4.1) THEOREM. *There is a collection of twisting cochains $\Phi_B \in C^*(S(B); S(\Omega(B)))$, one for each pathwise connected space B, satisfying the following conditions*:

(i) *If $T \in S(B)$ is a 1-simplex and $T_0 \in S(B)$ is the constant 1-simplex, $\Phi_B(T) = T - T_0$. (Note that by the above identifications T and T_0 are also 0-simplexes in $S(\Omega(B))$.)*

(ii) *If T is a constant simplex in $S(B)$, $\Phi_B(T) = 0$.*

(iii) *If $f: B \to B'$ and $\bar{f}: \Omega(B) \to \Omega(B')$ is induced by f, then $\bar{f}_\# \Phi_B = \Phi_{B'} f_\#$.*

[1] We will frequently use the fact that a map of one free DGA-module into another is a chain equivalence if and only if it induces isomorphisms in homology.

The proof of (4.1) is given in Section 5. We will sometimes denote Φ_B simply by Φ.

If (X, B, p, λ) is a weakly transitive fiber space, λ defines an action of $\Omega(B)$ on $F = p^{-1}(b_0)$ which in turn defines an $S(\Omega(B))$-module structure on $S(F)$. Hence we may form the twisted tensor product $S(B)_\Phi \otimes S(F)$. Suppose (f, g) is a map of (X, B, p, λ) into another weakly transitive fiber space $(X', B', p',, \lambda')$. Let $k = f|F: F \to F'$ and let $\bar{g}: \Omega(B) \to \Omega(B')$ be the map defined by g. Then, since k and g commute with the actions of $\Omega(B)$ on F and $\Omega(B')$ on F', $k_\#(TU) = \bar{g}_\#(T)k_\#(U)$ for $T \in S(\Omega(B))$ and $U \in S(F)$. Also by (4.1), $\bar{g}_\#\Phi = \Phi g_\#$. Therefore $g_\# \otimes k_\#$: $S(B)_\Phi \otimes S(F) \to S(B')_\Phi \otimes S(F')$ is a map.

(4.2) THEOREM. *Let Φ_B be a collection of twisting cochains as described in* (4.1). *For each weakly transitive fiber space (X, B, p, λ), where B is pathwise connected, there is a map $\psi: S(B)_{\Phi_B} \otimes S(F) \to S(X)$ such that*:

(i) *ψ is a chain equivalence.*

(ii) *If $(f, g): (X, B, p, \lambda) \to (X', B', p', \lambda')$ is a map, then $\psi(g_\# \otimes h_\#) = f_\#\psi$ where $k = f|F: F \to F'$.*

Theorem (4.2) is proved in Section 6.

Suppose B is $n-1$ connected. Let $S_{n-1}(B)$ denote the chains generated by singular simplexes taking the $n-1$ skeleton of the standard simplex into b_0 and let $j: S_{n-1}(B) \to S(B)$ be the inclusion map. Let $\Phi' = \Phi j \in C^*(S_{n-1}(B); S(\Omega(B)))$. It is well known that j is a chain equivalence and Φ' is obviously a twisting cochain. Let i be the identity map on $S(F)$. By (3.6) and ((4.1) (ii))

(4.3) COROLLARY. *$\psi(j \otimes i): S_{n-1}(B)_{\Phi'} \otimes S(F) \to S(X)$ is a chain equivalence and $\Phi'_q = 0$ for $q < n$.*

Suppose (X', B', p', λ') is the fiber space induced from (X, B, p, λ) by a map $g: B \to B'$ as in (1.3). Clearly $(\Phi_B)g_\# \in C^*(S(B'); S(\Omega(B)))$ is a twisting cochain. Also by (1.3) and (4.1), the identity maps $i_1: S(B') \to S(B')$ and $i_2: S(F) \to S(F)$ and $\bar{g}_\#: S(\Omega(B')) \to S(\Omega(B))$ define a map:

$$i_1 \otimes i_2: S(B')_{\Phi_{B'}} \otimes S(F) \to S(B')_{\Phi_B g_\#} \otimes S(F)$$

The fact that $i_1 \otimes i_2$ is an isomorphism yields:

(4.4) COROLLARY. *$S(B')_{\Phi_B g_\#} \otimes S(F)$ and $S(X)$ are chain equivalent.*

Suppose (X, p, B, F, G) is a Steenrod fiber bundle and B is pathwise connected. By (1.4), p admits a weakly transitive lifting function λ and a product preserving map $k: \Omega(B) \to G$ such that $\lambda(\alpha, x) = k(\alpha)x$, $\alpha \in \Omega(B)$, $x \in F$. Let $\varphi = k_\#\Phi_B \in C^*(S(B); S(G))$. Since k is product preserving φ is a twisting cochain. By the same argument as was given for (4.4):

(4.5) COROLLARY. *$S(B)_\varphi \otimes S(F)$ is chain equivalent to $S(X)$, when $S(F)$ is viewed as an $S(G)$-module defined by the action of G on F.*

REMARK. The above theory can be converted to a theory over an arbitrary commutative ring with unit Λ simply by tensoring everything in sight with Λ.

REMARK. If (X, B, p, λ) is a weakly transitive fiber space, the additive homology structure of X is completely determined by the action of $S(\Omega(B))$ on $S(F)$. This may be seen as follows: Let E be an acyclic, $S(\Omega(B))$ free DGA right $S(B))$-module. $E \otimes_{S(\Omega(B))} S(F)$ can be made into a DGA Z-module in the obvious fashion. By well-known arguments, the homology groups of $E \otimes_{S(\Omega(B))} S(F)$ are independent of the choice of E (see [2]). But $S(B)_\Phi \otimes S(\Omega(B))$ is an $S(\Omega(B))$ free $S(\Omega(B))$-module which is acyclic since it is chain equivalent to $S(E(B))$. Furthermore,

$$[S(B)_\Phi \otimes S(\Omega(B))] \otimes_{S(\Omega(B))} S(F) = S(B)_\Phi \otimes S(F)$$

Therefore,

$$H_q(E \otimes_{S(\Omega(B))} S(F)) \approx H_q(X).$$

REMARK. It $X = B \times F$, λ can be chosen so that $\Omega(B)$ acts trivially on F. In this case it can be easily shown that ((4.1) (ii), (iii)) imply that $(T \otimes S) \frown \Phi = 0$, $T \in S(B)$, $S \in S(F)$. Thus (4.2) is in fact a generalization of the Eilenberg-Zilber theorem.

5. Proof of the existence of Φ_B

Let $\bar{\Delta}_q$ be the identification space formed from the standard q-simplex Δ_q by identifying its vertices to a point. Let $\xi_q : \bar{\Delta}_q \to \Delta_q$ be the identification map and let $d \in \bar{\Delta}_q$ be the point consisting of the vertices of Δ_q. We take d as the base point for $\bar{\Delta}_q$. Note $\bar{\Delta}_q$ has the same homotopy type as a collection of circles joined at a point. Hence $H_n(\Omega(\bar{\Delta}_q)) = 0$ for $n > 0$.

Let $\Phi_B = \sum \Phi_q$, $\Phi_q \in C^q(S(B); S(\Omega(B)))$. We define Φ_q by induction on q. Φ_1 is defined in ((4.1) (i)), satisfies ((4.1) (ii), (iii)), and $a\Phi_1 = 0$ (a is the augmentation). We next define Φ_2. For each $s = s_0 d_0 + s_1 d_1 \in \Delta_1$ let $\alpha_s \in P(\Delta_2)$ be given by:

$$\begin{aligned} \alpha_s(t) &= (1-t)d_0 + td_1, && 0 \leqq t \leqq s_1 \\ &= (1 + s_1 - t)s + (t - s_1)d_2, && s_1 \leqq t \leqq s_1 + 1. \end{aligned}$$

If $T \in S(B)$ is a 2-simplex, let $k(T) : \Delta_1 \to \Omega(B)$ by $k(T)(s) = T\alpha_s$. Note $\partial k(T) = (\partial_2 T)(\partial_0 T) - \partial_1 T$ when $\partial_i T$ is viewed as a 0-simplex in $S(\Omega(B))$. Let $h_i : \Delta_{q+1} \to \Delta_q$ denote the simplicial map which is onto, monotone with respect to the order of the vertices and takes d_i and d_{i+1} into d_i. For each singular simplex U let $D_i U = U h_i$, that is, D_i is the usual degeneracy operator. Finally let Φ_2 be defined by :

$$\Phi_2(T) = k(T) - k(D_0\partial_0 T) - k(D_1\partial_1 T) + k(D_0^2\partial_0^2 T)$$

Then Φ_2 is clearly natural and if T is the constant 2-simplex, $\Phi_2(T) = 0$. Let $T_0 \in S(B)$ be the constant 1-simplex. Then,

$$\begin{aligned}\partial\Phi_2(T) &= (\partial_0 T - T_0) - (\partial_1 T - T_0) + (\partial_2 T - T_0) + (\partial_2 T - T_0)(\partial_0 T - T_0)\\ &= \Phi_1(\partial T) + \Phi_1 \smile \Phi_1(T)\end{aligned}$$

Thus Φ_2 has the desired form. Suppose that for $p < q > 1$, Φ_p has been defined, satisfies ((4.1) (i), (ii), (iii)) and

$$\partial\Phi_p = \Phi_{p-1}\partial - \sum (-1)^k \Phi_k \smile \Phi_{p-k} \tag{5.1}$$

Let ν_p denote the right side of (5.1). Then ν_p is defined and a straightforward calculation shows that $\partial\nu_q = 0$. Therefore $\nu_q(\xi_q) \in S(\Omega(\bar{\Delta}_q))$ is a $q-1$ cycle. Since $H_{q-1}(\Omega(\bar{\Delta}_q)) = 0$ for $q > 1$, there is a chain $c \in S(\Omega(\Delta_q))$ such that $\partial c = \nu_q(\xi_q)$. Let $T_0 \in S(B)$ be the constant q-simplex. For each q-simplex $T \in S(B)$ let $\Phi_p(T) = \bar{T}_\#(c) - \bar{T}_{0\#}(c)$ where "—" indicates the induced map of $\Omega(\bar{\Delta}_q) \to \Omega(B)$. Φ_q is clearly natural and $\Phi_q(T_0) = 0$.

$$\begin{aligned}\partial\Phi_q(T) &= (T_\# - T_{0\#})(\partial c)\\ &= (T_\# - T_{0\#})\nu_q(\xi_q)\\ &= \nu_q(T) - \nu_q(T_0)\\ &= \nu_q(T)\end{aligned}$$

Thus Φ_q satisfies (5.1) and the proof of (4.1) is complete.

6. Proof of Theorem (4.2)

We first define a set of acyclic models to be used in the proof of (4.1).

(6.1) There exists weakly transitive fiber spaces $\mathcal{F}_{n,m} = (X_{n,m}, B_{n,m}, p_{n,m}, \lambda_{n,m})$ and $\mathcal{F}_n = (X_n, B_n, p_n, \lambda_n)$ with fibers $F_{n,m}$ and F_n and singular n, m and n-simplexes $\xi_n \in S(B_{n,m})$, $\eta_m \in S(F_{n,m})$ and $\zeta_n \in S(X_n)$ such that :

(i) If $\mathcal{F} = (X, B, p, \lambda)$ is a weakly transitive fiber space and $T \in S(B)$, $S \in S(F)$ and $U \in S(X)$ are n, m and n simplexes, then there exists maps $(u, v) : \mathcal{F}_{n,m} \to \mathcal{F}$ and $(u', v') : \mathcal{F}_n \to \mathcal{F}$ such that $v_\#(\xi_n) = T$, $u_\#(\eta_m) = S$ and $v'_\#(\zeta_n) = U$.

(ii) $S(X_{n,m})$, $S(X_n)$, $S(B_{n,m})_\Phi \otimes S(F_{n,m})$ and $S(B_n)_\Phi \otimes S(F_n)$ are acyclic.

PROOF. Let $\bar{\Delta}_n$ and $d \in \bar{\Delta}_n$ be as in Section 5. $\mathcal{F}_{n,m}$, $\mathcal{F}_n$, ξ_n, η_m, ζ_n, (u, v) and (u', v') may be defined as follows :

$$\begin{aligned}X_{n,m} &= E(\bar{\Delta}_n) \times \Delta_m\\ B_{n,m} &= \bar{\Delta}_n\end{aligned}$$

$$\begin{aligned}
p_{n,m}(\alpha, t) &= \alpha(0)\\
\lambda_{n,m}(\alpha, (\beta, t)) &= (\alpha\beta, t)\\
\xi_n : \Delta_n &\to \bar{\Delta}_n, \text{ the identification map.}\\
\eta_n(t) &= (e_a, t)\\
v : \bar{\Delta}_n &\to B, \text{ the map induced by } T : \Delta_n \to B.\\
u(a, t) &= \lambda(v\alpha, S(t))\\
X_n &= \{(\alpha, t) \in E(\bar{\Delta}_n) \times \Delta_n \mid \varepsilon\alpha = \xi_n(t)\}\\
B_n &= \bar{\Delta}_n\\
p_n(\alpha, t) &= \alpha(0)\\
\lambda_n(\alpha, (\beta, t)) &= (\alpha\beta, t)\\
\zeta_n(t) &= (e_{\xi_n(t)}, t)\\
v' : \bar{\Delta}_n &\to B, \text{ the map defined by } pU : \Delta_n \to B.\\
u'(\alpha, t) &= (v'\alpha, U(t))
\end{aligned}$$

To write out the appropriate definitions is to check that $\mathcal{F}_{n,m}$ and $\mathcal{F}_n$ are transitive fiber spaces, that (u, v) and (u', v') are maps and that ((6.1)(i)) holds. $X_{n,m}$ is clearly contractible to a point. By shrinking paths to their end points, X_n may be contracted to $\{(e_{\xi_n(x)}, x) \mid x \in \Delta_n\}$ which is homeomorphic to Δ_n which in turn is contractible to a point. Therefore $S(X_{n,m})$ and $S(X_n)$ are acyclic.

Let $\mathcal{H}_n = (E(\bar{\Delta}_n), \bar{\Delta}_n, \bar{p}, \bar{\lambda})$ be the transitive fiber space defined by $\bar{p}(\alpha) = \alpha(0)$ and $\bar{\lambda}(\alpha, \beta) = \alpha\beta$. Let $(r, s) : \mathcal{H}_n \to \mathcal{F}_{n,m}$ and $(r', s') : \mathcal{H}_n \to \mathcal{F}_n$ be the maps defined by $r(\alpha) = (\alpha, d_0)$, $d_0 \in \Delta_m$, $r'(\alpha) = (\alpha, d)$ and $s = s' =$ identity. The fiber of $\mathcal{H}_n$ is $\Omega(\bar{\Delta}_n)$. Let $k : \Omega(\bar{\Delta}_n) \to F_{n,m}$ and $k' : \Omega(\bar{\Delta}_n) \to F_n$ be the maps defined by r and r'. Since $E(\bar{\Delta}_n)$, $X_{n,m}$ and X_n are each contractible to a point and s and s' are identity maps, the homotopy sequence of a fiber space yields $k_* : \pi_q(\Omega(\bar{\Delta}_n)) \approx \pi_q(F_{n,m})$ and similarly for k'. Therefore k and k' induce isomorphisms in homology. Therefore by (3.7), $S(\bar{\Delta}_n)_\Phi \otimes S(\Omega(\bar{\Delta}_n))$, $S(B_{n,m})_\Phi \otimes S(F_{n,m})$ and $S(B_n)_\Phi \otimes S(F_n)$ have isomorphic homology groups. We complete the proof of (6.1) by showing :

(6.2) $S(\bar{\Delta}_n)_\Phi \otimes S(\Omega(\bar{\Delta}_n))$ is acyclic.

PROOF. Let $\pi = \pi_1(\bar{\Delta}_n)$. We make $Z(\pi)$ (the group ring of π) into a DGA Z-module by assigning to each of its elements dimension zero, zero derivative and the usual augmentation. Let $h : S(\Omega(\bar{\Delta}_n)) \to Z(\pi)$ be the map defined as follows : Let $T \in S(\Omega(\bar{\Delta}_n))$ be a q-simplex. If $q > 0$, $h(T) = 0$, if $q = 0$, $h(T) =$ the element of π represented by T. Note h is a DGA algebra map. Since $\bar{\Delta}_n$ has the same homotopy type as a collection of circles joined at a point, h is a chain equivalence. Let $\varphi : S(\bar{\Delta}_n) \to Z(\pi)$ be defined for each q-simplex T by $\varphi(T) = 0$ if $q \neq 1$ and if $q = 1$, $\varphi(T) =$

$[T] - e$ where e is the identity element of π and $[T]$ is the element of π represented by T. By ((4.1) (i)) $\varphi = h\Phi_{\bar{\Delta}_n}$ and hence φ is a twisting cochain. Let j be the identity map on $S(\bar{\Delta}_n)$. $j \otimes h : S(\bar{\Delta}_n)_\Phi \otimes S(\Omega(\bar{\Delta}_n)) \to S(\bar{\Delta}_n)_\Phi \otimes Z(\pi)$ is clearly a map and by (3.7) it induces isomorphisms in homology. Note $S(\bar{\Delta}_n) \otimes Z(\pi) = S(\bar{\Delta}_n) \otimes \pi$. If $T \in S(\bar{\Delta}_n)$ is a q-simplex and $\gamma \in \pi$, then

$$\begin{aligned} \partial_\varphi(T \otimes \gamma) &= (\partial T) \otimes \gamma + (-1)^q(\partial_q T) \otimes \varphi(\partial_0^{q-1} T)\gamma \\ &= \textstyle\sum_{i=0}^{q-1} (-1)^i \partial_i T \otimes \gamma + (-1)^q \partial_q T \otimes [\partial_0^{q-1} T]\gamma \end{aligned}$$

But by a well-known theorem $S(\bar{\Delta}_n) \otimes \pi$, with ∂_φ as above, is chain equivalent to the total singular complex of the universal cover U_n of $\bar{\Delta}_n$. $\pi_1(U_n) = 0$ and for $q > 1$, $\pi_q(U_n) \approx \pi_q(\bar{\Delta}_n) = 0$. Hence $S(U_n)$ is acyclic. Therefore $S(\bar{\Delta}_n)_\varphi \otimes Z(\pi)$ is acyclic and the proof of (6.2) is complete.

The proof of (4.2) now follows from (6.1) by standard arguments. We define by induction on dimension, maps $\psi : S(B)_\Phi \otimes S(F) \to S(X)$ and $\theta : S(X) \to S(B)_\Phi \otimes S(F)$ and chain homotopies D_1 and D_2 such that ψ, θ, D_1 and D_2 are natural and $\partial D_1 + D_1\partial = \psi\theta$-identity and $\partial D_2 + D_2\partial = \theta\psi$-identity.

We first define ψ. Let $T \in S(B)$ and $S \in S(F)$ be n and m simplexes. If $n = 0$ and $m = 0$ or 1, let $\psi(T \otimes S) = S$. For these values of n and m it is clear that ψ is natural and commutes with the differentiations. Suppose that $n = 1$ and $m = 0$. For each $s = s_0 d_0 + s_1 d_1 \in \Delta_1$ let $\alpha_s \in P(\Delta_1)$ be given by:

$$\begin{aligned} \alpha_s(t) &= s, & 0 \leqq t \leqq s_1 \\ &= (1-t)d_0 + td_1, & s_1 \leqq t \leqq 1 \end{aligned}$$

Let $T' : \Delta_1 \to X$ by $T'(s) = \lambda(T\alpha_s, S)$, (Recall we have identified S with its image in F). Let T_0 be the constant 1-simplex of B. Then $\partial T' = T_0 S - TS$ when T and T_0 are viewed as 0-simplexes in $S(\Omega(B))$. Let $\psi(T \otimes S) = T'$.

$$\begin{aligned} \psi(\partial_\Phi(T \otimes S)) &= \psi(-(b_0 \otimes \Phi(T)S)) \\ &= \psi(b_0 \otimes (T_0 - T)S) \\ &= T_0 S - TS \\ &= \partial\psi(T \otimes S) \end{aligned}$$

Thus we have defined $\psi(T \otimes S)$ for $n + m \leqq 1$ so that it is natural and commutes with the differentiations. Suppose this has been done for $n + m < q > 1$. Suppose $n + m = q$. Consider the model $\mathscr{F}_{n,m}$ given in (6.1). Since $S(X_{n,m})$ is acyclic there exists a chain $\psi(\xi_n \otimes \eta_m) \in S(X_{n,m})$ such that $\partial\psi(\xi_n \otimes \eta_m) = \psi(\partial_\Phi(\xi_n \otimes \eta_m))$. Let $\psi(T \otimes S) = u_\sharp(\psi(\xi_n \otimes \eta_m))$ where $u : X_{n,m} \to S$ is the map given in (6.1). Then ψ is natural for $n + m = q$ and

$$\begin{aligned}\partial\psi(T\otimes S) &= u_\#(\psi(\partial_\Phi(\xi_n\otimes\eta_m)))\\ &= \psi(\partial_\Phi(v_\#(\xi_n)\otimes u_\#(\eta_m)))\\ &= \psi(\partial_\varphi(T\otimes S))\end{aligned}$$

We next define θ. Let $U\in S(X)$ be an n-simplex. If $n=0$ let $\theta(U)=b_0\otimes U$, (Recall the 0-simplexes of $S(X)$ are in $S(F)$.) Suppose $n=1$. For each $s=s_0d_0+s_1d_1\in\Delta_1$ let $\beta_s\in P(\Delta_1)$ be given by $\beta_s(t)=(1-t)d_0+td_1$, $0\leqq t\leqq s_1$. Let U': $\Delta_1\to F$ by $U'(s)=\lambda(pU\beta_s,\ U(s))$. Then $\partial U'=(pU)\partial_0U-\partial_1U$. Let T_0 be the constant 1-simplex in $S(B)$ and let V: $\Delta_1\to\Omega(B)$ by $V(s)=T_0\beta_s$. Then $\partial V=T_0-e_{a_0}$. Finally let

$$\theta(U)=pU\otimes\partial_0U-b_0\otimes(U'-V\partial_0U)$$

Then

$$\begin{aligned}\partial_\Phi\theta(U) &= -b_0\otimes(pU-T_0)(\partial_0U)+b_0\otimes((pU)\partial_0U-\partial_1U-T_0\partial_0U+\partial_0U)\\ &= b_0\otimes(\partial_0U-\partial_1U)\\ &= \theta(\partial U)\end{aligned}$$

The definition of θ for $n>1$ now proceeds as in the case of ψ.

$\theta\psi$ and $\psi\theta$ are the identity maps in dimension zero; so we may take $D_1=D_2=0$ in this dimension. For higher dimensions D_1 and D_2 can be defined by means of the models $\mathcal{F}_{n,m}$ and $\mathcal{F}_n$ in the same fashion as ψ was defined. This completes the proof of (4.2).

7. Spectral sequences

In Section 3 a spectral sequence was associated with a twisted tensor product. In this section we investigate the relation between the spectral sequence obtained from $S(B)_\Phi\otimes S(F)$ and a more conventionally defined spectral sequence for the fibering $p: X\to B$.

Let T be a singular q-simplex and D_i be the usual degeneracy operator. We define the *real dimension*, RdT, of T to be the least integer n such that

$$T=D_{i_1}D_{i_2}\cdots D_{i_{q-n}}T'$$

for some singular n-simplex T'.

Let (X, B, p, λ) be a weakly transitive fiber space and suppose B is pathwise connected. We filter $S(X)$ as follows: Let $A^q(X)\subset S(X)$ be generated by all simplexes $T\in S(X)$ such that Rd$pT\leqq q$. It is easily checked that $A^q(X)$ is a filtration of $S(X)$ in the sense of Serre [5]. Let $E_r(X)$ denote the spectral sequence obtained from $A^q(X)$. Let $\mathcal{A}^q(X)$ and $\mathcal{E}_r(X)$ denote the filtration and spectral sequence obtained from $S(B)_\Phi\otimes S(F)$ as in Section 3. Let $\psi: S(B)_\Phi\otimes S(F)\to S(X)$ be the map given by (4.2).

(7.1) THEOREM. *ψ can be chosen such that $\psi(\mathcal{A}^q(X)) \subset A^q(X)$ and $\psi_* : H(\mathcal{A}^q(X)) \approx H(A^q(X))$, $q \leqq 0$.*

Recall $E_1(X) = H(A^q(X), A^{p-1}(X))$ and similarly for $\mathcal{E}_1$. Hence (7.1) implies :

(7.2) COROLLARY. *$\mathcal{E}_r(X) \approx E_r(X)$, $r \geqq 1$.*

PROOF of (7.1). Let $C(\bar{\Delta}_n) \subset S(\bar{\Delta}_n)$ be generated by all $T: \Delta_m \rightarrow \bar{\Delta}_n$ such that $T = T'\xi_n$ where $\xi_n : \Delta_n \rightarrow \bar{\Delta}_n$ is the identification map and $T' : \Delta_m \rightarrow \Delta_n$ is a simplicial map. If $(X, \bar{\Delta}_n, p, \lambda)$ is a fiber space, let $C(X) = p_\#^{-1}(C(\bar{\Delta}_n))$. We first prove :

(7.3) If $(X, \bar{\Delta}_n, p, \lambda)$ is one of the model fiber spaces given in (6.1), then $C(X)$ and $C(\bar{\Delta}_n)_{\Phi'} \otimes S(F)$, $\Phi' = \Phi \mid C(\bar{\Delta}_n)$, are acyclic.

It is easily checked that for each of the model fiber spaces $E(\bar{\Delta}_n)$ is a deformation retract of X by a deformation F_t such that pF_t is the identity map. Hence $C(X)$ and $C(E(\bar{\Delta}_n))$ are chain equivalent. Let π: $U_n \rightarrow \bar{\Delta}_n$ be the universal cover of $\bar{\Delta}_n$. Then π induces a homeomorphism π': $E(U_n) \rightarrow E(\bar{\Delta}_n)$ such that $p\pi' = p'$ where p and p' are the usual projections. Therefore $C(E(U_n))$ and $C(E(\bar{\Delta}_n)$ are isomorphic. $\pi_i(U_n) = 0$ for $i \geqq 0$ and hence U_n is contractible. (In fact, U_n is euclidean n space.) Therefore, U_n can be embedded in $E(U_n)$, and a little geometry shows that U_n is a deformation retract of $E(U_n)$ by a deformation F_t such that $p'F_t$ is the identity. Hence $C(E(U_n))$ and $C(U_n)$ are chain equivalent. But all possible liftings of $\xi_n : \Delta_n \rightarrow \bar{\Delta}_n$ define a simplicial subdivision of U_n, and $C(U_n)$ is simply the simplicial chains with respect to this subdivision. Hence $C(U_n)$ is acyclic.

Just as in the proof of ((6.1) (ii)), $C(\Delta_n)_{\Phi'} \otimes S(F)$ is chain equivalent to $C(\bar{\Delta}_n)_\varphi \otimes Z(\pi_1(\bar{\Delta}_n))$ where $\varphi_q = 0$, $q \neq 1$ and $\varphi_1(T) = [T] - e \in \pi_1(\bar{\Delta}_n)$ for 1-simplexes $T \in C(\bar{\Delta}_n)$. But the latter complex is isomorphic to $C(U_n)$.

Let $\mathcal{M}$ be the category whose objects are weakly transitive fiber spaces with base space $\bar{\Delta}_n$ for some $n \geqq 0$, and whose maps are fiber space maps (f, g) where $g : \bar{\Delta}_n \rightarrow \bar{\Delta}_m$ is defined by a simplicial map $g' : \Delta_n \rightarrow \Delta_m$. By virtue of (7.3), Theorem (4.2) and its proof remain valid on $\mathcal{M}$ when $S(B)$ and $S(X)$ are replaced $C(\bar{\Delta}_n)$ and $C(X)$. That is, there are maps $\psi : C(\bar{\Delta}_n)_{\Phi'} \otimes S(F) \rightarrow C(X)$ and $\theta : C(X) \rightarrow C(\bar{\Delta}_n)_{\Phi'} \otimes S(F)$ and chain homotopies D_1 D_2 such that $\partial D_1 + D_1\partial = \psi\theta - \mathrm{id}$, $\partial D_2 + D_2\partial = \theta\psi - \mathrm{id}$ and ψ, θ, D_1, and D_2 are natural with respect to the maps of $\mathcal{M}$.

Using ψ, θ, D_1, and D_2 on $\mathcal{M}$, we define ψ, θ, D_1, and D_2 for (X, B, p, λ) by means of the model fiber spaces just as in the proof of (4.2). For example, $\psi : S(B)_\Phi \otimes S(F) \rightarrow S(X)$ is defined as follows : Let $T \in S(B)$ and

$S \in S(F)$ be n and m simplexes.

$$\psi(T \otimes S) = u_{\#}\psi(\xi_n \otimes \eta_m)$$

where ξ_n, η_m, and u are as in (6.1) and $\psi(\xi_n \otimes \eta_m)$ is as defined above. But $\psi(\xi_n \otimes \eta_m) \in C(X_{n,m}) \subset A^n(X_{n,m})$. Hence $\psi(T \otimes S) \in A^n(X)$. θ, D_1, and D_2 are defined in a similar fashion so as to preserve the filtrations. It then follows that $\psi_* : H(\mathcal{A}^n(X)) \approx H(A^n(X))$ and the proof of (7.1) is complete.

Let $i : F \to X$ be the inclusion map and let $j : S(F) \to S(B)_\Phi \otimes S(F)$ and $\pi : S(B)_\Phi \otimes S(F) \to S(B)$ be defined as follows: Let T_0 be the zero simplex of $S(B)$ and let a be the augmentation of $S(F)$.

$$j(S) = T_0 \otimes S, \qquad S \in S(F)$$
$$\pi(T \otimes S) = a(S)T, \qquad T \in S(B),\ S \in S(F)$$

(7.4) $\psi : S(B)_\Phi \otimes S(F) \to S(X)$ can be chosen so that $\psi j = i_{\#}$ and $p_{\#}\psi = \pi \bmod D(B)$ where $D(B)$ is the degenerate chains of $S(B)$.

PROOF. The first part of (7.4) is immediate from the proof of (4.2). Let ψ be chosen as in the proof of (7.1). Consider the model $\mathcal{F}_{n,0}$. It is sufficient to show that $p_{n,0\#}\psi(\xi_n \otimes \eta_0) = \xi_n \bmod D(\bar{\Delta})$. We prove this by induction on n. For $n = 0, 1$, it is true by the way ψ was defined in the proof of (4.2). Suppose it is true for $n - 1$. Then,

$$\begin{aligned}\partial p_{n,0\#}\psi(\xi_n \otimes \eta_0) &= p_{n,0\#}(\partial\xi_n \otimes \eta_0 + (-1)^n \xi_n \otimes \eta_0 \frown \Phi \\ &= \partial\xi_n \bmod D(\bar{\Delta}_n)\end{aligned}$$

Suppose $c \in C(\bar{\Delta}_n)$, $n > 1$, and $\partial c = \xi_n \bmod D(\bar{\Delta}_n)$. $c = c' + c''$ where $c'' \in D(\bar{\Delta}_n)$ and c' does not involve degenerate simplexes. Then $\partial(c' - \xi_n) = 0 \bmod D(\bar{\Delta}_n)$. Therefore $\partial(c' - \xi_n) = 0$. But $C(\bar{\Delta}_n)$ is acyclic and $C_{n+1}(\bar{\Delta}_n) \subset D(\Delta_n)$. Hence $c' = \xi_n$.

We next investigate the relation between Φ and the transgression. Suppose B is $n - 1$ connected, $n > 1$, and F is pathwise connected. Let $S_{n-1}(B) \subset S(B)$ be as defined in Section 4 and let $D \subset S(B)$ be generated by the constant p-simplexes, $q > 0$. Let $C(B) = S_{n-1}(B)/D$. By (4.1), $\Phi(D) = 0$ and hence Φ defines a twisting cochain $\varphi : C(B) \to S(\Omega(B))$. By (3.7), $C(B)_\varphi \otimes S(F)$ and $S(B)_\Phi \otimes S(F)$ are chain equivalent. Since $\varphi_q = 0$ for $q < n$, (3.2) yields

$$\partial\varphi_q = \varphi_{q-1}\partial \qquad \text{for } q < 2n$$

Let $S_0 \in S(F)$ be a 0-simplex and let $t : C_q(B) \to S(F)_{q-1}$, $q < 2n$, be defined by $t(T) = \varphi_q(T)S_0$, $T \in C_q(B)$. Then t defines a homomorphism $t_* : H_q(B) \to H_{q-1}(F)$, $q < 2n - 1$. t_* is obviously independent of the choice of S_0.

(7.5) THEOREM. *For each* $u \in H_q(B)$, $q < 2n-1$, *there is* $v \in H_q(X, F)$ *such that* $p_*v = u$ *and* $\partial_* v = (-1)^q t_* u$ *where* $p_* : H_q(X, F) \to H_q(B)$ *and* $\partial_* : H_q(X, F) \to H_{q-1}(F)$.

PROOF. By (7.4) and (4.2), we may use $C(B)_\varphi \otimes S(F)$ and $C_0(B) \otimes S(F)$ in place of $S(X)$ and $S(F)$. Let $V \in Z_q(B)$ represent u.

$$\partial(U \otimes S_0) = (-1)^q(T_0 \otimes \varphi(U)S_0).$$

Hence $U \otimes S_0 \in Z(C(B)_\varphi \otimes S(F),\ C_0(B) \otimes S(F))$. Let v be the class of $U \otimes S_0$. Then $\partial_* v = (-1)^q t_* u$.

Continuing the notation and hypotheses on B as given in the previous paragraph, suppose $X = E(B)$ and p is the usual projection. $\partial\varphi_n = \varphi_{n-1}\partial = 0$ and $\delta\varphi_n = \varphi_n\partial = \partial\varphi_{n+1}$. Therefore φ_n defines a cocycle $V \in C^n(\Omega(B))$. Let $k : H_{n-1}(\Omega(B)) \approx \pi_n(B)$ be the well-known isomorphism. Let $\gamma \in H^n(B\,;\pi_n(B))$ be represented by $(-1)^n kV$.

(7.6) γ is the characteristic class of B.

PROOF. Let t be the homomorphism given above. By the same argument as was given by (7.5),

$$(-1)^n t^* :\ H^{n-1}(\Omega(B)\,;\ \pi_n(B)) \to H^n(B\,;\ \pi_n(B))$$

is the transgression. Let W be an extension of the natural map $Z_{n-1}(\Omega(B))$ onto $H_{n-1}(\Omega(B))$. Let w be represented by kW. It is well known that the transgression of w is the characteristic class of B. On the other hand, $t^* w = (-1)^n \gamma$ because $t^\# w = V$ on $Z_n(B\,;\ H_{n-1}(\Omega(B)))$ and $H^{n-1}(B) = 0$.

(7.7) THEOREM. *If* (X, B, p, λ) *is a weakly transitive fiber space and* B *is pathwise connected, then*

$$\mathcal{E}_2(X) = H(B\,; H(F))$$

where $H(B\,;\ H(F))$ *is formed using local coefficients defined by the action of* $\pi_1(B)$ *on* $H(F)$.

PROOF. If $T \otimes S \in {}^q(X\mathcal{A})$,

$$\partial_\Phi(T \otimes S) = \partial T \otimes S + (-1)^q(T \otimes \partial S + T \otimes S \otimes \frown \varphi_1) \bmod \mathcal{A}^{q-2}(X)\,.$$

By (4.1),

$$T \otimes S \frown \varphi_1 = \partial_q T \otimes ([\partial_0^{q-1} T]S - S)$$

where $[\partial_0^{q-1}] \in S(\Omega(B))_0$. Theorem (7.7) then follows from the definition of $\mathcal{E}_2$.

If B is $n-1$ connected and $\Omega(B)$ acts on F, a pairing $\pi_n(B) \otimes H(F) \to H(F)$ is defined by $\pi_n(B) \approx H_{n-1}(\Omega(B))$ and the pairing $H(\Omega(B)) \otimes H(F) \to H(F)$.

(7.8) THEOREM (Hurewicz and Fadell). *If* (X, B, p, λ) *is a weakly transi-*

tive fiber space, B is $n-1$ connected and F is pathwise connected, then

$$H(B;\ H(F)) = \mathcal{E}_2(X) = \cdots = \mathcal{E}_n(X)$$

Also $d_n : \mathcal{E}_n(X) \to \mathcal{E}_n(X)$ is given by

$$d_n u = (-1)^{q+n} u \frown \gamma$$

where γ is the characteristic class of B and the pairing $\pi_n(B) \otimes H(F) \to H(F)$ is defined by the action of $\Omega(B)$ on F.

PROOF. Theorem (7.8) follows from the definition of $\mathcal{E}_r$, (7.6) and

$$\partial_\varphi(T \otimes S) = \partial T \otimes S + (-1)^q(T \otimes \partial S + T \otimes S \frown \varphi_n) \bmod \mathcal{A}^{q-n-1}(X)$$

for $T \otimes S \in \mathcal{A}^q(X)$, $T \in C_q(B)$.

8. Some exact sequences

Using $S(B)_\Phi \otimes S(F)$, one can derive various exact sequences involving the homology groups of B, F, and X, i.e., Wang, Gysin, and Serre sequences. In this section we derive a few less known exact sequences by a technique that will also give the above mentioned sequences.

Let $\mathcal{F}' = (X', B', p', \lambda')$ be a weakly transitive fiber space, let $f: B \to B'$, and let (X, B, p, λ) be the fiber space induced from $\mathcal{F}'$ by f.

(8.1) THEOREM. *Suppose B is $m-1$ connected, $F = p^{-1}(b_0)$ is $n-1$ connected, and B' is n connected, $n, m > 1$. Then there exist exact sequences:*

$$H_{m+2n-1}(B \times F, B \vee F) \to \cdots$$
$$\cdots \to H_{q+1}(B) \to H_q(B \times F, B \vee F) \to H_q(X, F) \to H_q(B) \to$$
$$H_{2m-1}(F) \to \cdots$$
$$\cdots \to H_q(X) \to H_q(B \times F, b_0 \times F) \to H_{q-1}(F) \to H_{q-1}(X) \to$$
$$H_{2n-1}(B \times F, B \times x_0) \to \cdots$$
$$\cdots \to H_q(B) \to H_{q-1}(B \times F, B \times x_0) \to H_{q-1}(F) \to H_{q-1}(X) \to$$

The above sequences are exact for any coefficient group and similar exact sequences hold for cohomology.

PROOF. Let M and N be free DGA Z-modules and let $g: M_q \to N_{q-1}$ be a homomorphism defined for $q \leqq r$. Let $\hat{g}$ be the DGA Z-module defined as follows:

$$\begin{aligned} \hat{g}_q &= M_q + N_q & q &\leqq r \\ &= 0 & q &> r \end{aligned}$$

$$\hat{\partial}(x + y) = \partial x + gx + \partial y$$

$x \in M$, $y \in N$. Then by well-known arguments:

(8.2) If $\hat{\partial}^2 = 0$, $g\partial = -\partial g$ and there is an exact sequence

$$H_{r-1}(N) \to \cdots \to H_{q+1}(M) \to H_q(N) \to H_q(g) \to H_q(M) \to$$

The above sequence is exact for any coefficient groups and a similar sequence is exact for cohomology.

By well-known constructions we may assume f is a fiber map. Let $F' = f^{-1}(b_0)$. Let K be the subcomplex of $S(B)$ generated by simplexes taking the n and $m-1$ skeletons of Δ_q into F' and b_0 respectively. Let $D \subset S(B)$ be generated by the constant q-simplexes, $q > 0$ and let $C(B) = K/D$. Since B is $m-1$ connected and $\pi_i(B, F') \approx \pi_i(B') = 0$, $i \leqq n$, $C(B)$ and $S(B)$ are chain equivalent. Let $C(F)$ be the subcomplex of $S(F)$ defined as follows : Let S_0 be a 0-simplex in $S(F)$

$$\begin{aligned} &C_0(F) \text{ is generated by } S_0 \\ &C_q(F) = 0 && 0 < q < n \\ &\qquad\;\; = Z_n(S(F)) && q = n \\ &\qquad\;\; = S(F)_q && q > n \end{aligned}$$

Let $C(\Omega(B')) \subset S(\Omega(B'))$ be defined just as $C(F)$ was defined. Since $H_i(F) = H_i(\Omega(B_i)) = 0$, $i < n$, $C(F)$ and $C(\Omega(B'))$ are chain equivalent to $S(F)$ and $S(\Omega(B'))$ respectively. $C(\Omega(B))$ inherits an algebra structure from $S(\Omega(B))$ and $C(F)$ is a $C(\Omega(B))$-module. Let $\varphi : C(B) \to C(\Omega(B))$ be the twisting cochain defined by $\varphi(c + D) = \Phi_{B'}(f_\# c)$, $c \in K$. Note $\Phi_{B'}(f_\# D) = 0$ and $\varphi_q = 0$ for $p \leqq \max(n, m\text{-}1)$ because $\Phi_{B'}$ is zero on constant simplexes. By (3.7) $C(B)_\varphi \otimes C(F)$ and $S(B)_{\Phi_{B'} f_\#} \otimes S(F)$ are chain equivalent and by (4.3) the later complex is chain equivalent to $S(X)$.

Suppose $T \in C_q(B)$ and $S \in C(F)$. Then because $\varphi_q = 0$ for $q \leqq \max(n, m-1)$ and $C_q(B) = 0$, $0 < q < m$,

$$\begin{aligned} T \otimes S \frown \varphi &= T_0 \otimes \varphi(T) S, && q \leqq \max(2m\text{-}1, m+n) \\ &= 0\,, && q \leqq n \end{aligned} \tag{8.3}$$

Theorem (8.1) now follows from (8.2), and (8.2) by the following choices for M, N, g, and r. Let $C(B) \otimes C(F)$ denote the complex with the usual product differentiation. For the first sequence let

$$\begin{aligned} M &= C(B) \\ N &= C(B) \otimes C(F)/C_0(B) \otimes C(F) + C(B) \otimes C_0(F) \\ r &= m + 2n \\ g(T) &= (-1)^q T \otimes S_0 \frown \varphi && T \in C_q(B)\ q \leqq r \end{aligned}$$

For the second let

$$\begin{aligned} M &= C(B) \otimes C(F)/C_0(B) \otimes C(F) \\ N &= C(F) \\ r &= 2m \\ g(T \otimes S) &= (-1)^q \varphi(T) S && T \in C_q(B),\ S \varepsilon C(F) \end{aligned}$$

For the third let

$$M = C(B)$$
$$N = C(B) \otimes C(F)/C(B) \otimes C_0(F)$$
$$r = 2n$$
$$g(T) = (-1)^q T \otimes S_0 \frown \varphi \qquad T \in C_q(B)$$

In each case $\hat{g}_q = (C(B)_\varphi \otimes C(F))_q$ for $q \leqq r$.

9. Theory of Hirsch[2]

Let K, A, and L be a DGA coalgebra, algebra, and A module. Let $\varphi \in C^*(K; A)$ be a twisting cochain. Let L^* be the DGA algebra defined as follows:

$$L_q^* = \{h \in \mathrm{Hom}\,\{H(L), H(L)\} \mid h(H_p(L)) \subset H_{p+q}(L),\ p \geqq 0\}$$

L^* has trivial differentiation, multiplication by composition, and augmentation given by $a(h) = ah(1)$, $h \in L^*$, $1 \in L$. $H(L)$ is then a DGA L^* module with trivial differentiation.

(9.1) THEOREM. *If Λ is a principal ideal ring and L, K, and $H(L)$ are Λ free, then there exists a twisting cochain $\varphi^* \in C^*(K; L^*)$ and a chain equivalence $\mu: K_\varphi \otimes L \to K_{\varphi^*} \otimes H(L)$.*

(9.2) COROLLARY. *If (X, B, p, λ) is a weakly transitive fiber space, B is pathwise connected and $H(F; \Lambda)$ is free where Λ is a principal ideal ring, there is a twisting cochain $\varphi^*: S(B) \otimes \Lambda \to (S(F) \otimes \Lambda)^*$ such that $S(B) \otimes \Lambda_{\varphi^*} \otimes H(F; \Lambda)$ and $S(X) \otimes \Lambda$ are chain equivalent.*

Corollary (9.2) stated in terms of cohomology gives:

(9.3) COROLLARY (Hirsch). *Under the hypotheses of* (9.2) *there is a differentiation on $C^*(B; H^*(F; \Lambda))$ which makes it chain equivalent to $C^*(X; \Lambda)$.*

PROOF OF (9.1). Let $\mu': K_\varphi \otimes L \to H(L)$ be a dimension preserving homomorphism and let $\mu: K_\varphi \otimes L \to K \otimes H(L)$ be defined by $\mu = (\mathrm{id} \otimes \mu')(\Delta \otimes \mathrm{id})$ where Δ is the coproduct on K. Let A^q and A'^q be the filtrations on $K_\varphi \otimes L$ and $K \otimes H(L)$ as given in Section 3. Suppose $\varphi^*: K \to L^*$ is a homomorphism such that $\varphi^*(K_q) \subset L^*_{q-1}$. Simple calculations show that:

(9.4) $\partial_{\varphi^*}^2 = 0$ on A'^p if and only if

$$\delta\varphi^*_{q-1} = \sum (-1)^k \varphi^*_k \smile \varphi^*_{q-k} \qquad \text{for } q \leqq p$$

(Note this is the twisting cochain identity for φ^*).

[2] The main theorem of this section implies a theorem (see (9.2)) proved by Hirsch in [8]. The proof of the main theorem was suggested by a proof of Hirsch's theorem given by Cockcroft in a paper (as yet unpublished) on the homology groups of spaces with two non-trivial homotopy groups.

(9.5) $\mu\partial_\varphi(c) = \varphi_{\varphi^*\mu(c)}$ for $c \in A^p$ if and only if

$$\mu'\partial_\varphi(c) = \nu(\nabla \otimes \mathrm{id})\,(c) \qquad \text{for } c \in A^p$$

where $\nu : K \otimes K \otimes L \to H(L)$ is given by $\nu(k \otimes k' \otimes l) = (-1)^q \varphi^*(k)\mu'(k' \otimes l)$, $k \in K_q$, $k' \in K$, $l \in L$.

Let $\eta : L \to H(L)$ be an extension of the natural map of $Z(L)$ onto $H(L)$. η exists because Λ is principal. Since L and $H(L)$ are free, there is a map $\xi : H(L) \to L$ and a chain homotopy $D : L \to L$ such that $\partial D + D\partial = \xi\eta - \mathrm{id}$.

We define μ' on A^p and φ_p^* by induction on p such that:

(9.6) $\mu\partial_\varphi = \partial_{\varphi^*}\mu$ on A^p and $\mu(A^p) = A'p$. (Note if μ' is defined on A^p, so is μ.)

Let $\mu'(k \otimes l) = a(k)\eta(l)$, $k \otimes l \in A^0$ and let $\varphi_0^* = 0$. Suppose φ_q^*, $q \leqq p$ and μ' on A^p have been defined and satisfy (9.6). (9.6) implies that $\partial_{\varphi^*}^2 = 0$ on A'^p and hence φ^* satisfies the identity given in (9.4). (9.6) also implies that μ' satisfies the identity given in (9.5). Consider this latter identity for $p + 1$. Rearranging its terms, it may be written:

(9.7) If $k \in K_{p+1}$ and $l \in L$

$$\begin{aligned} &\nu(k \otimes 1 \otimes l) + (-1)^q \mu'(k \otimes \partial l) \\ &\qquad = \mu'(\partial k \otimes l + (-1)^{p+1} k \otimes l \frown \varphi) - \nu(\nabla' \otimes \mathrm{id})(k \otimes l) \end{aligned}$$

where $\nabla'(k) = \nabla(k) - k \otimes 1$. Note $\nu(k \otimes 1 \otimes l) = (-1)^{p+1}\varphi^*(k)\eta(l)$. Let $U(k \otimes l)$ denote the right side of (9.7). We construct $\varphi^*(k)$ and $\mu'(k \otimes l)$ so as to satisfy (9.7). A tedious calculation shows that $U(k \otimes \partial l) = 0$. Let

$$\begin{aligned} \varphi^*(k)(x) &= (-1)^{p+1} U(k \otimes \xi(x)) \qquad k \in K_{p+1},\ x \in H(L) \\ \mu'(k \otimes l) &= (-1)^{p+1} U(k \otimes D(l)) \end{aligned}$$

Then

$$\begin{aligned} \nu(k \otimes 1 \otimes l) + (-1)^p \mu'(k \otimes \partial l) &= (-1)^{p+1}(\varphi^*(k)\eta(l) - \mu'(k \otimes l)) \\ &= U(k \otimes (\xi\eta l - D\partial l) \\ &= U(k \otimes (l + \partial D l) \\ &= U(k \otimes l) \end{aligned}$$

Therefore $\mu' \mid A^{p+1}$ and φ_{p+1}^* have been defined and satisfy (9.7). By (9.5) $\mu\partial_\varphi = \partial_{\varphi^*}\mu$ on A^{p+1}. If $k \in K_{p+1}$ and $l \in L$,

$$\begin{aligned} \mu(k \otimes l) &= k \otimes \mu'(1 \otimes l) && \mod A'^p \\ &= k \otimes \eta(l) && \mod A'^p \end{aligned}$$

The fact that η is onto and the inductive hypothesis that $\mu(A^p) = A'^p$ then implies that $\mu(A^{p+1}) = A'^{p+1}$. Thus μ and φ^* have been defined such that $\mu\partial_\varphi = \partial_{\varphi^*}\mu$ and such that μ is onto. This implies $\partial_{\varphi^*}^2 = 0$ and by (9.5) that φ^* is a twisting cochain.

Let E_r and E'_r be the spectral sequences obtained from A^p and A'^p. $\mu_* : E_0 = K \otimes L \to E'_0 = K \otimes H(L)$ is given by $\mu_*(k \otimes l) = k \otimes \eta(l)$. Hence $\mu_* : E_1 \approx E'_1$. Therefore, by the usual spectral sequence argument $\mu_* : H(K_\varphi \otimes L) \to H(K_{\varphi *} \otimes H(L))$ and since everything is free, μ is a chain equivalence.

10. The bar and cobar constructions

Let K be a DGA algebra or coalgebra. We embed Λ in K_0 by identifying $1 \in \Lambda$ with $1 \in K$. Let $T(K)$ be the DGA Λ-module with differentiation ∂_t, grading $\dim_t$, coproduct ∇_t and product μ_t as follows:

$$\bar{K} = K/\Lambda$$

$$\bar{K}^n = \bar{K} \otimes \cdots \otimes \bar{K} \qquad n \text{ factors}$$

$$\bar{K}^0 = \Lambda$$

$$T(\bar{K}) = \sum K^n \qquad \text{direct sum}$$

We will denote $k_1 \otimes k_2 \otimes \cdots \otimes k_n$ by $[k_1, k_2, \cdots, k_n]$ and 1 by $[\ \ ]$.

$$\dim_t[k_1, \cdots, k_n] = \sum \dim k_i$$

$$\partial_t[k_1, \cdots, k_n] = \sum (-1)^{n_i}]k_1, \cdots, \partial k_i, \cdots, k_n]$$

where $n_i = \dim_t[k_1, \cdots, k_{i-1}]$.

$$\nabla_t[k_1, \cdots, k_n] = \sum_{i=0}^{n}[k_1, \cdots, k_i] \otimes [k_{i+1}, \cdots, k_n]$$

$$\mu_t[k_1, \cdots, k_n] \otimes [k'_1, \cdots, k'_m] = [k_1, \cdots, k_n, k'_1, \cdots, k'_m]$$

Let $\dim_s[k_1, \cdots, k_n] = n$.

Suppose A is a DGA algebra. The bar construction [7] $\mathscr{B}(A)$ is a DGA coalgebra defined as follows: $\mathscr{B}(A) = T(A)$ with coproduct ∇_t, grading $\dim_B = \dim_t + \dim_s$ and differentiation $\partial_B = -\partial_t + \partial_s$ where

$$\partial_s[a_1, \cdots, a_n] = \sum (-1)^{n_i}[a_1, \cdots, a_{i-1}a_i, \cdots, a_n]$$

$$n_i = \dim_B[a_1, \cdots, a_{i-1}]$$

In [7] it is shown that $\mathscr{B}(A)$ is a DGA coalgebra.

Suppose K is a DGA coalgebra such that $K_0 = \Lambda$. The cobar construction [1] $\mathscr{F}(K)$ is a DGA algebra defined as follows: $\mathscr{F}(K) = T(A)$ with product μ_t, grading $\dim_F = \dim_t - \dim_s$, and differentiation ∂_F given as follows. Let $\mathscr{P} : K \to K$ by $\mathscr{P}(K) = (-1)^q k$ for $k \in K_q$.

$$\partial_F[k] = -[\partial k] + \mu_t(\mathscr{P} \otimes \mathrm{id})\nabla' \qquad k \in K$$

where $\nabla'(k) = \nabla(k) - k \otimes 1 - 1 \otimes k$. ∂_F is then defined on $\mathscr{F}(K)$ by the requirement that $\mathscr{F}(K)$ be a DGA algebra. In [1] it is shown that $\mathscr{F}(K)$ is a DGA algebra.

Let K be a DGA coalgebra such that $K_0 = \Lambda$, let A be a DGA algebra, and let $\varphi \in C^*(K ; A)$ be a twisting cochain. Let $\omega^n : K \to \bar{K}^n$, $\tilde{\nabla} : K \to$

$T(K)$, $\tilde{\mu}: T(A) \to A$, and $\varphi_*: T(K) \to T(A)$ be homomorphism defined as follows :

$$\omega^0 = a$$
$$\omega^1 = \text{natural map of } K \text{ onto } \overline{K}$$
$$\omega^n = \mu_t(\omega^{n-1} \otimes \omega^1)\nabla$$
$$\tilde{\nabla} = \sum_{n=0}^{\infty} \omega^n$$

Note $\tilde{\nabla}$ is well defined for $\omega^n(k) = 0$ if $\dim k < n$.

$$\tilde{\mu}[a_1, a_2, \cdots, a_n] = a_1 a_2 \cdots a_n$$
$$\varphi_*[k_1, \cdots, k_n] = [\varphi' k_1, \cdots, \varphi' k_n]$$

where $\varphi'(k) = (-1)^q \varphi(k)$ for $k \in K_q$.

Finally let $\varphi^B = \varphi_* \tilde{\nabla}: K \to \mathscr{B}(A)$ and $\varphi^F = \tilde{\mu}\varphi_*: \mathscr{F}(K) \to A$.

(10.1) φ^B is a DGA coalgebra map and φ^F is a DGA algebra map.

PROOF. Most of the necessary verifications are straight forward. The only difficult ones are $\varphi^B \partial = \partial_B \varphi^B$ and $\nabla_t \varphi^B = (\varphi^B \otimes \varphi^B)\nabla$. The latter follows from :

$$\nabla_t \omega^n = \sum_{k=0}^{n} (\omega^k \otimes \omega^{n-k})$$

which may be proved by induction on n. $\varphi^B \partial = \partial_B \varphi^B$ follows from :

$$\varphi_* \omega^n \partial = \partial_t \varphi_* \omega^n + \partial_s \varphi_* \omega^{n+1}$$

This also may be fairly easily proved by induction on n.

(10.2) THEOREM. *If K and A are Λ free, $K_0 = H_0(A) = \Lambda$, $H_1(K) = 0$, and $K_\varphi \otimes L$ is acyclic, then φ^B and φ^F are chain equivalences.*

(10.3) COROLLARY. *If Y is a simply connected topological space, $S(Y)$ and $S(\Omega(Y))$ are chain equivalent to $\mathscr{B}(S(\Omega(Y)))$ and $\mathscr{F}(S(Y))$ respectively.*

(This result was proved for $\mathscr{F}$ in [1] and for $\mathscr{B}$ when Y is an Eilenberg-MacLane space in [7]).

PROOF OF (10.2). Let $\varphi_B \in C^*(\mathscr{B}(A); A)$ and $\varphi_F \in C^*(K; \mathscr{F}(A))$ be defined by

$$\varphi_B[a_1, \cdots, a_n] = (-1)^{q-1} a_1 \quad \text{if } n = 1 \text{ and } a_1 \in A_q$$
$$= 0 \quad \text{otherwise}$$
$$\varphi_F(k) = (-1)^{q-1}[k] \quad \text{if } k \in K_q\,.$$

It is easily checked that φ_F and φ_B are twisting cochains and that $\varphi_B \varphi^B = \varphi$ and $\varphi^F \varphi_F = \varphi$. Thus we have DGA maps

$$(\varphi^B \otimes \mathrm{id}): K_\varphi \otimes A \to \mathscr{B}(A)_{\varphi_B} \otimes A$$
$$(\mathrm{id} \otimes \varphi^F): K_{\varphi_F} \otimes \mathscr{F}(K) \to K_\varphi \otimes A$$

In [2] and [1] it is shown that $\mathscr{B}(A)_{\varphi_B} \otimes A$ and $K_{\varphi_F} \otimes \mathscr{F}(K)$ are acyclic. The desired result then follows from (3.7).

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