On the Signature of a Quadratic Form

E. T. Browne

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ON THE SIGNATURE OF A QUADRATIC FORM.

BY E. T. BROWNE.*

1. **Introduction.** A real quadratic form in n variables $x_1, \cdots, x_n$

$$\sum_{i,j}^{1,\cdots,n} a_{ij}\, x_i\, x_j \qquad (a_{ij} = a_{ji}) \tag{1}$$

of rank r can always be reduced by a real non-singular linear transformation to an expression of the type

$$\sum_{i=1}^{r} c_i\, x_i^2. \tag{2}$$

Although the (real) transformation by which (1) is reduced to the form (2) is not unique, it is well known that the number N of negative coefficients in (2) is the same whatever be the particular transformation employed. A similar remark holds for the number P of positive coefficients in (2). Thus, P and N, and therefore their difference $s = P - N$, called the *signature* of the quadratic form, are arithmetic invariants under real non-singular linear transformations. Since $P + N = r$, manifestly s is determined when N and r are known.

Frobenius has shown† that if (1) is of rank r the variables can always be so numbered that no consecutive two of the numbers in the sequence

$$A_0 = 1, \quad A_1 = a_{11}, \quad A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \cdots, \quad A_r = \begin{vmatrix} a_{11}, & \cdots, & a_{1r} \\ \cdot & \cdot \quad \cdot & \cdot \\ a_{r1}, & \cdots, & a_{rr} \end{vmatrix} \tag{3}$$

are zero and $A_r \neq 0$. (1) is then said to be *regular*, and its matrix *regularly arranged*. Gundelfinger has shown‡ that in this case the number N of negative coefficients in (2) is equal to the number of variations of sign in the sequence (3), where a vanishing term may be given a sign at pleasure.

In the paper of Frobenius it was shown that for certain particular types of forms, for example, recurring forms in which $a_{ij} = a_{i+j-2}$ even if $A_r = 0$ and an arbitrary number of consecutive A's in the sequence (3) vanish, by replacing the vanishing A_r by a certain new determinant A_r', the number of negative coefficients in (2) can always be determined from the new

* Received September 16, 1928.

† Frobenius, *Über das Trägheitsgesetz der quadratischen Formen,* Crelle, vol. 114 (1895), p. 193.

‡ Gundelfinger, *Zur Theorie der quadratischen Formen,* Crelle, vol. 91 (1881), p. 225.

sequence. Moreover, it was shown by Frobenius (and more recently by Franklin)* that if (1) is a general real quadratic form of rank r in which $A_r \neq 0$, N can always be determined when in the sequence (3) not more than two consecutive A's vanish. The interesting demonstration by Franklin was based on certain properties of the secular equation. In this paper we shall show that Gundelfinger's rule and the scheme of rearrangement, mentioned previously as used by Frobenius, furnish a very simple method of arriving at these facts.

2. **Quadratic forms which are not regular.** In the quadratic form (1) of rank r let us suppose that $A_r \neq 0$. Let us suppose also that

$$(4)\qquad \begin{aligned} &A_\tau A_{\tau+\sigma+1} \neq 0 \qquad && (0<\tau;\ 1 \leqq \sigma \leqq r-\tau-1),\\ &A_{\tau+i} = 0 && (i = 1, \cdots, \sigma). \end{aligned}$$

Since $A_\tau \neq 0$ while $A_{\tau+1} = 0$ there exists a real set $(x_1, \cdots, x_\tau)$ such that

$$(5)\qquad a_{i,\tau+1} = \sum_{j=1}^{\tau} a_{ij}\, x_j \qquad (i = 1, \cdots, \tau+1).$$

If from the elements of the $(\tau+1)$th column of $A_{\tau+2}$ we subtract the sum of the products of the elements of the first τ columns by $x_1, \cdots, x_\tau$ every element in this column is reduced to zero except the last one which is

$$k = a_{\tau+2,\tau+1} - \sum_{j=1}^{\tau} a_{\tau+2,j}\, x_j.$$

Performing the same operation on the $(\tau+1)$th row and expanding, we have

$$(6)\qquad A_{\tau+2} = -k^2 A_\tau.$$

If, therefore, $\sigma = 1$ in (4) so that $A_{\tau+2} \neq 0$, $A_{\tau+2}$ is evidently of opposite sign to A_τ, a fact which is well known. If, however, $\sigma > 1$ so that $A_{\tau+2} = 0$, (5) holds for $i = \tau+2$.

Now holding the first τ columns fixed in $A_{\tau+\sigma+1}$ let us transfer the $(\tau+\alpha)$th column over the $\alpha-1$ columns immediately preceding it and put it in the $(\tau+1)$th place. Proceeding similarly with the rows, we obtain a new sequence of A's which we shall denote by

$$(3)\qquad A_0, A_1, \cdots, A_\tau, A_{\tau+1}^{(\alpha)}, \cdots, A_{\tau+\sigma}^{(\alpha)}, A_{\tau+\sigma+1}, \cdots, A_r.$$

On making use of the conditions (5) for $i = 1, \cdots, \tau+2$, and expanding $A_{\tau+2}^{(\alpha)}$, we have

$$A_{\tau+2}^{(\alpha)} = -k_\alpha^2 A_\tau,$$

* Franklin, *A theorem of Frobenius on quadratic forms*, Bull. Amer. Math. Soc., vol. 33 (1927), pp. 447–452.

where

$$k_\alpha = a_{\tau+\alpha,\tau+1} - \sum_{j=1}^{\tau} a_{\tau+\alpha,j}\, x_j.$$

Now $A_{\tau+2}^{(\alpha)}$ cannot be zero for every choice of α $(3 \leqq \alpha \leqq \sigma+1)$ for in that case we would have

$$a_{\tau+\alpha,\tau+1} = \sum_{j=1}^{\tau} a_{\tau+\alpha,j}\, x_j \qquad (\alpha = 3, \cdots, \sigma+1)$$

which in connection with (5) would give $A_{\tau+\sigma+1} = 0$. Hence, for some choice of α $(3 \leqq \alpha \leqq \sigma+1)$ $A_{\tau+2}^{(\alpha)} \neq 0$ and is of *opposite* sign to A_τ. By a similar argument it follows that if in the subsequence

$$A_\tau,\ A_{\tau+1}^{(\alpha)}, \cdots, A_{\tau+\sigma}^{(\alpha)},\ A_{\tau+\sigma+1}$$

two or more consecutive terms vanish, without affecting those terms which do not vanish, the last $\sigma - 1$ rows and columns of $A_{\tau+\sigma+1}$ can be so arranged that in the new subsequence

$$A_\tau,\ A_{\tau+1}^{(\alpha)}, \cdots, A_{\tau+\sigma}^{(\alpha)},\ A_{\tau+\sigma+1} \tag{7}$$

no two consecutive A's vanish.

Assuming that the number of variations of sign in the remainder of the sequence (3) can be determined we proceed to the determination of the number ν of variations in the subsequence (7). Since A_τ and $A_{\tau+2}^{(\alpha)}$ have opposite signs, ν evidently satisfies the inequalities

$$1 \leqq \nu \leqq \sigma, \tag{8}$$

ν being even or odd according as A_τ and $A_{\tau+\sigma+1}$ have the same sign or opposite signs.

In the preceding discussion we assumed that $A_\tau \neq 0$ for $\tau > 0$. This restriction may easily be removed. For if $A_1 = A_2 = \cdots = A_\sigma = 0$, we consider the quadratic form

$$x_0^2 + \sum_{i,j}^{1,\cdots,n} a_{ij}\, x_i\, x_j, \tag{9}$$

the sequence (3) corresponding to which is

$$A_0 = 1,\ A_1 = 1,\ A_2 = a_{11}, \cdots, A_{r+1} = \begin{vmatrix} a_{11}, & \cdots, & a_{1r} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ a_{r1}, & \cdots, & a_{rr} \end{vmatrix}. \tag{3'}$$

The canonical forms of (9) and (1) have exactly the same number of negative coefficients. Moreover, the number of variations of sign in the sequence (3′) is exactly equal to the number of variations in the sequence (3), and in the former $A_\tau \neq 0$ with $\tau > 0$.

We may draw the following immediate conclusions:

If $\sigma = 2$, $\nu = 2$ or 1 according as A_τ and $A_{\tau+3}$ have the same sign or opposite signs.

If $\sigma = 3$ so that $1 \leqq \nu \leqq 3$, if A_τ and $A_{\tau+4}$ have the same sign, ν is even and hence is equal to 2; but if A_τ and $A_{\tau+4}$ have opposite signs, ν is odd and we cannot distinguish between the cases $\nu = 1$ and $\nu = 3$. Indeed, in this latter case the signature of (1) cannot be determined from the sequence (3) alone. Cf. the classical example of Frobenius as quoted by Franklin.*

If $\sigma = 4$ so that $1 \leqq \sigma \leqq 4$, if A_τ and $A_{\tau+5}$ have the same sign, ν is even and we cannot distinguish between the cases $\nu = 2$ and $\nu = 4$; while if A_τ and $A_{\tau+5}$ have opposite signs, ν is odd and we cannot distinguish between the cases $\nu = 1$ and $\nu = 3$.

That all of these cases actually arise is clear from a single example. Thus, consider the form

$$2\,a x_1 x_5 + 2\,b x_2^2 + 2\,b x_2 x_4 + a x_3^2 + 2\,b x_4^2$$

the sequence of A's corresponding to which is

$$1, \quad 0, \quad 0, \quad 0, \quad 0, \quad -3\,a^3 b^2.$$

However, on renumbering the variables

$$x_1 = y_5, \quad x_2 = y_1, \quad x_3 = y_2, \quad x_4 = y_3, \quad x_5 = y_4,$$

the form becomes

$$2\,b y_1^2 + 2\,b y_1 y_3 + a y_2^2 + 2\,b y_3^2 + 2\,a y_4 y_5,$$

the sequence of A's corresponding to which is

$$1, \quad 2b, \quad 2ab, \quad 3ab^2, \quad 0, \quad -3\,a^3 b^2.$$

The form is now regular and ν can be determined by Gundelfinger's rule. In fact, if $a > 0$ so that A_0 and A_5 have opposite signs, $\nu = 1$ or 3 according as $b > 0$ or $b < 0$; while if $a < 0$ so that A_0 and A_5 have the same sign, $\nu = 2$ or 4 according as $b > 0$ or $b < 0$.

For $\sigma > 4$ it is apparent that the number of cases which may arise is even greater.

We may therefore state the following theorem.

THEOREM 1. *If for a real quadratic form* (1) *of rank r we set up the sequence* (3) *in which $A_r \neq 0$, then the number of negative coefficients in the canonical form* (2) *is equal to the number of variations of sign in the sequence* (3), *where*

* Franklin, *loc. cit* pp. 451–452.

(i) *if* $A_\tau A_{\tau+2} \neq 0$ *while* $A_{\tau+1} = 0$, *the subsequence* $A_\tau, 0, A_{\tau+2}$ *gives rise to one variation*;

(ii) *if* $A_\tau A_{\tau+3} \neq 0$ *while* $A_{\tau+1} = A_{\tau+2} = 0$, *we assign to the subsequence* $A_\tau, 0, 0, A_{\tau+3}$ *two variations or one variation according as* A_τ *and* $A_{\tau+3}$ *have the same sign or opposite signs*;

(iii) *if* $A_\tau A_{\tau+4} \neq 0$ *while* $A_{\tau+1} = A_{\tau+2} = A_{\tau+3} = 0$, *we assign to the subsequence* $A_\tau, 0, 0, 0, A_{\tau+4}$ *two variations if* A_τ *and* $A_{\tau+4}$ *have the same sign, while if* A_τ *and* $A_{\tau+4}$ *have opposite signs the number of variations is undetermined.*

(iv) *For* $\sigma \geqq 4$, *if* $A_\tau A_{\tau+\sigma+1} \neq 0$ *while* $A_{\tau+i} = 0$ $(i = 1, \cdots, \sigma)$ *the number of variations in the subsequence cannot be determined by the signs of the* A's *alone.*

3. **Recurring forms.** The scheme of rearrangement which was employed in the preceding section on the general quadratic form will now be applied to a recurring form, i. e., a form (1) in which

$$a_{ij} = a_{i+j-2}.$$

We first prove the following theorem.

THEOREM 2. *If for a real recurring form* (1) *we have* $A_\tau \neq 0$ $(\tau > 0)$ *while* $A_{\tau+i} = 0$ $(i = 1, \cdots, \sigma)$, *there exist* $\sigma + 1$ *real sets* $X^{(k)} = (x_0^{(k)}, \cdots, x_{\tau-1}^{(k)})$ $(k = 1, \cdots, \sigma+1)$ *such that*

$$a_{\tau+k-1+i} = \sum_{j=0}^{\tau-1} a_{i+j}\, x_j^{(k)} \qquad \begin{cases} (k = 1, \cdots, \sigma+1) \\ (i = 0, \cdots, \tau+\sigma-k), \end{cases} \tag{10}$$

and satisfying the conditions

$$\sum_{j=0}^{\tau-1} a_{\tau+i-1+j}\, x_j^{(k)} = \sum_{j=0}^{\tau-1} a_{\tau+i-2+j}\, x_j^{(k+1)} \qquad \begin{cases} (k = 1, \cdots, \sigma) \\ (i = 1, \cdots, \sigma+1). \end{cases} \tag{11}$$

In order to prove this theorem we shall need first to establish two lemmas.

LEMMA 1. *If for a recurring form* (1) *there exists a single set* $X' = (x'_0, \cdots, x'_{\tau-1})$ *satisfying*

$$a_{\tau+i} = \sum_{j=0}^{\tau-1} a_{i+j}\, x'_j \qquad (i = 0, \cdots, \tau+\sigma-1), \tag{10_1}$$

there exist σ *additional sets* $X^{(k)}$ *satisfying the conditions* (10) *of the theorem.*

In (10_k) we have on replacing i by $i+1$,

$$\begin{aligned} a_{\tau+k+i} &= \sum_{j=0}^{\tau-1} a_{i+j+1}\, x_j^{(k)} \\ &= \sum_{j=0}^{\tau-2} a_{i+j+1}\, x_j^{(k)} + a_{\tau+i}\, x_{\tau-1}^{(k)} \quad (i = 0, \cdots, \tau+\sigma-k-1). \end{aligned}$$

In the first term on the right replace j by $j-1$ and in the second term replace $a_{\tau+i}$ by its value from (10_1). We then have

$$(10_{k+1}) \qquad a_{\tau+k+i} = \sum_{j=1}^{\tau-1} a_{i+j}\, x_{j-1}^{(k)} + x_{\tau-1}^{(k)} \sum_{j=0}^{\tau-1} a_{i+j}\, x_j' = \sum_{j=0}^{\tau-1} a_{i+j}\, x_j^{(k+1)} \qquad (i = 0, \cdots, \tau+\sigma-k-1),$$

where

$$(12) \qquad x_0^{(k+1)} = x_0'\, x_{\tau-1}^{(k)}; \quad x_j^{(k+1)} = x_j'\, x_{\tau-1}^{(k)} + x_{j-1}^{(k)} \qquad (j = 1, \cdots, \tau-1).$$

Since for $k = 1$, (10_k) becomes (10_1), we have shown that (10_1) implies (10_2); this in turn implies (10_3); and in general (10_k) implies (10_{k+1}) for $k \leqq \tau+\sigma-1$ and therefore for $k \leqq \sigma$. Hence the lemma is proved.

LEMMA 2. *The sets $X^{(k)}$ of Lemma* 1 *satisfy the conditions* (11) *of the theorem.*

For, on replacing in the right hand member of (11) $X^{(k+1)}$ by its value from (12) and using (10_1) with i replaced by $\tau+i-2$ $(i = 1, \cdots, \sigma+1)$ the equality follows at once.

In view of these two lemmas in order to prove the theorem completely we have merely to show the existence of a single set X' satisfying (10_1). That such a set exists for $\sigma = 1$ follows directly from (5) after making obvious changes in notation. To proceed by induction we assume that the theorem is true for σ and prove that it is true for $\sigma+1$.

From the elements of the $(\tau+k)$th column of $A_{\tau+\sigma+1}$ subtract the sum of the products of the elements of the first τ columns by $x_0^{(k)}, \cdots, x_{\tau-1}^{(k)}$ $(k = 1, \cdots, \sigma+1)$. Proceeding similarly with the rows it follows from (10) that $A_{\tau+\sigma+1}$ is reduced to the form

$$\begin{vmatrix} A_\tau, & 0 \\ 0, & B \end{vmatrix}$$

where B is a symmetric determinant of order $\sigma+1$. Indeed, B is recurring, for if b_{ik} denote the element in its ith row and kth column evidently

$$(13) \qquad b_{ik} = b_{ki} = a_{2\tau+i+k-2} - \sum_{j=0}^{\tau-1} a_{\tau+i-1+j}\, x_j^{(k)} \qquad \begin{cases} (k = 1, \cdots, \sigma+1) \\ (i = 1, \cdots, \sigma+1), \end{cases}$$

which in view of (11) equals $b_{i-1,k+1}$. We may therefore write

$$b_{ik} = b_{i+k-2} \qquad (i+k = 2, \cdots, 2\sigma+2).$$

Moreover, from (10_k) with i replaced by $\tau+i-1$,

$$a_{2\tau+i+k-2} - \sum_{j=0}^{\tau-1} a_{\tau+i-1+j}\, x_j^{(k)} = 0 \qquad (i+k \leqq \sigma+1)$$

so that

$$(14) \qquad b_{ik} = 0 \qquad (i+k \leqq \sigma+1).$$

Hence, on expanding $A_{\tau+\sigma+1}$ we have

(15) $$A_{\tau+\sigma+1} = (-1)^{\sigma(\sigma+1)/2}(b_\sigma)^{\sigma+1} A_\tau,$$

where

$$b_\sigma = a_{2\tau+\sigma} - \sum_{j=0}^{\tau-1} a_{\tau+\sigma+j}\, x'_j.$$

If, therefore, $A_{\tau+\sigma+1} = 0$, $b_\sigma = 0$ so that (10_1) holds for $i = \tau+\sigma$ and the induction is complete.

COROLLARY 1. *If* $A_\tau A_{\tau+\sigma+1} \neq 0$ $(\tau > 0)$ *while* $A_{\tau+i} = 0$ $(i = 1, \cdots, \sigma)$ *then if* σ *is odd,* A_τ *and* $A_{\tau+\sigma+1}$ *have the same sign or opposite signs according as* σ *is of the form* $4n-1$ *or* $4n+1$.

We may now state a corollary to Theorem 1.

COROLLARY 2. *If in the sequence* (3) *for a real recurring form of rank* r, $A_r \neq 0$ *and three consecutive terms vanish, the subsequence of five terms containing these vanishing terms should be considered as presenting exactly two variations of sign.*

Now let us suppose that $A_\tau A_{\tau+\sigma+1} \neq 0$ while $A_{\tau+i} = 0$ $(i = 1, \cdots, \sigma)$. In $A_{\tau+\sigma+1}$ let us place the $(\tau+1)$th row and column last. From the elements of the last column of the new determinant subtract the sum of the products of the elements of the first τ columns by $x'_0, \cdots, x'_{\tau-1}$. Proceeding similarly with the last row, we have in view of (10)

$$A_{\tau+\sigma+1} = -(a_{2\tau+\sigma} - \sum a_{\tau+\sigma+i}\, x'_i)^2 D_{\tau+\sigma-1}$$

where $D_{\tau+\sigma-1}$ is the principal minor determinant of order $\tau+\sigma-1$ standing in the upper left hand corner of the rearranged determinant $A_{\tau+\sigma+1}$. Evidently $D_{\tau+\sigma-1} \neq 0$ and is of opposite sign to $A_{\tau+\sigma+1}$.

In $D_{\tau+\sigma-1}$ place the $(\tau+1)$th row and column last. Using the multipliers $x''_0, \cdots, x''_{\tau-1}$ and proceeding in a manner similar to that just indicated, we have

$$D_{\tau+\sigma-1} = -(a_{2\tau+\sigma} - \sum a_{\tau+\sigma-1+i}\, x''_i)^2 D_{\tau+\sigma-3}.$$

This process may be continued using the multipliers $x_0^{(k)}, \cdots, x_{\tau-1}^{(k)}$ $(k = 3, \cdots)$ until we arrive at one or the other of the conclusions according as σ is even or odd.

Either $D_{\tau+3} = -c^2 D_{\tau+1}$ $(c \neq 0)$; or $D_{\tau+2} = -d^2 A_\tau$ $(d \neq 0)$. In order to estimate the number of variations of sign due to the vanishing of the σ consecutive A's, we may employ the subsequence consisting of the $\sigma+2$ quantities

(16) $$A_\tau,\quad D_{\tau+1},\quad D_{\tau+2}, \cdots, D_{\tau+\sigma-1},\quad D_{\tau+\sigma},\quad A_{\tau+\sigma+1}.$$

Denoting by ϵ the sign of $A_{\tau+\sigma+1}$ and recalling Corollary 1, it is clear that according as σ is of the form $4s-1$, $4s$, $4s+1$ or $4s+2$ the signs of the numbers in (16) are as follows:

$$\begin{array}{ll} \sigma = 4s-1, & \epsilon \,|\, (\ )-\epsilon(\ )\epsilon \,|\, \cdots \text{to } s \text{ terms} \cdots |\, (\ )-\epsilon(\ )\epsilon \,|; \\ \sigma = 4s, & \pm\epsilon\epsilon \,|\, (\ )-\epsilon(\ )\epsilon \,|\, \cdots \text{to } s \text{ terms} \cdots |\, (\ )-\epsilon(\ )\epsilon \,|; \\ \sigma = 4s+1, & -\epsilon(\ )\epsilon \,|\, (\ )-\epsilon(\ )\epsilon \,|\, \cdots \text{to } s \text{ terms} \cdots |\, (\ )-\epsilon(\ )\epsilon \,|; \\ \sigma = 4s+2, & \pm\epsilon-\epsilon(\ )\epsilon \,|\, (\ )-\epsilon(\ )\epsilon \,|\, \cdots \text{to } s \text{ terms} \cdots |\, (\ )-\epsilon(\ )\epsilon \,|. \end{array}$$

The number of variations of sign in each of these subsequences is easily estimated and we may summarize the results in the form of a theorem.

THEOREM 3. *If for a real recurring form* (1) *we set up the sequence* (3) *in which* $A_\tau A_{\tau+\sigma+1} \neq 0$ *while* $A_{\tau+i} = 0$ $(i = 1, \cdots, \sigma)$, *then to the subsequence*

$$A_\tau,\ 0,\ 0,\ \cdots,\ 0,\ A_{\tau+\sigma+1}$$

we assign exactly $(\sigma+1)/2$ *variations of sign if* σ *is odd. If* σ *is even of the form* $4s$ $(4s+2)$ *we assign* $\sigma/2$ *or* $\sigma/2+1$ $(\sigma/2+1$ *or* $\sigma/2)$ *variations according as* A_τ *and* $A_{\tau+\sigma+1}$ *are of the same sign or of opposite signs.*

Although Theorem 3 was proved on the assumption that $A_\tau \neq 0$ for $\tau > 0$ the results are still true for $\tau = 0$ as is clear from the discussion in § 2. However, this might have been proved directly. For, let $A_\tau = A_0 = 1$. Then $A_1 = 0$, $A_2 = 0, \cdots, A_\sigma = 0$ imply in turn that $a_0 = 0$, $a_1 = 0, \cdots, a_{\sigma-1} = 0$, and we have

$$A_{\sigma+1} = (-1)^{\sigma(\sigma+1)/2}\, a_\sigma^{\sigma+1} \tag{15'}$$

which is (15) with $\tau = 0$ and $X^{(k)} = 0$. If now we consider each set $X^{(k)}$ as zero and proceed with $A_{\sigma+1}$ in the same manner as we proceeded with $A_{\tau+\sigma+1}$ the result is precisely the same as that previously arrived at. Indeed this is exactly the scheme employed by Frobenius in this special case.

4. **Recurring forms for which $A_r = 0$.** In the preceding section it was shown that if (1) is a real recurring form of rank r for which $A_r \neq 0$ the signature of the form was completely determined by the sequence of A's alone even when an arbitrary number of consecutive A's vanish. The rule here given for determining the signature is easily identified with that given by Frobenius.

We now consider the case in which $A_r = 0$. Then $A_\tau \neq 0$ $(\tau < r)$ while $A_i = 0$ $(i = \tau+1, \cdots, n)$. Writing $\tau+\sigma+1 = n$ we may adopt the notation of the preceding section. Manifestly the signature of the original form is the same as the signature of the form whose matrix is (12), and hence is the sum of the signatures of the two forms whose matrices are A_τ and B. If ϱ denote the rank of B, evidently $\varrho < \sigma+1$ (since $A_r = 0$). In view of the fact that in the recurring matrix B $b_{ij} = 0$ for $i+j \leqq \sigma+1$, B is of rank exactly σ if $b_\sigma = 0$ while $b_{\sigma+1} \neq 0$. Similarly, B is of rank

$\sigma - 1$ if $b_\sigma = b_{\sigma+1} = 0$ while $b_{\sigma+2} \neq 0$, and in general, B is of rank $\varrho = \sigma + 1 - \mu$ if $b_\sigma = b_{\sigma+1} = \cdots = b_{\sigma-1+\mu} = 0$ while $b_{\sigma+\mu} \neq 0$. In this case manifestly the principal minor determinant B_ϱ of order ϱ standing in the lower right hand corner of B is different from zero, and therefore in view of the manner in which B was built up, it is clear that the determinant of order $r = \tau + \varrho$ formed by bordering A_τ with the last ϱ rows and columns of A is different from zero. Following Frobenius we shall call this determinant A'_r.

In B let us, while maintaining their positions relative to one another, transfer the last ϱ rows and colums over the μ rows and columns preceding them thus bringing the determinant B_ϱ into the upper left hand corner of B. The number N of negative coefficients in the form B is then easily determined by the rule given in Theorem 3 for a recurring form of rank ϱ in which $B_1 = B_2 = \cdots = B_{\varrho-1} = 0$ while $B_\varrho \neq 0$. Since N, which was determined from the sequence

$$1, 0, 0, \cdots \text{to } \varrho - 1 \text{ terms} \cdots, 0, 0, B_\varrho,$$

might have been determined just as well from the sequence

$$A_\tau, 0, 0, \cdots \text{to } \varrho - 1 \text{ terms} \cdots, 0, 0, A_\tau B_\varrho = A'_r,$$

we may formulate the rule as follows.

If for a real recurring form of rank r we set up the sequence (3) in which $A_\tau \neq 0$ $(\tau < r)$ while $A_i = 0$ $(i > \tau)$ and if we adjoin to this sequence $r - \tau - 1$ zeros and in addition a determinant A'_r formed by bordering A_τ with the last $r - \tau$ rows and columns of the matrix of the form

$$A_0, A_1, \cdots, A_\tau, 0, 0, \cdots \text{to } r - \tau - 1 \text{ terms} \cdots, 0, A'_r \tag{17}$$

then the number N of negative coefficients in the canonical form of (1) is given by the number of variations of sign in the sequence (17) where the number of variations corresponding to σ consecutive zeros in the sequence is determined by Theorem 3.

THE UNIVERSITY OF NORTH CAROLINA.