An Extension of Results of Novikov and Browder

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AN EXTENSION OF RESULTS OF NOVIKOV AND BROWDER.

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Let V^v be a closed, oriented, simply-connected, smooth manifold. According to Whitney [20], V^v can be smoothly imbedded in spheres S^{N+v} of sufficiently high dimension ($N \geqq v$ is certainly adequate). Let $z \in H_v(V)$ be the fundamental class of V, ξ^N the normal bundle of V in S^{N+v}, V^ξ the Thom complex of ξ, and $\alpha \in \pi_{N+v}(V^\xi)$ the element derived by the Thom construction ("shrinking the complement of an open tubular neighbourhood of V in S^{N+v} to a point"). The quadruple (V, z, ξ, α) has certain properties: e.g. (V, z) satisfies Poincaré duality, and the Hurewicz image of α, $h(\alpha)$, equals the image, $\phi(z)$, of z under the Thom isomorphism. In a remarkable paper [10], Novikov indicated a proof that in the case $N > v \geqq 5$, the invariants above almost characterised the diffeomorphism class of V; and Browder showed in [1] that most quadruples (V, z, ξ, α) satisfying the conditions could so arise.

The main object of this paper is to extend these results to the case of manifolds with boundary, where, somewhat surprisingly, a stronger result can be obtained. We also use a paper of Haefliger [2] (the methods of which were used in [5]) to extend the results to the metastable range: this extension was first investigated by Levine [7].

Since no proofs of any of the above results have appeared in print at the time of writing, we shall give them all here, making use only of the work of Milnor and Kervaire [6], [8].

1. Invariant systems. We first list the data which will be required for the construction. For a closed manifold, this was done above. It is convenient to have a name for them: we choose the name 'invariant systems,' but will always abbreviate it to I. S.; we distinguish the closed case from the bounded case by writing B. I. S. or C. I. S. We call the quadruple (X, z, ξ^N, α) a C. I. S. if X is a finite, simply-connected C. W. complex; $z \in H_v(X)$ has the property that for any integer r and abelian group G,

$$z \cap : H^r(X; G) \to H_{v-r}(X; G)$$

is an isomorphism (we call z a fundamental class for X); ξ^N is an N-vector

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bundle over X with Thom space X^{ξ}; and $\alpha \in \pi_{N+v}(X^{\xi})$ has the property $h(\alpha) = \Phi(z)$.

The definition in the relative case is entirely analogous. We call $(X, \partial X, z, \xi, \alpha)$ a B. I. S. if $(X, \partial X)$ is a finite C.W.-pair, with each of X, ∂X connected and simply-connected; $z \in H_v(X, \partial X)$ induces isomorphisms

$$z \cap : H^r(X; G) \to H_{v-r}(X, \partial X; G)$$

and

$$z \cap : H^r(X, \partial X; G) \to H_{v-r}(X; G)$$

for all r, G; ξ^N is as above, and we write ∂X^{ξ} for the Thom complex of $\xi \mid \partial X$; and $\alpha \in \pi_{N+v}(X^{\xi}, \partial X^{\xi})$ again satisfies $h(\alpha) = \Phi(z)$.

Since, by Whitney [20], any v-manifold has an essentially unique imbedding in D^{N+v} if $N > v$, we note that a smooth compact manifold determines an essentially unique B. I. S. or C. I. S., again provided $N > v$.

LEMMA 1. *Let* $(X, \partial X, z, \xi, \alpha)$ *be a B. I. S. Then* $(\partial X, \partial_* z, \xi \mid \partial X, \partial_* \alpha)$ *is a C. I. S.*

Proof. By a standard property of cap products, the following diagram is commutative up to sign (with arbitrary coefficient group G):

$$\begin{array}{ccccccccc} H^r(X,\partial X) & \to & H^r(X) & \to & H^r(\partial X) & \longrightarrow & H^{r+1}(X,\partial X) & \to & H^{r+1}(X) \\ \downarrow z\cap & & \downarrow z\cap & & \downarrow \partial_* z\cap & & \downarrow z\cap & & \downarrow z\cap \\ H_{v-r}(X) & \to & H_{v-r}(X,\partial X) & \to & H_{v-r-1}(\partial X) & \longrightarrow & H_{v-r-1}(X) & \to & H_{v-r-1}(X,\partial X). \end{array}$$

Since both rows are exact, we can use the Five Lemma to conclude that

$$\partial_* z \cap : H^r(\partial X) \to H_{v-r-1}(\partial X)$$

is an isomorphism. It now only remains to observe

$$h(\partial_* \alpha) = \partial_* h(\alpha) = \partial_* \Phi(z) = \Phi(\partial_* z).$$

2. Constructing a manifold. What would be nice to prove is that there is a (1-1) correspondence between appropriately defined equivalence classes of B. I. S. and C. I. S. and embeddings of manifolds in spheres. This is not true, however, and one can only obtain approximations to it. We shall start by constructing an imbedded manifold which has the wrong homotopy type; this we shall try to rectify in subsequent paragraphs.

First, given a C. I. S., we can replace X by a homotopy equivalent simplicial complex, imbed this in Euclidean space of large dimension, and take the interior of a regular neighbourhood—this is a homotopy equivalent open differentiable manifold. For a B. I. S., we take $(X, \partial X)$ as a simplicial pair, and imbed in a simplex Δ^k in such a way that ∂X is imbedded in one face, and the remainder of X in the interior. Taking a neighbourhood in Δ^k now allows us to replace X by a (non-compact) manifold, ∂X by its boundary.

We can now regard the Thom complex X^ξ as a manifold (except at the 'point at infinity') with boundary ∂X^ξ. In the case of a C. I. S. we now take a representative $f: S^{N+v} \to X^\xi$ of α, make it t-regular on X (c. f. Thom [14]), and write $M^v = f^{-1}(X)$; this is a smooth submanifold of S^{N+v}, with normal bundle $f^*(\xi)$ (by the t-regularity). For a B. I. S., we represent α by $f: (D^{N+v}, \partial D^{N+v}) \to (X^\xi, \partial X^\xi)$, and make this t-regular on $(X, \partial X)$. Observe particularly that if a map from ∂D^{N+v} to ∂X^ξ, already t-regular on ∂X, is in the homotopy class $\partial_* \alpha$, we can take this as the restriction of f to ∂X, by another result of Thom (loc. cit.). Again, when f is t-regular on X, $M^v = f^{-1}(X)$ is a smooth submanifold of D^{N+v}, meeting the boundary transversely in $\partial M = f^{-1}(\partial X)$, and with normal bundle in D^{N+v} given by $f^*(\xi)$.

The construction, which is due to Browder [1], gives us a manifold M^v, and a map $f: M \to X$ with $f^{-1}(\partial X) = \partial M$. We assert that f has degree 1. For let $\eta = f^* \xi$ be the normal bundle of M. Then we have maps $S^{N+v} \to M^\eta \to X^\xi$ (in the closed case) or

$$(D^{N+v}, \partial D^{N+v}) \to (M^\eta, \partial M^\eta) \to (X^\xi, \partial X^\xi);$$

the first has degree 1 by construction, and the composite also, by the assumption $h(\alpha) = \Phi(z)$. Hence the map from M^η to X^ξ has degree 1; desuspending by the Thom isomorphism, so has f.

LEMMA 2. *Let M, X satisfy Poincaré duality with fundamental classes y, z. Let $f: M \to X$ have degree* 1. *Write $K^r(M; G)$ for the cokernel of $f^*: H^r(M; G) \leftarrow H^r(X; G)$. Then f_* induces a map from $K^r(M; G)$ to $H^r(M; G)$ so that*

$$H^r(M; G) \cong K^r(M; G) \oplus f^* H^r(X; G).$$

f^* is a monomorphism of rings, and

$$f^* H^r(X; G) \cup K^s(M; H) \subset K^{r+s}(M; G \otimes H).$$

An analogous splitting occurs in homology.

Proof. Since $f_*(y) = z$, the diagram

$$\begin{array}{ccc} H^r(M;G) & \xleftarrow{f^*} & H^r(X;G) \\ \downarrow y\cap & & \downarrow z\cap \\ H_{v-r}(M;G) & \xrightarrow{f_*} & H_{v-r}(X;G) \end{array}$$

is commutative, for by a standard property of cap products,

$$z \cap c = f_*y \cap c = f_*(y \cap f^*c).$$

Since $z\cap$ is an isomorphism, $(z\cap)^{-1}f_*(y\cap)$ splits f^*. We have $K^r(M;G) = (y\cap)^{-1}\operatorname{Ker} f_*$. If $c_1 \in H^r(X;G)$, and $c_2 \in K^s(M;H)$, then

$$\begin{aligned} f_*(y \cap c_2 \cdot f^*c_1) &= f_*((y \cap c_2) \cap f^*c_1) \\ &= f_*(y \cap c_2) \cap c_1 = 0 \cap c_1 = 0, \end{aligned}$$

so $c_2 \cdot f^*c_1$ is in $K^{r+s}(M;G\otimes H)$.

We shall also need the corresponding results for bounded manifolds. Let $y \in H_v(M,\partial M)$ and $z \in H_v(X,\partial X)$ be fundamental classes; $f:(M,\partial M) \to (X,\partial X)$ have degree 1 (i.e. $f_*y = z$); $K_r(M) = \operatorname{Ker} f_*: H_r(M) \to H_r(X)$; similarly for $K_r(\partial M)$ and $K_r(M,\partial M)$.

LEMMA 3. *With the above assumptions, the exact homology sequence of* $(M,\partial M)$ *splits as the direct sum of an isomorphic copy of the exact sequence of* $(X,\partial X)$ *with the sequence of the* K_r. *The intersection of any* $x \in K_r(M;G)$ *with any* $y \in H_{v-r}(X,\partial X;H)$ *vanishes.*

Proof. Consider the diagram

$$\begin{array}{ccccccc} H^{v-r}(X,\partial X;G) & \to & H^{v-r}(X;G) & \to & H^{v-r}(\partial X;G) & \to & H^{v-r+1}(X,\partial X;G) \\ \downarrow f^* & & \downarrow f^* & & \downarrow f^* & & \downarrow f^* \\ H^{v-r}(M,\partial M;G) & \to & H^{v-r}(M;G) & \to & H^{v-r}(\partial M;G) & \to & H^{v-r+1}(M,\partial M;G) \\ \downarrow y\cap & & \downarrow y\cap & & \downarrow \partial_*y\cap & & \downarrow y\cap \\ H_r(M;G) & \to & H_r(M,\partial M;G) & \to & H_{r-1}(\partial M;G) & \to & H_{r-1}(M;G) \\ \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ H_r(X;G) & \to & H_r(X,\partial X;G) & \to & H_{r-1}(\partial X;G) & \to & H_{r-1}(X;G) \end{array}$$

As in Lemma 2, the composite map of the upper row on the lower is the isomorphism induced by $z\cap$. Since the maps induced by $y\cap$ are isomorphisms, f^* again splits f_* additively. By the commutativity of the lower squares, the K's form a subcomplex of the exact sequence of $(M, \partial M)$; since $(y \cap) f^* (z \cap)^{-1}$ also imbeds the exact sequence of $(X, \partial X)$ as a subcomplex, and the groups in the third row all split, the whole sequence splits as a direct sum.

For the last part, recall that intersections can be computed as cup products; we wish to show

$$K^{v-r}(M, \partial M; G) \cdot f^* H^r(X; H) = 0.$$

But by an argument as in Lemma 2, we see that the product lies in $K^v(M, \partial M; G \otimes H) = 0$.

We remark that the sequence of K's looks rather like the sequence of homology groups of a manifold, except that the extreme terms are absent. In particular, the direct sum splitting shows that the K's satisfy Poincaré or Lefschetz duality (appropriately interpreted) and the universal coefficient theorem.

3. Surgery. In § 2 we constructed from a C. I. S. or B. I. S. a manifold M^v embedded in S^{N+v} or D^{N+v} and a map of degree 1 $f: M^v \to X^v$ such that $f^*\xi$ is isomorphic to the normal bundle η of M, and if the map from S^{N+v} resp. D^{N+v} to M^η given by the Thom construction is followed by the map from M^η to X^ξ determined by an isomorphism above, the composite has homotopy class α. Under these circumstances we say M *corresponds* to the C. I. S. or B. I. S. If also f is a homotopy equivalence, we say M *realises* the invariant system.

Let M correspond to an I. S.; we shall simplify M by surgery, making sure that at each stage it corresponds to the I. S., and try to make f a homotopy equivalence. We shall find it necessary to impose arithmetic restrictions on v and N as we proceed.

Suppose inductively f r-connected $(r \geqq 0)$, and $x \in K_r(M; \mathbf{Z})$.

(3.1) *If $r > 0$, x is spherical.*

For $r = 0$, $K_0(M) = \tilde{H}_0(M)$ is generated by spherical classes, though it does not consist solely of them. If $r = 1$, M is connected, and $x \in K_1(M) = H_1(M)$ is always spherical. If $r \geqq 2$, M and X are simply-connected, and by the relative Hurewicz theorem, $h: \pi_{r+1}(f) \to H_{r+1}(f)$ is an isomorphism. But ∂_* induces an isomorphism of $H_{r+1}(f)$ on $K_r(M)$, so we can write $x = \partial_* h(y) = h \partial_*(y)$, and x is spherical.

(3.2) *If $v \geqq 2r + 1$, x can be represented by an imbedded sphere S^r.*

By (3.1), we can represent x by a map of S^r to M, and by a general position argument which stems from Whitney [20], we can deform such a map slightly to an imbedding, i say. We can also obtain an imbedding if $v=2r$, $r>2$ and M is simply-connected [3].

We now use $i(S^r)$ to perform surgery to kill x; however, we must be careful that our construction gives an M' which corresponds to the given I. S. We shall in fact construct a submanifold W^{v+1} of $S^{N+v}\times I$ or $D^{N+v}\times I$ and a homotopy $F\colon S^{N+v}\times I\to X^{\xi}$ or $(D^{N+v},\partial D^{N+v})\times I\to(X^{\xi},\partial X^{\xi})$ which is t-regular on X, has $F^{-1}(X)=W$, and $F(P,0)=f(P)$ for $P\in S^{N+v}$ or D^{N+v}. Then, clearly, if M' is the 'upper end' $W\cap(S^{N+v}\times 1)$, M' corresponds to the I. S., and if W is diffeomorphic to $M\times I$ with an $(r+1)$-handle attached to kill $x\in K_r(M)$, we are one step forward in the surgery.

(3.3) *If $v\geqq 2r$ and $N>r+1$, we can perform surgery as above to kill $x\in K_r(M)$ (if $r\geqq 2$), $x\in\pi_r(M)$ (if $r=0,1$), provided if $v=2r$ that $r\geqq 4$, and $r\neq 7$ and the sphere representing x has trivial normal bundle in M.*

Since $fi\colon S^r\to X$ is nullhomotopic, we can extend to a map j of D^{r+1}. The bundle $j^*\xi$ is trivial: we choose a framing $f_1,\cdots,f_N$ for it. As $j^*\xi\mid S^{r-1}=i^*f^*\xi=i^*\eta$, we have now a framing along $i(S^r)$ of the normal bundle η of M in S^{N+v}. Deform $i(S^r)$ along the normal vector f_1 into the complement of M, and then span by a disc D^{r+1}, which we can suppose smooth. Since $N+v\geqq 3r+2\geqq 2r+3$, we may suppose D^{r+1} imbedded. As also $N+v>(r+1)+v$, we can suppose D^{r+1} disjoint from M except along the boundary.

The vectors $f_2,\cdots,f_N$ give a normal $(N-1)$-field to D^{r+1} along its boundary. Since the normal bundle of D^{r+1} is trivial, we can regard them as defining a map from S^r to the Stiefel manifold $V_{N+v-r-1,N-1}$. If $r<v-r$, all such maps are trivial, so we can extend $f_2,\cdots,f_N$ to a framing $f_2,\cdots,f_{N+v-r}$ over all D^{r+1}. If $r=v-r$, and $r\neq 1,3,7$ since

$$\partial_*\colon\pi_r(V_{N+v-r-1,N-1})\to\pi_{r-1}(SO_{v-r})$$

is then a monomorphism, and the image of the element above is clearly the characteristic class of the normal bundle of $i(S^r)$, which vanishes by hypothesis, we can again extend. Use the vectors $f_{N+1},\cdots,f_{N+v-r}$ to define a tube $D^{r+1}\times D^{v-r}$ round D^{r+1}, meeting M in $S^r\times D^{v-r}$.

Now take $M\times I\subset S^{N+v}\times I$, and attach $D^{r+1}\times D^{v-r}\times 1$ in $S^{N+v}\times 1$. There is a right-angle corner at $S^r\times S^{v-r}$; this is to be straightened, and $D^{r+1}\times\operatorname{Int}(D^{v-r})$ pushed into $S^{N+v}\times(0,1)$. For more details of this construction, see Haefliger [2], p. 460. This defines W. We now define the homotopy F. We start with $f\colon M\to X$, the given map extended (as given) over a tubular neighbourhood of M to X^{ξ} and (we may suppose) mapping the remainder of S^{N+v} to the point at ∞. Extend over W by mapping $M \times I$ via projection on M, D^{r+1} by j, and the remainder of $D^{r+1} \times D^{v-r}$ using a retraction on $(D^{r+1} \times 0) \cup (S^r \times D^{n-r})$, which is already mapped. Over a tubular neighbourhood of W, for $M \times I$ we again use projection; over D^{r+1} the normal framing $f_0, f_2, \cdots, f_N$ (where f_0 is the direction of the second factor in $S^{N+v} \times I$) defines a map (recall that $f_1, \cdots, f_N$ framed $j^*\xi$). When the corner is straightened, f_0 and f_1 will be forced to fit together there; we then have covered $h: W \to X$ by a map in ξ of the normal bundle of W in $S^{N+v} \times I$. Take the induced map of a tubular neighbourhood of W to the Thom complex of ξ, and map the rest of $S^{N+v} \times I$ to the point ∞. The constructed map has all desired properties and we have completed the proof of (3.3).

(3.4) *Let $v = 2k$ or $2k+1$, $N > k \geqq 2$. Then we can perform surgery to make M simply-connected, and $f: M \to X$ k-connected, so that $K_r(M) = 0$ for $r < k$.*

We first add 1-handles to connect up the components of M. Next kill in turn the generators of the fundamental group of M; under the assumption $k \geqq 2$ above, it follows from ([6] Lemma 5.2) that this reduces the group to zero. The argument above will now permit us to kill in turn the $K_i(M)$ for $2 \leqq i < k$.

In the case when M is bounded, we can perform surgery both on ∂M and on M. By an observation made in § 2, when V is given, and $f: S^{N+v-1} \to \partial X^{\xi}$ represents α, is t-regular on ∂X, and has $V = f^{-1}(\partial X)$, we can take M as above with boundary V, and then perform surgery on M.

(3.5) *Let $v = 2k+1$, $N > k \geqq 2$. Then we can perform surgery to make M and ∂M simply-connected, and kill all the K_r except those lying in the exact sequence*

$$0 \to K_{k+1}(M) \to K_{k+1}(M, \partial M) \to K_k(\partial M) \to K_k(M) \to K_k(M, \partial M) \to 0.$$

We may first perform surgery on ∂M to make $K_r(\partial M) = 0$ for $r < k$, then the same for M. The higher $K_i(\partial M)$ and $K_i(M, \partial M)$ vanish by duality (Lemma 3) and the exact sequence shows that the other K's are also zero.

(3.6) *Let $v = 2k+2$, $N - 1 > k \geqq 2$. Then we can perform surgery to make M and ∂M simply-connected and kill all the K_r except*

$$0 \to K_{k+1}(\partial M) \to K_{k+1}(M) \to K_{k+1}(M, \partial M) \to K_k(\partial M) \to 0.$$

The proof is as for (3.5).

4. Surgery in middle dimensions.

(4.1) *Suppose M closed, $v = 2k + 1$, $N - 1 > k \geqq 2$. Then we can perform surgery to kill K_k also and so make $f: M \to X$ a homotopy equivalence.*

By (3.3), the above hypotheses are sufficient to ensure that for any $x \in K_k(M)$, we can perform surgery to kill it. Now by § 5 of [6] (particularly Lemma 5.7) or § 2 of [15] the result follows if k is even. When k is odd, the argument of § 6 of [6] gives the result[1] provided that (3.3) can be sharpened to state that the normal framing of $i(S^r)$ in M induced by the modification can be altered by any element in the image of

$$\partial: \pi_{k+1}(S^{k+1}) \to \pi_k(SO_{k+1}).$$

But this is not difficult (c.f. [2], end of § 3). For the extension of the partial frame $f_2, \cdots, f_N$ over D^{r+1} may be varied by any element α of $\pi_{r+1}(V_{N+v-r-1,N-1})$ (it is always possible to extend the resulting partial frame over the contractible D^{r+1} to a complete frame). Substituting for v and r in terms of k, this group is $\pi_{k+1}(V_{N+k,N-1})$. Now the Stiefel manifold is the homogeneous space SO_{N+k}/SO_{k+1}, hence the above lies in the exact sequence

$$\pi_{k+1}(V_{N+k,N-1}) \xrightarrow{\partial_*} \pi_k(SO_{k+1}) \xrightarrow{i_*} \pi_k(SO_{N+k}).$$

The varying by α of the partial frame over D^{r+1} alters the normal framing of $i(S^r)$ in M by $\partial_*\alpha$; this can be any element of $\operatorname{Im}\partial_* = \operatorname{Ker} i_*$. But as $N \geqq 2$, $\pi_k(SO_{N+k})$ is stable; $\operatorname{Ker} i_*$ is the same as for $N = 2$, hence equals the image of $\pi_{k+1}(V_{k+2,1}) = \pi_{k+1}(S^{k+1})$.

(4.2) *Suppose M bounded, $f \mid \partial M$ already a homotopy equivalence, and f k-connected ($v = 2k + 1 \geqq 5$, $N > k + 1$). Then we can perform surgery to make f a homotopy equivalence.*

By a remark at the end of § 3, the K_r satisfy duality and the universal coefficient theorem. Hence in the middle dimension, they have linking numbers with the usual properties. The argument of Kervaire and Milnor is now again valid.

We can now give

THEOREM 1A. (Browder [1]). *Let $v > 4$, $v \neq 2$ (mod 4), $2N > v + 2$.*

[1] It is possible to show that the argument on p. 525 of [6], using Lemma 5.10, can be applied here, by obtaining Wu formulae for X. However, this argument of [6] is irrelevant. For since L is nonsingular, to λ corresponds a μ with $L(\lambda, \mu) \neq 0$, and hence the class in M' determined by μ has order $\neq 2$. This rules out the second possibility on p. 526, the only one which causes trouble.

Then a C.I.S. can be realised by a manifold, provided, if $v=4j$, that $\sigma=\tau$, where σ is the signature of X and τ the (Hirzebruch) L-genus of the bundle inverse to ξ.

Proof. Only the case $v=4j$ remains, and we can mimic the proof in [6] provided that the signature of K_{2j} vanishes. But by Hirzebruch's theorem, τ is the signature of M, so this condition is equivalent to $\sigma=\tau$.

For a more detailed discussion of the case $v=4j+2$, we refer the reader to work of S. P. Novikov [11].

COMPLEMENT 1B. *If X is a closed smooth manifold, the result holds also if $v=4$.*
The above proof breaks down, as we do not have an adequate imbedding theorem for S^2 in M^4. However, we know that M and X have the same signature, and that $w_2(M)$ and $w_2(X)$ are both zero or both nonzero. By (4.2) of [18], if we take the connected sum of M with sufficiently many copies of $S^2\times S^2$, the result is diffeomorphic to a connected sum of X with copies of $S^2\times S^2$. This can be achieved by performing spherical modifications on circles which bound discs in M. Now using Theorem 2 of [19], we may suppose that M is the connected sum of X with $k\geqq 2$ copies of $S^2\times S^2$, and that the map of M to X is induced by shrinking these to a point. The remainder of the proof now proceeds as usual.

Our main result is

THEOREM 2. *Let $v\geqq 5$, $2N>v+2$. Then any B.I.S. can be realised by a manifold (provided if $v=5$ that ∂X is a closed smooth manifold) $V^v\subset D^{N+v}$. If $v\geqq 6$, $2N>v+3$, the pair (D^{N+v}, V^v) is unique up to diffeomorphism.*

Proof. First suppose $v=2k+1$, $k\geqq 2$, $N>k+1$. Then it follows by (4.2) that any manifold ∂V realising the boundary of the B.I.S. is the boundary of a manifold V realising the B.I.S. So we must check existence of ∂V. If k is even, we can observe that for ∂X, $\sigma=\tau=0$; the proof of this is identical with the proof (Thom [13]) valid for the boundary of a smooth manifold. The result then follows from 1A and 1B. In fact we can give a geometrical argument which covers also the case when k is odd.

Use (3.5) to kill all the K_i except those in the middle dimensions. Next, we will kill $K_k(M,\partial M)$ (c.f. [18], Theorem 1 for the proof). By (4.2) we may suppose that $K_k(\partial M)\neq 0$, so is free abelian. Since the homomorphisms $K_{k+1}(M,\partial M)\rightarrow K_k(\partial M)$ and $K_k(\partial M)\rightarrow K_k(M)$ are dual, $K_k(M)$ has nonzero rank. So it is generated by primitive (i.e. indivisible) elements.

Now if $K_k(M, \partial M)$ is nonzero, we choose $x \in K_k(M)$ primitive and with non-zero image in $K_k(M, \partial M)$, and perform surgery starting with x. Since x is primitive, no new element is introduced in $K_k(M, \partial M)$, the effect is simply to kill the image of x. Each such step decreases either the rank or the order of the torsion subgroup of $K_k(M, \partial M)$; making an induction on these, we see that we can reduce the group to zero.

By duality, $K_{k+1}(M) = 0$, so only the terms of

$$0 \to K_{k+1}(M, \partial M) \xrightarrow{b} K_k(\partial M) \to K_k(M) \to 0$$

survive.

Now these groups are all free, so the extension splits, and it follows (as in [6] Lemma 7.1), that if we can kill in turn the elements of a basis of $\operatorname{Im} b$, we reduce $K_k(\partial M)$ to zero, and the result follows. To prove these elements spherical, we need a 'relative relative' Hurewicz theorem for the quadruple

$$\Phi: \begin{array}{ccc} \partial M & \to & M \\ \downarrow & & \downarrow \\ \partial X & \to & X. \end{array}$$

Since all four spaces are 1-connected (hence the map $\partial X \to X$ also is), and the map $M \to X$ is k-connected, we can apply Theorem 2.4 of [9] with $m = 2$, $n = k + 1$, $r = k + 1$. For $q \leqq k + 1$, $H_q(\Phi) = K_{q-1}(M, \partial M) = 0$, so Φ is $(k+1)$-connected, and $\pi_{k+2}(\Phi)$ maps onto $H_{k+2}(\Phi) \cong K_{k+1}(M, \partial M)$. So the (isomorphic) image of $K_{k+1}(M, \partial M)$ in $H_{k+1}(M, \partial M)$ consists of spherical classes.

Take any element of $\operatorname{Im} b$, and lift it to an element of $K_{k+1}(M, \partial M)$. By the above, we may represent this by a map of a disc $f: D^{k+1} \to M$ with $f(\partial D^{k+1}) \subset \partial M$, and we may suppose the boundary k-sphere imbedded. Take the map of the disc in general position (c.f. [21]): then the singularity set Σ of f is 1-dimensional, and $f(\Sigma)$ is a union of arcs and simple loops. The map f is an immersion except at a finite set of points corresponding to the end-points of these arcs. We assert that f is homotopic (rel ∂D^{k+1}) to an immersion. For by [4], the only obstruction to this lies in $H^{k+1}(D^{k+1}, \partial D^{k+1}; \pi_k(V_{2k+1,k+1})) \cong \pi_k(V_{2k+1,k+1})$, a group isomorphic either to $\boldsymbol{Z}$ or to $\boldsymbol{Z}_2$. But each singularity point (being generic), contributes a unit to the obstruction: since there are an even number of them (each are of $f(\Sigma)$ contains just 2), the obstruction is zero in the $\boldsymbol{Z}_2$ case: it vanishes also in the integer case, as it is not difficult to verify that the singularities at the 2 ends of an arc have opposite signs. Thus we may suppose f an immersion:

then $f(D^{k+1})$ has trivial normal bundle, and $f(\partial D^{k+1})$ also has. Thus by (3.3) we can perform surgery to kill elements of $\mathrm{Im}\, b$, and the result follows. The appeal to (3.3) fails if $r=3, 7$. However, the existence of the immersed disc shows directly that the obstruction there obtained in $\pi_k(V_{N+k-1,N-1})$ vanishes. For this obstruction is identified (by suspension) with the obstruction in $\pi_k(V_{2k+1,k+1})$ to immersing the disc. Geometrically, if the disc is immersed in M^v, $\tau_M \mid D^{k+1}$ gives a $(2k+1)$-frame, and the normal vectors a complementary N-frame. So the isomorphic image of the obstruction in $\pi_k(V_{N+k,N})$ vanishes.

Now we let $v=2k$, $k \geqq 3$, $N>k$. By (4.1), there is a manifold ∂M realising the boundary C.I.S.; this bounds some M, and by (3.6) we can kill all K's except $K_k(M) \cong K_k(M, \partial M) \cong K^k(M)$. If k is even, this is a free abelian group, with nonsingular even quadratic form (c.f. [8], Lemma 7 and Remark), hence has signature divisible by 8. If k is odd, then by the method of [6, § 8] we can define an Arf invariant c of $K_k(M)$. According to [16], there exists an almost-closed $(k-1)$-connected π-manifold H of dimension $2k$, with signature minus that of $K_k(M)$ when k is even; with Arf invariant c when k is odd, moreover, H is a handlebody. $H \times I$, with corners rounded, is then also a handlebody, so by the classification ([17] Theorem 2) is diffeomorphic to a sum of copies of $S^k \times D^{k+1}$. So it imbeds in $\boldsymbol{R}^{2k+1}$, and H imbeds in this, so also in S^{N+2k-1}, with trivial normal bundle (this result is due to Milnor). Take such an imbedding, and deform the interior of H into that of D^{N+2k}. Now form the (boundary-connected) sum of (D^{N+2k}, M) with (D^{N+2k}, H) to obtain a pair (D^{N+2k}, N) with normal bundle η'. Since the normal bundle of H in D^{N+2k} is trivial, we can extend $f\colon M^\eta \to X^\xi$ to a map of $N^{\eta'}$, using the framing of H to map the second part to S^N, the Thom space of the base point $*$ in ∂X. Clearly the new map of D^{N+2k} is still in the homotopy class $\alpha \in \pi_{N+2k}(X^\xi, \partial X^\xi)$, and it is t-regular on X, with inverse image N. However, as H is mapped to $*$, $K_k(N) = K_k(M) \oplus H_k(H)$, so by construction it has signature (resp. Arf invariant) zero. The proof is concluded as before by appealing to Lemma 7.3 or 8.4 of [6].

Now we consider uniqueness. Let $(D^{N+v}, V_1{}^v)$ and $(D^{N+v}, V_2{}^v)$ be two pairs, realising the given I.S. with maps $f_i\colon (D^{N+v}, \partial D^{N+v}) \to (X^\xi, \partial X^\xi)$. Take a homotopy F of f_1 to f_2, and make it also t-regular on X (keeping the ends fixed), so that $F^{-1}(X) = W^{v+1}$ is a smooth manifold with corner, and boundary in 3 parts—V_1, V_2, and $\partial_c W = W \cap (\partial D^{N+v} \times I)$. We now perform surgery on W as above.

First consider a C.I.S. If $v=2k$, $N-1>k\geqq 2$, then as in (4.1) we can perform surgery to make W, too, map by a homotopy equivalence. We then have an h-cobordism of pairs (S^{N+v}, V^v); by a result of Smale [12], since $v \geqq 5$, $N \geqq 3$, this is diffeomorphic to a product, so the pairs are diffeomorphic. If $v = 2k+1$, there is the usual obstruction to performing the surgery, and we find

THEOREM 1C. (Novikov [10]). *With the assumptions of* 1A, *and if* $2N > v+3$, *the corresponding manifold is unique up to diffeomorphism if* v *is even, and to connected sum with a pair* (S^{N+v}, Σ^v) *(where* Σ^v *bounds a* π*-manifold) if* v *is odd.*

We return to the uniqueness part of Theorem 2. If v is odd we perform surgery on $\partial_c W$ to obtain an h-cobordism as above. If v is even, we again avoid the obstruction by taking a boundary-connected sum with a pair (D^{N+v+1}, H), as in the existence part of the proof. So we may suppose $\partial_c W$ an h-cobordism. Now if v is even, we can perform surgery on W (leaving ∂W fixed) to obtain a homotopy equivalence. If v is odd, there appears to be an obstruction, but the same proof as above shows that it vanishes, so we can make F a homotopy equivalence then also. We now apply Smale's theorem twice, first to see that $(S^{N+v-1} \times I, \partial_c W)$ is diffeomorphic to a product, so that a neighbourhood of it also is, and now again to $W - \partial_c W$, using the product structure just obtained in the complement of a compact set, to say that it extends over W. Thus $(D^{N+v} \times I, W)$ is diffeomorphic to the product $(D^{N+v}, V_1{}^v) \times I$, and the two pairs $(D^{N+v}, V_1{}^v)$ and $(D^{N+v}, V_2{}^v)$ are diffeomorphic, as asserted.

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