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ALGEBRAIC BORDISM GROUPS

BY EDGAR H. BROWN, JR. AND FRANKLIN P. PETERSON*

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1. Introduction

In [7], Thom defined the unoriented cobordism groups N_n. The elements of these groups are equivalence classes of n-dimensional, compact, closed, C^∞-manifolds, two manifolds being equivalent if their disjoint union is the boundary of a compact C^∞-manifold. In this paper we define groups ${}^aN_n$ which are an algebraic analog of the groups N_n. The elements of ${}^aN_n$ will be equivalence classes of algebras having properties satisfied by the mod two cohomology algebra of a compact closed n-manifold, namely, algebras which satisfy Poincaré duality, are of finite type, and are unstable (see § 2) left algebras over the mod two Steenrod algebra. The equivalence relation will be obtained from the conditions fulfilled when a manifold is a boundary, e.g., Lefschetz duality. By analogy with the Wu formulas, we define Stiefel-Whitney classes of such algebras and prove the algebraic analog of Thom's theorem that a manifold is a boundary if and only if all its Stiefel-Whitney numbers vanish.

Atiyah [2] and Conner and Floyd [5] have defined bordism groups $N_*(K)$ for a space K in such a way that $N_n = N_n(\text{pt.})$. In § 3 we define algebraic bordism groups ${}^aN_*(X)$, where X is an unstable left algebra over the mod two Steenrod algebra in such a way that ${}^aN_n = {}^aN_n(Z_2)$. Using the ideas developed in [4], we show that there is a natural isomorphism $\varphi\colon N_*(K) \to {}^aN_*(H^*(K))$, and hence, in particular, that N_n and ${}^aN_n$ are isomorphic. Finally we prove that the functors $({}^aN_*(X))^*$ and $H^*(BO) \otimes_A X$, defined on the category of unstable left algebras over the Steenrod algebra A, are equivalent.

2. H bordism to zero

Throughout this paper all algebras and modules will be graded, will be of finite type, and will have Z_2 as ground field. If X is a module or an algebra, X^* will denote $\mathrm{Hom}(X, Z_2)$. All cohomology groups will have Z_2 coefficients. A will denote the mod two Steenrod algebra.

In [6] a left A module X is defined to be unstable if $\mathrm{Sq}^i x = 0$ for $x \in X^j$ and $j < i$. X is defined to be a left algebra over the Hopf algebra A if

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X is a commutative algebra and a left A module such that the Cartan formulas hold. We define X to be an unstable left algebra over A if it is an unstable left A module, a left algebra over the Hopf algebra A, has a unit, and if $\mathrm{Sq}^i x = x^2$ if $x \in X^i$.

Let H be an n-dimensional Poincaré algebra [1], [3], i.e., H is an unstable left algebra over A together with an element $\mu_H \in (H^n)^*$ such that $\mu_H(hh')$, $h \in H^q$, $h' \in H^{n-q}$, defines a non-singular pairing of H^q and H^{n-q} for all q. In [1] H is made into a right A module by the formula:

$$\mu_H(ha \cdot h') = \mu_H(h \cdot ah') ,$$

where $a \in A$, h, $h' \in H$ and $\dim a + \dim h + \dim h' = n$. In [4] it is shown that an unstable algebra X over A may be made into a right A module by the formulas:

$$h\chi(\mathrm{Sq}^i) = \sum \bar{w}_j \cdot \mathrm{Sq}^{i-j} h ,$$

where χ is the canonical anti-automorphism of A, and $\bar{w}_j \in X^j$ are any collection of elements satisfying the Wu formulas giving $\mathrm{Sq}^i \bar{w}_j$ as polynomials in the $\bar{w}_j$'s. X will be called a right-left A algebra if A acts on right and left of X as above. A Poincaré algebra is a right-left algebra if one takes $\bar{w}_j = (1)\chi(\mathrm{Sq}^j)$. $H^*(BO)$ is a right-left algebra if one takes $\bar{w}_j = \bar{w}_j$. In [4] it is shown that, if X is a right-left A algebra, there is a unique right-left algebra homomorphism $\tau_X\colon H^*(BO) \to X$.

An n-dimensional Poincaré algebra H is defined to be the boundary of (H', H''), written $H = \partial(H', H'')$, if the following conditions are satisfied: H' is an unstable left algebra over A, and H'' is an unstable left A module and an H' module such that

$$\mathrm{Sq}^i(h'h'') = \sum (\mathrm{Sq}^j h')(\mathrm{Sq}^{i-j} h'')$$

for $h' \in H'$ and $h'' \in H''$. There are left A homomorphisms $j\colon H'' \to H'$, $j\colon H' \to H$ and $\delta\colon H \to H''$ of degrees 0, 0 and 1, respectively, such that

$$\begin{tikzcd} H' \arrow[rr, "i"] & & H \arrow[dl, "\delta"] \\ & H'' \arrow[ul, "j"] & \end{tikzcd}$$

is exact. Furthermore, i is an algebra homomorphism, $j(h'h'') = h'j(h'')$ and $\delta(h \cdot jh') = \delta h \cdot h'$. Finally, there is an element $\mu_{H''} \in (H''^{n+1})^*$ such that $\mu_H = \mu_{H''}\delta$, and $\mu_{H''}(h'h'')$ defines a dual pairing between H' and H''.

If $H = \partial(H', H'')$ as above, we make H' into a right A module as follows: If $a \in A_i$ and $h' \in H'^j$,

$$\mu_{H''}(h'a \cdot h'') = \mu_{H''}(h' \cdot ah'')$$

for all $h'' \in H''^{n+1-i-j}$.

LEMMA 2.1. *H' is a right-left A module with $\bar{w}_i = (1)\chi(\mathrm{Sq}^i)$, and $i\colon H' \to H$ is a right-left A homomorphism.*

PROOF. As in [4] one may show by induction on i that

$$h'\chi(\mathrm{Sq}^i) = \sum \bar{w}_j \cdot \mathrm{Sq}^{i-j} h' ,$$

where $\bar{w}_i = (1)\chi(\mathrm{Sq}^i)$. As is proved in [4], the fact that A acts on the right of H'' implies the $\bar{w}_i$'s satisfy the Wu formulas. By definition, i is a left A homomorphism. Let $a \in A_i$, $h' \in H'^j$ and $h \in H^{n-i-j}$

$$\begin{aligned}
\mu_H\big(i(h')a \cdot h\big) &= \mu_{H''}\delta\big(i(h')a \cdot h\big) \\
&= \mu_{H''}(h' \cdot \delta a h) \\
&= \mu_{H''}(h'a \cdot \delta h) \\
&= \mu_{H''}\delta\big(i(h'a) \cdot h\big) \\
&= \mu_H\big(i(h'a) \cdot h\big) .
\end{aligned}$$

The above is true for all h, hence $i(h')a = i(h'a)$.

In § 4 we will prove the following theorem:

THEOREM 2.2. *If H is a Poincaré algebra, $H = \partial(H', H'')$ if and only if $\mu_H \tau_H = 0$, where $\tau_H\colon H^*(BO) \to H$ is the unique right-left A homomorphism.*

Let H_1 and H_2 be n-dimensional Poincaré algebras. One may easily check that $H = H_1 + H_2$ (direct sum) becomes a Poincaré algebra if μ_H is defined by $\mu_H(h_1, h_2) = \mu_{H_1}(h_1) + \mu_{H_2}(h_2)$. The following is immediate.

LEMMA 2.3. $\tau_{H_1+H_2} = \tau_{H_1} + \tau_{H_2}$.

Define H_1 and H_2 to be cobordant, $H_1 \sim H_2$, if $H_1 + H_2 = \partial(H', H'')$. Theorem 2.2 and Lemma 2.3 then yield:

THEOREM 2.4. *H_1 and H_2 are cobordant if and only if $\mu_{H_1}\tau_{H_1} = \mu_{H_2}\tau_{H_2}$.*

Theorem 2.4 shows that $\sim$ is an equivalence relation on the set of n-dimensional Poincaré-algebras. Let ${}^aN_n$ be the set of equivalence classes. ${}^aN_n$ becomes a Z_2 vector space under the addition defined by $[H_1] + [H_2] = [H_1 + H_2]$. In analogy with the geometric case, one may make ${}^aN = \sum {}^aN_n$ into a ring by defining $[H_1] \cdot [H_2] = [H_1 \otimes H_2]$. The details are left to the reader as we will not need this fact.

3. Bordism groups

Let X be an unstable left algebra over A. Let H be an n-dimensional Poincaré algebra and let $f\colon X \to H$ be an algebra and a left A homomorphism. Define $(H, f) \sim 0$ if $H = \partial(H', H'')$ and if there exists a homomorphism $f'\colon X \to H'$ such that $if' = f$. In § 4 we prove the following theorem which generalizes Theorem 2.1 and is the analog of the unoriented version of [5, Th. 1.11].

THEOREM 3.1. *$(H, f) \sim 0$ if and only if, for every $x \in X^i$ and $u \in H^{n-i}(BO)$, $\mu_H(f(x) \cdot \tau_H(u)) = 0$.*

Define $(H_1, f_1) \sim (H_2, f_2)$ if $(H_1 + H_2, f_1 + f_2) \sim 0$. By 3.1 this is an equivalence relation, and we define ${}^aN_n(X)$ to be the set of equivalence classes. Define ${}^aN_*(X) = \sum {}^aN_n(X)$. ${}^aN_*(X)$ is then a contravariant functor from the category of unstable left algebras over A to the category of graded vector spaces over Z_2.

Let K be a CW-complex with a finite number of cells in each dimension. Let $N_*(K)$ denote the differentiable cobordism groups of K as defined in [5]. Define a natural transformation of functors $\varphi(K)\colon N_*(K) \to {}^aN_*(H^*(K))$ by $\varphi(K)(M, f) = (H^*(M), f^*)$, where M is a closed, compact C^∞ manifold and $f\colon M \to X$.

There is constructed in [4] a natural equivalence

$$\theta(K)\colon H^*(BO) \otimes_A H^*(K) \to N_*(K)^* .$$

We next generalize this to the case of algebras.

Let X be an unstable left algebra over A, and let

$$\theta(X)\colon H^*(BO) \otimes_A X \to ({}^aN_*(X))^*$$

be the map of degree zero defined by $\theta(X)(h \otimes x)(H, f) = \mu_H(\tau_H(h) \cdot f(x))$. By Theorem 3.1, $\theta(X)$ is well defined and is an epimorphism. In § 5 we prove the following theorem.

THEOREM 3.2. *θ is a natural equivalence of covariant functors defined on the category of unstable left algebras over A.*

COROLLARY 3.3. *$\varphi(K)\colon N_*(K) \to {}^aN_*(H^*(K))$ is a natural equivalence.*

COROLLARY 3.4. *$\varphi(\text{pt.})\colon N_*(\text{pt.}) \to {}^aN_*(H^*(\text{pt.})) = {}^aN$ is an isomorphism.*

PROOF OF 3.3. Consider the commutative diagram:

$$\begin{tikzcd}
H^*(BO) \otimes_A H^*(K) \arrow[rr, "\theta(K)"] \arrow[dr, "\theta(H^*(K))"'] & & N_*(K)^* \\
& ({}^aN_*(H^*(K)))^* \arrow[ur, "\varphi(K)^*"'] &
\end{tikzcd} .$$

Since $\theta(K)$ and $\theta(H^*(K))$ are isomorphisms, so is $\varphi(K)^*$, and hence $\varphi(K)$ is.

4. Proof of Theorems 2.2 and 3.1

Since 2.2 is a corollary of 3.1 obtained by setting $X = H^*(\text{pt.})$, we prove only the latter.

Suppose $(H, f) \sim 0$. Since i is a left-right A homomorphism, $i\tau_{H'} = \tau_H$. If $x \in X^i$ and $u \in H^{n-i}(BO)$,

$$\begin{aligned} \mu_H\big(f(x) \cdot \tau_H(u)\big) &= \mu_H i\big(f'(x) \cdot \tau_{H'}(u)\big) \\ &= \mu_{H''}\delta i\big(f'(x) \cdot \tau_{H'}(u)\big) \\ &= 0 , \end{aligned}$$

since $\delta i = 0$.

Conversely, suppose (H, f) is such that $\mu_H(f(x) \cdot \tau_H(u)) = 0$ for all $x \in X^i$ and $u \in H^{n-i}(BO)$. If $Y \subset H$, $\{Y\}$ will denote the subalgebra of H generated by Y and closed under the left action of A. Note that $\mu_H\{\operatorname{Im} f, \operatorname{Im} \tau_H\} = 0$ by assumption, as both $\operatorname{Im} f$ and $\operatorname{Im} \tau_H$ are subalgebras with unit closed under left operations of A. Let H' be the maximal subalgebra of H closed under the left action of A such that $H' \supset \{\operatorname{Im} f, \operatorname{Im} \tau_H\}$ and $\mu_H H' = 0$. This means that, if $h \notin H'$, $\mu_H\{h, H'\} \neq 0$. Let

$$(H'')^j = H^{j-1}/(H')^{j-1} .$$

Let $i\colon H' \to H$ be the inclusion map, $j\colon H'' \to H'$ be the zero map, and $\delta\colon H \to H''$ be the projection.

Since $\mu_H(H') = 0$, μ_H induces a map $\mu_{H''} \in (H''^{n+1})^*$ such that $\mu_H = \mu_{H''}\delta$. Let H' act on H'' by $h' \cdot [h] = [h'h]$ where $[h] = \delta h$. Since H' is closed under left action by A, H'' inherits a left A module structure. All the conditions necessary for $H = \partial(H', H'')$ are immediate except Lefschetz duality, which we prove below. Note that since $\operatorname{Im} f \subset H'$, there is a map $f'\colon X \to H'$ such that $if' = f$, and hence $(H, f) \sim 0$.

We now prove that H', H'' satisfy Lefschetz duality. Suppose $\mu_{H''}(h'h'') = 0$ for $h' \in H'$ and all $h'' \in H''$. Then $\mu_H(h'h) = \mu_{H''}(h'[h]) = 0$ for all $h \in H$. Therefore by Poincaré duality in H, $h' = 0$.

Conversely, suppose $\mu_{H''}(h'h'') = 0$ for $h'' \in H''$ and all $h' \in H'$. Let $h'' = [h]$. Then $\mu_H(h'h) = 0$ for all $h' \in H'$, and we wish to show that $h \in H'$. Assume $h \notin H'$. Then $\mu_H\{h, H'\} \neq 0$. We first show that $\mu_H(\mathrm{Sq}^I(h) \cdot H') = 0$ by induction on $n(I)$ (see [6]). If $n(I) = 0$, this is true by assumption. Suppose $h \in H^i$ and $h' \in H'^{n-i-n(I)}$.

$$\mathrm{Sq}^I(h) \cdot h' = \mathrm{Sq}^I(h \cdot h') + \textstyle\sum_{n(I') < n(I)} \mathrm{Sq}^{I'}(h)\, \mathrm{Sq}^{I''}(h') .$$

Also, $\mathrm{Sq}^I(hh') = (1)\mathrm{Sq}^I \cdot hh'$ and $(1)\mathrm{Sq}^I \in \operatorname{Im} \tau_H \subset H'$. Thus by inductive hypothesis, $\mu_H(\mathrm{Sq}^I(h) \cdot h') = 0$. $\{h, H'\}$ consists of sums of products of elements in H' and elements of the form $\theta(h)$ where $\theta \in H^*(Z_2, i)$. Since

$\mu_H\{h, H'\} \neq 0$, $\mu_H(\theta(h)h') \neq 0$ for some h' and θ. θ is a linear combination Sq^I's if $\dim h \geqq n/2$. Therefore $\dim h < n/2$. But, by Poincaré duality, there is a $g \in H$ such that $\mu_H(g \cdot H') = 0$ and $\mu_H(g \cdot h) \neq 0$. Since $\mu_H(g \cdot h) \neq 0$, $g \notin H'$. Applying the above argument to g, $\dim g < n/2$. But this is a contradiction since $\dim g + \dim h = n$.

5. Proof of Theorem 3.2

Let $\psi\colon H^*(BO) \otimes X \to H^*(BO) \otimes_A X$ be the natural map. $H^*(BO) \otimes X$ is a right-left algebra over A if we take $\bar{w}_i = \bar{w}_i \otimes 1$ (see § 2). Let $F(X) = (H^*(BO) \otimes X)\bar{A}$ where $\bar{A}$ is the set of elements of A of positive degree. We first note that $F(X) \subset \operatorname{Ker} \psi$. In [4] it is shown that for any right-left A algebra X, if $x, y \in X$, then $(x \cdot y)\,\mathrm{Sq}^i = \sum (x)\mathrm{Sq}^j \cdot \chi(\mathrm{Sq}^{i-j})(y)$. Hence

$$\begin{aligned}\psi((u \otimes x)\mathrm{Sq}^i) &= \psi\left(\sum (u)\mathrm{Sq}^j \otimes \chi(\mathrm{Sq}^{i-j})(x)\right) = \sum (u)\mathrm{Sq}^j \otimes_A \chi(\mathrm{Sq}^{i-j})(x) \\ &= \sum u \otimes_A \mathrm{Sq}^j\, \chi(\mathrm{Sq}^{i-j})(x) = 0\end{aligned}$$

because $\sum \mathrm{Sq}^j\, \chi(\mathrm{Sq}^{i-j}) = 0$ if $i > 0$. Next suppose $u \notin F(X)^n$. Then we will construct an (H, f) such that $\theta\psi(u)(H, f) \neq 0$. This will prove that $\theta(X)$ is a monomorphism and conclude the proof of 3.2. Let $U = H^*(BO) \otimes X$, and let $z \in U^{n*}$ be such that $z(F(X)) = 0$ and $z(u) \neq 0$. Let $J = \{y \in U^k \mid z(y \cdot U^{n-k}) = 0\}$. J is clearly an ideal. We show that, if $y \in J$, $\chi(\mathrm{Sq}^i)(y) \in J$ by induction on i. For $i = 0$ this is true by assumption. Let $k = \dim y$ and let $v \in U^{n-k-i}$. As noted above,

$$v \cdot \chi(\mathrm{Sq}^i)(y) = (v \cdot y)\mathrm{Sq}^i + \sum_{j>0} (v)\mathrm{Sq}^j \cdot \chi(\mathrm{Sq}^{i-j})(y)\ .$$

Hence by the inductive hypothesis and the fact that $z(F(X)) = 0$, $z(\chi(\mathrm{Sq}^i)(y)) = 0$. Let $H = U/J$ and let $\eta\colon U \to H$ be the projection. Let $f\colon X \to H$ be given by $f(x) = \eta(1 \otimes x)$ and let $\mu_H \in (H^n)^*$ be defined by $\mu_H \eta = z$. Then H is an unstable left algebra over A because J is an ideal closed under left operations by A, and H satisfies Poincaré duality by construction. Note that $\theta\psi(v \otimes x)(H, f) = \mu_H(\tau_H(v) \cdot f(x)) = z(v \otimes x)$. Hence $\theta\psi(u)(H, f) = z(u) \neq 0$.

BRANDEIS UNIVERSITY AND INSTITUTE FOR ADVANCED STUDY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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