AXIOMS FOR THE GENERALIZED PONTRYAGIN COHOMOLOGY OPERATIONS

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1. The axioms

FOR any space X and any abelian group G denote by $H^*(X; G)$ the singular cohomology groups of X with coefficients in the group G. Let Z denote the group of the integers and Z_q $(q \geqslant 2)$ the integers mod q. Let p be an odd prime number and r a fixed positive integer. Denote by ϕ and η the canonical homomorphisms

$$Z_{p^r} \xrightarrow{\phi} Z_{p^{r+1}}, \quad Z_{p^{r+1}} \xrightarrow{\eta} Z_{p^r},$$

given respectively by the inclusion and the factor map.

Let n be a fixed, positive integer and consider cohomology operations C that satisfy the following axioms:†

$$C\colon H^{2n}(X; Z_{p^r}) \to H^{2pn}(X; Z_{p^{r+1}}); \tag{1.1}$$

$$\eta C(u) = u^p \quad (p\text{-fold cup-product}); \tag{1.2}$$

$$C\eta(v) = v^p; \tag{1.3}$$

$\sigma C = 0$, *where σ denotes the suspension of cohomology operations*; (1.4)

$$C(u_1+u_2) = C(u_1)+C(u_2)+\phi\Big[\sum_{0<i<p} \frac{1}{p}\binom{p}{i} u_1^i \cup u_2^{p-i}\Big], \tag{1.5}$$

where $\binom{p}{i}$ *denotes the binomial coefficient* $p!/i!(p-i)!$.

Here X is any space; u, u_1, $u_2 \in H^{2n}(X; Z_{p^r})$; and $v \in H^{2n}(X; Z_{p^{r+1}})$. We prove the theorem:

THEOREM 1. *There exists a cohomology operation C satisfying axioms* (1.1)–(1.5). *Furthermore, the operation C is unique.*

The existence of such an operation is given by the operation $\mathfrak{P}_p$, defined in (5). Axiom (1.4) follows from Theorem I in (6). We prove the uniqueness of the operation $\mathfrak{P}_p$ in § 3.

† If α is any coefficient group homomorphism, we denote by the same symbol the induced cohomology operation.

2. Cohomology of Eilenberg-MacLane spaces

Let p be an odd prime number and m, r positive integers. We state some facts about the cohomology ring $H^*(Z_{p^r}, m; Z_p)$. The proofs are given in the work of H. Cartan (**2**).

Let $u \in H^m(Z_{p^r}, m; Z_p)$ denote the (mod p)-reduction of the fundamental class $\tilde{u} \in H^m(Z_{p^r}, m; Z_{p^r})$. Set

$$v = \beta_r(\tilde{u}) \in H^{m+1}(Z_{p^r}, m; Z_p),$$

where β_r is the Bockstein coboundary associated with the exact coefficient sequence

$$0 \to Z_p \to Z_{p^{r+1}} \to Z_{p^r} \to 0.$$

Then $H^*(Z_{p^r}, m; Z_p)$ is a tensor product of polynomial algebras and exterior algebras whose generators are obtained from u and v by applying certain canonical compositions of the Steenrod operations $\mathscr{P}^i$ $(i \geqslant 1)$ together with the Bockstein coboundary $\beta = \beta_1$ (going from coefficients Z_p to Z_p): that is, each generator may be written

$$C_q \circ C_{q-1} \circ \dots \circ C_2 \circ C_1(w) \quad (w = u \text{ or } v;\ q \geqslant 1),$$

where each operation C_i is either a certain Steenrod operation $\mathscr{P}^{j_i}$ or is β. We divide the generators into three types:

Type (1): the terminal operation C_q is a Steenrod operation $\mathscr{P}^{j_q}$.

Type (2): the terminal operation C_q is the Bockstein coboundary β.

Type (3): u and v.

If we apply β to a generator of Type (1), we obtain a generator of Type (2). Denote by V_i $(i = 1, 2, 3)$ the linear subspace of $H^*(Z_{p^r}, m, Z_p)$ spanned by the generators of Type (i). Then,

$$\beta\colon V_1 \approx V_2. \tag{2.1}$$

Define P to be the ideal generated by the decomposable elements. Then

$$H^*(Z_{p^r}, m; Z_p) = P \oplus V_1 \oplus V_2 \oplus V_3, \tag{2.2}$$

as a (mod p)-vector space. Suppose now that $m > 1$. Again denote by σ the suspension of cohomology operations, thought of here as a homomorphism (of degree -1) from $H^*(Z_{p^r}, m)$ to $H^*(Z_{p^r}, m-1)$. In the splitting (2.2) we have

$$\sigma(P) = 0; \quad \sigma \mid (V_1 \oplus V_2 \oplus V_3) \textit{ is a monomorphism.} \tag{2.3}$$

Suppose now that $m = 2n$. We shall need the following lemma about the (mod p^2) cohomology of $K(Z_{p^r}, 2n)$:

[2.4] LEMMA. *$H^{2pn}(Z_{p^r}, 2n; Z_{p^2}) \approx Z_{p^2} \oplus M$ (group direct sum), where $pM = 0$. Furthermore, a generator w can be chosen for the summand Z_{p^2} such that $\zeta(w) = u^p$, where ζ is induced by the factor homomorphism $Z_{p^2} \to Z_p$.*

This also is contained in (**2**). Another proof can be given by combining Theorem 5.5 and Proposition 1.2 in (**1**).

3. Proof of uniqueness

Suppose that C and C' are two cohomology operations satisfying axioms (1.1)–(1.5). Set $D = C - C'$. The proof of Theorem 1 is completed when we show that $D \equiv 0$. The operation D has the following properties:

$$D\colon H^{2n}(X; Z_{p^r}) \to H^{2pn}(X; Z_{p^{r+1}}); \tag{3.1}$$

$$\eta \circ D = 0; \tag{3.2}$$

$$D \circ \eta = 0; \tag{3.3}$$

$$\sigma D = 0; \tag{3.4}$$

$$D(u_1 + u_2) = D(u_1) + D(u_2). \tag{3.5}$$

To prove Theorem 1 it suffices to show that $D(\bar{u}) = 0$, where $\bar{u}$ is the fundamental class of $K(Z_{p^r}, 2n)$. Consider the following exact sequence of coefficient groups.

$$0 \to Z_p \xrightarrow{\theta} Z_{p^{r+1}} \xrightarrow{\eta} Z_{p^r} \to 0, \tag{3.6}$$

where θ is the inclusion. By (3.2) and the exactness of (3.6), there is a cohomology class $x \in H^{2pn}(Z_{p^r}, 2n; Z_p)$ such that

$$D(\bar{u}) = \theta(x). \tag{3.7}$$

Using the splitting given in (2.2) we may set†

$$x = y + v_1 + v_2, \tag{3.8}$$

where $y \in P$ and $v_i \in V_i$ $(i = 1, 2)$. By (2.1), $V_2 = \beta V_1$ and therefore $\theta V_2 = \theta\beta V_1 = 0$. Thus, in view of (3.7), we may assume without loss of generality that $v_2 = 0$ in (3.8). Now by (2.3), $\sigma(y) = 0$. Since $\theta\sigma = \sigma\theta$, we obtain from (3.4) that

$$\theta\sigma(v_1) = \theta\sigma(x) = \sigma\theta(x) = \sigma D(\bar{u}) = 0.$$

Therefore, from (3.6),
$$\sigma v_1 = \beta_r(z_1)$$
for some element $z_1 \in H^{2pn-2}(Z_{p^r}, 2n-1; Z_{p^r})$. Since $\beta \circ \beta_r = 0$ and since $\sigma\beta = \beta\sigma$, we obtain that

$$\sigma\beta v_1 = 0.$$

† Since $2pn > 2n+1$, the component of x in V_3 is zero.

Therefore, by (2.3) and (2.1), $v_1 = 0$: that is,

$$D(\bar{u}) = \theta(y) \quad (y \in P).$$

Set $K = K(Z_{p^r}, 2n)$ and recall that K may be taken to be a group. Denote by μ and π_i $(i = 1, 2)$ the maps from $K \times K$ to K given respectively by the multiplication and the projection on the ith factor. Set

$$\psi = \mu^* - \pi_1^* - \pi_2^*: \quad H^*(K; G) \to H^*(K \times K; G),$$

where G is any coefficient group and μ^*, π_i^* denote the cohomology homomorphisms induced by the maps μ and π_i $(i = 1, 2)$. By definition an element $w \in H^*(K; G)$ is primitive if $\psi(w) = 0$. Therefore, if E denotes a cohomology operation defined on $\bar{u}$, then by Steenrod [(**4**) 6.7], E is additive if and only if $\psi E(\bar{u}) = 0$. Now ψ commutes with operations induced by coefficient group homomorphisms, and therefore by (3.5),

$$\theta\psi(y) = \psi\theta(y) = \psi D(\bar{u}) = 0.$$

Consequently, by the exactness of (3.6),

$$\psi(y) = \beta_r(z)$$

for some element $z \in H^{2pn-1}(K \times K; Z_{p^r})$. Again, since $\beta \circ \beta_r = 0$ and since $\psi\beta = \beta\psi$, we obtain that

$$\psi(\beta y) = 0.$$

Thus, βy is primitive. But $\beta y \in P$ since $y \in P$ and β is a derivation. Hence, by Proposition 4.23 of (**3**), $\beta y = 0$ since $\dim \beta y$ is odd.

Now consider the exact coefficient sequence

$$0 \to Z_p \xrightarrow{\alpha} Z_{p^2} \xrightarrow{\zeta} Z_p \to 0, \tag{3.9}$$

where α is the inclusion and ζ is the factor map. Since β is the Bockstein coboundary associated with (3.9) and since $\beta y = 0$, there is a class $Y \in H^{2pn}(K; Z_{p^2})$ such that $y = \zeta(Y)$. By (2.4) we may write

$$Y = aw + z \quad (a \in Z_p),$$

where $\zeta(w) = u^p$ and $pz = 0$. One can easily show that there are classes $s \in H^{2pn}(K; Z_p)$ and $t \in H^{2pn-1}(K; Z_p)$ such that

$$z = \alpha(s) + \delta(t),$$

where δ is the Bockstein coboundary from coefficients Z_p to Z_{p^2}. Therefore

$$y = \zeta(Y) = \zeta(aw + \alpha(s) + \delta(t)) = au^p + \beta(t)$$

since $\zeta\alpha = 0$ and $\zeta\delta = \beta$. Now $\theta\beta = 0$ and hence, if we set

$$\hat{y} = y - \beta(t) = au^p, \tag{3.10}$$

we continue to have $\theta(\hat{y}) = \theta(y) = D(\bar{u})$.

Thus the proof of Theorem 1 is completed when we show that $a = 0$.

To do this we use property (3.3). Let ι denote the fundamental class of $H^{2n}(Z, 2n; Z)$, and set $\iota_s = j_s(\iota)$, where $j_s\colon Z \to Z_{p^s}$ ($s \geqslant 1$). Then, by (3.3),

$$D(\iota_r) = D\eta(\iota_{r+1}) = 0.$$

Let ρ denote the homomorphism $Z_{p^{r+1}} \to Z_p$. Then, by (3.10),

$$\hat{y}(\iota_r) = a\iota_1^p = a\rho(\iota_{r+1})^p,$$

since ρ is a multiplicative homomorphism. Now $\theta\rho(x) = p^r(x)$ for any class $x \in H^*(Z, 2n; Z_{p^{r+1}})$, and consequently

$$0 = D(\iota_r) = \theta\hat{y}(\iota_r) = \theta(a\rho\iota_{r+1}^p) = ap^r\iota_{r+1}^p.$$

But ι_{r+1}^p has order p^{r+1}, as will be shown in a moment. Thus $a = 0$, which completes the proof of Theorem 1.

Denote by X the infinite-dimensional complex projective space. Then, $H^*(X; Z)$ is a polynomial ring on a 2-dimensional generator x. Let f be a map from X to $K(Z, 2n)$ such that $f^*\iota = x^n$, where f^* is the homomorphism induced by f. By naturality,

$$f^*\iota_{r+1}^p = j_{r+1}f^*(\iota^p) = j_{r+1}(x^{pn}).$$

Since $j_{r+1}(x^{pn})$ has order p^{r+1}, the same is then true of ι_{r+1}^p, as asserted.

4. The Pontryagin square

The axioms given in § 1 were relative to an arbitrary *odd* prime p. There is a corresponding set of axioms for the prime 2: that is, consider cohomology operations C which satisfy axioms (1.1)–(1.3) (with $p = 2$), together with the axiom

$$\sigma C = \mathfrak{p}, \text{ the Postnikov square.} \tag{4.1}$$

The Postnikov square $\mathfrak{p}\colon H^{2n}(X; Z_{2^r}) \to H^{4n+1}(X; Z_{2^{r+1}})$, is completely characterized by

$$\mathfrak{p}(u) = \phi(u \cup \delta(u)) \qquad (u \in H^{2n}(X; Z_{2^r})),$$

where ϕ is induced by the inclusion $Z_{2^r} \subset Z_{2^{r+1}}$ and δ is the Bockstein coboundary from coefficients Z_{2^r} to Z_{2^r}. One then has the theorem:

Theorem 2. *There exists a cohomology operation satisfying axioms* (1.1)–(1.3) (*with* $p = 2$), *together with axiom* (4.1). *Furthermore, this operation is unique.*

The operation is the Pontryagin square. Notice that it is not necessary to include the analogue of axiom (1.5). This is because the only decomposable element in $H^{4n}(Z_{2^r}, 2n; Z_2)$ is u^2, where u is the (mod 2) reduction of the fundamental class. The proof of Theorem 2 is quite similar to that of Theorem 1, and we leave the details to the reader.

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