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Quadratic K-Theory and Geometric Topology

II

Bruce Williams*

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Introduction

Suppose R is a ring with an (anti)-involution $-\colon R \to R$, and with choice of central unit ε such that $\bar{\varepsilon}\varepsilon = 1$. Then one can ask for a computation of $\mathbb{K}Quad(R,-,\varepsilon)$, the K-theory of quadratic forms. Let $H\colon \mathbb{K}R \to \mathbb{K}Quad(R,-,\varepsilon)$ be the hyperbolic map, and let $F\colon \mathbb{K}Quad(R,-,\varepsilon) \to \mathbb{K}R$ be the forget map. Then the Witt groups

$$W_0(R,-,\varepsilon) = coker\left(K_0R \xrightarrow{H} K_0Quad(R,-,\varepsilon)\right)$$

$$W_1(R,-,\varepsilon) = ker\left(F\colon KQuad_1(R,-,\varepsilon) \xrightarrow{F} K_1R\right)$$

have been highly studied. See [6], [29,32,42,46,68], and [86]–[89]. However, the higher dimensional quadratic K-theory has received considerably less attention, than the higher K-theory of f.g. projective modules. (See however, [34,35,39], and [36].)

Suppose M is an oriented, closed topological manifold of dimension n. We let

$$G(M) = \text{simplicial monoid of homotopy automorphisms of } M\,,$$

$$Top(M) = \text{sub-simplicial monoid of self-homeomorphisms of } M\,, \text{and}$$

$$\mathcal{S}(M) = \sqcup\, G(N)/Top(N)\,,$$

where we take the disjoint union over homeomorphisms classes of manifolds homotopy equivalent to M. Then $\mathcal{S}(M)$ is called the *moduli space* of manifold structures on M.

In classical surgery theory (see Sect. 3.3) certain subquotients of $K_jQuad(R,-,\varepsilon)$ with $j = 0, 1$; $R = \mathbb{Z}\pi_1M$; and $\varepsilon = \pm 1$ are used to compute $\pi_0\mathcal{S}(M)$.

The main goals of this survey article are as follows:

1. Improve communication between algebraists and topologists concerning quadratic forms
2. Call attention to the central role of *periodicity*.
3. Call attention to the connections between $KQuad(R,-,\varepsilon)$ and $\mathcal{S}(M)$, not just $\pi_0\mathcal{S}(M)$.
4. Stimulate interest in the higher dimensional quadratic K-groups.

The functor which sends a f.g. projective module P to $Hom_R(P,R)$ induces an involution T on $\mathbb{K}(R)$. For any $i, j \in \mathbb{Z}$ and any T-invariant subgroup $X \subset K_j(R)$, topologists (see Sect. 3.5) have defined groups $L_i^X(R)$. The subgroup X is called the *decoration* for the L-group. Here are a few properties:

1. *Periodicity* $L_i^X(R) \simeq L_{i+4}^X(R)$
2. $L_{2i}^{K_0}(R) \simeq W_0\left(R,-,(-1)^i\right)$
3. $L_{2i+1}^{K_2}(R) \simeq W_1\left(R,-,(-1)^i\right)$
4. $L_i^{K_j}(R) \simeq L_i^{O_{j-1}}(R)$, where O_{j-1} is the trivial subgroup of $K_{j-1}(R)$

5. *Rothenberg Sequences* If $X \subset Y \subset K_j(R)$, we get an exact sequence

$$\cdots \to L_i^X(R) \to L_i^Y(R) \to \hat{H}^i\left(\mathbb{Z}/2, Y/X\right) \to \cdots .$$

6. *Shaneson Product Formula* For all $i, j \in \mathbb{Z}$,

$$L_{i+1}^{K_{j+1}}\left(R[t,t^{-1}]\right) \cong L_{i+1}^{K_{j+1}}(R) \oplus L_i^{K_j}(R) ,$$

where we extend the involution on R to the Laurent ring $R[t,t^{-1}]$ by $\bar{t} = t^{-1}$.

Notice that $L_i^X(R) \otimes \mathbb{Z}[\frac{1}{2}]$ is independent of X. Different choices for X are used to study various geometric questions. The classification of compact topological manifolds uses $X \subset K_1$ (See Sect. 3.3). The study of open manifolds involves $X \subset K_j$, with $j < 1$, (See [25,31,50,54,65] and [71]). The study of homeomorphisms of manifolds involves $X \subset K_j$, with $j > 1$, (see Sect. 3.6).

Localization Sequences: Suppose S is a multiplicative system in the ring R. Then we get an exact sequence

$$\cdots \to K_i(R,S) \to K_i(R) \to K_i(S^{-1}R) \to \cdots ,$$

where $K_i(R,S)$ is the K-theory of the exact category of S-torsion R-modules of homological dimension 1. In the case of L-theory(with appropriate choice of decorations) one gets an analogous exact sequence using linking forms on torsion modules (see [61,64], and [53]). However, what is striking about the L-theory localization is that it is gotten by splicing together two 6-term exact sequences. One of these involves $(R,-,+1)$ quadratic forms and the other involves $(R,-,-1)$ quadratic forms. The resulting sequence is then 12-fold *periodic*. In fact L-theory satisfies many other such periodic exact sequences (see [64]).

Let $L_i^{<-\infty>}(R)$ be the direct limit of $L_i^{K_0}(R) \to L_i^{K_{-1}}(R) \to \cdots$.

Let $\mathbb{K}(R)$ be the K-theory spectrum constructed by Wagoneer [78] where for all $i \in \mathbb{Z}$, $K_i(R) \simeq \pi_i(\mathbb{K}(R))$. Similarly, let $\mathbb{K}Quad(R,-,\varepsilon)$ be the spectrum where for all $i \in \mathbb{Z}$, $KQuad_i(R,-,\varepsilon) \simeq \pi_i(\mathbb{K}Quad(R,-,\varepsilon))$. Similarly, let $\mathbb{K}Herm(R,-,\varepsilon)$ be the K-theory spectrum for Hermitian forms (see Sects. 3.2 and 3.4). There is a functor $Quad(R,-,\varepsilon) \to Herm(R,-,\varepsilon)$ which induces a homotopy equivalence on K-theory when 2 is a unit in R.

Given a spectrum $\mathbb{K}$ equipped with an action by a finite group G we get the norm homotopy fibration sequence

$$\mathbb{H}_*(G,\mathbb{K}) \xrightarrow{N} \mathbb{H}^*(G,\mathbb{K}) \to \hat{\mathbb{H}}^*(G,\mathbb{K}) ,$$

where $\mathbb{H}_*(G,\mathbb{K})$ is the homotopy orbit spectrum of G acting on $\mathbb{K}$, $\mathbb{H}^*(G,\mathbb{K})$ is the homotopy fixed spectrum, and N is the *norm* map.

The key example for us is $\mathbb{K} = \mathbb{K}(R)$, $G = \mathbb{Z}/2$, and the action is given by the involution T.

Theorem 1: *(Hermitian K-theory Theorem)* There exists a homotopy cartesian diagram

$$\begin{array}{ccc} \mathbb{K}Herm(R,-,\varepsilon) & \to & \mathcal{L}(R,-,\varepsilon) \\ \tilde{F}\downarrow & & \downarrow \\ \mathbb{H}^*\left(\mathbb{Z}/2,\mathbb{K}(R)\right) & \to & \hat{\mathbb{H}}^*\left(\mathbb{Z}/2,\mathbb{K}(R)\right) \end{array}$$

with the following properties.

1. If 2 is a unit in R, then for $i = 0$ or 1,

$$\pi_i\mathcal{L}(R,-,+1) \simeq L_i^{<-\infty>}(R)$$

$$\pi_i\mathcal{L}(R,-,-1) \simeq L_{i+2}^{<-\infty>}(R)$$

2. **Periodicity:** If 2 is a unit in R, then $\Omega^2\mathcal{L}(R,-,\varepsilon) \simeq \mathcal{L}(R,-,-\varepsilon)$.
3. The composition $\mathbb{K}Herm(R,-,\varepsilon) \xrightarrow{\tilde{F}} \mathbb{H}^*(\mathbb{Z}/2,\mathbb{K}(R)) \to \mathbb{K}(R)$ is the forgetful map F.
4. The homotopy fiber of $\mathbb{K}Herm(R,-,\varepsilon) \to \mathcal{L}(R,-,\varepsilon)$ is a map $\tilde{H}\colon \mathbb{H}_*\left(\mathbb{Z}/2,\mathbb{K}(R)\right) \to \mathbb{K}Herm(R,-,\varepsilon)$ such that the composition $\mathbb{K}(R) \to \mathbb{H}_*\left(\mathbb{Z}/2,\mathbb{K}(R)\right) \xrightarrow{\tilde{H}} \mathbb{K}Herm(R,-,\varepsilon)$ is the hyperbolic map.

We call $\tilde{F}$ the *enhanced* forgetful map, and $\tilde{H}$ the *enhanced* hyperbolic map.

Before we state an analogous theorem for $\mathcal{S}(M)$ we need to introduce some more background.

Let $hcob(M)$ be the simplicial set of h-cobordisms on M and let $hcob(M) \to \mathbb{S}(M)$ be the map which sends an h-cobordism $h\colon (W,\partial W) \to (M \times I, M \times \partial I)$ to $h|_{M_1}\colon M_1 \to M \times 1$ where $\partial W = M \sqcup M_1$. Let $HCOB(M)$ be the homotopy colimit of

$$hcob(M) \to hcob(M \times I) \to hcob(M \times I^2) \cdots$$

Igusa has shown that if M is smoothable, then the map $hcob(M) \to HCOB(M)$ is at least $k+1$-connected where $n = dimM \geq max(2k+7, 3k+4)$.

Let $\Omega WH\left(\mathbb{Z}\pi_1(M)\right)$ be the homotopy fiber of the assembly map $\mathbb{H}^*(M,\mathbb{K}\mathbb{Z}) \to \mathbb{K}\mathbb{Z}\pi_1(M)$. For $n > 4$, the s-cobordism theorem yields a bijection $\pi_0\left(hcob(M)\right) \to \pi_0\left(\Omega WH\left(\mathbb{Z}\pi_1(M)\right)\right)$. Waldhausen [80] and Vogel [76,77] has shown how in the definition of $\mathbb{K}\mathbb{Z}\pi_1(M)$ we can replace $\pi_1(M)$ with the loop space of M and $\mathbb{Z}$ with the sphere spectrum. This yields $\mathbb{A}(M)$, the K-theory of the space M. There exists a linearization map $\mathbb{A}(M) \to \mathbb{K}\mathbb{Z}\pi_1(M)$ which is 2-connected. Furthermore, there exists a homotopy equivalence $HCOB(M) \to \Omega WH(M)$, where $\Omega WH(M)$ is the homotopy fiber of the assembly map $\mathbb{H}_*\left(M,\mathbb{A}(*)\right) \to \mathbb{A}(M)$.

Constructions of Ranicki yield spectrum $\mathbb{L}^X(R)$ such that $\pi_i(\mathbb{L}^X(R)) \simeq L_i^X(R)$. Let $\mathbb{L}$ be the 1-connected cover of $\mathbb{L}^{K_0}(\mathbb{Z})$.

Let $iso\mathcal{P}(R)$ be the category with objects finitely generated projective modules, and maps R-linear isomorphisms. The *hyperbolic functor* $H: iso\mathcal{P}(R) \to Quad(R,-,\varepsilon)$ is defined as follows:

$$objects : P \mapsto \left(P \oplus P^*, \begin{bmatrix} 0 & 0 \\ \text{eval} & 0 \end{bmatrix}\right) ,$$

$$maps : (f\colon P \to Q) \mapsto f \oplus (f^{-1})^* ,$$

where $eval(p,\alpha) = \alpha(p)$ for $P \in P$ and $\alpha \in P^*$.

We let $GQ_{2l}(R,-,\varepsilon) := Aut\left(H(R^l)\right)$, and $GQ(R,-,\varepsilon)$ is the direct limit of the direct system $\{GQ_{2l}(R,-,\varepsilon), \theta_{2l}\}$ where $\theta_{2l}\colon GQ_{2l}(R,-,\varepsilon) \to GQ_{2l+2}(R,-,\varepsilon)$ is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \to \begin{pmatrix} A & 0 & B & 0 \\ 0 & 1 & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

Recall that if X is a connected topological space and $\pi_1(X)$ is a quasi-perfect group, i.e. $[\pi_1(X), \pi_1(X)]$ is perfect, then the Quillen plus construction is a map $X \to X^+$ which abelianizes π_1 and which induces an isomorphism on homology for all local coefficient systems on X^+.

4 **Theorem 4** The group $GQ(R,-,\varepsilon)$ is quasi-perfect and $KQuad(R,-,\varepsilon)$ is homotopy equivalent to $KQuad_0(R,-,\varepsilon) \times BGQ(R,-,\varepsilon)^+$.

See [6, 57], and [29] for information on the group $GQ(R,-,\varepsilon)$, in particular about generators for the commmutator subgroup.

The hyperbolic functor induces a map of infinite loop spaces $H : KR \to KQuad(R,-,\varepsilon)$ and we let $KQuad^{(-1)}(R,-,\varepsilon)$ be the homotopy fiber of the deloop of H. Thus we get a homotopy fibration sequence $KR \xrightarrow{H} KQuad(R,-,\varepsilon) \to KQuad^{(-1)}(R,-,\varepsilon)$.

The *forgetful functor* $F : Quad(R,-,\varepsilon) \to iso\mathcal{P}(R)$ is given as follows:

$$objects : (P,\alpha) \mapsto P ,$$

$$maps : [f : (P,\alpha) \to (P',\alpha')] \mapsto f .$$

This induces a map of infinite loop spaces $F\colon KQuad(R,-,\varepsilon) \to K(R)$ and we let $KQuad^{(1)}(R,-,\varepsilon)$ denote the homotopy fiber.

Let $T : iso\mathcal{P}(R) \to iso\mathcal{P}(R)$ be functor which sends an object P to P^* and which sends a map f to $(f^{-1})^*$. Then T induces a homotopy involution on K_iR for all i. Also the composition $F \circ H\colon K_i(R) \to K_i(R)$ equals $1 + T$.

Theorem 5: *(Karoubi Periodicity)*
Asume 2 is a unit in R. Then the 2nd loop space of $KQuad^{(-1)}(R,-,\varepsilon)$ is homotopy equivalent to $KQuad^{(1)}(R,-,-\varepsilon)$.

In Sect. 3.3 we'll see that part (2) of the Hermitian K-theory Theorem follows from Karoubi Periodicity.

Giffen has suggested that there should be a version of Karoubi Periodicity without the assumption that 2 is a unit. Let $Even(R,-,\varepsilon)$ be the category of even hermitian forms and let $Split(R,-,\varepsilon)$ be the category of split quadratic forms (see [64] for definitions). Then we get forgetful functors

$$Split(R,-,\varepsilon) \to Quad(R,-,\varepsilon) \to Even(R,-,\varepsilon) \to Herm(R,-,\varepsilon) ,$$

which are equivalences when 2 is a unit in R. We can define analogues of $KQuad^{(1)}(R,-,-\varepsilon)$ and $KQuad^{(-1)}(R,-,\varepsilon)$ for each of these categories, Giffen's idea is that the 2nd loop space of $KQuad^{(-1)}(R,-,\varepsilon)$ should be homotopy equivalent to $KEven^{(1)}(R,-,-\varepsilon)$. There should also be a similar result for each adjacent pair of categories.

L-Theory of Quadratic Forms

When we are using only one involution on our ring R we'll write (R,ε) as short for $(R,-,\varepsilon)$.

Let $F_j\colon KQuad_j(R,\varepsilon) \to K_jR$ be the map induced by the forgetful functor F.

Let $H_j\colon K_jR \to KQuad_j(R,\varepsilon)$ be the map induced by the hyperbolic functor.

Based L-Groups Following [88]

$$L^S_{2i}(R) = L^{K_2}_{2i}(R) := ker\left(disc : \pi_0\left(KQuad^{(1)}(R,(-1)^i) \to K_1R\right)\right)$$

$$L^S_{2i+1}(R) = L^{K_2}_{2i+1}(R) := ker\left(F_1 : KQuad_1\left(R,(-1)^i\right) \to K_1R\right) ,$$

where $\pi_0(KQuad^{(1)}(R,(-1)^i)$ can be identified as the K_0 of the category of based, even rank quadratic forms and "*disc*" is the discriminate map.

Free L-Groups

$$L^{K_1}_{2i}(R) := ker\left(F_0 : KQuad_0\left(R,(-1)^i\right) \to K_0R\right)$$

$$L^{K_1}_{2i+1}(R) := coker\left(H_1 : K_1R \to KQuad_1\left(R,(-1)^i\right)\right)$$

Remarks: In the next section we'll explain how these free L-groups are used to classify compact manifolds.

Projective L-Groups
For $i \in \mathbb{Z}$, $L^{K_0}_{2i}(R) = L^p_{2i}(R) := coker\left(K_0R \xrightarrow{H_0} KQuad_0\left(R,(-1)^i\right)\right)$. If 2 is a unit in R,

then $L_0^p(R)$ is often denoted by $W(R)$. If R is also commutative, then tensor product of forms makes $W(R)$ into a ring called the *Witt ring*.(See [7,46,51], and [68].)
Remarks: the letter "p" stands for "projective".

See [57] where Ranicki defined $L_{2i-1}^p(R)$. in terms of "formations".

Also he proved the following *Shaneson Product Formula*

$$L_i^{K_j}(R) \cong coker\left(L_{i+1}^{K_{j+1}}(R) \to L_{i+1}^{K_{j+1}}\left(R[t,t^{-1}]\right)\right)$$

for all $i, j = 0$ or 1 and where we extend the involution on R to the Laurent ring $R[t,t^{-1}]$ by $\bar{t} = t^{-1}$.

Ranicki also constructed the Rothenberg Sequence

$$\cdots \to L_i^{K_{j+1}}(R) \to L_i^{K_j}(R) \to \hat{H}^i\left(\mathbb{Z}/2, K_j(R)\right) \to \cdots$$

for all $i \in \mathbb{Z}$, and $j = 0$ or 1.

"Lower" L-Groups

This product formula suggests the following downward inductive definition:
For $j < 0$ and any $i \in \mathbb{Z}$,

$$L_i^{K_j}(R) := coker\left(L_{i+1}^{K_{j+1}}(R) \to L_{i+1}^{K_{j+1}}\left(R[t,t^{-1}]\right)\right)$$

Notice how this is analogous to Bass's definition of $K_j(R)$ for $j < 0$ (see[8]).

By using the fundamental theorem of algebraic K-theory, and the fact that the involution T interchanges the two Nil terms in $K_j(R[t,t^{-1}]$ it is to easy that

$$\hat{H}^{i+1}\left(\mathbb{Z}/2, K_{j+1}\left(R[t,t^{-1}]\right)\right) \cong \hat{H}^{i+1}\left(\mathbb{Z}/2, K_{j+1}(R)\right) \oplus \hat{H}^i\left(\mathbb{Z}/2, K_j(R)\right)$$

for all $i, i \in \mathbb{Z}$.

Then one can deduce the following Rothenberg exact sequence

$$\cdots \to L^{K_{j+1}}(R) \to L^{K_j}R \to \hat{H}\left(\mathbb{Z}/2, K_j(R)\right) \to \cdots$$

for all $i \in \mathbb{Z}$ and all $j < 0$.

Recall that it was much harder to find the "correct" definition for high dimensional K-theory than for low dimensional K-theory. Similarly, the definition of $L_i^{K_j}(R)$ for $j \geq 1$ is harder than for $j \leq 1$. See Sect. 3.5 for the definition of L-groups with "higher" decorations for all Hermitian rings. In Sect. 3.4 we use Karoubi periodicity to give another description when 2 is a unit in the Hermitian ring.

3.3 Classification of Manifolds up to Homeomorphism

Surgery theory was invented by Kervaire–Milnor, Browder, Novikov, Sullivan, and Wall [87]. The reader is encouraged to look at the following new introductions to the subject [48,66], and [30].

Poincaré Complexes

We first introduce the homotopy theoretic analogue of a closed manifold. A connected finite CW complex X is an (oriented) n-dimensional *Poincaré complex* with fundamental class $[X] \in H_n(X)$ if $[X] \cap -: H^*(X;\Lambda) \to H_{n-*}(X;\Lambda)$ is an isomorphism for every $\mathbb{Z}\pi$-module Λ, where $\pi = \pi_1(X)$. Assume $q >> n$, then X has a preferred S^{q-1} spherical fibration S_X such that $Thom(S_X)$ has a *reduction*, i.e. a map $c_X: S^{n+q} \to Thom(S_X)$ which induces an isomorphism on H_{n+q}. We call S_X the *Spivak fibration* for X, and given any S^{q-1}-fibration η equipped with a reduction c, there exists a map of spherical fibrations $\gamma : \eta \to S_X$ which sends c to c_X. The map γ is unique up to fiber homotopy. Notice that if X is a closed manifold embedded in S^{n+q} with normal bundle ν_X, then the map $c_X: S^{n+q} \to \left(S^{n+q}/\left(S^{n+q} - \text{tubular nghd}\right)\right) \simeq Thom(\nu_X)$ is a reduction.

Manifold Structures

If X is an (oriented) n-dimensional Poincaré complex, we let $\mathcal{S}(X)$ denote the simplicial set of *topological manifold structures* on X. An element in $\pi_0(\mathcal{S}(X))$ is represented by a homotopy equivalence $h: M \to X$ where M is a closed topological manifold. A second homotopy equivalence $h_1: M_1 \to X$ represents the same element if there exist a homeomorphism $\alpha: M \to M_1$ such that h is homotopic to $h_1 \circ \alpha$. A k-simplex in $\mathcal{S}(X)$ is given by a fiber homotopy equivalence $M \times \Delta^k \to X \times \Delta^k$ over Δ^k.
Let's consider the following two questions.

(Existence) When is $\mathcal{S}(X)$ nonempty?
(Classification) Suppose $h: M \to X$ and $h_1: M_1 \to X$ represent $[h]$ and $[h_1]$ in $\pi_0(\mathcal{S}(X))$. How do we decide when $[h] = [h_1]$?

We break these two questions into a series of subquestions.

Existence Step I: Euclidean bundle structure on the Spivak fibration

Question 1E: (Homotopy Theory) Does there exist a topological $\mathbb{R}^q$ bundle η over X with a reduction $c: S^{n+q} \to Thom(\eta)$?

Notice the answer to 1E is yes iff the map $\overline{S_X}: X \to BG$ which classifies S_X factors thru $BTop$, the classifying space for stable Euclidean bundles. Also suppose $h : M \to X$ is a homotopy equivalence where M is a closed topological manifold. If g is a homotopy inverse to h, then $\eta_h = g^*\nu_M$ has a reduction $c_h: S^{n+q} \xrightarrow{c_M} Thom(\nu_M) \to Thom(\eta_h)$. We call the pair (η_h, c_h) the *normal invariant* of the manifold structure h.

This construction yields a map of simplicial sets $n : \mathcal{S}(X) \to Lift(S_X)$, where $Lift(S_X)$ is the simplicial set of lifts of $\overline{S_X}$ thru $BTop$.

Existence Step II: Surgery Problem

Suppose the answer to 1E is yes. Then by replacing c by a map transversal to the copy of X given by the zero section in $Thom(\eta)$, we get a pair $(f: M \to X, \hat{f})$, where $M = c^{-1}(\text{zero section}), f = c|_M, and\, \hat{f}: \nu_M \to \eta$ is the bundle map covering f given by transversality. The pair $(f, \hat{f})$ is an example of a *surgery map*. Notice that f might not be a homotopy equivalence, but we can assume that f induces an isomorphism on H_n. A second surgery map $(f_1: M_1 \to X, \hat{f}_1)$ is *normal cobordant* to $(f, \hat{f})$ iff there exists a manifold $(W, \partial W) \subset (S^{n+q} \times I, S^{n+q} \times \partial I)$ with $\partial W = M \sqcup M_1$, and maps $F: W \to X \times I, \hat{F}: \nu_W \to \eta \times I$ such that $F|_M = f, f|_{M_1} = f_1, \hat{F}|_{\nu_M} = \hat{f}$, and $\hat{F}|_{\nu_{M_1}} = \gamma \circ \hat{f}_1$, where $\gamma: \eta \to \eta_1$ is a bundle isomorphism. The pair $(\eta, c: S^{n+q} \to Thom(\eta))$ determines $(f, \hat{f})$ up to normal cobordism.

Question 2E: (Surgery Theory) Is $(f, \hat{f})$ normal cobordant to a homotopy equivalence?

Given a group ring $\mathbb{Z}\pi$ we let $-: \mathbb{Z}\pi \to \mathbb{Z}\pi$ be the anti-involution

$$\sum_{g \in \pi} n_g g \mapsto \sum_{g \in \pi} n_g g^{-1}.$$

We need the following minor variation of the free L-groups. Let

$$L^h_{2i}(\mathbb{Z}\pi) := L^{K_1}_{2i}(\mathbb{Z}\pi), \text{ and}$$

$$L^h_{2i+1}(\mathbb{Z}\pi) := L^{K_1}_{2i}(\mathbb{Z}\pi) \text{ modulo the subgroup generated by } \begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}.$$

Here "h" stands for homotopy equivalence.

6 **Theorem 6:** *(Surgery Theorem)* Assume $n > 4$. An n-dimensional surgery problem $(f: M^n \to X, \hat{f})$ determines an element $\sigma(f, \hat{f}) \in L^h_n(\mathbb{Z}\pi)$ such that $\sigma(f, \hat{f}) = 0$ iff $(f, \hat{f})$ is normal cobordant to a homotopy equivalence.

There is also a relative version of this where the closed manifold M is replaced by a compact manifold with boundary $(M, \partial M)$, the Poincaré complex X is replaced by a Poincaré pair (X, Y) i.e. $[X, Y] \cap -: H^*(X; \Lambda) \to H_{n-*}(X, Y; \Lambda)$ is an isomorphism, and $f: (M, \partial M) \to (X, Y)$ is such that $f|\partial M: \partial M \to Y$ is a homotopy equivalence. Then $(f, \hat{f})$ still determines an element in $L^h_n(\mathbb{Z}\pi_1(X))$ which is trivial iff $(f, \hat{f})$ is normal cobordant (rel the boundary) to a homotopy equivalence of pairs.

The first paragraph of the Surgery Theorem yields a map $\sigma: \pi_0(Lift(S_X)) \to L^h_n(\mathbb{Z}\pi)$ such that $\pi_0 \mathcal{S}(X) \xrightarrow{n} \pi_0 Lift(S_X) \xrightarrow{\sigma} L^h_n(\mathbb{Z}\pi)$ is exact at $\pi_0 Lift(S_X)$.

From Geometry to Quadratic Forms

A detailed explanation of the Surgery Theorem is given in [48] and [66]. Here we'll just give a brief outline.
First we'll introduce some terminology.

Suppose V^m is a cobordism from $\partial_- V$ to $\partial_+ V$, i.e. $\partial V = \partial_- V \sqcup \partial_+ V$. Given an embedding $g: \sqcup (S^{i-1} \times D^{m-i}) \to \partial_+ V$, we let V' be the result of using g to *attach handles* of index i to V, i.e. $V' = V \cup_g (\sqcup (D^i \times D^{m-i})$. Then V' is a cobordism from $\partial_- V$ to a new manifold $\partial_+ V'$, and we say that $\partial_+ V'$ is the result of *doing surgery on g*.

Suppose $(F: W \to X \times I, \hat{F}))$ is a normal cobordism from $(f: M \to X, \hat{f})$ to some other surgery problem. Then W has a filtration $(M \times I) = W_0 \subset W_1 \subset \cdots \subset W_n = W$ where for each i, $W_{i+1} = W_i \cup$ (handles of index $i + 1$).

Our surgery problem $(f: M \to X, \hat{f})$ is a homotopy equivalence iff f induces an isomorphism on π_1, and $\tilde{f}_*: H_j(\tilde{M}) \to H_j(\tilde{X})$ is an isomorphism for each j, where $\tilde{f}: \tilde{M} \to \tilde{X}$ is a π-equivariant map of universal covers over f.

Special case: $n = 2i > 4$
Then up to normal cobordism there is no obstruction to arranging that $(f, \hat{f})$ induces an isomorphism on π_1, $\tilde{f}_*: H_j(\tilde{M}) \to H_j(\tilde{X})$ is an isomorphism for $j \neq i$, and $kernel(H_i(\tilde{M}) \to H_i(\tilde{X})) \simeq \pi_{i+1}(f)$ is a free $\mathbb{Z}\pi$-module of even rank $2l$. Any element $a \in \pi_{i+1}(f)$ is presented by a continuous map $\partial a: S^i = \partial D^{i+1} \to M$ plus an extension of $f \circ \partial a$ to D^{i+1}. This extension plus the bundle map $\hat{f}$ determines a regular homotopy class of immersions $\hat{a}: S^i \times D^i \to M$. Notice that if this immersion is in fact an embedding then one can do surgery on $\hat{a}$. By considering transversal intersections of these immersions one gets a nonsingular $(-1)^i$-Hermitian form $\beta: \pi_{i+1}(f) \times \pi_{i+1}(f) \to \mathbb{Z}\pi$. By considering transversal *self* intersections one gets a quadratic form α such that $(1 + T_{-1^i})\alpha = \beta$. (See [85] and [48].)
Suppose we have an isomorphism of quadratic forms $H((\mathbb{Z}\pi)^l) \to \pi_{i+1}(f)$. Let $(a_k, k = 1, \ldots l)$ be a basis for the image of $(\mathbb{Z}\pi)^l \subset H((\iota)^l) \to \pi_{i+1}(f)$. Then there exists an embedding $g = \sqcup \hat{a}_k: \sqcup S^i \times D^i \to M$ such that if we do surgery on g we get a normal cobordism to a homotopy equivalence.

Special case: $n = 2i + 1 > 3$
Suppose one has a nonsingular $(-1)^i$-quadratic form (P, α), plus two isomorphisms $A_1, A_2: H((Z\pi)^l) \to (P, \alpha)$. Then $A_2^{-1} \circ A_2$ is an element in $GQ_{2l}(\mathbb{Z}\pi, (-1)^i$ which maps to $L^h_{2i+1}(\mathbb{Z}\pi)$. Roughly speaking this is what one gets from a $2i+1$-dimensional surgery problem after it is made highly connected. To make this precise it is best to introduce the notion of *formations*. See [57] and [66].

Classification:
Suppose $h: M \to X$ and $h_1: M_1 \to X$ represent $[h]$ and $[h_1]$ in $\pi_0(\mathcal{S}(X))$.

Classification Step I: Normal invariant
Question 1C: (Homotopy Theory) Are the normal invariants (η_h, c_h) and (η_{h_1}, c_{h_1}) equivalent?

(To simplify notation let $(\eta, c) := (\eta_h, c_h)$ and $(\eta_1, c_1) := (\eta_{h_1}, c_{h_1})$.)

In other words does there exist a bundle isomorphism $\gamma: \eta \to \eta_1$ and a homotopy $H: S^{n+q} \times I \to Thom(\eta_1)$ such that $H|S^{n+q} \times 0 = Thom(\gamma) \circ c$, and $H|S^{n+q} \times 1 = c_1$. If γ and H exist, then we can choose H so that it is transversal to $X \times I$. This then yields a normal cobordism $(F: W \to X \times I, \hat{F}: \nu_W \to \eta_1)$ from $h: M \to X$ to $h_1: M_1 \to X$.

Suppose $\overline{n(h)}, \overline{n(h_1)} : X \to BTop$ are the lifts of $\overline{S_X}$: $X \to BG$ which classify (η_h, c_h) and (η_{h_1}, c_{h_1}) respectively. Then $\overline{n(h)}$ and $\overline{n(h_1)}$ are homotopic as lifts iff (η_h, c_h) and (η_{h_1}, c_{h_1}) are equivalent. Furthermore, the group $[X, G/Top]$ acts simply transitively on the set of homotopy class of lifts of $\overline{S_X}$. See [49] for results of Sullivan and others on the space G/Top.

Classification Step II: Relative surgery problem

Question 2C: (Surgery Theory) Suppose $(F\colon W \to X \times I, \hat{F}\colon \nu_W \to \eta_1)$ is a solution to 1C. Is $(F, \hat{F})$ normal cobordant (rel boundary) to an *h-cobordism*?

Here W is an h-cobordism from M to M_1 iff the two inclusion maps $M \subset W$ and $M_1 \subset W$ are homotopy equivalences.

Notice that the second paragraph of the Surgery Theorem yields an element $\sigma(F, \hat{F}) \in L^h_{n+1}(\mathbb{Z}\pi)$ which is 0 iff the answer to 2C is yes.

Classification Step III: H-cobordism problem

Question 3C: (Product Structure on H-cobordisms) Suppose W is an h-cobordism from M to M_1. When is W homeomorphic to $M \times I$?

Let $hcob(M) = \mathcal{S}(M \times I, M \times 0)$ be the simplicial set of topological manifold structures on $M \times I$ rel $M \times 0$. Thus an element in $\pi_0(hcob(M)$ is represented by an h-cobordism from M to some other manifold. Two such h-cobordism represent the same element iff there exists a diffeomorphisms between them which is the identity on M.

We let $Wh_1(\pi) := coker\left(\{\pm\pi\} \to Gl_1(\mathbb{Z}\pi) \to K_1(\mathbb{Z}\pi)\right)$.

7 **Theorem 7:** *(S-Cobordism Theorem)* Assume $n > 4$. There exists a bijection $\tau\colon \pi_0(hcob(M)) \to Wh_1(\pi_1(M))$ such that the product h-cobordism $M \times I$ maps to the unit element.

Suppose R is a ring such that $R^n \simeq R^m$ implies that $n = m$. Let B be a nontrivial subgroup of $K_1(R)$. Let $g\colon R^n \xrightarrow{\sim} P$ and $g_1 : R^n \xrightarrow{\sim} P$ be two bases for a f.g. R-module P. The bases g and g_1 are said to be B-equivalent iff the map $GL_n(R) \to K_1(R)$ sends $g^{-1} \circ g_1$ to an element in B. We say that P is B-based if it is equipped with an B-equivalence class of basis. Notice that an isomorphism between two f.g. B-based modules determines an element in $K_1(R)/B$. More generally an R-chain homotopy equivalence g between two f.g. B-based, R-chain complexes determines an element, $\tau(g) \in K_1(R)/B$ called the *torsion* of g, (See [48, 2.2]).

Geometric Example: Suppose $f\colon A_1 \to A_2$ is a homotopy equivalence between finite CW complexes with fundamental groups isomorphic to π. Then the universal covers $\widetilde{A_1}$ and $\widetilde{A_2}$ are also CW complexes, and the CW -chain complexes $C(\widetilde{A_1})$ are $C(\widetilde{A_2})$ are B-based, where $B = im\left(\{\pm\pi\} \to Gl_1(\mathbb{Z}\pi) \to K_1(\mathbb{Z}\pi)\right)$. If $\widetilde{f}\colon \widetilde{A_1} \to \widetilde{A_2}$ is a π-equivariant map covering f, we let $\tau(f) = \tau\left(C(\widetilde{f})\right) \in Wh_1(\pi)$.

The map τ in the s-cobordism theorem sends an h-cobordism $M \subset W \supset M_1$ to $\tau(M \subset W)$.

Let $S^h(M) = \left(\pi_0\left(\mathcal{S}(M)\right)/\text{h-cobordisms}\right)$ = orbit set of the action of $Wh_1\left(\pi_1(M)\right)$ on $\pi_0\left(\mathcal{S}(M)\right)$.

Theorem 8: *(Wall's h-Realization Theorem)* Assume $n > 4$. There is an action of $L^h_{n+1}(\mathbb{Z}\pi)$ on $S^h(M)$ such that the normal invariant map $\pi_0\mathcal{S}(M) \to \pi_0 Lift(S_M)$ factors thru an injection $S^h(M)/L^h_{n+1}(\mathbb{Z}\pi) \to \pi_0 Lift(S_M)$.

Let $\tau\colon \pi_0\mathcal{S}(M) \to Wh_1(\pi)$ be the map which sends $h\colon M_1 \to M$ to $\tau(h)$. Let $S^s(M) = ker\left(\tau\colon \pi_0\mathcal{S}(M) \to Wh_1(\pi)\right)$.

Let $L^s_{2i}(\mathbb{Z}\pi) = L^B_{2i}(\mathbb{Z}\pi)$, and let $L^s_{2i+1}(\mathbb{Z}\pi) = L^B_{2i+1}(\mathbb{Z}\pi)$ modulo the subgroup generated by $\begin{pmatrix} 0 & 1 \\ \pm 1 & 0 \end{pmatrix}$, where $B = im\left(\{\pm\pi\} \to Gl_1(\mathbb{Z}\pi) \to K_1(\mathbb{Z}\pi)\right)$.

Theorem 9: *(Wall's s-Realization Theorem)* Assume $n > 4$. There is an action of $L^s_{n+1}(\mathbb{Z}\pi)$ on $S^s(M)$ such that the restriction of the normal invariant map $S^s(M) \subset \pi_0\mathcal{S}(M) \to \pi_0 Lift(S_M)$ factors thru an injection $S^s(M)/L^s_{n+1}(\mathbb{Z}\pi) \to \pi_0 Lift(S_M)$.

Higher Hermitian K-Theory

Homotopy Fixed Spectrum, Homotopy Orbit Spectrum, and the Norm Fibration Sequence

See [2,28,72–74], and [41]. Suppose $\mathbb{K}$ is an Ω-spectrum equipped with an action by a finite group G.

Classical Example: Suppose $\mathbb{K}$ is the Eilenberg–MacLane spectrum $\mathbb{H}(A)$ where $\pi_0(\mathbb{H}(A)) = A$, a G-module.

Let

$$\mathbb{H}^*(G;\mathbb{K}) = \mathbb{K}^{hG} = F_G(\Sigma^\infty EG_+, \mathbb{K})\,, \text{ and}$$

$$\mathbb{H}_*(G;\mathbb{K}) = \mathbb{K}_{hG} = \Omega^\infty(\Sigma^\infty EG_+ \wedge_G \mathbb{K})\,;$$

where F_G is the function spectrum of G-equivariant maps, and where Ω^∞ is the functor which converts a spectrum to a homotopy equivalent Ω-spectrum.

Notice that

$$\pi_i\left(\mathbb{H}^*\left(G;\mathbb{H}(A)\right)\right) = H^{-i}(G;A)\,, \text{ and}$$

$$\pi_i\left(\mathbb{H}_*\left(G;\mathbb{H}(A)\right)\right) = H_i(G;A)\,.$$

For general $\mathbb{K}$, there exist spectral sequences which abut to $\pi_*(\mathbb{H}^*(G;\mathbb{K}))$ and to $\pi_*(\mathbb{H}^*(G;\mathbb{K}))$, where E_2 is $H^*(G;\pi_*\mathbb{K})$ and $H_*(G;\pi_*\mathbb{K})$ respectively.

The map $EG \to pt$ induces maps $\mathbb{H}^*(G;\mathbb{K}) \to \mathbb{K}$ and $\mathbb{K} \to \mathbb{H}_*(G;\mathbb{K})$. Let $n\colon \mathbb{K} \to \mathbb{K}$ be the map given by $\prod_{g\in G} g$.

Then Adem–Dwyer–Cohen [2], and May–Greenlees [28] have constructed a norm fibration sequence

$$\mathbb{H}_*(G;\mathbb{K}) \xrightarrow{N} \mathbb{H}^*(G;\mathbb{K}) \to \hat{\mathbb{H}}^*(G;\mathbb{K}),$$

where the following diagram is homotopy commutative

$$\begin{array}{ccc} \mathbb{K} & \xrightarrow{n} & \mathbb{K} \\ \downarrow & & \uparrow \\ \mathbb{H}_*(G;\mathbb{K}) & \xrightarrow{N} & \mathbb{H}^*(G;\mathbb{K}), \end{array}$$

see also [91]. Furthermore, $\pi_i\left(\hat{\mathbb{H}}^*(G;\mathbb{H}(A))\right) \simeq \hat{H}^i(G;A)$ in the sense of Tate, see [69]. Thus $\hat{\mathbb{H}}^*(G;\mathbb{K})$ is called the Tate spectrum for G acting on $\mathbb{K}$.

3.4.2 Thomason's Homotopy Limit and Homotopy Colimit Problems

If G is a finite group, then $\underline{G}$ is the category with a single object, and maps are elements of the group G. Composition of maps is given by multiplication in G. Let *Cat* be the category of small categories. An action of G on a category $\mathcal{C}$ is a functor $\underline{G} \to Cat$ which sends the single object in $\underline{G}$ to $\mathcal{C}$. Let *SymMon* be the category with objects small symmetric monoidal categories, and maps symmetric monoidal functors. The category of G symmetric monoidal categories, $G - SymMon$, is then the category of functors from $\underline{G}$ into *SymMon*.

Suppose $\mathcal{C}$ is a G-symmetric monoidal category. Constructions of Thomason, then yield the following commutative diagram which commutes up to a preferred homotopy. (See [72–74], [43] and the next two page of this paper.)

$$\begin{array}{ccc} K(\mathcal{C}_{hG}) & \xrightarrow{\widetilde{Tr}} & K(\mathcal{C}^{hG}) \\ \downarrow & & \tilde{F}\downarrow \\ \mathbb{H}_*(G,K\mathcal{C}) & \xrightarrow{N} & \mathbb{H}^*(G,K\mathcal{C}) \end{array}$$

In [74] Thomason showed that the left vertical map is a homotopy equivalence, and in [72] he observed that many fundamental questions can be viewed as asking when the right vertical map becomes an equivalence after some sort of completion.

Examples: We'll ignore the complication that each of the following categories should be replaced with equivalent small categories,

1. (Segal Conjecture) Let $\mathcal{C}$ be the category of finite sets, equipped with the trivial action by G. Then $K(\mathcal{C}^{hG}) \simeq K(\text{finite } G\text{–sets})$ is equivalent to $\vee\Sigma^\infty B(N_G H/H)_+$, where we wedge over the set of conjugacy classes of subgroups of G. Also $\mathbb{H}^*(G,K\mathcal{C})$ is equivalent to the function spectrum $F(\Sigma^\infty BG_+,\mathbb{S})$, where $\mathbb{S}$ is the sphere spectrum. The Segal Conjecture as proved by Carlsson states that in this case the map $K(\mathcal{C}^{hG}) \to \mathbb{H}^*(G,K\mathcal{C})$ becomes an equivalence after completion with respect to $I(G) = kerK_0(\text{finite } G - \text{sets}) \xrightarrow{rank} \mathbb{Z}$. (See [16])
2. (Quillen–Lichtenbaum) Let $\mathcal{C} = \mathcal{P}(F)$ where the field F is a finite, Galois extension of a field f. Let $G = Gal(F/f)$. If $g \in G$ and V is a F-module with multiplication $m\colon F \times V \to V$, then $F \times V \xrightarrow{g\times id} F \times V \xrightarrow{m} V$ is a new F-module structure on V. This yields an action of G on $\mathcal{P}(F)$ such that $K(\mathcal{P}(F)^{hG}) \simeq K(f)$. Then Thomason [73] has shown that a version of the Quillen–Lichtenbaum Conjecture can be reduced to showing that the map $Kf \to \mathbb{H}^*(G,KF)$ is an equivalence after profinite completion.
3. (Hermitian K-theory) Suppose $(R,-,\varepsilon)$ is a hermitian ring. Then T_ε is "almost" an involution on $\mathcal{P}(R)$ in that there exists a natural equivalence between T_ε^2 and *id*. We can *rectify* this to get an honest action by $\mathbb{Z}/2$ via the following construction. Let $\widetilde{P}(R,-,\varepsilon)$ be the category where an object is a triple $(P,Q,h\colon P \xrightarrow{\sim} Q^*)$, where P and Q are objects in $\mathcal{P}(R)$ and h is an isomorphism. A map from $(P,Q,P \xrightarrow{h} Q^*)$ to $(P_1,Q_1,P_1 \xrightarrow{h_1} Q_1^*)$. is given by a pair of R-module isomorphisms $f\colon P \to P_1$ and $g\colon Q \to Q_1$ such that $h = g^* \circ h_1 \circ f$. Then $\widetilde{P}(R,-,\varepsilon)$ is equivalent to $iso\mathcal{P}(R)$. Furthermore we get an involution $\widetilde{T}_\varepsilon\colon \widetilde{P}(R,-,\varepsilon) \to \widetilde{P}(R,-,\varepsilon)$ that sends (P,Q,h) to $(Q,P,Q \xrightarrow{\eta_{-,\varepsilon}} Q^{**} \xrightarrow{h^*} P*)$, where $Q \xrightarrow{\eta_{-,\varepsilon}} Q^{**}$ is the natural equivalence $\eta_{-,\varepsilon}(q)(f) = \varepsilon\overline{f(q)}$ for all $q \in Q$ and all $f \in Q^*$. Then $\widetilde{P}(R,-,\varepsilon)^{h\mathbb{Z}/2}$ is equivalent to $KHerm(R,-,\varepsilon)$.

Conjecture: The map $\widetilde{F}\colon KHerm(R,-,\varepsilon) \to \mathbb{H}^*(\mathbb{Z}/2,KR)$ becomes an equivalence under profinite completion. (See [23] and [9].)

Let $\mathcal{E}G$ be the transport category for the group G. Thus $Obj(\mathcal{E}G) = G$, and $Map_{\mathcal{E}G}(g_1,g_2)$ has just one element for each pair ordered (g_1,g_2). Then G acts on $\mathcal{E}G$ via multiplication in G. Notice that the classifying space $B\mathcal{E}G$ is contractible and the induced action of G on $B\mathcal{E}G$ is free. Thomason defines $\mathcal{C}^{hG}$ as $Fun_G(\mathcal{E}G,\mathcal{C})$, the category of G-equivariant functors from $\mathcal{E}G$ to $\mathcal{C}$. Notice that an object in $\mathcal{C}^{hG}$ can be viewed as a pair (x,α) where x is an object in $\mathcal{C}$ and α is a function assigning to each $g \in G$ an isomorphism $\alpha(g)\colon x \to gx$. The function α must satisfy the identities $\alpha(1) = 1$ and $\alpha(gh) = g\alpha(h)\cdot\alpha(g)$. Then we get the *transfer functor*:

$$Tr\colon \mathcal{C} \to \mathcal{C}^{hG}$$
$$x \mapsto \left(\sum_{g\in G} gx, \alpha\right),$$

where $\alpha(h)\colon \sum gx \xrightarrow{\sim} h\sum gx$ is the obvious permutation isomorphism.

See [74] and [41] for the construction of $\mathcal{C}_{hG}$ and the factorization

$$Tr\colon \mathcal{C} \to \mathcal{C}_{hG} \stackrel{\widetilde{Tr}}{\to} \mathcal{C}^{hG} .$$

When $\mathcal{C} = \mathcal{P}(R, -, \varepsilon)$, Tr is the hyperbolic functor.

3.4.3 Karoubi Periodicity

See [39] and [43–45].

Let $\widetilde{H}$ be the composition

$$\mathbb{H}_*(\mathbb{Z}/2, KR) \simeq K(\widetilde{\mathcal{P}}(R, -, \varepsilon))_{h\mathbb{Z}/2} \stackrel{\widetilde{Tr}}{\to} KHerm(R, -, \varepsilon) .$$

We want to improve the following homotopy commutative diagram in a couple of ways:

$$\begin{array}{ccc} \mathbb{H}_*(\mathbb{Z}/2, KR) & \stackrel{\widetilde{H}}{\to} & KHerm(R,-,\varepsilon) \\ id \downarrow & & \widetilde{F} \downarrow \\ \mathbb{H}_*(\mathbb{Z}/2, KR) & \stackrel{N}{\to} & \mathbb{H}^*(\mathbb{Z}/2, KR) \end{array}$$

1. We want to replace the (-1)-connective spectra KR and $KHerm(R, -, \varepsilon)$ with spectra $\mathbb{K}R$ and $\mathbb{K}Herm(R, -, \varepsilon)$ where for all $i \in \mathbb{Z}$, $K_i(R) = \pi_i(\mathbb{K}R)$ and $KHerm_i(R, -, \varepsilon) = \pi_i(\mathbb{K}(R, -, \varepsilon))$.
2. We want to use Karoubi periodicity to show that when 2 is a unit in R, then $\Omega^2\mathcal{L}(R, -, \varepsilon) \simeq \mathcal{L}(R, -, -\varepsilon)$ where $\mathcal{L}(R, -, \varepsilon)$ is the deloop of the homotopy fiber of the map $\mathbb{H}_*\left(\mathbb{Z}/2, \mathbb{K}R\right) \stackrel{\widetilde{H}}{\to} \mathbb{K}Herm(R, -, \varepsilon)$.

Disconnected K-theory

For any ring R we let CR, the *cone* of R, be the ring of infinite matrices (a_{ij}), $(i,j) \in \mathbb{N} \times \mathbb{N}$ such that each row and each column has only a finite number of nonzero entries. let SR, the *suspension* of R, be CR modulo the ideal of matrices with only a finite number of nonzero rows and columns. Gersten and Wagoneer [78] have shown that $KCR \simeq *$ and that $\Omega KSR \simeq KR$. This yields a spectrum $\mathbb{K}R$ such that KR is the (-1)-connected cover of $\mathbb{K}R$ and for $i < 0$, $\pi_i(\mathbb{K}R) \simeq K_iR$ in the sense of Bass.

If $\phi\colon R_1 \to R_2$ is a ring homomorphism, we let

$$\Gamma(\phi) = lim\left(SR_1 \stackrel{S\phi}{\to} SR_2 \leftarrow CR_2\right)$$

and following Wagoneer [78] we get a homotopy fibration sequence

$$\mathbb{K}R_1 \to \mathbb{K}R_2 \to \mathbb{K}\left(\Gamma(\phi)\right) .$$

Suppose $(R, -, \varepsilon)$ is a hermitian ring. We then get hermitian rings $C(R, -, \varepsilon)$ and $S(R, -, \varepsilon)$ with underlying rings CR and SR respectively. The anti-involution of the matrix rings CR and SR is given by $M \mapsto \overline{M}^t$, i.e. apply $-$ componentwise and then take the matrix transpose. The choice of central unit is εI where I is the identity matrix. Then Karoubi has shown that $KHermC(R, -, \varepsilon) \simeq *$ and that $\Omega KHermS(R, -, \varepsilon) \simeq KHerm(R, -, \varepsilon)$. This yields the spectrum $\mathbb{K}Herm(R, -, \varepsilon)$ with (-1)-connected cover
$KHerm\left(R, -, \varepsilon/\right.$.

If $\phi\colon (R_1, -_1, \varepsilon_1) \to (R_2, -_2, \varepsilon_2)$ is a map of hermitian rings, then $\Gamma(\phi)$ inherits hermitian structure and we get a homotopy fibration sequence

$$\mathbb{K}Herm(R_1, -_1, \varepsilon_1) \to \mathbb{K}Herm(R_2, -_2, \varepsilon_2) \to \mathbb{K}Herm\left(\Gamma(\phi)\right) .$$

Karoubi's Hyperbolic and Forgetful Tricks

For any hermitian ring $(R, -, \varepsilon)$ we let $(R \times R^{op}, s, \varepsilon \times \overline{\varepsilon})$ be the hermitian ring where $s(a, b) = (b, a)$.

Theorem 10: *(Forgetful Trick)* Let $d\colon R \to R \times R^{op}$ send r to $(r, \overline{r})$. Then we get the following commutative diagram

$$\begin{array}{ccc} \mathbb{K}Herm(R,-,\varepsilon) & \stackrel{d}{\to} & \mathbb{K}Herm\left(R \times R^{op}, s, \varepsilon \times \overline{\varepsilon}\right) \\ id \downarrow & & \simeq \downarrow \\ \mathbb{K}Herm(R,-,\varepsilon) & \stackrel{F}{\to} & \mathbb{K}R\,, \end{array}$$

where F is the forgetful map.

Thus if $V(R, -, \varepsilon) = \Gamma(d)$, we get a homotopy fibration

$$\mathbb{K}Herm(R, -, \varepsilon) \stackrel{F}{\to} \mathbb{K}R \to \mathbb{K}Herm\left(V(R, -, \varepsilon)\right)$$

with connecting homomorphism $\partial\colon \Omega\mathbb{K}Herm\left(V(R, -, \varepsilon)\right) \to \mathbb{K}Herm(R, -, \varepsilon)$.

Let $\mathbb{K}Herm^{(1)}(R, -, \varepsilon) = \Omega\mathbb{K}Herm\left(V(R, -, \varepsilon)\right)$.

We can iterate the construction of V and let

$$\mathbb{K}Herm^{(j)}(R, -, \varepsilon) = \Omega^j\mathbb{K}Herm\left(V^j(R, -, \varepsilon)\right) , \text{ for } j = 1, 2 \cdots .$$

Also we let $\mathbb{K}Herm^{(\infty)}(R, -, \varepsilon)$ be the homotopy limit of the diagram

$$\cdots \to \mathbb{K}Herm^{(j)}(R, -, \varepsilon) \to \mathbb{K}Herm^{(j-1)}(R, -, \varepsilon) \to \cdots \mathbb{K}Herm(R, -, \varepsilon) .$$

Theorem 11: *(Kobal's Forgetful Theorem)* There exists a homotopy fibration

$$\mathbb{K}Herm^{(\infty)}(R, -, \varepsilon) \to \mathbb{K}Herm(R, -, \varepsilon) \stackrel{\widetilde{F}}{\to} \mathbb{H}^*(\mathbb{Z}/2, \mathbb{K})$$

such that the following diagram commutes

$$\begin{array}{ccc} KHerm(R,-,\varepsilon) & \stackrel{\widetilde{F}}{\to} & \mathbb{H}^*(\mathbb{Z}/2, K(R)) \\ \downarrow & & \downarrow \\ \mathbb{K}Herm(R,-,\varepsilon) & \stackrel{\widetilde{F}}{\to} & \mathbb{H}^*(\mathbb{Z}/2, \mathbb{K}(R))\,. \end{array}$$

For any hermitian ring $(R, -, \varepsilon)$ we let $(M_2(R), \gamma, \varepsilon I)$ be the hermitian ring where $\gamma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix}$.

12 **Theorem 12:** *(Hyperbolic Trick)* If $e\colon (R \times R^{op}, s, \varepsilon \times \bar{\varepsilon}) \to (M_2(R), \gamma, \varepsilon I)$ is given by $e(a,b) = \begin{pmatrix} a & o \\ 0 & \bar{b} \end{pmatrix}$, then we get the following commutative diagram

$$\begin{array}{ccc} \mathbb{K}Herm(R \times R^{op}, s, \varepsilon \times \bar{\varepsilon}) & \xrightarrow{e} & \mathbb{K}Herm(M_2(R), \gamma, \varepsilon I) \\ \simeq \downarrow & & \simeq \downarrow \\ \mathbb{K}R & \xrightarrow{H} & \mathbb{K}Herm(R, -, \varepsilon) \end{array}$$

where H is the hyperbolic map.

Thus if $U(R, -, \varepsilon) = \Gamma(e)$, we get a homotopy fibration

$$\mathbb{K}R \xrightarrow{H} \mathbb{K}Herm(R, -, \varepsilon) \to \mathbb{K}U(R, -, \varepsilon) .$$

Let $\mathbb{K}Herm^{(-1)}(R, -, \varepsilon) = \mathbb{K}Herm\left(U(R, -, \varepsilon)\right)$.
We can iterate the construction of U and let

$$\mathbb{K}Herm^{(-j)}(R, -, \varepsilon) = \mathbb{K}Herm\left(U^j(R, -, \varepsilon)\right) , \quad \text{for } j = 1, 2, \cdots .$$

Also we let $\mathcal{L}(R, -, \varepsilon)$ be the homotopy colimit of the diagram

$$\mathbb{K}Herm(R, -, \varepsilon) \to \mathbb{K}Herm^{(-1)}(R, -, \varepsilon) \to \cdots \mathbb{K}Herm^{(-j)}(R, -, \varepsilon) \cdots$$

13 **Theorem 13:** *(Kobal's Hyperbolic Theorem)* There exists a homotopy fibration sequence

$$\mathbb{H}_*(\mathbb{Z}/2, \mathbb{K}R) \xrightarrow{\tilde{H}} \mathbb{K}Herm(R, -, \varepsilon) \to \mathcal{L}(R, -, \varepsilon)$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{H}_*(\mathbb{Z}/2, KR) & \xrightarrow{\tilde{H}} & KHerm(R, -, \varepsilon) \\ \downarrow & & \downarrow \\ \mathbb{H}_*(\mathbb{Z}/2, \mathbb{K}R) & \xrightarrow{\tilde{H}} & \mathbb{K}Herm(R, -, \varepsilon) \end{array}$$

where the top horizontal map was described earlier using results of Thomason.

Let $\mathbb{K}Herm^{(0)}(R, -, \varepsilon) = \mathbb{K}Herm(R, -, \varepsilon)$, and for all $j \in \mathbb{Z}$ we let $\mathbb{K}H^{(j)} = \mathbb{K}Herm^{(j)}(R, -, \varepsilon)$.

Consider the short exact sequence of $\mathbb{Z}[\mathbb{Z}/2]$-modules

$$\mathbb{Z}^{<-1>} \to \mathbb{Z}[\mathbb{Z}/2] \xrightarrow{\varepsilon} \mathbb{Z}$$

where $\varepsilon(n + mT) = n + m$. Here $T \in \mathbb{Z}/2$ acts trivially on $\mathbb{Z}$ and $\mathbb{Z}^{<-1>}$ is a copy of $\mathbb{Z}$ with the nontrivial action by $\mathbb{Z}/2$.

Let $J\colon \mathbb{Z}[\mathbb{Z}/2]$-modules $\to \mathbb{Z}[\mathbb{Z}/2]$-modules be the functor which sends a module P to $P \otimes_{\mathbb{Z}} \mathbb{Z}^{<-1>}$ where $\mathbb{Z}/2$ acts diagonally. Then for $j = 1, 2, \cdots$ we let $P^{<-j>} = J^j(P)$. We let $P^{<0>} = P$. Notice that $P^{<-2>} \cong P$ as $\mathbb{Z}[\mathbb{Z}/2]$-modules.

If we apply $H^*(\mathbb{Z}/2; P \otimes_{\mathbb{Z}} ?)$ to the above sequence we get a long exact sequence with connecting homomorphism $\partial\colon H^*(\mathbb{Z}/2; P) \to H^{*-1}(\mathbb{Z}/2; P^{<-1>})$.

If $\mathbb{K}$ is a spectrum with an action by $\mathbb{Z}/2$ we can perform an analogous construction by replacing $\mathbb{Z}$ by the sphere spectrum. In particular we get a connecting homomorphisms $\mathbb{H}^*(\mathbb{Z}/2; \mathbb{K}) \to \mathbb{H}^*(\mathbb{Z}/2; \Omega^{-1}\mathbb{K}^{<-1>})$.

Warning: $\mathbb{K}^{<-2>}$ is not necessarily equivariantly equivalent to $\mathbb{K}$. Consider the special case when $\mathbb{K}$ is the sphere spectrum with the trivial action and compare homotopy orbits.

There is a homotopy equivalence between $\hat{\mathbb{H}}^*(\mathbb{Z}/2; \mathbb{K})$ and the homotopy colimit of the diagram

$$\mathbb{H}^*\left(\mathbb{Z}/2; \mathbb{K}\right) \to \mathbb{H}^*\left(\mathbb{Z}/2; \Omega^{-1}\mathbb{K}^{<-1>}\right) \to \cdots \mathbb{H}^*\left(\mathbb{Z}/2; \Omega^{-j}\mathbb{K}^{<-j>}\right) \to \cdots ,$$

such that the map $\mathbb{H}^*(\mathbb{Z}/2; \mathbb{K}) \to \hat{\mathbb{H}}^*(\mathbb{Z}/2; \mathbb{K})$ from the norm fibration sequence gets identified with the map

$$\mathbb{H}^*(\mathbb{Z}/2; \mathbb{K}) \to hcolim_j \mathbb{H}^*(\mathbb{Z}/2; \Omega^{-j}\mathbb{K}^{<-j>}) .$$

Consider the following commutative diagram

$$\begin{array}{ccccc} KHerm^{(-j)}(R, -, \varepsilon) & \to & KHerm^{(-j-1)}(R, -, \varepsilon) & \to & \cdots \\ \tilde{F} \downarrow & & \tilde{F} \downarrow & & \\ H^*\left(\mathbb{Z}/2, KU^j(R, -, \varepsilon)\right) & \to & H^*\left(\mathbb{Z}/2, KU^{j+1}(R, -, \varepsilon)\right) & \to & \cdots . \end{array}$$

Notice that each square in this diagram is homotopy cartesian (compare the horizontal homotopy fibers.) One can than conclude that the square in the Hermitian K-theory Theorem is homotopy cartesian by observing that it is equivalent to the following homotopy cartesian square.

$$\begin{array}{ccc} KHerm\left(U^0(R, -, \varepsilon)\right) & \to & hcolim_j\, KHerm\left(U^j(R, -, \varepsilon)\right) \\ \downarrow & & \downarrow \\ H^*\left(\mathbb{Z}/2, KU^0(R, -, \varepsilon)\right) & \to & hcolim_j H^*\left(\mathbb{Z}/2, KU^j(T, -, \varepsilon)\right) . \end{array}$$

If we replace $(R, -, \varepsilon)$ by $U^j(R, -, \varepsilon)$ we get the same Karoubi tower, but shifted to the left j steps. Similarly, if we replace $(R, -, \varepsilon)$ by $V^j(R, -, \varepsilon)$ we get the same

Karoubi tower, but shifted to the right j steps. This observation plus the Karoubi Periodicity theorem in Sect. 3.2 yields the following.

14 **Theorem 14:** *(Generalized Karoubi Periodicity)* Assume 2 is a unit in R. Then the 2nd loop space of $KHerm^{(j)}(R,-,\varepsilon)$ is homotopy equivalent to $KHerm^{(j+2)}(R,-,-\varepsilon)$. Thus $\Omega^2\mathcal{L}(R,-,\varepsilon) \simeq \mathcal{L}(R,-,-\varepsilon)$.

3.4.6 General Definition of L-Groups (when 2 is a Unit)

Let $\mathbb{K}Herm^{(0)}(R,-,\varepsilon) = \mathbb{K}Herm(R,-,\varepsilon)$, and for all $j \in \mathbb{Z}$ we let $\mathbb{K}H^{(j)} = \mathbb{K}Herm^{(j)}(R,-,\varepsilon)$.

The following diagram is called the *Karoubi Tower*.

$$\begin{array}{ccccccc}
 & & \Omega^{j+1}\mathbb{K}R & & \Omega^{j}\mathbb{K}R & & \\
 & & {\scriptstyle H^{(j+1)}}\downarrow & & {\scriptstyle H^{(j)}}\downarrow & & \\
\cdots \to & & \mathbb{K}H^{(j+1)} & \to & \mathbb{K}H^{(j)} & \to & \cdots \\
 & & {\scriptstyle F^{(j+1)}}\downarrow & & {\scriptstyle F^{(j)}}\downarrow & & \\
 & & \Omega^{j+1}\mathbb{K}R & & \Omega^{j}\mathbb{K}R & . &
\end{array}$$

where for each $j \in \mathbb{Z}$

$$\Omega^{(j+1)}\mathbb{K}R \overset{H^{(j+1)}}{\to} \mathbb{K}H^{(j+1)} \to \mathbb{K}H^{(j)} \overset{F^{(j)}}{\to} \Omega^{(j)}\mathbb{K}R$$

is a homotopy fibration sequence.

Furthermore, $\Omega^j\mathbb{K}R \overset{F^{(j)}\circ H^{(j)}}{\to} \Omega^j\mathbb{K}R$ is homotopic to Ω^j of $I \pm T_\varepsilon$.

The $F^{(j)}$ for $j > 0$ can be viewed as *higher order forgetful maps*. The $H^{(j)}$ for $j < 0$ can be viewed as *higher order hyperbolic maps* .

Let $\pi_k F^{(j)}$ and $\pi_k H^{(j)}$ be the induced maps on the k-th homotopy groups.

For any T_ε-invariant subgroup $X \subset K_j(R)$ we let

$$\mathcal{L}^X_{2i}(R) := \frac{\left(\pi_0 F^{(j)}\right)^{-1}(X)}{\left(\pi_0 H^{(j)}\right)(X)}\,, \text{ where } \varepsilon = (-1)^i$$

$$\mathcal{L}^X_{2i+1}(R) := \frac{\left(\pi_1 F^{(j-1)}\right)^{-1}(X)}{\left(\pi_1 H^{(j-1)}\right)(X)}\,, \text{ where } \varepsilon = (-1)^i\,.$$

15 **Proposition 15:** *(Rothenberg Sequence)* (Assume 2 is a unit in R.) For any $i, j \in \mathbb{Z}$ we get an exact sequence

$$\cdots \to \mathcal{L}^{K_{j+1}}_i(R) \to \mathcal{L}^{K_j}_i(R) \to \hat{H}^i\left(\mathbb{Z}/2, K_j\right) \to \cdots$$

Following [88] the proof of this is an easy diagram chase except for exactness at the middle term of

$$\mathcal{L}^{K_j}_{4k+2}(R) \to \hat{H}^0\left(\mathbb{Z}/2, K_j\right) \to \mathcal{L}^{K_{j+1}}_{4k+1}(R)\,.$$

The proof of this step uses the commutativity of the following diagram

$$\begin{array}{ccc}
\Omega^2 K\mathbb{H}erm^{(j-2)}(R,\varepsilon) & \overset{\Omega^2 F^{(j-2)}}{\to} & \Omega^2\Omega^{j-2}\mathbb{K}R \\
{\scriptstyle Periodicity}\downarrow & & \cong\downarrow \\
K\mathbb{H}erm^{(j)}(R,-\varepsilon) & \overset{F^{(j)}}{\to} & \Omega^j\mathbb{K}R\,.
\end{array}$$

It is fairly easy to see that when 2 is a unit in R, Theorem 1.1 implies that $L^{K_j}_i(R) \cong \mathcal{L}^{K_j}_i(R)$ for all $i \in \mathbb{Z}$ and $j = 1$ or 2 (see [40] for details).

16 **Proposition 16:** *(Shaneson Product Formula)* Assume 2 is a unit in R. For all $i \in \mathbb{Z}$, and $j \leq 1$

$$\mathcal{L}^{K_{j+1}}_{i+1}(R) \oplus \mathcal{L}^{K_j}_i(R) \cong \mathcal{L}^{K_{j+1}}_{i+1}\left(R[t,t^{-1}]\right)\,,$$

The map $\mathcal{L}^{K_{j+1}}_{i+1}(R) \to \mathcal{L}^{K_{j+1}}_{i+1}\left(R[t,t^{-1}]\right)$ is induced by a map of Hermitian rings.

Karoubi [39] has constructed pairings $\mathbb{K}H^{(j_1)}(R_1,\varepsilon_1) \times \mathbb{K}H^{(j_2)}(R_2,\varepsilon_2) \to \mathbb{K}H^{(j_1+j_2)}(R_1 \otimes R_2, \varepsilon_1 \otimes \varepsilon_2)$. There exists an element $\sigma \in KHerm_1\left(\mathbb{Z}[\frac{1}{2}][t,t^{-1}]\right)$ such that when i is even, the map $\mathcal{L}^{K_j}_i(R) \to \mathcal{L}^{K_{j+1}}_{i+1}\left(R[t,t^{-1}]\right)$ is induced by pairing with σ. When i is odd, the map uses periodicity plus pairing with σ. The element σ can be viewed as the "round" symmetric signature of the circle (see [65]).

When $j = 1$ one can see that the sum of these two maps is an isomorphism by using the Shaneson product formula from Sect. 3.2. One then does downward induction on j using the Rothenberg sequences.

17 **Theorem 17** Assume 2 is a unit in R. Then

$$L^{K_j}_i(R) \cong \mathcal{L}^{K_j}_i(R)$$

for all $i \in \mathbb{Z}$ and $j \leq 1$.

We already noted this is true when $j = 1$. We then do downward induction of j by using the fact that both sides satisfy a Shaneson Product formula.

3.5 Symmetric and Quadratic Structures on Chain Complexes

See [62–64, 67], and [58]. Connections between geometric topology and algebra can be greatly enhanced by using chain complex descriptions of K-theory and L-theory. Also we want a version of periodicity without the assumption that 2 is a unit.

For example, a *parameterized version* of Whitehead torsion is gotten by applying Waldhausen's $S.$ construction to the category of f.g. projective R-chain complexes to get a more "geometric" model for KR. (See [79–84], and [22])

Our goal in this section is to give a quick introduction to some of the key ideas from the work of Ranicki on L-theory (see also [52]).

Let $(R, -, +1)$ be a hermitian ring. Recall that in Sect. 3.2, symmetric (i.e. hermitian) forms on a module P were defined using the group $Sesq(P)$ equipped with the involution T_ε. Quadratic forms were defined using the map $N_\varepsilon = I + T_\varepsilon\colon Sesq(P) \to Sesq(P)$.

3.5.1 Symmetric Complexes

Given a chain complex

$$\mathcal{C}\colon \cdots \to C_{r+1} \overset{d}{\to} C_r \overset{d}{\to} C_{r-1} \to \cdots C_0 \to 0$$

of f.g. projective R modules write $C^r = (C_r)^*$. Let $\mathcal{C}^{n-*}$ be the chain complex with $\mathcal{C}^{n-*}_r = C^{n-r}$ and $d^{n-*}_{\mathcal{C}} = (-1)^r d_C\colon C^{n-r} \to C^{n-r+1}$.

The duality isomorphisms

$$T\colon Hom_R(C^p, C_q) \to Hom_R(C^q, C_p); \phi \mapsto (-1)^{pq}\phi^*$$

are involutions with the property that the dual of a chain map $f\colon \mathcal{C}^{n-*} \to \mathcal{C}$ is a chain map $Tf\colon \mathcal{C}^{n-*} \to \mathcal{C}$, with $T(Tf) = f$.

Let

$$W\colon \cdots \to \mathbb{Z}[\mathbb{Z}/2] \overset{1+T}{\to} \mathbb{Z}[\mathbb{Z}/2] \overset{1-T}{\to} \mathbb{Z}[\mathbb{Z}/2] \overset{1+T}{\to} \mathbb{Z}[\mathbb{Z}/2] \overset{1-T}{\to} \mathbb{Z}[\mathbb{Z}/2]$$

be the free $\mathbb{Z}[\mathbb{Z}/2]$-module resolution of $\mathbb{Z}$.

A *n-dimensional symmetric chain complex* is a pair $(\mathcal{C}, \phi)$ where $\mathcal{C}$ is an n-dimensional f.g. projective chain complex and ϕ is an n-dim cycle in the $\mathbb{Z}$-module chain complex

$$Hom_{\mathbb{Z}[\mathbb{Z}/2]}\left(W, Hom_R(\mathcal{C}^*, \mathcal{C})\right) .$$

The element ϕ can be viewed as a chain map $\phi_0\colon \mathcal{C}^{n-*} \to \mathcal{C}$, plus a chain homotopy ϕ_1 from ϕ_0 to $T\phi_0$, plus a second order homotopy from ϕ_1 to $T\phi_1$, etc. The pair $(\mathcal{C}, \phi)$ is *Poincaré* i.e. nonsingular, if ϕ_0 is a homotopy equivalence.

Model Example: (Miščenko) Let X^n be an oriented Poincaré complex with universal cover $\widetilde{X}$ and cellular $\mathbb{Z}\pi$-chain complex $C(\widetilde{X})$, where $\pi = \pi_1(X)$. Then capping with the fundamental class $[X]$ yields a chain homotopy equivalence $\phi_0\colon C(\widetilde{X})^{n-*} \to C(\widetilde{X})$. The higher chain homotopies $\phi_1, \phi_2, \cdots$ are given by an analogue of the construction of the Steenrod squares.

By using Poincaré duality for a compact manifold with boundary $(W, \partial W)$ as a model, Ranicki also introduced the notion of *Poincaré symmetric pairs* of complexes and *bordism* of Poincaré symmetric complexes. Then the n-th (projective) symmetric L-group, $L^n_p(R)$, is defined as the group of bordism classes of n-dimensional Poincaré symmetric chain complexes. One also gets symmetric L-groups with other decorations such as $L^n_h(\mathbb{Z}\pi)$ and $L^n_s(\mathbb{Z}\pi)$ by using free or based chain complexes.

An oriented Poincaré complex X, then determines an element $\sigma^*_h(X) \in L^n_h(\mathbb{Z}\pi)$ called the *symmetric signature* of X. If n = $4k$, then the image of $\sigma^*_h(X)$ under the map $L^{4k}_s(\mathbb{Z}\pi) \to L^{4k}_s(\mathbb{Z}) \simeq \mathbb{Z}$ is just the signature of the the pairing.

$$H^{2k}(X, \mathbb{R}) \times H^{2k}(X, \mathbb{R}) \to H^{4k}(X, \mathbb{R}) \simeq \mathbb{R}$$

given by cup products. If X is a manifold M, then $\sigma^*_h X)$ has a preferred lifting to $\sigma^*_s(M) \in L^n_s(\mathbb{Z}\pi)$.

It is easy to see that $L^0_p(R, -, +1) \simeq K_0 Herm(R, -, +1)$/metabolic forms, (see [64] p.66, [64] p.74, and [6] p.12).

Quadratic Chain Complexes

Recall from Sect. 3.4 that given a spectrum $\mathbb{K}$ with action by $\mathbb{Z}/2$ we get a norm map

$$N\colon \Omega^\infty(\Sigma^\infty E\mathbb{Z}/2_+ \wedge_{\mathbb{Z}/2} \mathbb{K}) \to F_{\mathbb{Z}/2}\left(\Sigma^\infty E\mathbb{Z}/2_+, \mathbb{K}\right) .$$

Similarly, we get a norm map for the $\mathbb{Z}[\mathbb{Z}/2]$-chain complex $Hom_R(\mathcal{C}^*, \mathcal{C})$.

$$N\colon W \otimes_{\mathbb{Z}} [\mathbb{Z}/2]\, Hom_R(\mathcal{C}^*, \mathcal{C}) \to Hom_{\mathbb{Z}}[\mathbb{Z}/2]\left(W, Hom_r(\mathcal{C}^*, \mathcal{C})\right) .$$

(Notice that W is the cellular chain complex for $E\mathbb{Z}/2$.)

An *n-dimensional quadratic chain complex* is a pair $(\mathcal{C}, \psi)$ where $\mathcal{C}$ is an n-dimensional f.g. projective chain complex and ψ is an n-cycle in $W \otimes_{\mathbb{Z}} [\mathbb{Z}/2]\, Hom_R(\mathcal{C}^*, \mathcal{C})$. Notice that then $(\mathcal{C}, N(\psi))$ is an n-dim symmetric chain complex. If $(\mathcal{C}, N(\psi))$ is Poincaré, we say $(\mathcal{C}, \psi)$ is Poincaré. Similarly there are notions of *quadratic pairs* and *quadratic bordism*. The n-th (projective) quadratic L-group, $L^p_n(R)$, is the the bordism group of n-dimensional Poincaré quadratic chain complexes. One also gets quadratic L-groups with other decorations such as $L^h_n(\mathbb{Z}\pi)$ and $L^s_n(\mathbb{Z}\pi)$ by using free or based chain complexes.

The norm map N induces a map $1 + T : L_n(R) \to L^n(R)$ for any choice of decoration. If 2 is a unit in R, then $1 + T$ is an isomorphism. Furthermore for all rings R, $1 + T : L_n(R) \otimes \mathbb{Z}[\frac{1}{2}] \to L^n(R) \otimes \mathbb{Z}[\frac{1}{2}]$ is an isomorphism.

Suppose ε is any central unit in R such that $\bar{\varepsilon}\varepsilon = 1$. If we replace T by $T_\varepsilon\colon Hom_R(C^p, C_q) \to Hom_R(C^q, C_p); \phi \mapsto (-1)^{pq}\varepsilon\phi^*$ we get the quadratic groups $L_n(R,\varepsilon)$.

It is easy to see that if $n = 0$ or 1, then these quadratic chain complex descriptions of the quadratic L-groups are consistent with the definitions in Sect. 3.2. The following result implies consistency for all n.

18 **Theorem 18:** *(Ranicki Periodicity)* For all $n \geq 0$, and for all rings R, $L_n^p(R,-\varepsilon) \simeq L^p_{n+2}(R,\varepsilon)$.

Model Example: Suppose $(f : M^n \to X, \hat{f})$ is a surgery problem where f induces an isomorphism on π_1. Let $C(f)$ be the mapping cone of $C(\widetilde{M}) \to C(\widetilde{X})$. It is easily seen that $C(f)$ admits Poinare symmetric structure which represents $\sigma_h^*(X) - \sigma_h^*(M)$ in $L_h^n(\mathbb{Z}\pi_1(M))$. However, Ranicki [63] has shown that the bundle map $\hat{f}$ determines an element $\sigma_*^h(f,\hat{f}) \in L_n^h(\mathbb{Z}\pi_1(X))$ such that $N(\sigma_*(f,\hat{f})) = \sigma_h^*(X) - \sigma_h^*(M)$. Under Ranicki Periodicity, $\sigma_*^h(f,\hat{f})$ gets identified with the surgery obstruction discussed in Sect. 3.3.

There are operations on symmetric and quadratic chain complexes which are algebraic analogs of surgery on a manifold. This *algebraic surgery* is what is used to prove the Ranicki Periodicity Theorem. It would be good to have a better understanding of the relationship between Karoubi and Ranicki Periodicity (Also see Sharpe Periodicity [70] [40].)

Applications of Quadratic Chain Complexes

Besides bordism and surgery there are other geometric operations such as *transversality* which have quadratic chain complex analogues. Ranicki's chain complex description of L-theory has helped to yield many important results.

1. (Instant descriptions of the surgery obstruction)
   Given a surgery problem $(f,\hat{f})$, $\sigma_*(f,\hat{f}) \in L_n(\mathbb{Z}\pi_1(X))$ is defined without first making f highly connected.
2. (Product Formula)
   Suppose N^k is a k-dimensional manifold. There exists a pairing
   $$\mu\colon L^k\left(\mathbb{Z}\pi_1(N)\right) \times L_n\left(\mathbb{Z}\pi_1(X)\right) \to L_{k+n}\left(\mathbb{Z}[\pi_1 N \times \pi_1 X]\right)$$
   such that the surgery obstruction for $id_N \times (f,\hat{f})$ is $\mu\left(\sigma^*(M), \sigma_*(f,\hat{f})\right)$.
3. (Relative L-groups)
   Suppose $f\colon R_1 \to R_2$ is a map of rings with involution. Then there exist 4-periodic *relative* L-groups, $L_n(f)$ such that with appropriate choice of decorations there exists a long exact sequence
   $$\cdots \to L_n(R_1) \to L_n(R_2) \to L_n(f) \to L_{n-1}(R_1) \to \cdots .$$

   Here $L_n(f)$ is defined in terms of n-Poincaré quadratic R_2-pairs where the "boundary" is induced by f from a $(n-1)$-dim Poincaré quadratic R_1-chain complex. When f is a localizing map, then $L_n(f)$ has a description in terms of quadratic linking pairings [53, 61, 64]. When a group G is the result of an amalgamated product or a HNN construction, one gets Mayer–Vietoris sequences for L-theory analogous to those given by Waldhausen for K-theory [15, 59, 60].
4. (L-theory Spectrum)
   Quinn [55, 56] and Ranicki [58, 65] have constructed Ω-spectra $\mathbb{L}^X(\mathbb{Z}\pi)$ and $\mathbb{L}_X(\mathbb{Z}\pi)$ with decorations $X \subset K_j(\mathbb{Z}\pi), j < 2$ such that $\pi_n\mathbb{L}^X\left(\mathbb{Z}\pi)\right) \simeq L_n^X(\mathbb{Z}\pi)$ and $\pi_n\left(\mathbb{L}_X(\mathbb{Z}\pi)\right) \simeq L_X^n(\mathbb{Z}\pi)$. A k-simplex in the infinite loop space associated to $\mathbb{L}^p(\mathbb{Z}\pi)$ is given by a pair $(\mathcal{C}, \phi)$ where $\mathcal{C}$ is a functor from the category of faces of the standard k-simplex Δ^k to the category of f.g. proj. chain complexes of $\mathbb{Z}\pi$-modules, and where ϕ is a Poincaré symmetric structure on such a functor. Thus a 1-simplex is a symmetric bordism, a 2-simplex is a second order symmetric bordism,etc. The definitions of $\mathbb{L}^h(\mathbb{Z}\pi)$ and $\mathbb{L}^s(\mathbb{Z}\pi)$ are similar except projective is replaced by free and based respectively.
   Suppose we let $\mathbb{L}^s = \mathbb{L}^{<2>}, \mathbb{L}^h = \mathbb{L}^{<1>}, \mathbb{L}^p = \mathbb{L}^{<0>}$, and $\mathbb{L}^{K_j} = \mathbb{L}^{<j>}$ for $j = -1,-2,-3,\cdots$. Let C_∞ be the infinite cyclic group Then for $j = 2,1,0,\cdots$, we get that $\mathbb{L}^{<j>}(\mathbb{Z}\pi)$ is homotopy equivalent to the homotopy fiber of $\mathbb{L}^{<j+1>}(\mathbb{Z}\pi) \to \mathbb{L}^{<j+1>}\left(\mathbb{Z}[\pi \times C_\infty]\right)$. Notice that the map $\mathbb{L}^h(\mathbb{Z}\pi) \to \mathbb{L}^p(\mathbb{Z}\pi)$ is induced by commutativity of the following diagram
   $$\begin{array}{ccc} \mathbb{L}^s\left(\mathbb{Z}\pi\right) & \to & \mathbb{L}^s\left(\mathbb{Z}[\pi \times C_\infty]\right) \\ \downarrow & & \downarrow \\ \mathbb{L}^h\left(\mathbb{Z}\pi\right) & \to & \mathbb{L}^h\left(\mathbb{Z}[\pi \times C_\infty]\right) \end{array} .$$
   Then by downward induction on j we get maps $\mathbb{L}^{<j>}(\mathbb{Z}\pi) \to \mathbb{L}^{<j-1>}(\mathbb{Z}[\pi])$ for $j = 2, 1, 0, \cdots$.
   Let $\mathbb{L}^{<-\infty>}(\mathbb{Z}\pi)$ be the homotopy colimit of
   $$\mathbb{L}^p(\mathbb{Z}\pi) \to L^{<-1>}(\mathbb{Z}\pi) \to \cdots L^{<-j>}(\mathbb{Z}\pi) \to \cdots .$$
   Open Question: Are $\mathbb{L}^{<-\infty>}(\mathbb{Z}[\frac{1}{2}]\pi)$ and $\mathcal{L}(\mathbb{Z}[\frac{1}{2}]\pi, -, +1)$ homotopy equivalent?
5. (Block Space of Homeomorphisms)
   Suppose M is a compact manifold, and $Top(M)$ is the singular complex of the topological group of homeomorphisms of M. A k-simplex in $Top(M)$ is given by a homeomorphism $h\colon \Delta^k \times M \to \Delta^k \times M$ which commutes with projection to Δ^k. Classical surgery is not strong enough to determine $Top(M)$ itself so we introduce a pseudo or block version $\widetilde{Top}(M)$; where a k-simplex is a homeomorphism $h\colon \Delta^k \times M \to \Delta^k \times M$ such that for any face $\tau \subset \Delta^k$, $h(\tau \times M) \subset (\tau \times M)$. Notice that we get an inclusion of simplicial groups $Top(M) \subset \widetilde{Top}(M)$. If $G(M)$ is the simplicial monoid of homotopy automorphisms we get

a similar inclusion $G(M) \subset \widetilde{G}(M)$ but in this case the inclusion is a homotopy equivalence. For a Poincaré complex X we let

$$\widetilde{\mathcal{S}}(X) = \sqcup \widetilde{G}(N)/\widetilde{Top}(N) ,$$

where we take the disjoint union over homeomorphism classes of manifolds homotopy equivalent to X.
Notice that a component of $\widetilde{S}(X)$ is represented by a homotopy equivalence $N \to X$.
If X is a manifold M, we let $\widetilde{S}^s(M)$ be the union of the components of $\widetilde{S}(M)$ represented by simple homotopy equivalences. The ideas described in Sect. 3.3 can be used to prove the following theorem. (See [5,58], and [14].)

19 **Theorem 19:** *(Surgery Exact Sequence)* Assume $n > 4$. Suppose M is a n-dimensional closed oriented manifold. There exists a homotopy equivalence between $\widetilde{\mathcal{S}}^s(M)$ and the union of certain components of the -1-connected cover of Ω^n of the homotopy fiber of the assembly map

$$\mathbb{H}_*(M, \mathbb{L}) \to \mathbb{L}^s(\mathbb{Z}\pi_1(M)) ,$$

where $\mathbb{L}$ is the 1-connected cover of $\mathbb{L}^p(\mathbb{Z})$.

The question of which components involves resolving homology manifolds (see [10] and [58]).
The homotopy fiber of $\mathcal{S}(M) \to \widetilde{\mathcal{S}}(M)$ over the "identity vertex" $id\colon M \to M$ is equivalent to $\widetilde{Top}(M)/Top(M)$. It is easy to see that there is an exact sequence $\pi_1 hcob(M) \to \pi_0 Top(M) \to \pi_0 \widetilde{Top}(M)$. Hatcher [33] has shown that here exists a spectral sequence which abuts to $\pi_*(\widetilde{Top}(M)/Top(M))$ and E_2 of the spectral sequence is given in terms of $\pi_*\left(hcob(M \times I^j)\right)$. Recall from the introduction that $HCOB(M)$ is the homotopy colimit of

$$hcob(M) \to hcob(M \times I) \to hcob(M \times I^2) \cdots .$$

In [90] it is shown that there exists an involution on the infinite loop space $HCOB(M)$ such that if $HCOB_s(M)$ is the 0-connected cover of $HCOB(M)$ then there exists a map $\widetilde{Top}(M)/Top(M) \to \mathbb{H}_*(\mathbb{Z}/2, HCOB_s(M))$ which is at least $k+1$ connected where $\dim M \geq \max(2k+7, 3k+4)$ and M is smoothable.

6. (Map from L-theory to Tate of K-theory)
In order to study $S(M)$ instead of $\widetilde{S}(M)$ in the next section we need to understand how to "glue together" L-theory with higher K-theory. Suppose R is any ring with involution, and $X \subset K_j(R)$ is an involution invariant subgroup. We let $c_X\colon \mathbb{K}^X(R) \to \mathbb{K}(R)$ be such that $\pi_i(\mathbb{K}^X(R)) = 0$, for $i < j$, $\pi_j(\mathbb{K}^X(R)) = X$, and c_X induces an isomorphism on π_i for $i > j$. Then one can construct the following homotopy cartesian square,

$$\begin{array}{ccc} \mathbb{L}^X(R) & \to & \mathbb{L}^{<-\infty>}(R) \\ {\scriptstyle \Xi^X}\downarrow & & {\scriptstyle \Xi}\downarrow \\ \hat{\mathbb{H}}(\mathbb{Z}/2, \mathbb{K}^X(R)) & \to & \hat{\mathbb{H}}(\mathbb{Z}/2, \mathbb{K}(R)) . \end{array}$$

It is then very easy to see that we get the Rothenberg sequences and Shaneson formulae mentioned in the introduction. If $j < 2$, this $\mathbb{L}^X(R)$ is consistent with the one constructed by Quinn and Ranicki.
The map Ξ is constructed by using the Thomason homotopy limit problem map $K(\mathcal{C}^{hG}) \to \mathbb{H}^*(G, K\mathcal{C})$ plus a "bordism-like" model for $\hat{\mathbb{H}}(\mathbb{Z}, \mathbb{K}(R))$, see [91,92]. It would be good to have a better understanding of the relationship between Ξ and the right vertical map in the Hermitian K-theory Theorem.

Manifold Structures

Let M^n be a connected, oriented, closed manifold.
Our first goal is to explain the following tower of simplicial sets

$$\mathcal{S}(M) \to \mathcal{S}^b(M \times \mathbb{R}^1) \to \cdots \to \mathcal{S}^b(M \times \mathbb{R}^j) \to \cdots .$$

Given two spaces over $\mathbb{R}^j$, $X \xrightarrow{p} \mathbb{R}^j$ and $Y \xrightarrow{q} \mathbb{R}^j$, we say that a continuous map $f\colon X \to Y$ is bounded if there exists $K \in \mathbb{R}$ such that for all $x \in X$, $|p(x) - q(h(x))| < K$. When we write $M \times \mathbb{R}^j$ we mean the space over $\mathbb{R}^j$ given by the projection map $M \times \mathbb{R}^j \to \mathbb{R}^j$.
Given $p\colon X \to \mathbb{R}^j$ we get the following diagram of simplicial monoids

$$\begin{array}{ccc} Top^b(p) & \to & G^b(p) \\ \downarrow & & \downarrow \\ \widetilde{Top}^b(p) & \to & \widetilde{G}^b(p) , \end{array}$$

where the superscript "b" denotes the fact we are using bounded versions of the simplicial monoids defined in previous sections. The map $G^b(p) \to \widetilde{G}^b(p)$ is a homotopy equivalence. Furthermore, the map $G(M) \to G^b(M \times \mathbb{R}^j)$ gotten by crossing with $id_{\mathbb{R}^j}$ is a homotopy equivalence.
We say that $p\colon V^m \to \mathbb{R}^j$ is an *m-dimensional manifold approximate fibration* if V is an m-dimensional manifold, p is proper, and p satisfies the ε-homotopy lifting property for all $\varepsilon \geq 0$, (see [37]).

Key Example: (Siebenmann and Hughes–Ranicki [37, Chap.16]) Assume $n > 4$. Let W^n be a manifold with a tame end ε, Then ε has a neighborhood which is the total space of a manifold approximate fibration over R^1.

Let

$$\mathcal{S}^b(M^n \times \mathbb{R}^j) := \sqcup\, G^b\left(p\colon V^{n+j} \to \mathbb{R}^j\right) / Top\left(p\colon V^{n+j} \to \mathbb{R}^j\right) ,$$

where we take the disjoint union over bounded homeomorphism classes of $(n+j)$-dimensional manifold approximate fibrations homotopy equivalent to M. Notice that if $j = 0$, then $\mathcal{S}^b(M \times \mathbb{R}^j) = \mathcal{S}(M)$.

Crossing with the identity map on $\mathbb{R}^1$ gives maps $\mathcal{S}^b(M \times \mathbb{R}^j) \to \mathcal{S}^b(M \times \mathbb{R}^{j+1})$, and $Top^b(M \times \mathbb{R}^j) \to Top^b(M \times \mathbb{R}^{j+1})$. Let $\mathcal{S}^{-\infty}(M) = hcolim_j \mathcal{S}^b(M \times \mathbb{R}^j)$.

Let $hcob^b(M \times \mathbb{R}^j)$ be the simplicial set of bounded h-cobordisms on $M \times \mathbb{R}^j$. Then there exists a homotopy fibration (see [3,4])

$$hcob^b(M \times \mathbb{R}^j) \xrightarrow{\mathcal{P}} \mathcal{S}^b(M \times \mathbb{R}^j) \to \mathcal{S}^b(M \times \mathbb{R}^{j+1}) .$$

Furthermore $\Omega hcob^b(M \times \mathbb{R}^j) \simeq hcob^b(M \times I \times \mathbb{R}^{j-1})$. This makes $HCOB(M) = hcolim_i hcob^b(M \times I^i)$ into the 0-th space of an Ω-spectrum with j-th delooping given by $HCOB(M \times \mathbb{R}^j) = hcolim_i hcob^b(M \times I^i \times \mathbb{R}^j)$.

Let $\pi = \pi_1(M)$, then (see [3])

$$\pi_k(HCOB(M)) = \begin{cases} Wh_1(\pi), & \text{for } k = 0 \\ \widetilde{K}_0(\mathbb{Z}\pi), & \text{for } k = -1 \\ K_{k+1}(\mathbb{Z}\pi), & \text{for } k < -1 . \end{cases}$$

Anderson and Hsiang have conjectured that for $k < 1$, $K_k(\mathbb{Z}\pi)$ is trivial. Carter [17] has proved this for finite groups. Farrell and Jones [24] have proved this for virtually infinite cyclic groups

Let $A(X)$ be Waldhausen's algebraic K-theory of the connected space X, see [80]. Let $\mathbb{A}(X)$ be the disconnected Ω-spectrum constructed by Vogel [76] [77] such that $A(X) \to \mathbb{A}(X)$ induces an isomorphism on homotopy groups in positive dimensions. Also there exists a linearization map $\mathbb{A}(X) \to \mathbb{K}(\mathbb{Z}\pi_1(X))$ which is 1-connected. Let $\Omega WH(X)$ be the homotopy fiber of the assembly map $\mathbb{H}^*(X, \mathbb{A}(*)) \to \mathbb{A}(X)$. Then we get

(1:) There exists a homotopy equivalence $HCOB(M) \to \Omega WH(M)$. (See [79–84], and [22, §9].)

(2:) There exists a homotopy fibration sequence

$$Top^{-\infty}(M)/Top(M) \to \mathcal{S}(M) \to \mathcal{S}^{-\infty}(M) ,$$

where $Top^{-\infty}(M) = hcolim^j Top^b(M \times \mathbb{R}^j)$.

(3:) There exists an involution T on $\Omega WH(M)$ and a map

$$\psi : Top^{-\infty}(M)/Top(M) \to \mathbb{H}_*\left(\mathbb{Z}/2, \Omega WH(M)\right)$$

which is at least $k+1$ connected where k satisfies $\dim M \geq \max(2k+7, 3k+4)$ and M is smoothable. (See [90].)

3.6.1 Higher Whitehead Torsion

Recall the Whitehead torsion map τ: $\pi_0(\mathcal{S}(M)) \to Wh_1(M)$. We want to promote τ to a map of spaces $\mathcal{S}(M) \to \Omega WH(M)$ and analogous maps for $\mathcal{S}(M \times \mathbb{R}^j)$. (See [18–21] and [38].)

Let Q be the Hilbert cube, and let $\mathcal{S}^b_Q(M \times \mathbb{R}^j)$ be the same as $\mathcal{S}^b(M \times \mathbb{R}^j)$ but instead of using finite dimensional manifolds we use Hilbert cube manifolds. Then $\Omega\mathcal{S}^b_Q(M \times \mathbb{R}^j) \simeq \mathcal{S}^b_Q(M \times \mathbb{R}^{j-1})$. Thus $\mathcal{S}^b_Q(M \times \mathbb{R}^j)$ is an infinite loop space for all j. We'll abuse notation and let $\mathcal{S}^b_Q(M \times \mathbb{R}^j)$ also denote the associated Ω-spectrum. We get the following properties.

(1:) $\mathcal{S}^b_Q(M \times \mathbb{R}^j) \simeq HCOB(M \times \mathbb{R}^j) \simeq$ the j-th delooping of $\Omega WH(M)$

(2:) The map $\mathcal{S}(M) \xrightarrow{\times Q} S_Q(M)$ induces the torsion map τ when we apply π_0.

(3:) The map $\mathcal{S}^b(M \times \mathbb{R}^{j-1}) \to \mathcal{S}^b(M \times \mathbb{R}^j)$ has a lifting to the homotopy fiber of $\mathcal{S}^b(M \times \mathbb{R}^j) \xrightarrow{\times Q} \mathcal{S}^b_Q(M \times \mathbb{R}^j)$ which is at least $j+k+1$ connected where $\dim M \geq \max(2k+7, 3k+4)$ and M is smoothable.

(4:) There exists a homotopy commutative diagram

$$\begin{array}{ccc} hcob(M \times \mathbb{R}^j) & \xrightarrow{\mathcal{P}} & \mathcal{S}^b(M \times \mathbb{R}^j) \\ \downarrow & & \downarrow \\ \Omega^{1-j} WH(M) & \xrightarrow{1+(-1)^j T} & \Omega^{1-j} WH(M) \ ; \end{array}$$

where the left vertical map is the composition $hcob(M \times \mathbb{R}^j) \to HCOB(M \times \mathbb{R}^j) \simeq \Omega^{1-j} WH(M)$. and the right vertical map is the composition $\mathcal{T} : \mathcal{S}^b(M \times \mathbb{R}^j) \xrightarrow{\times Q} \mathcal{S}^b_Q(M \times \mathbb{R}^j) \simeq \Omega^{1-j} WH(M)$.

Notice the analogy between the following tower and the right half of the Karoubi Tower described in Sect. 3.4.

$$\begin{array}{ccccc} hcob(M) & & hcob^b(M \times \mathbb{R}^1) & & hcob^b(M \times \mathbb{R}^j) \\ \mathcal{P}\downarrow & & \mathcal{P}\downarrow & & \mathcal{P}\downarrow \\ \mathcal{S}(M) & \to & \mathcal{S}^b(M \times \mathbb{R}^1) & \to \cdots & \mathcal{S}^b(M \times \mathbb{R}^j) \to \cdots \\ \mathcal{T}\downarrow & & \mathcal{T}\downarrow & & \mathcal{T}\downarrow \\ \Omega WH(M) & & WH(M) & & \Omega^{1-j} WH(M) \end{array}$$

Bounded Block Structure Spaces

In order to use surgery theory to compute $\mathcal{S}^{-\infty}(M)$ we need to introduce the block or pseudo version of $\mathcal{S}^b(M \times \mathbb{R}^j)$.

Let

$$\widetilde{\mathcal{S}}^b(M^n \times \mathbb{R}^j) := \sqcup\, \widetilde{G}^b(p \colon V^{n+j} \to \mathbb{R}^j)/\widetilde{Top}(p \colon V^{n+j} \to \mathbb{R}^j) ,$$

where we take the disjoint union over bounded homeomorphism classes of $(n+j)$-dimensional manifold approximate fibrations homotopy equivalent to M. Notice that if $j = 0$, then $\widetilde{\mathcal{S}}^b(M \times \mathbb{R}^j) = \widetilde{\mathcal{S}}(M)$.

Notice that crossing with the identity map on $\mathbb{R}^1$ gives a map $\widetilde{\mathcal{S}}^b(M \times \mathbb{R}^j) \to \widetilde{\mathcal{S}}^b(M \times \mathbb{R}^{j+1})$. Let $\widetilde{\mathcal{S}}^{-\infty}(M) = hcolim_j \widetilde{\mathcal{S}}^b(M \times \mathbb{R}^j)$.

20 **Theorem 20:** *(Stabilization Kills the Difference Between Honest and Pseudo)* The maps $\mathcal{S}^b(M \times \mathbb{R}^j) \to \widetilde{\mathcal{S}}^b(M \times \mathbb{R}^j)$, for $j = 0, 1, \cdots$ induce a homotopy equivalence $\mathcal{S}^{-\infty}(M) \simeq \widetilde{\mathcal{S}}^{-\infty}(M)$. (See [90].)

Since $\pi_0(\mathcal{S}^b(M \times \mathbb{R}^j)) \simeq \pi_0(\widetilde{\mathcal{S}}^b(M \times \mathbb{R}^j))$ we get a "torsion map" $\pi_0(\widetilde{\mathcal{S}}^b(M \times \mathbb{R}^j)) \to \pi_{1-j}(\Omega WH(M))$. Let $\widetilde{\mathcal{S}}^{b,s}(M \times \mathbb{R}^j)$ be the union of the components of $\widetilde{\mathcal{S}}^b(M \times \mathbb{R}^j)$ with trivial torsion.

21 **Theorem 21:** *(Bounded Surgery Exact Sequence)* Assume $n + j > 4$. Suppose M is a n-dimensional closed oriented manifold. There exists a homotopy equivalence between $\widetilde{\mathcal{S}}^{b,s}(M \times \mathbb{R}^j)$ and the -1-connected cover of Ω^n of the homotopy fiber of the assembly map

$$\mathbb{H}^*(M, \mathbb{L}) \to \mathbb{L}^{<2-j>}\left(\mathbb{Z}\pi_1(M)\right) ,$$

where $\mathbb{L}$ is the 1-connected cover of $\mathbb{L}^p(\mathbb{Z})$.

Thus we get that $\mathcal{S}^{-\infty}(M) \simeq \widetilde{\mathcal{S}}^{-\infty}(M)$ is homotopy equivalent to the -1-connected cover of Ω^n of the homotopy fiber of the assembly map

$$\mathbb{H}_*(M, \mathbb{L}) \to \mathbb{L}^{<-\infty>}(\mathbb{Z}\pi_1(M)) .$$

Notice that so far we have explained the top horizontal map in the Manifold Structure Theorem from the introduction. Also we have outlined the proofs of the following parts of that theorem: (1), (2), (3) , and (4).

The diagram in the Manifold Structure Theorem is then a consequence of constructing an involution T on $\Omega WH(M)$ and factorizations $\widetilde{\mathcal{T}}$ of $\mathcal{T}: \mathcal{S}(M \times \mathbb{R}^j) \to \Omega^{1-j}WH(M)$ thru $\mathbb{H}^*(\mathbb{Z}/2, \Omega^{1-j}WH(M))$ for $j = 0, 1, 2, \cdots$ such that we get commutative diagrams

$$\begin{array}{ccc}
\mathcal{S}(M \times \mathbb{R}^j) & \to & \mathcal{S}(M \times \mathbb{R}^{j+1}) \to \cdots \\
\widetilde{\mathcal{T}} \downarrow & & \widetilde{\mathcal{T}} \downarrow \\
\mathbb{H}^*(\mathbb{Z}/2, \Omega^{1-j}WH(M)^{<j>}) & \to & \mathbb{H}^*(\mathbb{Z}/2, \Omega^{-j}WH(M)^{<j+1>}) \to \cdots .
\end{array}$$

3.6.3 More About Torsion

First we'll recall more about the construction of the map $\mathcal{T}: \mathcal{S}(M) \to \Omega WH(M)$.

Recall that $\Omega Wh(M)$ is the homotopy fiber of the assembly map $\mathbb{H}_*(M; A(*)) \to A(M)$, and that $\Omega Wh(M)$ is the (-1)-connected cover of $\Omega WH(M)$.

Suppose G is a simplicial monoid and A is a simplicial G-set. Then $A^{hG} = Map_G(EG, A) = Sec(EG \times_G A \to BG)$, where $Sec(\ )$ denotes the simplicial set of sections.

Notice that $\Omega Wh(M)$ is also the homotopy fiber of

$$EG(M) \times_{G(M)} \mathbb{H}_*(M; A(*)) \to EG(M) \times_{G(M)} A(M) ,$$

where $G(M)$ is the simplicial monoid of homotopy automorphisms of M. The map $\mathcal{T}$ is constructed by first constructing $\chi \in A(X)^{hG(X)}$ and then a lifting $\chi^{\%}: BTop(M) \to EG(M) \times_{G(M)} \mathbb{H}_*(M; A(*))$ of the composition $BTop(M) \to BG(M) \xrightarrow{\chi} EG(M) \times_{G(M)} A(M)$.

Thus $\chi^{\%} \in H^*(M; A(*))^{hTop(M)}$.

Construction of of χ :
For any space X, $\mathcal{R}(X)$ is the category of retractive spaces over the topological space X. Thus an object in $\mathcal{R}(X)$ is a diagram of topological spaces $W \underset{s}{\overset{r}{\rightleftarrows}} X$ such that $rs = id_X$ and s is a closed embedding having the homotopy extension property. The morphisms in $\mathcal{R}(X)$ are continuous maps over and relative to X. A morphism is a *cofibration* if the underlying map of spaces is a closed embedding having the homotopy extension property. A morphism is a *weak equivalence* if the underlying map of spaces is a homotopy equivalence.
Let $\mathcal{R}^{fd}(X)$ be the full subcategory of homotopy finitely dominated retractive spaces over X (see [22, +II,Sec.6] for details). Then $\mathcal{R}^{fd}(X)$ is a category with cofibrations and weak equivalences, i.e. a Waldhausen category, and $A(X)$ is the K-theory of $\mathcal{R}^{fd}(X)$.

If X is a finitely dominated CW complex we let $\chi(X)$ be the vertex in $A(X)$ represented by the retraction space $X \sqcup X \underset{s}{\overset{r}{\rightleftarrows}} X$ where r is the identity on each copy of X, and s is the inclusion into the first copy of X. Suppose $p: E \to B$ is a fibration with finitely dominated fibers, then it is shown in [22] that the rule $b \mapsto \chi(p^{-1}(b)) \in A(p^{-1}(b))$ for each $b \in B$ is continuous. If we apply this to the universal M-fibration over $BG(M)$, this continuous rule is the desired map $\chi: BG(M) \to EG(M) \times_{G(M)} A(M)$.

The lifting $\chi^{\%}$ is constructed using controlled topology in [22].

The construction of $\mathcal{T}: \mathcal{S}(M \times \mathbb{R}^j) \to \Omega^{1-j}WH(M)$ for $j > 0$ is similar except $A(X)$ is replaced by Vogell's $A^b(X \times \mathbb{R}^j)$ where $\Omega^j A^b(X \times \mathbb{R}^j) \simeq A(X)$.

Poincaré Duality and Torsion

Recall Thomason's map $K\left(\mathcal{C}^{h\mathbb{Z}/2}\right) \to \mathbb{H}^*\left(\mathbb{Z}/2; K(\mathcal{C})\right)$ where $\mathcal{C}$ is a $\mathbb{Z}/2$-symmetric monoidal category.

Recall that Waldhausen used his $S_.$ construction to define K-theory for Waldhausen, i.e. categories with cofibrations and weak equivalences. In [92] axioms are given for the notion of duality D in a Waldhausen category $\mathcal{C}$. Duality in $\mathcal{C}$ can be used to define non-singular pairings in $\mathcal{C}$ and an involution so that one gets a map $K\left(\text{non-singular } D\text{-pairings in } \mathcal{C}\right) \to \mathbb{H}^*\left(\mathbb{Z}/2; K(\mathcal{C})\right)$

Examples:
(1): Suppose $\mathcal{C} = Ch(R)$ is the category of f.g. projective chain complexes over R which is equipped with an (anti)-involution. The weak equivalences are the chain homotopy equivalences. The cofibrations are the chain maps which are split mono in each dimensional For each $n = 0, 1, 2, \cdots$ there exists a duality D_n-such that the non-singular D_n-pairings are n-dimensional Poincaré symmetric complexes in the sense of Ranicki.
(2): Suppose $\mathcal{C} = \mathcal{R}^{fd}(X)$, where X is equipped with a spherical fibration η. Then by essentially just following Vogel [75,91,92,94] one gets dualities D_n for $n = 0, 1, 2, \cdots$ such that if X is an n-dimensional Poincaré complex and η is the Spivak fibration of X, then the retractive space $X \sqcup X \underset{s}{\overset{r}{\rightleftarrows}} X$ has a preferred non-singular self D_n-pairing. The homotopy invariance of the Spivak fibration and this preferred pairing implies that we have a lifting of of χ to $\mathbb{H}^*(\mathbb{Z}/2; A(X))^{hG(X)}$.
(3): Same as (2) except weak equivalence are controlled (see [22, §2 and §7]) and $X = M$ is a closed manifold. Then we get the desired $\chi^{\%} \in \mathbb{H}^*(\mathbb{Z}/2; \mathbb{H}_*(M; A(*))^{hTop(M)}$.

The construction of $\widetilde{\mathcal{T}} : \mathcal{S}(M \times \mathbb{R}^j) \to \mathbb{H}^*\left(\mathbb{Z}/2; \Omega^{1-j} WH(M)\right)$ for $j > 0$ is similar.

With the exception of showing that the square in the Manifold Structure Theorem is homotopy cartesian for a certain range, we are now done.

3.6.5 Homotopy Cartesian for a Range

For $j = 0, 1, 2, \cdots$ we get the following homotopy commutative diagram.

$$\begin{array}{ccc} \mathcal{S}(M \times \mathbb{R}^j) & \to & \mathcal{S}(M \times \mathbb{R}^{j+1}) \\ \widetilde{\mathcal{T}} \downarrow & & \widetilde{\mathcal{T}} \downarrow \\ \mathbb{H}^*(\mathbb{Z}/2, \Omega^{1-j} WH(M)^{<j>}) & \to & \mathbb{H}^*(\mathbb{Z}/2, \Omega^{-j} WH(M)^{<j+1>}) \ . \end{array}$$

The top horizontal homotopy fiber is $hcob(M \times \mathbb{R}^j)$, the bottome horizontal homotopy fiber is $\Omega^{1-j} WH(M)^{<j>}$). The induced map Σ between them is the composition of the stabilization map $hcob(M \times \mathbb{R}^j)) \to HCOB(M \times \mathbb{R}^j))$ and the equivalence $HCOB(M \times \mathbb{R}^j)) \simeq \Omega^{1-j} WH(M)$. By Anderson–Hsiang [3] $hcob(M \times \mathbb{R}^j)) \to HCOB(M \times \mathbb{R}^j))$ induces an isomorphism on π_k for $k \le j$. Also if we loop this map j times we get the stabilization map $hcob(M) \to HCOB(M)$ which Igusa has shown is at least $k + 1$ connected where k satisfies $\dim M \ge \max(2k + 7, 3k + 4)$ and M is smoothable. The connectivity of $hcob(M) \to HCOB(M)$ is called the *h-cobordism stable range.*

By combining this with Sect. 3.4.5 we get part (5) of the Manifold Structure Theorem.

3.6.6 More on Connections Between Quadratic Forms and Manifold Structures

Let $\mathcal{S}^s(M)$ be the components of $\mathcal{S}(M)$ with trivial torsion. Let $\Omega Wh_s(M)$ be the 0-connected cover of $\Omega WH(M)$ Then we get the following homotopy commutative diagram which is homotopy cartesian for the same range as the diagram in the Manifold Structure Theorem.

$$\begin{array}{ccc} \mathcal{S}^s(M) & \to & \widetilde{\mathcal{S}}^s(M) \\ \downarrow & & \downarrow \\ \mathbb{H}^*\left(\mathbb{Z}/2; \Omega Wh_s(M)\right) & \to & \hat{\mathbb{H}}^*\left(\mathbb{Z}/2; \Omega Wh_s(M)\right) \ . \end{array}$$

It is natural to ask for a so-called "super simple" form of surgery theory such that its assembly map determines $\mathcal{S}^s(M)$ (at least in the h-cobordism stable range) in the same way that $\mathbb{L}^s$ determines $\widetilde{\mathcal{S}}^s(M)$ via the surgery exact sequence. This leads one to ask for an algebraic description of the the right vertical map in the above diagram. In particular one might ask how this map is related to the right vertical map in the Hermitian K-theory Theorem, or the map $\Xi^p : \mathbb{L}^p(\mathbb{Z}\pi_1(M)) \to \hat{\mathbb{H}}^*\left(\mathbb{Z}/2, K\left(\mathbb{Z}\pi_1(M)\right)\right)$ from Sect. 3.5.

Suppose $(\mathcal{C}, D)$ is a Waldhausen category with duality such as examples (1), (2), and (3). Then we get a quadratic L-theory spectrum $\mathbb{L}_*(\mathcal{C}, D)$, a symmetric L-theory spectrum $\mathbb{L}^*(\mathcal{C}, D)$, a $1 + T$ map $\mathbb{L}_*(\mathcal{C}, D) \to \mathbb{L}^*(\mathcal{C}, D)$, an involution on $K\mathcal{C}$, and a map Ξ: $\mathbb{L}^*\left(\mathcal{C}, D\right) \to \hat{\mathbb{H}}^*(\mathbb{Z}; K(\mathcal{C}))$. (It might be interesting to compare this L-theory of Waldhausen categories with duality with Balmer's notion of Witt groups for triangulated categories [7].) Examples:
(1): Suppose $\mathcal{C} = Ch_R$ and $D = D_n$, then $\mathbb{L}_*(Ch_R C, D_n) = \mathbb{L}^p_n(R) = \Omega^n \mathbb{L}^p(R)$. Similarly $\mathbb{L}^*(Ch_R C, D_n) = \Omega^n \mathbb{L}_p(R)$.
(2): Suppose $\mathcal{C} = \mathcal{R}^{fd}(X)$ where X is equipped with the oriented spherical fibration η. Then for $n = 0, 1, \cdots$ we get a homotopy equivalence $\mathbb{L}_*\left(\mathcal{R}^{fd}(X), D_n\right) \to \mathbb{L}^p_n\left(\mathbb{Z}\pi_1(X)\right)$, but the analogous map for symmetric L-theory is not an equivalence. Thus we get a map $\Xi : \mathbb{L}^p_n\left(\mathbb{Z}\pi_1(X)\right) \to \hat{\mathbb{H}}^*\left(\mathbb{Z}; A(X)\right)$.
(3): By using the controlled version of example (2) we get that Ξ is natural with respect to assembly maps, i.e. we get the following diagram which commutes up to a preferred homotopy.

$$\begin{array}{ccc} \mathbb{H}_*\left(M; \mathbb{L}^p_n(\mathbb{Z})\right) & \to & \mathbb{L}^p\left(\mathbb{Z}\pi_1(X)\right) \\ \downarrow & & \downarrow \\ \hat{\mathbb{H}}^*\left(\mathbb{Z}/2; \mathbb{H}_*\left(M; A(*)\right)\right) & \to & \hat{\mathbb{H}}^*\left(\mathbb{Z}/2; A(M)\right) \ . \end{array}$$

There is an analogous s-version of this diagram. By the Surgery Exact Sequence the induced map on the horizontal homotopy fibers of the s-version is a map

$$\widetilde{\mathcal{S}}^s(M) \to \hat{\mathbb{H}}^*\left(\mathbb{Z}/2; \Omega Wh_s(M)\right)$$

which can be identified with the right vertical map in the previous diagram.

Localize at Odd Primes

If we localize at odd primes, it is easy to see that $\mathcal{S}^s(M) \simeq \widetilde{\mathcal{S}}^s(M) \times \widetilde{Top}(M)/Top(M)$, see [12, 13], or [93, 1.5.2]. Burghelea and Fiedorowicz [11, 12] have used this to show that in the h-cobordism stable range $\mathcal{S}^s(M)$ can be rationally computed using $KHerm(\mathbb{Z}\Omega M)$, where ΩM is the simplicial group gotten by applying Kan's G-functor to the singular complex of M. In order to get a similar result at odd primes one needs to replace $\mathbb{Z}$ with the sphere spectrum, see [26] and [27].

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