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DIFFERENTIABLE S^1 ACTIONS ON HOMOTOPY SPHERES

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§1. Introduction

In this paper we study fixed point free, differentiable actions of the circle on homotopy spheres, using machinery developed by Sullivan. The paper is largely expository and, in part, is devoted to expanding the remarks in [23] and [24] on free circle actions. In addition some new results, computations, and examples are included. The author felt that this longer, partially expository paper, detailing the application of Sullivan's ideas, would be significantly more useful to those interested in circle actions than would be a paper which described only the rather technical, new results.

Denote an action $T: S^1 \times \Sigma^{2n+1} \to \Sigma^{2n+1}$ by (Σ^{2n+1}, T). It is well-known that the orbit space $P^{2n} = \Sigma^{2n+1}/T$ is a differentiable manifold homotopy equivalent to complex projective space [14]. In fact, the principal S^1 bundle, $T: \Sigma^{2n+1} \to P^{2n}$, is classified by a map to the universal bundle, H, over $\mathbb{C}P(n)$. That is, there is a commutative diagram

$$\begin{array}{ccc} \Sigma^{2n+1} & \longrightarrow & S^{2n+1} \\ \downarrow T & & \downarrow H \\ P^{2n} & \xrightarrow{f} & \mathbb{C}P(n) \end{array}$$

with f a homotopy equivalence.

Remark. If $\mathbb{C}P(n)$ is regarded as the space of lines in $\mathbb{C}^{n+1}$ then there is an obvious "canonical line bundle" over $\mathbb{C}P(n)$. By H, or the Hopf bundle, we mean the dual of this canonical bundle. With this notation the normal bundle of $\mathbb{C}P(n)$ in $\mathbb{C}P(n+1)$ is H. This will be convenient later.

Let $z = c_1(H) \in H^2(\mathbb{C}P(n), \mathbb{Z})$ be the first Chern class of H. Then $f^*z \in H^2(P^{2n}, \mathbb{Z})$ is the first Chern class of the S^1 bundle $T\colon \Sigma^{2n+1} \to P^{2n}$. Further, $(f^*z)^n \in H^{2n}(P^{2n}, \mathbb{Z})$ defines an orientation of P^{2n}, and this, together with the S^1 action on Σ^{2n+1}, determines an orientation of Σ^{2n+1}. For instance, Σ^{2n+1} is the boundary of an oriented D^2 bundle over the oriented manifold P^{2n}. When we write (Σ^{2n+1}, T), it will be understood that the orientation on $\Sigma^{2n+1} \in \Gamma_{2n+1}$ (Γ_{2n+1} is the group of oriented differentiable structures on the sphere) agrees with the orientation determined by the S^1 action on the manifold Σ^{2n+1}.

It is easy to see that two S^1 actions, (Σ_o^{2n+1}, T_o) and (Σ_1^{2n+1}, T_1) are equivariantly diffeomorphic if and only if there is a diffeomorphism $h\colon P_o^{2n} \xrightarrow{\approx} P_1^{2n}$ such that the diagram

$$\begin{array}{ccc} P_o^{2n} & & \\ & \searrow^{f_o} & \\ h \downarrow \wr & & \mathbb{C}P(n) \\ & \nearrow_{f_1} & \\ P_1^{2n} & & \end{array}$$

commutes up to homotopy. Here P_j^{2n} is Σ_j^{2n+1}/T_j and f_j is the classifying map for $T_j\colon \Sigma_j^{2n+1} \to P_j^{2n}$, $j = 0,1$. Since $\mathbb{C}P(n)$ is a skeleton of $K(\mathbb{Z},2)$, homotopy commutativity holds if and only if $h^* f_1^*(z) = f_o^*(z)$. It follows that if $\tilde{h}\colon \Sigma_o^{2n+1} \xrightarrow{\simeq} \Sigma_1^{2n+1}$ is an equivariant diffeomorphism covering h, then $\tilde{h}$ preserves orientations.

Given a differentiable manifold M^k, Sullivan defines a homotopy smoothing of M^k to be a homotopy equivalence of pairs, $f\colon L^k, \partial L^k \to M^k, \partial M^k$, where L^k is a smooth manifold. Two homotopy smoothings (L_o^k, f_o) and (L_1^k, f_1) are called equivalent if there is a diffeomorphism $h\colon L_o^k \xrightarrow{\simeq} L_1^k$ such that the diagram

$$\begin{array}{ccc} L_o^k, \partial L_o^k & & \\ h \downarrow \wr & \searrow^{f_o} & M^k, \partial M^k \\ L_1^k, \partial L_1^k & \nearrow_{f_1} & \end{array}$$

commutes up to homotopy. The set of equivalence classes of homotopy smoothings of M^k is denoted $hS(M^k)$. From the discussion above, we see that there is a natural 1-1 correspondence between equivariant diffeomorphism classes of free S^1 actions on homotopy $2n+1$ - spheres and elements of $hS(\mathbb{C}P(n))$.

We next outline briefly Sullivan's homotopy theoretic description of $hS(M^k)$. We assume that M^k and ∂M^k are simply connected and that $k \geq 6$.

Let F/O be the fibre of the natural map $BO \to BF$.

A map $g: X \to F/O$ consists of a stable vector bundle $\xi = \xi(g)$ over X (The composition $X \xrightarrow{g} F/O \xrightarrow{i} BO$) together with a fibre homotopy trivialization of the sphere bundle $S(\xi)$ associated to ξ, that is, a map $G: S(\xi) \to S^N$, $N = \dim(\xi)$, which is of degree one on each fibre. Two F/O bundles (ξ_o, G_o) and (ξ_1, G_1) are equivalent if there is an F/O bundle (ξ, G) over $X \times I$ and bundle isomorphisms $f_j: \xi_j \xrightarrow{\sim} \xi|_{X \times j}$, $j = 0,1$, such that $Gf_j \sim G_j$.

Suppose that $X = M^k$ is a manifold. By making $G: S(\xi) \to S^N$ transverse regular to a point in S^N, we may assume that there is a framed submanifold $L^k \times \mathbb{R}^N \subset S(\xi)$ such that $G|_{L^k \times \mathbb{R}^N}: L^k \times \mathbb{R}^N \to S^N - p = \mathbb{R}^N$ is projection onto the second factor and $G(S(\xi) - L^k \times \mathbb{R}^N) = p$, and such that the map $\pi: L^k \to M^k$, induced by the projection $\pi: S(\xi) \to M^k$, is a map of degree one. If $W^{k+1} \times \mathbb{R}^N \subset S(\xi) \times I$ is a framed cobordism between the two ends $L_j^k \times \mathbb{R}^N = W^{k+1} \times \mathbb{R}^N \cap S(\xi) \times j$, $j = 0,1$, then the two F/O bundles given by $L_j^k \times \mathbb{R}^N \subset S(\xi)$, $j = 0,1$, are equivalent. Thus given $L^k \times \mathbb{R}^N \subset S(\xi)$ we can do framed surgery on $L^k \times \mathbb{R}^N$ in $S(\xi)$ and try to make $\pi: L^k \to M^k$ a homotopy equivalence, without changing the associated F/O bundle over M^k. There is a single obstruction to making $\pi: L^k \to M^k$ a homotopy equivalence, associated to middle dimensional surgery, in the group $P_k = \mathbb{Z}, 0, \mathbb{Z}_2, 0$ for $k \equiv 0,1,2,3 \bmod 4$, respectively.

Conversely, given a homotopy equivalence $f: L^k \to M^k$, cover f up to homotopy by an embedding of L^k in the total space of the stable vector bundle $\xi = \bar{f}^*(\tau_L) - \tau_M$ over M^k, where τ_L and τ_M are the tangent bundles and $\bar{f}$ is a homotopy inverse of f. One sees by an easy computation that L^k has a trivial normal bundle in ξ. Choosing a framing $L^k \times \mathbb{R}^N \subset \xi$ determines a map $\theta(L^k,f)\colon M^k \to F/O$. The notion of equivalence of F/O bundles guarantees that the homotopy class of $\theta(L^k,f)$ is independent of all choices made. Thus we have constructed a sequence

$$hS(M^k) \xrightarrow{\theta} [M^k,F/O] \xrightarrow{S} P_k$$

which is exact as a sequence of pointed sets.

There is an action of Γ_k on $hS(M^k)$ defined as follows. Given $(L^k,f) \in hS(M^k)$, we may assume that f maps a disc in L^k to a point in M^k. Then if $\Sigma^k \in \Gamma_k$, define $(L^k,f) \# \Sigma^k$ to be $(L^k \# \Sigma^k, f \# pt)$, where $L^k \# \Sigma^k$ denotes the connected sum.

Theorem 1.0. If $\partial M^k \neq \emptyset$ then in the exact sequence above $s = 0$ and θ is an isomorphism. If $\partial M^k = \emptyset$ and $\theta(L^k,f) = 0$ then $(L^k,f) \simeq (M^k,Id) \# \Sigma^k$ for some $\Sigma^k \in bP_{k+1}$, the subgroup of exotic spheres which bound π-manifolds. In particular, θ is a monomorphism if k is even.

The surgery obstruction s need not be zero if $\partial M^k = \emptyset$. If $k = 4n$ then the surgery obstruction of a map $f\colon M^{4n} \to F/O$

is given by

$$s(f) = (\tfrac{1}{8})(\mathrm{index}(L^{4n}) - \mathrm{index}(M^{4n}))$$
$$= (\tfrac{1}{8})\langle L(\tau_M)\cdot(L(\xi) - 1),[M^{4n}]\rangle \in \mathbb{Z}$$

where $L^{4n} \times \mathbb{R}^N \subset S(\xi)$ defines f and $L = 1 + L_1 + L_2 + \cdots$ is the total Hirzebruch L-polynomial. This provides a formula for $s(f)$ in terms of the Pontryagin classes of τ_M and ξ. If $k = 4n + 2$, the surgery obstruction is much harder to compute in terms of invariants of M^{4n+2} and $f\colon M^{4n+2} \to F/O$.

We will study the exact sequence of sets

$$0 \to hS(\mathbb{C}P(n)) \xrightarrow{\theta} [\mathbb{C}P(n),F/O] \xrightarrow{S} P_{2n},$$

$n > 2$. Let $\mathbb{C}P(n+1)_o$ be $\mathbb{C}P(n+1)$ with a disc D^{2n+2} removed. $\mathbb{C}P(n+1)_o$ is the total space of the D^2 bundle H over $\mathbb{C}P(n)$. Thus there are maps

$$S^{2n+1} = \partial\mathbb{C}P(n+1)_o \xrightarrow{i} \mathbb{C}P(n+1)_o \xrightarrow{H} \mathbb{C}P(n)$$

which induce

$$[\mathbb{C}P(n),F/O] \xrightarrow[\sim]{H^*} [\mathbb{C}P(n+1)_o,F/O] \xrightarrow[\sim]{\theta} hS(\mathbb{C}P(n+1)_o)$$
$$\xrightarrow{i^*} \Gamma_{2n+1}$$

where i^* is the map which assigns to a homotopy smoothing of $\mathbb{C}P(n+1)_o$ its boundary, which is a homotopy sphere. Denote by σ the composite $\sigma = i^*\theta H^*\colon [\mathbb{C}P(n),F/O] \to \Gamma_{2n+1}$.

Proposition 1.1. Let $f\colon P^{2n} \to \mathbb{C}P(n)$ in $hS(\mathbb{C}P(n))$ correspond to the S^1 action (Σ^{2n+1},T). Then $\Sigma^{2n+1} = \sigma\theta(P^{2n},f)$.

Proof. $\theta(P^{2n},f) = g\colon \mathbb{C}P(n) \to F/O$ corresponds to a framing $P^{2n} \times \mathbb{R}^N \subset S(\xi)$, where ξ is a bundle over $\mathbb{C}P(n)$. Since $H\colon \mathbb{C}P(n+1)_o \to \mathbb{C}P(n)$ is a disc bundle, it is easy to see that $gH\colon \mathbb{C}P(n+1)_o \to F/O$ corresponds to a framing of the total space of the disc bundle f^*H over P^{2n}, in the sphere bundle $S(H^*\xi)$ over $\mathbb{C}P(n+1)_o$. But f^*H is the disc bundle associated to the principal S^1 bundle $T\colon \Sigma^{2n+1} \to P^{2n}$, hence $\Sigma^{2n+1} = \partial f^*H = \sigma g$.

The author is particularly interested in characterizing those homotopy spheres which admit free S^1 actions. Proposition 1.1 gives us a homotopy theoretic hold on this problem.

Summarizing, our approach to the problem of classifying free circle actions on homotopy spheres will be to attempt (i) computation of $[\mathbb{C}P(n),F/O]$ (ii) computation of the surgery obstruction $s\colon [\mathbb{C}P(n),F/O] \to P^{2n}$, hence also $\theta(hS(\mathbb{C}P(n))) \subset [\mathbb{C}P(n),F/O]$, and (iii) computation of the "boundary" map $\sigma\colon [\mathbb{C}P(n),F/O] \to \Gamma_{2n+1}$. A complete solution of these three problems by homotopy theoretic methods is not feasible. However, we do reduce (i), (ii) if $n \neq 4k+3$, and (iii) if $n \neq 4k+2$, to much more familiar problems in homotopy theory and we obtain complete solutions of (i), (ii), (iii) for $n \leq 6$.

We also study maps of $\mathbb{C}P(n)$ into a number of other useful spaces. There is a diagram with rows fibrations

$$\begin{array}{ccccc} PL & \to & F & \to & F/PL \\ \downarrow & & \downarrow & & \| \\ PL/O & \to & F/O & \to & F/PL \end{array} \tag{1.2}$$

F/PL plays the same role in the PL category that F/O plays in the differentiable category. Namely, define a homotopy triangulation of a PL manifold M^k to be a homotopy equivalence $f: L^k, \partial L^k \to M^k, \partial M^k$, where L^k is a PL manifold. (L_o^k, f_o) and (L_1^k, f_1) are equivalent if there is a PL isomorphism $h: L_o^k \xrightarrow{\simeq} L_1^k$ such that $f_1 h \sim f_o$. Denote the set of equivalence classes by $hT(M^k)$. The following is, of course, due to Sullivan.

1.3. There is an exact sequence of sets

$$0 \to hT(M^k) \xrightarrow{\theta} [M^k, F/PL] \xrightarrow{S} P_k$$

For a smooth manifold M^k, we have the following additional, well-known statements concerning maps of M^k into the spaces in diagram 1.2.

1.4. $[M^k, PL/O]$ is isomorphic to the set of concordance classes of smoothings of M^k [13], [17].

1.5. $\mathrm{Image}([M^k, PL] \to [M^k, PL/O])$ consists of those smoothings of M^k which preserve the stable tangent bundle.

1.6. $[M^k,F]$ consists of equivalence classes of embeddings $L^k \times \mathbb{R}^N \subset M^k \times S^N$ such that $\pi: L^k \to M^k$ is a map of degree one, where $L_j^k \times \mathbb{R}^N \subset M^k \times S^N$, $j = 0,1$, are equivalent if there is a framed cobordism $W^{k+1} \times \mathbb{R}^N \subset M^k \times S^N \times I$ between them, that is, $W^{k+1} \times \mathbb{R}^N \cap M^k \times S^N \times j = L_j^k \times \mathbb{R}^N$, $j = 0,1$.

In particular, π is a tangential map, that is, $\pi^* \tau_M = \tau_L$. $[M^k,F]$ is thus useful for studying tangential homotopy equivalences.

Finally, we use our homotopy theoretic methods to study diffeomorphisms of $\mathbb{C}P(n)$ modulo pseudo-isotopy. Given a diffeomorphism $h: M^k \xrightarrow{\simeq} M^k$ of a smooth manifold M^k and a homotopy of h to the identity, there is an associated element of $[M^k,\Omega(F/O)]$ which vanishes if the homotopy can be deformed, rel $M^k \times \partial I$, to a pseudo-isotopy of h to the identity [23], [24]. We give in §7 some non-trivial examples of diffeomorphisms of $\mathbb{C}P(n)$. On the other hand $[\mathbb{C}P(n),\Omega(F/PL)] = 0$ and every PL isomorphism of $\mathbb{C}P(n)$ homotopic to the identity is actually PL pseudo-isotopic to the identity [24].

The paper is arranged as follows. In §2 we give some general results on the structure of $[\mathbb{C}P(n),F/O]$, using results of Adams on $J(\mathbb{C}P(n))$. In §3 we study the maps s and σ. This section contains nearly all the new results of the paper. In §4 we discuss " conjugation", a natural involution defined on $hS(\mathbb{C}P(n))$, which assigns to an S^1 bundle over a homotopy complex projective space its dual bundle. A spectral sequence, useful for computing $[\mathbb{C}P(n),Y]$ for any space Y, is constructed in §5. In §6 a geometric interpretation of the differentials is given for $Y = PL/O$ (or essentially any of the other spaces in diagram 1.2). This spectral sequence is used in §7 to obtain much information on the torsion subgroup of $[\mathbb{C}P(n),F/O]$ for small n. In §8 we determine $[\mathbb{C}P(n),F/O]$ and $hS(\mathbb{C}P(n))$ for $n \leq 6$ and give examples of various other geometric phenomena based on our computations.

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§2. Structure of $[\mathbb{C}P(n),F/O]$

In this section we study $[\mathbb{C}P(n),F/O]$ using the sequence of fibrations

$$SO \to SF \to F/O \to BSO \to BSF,$$

knowledge of $KO^*(\mathbb{C}P(n))$, and Adams' results on $J(\mathbb{C}P(n))$. All the results are implicit in the discussion in [24]. We summarize the needed facts in the following Theorem [2]. Let H be the dual of the canonical line bundle over $\mathbb{C}P(n)$ and let $\omega = r(H - 1)$ be the realification of $H - 1$. We use the notation $KO^o(X)$ for the reduced K-theory group $\widetilde{KO}(X) = [X,BSO]$.

Theorem 2.0

(1) The ring $KO^o(\mathbb{C}P(n))$ is generated by the element ω, subject to the relations

$$\begin{aligned} \omega^{2k+1} &= 0 && \text{if } n = 4k \\ 2\omega^{2k+1} = \omega^{2k+2} &= 0 && \text{if } n = 4k+1 \\ \omega^{2k+2} &= 0 && \text{if } n = 4k+2,\ 4k+3. \end{aligned}$$

In particular, $KO^o(\mathbb{C}P(n)) = \oplus\, \mathbb{Z}$ if $n \equiv 0,2,3 \bmod 4$

$= \oplus\, \mathbb{Z} + \mathbb{Z}_2$ if $n \equiv 1 \bmod 4$.

(2) $KO^{-1}(\mathbb{C}P(n)) = [\mathbb{C}P(n),SO] = 0$

(3) The "Adams conjecture" holds for $\mathbb{C}P(n)$. That is, $\text{kernel}(KO^o(\mathbb{C}P(n)) \to [\mathbb{C}P(n),BF])$ is generated additively by elements $p^e(\psi^p(\xi) - \xi)$, where p and e are integers, e sufficiently large, ψ^p is the Adams operation, and $\xi \in KO^o(\mathbb{C}P(n))$.

The Adams operation ψ^p in $KO^o(\mathbb{C}P(n))$ is given by $\psi^p(\omega) = T_p(\omega)$, where T_p is the polynomial such that $T_p(z - 2 + z^{-1}) = z^p - 2 + z^{-p}$. In particular, it follows that the torsion element $\omega^{2k+1} \in KO^o(\mathbb{C}P(4k+1))$ is non-zero in $J(\mathbb{C}P(4k+1)) = \text{image}(KO^o(\mathbb{C}P(4k+1)) \to [\mathbb{C}P(4k+1),BF])$.

Corollary 2.1. There is an exact sequence

$$0 \to [\mathbb{C}P(n),F] \to [\mathbb{C}P(n),F/O] \to \oplus\, \mathbb{Z} \to 0,$$

where $\oplus \mathbb{Z} \subset KO^0(\mathbb{C}P(n))$ is a free subgroup of maximal rank generated by elements $p^e(\psi^p(\xi) - \xi)$.

Proof. There is an exact sequence

$$[\mathbb{C}P(n),SO] \to [\mathbb{C}P(n),F] \to [\mathbb{C}P(n),F/O] \to [\mathbb{C}P(n),BSO] \to [\mathbb{C}P(n),BF].$$

Since $[\mathbb{C}P(n),BF]$ is finite and since the only torsion in $[\mathbb{C}P(n),BSO]$ is non-zero in $[\mathbb{C}P(n),BF]$, the corollary follows immediately from Theorem 2.0 (2) and (3).

Thus the torsion subgroup of $[\mathbb{C}P(n),F/O]$ is identified with $[\mathbb{C}P(n),F]$. Since $F = \lim_{m\to\infty} \Omega^m S^m$, we see that as a set (the group structures are different) $[\mathbb{C}P(n),F] = \lim_{m\to\infty}[S^m \wedge \mathbb{C}P(n),S^m]$ $= \pi_s^0(\mathbb{C}P(n))$, the 0^{th} stable cohomotopy group of $\mathbb{C}P(n)$. Also, from the information in Theorem 2.0 the Pontryagin classes of all F/O bundles over $\mathbb{C}P(n)$ are theoretically computable.

The inclusion $i: \mathbb{C}P(n) \to \mathbb{C}P(n + 1)$ induces a map $i^*: [\mathbb{C}P(n + 1),F/O] \to [\mathbb{C}P(n),F/O]$.

Corollary 2.2. $i^*: [\mathbb{C}P(n + 1),F/O] \to [\mathbb{C}P(n),F/O]/(\text{torsion})$ is surjective.

Proof. This is immediate from Theorem 2.0 (1) and (3).

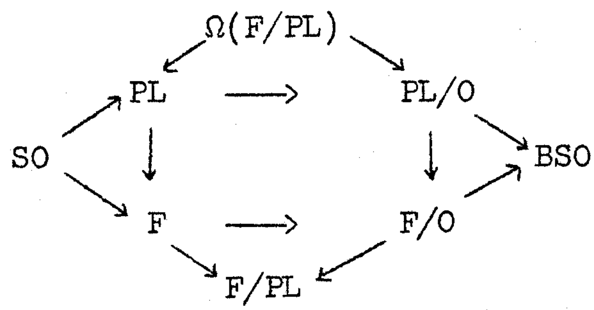
Corollary 2.3. There is a splitting

$$[\mathbb{C}P(n),F/O] = [\mathbb{C}P(n),F] + \oplus \mathbb{Z}$$

such that the subgroup $\oplus \mathbb{Z}$ is contained in $i^*([\mathbb{C}P(n+1),F/O])$ $\subset [\mathbb{C}P(n),F/O]$.

Proof. This is immediate from Corollaries 2.1 and 2.2.

Next, we consider maps of $\mathbb{C}P(n)$ into the diagram



Since $\pi_k(F/PL) = P_k = \mathbb{Z},0,\mathbb{Z}_2,0$ for $k \equiv 0,1,2,3 \bmod 4$, respectively, it follows that $[\mathbb{C}P(n),\Omega(F/PL)] = 0$.

Proposition 2.4.

(1) The map $[\mathbb{C}P(n),PL] \xrightarrow{\simeq} [\mathbb{C}P(n),PL/O]$ is an isomorphism.

(2) The inclusion $0 \to [\mathbb{C}P(n),PL] \to [\mathbb{C}P(n),F]$ is an isomorphism on odd torsion components.

Proof. The homotopy groups $\pi_k(PL/O) = \Gamma_k$ are finite, hence $[\mathbb{C}P(n),PL/O]$ is finite. Since the only torsion in $[\mathbb{C}P(n),BSO]$ is non-zero in $[\mathbb{C}P(n),BPL]$ (in fact, in $[\mathbb{C}P(n),BF]$), the map $[\mathbb{C}P(n),PL/O] \to [\mathbb{C}P(n),BSO]$ is zero. Statement (1) follows from this and Theorem 2.0 (2). Statement (2) follows from the fact that there is no odd torsion in $[\mathbb{C}P(n),F/PL]$.

Corollary 2.5. The stable tangent bundle of a smoothing of $\mathbb{C}P(n)$ is independent of the smoothing.

Proof. This follows from Proposition 2.4 (1) and 1.5 of the introduction.

§3. The maps s and σ

Recall from the introduction that $hS(\mathbb{C}P(n))$ is identified with kernel($[\mathbb{C}P(n),F/O] \xrightarrow{S} P_{2n}$). If n is odd, $P_{2n} = \mathbb{Z}_2$ and s is actually a group homomorphism [25]. If n is even, $P_{2n} = \mathbb{Z}$. In this case s is not a homomorphism, but by kernel(s) we still mean $s^{-1}(0)$.

From Corollary 2.3 write $[\mathbb{C}P(n),F/O] = \oplus\mathbb{Z} + [\mathbb{C}P(n),F]$, where the summand $\oplus\mathbb{Z}$ is contained in $i^*([\mathbb{C}P(n+1),F/O])$ $\subset[\mathbb{C}P(n),F/O]$. We first recall the formula for the surgery obstruction when $n = 2k$ is even.

Proposition 3.1. Let $g \in [\mathbb{C}P(2k),F/O]$. Then $s(g) = (\frac{1}{8})\langle L(\mathbb{C}P(2k))\cdot(L(\xi) - 1),[\mathbb{C}P(2k)]\rangle \in \mathbb{Z}$, where $\xi = \xi(g)$ is the bundle given by $\mathbb{C}P(2k) \xrightarrow{g} F/O \xrightarrow{i} BSO$. In particular, s vanishes on $[\mathbb{C}P(n),F]$.

Since the Pontryagin classes of all such bundles ξ are computable (though the computation is not practical), this determines the possible Pontryagin classes which can occur for manifolds homotopy equivalent to $\mathbb{C}P(n)$, n even. For if $f\colon P^{2n} \to \mathbb{C}P(n)$ in $hS(\mathbb{C}P(n))$ corresponds to

$\theta(f) = g \in [\mathbb{C}P(n),F/O]$ then P^{2n} has a trivial normal bundle in the total space of the bundle $\xi = \xi(g)$ over $\mathbb{C}P(n)$. Thus the Pontryagin classes of P^{2n} are given by $p_m(P^{2n}) = \sum_{i+j=m} p_i(\mathbb{C}P(n))p_j(\xi) \in H^{4m}(P^{2n},\mathbb{Z})$.

For n odd, the surgery obstruction is very difficult to compute. Later in this section we give a KO-theory formula for s when $n = 4k + 1$.

Next we consider the map $\sigma\colon [\mathbb{C}P(n),F/O] \to \Gamma_{2n+1}$. Recall from Proposition 1.1 that if $g \in [\mathbb{C}P(n),F/O]$ is in $hS(\mathbb{C}P(n))$, corresponding to $f\colon P^{2n} \to \mathbb{C}P(n)$, then $\sigma(g) \in \Gamma_{2n+1}$ is the sphere Σ^{2n+1} on which S^1 acts freely over P^{2n}, that is, the total space of the S^1 bundle over P^{2n} induced by f. Before stating our results we need to recall some facts about the structure of Γ_{2n+1} from [16] and [11], [12].

First, $\Gamma_{4n-1} = bP_{4n} \oplus (\pi^s_{4n-1}/\text{im } J)$, where $bP_{4n} = \mathbb{Z}_{\theta_n}$, $\theta_n = a_n 2^{2n-2}(2^{2n-1} - 1)\text{num}(B_n/4n)$, $a_n = 1$ if n is even, 2 if n is odd, and B_n is the Bernoulli number [16], [11].

Remark. Strictly speaking, for very large even n, if $\text{im } J = \mathbb{Z}_{2j_n}$, where $j_n = \text{denom}(B_n/4n)$, then $bP_{4n} = \mathbb{Z}_{2\theta_n}$. For neatness of exposition, we consider only those n for which $\text{im } J = \mathbb{Z}_{j_n}$ and hence $bP_{4n} = \mathbb{Z}_{\theta_n}$. This is the case for $n \leq 2^{10}$ [18] and is presumably the case for all n.

There is an invariant $f_c\colon \Gamma_{4n-1} \to \mathbb{Z}_{\theta_n/a_n}$ defined as follows. Every $\Sigma^{4n-1} \in \Gamma_{4n-1}$ bounds a smooth manifold M^{4n} with a complex structure on its stable tangent bundle such that all Chern numbers of M^{4n}, except possibly c_{2n}, vanish. (More precisely, we should say Chern numbers of $\hat{M}^{4n} = M^{4n} \bigcup_{\Sigma^{4n-1}} D^{4n}$.) Set $f_c(\Sigma^{4n-1}) = (\frac{1}{8})\text{index}(M^{4n}) \in \mathbb{Z}$. This is well-defined, that is, independent of M^{4n}, modulo $(\theta_n/a_n)\mathbb{Z}$, hence we get a homomorphism $f_c\colon \Gamma_{4n-1} \to \mathbb{Z}_{\theta_n/a_n}$. Moreover, $f_c\colon bP_{4n} = \mathbb{Z}_{\theta_n} \to \mathbb{Z}_{\theta_n/a_n}$ is the natural projection. Thus for even n, $\text{kernel}(f_c) \simeq \pi^s_{4n-1}/\text{im } J$. For odd n, $\text{kernel}(f_c) \simeq \mathbb{Z}_2 \oplus (\pi^s_{4n-1}/\text{im } J)$ where $\mathbb{Z}_2 \subset bP_{4n}$ is the subgroup of order 2.

Secondly, $\Gamma_{8n+1} = \mathbb{Z}_2 \oplus b\ \text{spin}_{8n+2}$ where $b\ \text{spin}_{8n+2}$ is the subgroup of exotic spheres which bound spin manifolds [12]. There is a homomorphism $f_R\colon b\ \text{spin}_{8n+2} \to \mathbb{Z}_2$ defined by taking the Brown-Kervaire invariant [9] of a spin manifold M^{8n+2} where $\partial M^{8n+2} = \Sigma^{8n+1} \in b\ \text{spin}_{8n+2}$ and all Stiefel-Whitney numbers and KO characteristic numbers of M^{8n+2} vanish. On the subgroup $bP_{8n+2} = \mathbb{Z}_2 \subset b\ \text{spin}_{8n+2}$, f_R is the identity. Thus $b\ \text{spin}_{8n+2} = b\ P_{8n+2} \oplus \text{kernel}(f_R) = \mathbb{Z}_2 \oplus (\pi^s_{8n+1}/\mathbb{Z}_2+\mathbb{Z}_2)$.

The following result was pointed out to me by Sullivan.

<u>Proposition 3.2</u>. Let $\xi \in \oplus\, \mathbb{Z} \subset [\mathbb{C}P(n),F/O]$ with $\xi = i^*(\hat{\xi})$, $\hat{\xi} \in [\mathbb{C}P(n+1),F/O]$. Then $\sigma(\xi) \in bP_{2n+2}$. Specifically $\sigma(\xi) = s(\hat{\xi}) \in \mathbb{Z} \bmod \theta_k\mathbb{Z}$ if $n = 2k-1$ and $\sigma(\xi) = s(\hat{\xi}) \in \mathbb{Z}_2$ if $n = 2k$, $k \neq 2^j - 1$.

Proof. There is the cofibration $S^{2n+1} \xrightarrow{H} \mathbb{C}P(n) \xrightarrow{i} \mathbb{C}P(n+1)$. Since $\xi = i^*(\hat{\xi})$, the composition $S^{2n+1} \xrightarrow{H} \mathbb{C}P(n) \xrightarrow{\xi} F/O$ is zero. Recall that $\pi_{2n+1}(F/O) = \pi^s_{2n+1}/\text{im } J$. Also recall the Kervaire-Milnor map [16], $\varphi: \Gamma_{2n+1} \to \pi^s_{2n+1}/\text{im } J$, defined by choosing framings of homotopy $2n+1$-spheres embedded in some high dimensional sphere. From the construction of σ given in the introduction, one can see that the composition $[\mathbb{C}P(n),F/O] \xrightarrow{\sigma} \Gamma_{2n+1} \xrightarrow{\varphi} \pi^s_{2n+1}/\text{im } J$ coincides with the map $[\mathbb{C}P(n),F/O] \xrightarrow{\circ H} [S^{2n+1},F/O] = \pi^s_{2n+1}/\text{im } J$ induced by $H: S^{2n+1} \to \mathbb{C}P(n)$. Thus $0 = \xi \circ H = \varphi\sigma(\xi)$. But $\text{kernel}(\varphi) = bP_{2n+2}$ so $\sigma(\xi) \in bP_{2n+2}$.

Now $\sigma(\xi) \in bP_{2n+2}$ is the boundary of a homotopy $\mathbb{C}P(n+1)_o$, say L_o^{2n+2}, obtained from a fibre homotopy trivialization of the bundle $H^*\xi$, where $H: \mathbb{C}P(n+1)_o \to \mathbb{C}P(n)$. $\hat{\xi} \in [\mathbb{C}P(n+1),F/O]$ is constructed by extending the fibre homotopy trivialization of $H^*\xi\big|_{\partial\mathbb{C}P(n+1)_o = S^{2n+1}}$ over a disc D^{2n+2}. Thus $\hat{\xi}$ corresponds to a framing of a manifold $N^{2n+2} = L_o^{2n+2} \bigcup_{\sigma(\xi)} M_o^{2n+2}$ in a bundle over $\mathbb{C}P(n+1)$, where M_o^{2n+2} is a parallelizable manifold with $\partial M_o^{2n+2} = \sigma(\xi)$. If $n = 2k-1$ is odd then $\sigma(\xi) = (\frac{1}{8})\text{index}(M_o^{4k}) \pmod{\theta_k} = s(\hat{\xi}) \pmod{\theta_k}$. If $n = 2k$ is even, $k \neq 2^j - 1$, then $\sigma(\xi) \in bP_{4k+2} = \mathbb{Z}_2$ coincides with the Kervaire invariant of M_o^{4k+2}, which in turn gives the surgery obstruction $s(\hat{\xi})$. This completes the proof.

Proposition 3.3. Let $g \in [\mathbb{C}P(n),F] \subset [\mathbb{C}P(n),F/O]$, $n = 2k-1$. Then $\sigma(g) \in \text{kernel}(f_c)$.

Proof. $\sigma(g)$ is the boundary of a tangential homotopy $\mathbb{C}P(2k)_o$, say P_o^{4k}, associated to $\mathbb{C}P(2k)_o \xrightarrow{H} \mathbb{C}P(2k-1) \xrightarrow{g} F$. In particular, the Chern classes of P_o^{4k} coincide with those of $\mathbb{C}P(2k)_o$. Thus the decomposable Chern numbers of $M^{4k} = P_o^{4k} \# (-\mathbb{C}P(2k))$ vanish. Since $\partial M^{4k} = \sigma(g)$, we see that $f_c(\sigma(g)) = \text{index}(M^{4k}) = 0$.

Summarizing, we have shown the following. With respect to the decompositions $[\mathbb{C}P(2k-1),F/O] = \oplus \mathbb{Z} + [\mathbb{C}P(2k-1),F]$ and $\Gamma_{4k-1} = bP_{4k} + (\pi_{4k-1}^s/\text{im } J)$, we have $\sigma(\oplus \mathbb{Z}) \subseteq bP_{4k}$. Further, $\sigma(\oplus \mathbb{Z})$ is explicitly computable in terms of Pontryagin classes, using the fomulas in Propositions 3.1 and 3.2. We also have $\sigma([\mathbb{C}P(2k-1),F]) \subseteq \text{kernel}(f_c)$. For k even, $\text{kernel}(f_c) = \pi_{4k-1}^s/\text{im } J$ and thus $\sigma([\mathbb{C}P(2k-1),F])$ is computable as the composition

$$[\mathbb{C}P(2k-1),F] \xrightarrow{\circ H} [S^{4k-1},F] = \pi_{4k-1}^s \to \pi_{4k-1}^s/\text{im } J.$$

Because of the identification $[\mathbb{C}P(n),F] = [S^m \wedge \mathbb{C}P(n),S^m]$, m large, this reduces to the problem in stable homotopy theory of computing the map $[S^m \wedge \mathbb{C}P(2k-1),S^m] \xrightarrow{\circ S^m \wedge H} [S^{m+4k-1},S^m]$. For k odd, $\text{kernel}(f_c) = \mathbb{Z}_2 \oplus (\pi_{4k-1}^s/\text{im } J)$, where $\mathbb{Z}_2 \subset bP_{4k}$. Later in this section we will show how to describe the map $\sigma\colon [\mathbb{C}P(2k-1),F] \to \mathbb{Z}_2 \oplus (\pi_{4k-1}^s/\text{im } J)$ in terms of homotopy theory.

First, we show how σ behaves on sums of elements in the decomposition $[\mathbb{C}P(2k-1),F/O] = \oplus \mathbb{Z} + [\mathbb{C}P(2k-1),F]$. σ is not a homomorphism on the summand $\oplus \mathbb{Z}$. However, Proposition 3.4. Let $f\colon \mathbb{C}P(2k-1) \to F/O$ be a map which extends to $\hat{f}\colon \mathbb{C}P(2k) \to F/O$ (for example, $f \in \oplus \mathbb{Z} \subset [\mathbb{C}P(2k-1), F/O]$) and let $g\colon \mathbb{C}P(2k-1) \to F$ be any map. Then $\sigma(f+g) = \sigma(f) + \sigma(g)$.

Proof. The map $\hat{f}$ corresponds to an embedding $Q^{4k} \times \mathbb{R}^N \subset \hat{\xi}^N$, where $\hat{\xi}^N$, N large, is the bundle $\mathbb{C}P(2k) \xrightarrow{\hat{f}} F/O \xrightarrow{i} BSO$. By performing framed surgery, we may assume that $Q^{4k} = M_o^{4k} \underset{\sigma(f)}{\cup} W_o^{4k}$, where W_o^{4k} is a parallelizable manifold with $\mathrm{index}(W_o^{4k}) = s(\hat{f})$ and the projection $M_o^{4k} \to \mathbb{C}P(2k)_o$ is a homotopy equivalence.

The composition $\mathbb{C}P(2k)_o \xrightarrow{H} \mathbb{C}P(2k-1) \xrightarrow{g} F$ corresponds to an embedding $P_o^{4k} \times \mathbb{R}^N \subset e^N$, where e^N is the trivial bundle over $\mathbb{C}P(2k)_o$ and the projection $P_o^{4k} \to \mathbb{C}P(2k)_o$ is a tangential homotopy equivalence. Since $\partial P_o^{4k} = \sigma(g)$, this gives an embedding $\sigma(g) \times \mathbb{R}^N \subset S^{4k-1} \times \mathbb{R}^N$. This framed exotic sphere corresponds to a unique PL N-bundle $\bar{\sigma}(g)^N = \bar{\sigma}^N \in \pi_{4k-1}(PL) = \pi_{4k}(BPL)$. Let $j\colon \mathbb{C}P(2k) \to S^{4k}$ be a map of degree one and let P^{4k} be the almost smooth manifold $P_o^{4k} \underset{\sigma(g)}{\cup} D^{4k}$. Then there is an embedding $P^{4k} \times \mathbb{R}^{2N} \subset e^N + j^*(-\bar{\sigma}^N)$ over $\mathbb{C}P(2k)$, extending the embedding $P_o^{4k} \times \mathbb{R}^N \subset e^N$ over $\mathbb{C}P(2k)_o$.

The Pontryagin classes of P^{4k} coincide with those of $\mathbb{C}P(2k)$. For the classes $p_i(P^{4k})$, $i < k$, this is so because P_o^{4k} is a tangential homotopy $\mathbb{C}P(2k)_o$. Since $\mathrm{index}(P^{4k}) = \mathrm{index}(\mathbb{C}P(2k))$, we then see from the Hirzebruch index formula that $p_k(P^{4k}) = p_k(\mathbb{C}P(2k))$. It follows that $p_k(\bar{\sigma}^N) = 0$.

Now embed $Q^{4k} \times \mathbb{R}^N \times P^{4k} \times \mathbb{R}^{2N} \subset \hat{\xi}^N \times (e^N + j^*(-\bar{\sigma}^N))$ over $\mathbb{C}P(2k) \times \mathbb{C}P(2k)$ by the product map. By transversality, we may assume that the intersection $Q^{4k} \times P^{4k} \times \mathbb{R}^{3N} \cap (\hat{\xi}^N \times (e^N + j^*(-\bar{\sigma}^N)))|_{\Delta\mathbb{C}P(2k)}$ is an embedded, framed submanifold $N^{4k} \times \mathbb{R}^{3N} \subset \hat{\xi}^N + e^N + j^*(-\bar{\sigma}^N)$ over the diagonal $\Delta\mathbb{C}P(2k) = \mathbb{C}P(2k)$. If we restrict to $\mathbb{C}P(2k)_o \subset \mathbb{C}P(2k)$ we get an embedding $N_o^{4k} \times \mathbb{R}^{3N} \subset \hat{\xi}^N + e^{2N}$ which represents the Whitney sum $\mathbb{C}P(2k)_o \xrightarrow{H} \mathbb{C}P(2k-1) \xrightarrow{f+g} F/0$.

The framing $N^{4k} \times \mathbb{R}^{3N} \subset \hat{\xi}^N + e^N + j^*(-\bar{\sigma}^N)$ induces an almost smooth structure on N^{4k} for which the obstruction to extending the smoothing is $\sigma(g)$. Moreover, the surgery obstruction for the projection $N^{4k} \to \mathbb{C}P(2k)$ is clearly the same as that of $Q^{4k} \to \mathbb{C}P(2k)$ because of Proposition 3.1 and the fact that the Pontryagin classes of $\hat{\xi}^N$ are the same as those of $\hat{\xi}^N + e^N + j^*(-\bar{\sigma}^N)$. Again, by performing framed surgery, we may assume that $N^{4k} = \bar{L}_o^{4k} \cup_{\sigma(f)} W_o^{4k}$, where $\bar{L}_o^{4k}$ is an almost smooth, homotopy $\mathbb{C}P(2k)_o$ and W_o^{4k} is a paralleli-zable manifold with $\operatorname{index}(W_o^{4k}) = s(\hat{f})$. Pushing the singular point of $\bar{L}_o^{4k}$ out to $\partial\bar{L}_o^{4k} = \sigma(f)$ gives a smooth manifold, L_o^{4k}, with $\partial L_o^{4k} = \sigma(f) + \sigma(g)$, and a homotopy equivalence $L_o^{4k} \to \mathbb{C}P(2k)_o$. Since this corresponds to the map $\mathbb{C}P(2k)_o \xrightarrow{H} \mathbb{C}P(2k-1) \xrightarrow{f+g} F/0$, we have $\sigma(f+g) = \partial L_o^{4k} = \sigma(f) + \sigma(g)$ as desired.

We now return to the problem of describing the map $\sigma: [\mathbb{C}P(4k+1),F] \to \text{kernel}(f_c) = \mathbb{Z}_2 \oplus (\pi^s_{8k+3}/\text{im } J) \subset \Gamma_{8k+3}$. The projection of $\sigma([\mathbb{C}P(4k+1),F])$ onto the summand $\pi^s_{8k+3}/\text{im } J$ is, of course, computed as the composition $[\mathbb{C}P(4k+1),F] \xrightarrow{\circ H} [S^{8k+3},F] = \pi^s_{8k+3} \to \pi^s_{8k+3}/\text{im } J$, induced by the Hopf map $H: S^{8k+3} \to \mathbb{C}P(4k+1)$.

We recall from [11] that the decomposition $\Gamma_{8k+3} = bP_{8k+4} \oplus (\pi^s_{8k+3}/\text{im } J)$ is established by means of an invariant $f_R: \Gamma_{8k+3} \to \mathbb{Z}_{\theta_{2k+1}}$, which splits off $bP_{8k+4} \subseteq \Gamma_{8k+3}$. f_R is defined using spin manifolds with boundary a given exotic sphere, and improves f_c by a factor of 2 in dimensions $8k+4$. Thus we are interested in computing the composition $[\mathbb{C}P(4k+1),F] \xrightarrow{\sigma} \Gamma_{8k+3} \xrightarrow{f_R} \mathbb{Z}_{\theta_{2k+1}}$. Since $\sigma([\mathbb{C}P(4k+1),F]) \subseteq \text{kernel}(f_c)$, we know that $\text{image}(f_R\sigma)$ is contained in the subgroup of order 2 in $\mathbb{Z}_{\theta_{2k+1}}$. Let $e_R: \pi^s_{8k+3} \to \mathbb{Z}_{j_{2k+1}}$ be the Adams invariant which splits off $\text{im}(J) \subseteq \pi^s_{8k+3}$ as a direct summand [1].

<u>Proposition 3.5</u>.

(1) The homomorphism $e_R H^*: [\mathbb{C}P(4k+1),F] \to \mathbb{Z}_{j_{2k+1}}$, where $H^*: [\mathbb{C}P(4k+1),F] \to [S^{8k+3},F] = \pi^s_{8k+3}$ is induced by the Hopf map, satisfies $\text{image}(e_R H^*) \subseteq \mathbb{Z}_2 \subset \mathbb{Z}_{j_{2k+1}}$, and, with this identification of their images, $f_R\sigma = e_R H^*$.

(2) There is an element $\mu \in \pi_{8k+2}(PL)$ of order 2, such

that if $\mu_{8k+2} \in [\mathbb{C}P(4k+1),F]$ denotes the composition $\mathbb{C}P(4k+1) \xrightarrow{j} S^{8k+2} \xrightarrow{\mu} PL \to F$, where j is of degree one, then $\sigma(\mu_{8k+2}) \in bP_{8k+4}$ and $f_R\sigma(\mu_{8k+2}) \neq 0$.

<u>Proof</u>. Let $g \in [\mathbb{C}P(4k+1),F]$. The composition $\mathbb{C}P(4k+2)_o \xrightarrow{H} \mathbb{C}P(4k+1) \xrightarrow{g} F$ corresponds to a framing of a homotopy $\mathbb{C}P(4k+2)_o$ in the trivial bundle over $\mathbb{C}P(4k+2)_o$. Restricting to the boundary gives a framing of the homotopy sphere $\sigma(g)$ in the trivial bundle over S^{8k+3}. This framed homotopy sphere corresponds to an element $\overline{\sigma}(g) \in \pi_{8k+3}(PL)$. Clearly, $H^*(g) = J_{PL}(\overline{\sigma}(g)) \in \pi_{8k+3}(F)$, where $J_{PL}: \pi_{8k+3}(PL) \to \pi_{8k+3}(F)$ is the PL J-homomorphism, induced by $PL \to F$. It is shown in the proof of Proposition 3.4 that the Pontryagin class $p_{2k+1}(\overline{\sigma}(g)) = 0$, hence $\overline{\sigma}(g)$ is a torsion element of $\pi_{8k+3}(PL)$. From [11, Theorem 4.8] it follows that $e_R H^*(g) = e_R J_{PL}(\overline{\sigma}(g)) \in \mathbb{Z}_{j_{2k+1}}$ has the same order as $f_R\sigma(g) \in \mathbb{Z}_{\theta_{2k+1}}$. This proves (1).

To establish (2), recall from [11] that the map $PL \to F$ induces isomorphisms $\pi_{8k+2}(PL) \xrightarrow{\cong} \pi_{8k+2}(F)$ and ${}_2\pi_{8k+3}(PL) \xrightarrow{\cong} {}_2\pi_{8k+3}(F)$. (For a finitely generated abelian group G and a prime p, ${}_pG$ denotes the p-primary torsion summand of G.) Adams has constructed an element $\mu \in \pi^s_{8k+2}$ with $2\mu = 0$, $\mu\eta \in \operatorname{im} J \subseteq \pi^s_{8k+3}$, and $\mu\eta \neq 0$, where $\eta \in \pi^1_s = \mathbb{Z}_2$ [1]. Denote also by μ and $\mu\eta$ the corresponding elements in $\pi_{8k+2}(PL)$ and ${}_2\pi_{8k+3}(PL)$. Consider the composition $S^{8k+3} \xrightarrow{H} \mathbb{C}P(4k+1) \xrightarrow{j} S^{8k+2} \xrightarrow{\mu} PL \xrightarrow{\beta} PL/O$. It is easy to prove that $jH = \eta$ (See §5). Again from [11], the injection $\beta_*: {}_2\pi_{8k+3}(PL) \to {}_2\pi_{8k+3}(PL/O) = {}_2\Gamma_{8k+3}$ satisfies $\beta_*({}_2(\mathrm{im}\ J)) \subset bP_{8k+4}$. Thus $\sigma(\mu_{8k+2}) = \beta \circ \mu \circ j \circ H = \beta_*(\mu\eta)$ is non-zero in bP_{8k+4} as desired.

Remarks.

1. The relation $\mathrm{image}(e_R H^*) \subseteq \mathbb{Z}_2 \subset \mathbb{Z}_{j_{2k+1}}$ could be proved by an elementary argument, using the cofibration sequence $S^{8k+3} \to \mathbb{C}P(4k + 1) \to \mathbb{C}P(4k + 2) \to S^{8k+4}$, Theorem 2.0, and the fact that $[\mathbb{C}P(n),F]$ is the torsion subgroup of $[\mathbb{C}P(n),F/O]$. Results of [11] would still be required to show that $f_R\sigma = e_R H^*$, however.
2. If we regard μ_{8k+2} as an element of $[\mathbb{C}P(4k + 1),F/O]$ then $\mu_{8k+2} = i^*(\hat{\mu}_{8k+2})$, where $\hat{\mu}_{8k+2} \in [\mathbb{C}P(4k + 2),F/O]$, for $\sigma(\mu_{8k+2}) \in bP_{8k+4}$. Since $\sigma(\mu_{8k+2}) = s(\hat{\mu}_{8k+2})$ is non-zero, it follows that the Pontryagin class $p_{2k+1}(\hat{\mu}_{8k+2})$ is non-zero. That is, $\hat{\mu}_{8k+2}$ is not a torsion element (see Propositions 3.1 and 3.2). This also follows from the fact that the composition $S^{8k+3} \xrightarrow{H} \mathbb{C}P(4k + 1) \xrightarrow{\mu_{8k+2}} F$ represents $\mu\eta \in \pi^s_{8k+3}$ and since $\mu\eta \neq 0$, $\mu_{8k+2} \notin \mathrm{image}([\mathbb{C}P(4k + 2),F] \xrightarrow{i^*} [\mathbb{C}P(4k + 1),F])$.
3. Geometrically the homotopy $\mathbb{C}P(4k + 1)$ corresponding to $\mu_{8k+2} \in [\mathbb{C}P(4k + 1),F]$ is the connected sum $\mathbb{C}P(4k + 1) \# \Sigma^{8k+2}$, where $\Sigma^{8k+2} = \beta_*(\mu) \in \pi_{8k+2}(PL/O) = \Gamma_{8k+2}$.

Propositions 3.1 through 3.5 reduce the determination of the map $\sigma\colon [\mathbb{C}P(2k-1),F/O] \to \Gamma_{4k-1}$ to computations with Pontryagin classes and study of the stable cohomotopy of $\mathbb{C}P(2k-1)$. However, since the surgery obstruction $s\colon [\mathbb{C}P(2k-1),F/O] \to \mathbb{Z}_2$ is difficult to compute, these results only give partial information on $\sigma(hS(\mathbb{C}P(2k-1)))$, the collection of homotopy $(4k-1)$-spheres which admit free S^1 actions. Our next goal in this section will be to derive a KO-theory formula for $s\colon [\mathbb{C}P(2k-1),F/O] \to \mathbb{Z}_2$ when k is odd. This then gives a complete reduction of the problem of classifying free S^1 actions on homotopy $8n+3$-spheres to more familiar problems in homotopy theory.

First, Brown has defined a homomorphism $\Psi\colon \Omega^{spin}_{8k+2} \to \mathbb{Z}_2$ which extends the Kervaire invariant, $\Omega^{framed}_{8k+2} \to \mathbb{Z}_2$ [9]. It follows from the main theorem of [3] that $\Psi = \sum_J \alpha_J \pi^J + \sum_I \beta_I w^I$, where the $\pi^J = \pi_1^{j_1} \cdots \pi_r^{j_r}$ are KO characteristic numbers and the $w^I = w_1^{i_1} \cdots w_r^{i_r}$ are Stiefel-Whitney numbers of $8k+2$ spin manifolds, and $\alpha_J, \beta_I \in Z_2$ (see also [4]). Since Ψ vanishes on Ω^{framed}_{8k+2} [10], and since $\pi^{(0)}$ is the only KO characteristic number which is non-zero on Ω^{framed}_{8k+2} [3], it follows that $\alpha_{(0)} = 0$.

$\mathbb{C}P(4k+1)$ is a spin manifold and if $f \in [\mathbb{C}P(4k+1),F/O]$ corresponds to a framing $L^{8k+2} \times \mathbb{R}^N \subset \xi^N$ in a fibre homotopically trivial bundle ξ^N over $\mathbb{C}P(4k+1)$, then L^{8k+2} is also a spin manifold. In fact, if $\pi: L^{8k+2} \to \mathbb{C}P(4k+1)$ is the projection then $\pi^* w_j(\mathbb{C}P(4k+1)) = w_j(L^{8k+2})$, where w_j is the Stiefel-Whitney class. From [4], it follows that the surgery obstruction to making $\pi: L^{8k+2} \to \mathbb{C}P(4k+1)$ a homotopy equivalence is given by the formula

$$\begin{aligned} s(f) &= \Psi(L^{8k+2}) - \Psi(\mathbb{C}P(4k+1)) \\ &= \sum_J \alpha_J(\pi^J(L^{8k+2}) - \pi^J(\mathbb{C}P(4k+1))) \in \mathbb{Z}_2. \end{aligned}$$

The terms in Stiefel-Whitney numbers drop out because the Stiefel-Whitney numbers of L^{8k+2} and $\mathbb{C}P(4k+1)$ coincide.

Recall that the KO characteristic number $\pi^J(M^{8k+2})$ of a spin manifold M^{8k+2} is computed as the composition

$$S^{8N+8k+2} \xrightarrow{c} T(\nu^{8N}_{M^{8k+2}}) \to M\,\mathrm{spin}(8N) \xrightarrow{\Phi_K(\pi^J)} BSO$$

where $\nu^{8N}_{M^{8k+2}} = \nu$ is the normal bundle of M^{8k+2} embedded in a sphere $S^{8N+8k+2}$, c is the natural collapsing map, that is, c is the identity on ν and collapses $S^{8N+8k+2} - \nu$ to the point at infinity in $T(\nu)$, and $\Phi_K(\pi^J)$ is the image under the KO-theory Thom isomorphism of the operation π^J applied to the universal bundle over B spin. In the situation above we are interested in computing the characteristic number $\pi^J(L^{8k+2})$ in terms of invariants of the map $f: \mathbb{C}P(4k+1) \to F/O$.

Lemma 3.6. There is an element $\gamma \in KO(F/O)$ with $ph(\gamma) = \hat{A} \in H^{**}(F/O,Q) = Q[[p_1 p_2 \cdots]]$, where ph is the Pontryagin character and $\hat{A} = 1 + \hat{A}_1 + \hat{A}_2 + \cdots$ is the total $\hat{A}$ class of the universal bundle over F/O.

Proof. F/O is the universal space for stable vector bundles together with a fibre homotopy trivialization. Let η^{8N} be an $8N$ bundle representing the universal bundle restricted to some skeleton of F/O and let $t: T(\eta^{8N}) \to S^{8N}$ be the universal fibre homotopy trivialization.

There are two natural KO-theory orientations for $T(\eta^{8N})$. First, there is the element $t^*(\alpha) \in KO^o(T(\eta^{8N}))$, where $\alpha \in KO^o(S^{8N})$ is a generator, and, secondly, there is the canonical orientation $\Phi_K(1) \in KO^o(T(\eta^{8N}))$ of the spin bundle η^{8N}. (Since $w_2(\eta^{8N}) = 0$ and since F/O is simply connected, η^{8N} admits a unique spin structure.)

Define $\gamma \in KO(F/O)$ to be the element satisfying the formula $t^*(\alpha) = \gamma \cdot \Phi_K(1)$. Then $\Phi_H(1) = ph(t^*(\alpha)) = ph(\gamma) \cdot ph(\Phi_K(1))$ $= ph(\gamma)\Phi_H(\hat{A}^{-1}(\eta^{8N}))$ hence $ph(\gamma) = \hat{A}(\eta^{8N}) = \hat{A}$ as desired.

Strictly speaking, this defines γ only over a skeleton of F/O. But it is easy to check that one obtains a well-defined element $\gamma \in KO(F/O)$ by taking the limit over skeletons of all dimensions.

Proposition 3.7. Let $f \in [\mathbb{C}P(4k + 1),F/O]$ correspond to a framing $L^{8k+2} \times \mathbb{R}^{8N} \subset \xi^{8N}$, where ξ^{8N} is a bundle over $\mathbb{C}P(4k + 1)$. Then

$$\pi^J(L^{8k+2}) = c_1^*\Phi_K(\pi^J(\nu_{\mathbb{C}P(4k+1)} - \xi)\cdot f^*(\gamma))$$
$$\in KO^0(S^{16N+8k+2}) = \mathbb{Z}_2$$

where $S^{16N+8k+2} \xrightarrow{c_1} T(\nu^{16N}_{\mathbb{C}P(4k+1)})$ is the collapsing map and Φ_K is the Thom isomorphism Φ_K: $KO(\mathbb{C}P(4k+1)) \xrightarrow{\simeq} KO^0(T(\nu^{16N}_{\mathbb{C}P(4k+1)}))$.

Proof. Write the stable normal bundle of $\mathbb{C}P(4k+1)$ as a Whitney sum $\nu^{16N}_{\mathbb{C}P(4k+1)} = (\nu_{\mathbb{C}P(4k+1)} - \xi)^{8N} + \xi^{8N}$. Let π be the projection $\pi\colon \xi^{8N} \to \mathbb{C}P(4k+1)$ (we also denote by π the restriction of this projection to $L^{8k+2} \times \mathbb{R}^N$ and $L^{8k+2} = L^{8k+2} \times 0$) and let $\nu_{L\times\mathbb{R}^{8N}}$ be the normal bundle of the composite inclusion $L^{8k+2} \times \mathbb{R}^{8N} \subset \xi^{8N} \subset \nu^{16N}_{\mathbb{C}P(4k+1)}$. If $(p,x) \in L^{8k+2} \times \mathbb{R}^{8N}$ then the fibre of $\nu_{L\times\mathbb{R}^{8N}}$ at the point (p,x) is clearly the fibre $(\nu_{\mathbb{C}P(4k+1)} - \xi)_{\pi(p,x)}$ of the bundle $\nu_{\mathbb{C}P(4k+1)} - \xi$ over $\mathbb{C}P(4k+1)$ at the point $\pi(p,x)$. Let $h\colon \nu_{L\times\mathbb{R}^{8N}} \xrightarrow{\simeq} \pi^*(\nu_{\mathbb{C}P(4k+1)} - \xi) \times \mathbb{R}^{8N}$ be a bundle isomorphism, where $\pi\colon L^{8k+2} \to \mathbb{C}P(4k+1)$ and $\pi^*(\nu_{\mathbb{C}P(4k+1)} - \xi) \times \mathbb{R}^{8N}$ is regarded as a bundle over $L^{8k+2} \times \mathbb{R}^{8N}$. Let $h_{(p,x)}$ denote the induced isomorphism of fibres $h_{(p,x)}\colon (\nu_{\mathbb{C}P(4k+1)} - \xi)_{\pi(p,x)} \xrightarrow{\simeq} (\nu_{\mathbb{C}P(4k+1)} - \xi)_{\pi(p,0)}$. We may clearly assume that $h_{(p,0)} = \mathrm{Id}$.

Since the total space of $\nu_{L\times\mathbb{R}^{8N}}$ is also the total space of the stable normal bundle of L^{8k+2}, we see that the normal bundle of L^{8k+2} can be written as a sum $h\colon \nu^{16N}_L \xrightarrow{\simeq} \pi^*(\nu_{\mathbb{C}P(4k+1)} - \xi) + e^{8N}_L$, where e^{8N}_L is the trivial bundle over L^{8k+2}.

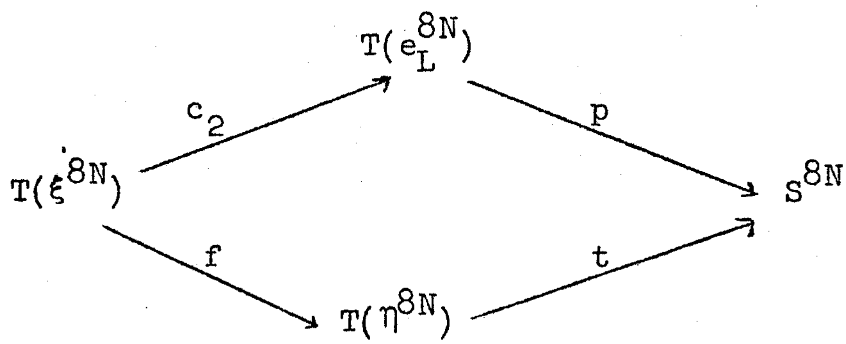
The framing $L^{8k+2} \times \mathbb{R}^{8N} \subset \xi^{8N}$ gives rise to a collapsing map $c_2\colon T(\xi^{8N}) \to T(e_L^{8N})$. Thus there is a diagram

$$
\begin{array}{ccc}
 & T(\nu_L^{16N}) \xrightarrow{\ \Delta h\ } T(\pi^*(\nu_{\mathbb{C}P(4k+1)} - \xi)) \wedge T(e_L^{8N}) & \\
 {}^{c_3}\nearrow & & \Big\downarrow \pi \wedge \mathrm{Id} \\
S^{16N+8k+2} & & T(\nu_{\mathbb{C}P(4k+1)} - \xi) \wedge T(e_L^{8N}) \\
 {}_{c_1}\searrow & & \Big\uparrow \mathrm{Id} \wedge c_2 \\
 & T(\nu^{16N}_{\mathbb{C}P(4k+1)}) \xrightarrow{\ \Delta\ } T(\nu_{\mathbb{C}P(4k+1)} - \xi) \wedge T(\xi^{8N}) &
\end{array}
$$

where the three spaces on the right side of the diagram are Thom spaces of bundles over the products $\mathbb{C}P(4k+1) \times \mathbb{C}P(4k+1)$, $\mathbb{C}P(4k+1) \times L^{8k+2}$, and $L^{8k+2} \times L^{8k+2}$, c_1, c_2, and c_3 are collapsing maps, and Δ is the diagonal.

This diagram commutes up to homotopy. For, first, if $z \in S^{16N+8k+2} - \nu_L^{16N}$, it is easy to see that $(\pi \wedge \mathrm{Id}) \circ \Delta h \circ c_3(z)$ and $(\mathrm{Id} \wedge c_2) \circ \Delta \circ c_1(z)$ are both the point at infinity in $T(\nu_{\mathbb{C}P(4k+1)} - \xi) \wedge T(e_L^{8N})$. Secondly, if $z \in \nu_L^{16N} = \nu_{L \times \mathbb{R}^{8N}}$ is described by coordinates (p,x,y), where $p \in L^{8k+2}$, $x \in \mathbb{R}^{8N}$, and $y \in (\nu_{\mathbb{C}P(4k+1)} - \xi)_{\pi(p,x)}$, then $(\mathrm{Id} \wedge c_2) \circ \Delta \circ c_1(z)$ has coordinates $(\pi(p,x),p,y,x)$ in the bundle $(\nu_{\mathbb{C}P(4k+1)} - \xi) \times e_L^{8N}$ over $\mathbb{C}P(4k+1) \times L^{8k+2}$, and $(\pi \wedge \mathrm{Id}) \circ \Delta h \circ c_3(z)$ has coordinates $(\pi(p,0),p,h_{(p,x)}(y),x)$. A homotopy between $(\pi \wedge \mathrm{Id}) \circ \Delta h \circ c_3$ and $(\mathrm{Id} \wedge c_2) \circ \Delta \circ c_1$ is thus given by $(\pi(p,tx),p,h^{-1}_{(p,tx)} \circ h_{(p,x)}(y),x$ $0 \leq t \leq 1$. The homotopy maps $S^{16N+8k+2} - \nu_L^{16N}$ to the point at infinity.

Now the characteristic number $\pi^J(L^{8k+2})$ is by definition the element $c_3^*(\Delta h)^*(\pi \wedge \mathrm{Id})^*(\Phi_K(\pi^J(\nu_{\mathbb{C}P(4k+1)} - \xi)) \cdot \Phi_K(1))$ $\in KO^0(S^{16N+8k+2}) = \mathbb{Z}_2$. (We use Φ_K to denote the KO-theory Thom isomorphism of all the spin bundles occuring in the argument.) By homotopy commutativity of the diagram above, this is equal to $c_1^*\Delta^*(\Phi_K(\pi^J(\nu_{\mathbb{C}P(4k+1)} - \xi)) \cdot c_2^*\Phi_K(1))$. Since F/O is the universal space for fibre homotopy trivial bundles, the framing $L^{8k+2} \times \mathbb{R}^{8N} \subset \dot{\xi}^{8N}$ associated to $f \in [\mathbb{C}P(4k+1), F/O]$ is constructed by transverse regularity from the fibre homotopy trivialization $T(\dot{\xi}^{8N}) \xrightarrow{f} T(\eta^{8N}) \xrightarrow{t} S^{8N}$, where, as in Lemma 3.6, $T(\eta^{8N}) \xrightarrow{t} S^{8N}$ is the universal example over F/O. It follows that the diagram



commutes up to homotopy, where p is the projection of the trivial bundle e_L^{8N} onto the fibre. Thus $c_2^*\Phi_K(1) = c_2^* p^*(\alpha)$ $= f^* t^*(\alpha) = f^*(\gamma \cdot \Phi_K(1)) = \Phi_K(f^*(\gamma))$, by naturality of the Thom isomorphisms Φ_K. Finally, then, $\Delta^*(\Phi_K(\pi^J(\nu_{\mathbb{C}P(4k+1)} - \xi))$ $\cdot\, \Phi_K(f^*(\gamma))) = \Phi_K(\pi^J(\nu_{\mathbb{C}P(4k+1)} - \xi) \cdot f^*(\gamma))$ hence $\pi^J(L^{8k+2})$ $= c_1^*\Phi_K(\pi^J(\nu_{\mathbb{C}P(4k+1)} - \xi) \cdot f^*(\gamma))$ as desired.

Proposition 3.8. Let $f \in [\mathbb{C}P(4k+1), F/O]$. Then

$$s(f) = \sum_J \alpha_J \cdot c^* \Phi_K(\pi^J(\nu_{\mathbb{C}P(4k+1)} - \xi) \cdot f^*(\gamma) - \pi^J(\nu_{\mathbb{C}P(4k+1)})) \in \mathbb{Z}_2.$$

Further, $s(f)$ depends only on the bundle $\xi = \xi(f)$, which represents the composition $\mathbb{C}P(4k+1) \xrightarrow{f} F/O \xrightarrow{i} BSO$. In particular, if $f \in [\mathbb{C}P(4k+1), F]$, then $s(f) = 0$.

Proof. The formula for $s(f)$ is an immediate corollary of Proposition 3.7 since by earlier remarks $s(f) = \Sigma\, \alpha_J(\pi^J(L^{8k+2}) - \pi^J(\mathbb{C}P(4k+1)))$.

By Lemma 3.6 $ph(\gamma) = \hat{A}(\eta)$, hence $ph(f^*(\gamma)) = f^*(ph(\gamma)) = f^*(\hat{A}(\eta)) = \hat{A}(f^*(\eta)) = \hat{A}(\xi)$. By Theorem 2.0(1), $f^*(\gamma) \in KO(\mathbb{C}P(4k+1))$ is determined modulo the torsion element $\omega^{2k+1} \in KO^0(\mathbb{C}P(4k+1))$ by $ph(f^*(\gamma)) = \hat{A}(\xi)$. Thus, also by Theorem 2.0(1), the product $\pi^J(\nu_{\mathbb{C}P(4k+1)} - \xi) \cdot f^*(\gamma)$ depends only on $\hat{A}(\xi)$. This proves that $s(f)$ depends only on ξ. The last statement is obvious from the formula for $s(f)$.

Remarks.

1. This formula for $s(f)$ is theoretically computable, although the coefficients α_J in the formula $\Psi = \Sigma\, \alpha_J \pi^J + \Sigma\, \beta_I w^I$ for the Brown-Kervaire invariant have not yet been determined. In §8 we make the computation for $k = 1$.

2. Sullivan has shown that there is a cohomology class $K = k_2 + k_6 + k_{10} + \cdots \in H^{4k+2}(F/O, \mathbb{Z}_2)$ such that if $f\colon M^{4k+2} \to F/O$ is a map then $s(f) = \langle W(M^{4k+2}) \cdot f^*(K), [M^{4k+2}]\rangle \in \mathbb{Z}_2$, where $W(M^{4k+2}) = 1 + w_1(M^{4k+2}) + w_2(M^{4k+2}) + \cdots$

$\in H^*(M^{4k+2},\mathbb{Z}_2)$ is the total Wu class. This formula is hard to apply because there is no procedure for computing the class $f^*(K)$.

3. It is obvious that if $f \in [\mathbb{C}P(n),PL] \subseteq [\mathbb{C}P(n),F]$ then $s(f) = 0$. The quotient $[\mathbb{C}P(4k+1),F]/[\mathbb{C}P(4k+1),PL]$ is very small. In fact, this quotient is $\mathbb{Z}_2$ for $k = 2$ and it is reasonable to conjecture that it is 0 for all other $k \geq 1$ (see §8 part (6)). Thus the last statement of Proposition 3.8 is not very strong.

4. $s([\mathbb{C}P(4k-1),F])$ is definitely non-zero for $k = 1,2,$ and 4 (and presumably for all $k = 2^j$ [6]). Thus $s\colon [\mathbb{C}P(4k-1),F/O] \to \mathbb{Z}_2$ seems extremely difficult to compute in terms of familiar invariants.

As another application of the machinery used in the proofs of Propositions 3.7 and 3.8, we describe the map $\sigma\colon [\mathbb{C}P(4k),F/O] \to \Gamma_{8k+1}$.

Proposition 3.9. $\sigma([\mathbb{C}P(4k),F/O]) \subseteq b\ \mathrm{spin}_{8k+2} \subset \Gamma_{8k+1}$. Moreover, with respect to the direct sum decompositions $[\mathbb{C}P(4k),F/O] = \oplus\mathbb{Z} + [\mathbb{C}P(4k),F]$ and $b\ \mathrm{spin}_{8k+2} = bP_{8k+2} + \mathrm{kernel}(f_R) = \mathbb{Z}_2 + (\pi^s_{8k+2}/\mathbb{Z}_2+\mathbb{Z}_2)$, we have $\sigma(\oplus\mathbb{Z}) \subseteq bP_{8k+2}$ and $\sigma([\mathbb{C}P(4k),F]) \subseteq \mathrm{kernel}(f_R)$. Finally, if $f \in \oplus\mathbb{Z}$ and $g \in [\mathbb{C}P(4k),F]$, then $\sigma(f+g) = \sigma(f) + \sigma(g)$.

Proof. The first statement is obvious, since by definition if $f \in [\mathbb{C}P(4k),F/O]$ then $\sigma(f)$ is the boundary of a homotopy $\mathbb{C}P(4k+1)_o$, which is a spin manifold.

The fact that $\sigma(\oplus \mathbb{Z}) \subseteq bP_{8k+2}$ follows from Proposition 3.2. In fact, if $f \in \oplus \mathbb{Z}$ then $\sigma(f) = s(\hat{f})$, where $\hat{f} \in [\mathbb{C}P(4k+1), F/O]$ extends f. Of course, $s(\hat{f})$ is given by the KO-theory formula of Proposition 3.8.

For $f \in [\mathbb{C}P(4k), F/O]$, $\sigma(f)$ is the boundary of a homotopy $\mathbb{C}P(4k+1)_o$, say P_o^{8k+2}. In [12], the KO characteristic numbers of P_o^{8k+2} are defined to be the KO numbers of the closed, almost smooth manifold $P^{8k+2} = P_o^{8k+2} \bigcup_{\sigma(f)} D^{8k+2}$ with respect to a KO orientation extending the canonical KO orientation of P_o^{8k+2}. The arguments of Propositions 3.7 and 3.8 show that these KO numbers depend only on the stable tangent bundle of P_o^{8k+2}. It follows that if $g \in [\mathbb{C}P(4k), F]$ then $f_R(\sigma(g)) = 0$. For the associated P_o^{8k+2} is tangentially homotopy equivalent to $\mathbb{C}P(4k+1)_o$ hence the KO numbers and Stiefel-Whitney numbers of $M^{8k+2} = P_o^{8k+2} \# \mathbb{C}P(4k+1)$ vanish. Since $\partial M^{8k+2} = \sigma(g)$, we have $f_R(\sigma(g)) = \Psi(M^{8k+2}) = 0$, where Ψ is the Brown-Kervaire invariant.

The argument in the paragraph above also shows that $f_R\sigma(f+g) = f_R\sigma(f)$ since the homotopy $\mathbb{C}P(4k+1)_o$'s corresponding to $f+g$ and f have the same tangent bundle. The formula $\sigma(f+g) = \sigma(f) + \sigma(g)$ is a consequence of this, the decomposition $b\,spin_{8k+2} = bP_{8k+2} \oplus \mathrm{kernel}(f_R) = \mathbb{Z}_2 \oplus (\pi^s_{8k+1}/\mathbb{Z}_2 + \mathbb{Z}_2)$ and the fact that the composition $[\mathbb{C}P(4k), F/O] \xrightarrow{\sigma} b\,spin_{8k+2} \xrightarrow{\pi_2} \pi^s_{8k+1}/\mathbb{Z}_2 + \mathbb{Z}_2$ coincides with the homomorphism $[\mathbb{C}P(4k), F/O] \xrightarrow{\circ H} [S^{8k+1}, F/O] = \pi^s_{8k+1}/\mathbb{Z}_2 \to \pi^s_{8k+1}/\mathbb{Z}_2 + \mathbb{Z}_2$ induced by the Hopf map $H: S^{8k+1} \to \mathbb{C}P(4k)$. (The subgroup $\mathbb{Z}_2+\mathbb{Z}_2 \subset \pi^s_{8k+1}$ which occurs is $\text{im } J = \mathbb{Z}_2$ plus the subgroup $\mathbb{Z}_2$ generated by the Adams element μ used earlier.)

Summarizing, $s: [\mathbb{C}P(4k), F/O] \to \mathbb{Z}$ is computable in terms of Pontryagin classes hence $hS(\mathbb{C}P(4k))$ is computable in terms of Pontryagin classes and stable homotopy theory. $\sigma: [\mathbb{C}P(4k), F/O] \to b\, spin_{8k+2} \subset \Gamma_{8k+1}$ is also computable, in terms of a KO-theory formula for $s: [\mathbb{C}P(4k+1), F/O] \to \mathbb{Z}_2$ and more stable homotopy theory. These results reduce the problem of classifying free S^1 actions on homotopy 8k+1-spheres to more familiar problems in homotopy theory.

The reduction of the classification of free S^1 actions on homotopy 8k+5 spheres to homotopy theory is made difficult by the problem of describing the group Γ_{8k+5} and the map $\sigma: [\mathbb{C}P(4k+2), F/O] \to \Gamma_{8k+5}$. For homotopy 8k+7 spheres, the difficulty is the computation of the surgery obstruction $s: [\mathbb{C}P(4k+3), F/O] \to \mathbb{Z}_2$.

§4. Conjugation

Let $T: S^1 \times \Sigma^{2n+1} \to \Sigma^{2n+1}$ be a free S^1 action on a homotopy 2n+1-sphere. Define another action $\overline{T}: S^1 \times \Sigma^{2n+1} \to \Sigma^{2n+1}$ by $\overline{T}(\alpha, x) = T(\overline{\alpha}, x)$, where $x \in \Sigma^{2n+1}$ and $\overline{\alpha}$ is the complex conjugate of $\alpha \in S^1$. We call $(\Sigma^{2n+1}, \overline{T})$ the conjugate of (Σ^{2n+1}, T). Clearly, the orbit spaces Σ^{2n+1}/T and $\Sigma^{2n+1}/\overline{T}$ coincide. Call this space P^{2n}. However, the principal S^1

bundles $T: \Sigma^{2n+1} \to P^{2n}$ and $\overline{T}: \Sigma^{2n+1} \to P^{2n}$ have different Chern classes. That is, if (Σ^{2n+1},T) corresponds to $f: P^{2n} \to \mathbb{C}P(n)$ in $hS(\mathbb{C}P(n))$ then $(\Sigma^{2n+1},\overline{T})$ corresponds to $\overline{f}: P^{2n} \to \mathbb{C}P(n)$, where $\overline{f}^*(z) = -f^*(z)$, $z \in H^2(\mathbb{C}P(n),\mathbb{Z})$.

Remark. In the paragraph above we have abused our convention with orientations discussed in the introduction. We should write the conjugate of (Σ^{2n+1},T) as $((-1)^{n+1}\Sigma^{2n+1},\overline{T})$. For, if n is even then $(f^*(z))^n = (\overline{f}^*(z))^n$ and the orientation of P^{2n} is unchanged. However, the orientation of the S^1 bundle, and hence of Σ^{2n+1}, is reversed. If n is odd, both the orientations of P^{2n} and the S^1 bundle are reversed by conjugation, hence the orientation of Σ^{2n+1} is unchanged.

Let $c: \mathbb{C}P(n) \xrightarrow{\sim} \mathbb{C}P(n)$ be the diffeomorphism defined by conjugation of complex coordinates. Then $c^*(z) = -z$. Thus under our identification of equivariant diffeomorphism classes of free S^1 actions on homotopy spheres with $hS(\mathbb{C}P(n))$, conjugation is simply the involution $c_*: hS(\mathbb{C}P(n)) \xrightarrow{\sim} hS(\mathbb{C}P(n))$ defined by $c_*(P^{2n},f) = (P^{2n},cf)$.

Clearly, (Σ^{2n+1},T) is equivariantly diffeomorphic to its conjugate if and only if the associated homotopy projective space, P^{2n}, admits a diffeomorphism $c: P^{2n} \xrightarrow{\sim} P^{2n}$ such that $c^* = -\mathrm{Id}: H^2(P^{2n},\mathbb{Z}) \to H^2(P^{2n},\mathbb{Z})$. A necessary condition is that $\Sigma^{2n+1} = (-1)^{n+1}\Sigma^{2n+1}$ in Γ_{2n+1}. In §8 we give examples of free S^1 actions on odd torsion elements of Γ_{2n+1}, n even (e.g. for $n = 6$), so certainly these are distinct from their conjugates. More interestingly, perhaps, we construct S^1 actions on the standard sphere S^{2n+1} which are distinct from their conjugates (e.g. for $n = 9, 10$).

We next show how conjugation can be computed in terms of homotopy theory.

Proposition 4.1. The following diagram commutes

$$\begin{array}{ccc} hS(\mathbb{C}P(n)) & \xrightarrow{\theta} & [\mathbb{C}P(n),F/O] \\ \downarrow c_* & & \downarrow c^* \\ hS(\mathbb{C}P(n)) & \xrightarrow{\theta} & [\mathbb{C}P(n),F/O] \end{array}$$

where $c^*(g) = gc$ if $g \in [\mathbb{C}P(n),F/O]$ and θ is the injection constructed in §1.

Proof. First, we point out that it follows from Theorem 2.0 that $c^*\colon KO^0(\mathbb{C}P(n)) \to KO^0(\mathbb{C}P(n))$ is the identity. For clearly $c^*(\omega) = \omega$. Also, since $c^2 = \mathrm{Id}$, we have $c = c^{-1}$.

Let $f\colon P^{2n} \to \mathbb{C}P(n)$ represent an element of $hS(\mathbb{C}P(n))$. Then $c_*(P^{2n},f) = (P^{2n},cf)$ and $\theta c_*(P^{2n},f) = \theta(P^{2n},cf)$ is given by a framing

$$P^{2n} \times \mathbb{R}^N \subset (f^{-1}c^{-1})^*\tau_{P^{2n}} - \tau_{\mathbb{C}P(n)} = (f^{-1})^*\tau_{P^{2n}} - \tau_{\mathbb{C}P(n)} = \xi^N$$

such that the projection $\pi\colon P^{2n} \to \mathbb{C}P(n)$ is homotopic to $cf\colon P^{2n} \to \mathbb{C}P(n)$.

On the other hand, $\theta(P^{2n},f)$ is given by a framing $P^{2n} \times \mathbb{R}^N \subset (f^{-1})^*\tau_{P^{2n}} - \tau_{\mathbb{C}P(n)} = \xi^N$ such that the projection $\pi\colon P^{2n} \to \mathbb{C}P(n)$ is homotopic to $f\colon P^{2n} \to \mathbb{C}P(n)$. From the diagram

$$\begin{array}{ccc} c^*(\xi^N) & \xrightarrow{c} & \xi^N \\ \pi\big\downarrow & & \big\downarrow\pi \\ \mathbb{C}P(n) & \xrightarrow{c} & \mathbb{C}P(n) \end{array}$$

it follows that $c^*\theta(P^{2n},f)$ is given by a framing $P^{2n}\times\mathbb{R}^N\subset c^*\xi^N=\xi^N$ such that the projection $\pi\colon P^{2n}\to\mathbb{C}P(n)$ is homotopic to $c^{-1}f=cf\colon P^{2n}\to\mathbb{C}P(n)$. Thus $c^*\theta(P^{2n},f)$ $=\theta c_*(P^{2n},f)$ as desired.

Sullivan has observed that $c^*\colon [\mathbb{C}P(n),F/PL]\to[\mathbb{C}P(n),F/PL]$ is the identity [24]. Thus any PL homotopy $\mathbb{C}P(n)$, P^{2n}, admits a PL isomorphism $c\colon P^{2n}\xrightarrow{\simeq}P^{2n}$ such that $c^*=-\mathrm{Id}\colon H^2(P^{2n},\mathbb{Z})\to H^2(P^{2n},\mathbb{Z})$.

§5. A Spectral Sequence

There is a well-known spectral sequence, which is useful for computing $[\mathbb{C}P(n),Y]$ for any space Y, constructed as follows. First, there are the cofibration sequences

5.1. $$S^{2q+1}\xrightarrow{H}\mathbb{C}P(q)\xrightarrow{i}\mathbb{C}P(q+1)\xrightarrow{j}S^1\wedge S^{2q+1}\xrightarrow{H}S^1\wedge\mathbb{C}P(q)\cdots$$

These fit together to give a diagram

5.2. $$\begin{array}{ccc} \bigvee\limits_{\substack{p\geq 0\\ q\geq 1}}(S^p\wedge\mathbb{C}P(q)) & \xrightarrow{i_{**}} & \bigvee\limits_{\substack{p\geq 0\\ q\geq 1}}(S^p\wedge\mathbb{C}P(q+1)) \\ {\scriptstyle H_{**}}\nwarrow & & \swarrow{\scriptstyle j_{**}} \\ & \bigvee\limits_{\substack{p\geq 0\\ q\geq 1}}(S^p\wedge S^{2q+1}) & \end{array}$$

where

$$H_{p,q}\colon S^p \wedge S^{2q+1} \to S^p \wedge \mathbb{C}P(q)$$
$$i_{p,q}\colon S^p \wedge \mathbb{C}P(q) \to S^p \wedge \mathbb{C}P(q+1)$$
$$j_{p,q}\colon S^p \wedge \mathbb{C}P(q+1) \to S^{p+1} \wedge S^{2q+1}$$

and the sequence of maps $\cdots H_{p,q}, i_{p,q}, j_{p,q}, H_{p+1,q} \cdots$ is the cofibration sequence 5.1.

Mapping diagram 5.2 into a space Y (in our applications Y will be an H-space) gives a bigraded exact couple

5.3.
$$\begin{array}{ccc} [S^* \wedge \mathbb{C}P(*), Y] & \xleftarrow{\;i^{**}\;} & [S^* \wedge \mathbb{C}P(*), Y] \\ & {\scriptstyle H^{**}}\searrow \quad \nearrow{\scriptstyle j^{**}} & \\ & [S^* \wedge S^{2*+1}, Y] & \end{array}$$

There is an associated spectral sequence with

$$E_1^{p,q} = [S^p \wedge S^{2q+1}, Y] = \pi_{p+2q+1}(Y)$$

and differentials

$$d_r^{p,q}\colon E_r^{p,q} \to E_r^{p-1,q+r}$$

where $E_r^{p,q}$ is a sub-quotient group of $E_1^{p,q} = \pi_{p+2q+1}(Y)$.

The differentials are computed as follows. Let $\alpha \in E_1^{p,q} = \pi_{p+2q+1}(Y)$ be represented by a map $\alpha\colon S^p \wedge S^{2q+1} \to Y$. Consider the composition $S^{p-1} \wedge \mathbb{C}P(q+1) \xrightarrow{j} S^p \wedge S^{2q+1} \xrightarrow{\alpha} Y$. The differential $d_r(\alpha)$ is defined if $\alpha \circ j$ extends to a map $\tilde{\alpha}\colon S^{p-1} \wedge \mathbb{C}P(q+r) \to Y$. The value $d_r(\alpha)$ is the composition

$\tilde{\alpha} \circ H$, which represents an element of $\pi_{p+2q+2r}(Y) = E_1^{p-1,q+r}$.

$$\begin{array}{ccccc} S^{p-1} \wedge S^{2q+2r+1} & \xrightarrow{H} & S^{p-1} \wedge \mathbb{C}P(q+r) & \xrightarrow{\tilde{\alpha}} & Y \\ & & \uparrow i & & \uparrow \alpha \\ & & S^{p-1} \wedge \mathbb{C}P(q+1) & \xrightarrow{j} & S^{p+2q+1} \end{array}$$

The indeterminacy in $d_r(\alpha)$ results from different choices of the extension $\tilde{\alpha}$.

Thus $d_r^{p,q}$ is a higher order homotopy operation defined on a subgroup of $\pi_{p+2q+1}(Y)$ with values in a quotient group of $\pi_{p+2q+2r}(Y)$. The universal example for computing $d_r^{p,q}$ is the space $S^{p-1} \wedge \mathbb{C}P(q+r)/S^{p-1} \wedge \mathbb{C}P(q) = S^{p-1} \wedge (\mathbb{C}P(q+r)/\mathbb{C}P(q))$, where $d_r^{p,q}$ is defined on the element represented by the bottom cell $S^{p+2q+1} \to S^{p-1} \wedge (\mathbb{C}P(q+r)/\mathbb{C}P(q))$ and has value represented by the suspension of the Hopf map $S^{p-1} \wedge S^{2q+2r+1} \to S^{p-1} \wedge (\mathbb{C}P(q+r)/\mathbb{C}P(q))$.

Of particular importance are the differentials $d_r^{1,q}$ and $d_r^{2,q}$, for these determine $[\mathbb{C}P(n),Y]$ up to group extensions. Specifically, let $D^{1,q} = \bigcup_s \text{image}(d_s^{1,q-s}) \subseteq \pi_{2q+1}(Y)$ and let $D^{2,q} = \bigcup_s \text{image}(d_s^{2,q-s}) \subseteq \pi_{2q+2}(Y)$. Then the cofibration sequence $S^{2q+1} \to \mathbb{C}P(q) \to \mathbb{C}P(q+1) \to S^{2q+2}$ gives rise to an exact sequence

$$0 \to \pi_{2q+2}(Y)/D^{2,q} \to [\mathbb{C}P(q+1),Y] \to [\mathbb{C}P(q),Y] \to D^{1,q} \to 0.$$

If $Y = F = \lim_{m\to\infty} \Omega^m S^m$, there is a simplification. For, as a set, $[S^p \wedge \mathbb{C}P(q),F] = [S^{m+p} \wedge \mathbb{C}P(q),S^m]$, m large. Thus the problem of computing the differential $d_r^{p,q}\colon \bar{\pi}_{p+2q+1}(F) \to \bar{\pi}_{p+2q+2r}(F)$ (The bars denote sub-quotient groups) as a homotopy operation with universal space $S^{p-1}\wedge(\mathbb{C}P(q+r)/\mathbb{C}P(q))$ reduces to the problem of computing the differential $d_r^{m+p,q}\colon \bar{\pi}_{m+p+2q+1}(S^m) \to \bar{\pi}_{m+p+2q+2r}(S^m)$, m large. This is a homotopy operation with universal space $S^{m+p-1}\wedge(\mathbb{C}P(q+r)/\mathbb{C}P(q))$, hence is purely a problem in stable homotopy theory. Denote by d_r^q this stable homotopy operation of degree $2r-1$. We can thus regard the spectral sequence for $[\mathbb{C}P(n),F]$ as having first term $E_1^{p,q} = \pi^s_{p+2q+1}$ and differentials $d_r^{p,q} = d_r^q\colon \bar{\pi}^s_{p+2q+1} \to \bar{\pi}^s_{p+2q+2r}$ independent of p.

The space $\mathbb{C}P(q+r)/\mathbb{C}P(q)$ is the Thom space of the bundle $(q+1)H$ over $\mathbb{C}P(r-1)$ [5]. Further, there is a least positive integer, M_r, computed by Adams [2], such that M_rH is fibre homotopically trivial over $\mathbb{C}P(r-1)$. It follows that for fixed r the stable homotopy type of $\mathbb{C}P(q+r)/\mathbb{C}P(q)$ is periodic in q with period M_r. Specifically, $\mathbb{C}P(q+M_r+r)/\mathbb{C}P(q+M_r)$ is the Thom space of $(q+1+M_r)H$, which is fibre homotopy equivalent to $(q+1)H + e^{2M_r}$, where e^{2M_r} is the trivial bundle. Hence $\mathbb{C}P(q+M_r+r)/\mathbb{C}P(q+M_r)$ is homotopy equivalent to $S^{2M_r}\wedge(\mathbb{C}P(q+r)/\mathbb{C}P(q))$. Thus the differentials in the spectral sequence for $[\mathbb{C}P(n),F]$ satisfy the periodicity relation

5.4. $$d_r^q = d_r^{q+M_r}.$$

Since the stable homotopy groups of spheres are finite, we can express the spectral sequence for $[\mathbb{C}P(n),F]$ as the direct sum of its p-primary components. The differentials in the p-primary component are determined by the p-primary homotopy type of $\mathbb{C}P(q+r)/\mathbb{C}P(q)$, $q >> r, p$. Imanishi has shown the following for odd primes p [15].

Lemma 5.5

(1) For $r < p^2 - 2$, $\mathbb{C}P(q+r)/\mathbb{C}P(q-1)$ has the mod p homotopy type of a wedge $\bigvee_{i=0}^{p-2} X_{q+i}$, where $X_{q+i} = S^{2q+2i} \cup e^{2q+2i+2(p-1)} \cup \cdots \cup e^{2q+2i+2t(p-1)}$, $t = [\frac{2r-2i}{2(p-1)}]$.

(2) The attaching map $S^{2q+2(p-1)-1} \to S^{2q}$ of the second cell of X_q has order

$$\begin{array}{lll} p & \text{if} & q \not\equiv 0 \mod p \\ 0 & \text{if} & q \equiv 0 \mod p. \end{array}$$

(3) The attaching map $S^{2q+4(p-1)-1} \to S^{2q} \cup e^{2q+2(p-1)}$ of the third cell of X_q has order

$$\begin{array}{lll} p^2 & \text{if} & q \not\equiv 0,1 \mod p \\ 0 & \text{if} & q \equiv -2p+1 \mod p^2 \\ p & \text{otherwise} & \end{array}$$

As a corollary we deduce

Proposition 5.6

(1) For $r < p^2 - 2$ the only possible non-zero differentials

d_r^q in the p-primary spectral sequence for $[\mathbb{C}P(n),F]$ are $d^q_{s(p-1)}$, $1 \leq s \leq p-1$.

(2) Up to multiplication by a unit in $\mathbb{Z}_p$ we have

$$
\begin{aligned}
d^{q-1}_{p-1} &= \alpha_1 \quad \text{if} \quad q \not\equiv 0 \bmod p \\
&= 0 \quad \text{if} \quad q \equiv 0 \bmod p
\end{aligned}
$$

where by α_1 we mean composition with $\alpha_1 \in {}_p\pi^S_{2(p-1)-1} = \mathbb{Z}_p$.

(3) Up to multiplication by a unit, the differential $d^{q-1}_{2(p-1)}$ is periodic in q with period p^2 and is given by

$$
\begin{aligned}
d^{q-1}_{2(p-1)} &= \langle \cdot\, ,\alpha_1,\alpha_1\rangle \quad \text{if} \quad q \not\equiv 0,1 \bmod p \\
&= 0 \quad \text{if} \quad q \equiv 0,-2p+1 \bmod p^2 \\
&= \alpha_2 \quad \text{otherwise}
\end{aligned}
$$

where $\langle \cdot\, ,\alpha_1,\alpha_1\rangle$ indicates the Toda bracket operation.

Proof.

(1) In the notation of Lemma 5.5(1) the Hopf map $S^{2q+2r-1} \to \mathbb{C}P(q+r-1)/\mathbb{C}P(q-1)$ is the attaching map of the top cell of X_{q+i}, where $i \equiv r \bmod (p-1)$, $0 \leq i \leq p-2$. The only possible non-zero differentials correspond to non-trivial attaching maps of X_q. That is, we must have $i = 0$. Thus these can only be the $d^{q-1}_{s(p-1)}$.

(2) This is obvious from Lemma 5.5(2).

(3) This follows from Lemma 5.5(3) and three additional facts. First, $\pi_{2q+4(p-1)-1}(S^{2q}\cup_{\alpha_1} e^{2q+2(p-1)}) = \mathbb{Z}_{p^2}$ with generator $\langle i_{2q},\alpha_1,\alpha_1\rangle$, where $i_{2q}\colon S^{2q} \to S^{2q}\cup_{\alpha_1} e^{2q+2(p-1)}$ is the

bottom cell [28], secondly, the bottom cell of the skeleton $S^{2q} \cup e^{2q+2(p-1)} \cup e^{2q+4(p-1)}$ of X_q splits off if and only if $q \equiv 0 \bmod p^2$ [2], and finally, $\langle \cdot ,\alpha_1,\alpha_1\rangle p = \alpha_2$ [26].

It is to be understood that in the formulas for the d_r^q above (and below) the range of definition of the operations on the right side is no less than that of d_r^q and the ambiguity is no greater.

More differentials could readily be computed from the information on X_q given in [15]. We need only the following.

Proposition 5.7. Let $q \equiv -1 \bmod p$. Then, up to multiplication by a unit, $d^{q-1}_{(p-1)(p-1)}$ is the p-fold Toda bracket $\langle \cdot \ \alpha_1,\alpha_1 \cdots \alpha_1\rangle$.

For $p = 2$, the determination of differentials in the spectral sequence for $[\mathbb{C}P(n),F]$ is more tedious. Computation of the 2-primary homotopy groups and attaching maps of the spaces $\mathbb{C}P(q+r)/\mathbb{C}P(q-1) = S^{2q} \cup e^{2q+2} \cup \cdots \cup e^{2q+2r}$ requires a hodge-podge of techniques (exact sequences, Toda bracket computations, K-theory invariants, etc.) and does not seem to be written up in the literature. We state without proof the following results.

Proposition 5.8. Up to multiplication by a unit, the first four 2-primary differentials are given by

(1). $d_1^{q-1} = \eta$ if $q \equiv 1 \bmod 2$

$= 0$ if $q \equiv 0 \bmod 2$

(2) $$d_2^{q-1} = \nu \quad \text{if} \quad q \equiv 2,3,6,7 \bmod 8$$
$$= 2\nu \quad \text{if} \quad q \equiv 1,4 \bmod 8$$
$$= 0 \quad \text{if} \quad q \equiv 0,5 \bmod 8$$

(3) $$d_3^{q-1} = \nu\eta^{-1}\nu \qquad \text{if} \quad q \equiv 2,6 \bmod 8$$
$$= \langle \cdot\ ,\eta,\nu\rangle \qquad \text{if} \quad q \equiv 5 \bmod 8$$
$$= \langle \cdot\ ,\nu,\eta\rangle \qquad \text{if} \quad q \equiv 7 \bmod 8$$
$$= \langle \cdot\ \eta,\nu\rangle + \langle \cdot\ ,2\nu,\eta\rangle \qquad \text{if} \quad q \equiv 1 \bmod 8$$
$$= \langle \cdot\ \eta,\nu\rangle 2 + \langle \cdot\ ,\nu,\eta\rangle \qquad \text{if} \quad q \equiv 3 \bmod 8$$
$$= 0 \qquad \text{if} \quad q \equiv 0,4 \bmod 8$$

(4) d_4^{q-1} is periodic in q with period 64. Partial information is given by

$$d_4^{q-1} = \sigma + \langle \cdot\ ,2\nu,\nu\rangle \qquad \text{if} \quad q \equiv 4 \bmod 8$$
$$= \sigma \qquad \text{if} \quad q \equiv 5 \bmod 8$$
$$= \sigma + \langle \cdot\ ,\nu\eta^{-1}\nu,\eta\rangle \qquad \text{if} \quad q \equiv 6 \bmod 8$$
$$= \sigma + \langle \cdot\ ,\nu,2\nu\rangle \qquad \text{if} \quad q \equiv 7 \bmod 8$$
$$= 2^{j_o}\sigma \qquad \text{if} \quad q \equiv 0 \bmod 8$$
$$= 2^{j_1}\sigma + \langle \cdot\ ,2\nu,\nu\rangle \qquad \text{if} \quad q \equiv 1 \bmod 8$$
$$= 2^{j_2}\sigma + \langle \cdot\ ,\nu,2\nu\rangle + \langle \cdot\ ,\nu\eta^{-1}\nu,\eta\rangle \qquad \text{if} \quad q \equiv 2 \bmod 8$$
$$= 2^{j_3}\sigma \qquad \text{if} \quad q \equiv 3 \bmod 8$$

where j_o, j_1, j_2, j_3 are greater than zero and depend on q modulo 64.

§6. The spectral sequence for $[\mathbb{C}P(n),PL/O]$

In the spectral sequence for computing $[\mathbb{C}P(n),PL/O]$, defined in §5, the differentials $d_r^{1,q}$ and $d_r^{2,q}$ have a very nice geometric interpretation. First, we need the following.

Lemma 6.1. Let $g \in [\mathbb{C}P(n),PL/O]$ correspond to the smoothing P^{2n} of $\mathbb{C}P(n)$. Then the composition $\mathbb{C}P(n+1)_o \xrightarrow{H} \mathbb{C}P(n) \xrightarrow{g} PL/O$ corresponds to a smoothing of $\mathbb{C}P(n+1)_o$ which coincides with the natural smooth structure, say P_o^{2n+2}, on the Hopf disc bundle over P^{2n}.

Proof. This requires some knowledge of the details of the identification of $[\mathbb{C}P(n),PL/O]$ with the set of concordance classes of smoothings of $\mathbb{C}P(n)$ [13], [17]. By Proposition 2.4(1), the map $g: \mathbb{C}P(n) \to PL/O$ corresponds to a map $G: \mathbb{C}P(n) \to PL$, that is, to a PL bundle isomorphism $G: \mathbb{C}P(n) \times \mathbb{R}^N \xrightarrow{\cong} \mathbb{C}P(n) \times \mathbb{R}^N$, N large. G is covered by a PL bundle isomorphism $G_o: \mathbb{C}P(n+1)_o \times \mathbb{R}^N \xrightarrow{\cong} \mathbb{C}P(n+1)_o \times \mathbb{R}^N$ such that the diagram

$$\begin{array}{ccc} \mathbb{C}P(n+1)_o \times \mathbb{R}^N & \xrightarrow{G_o} & \mathbb{C}P(n+1)_o \times \mathbb{R}^N \\ \downarrow H\times Id & & \downarrow H\times Id \\ \mathbb{C}P(n) \times \mathbb{R}^N & \xrightarrow{G} & \mathbb{C}P(n) \times \mathbb{R}^N \end{array}$$

commutes. G_o, regarded as a map $G_o: \mathbb{C}P(n+1)_o \to PL$, corresponds by the isomorphism of Proposition 2.4(1) to the composition $gH \in [\mathbb{C}P(n+1)_o,PL/O]$.

The standard smooth structure on $\mathbb{C}P(n) \times \mathbb{R}^N$ and the PL isomorphism G give rise to another differentiable structure, $(\mathbb{C}P(n) \times \mathbb{R}^N)_\alpha$, on $\mathbb{C}P(n) \times \mathbb{R}^N$. Local smooth coordinates $\mathbb{R}^{N+2n} \subset (\mathbb{C}P(n) \times \mathbb{R}^N)_\alpha$ are given by composing standard coordinates $\mathbb{R}^{N+2n} \subset \mathbb{C}P(n) \times \mathbb{R}^N$ with G^{-1}. Similarly, the PL isomorphism G_o induces a differentiable structure, $(\mathbb{C}P(n+1)_o \times \mathbb{R}^N)_\alpha$, on $\mathbb{C}P(n+1)_o \times \mathbb{R}^N$. Commutativity of the diagram above implies that $(\mathbb{C}P(n+1)_o \times \mathbb{R}^N)_\alpha$ coincides with the natural differentiable structure on the Hopf disc bundle over $(\mathbb{C}P(n) \times \mathbb{R}^N)_\alpha$. But $(\mathbb{C}P(n) \times \mathbb{R}^N)_\alpha$ is concordant to $P^{2n} \times \mathbb{R}^N$. The Hopf bundle over a concordance between $(\mathbb{C}P(n) \times \mathbb{R}^N)_\alpha$ and $P^{2n} \times \mathbb{R}^N$ is clearly itself a concordance between $(\mathbb{C}P(n+1)_o \times \mathbb{R}^N)_\alpha$ and the Hopf bundle over $P^{2n} \times \mathbb{R}^N$. Since the latter is obviously $P_o^{2n+2} \times \mathbb{R}^N$, the lemma is proved.

Now let $\Sigma = \Sigma^{2q+2} \in \Gamma_{2q+2} = \pi_{2q+2}(PL/O)$. Then $j^*(\Sigma) \in [\mathbb{C}P(q+1), PL/O]$, regarded as a smoothing of $\mathbb{C}P(q+1)$, is the connected sum $\mathbb{C}P(q+1) \# \Sigma^{2q+2}$. The total space of the Hopf disc bundle over $\mathbb{C}P(q+1) \# \Sigma^{2q+2}$ is a smoothing of $\mathbb{C}P(q+2)_o$, which by Lemma 6.1 corresponds to the composition $\mathbb{C}P(q+2)_o \xrightarrow{H} \mathbb{C}P(q+1) \xrightarrow{j^*(\Sigma)} PL/O$. Its boundary, an element of Γ_{2q+3}, is clearly $d_1^{1,q}(\Sigma) \in \pi_{2q+3}(PL/O)$. If $d_1^{1,q}(\Sigma) = 0$, the smoothing of $\mathbb{C}P(q+2)_o$ given by $j^*(\Sigma) \circ H$ extends to a smoothing of $\mathbb{C}P(q+2)$ by gluing on a disc, D^{2q+4}, along the boundary. This gluing is well-defined only up to diffeomorphisms of S^{2q+3}, hence the resulting smoothing of $\mathbb{C}P(q+2)$ is well-defined only up to connected sum with an element of Γ_{2q+4}. This is exactly the homotopy theoretic ambiguity in extending $j^*(\Sigma) \circ H\colon \mathbb{C}P(q+2)_o \to PL/O$ to a map $\mathbb{C}P(q+2) \to PL/O$. For each of these smoothings of $\mathbb{C}P(q+2)$, there is a Hopf disc bundle, which gives a smoothing of $\mathbb{C}P(q+3)_o$. The boundaries of these smoothings of $\mathbb{C}P(q+3)_o$ form a collection of elements $d_2^{1,q}(\Sigma) \subseteq \pi_{2q+5}(PL/O) = \Gamma_{2q+5}$. More precisely, $d_2^{1,q}(\Sigma)$ is a coset of $\text{image}(\pi_{2q+4}(PL/O) \xrightarrow{d_1^{1,q+1}} \pi_{2q+5}(PL/O))$.

Continuing this argument gives the following.

<u>Proposition 6.2</u>. Let $\Sigma = \Sigma^{2q+2} \in \Gamma_{2q+2} = \pi_{2q+2}(PL/O)$. Then a relation $\Sigma^{2q+2r+1} \in d_r^{1,q}(\Sigma)$ means that the smoothing $\mathbb{C}P(q+1) \mathbin{\#} \Sigma^{2q+2}$ of $\mathbb{C}P(q+1)$ extends to a smoothing of $\mathbb{C}P(q+r+1)_o$ with boundary $\Sigma^{2q+2r+1}$. Moreover, with this smoothing, $\mathbb{C}P(q+j) \subset \mathbb{C}P(q+r+1)_o$ is a smooth submanifold, $1 \leq j \leq r$.

Next, we want to give a geometric interpretation of the differentials $d_r^{2,q}$. Let $\Sigma = \Sigma^{2q+3} \in \Gamma_{2q+3}$. Then the composition $S^1 \times \mathbb{C}P(q+1) \xrightarrow{\pi} S^1 \wedge \mathbb{C}P(q+1) \xrightarrow{j} S^1 \wedge S^{2q+2} \xrightarrow{\Sigma} PL/O$, where π is the natural map, corresponds to the smoothing $(S^1 \times \mathbb{C}P(q+1)) \mathbin{\#} \Sigma^{2q+3}$. On the other hand, Σ^{2q+3} corresponds to a diffeomorphism, σ, of S^{2q+2}, which we may assume is the identity on the lower hemisphere, D_-^{2q+2}. Define a diffeomorphism $\Psi(\sigma)$ of $\mathbb{C}P(q+1)$ by

$$\Psi(\sigma) = \mathrm{Id} \quad \text{on} \quad \mathbb{C}P(q+1)_o$$
$$= \sigma \quad \text{on} \quad D_+^{2q+2} = \mathbb{C}P(q+1) - \mathbb{C}P(q+1)_o.$$

Then by [7,Lemma 1], the differentiable manifold $(S^1 \times \mathbb{C}P(q+1)) \# \Sigma^{2q+3}$ can be identified with the mapping torous $M_{\Psi(\sigma)} = \mathbb{C}P(q+1) \times I/(x,0) \equiv (\Psi(\sigma)x,1)$.

The diffeomorphism $\Psi(\sigma)$ is covered by a (fibre preserving) diffeomorphism, $\widetilde{\Psi(\sigma)}$, of the Hopf bundle $\mathbb{C}P(q+2)_o$ over $\mathbb{C}P(q+1)$. The mapping torous $M_{\widetilde{\Psi(\sigma)}}$ is a D^2 bundle over $M_{\Psi(\sigma)}$ and, further, the argument of Lemma 6.1 shows that $M_{\widetilde{\Psi(\sigma)}}$ is the smoothing of $S^1 \times \mathbb{C}P(q+2)_o$ corresponding to the composition $S^1 \times \mathbb{C}P(q+2)_o \xrightarrow{\mathrm{Id}\times H} S^1 \times \mathbb{C}P(q+1) \xrightarrow{\Sigma} PL/O$. Restricting the diffeomorphism $\widetilde{\Psi(\sigma)}$ of $\mathbb{C}P(q+2)_o$ to the boundary gives a diffeomorphism, $\tilde{\sigma}$, of S^{2q+3}. Then $\partial M_{\widetilde{\Psi(\sigma)}} = M_{\tilde{\sigma}}$ $= (S^1 \times S^{2q+3}) \# \Sigma^{2q+4}$, where $\Sigma^{2q+4} \in \Gamma_{2q+4}$ corresponds to the diffeomorphism $\tilde{\sigma}$. It follows from these observations that $\Sigma^{2q+4} = d_1^{2,q}(\Sigma)$.

Now $d_1^{2,q}(\Sigma) = 0$ if and only if the map $S^1 \wedge \mathbb{C}P(q+2)_o \xrightarrow{\Sigma \circ H} PL/O$ extends to a map $S^1 \wedge \mathbb{C}P(q+2) \to PL/O$. Equivalently, $d_1^{2,q}(\Sigma) = 0$ if and only if $\tilde{\sigma}$ is isotopic to the identity, that is, if the diffeomorphism $\widetilde{\Psi(\sigma)}$ of $\mathbb{C}P(q+2)_o$ extends to a diffeomorphism of $\mathbb{C}P(q+2)$. Of course, such an extension is not unique, but is well-defined only up to diffeomorphisms of S^{2q+4}, that is, up to elements of Γ_{2q+5}. Each such extension of $\widetilde{\Psi(\sigma)}$ induces a diffeomorphism of the Hopf bundle $\mathbb{C}P(q+3)_o$ over $\mathbb{C}P(q+2)$. Restricting these diffeomorphisms to the boundary, S^{2q+5}, gives a collection of elements $d_2^{2,q}(\Sigma) \subseteq \Gamma_{2q+6}$, which is a coset of image$(\pi_{2q+5}(PL/O) \xrightarrow{d_1^{2,q+1}} \pi_{2q+6}(PL/O))$.

Continuing this argument leads to the following.

<u>Proposition 6.3</u>. Let $\Sigma = \Sigma^{2q+3} \in \Gamma_{2q+3}$. Then a relation $\Sigma^{2q+2r+2} \in d_r^{2,q}(\Sigma)$ means that the diffeomorphism $\Psi(\sigma)$ of $\mathbb{C}P(q+1)$ constructed above extends to a diffeomorphism, $\widetilde{\Psi(\sigma)}$, of $\mathbb{C}P(q+r+1)_o$ such that the induced diffeomorphism $\tilde{\sigma}$ of $S^{2q+2r+1} = \partial\mathbb{C}P(q+r+1)_o$ corresponds to $\Sigma^{2q+2r+2} \in \Gamma_{2q+2r+2}$. Moreover, $\widetilde{\Psi(\sigma)}$ may be chosen so that it maps $\mathbb{C}P(q+j) \subset \mathbb{C}P(q+r+1)_o$ onto itself, $1 \leq j \leq r$.

If $Y = F$, F/O, or F/PL, we can describe the spectral sequence for $[\mathbb{C}P(n),Y]$ in an analogous geometric fashion. The details are somewhat more complicated because of the surgery obstruction to associating a homotopy $\mathbb{C}P(n)$ to a map of $\mathbb{C}P(n)$ into one of these spaces. However, the homotopy theory remains unchanged if $\mathbb{C}P(n)$ is replaced everywhere by $\mathbb{C}P(n+1)_o$, and for $\mathbb{C}P(n+1)_o$ the surgery obstruction always vanishes. Thus, computation with the spectral sequence for F, F/O, or F/PL can be interpreted as an investigation of the inclusion relations between tangential, smooth, or PL homotopy $\mathbb{C}P(q)_o$'s, respectively.

§7. Computation of $[\mathbb{C}P(n),F]$

In this section we use the spectral sequence of §5 to obtain much information on $[\mathbb{C}P(n),F]$, the torsion subgroup of $[\mathbb{C}P(n),F/O]$. Specifically, up to group extensions, we compute the p-primary component of $[\mathbb{C}P(n),F]$ for $n \leq (p^2 + 2p)(p - 1) - 3$, p odd, and the 2-primary component of $[\mathbb{C}P(n),F]$ for $n \leq 11$.

Of course, we are primarily interested in $[\mathbb{C}P(n),F/O]$. The map $F \to F/O$ induces a map of the (stable) spectral sequence for $[\mathbb{C}P(n),F]$ to the spectral sequence for $[\mathbb{C}P(n),F/O]$. Since on homotopy $\pi_k^s = \pi_k(F) \to \pi_k(F/O)$ is a surjection onto the torsion subgroup $\pi_k^s/\mathrm{im}(J)$ of $\pi_k(F/O)$, we can compute differentials in the spectral sequence for $[\mathbb{C}P(n),F/O]$ by naturality.

We begin by stating results of Toda on the p-primary stable homotopy groups of spheres, p odd [27].

Lemma 7.1.

(1) For $k < 2(p^2 + 2p)(p - 1) - 4$, ${}_p(\pi_k^s/\mathrm{im}(J)) = {}_p\pi_k(F/O)$ is given by the following:

$${}_p\pi_{r(2p(p-1)-2)}(F/O) = \mathbb{Z}_p = \{\beta_1^r\}, \quad 1 \leq r < p+3$$

$${}_p\pi_{\deg(\beta_1^r)+2p-3}(F/O) = \mathbb{Z}_p = \{\alpha_1\beta_1^r\}, \quad 1 \leq r < p$$

$${}_p\pi_{\deg(\beta_1^r)+2(sp+s-1)(p-1)-2}(F/O) = \mathbb{Z}_p = \{\beta_1^r\beta_s\}, \quad 0 \leq r, \quad 2 \leq s < p$$

$${}_p\pi_{\deg(\beta_1^r\beta_s)+2p-3}(F/O) = \mathbb{Z}_p = \{\alpha_1\beta_1^r\beta_s\}, \quad r+s < p+2$$

$${}_p\pi_{2(p-1)(p^2+2p-1)-4}(F/O) = \mathbb{Z}_p = \{\beta_2\beta_{p-1}\}$$
$${}_p\pi_{\deg(\beta_2\beta_{p-1})+2p-3}(F/O) = \mathbb{Z}_p = \{\alpha_1\beta_2\beta_{p-1}\}$$

$${}_p\pi_{2(p^2+i)(p-1)-2}(F/O) = \mathbb{Z}_p = \{\varepsilon_i\}, \quad 1 \leq i < p$$
$${}_p\pi_{\deg(\varepsilon_i)+2p-3}(F/O) = \mathbb{Z}_p = \{\alpha_1\varepsilon_i\}, \quad 1 \leq i < p-2$$

$${}_p\pi_{2(p^2+1)(p-1)-3}(F/O) = \mathbb{Z}_p = \{\varepsilon'\}$$
$${}_p\pi_{\deg(\varepsilon')+2p(p-1)-2}(F/O) = \mathbb{Z}_p = \{\beta_1\varepsilon'\}$$
$${}_p\pi_{2(p^2+p)(p-1)-3}(F/O) = \mathbb{Z}_{p^2} = \{\varphi\}$$

All other groups are zero.

(2) The following primary and secondary relations hold.

$$\alpha_1\varepsilon_{p-1} = p\varphi$$
$$\alpha_1\varepsilon' = \beta_1^4 \quad \text{if} \quad p = 3$$
$$= 0 \quad \text{if} \quad p > 3$$
$$\varepsilon_{i+1} \in \langle \varepsilon_i, p, \alpha_1 \rangle, \quad 1 \leq i < p-1$$
$$\varepsilon' \in \langle \beta_1^p, \alpha_1, \alpha_1 \rangle$$
$$\varphi \in \langle \varepsilon_{p-2}, \alpha_1, \alpha_1 \rangle$$

Proposition 7.2. All non-zero differentials, $d_r^{t,q}$, in the p-primary torsion of the spectral sequence for $[\mathbb{C}P(n), F/O]$, in the range $n \leq (p^2 + 2p)(p - 1) - 3$, $1 \leq t \leq 3$, are given by (up to a unit)

$$d_{p-1}^{1,*}(\beta_1^r) = \alpha_1\beta_1^r, \qquad r < p$$
$$d_{p-1}^{1,*}(\beta_1^r\beta_s) = \alpha_1\beta_1^r\beta_s, \quad r+s \neq p$$
$$d_{p-1}^{1,*}(\varepsilon_i) = \alpha_1\varepsilon_i, \qquad 1 \leq i < p-2$$

$$d_{2(p-1)}^{1,*}(\beta^{p+1}) = \beta_1\varepsilon'$$

$$d_{2(p-1)}^{1,*}(\varepsilon_{p-2}) = p\varphi$$

$$d_{p-1}^{2,*}(\beta_1\varepsilon') = \beta_1^5 \quad \text{if} \quad p = 3$$
$$= 0 \quad \text{if} \quad p > 3$$

$$d_{(p-1)(p-1)}^{2,*}(\alpha_1\beta_1^{p-1}) = \beta_1^p$$

$$d_{(p-1)(p-1)}^{2,*}(\alpha_1\beta_1^{r-1}\beta_s) = \beta_1^r\beta_s, \qquad r+s = p$$

$$d_{p-1}^{3,*}(\beta_1^r) = \alpha_1\beta_1^r, \qquad 1 \leq r < p-1$$

$$d_{p-1}^{3,*}(\beta_1^r\beta_s) = \alpha_1\beta_1^r\beta_s, \qquad r+s+1 \neq p$$

$$d_{p-1}^{3,*}(\varepsilon_{p-1}) = p\varphi$$

$$d_{2(p-1)}^{3,*}(\beta_1^p) = \varepsilon'$$

$$d_{2(p-1)}^{3,*}(\beta_1^{p+1}) = \beta_1\varepsilon' \quad \text{if} \quad p = 3$$
$$= 0 \quad \text{if} \quad p > 3$$

Proof. This is proved by combining Lemma 7.1 and Propositions 5.6 and 5.7. For example, by Proposition 5.7, $d_{(p-1)(p-1)}^{2,*}(\beta_1^{p-1}\alpha_1) = \langle\beta_1^{p-1}\alpha_1,\alpha_1\cdots\alpha_1\rangle \supseteq \beta_1^{p-1}\langle\alpha_1\cdots\alpha_1\rangle = \beta_1^p$ [28, Lemma 4.10]. We leave the other proofs to the reader.

Remark. It is not quite correct to speak of a spectral sequence for $_p[\mathbb{C}P(n),F/O]$ since $[\mathbb{C}P(n),F/O]$ has infinite summands and the differentials might relate these to the torsion. However, the results of §2 and §3 can be used to show that the infinite summands tie in only with the 2-torsion (precisely how will be clarified below). Hence, for odd p, we can speak of a spectral sequence for ${}_p[\mathbb{C}P(n),F/O]$.

Table 7.4 below illustrates the spectral sequence in the range above for $p = 3$. An arrow indicates a non-zero differential.

For $p = 2$ the spectral sequence is studied by combining results of Toda [26], May [20], and Mahowald [19] on ${}_2\pi_k^s$ with Proposition 5.8. We summarize our results in the following Proposition and Table 7.5.

<u>Proposition 7.3</u>. In the spectral sequence for ${}_2[\mathbb{C}P(n),F/O]$ the following differentials occur.

$$d_1^{1,6}(\kappa) = \eta\kappa$$

$$d_1^{1,10}(\eta^2\bar{\kappa}) = \eta^3\bar{\kappa} = 4\nu\bar{\kappa}$$

$$d_1^{1,14}(h_4^2) = \eta h_4^2$$

$$d_2^{1,2}(\nu^2) = \nu^3$$

$$d_2^{1,9}(\bar{\kappa}) = \nu\bar{\kappa}$$

$$d_4^{1,6}(\sigma^2) = \sigma^3$$

$$d_1^{2,10}(\langle\sigma^2,2,\varepsilon\rangle) = \eta\langle\sigma^2,2,\varepsilon\rangle$$

$$d_2^{2,10}(\nu\bar{\kappa}) = \nu^2\bar{\kappa}$$

$$d_1^{3,2}(\varepsilon) = \eta\varepsilon = \nu^3$$

$$d_1^{3,6}(\eta^*) = \eta\eta^*$$

$$d_1^{3,8}(\bar{\kappa}) = \eta\bar{\kappa}$$

$$d_2^{3,5}(\kappa) = \nu\kappa$$

$$d_2^{3,8}(2\bar{\kappa}) = 4\nu\bar{\kappa}$$

$$d_4^{3,5}(\sigma^2) = \sigma^3$$

Further, we may choose generators ι_8, ι_{16}, and ι_{24} of the summands $\mathbb{Z} \subset \pi_{8j}(F/O)$, $j = 1,2,3$, such that

$$d_1^{3,4j-2}(\iota_{8j}) = \mu_{8j+1}$$

In Table 7.5 all other differentials in the range $k \leq 10$ are zero. Thus the first unsettled questions are whether $d_8^{1,3}(\varepsilon) = 0$ or $\langle\sigma^2, 2, \varepsilon\rangle$ and whether $d_9^{3,1}(\nu^2) = 0$ or $2\nu\bar{\kappa}$.

In addition to the last statement of Proposition 7.3, the infinite summands of $[\mathbb{C}P(n), F/O]$ are related to the 2-torsion in the spectral sequence in the following manner. There is the element $\mu_{8n+2} \in [\mathbb{C}P(4n+1), F/O]$, defined in Proposition 3.5(2), which has order 2 but which lifts to an element of infinite order in $[\mathbb{C}P(4n+2), F/O]$. This shows up in the formula in the second column of Table 7.5 for order $({}_2[\mathbb{C}P(k), F/O])$ in that the cofibration sequence $S^{2k+1} \to \mathbb{C}P(k) \to \mathbb{C}P(k+1) \to S^{2k+2}$ does not induce an exact sequence of torsion groups, ${}_2[\cdot, F/O]$.

Table 7.4

k	$_3\pi_{2k+1}(F/O)$	$\lvert_3[CP(k),F/O]\rvert$	$_3\pi_{2k}(F/O)$	$\lvert_3[S^1\wedge CP(k-1),F/O]\rvert$	$_3\pi_{2k-1}(F/O)$
k = 27	$\mathbb{Z}_3 = \{\alpha_1\beta_2^2\}$	3^2	0	3	0
= 26	0	3^2	$\mathbb{Z}_3 = \{\beta_2^2\}$	3	0
= 25	0	3	$\mathbb{Z}_3 = \{\beta_1^5\}$	3^2	$\mathbb{Z}_3 = \{\alpha_1\beta_1^2\beta_2\}$
= 24	$\mathbb{Z}_3 = \{\alpha_1\beta_1^2\beta_2\}$	3^2	0	3^2	$\mathbb{Z}_3 = \{\beta_1\varepsilon'\}$
= 23	$\mathbb{Z}_3 = \{\beta_1\varepsilon'\}$	3^3	$\mathbb{Z}_3 = \{\beta_1^2\beta_2\}$	3	$\mathbb{Z}_9 = \{\varphi\}$
= 22	$\mathbb{Z}_9 = \{\varphi\}$	3^3	0	0	0
= 21	0	3^3	$\mathbb{Z}_3 = \{\varepsilon_2\}$	0	0
= 20	0	3^2	$\mathbb{Z}_3 = \{\beta_1^4\}$	0	$\mathbb{Z}_3 = \{\alpha_1\beta_1\beta_2\}$
= 19	$\mathbb{Z}_3 = \{\alpha_1\beta_1\beta_2\}$	3	$\mathbb{Z}_3 = \{\varepsilon_1\}$	0	$\mathbb{Z}_3 = \{\varepsilon'\}$
= 18	$\mathbb{Z}_3 = \{\varepsilon'\}$	0	$\mathbb{Z}_3 = \{\beta_1\beta_2\}$	3	0
= 17	0	0	0	3	0
= 16	0	0	0	3	0
= 15	0	0	$\mathbb{Z}_3 = \{\beta_1^3\}$	3^2	$\mathbb{Z}_3 = \{\alpha_1\beta_2\}$
= 14	$\mathbb{Z}_3 = \{\alpha_1\beta_2\}$	3	0	3	0
= 13	0	3	$\mathbb{Z}_3 = \{\beta_2\}$	3	0
= 12	0	0	0	3	$\mathbb{Z}_3 = \{\alpha_1\beta_1^2\}$
= 11	$\mathbb{Z}_3 = \{\alpha_1\beta_1^2\}$	3	0	0	0
= 10	0	3	$\mathbb{Z}_3 = \{\beta_1^2\}$	0	0
= 9	0	0	0	0	0
= 8	0	0	0	0	0
= 7	0	0	0	0	$\mathbb{Z}_3 = \{\alpha_1\beta_1\}$
= 6	$\mathbb{Z}_3 = \{\alpha_1\beta_1\}$	3	0	0	0
= 5	0	3	$\mathbb{Z}_3 = \{\beta_1\}$	0	0
$k \leq 4$	0	0	0	0	0

Table 7.5

$k = 15$	$\mathbb{Z}_2^2 = \{n, \eta h_4^2\}$		$\mathbb{Z}_2 = \{h_4^2\}$		0
$= 14$	0		$\mathbb{Z}_2 = \{\kappa^2\}$		0
$= 13$	0		$\mathbb{Z}_2^2 = \{\nu^2\bar{\kappa}, \mu_{26}\}$		$\mathbb{Z}_2 = \{\mu_{25}\}$
$= 12$	$\mathbb{Z}_2 = \{\mu_{25}\}$		$\mathbb{Z}_2 = \{\eta\langle\sigma^2, 2, \varepsilon\rangle\}$		$\mathbb{Z}_8+\mathbb{Z}_2=\{\nu\bar{\kappa}, \langle\sigma^2, 2, \varepsilon\rangle\}$
$= 11$	$\mathbb{Z}_8+\mathbb{Z}_2=\{\nu\bar{\kappa}, \langle\sigma^2, 2, \varepsilon\rangle\}$	$2^5 \cdot 8^2$	$\mathbb{Z}_2^2 = \{\nu\bar{\sigma}, \eta^2\bar{\kappa}\}$	2^2	$\mathbb{Z}_2^2 = \{\sigma^3, \eta\bar{\kappa}\}$
$= 10$	$\mathbb{Z}_2^2 = \{\sigma^3, \eta\bar{\kappa}\}$	$2^4 \cdot 8^2$	$\mathbb{Z}_8 = \{\bar{\kappa}\}$	2^2	$\mathbb{Z}_2 = \{\bar{\sigma}\}$
$= 9$	$\mathbb{Z}_2 = \{\bar{\sigma}\}$	$2^4 \cdot 8$	$\mathbb{Z}_8+\mathbb{Z}_2 = \{\nu^*, \mu_{18}\}$	2	$\mathbb{Z}_2^3 = \{\eta\eta^*, \nu\kappa, \mu_{17}\}$
$= 8$	$\mathbb{Z}_2^3 = \{\eta\eta^*, \nu\kappa, \mu_{17}\}$	2^3	$\mathbb{Z}_2 = \{\eta^*\}$	2	$\mathbb{Z}_2 = \{\eta\kappa\}$
$= 7$	$\mathbb{Z}_2 = \{\eta\kappa\}$	2^3	$\mathbb{Z}_2^2 = \{\sigma^2, \kappa\}$	0	0
$= 6$	0	2	0	0	0
$= 5$	0	2^2	$\mathbb{Z}_2 = \{\mu_{10}\}$	0	$\mathbb{Z}_2^2 = \{\nu^3, \mu_9\}$
$= 4$	$\mathbb{Z}_2^2 = \{\nu^3, \mu_9\}$	2^2	$\mathbb{Z}_2 = \{\varepsilon\}$	0	0
$= 3$	0	2	$\mathbb{Z}_2 = \{\nu^2\}$	0	0
$= 2$	0	0	0	0	0
$k = 1$	0	2	$\mathbb{Z}_2 = \{\eta^2\}$	0	0
	$E_1^{0,k} = {}_2\pi_{2k+1}(F/O)$	order ${}_2[\mathbb{C}P(k), F/O]$	$E_1^{1,k-1} = {}_2\pi_{2k}(F/O)$	order ${}_2[S^1 \wedge \mathbb{C}P(k-1), F/O]$	$E_1^{2,k-2} = {}_2\pi_{2k-1}(F/O)$

§8. Examples

(1) Computation of $[\mathbb{C}P(n),F/O]$, $s: [\mathbb{C}P(n),F/O] \to P_{2n}$, and $\sigma: [\mathbb{C}P(n),F/O] \to \Gamma_{2n+1}$ for $n \leq 6$.

First, using results of §2, §5, and §7, the spectral sequence for $[\mathbb{C}P(n),F/O]$, $n \leq 6$, is given by the following table.

$k = 6$	$\mathbb{Z}_3 = \{\alpha_1\beta_1\}$	$\mathbb{Z}^3 + \mathbb{Z}_2 + \mathbb{Z}_3$	$\mathbb{Z} = \{\iota_{12}\}$
$= 5$	0	$\mathbb{Z}^2 + \mathbb{Z}_2^2 + \mathbb{Z}_3$	$\mathbb{Z}_2 + \mathbb{Z}_3 = \{\mu_{10},\beta_1\}$
$= 4$	$\mathbb{Z}_2^2 = \{\nu^3,\mu_9\}$	$\mathbb{Z}^2 + \mathbb{Z}_2^2$	$\mathbb{Z} + \mathbb{Z}_2 = \{\iota_8,\varepsilon\}$
$= 3$	0	$\mathbb{Z} + \mathbb{Z}_2$	$\mathbb{Z}_2 = \{\nu^2\}$
$= 2$	0	$\mathbb{Z}$	$\mathbb{Z} = \{\iota_4\}$
$k = 1$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2 = \{\eta^2\}$
	$\pi_{2k+1}(F/O)$	$[\mathbb{C}P(k),F/O]$	$\pi_{2k}(F/O)$

Table 8.1

The only non-zero differentials are $d_2^{1,2}(\nu^2) = \nu^3$ and $d_2^{1,4}(\beta_1) = \alpha_1\beta_1$.

The following will be crucial in our computations.

<u>Lemma 8.2</u>. $\text{image}([\mathbb{C}P(6),F/O] \to [\mathbb{C}P(6),BSO]) = \mathbb{Z}^3$ has generators $\xi_1 = 24\omega + 98\omega^2 + 111\omega^3$, $\xi_2 = 240\omega^2 + 380\omega^3$, and $\xi_3 = 504\omega^3$. Thus, the summand $\mathbb{Z}^2 \subset [\mathbb{C}P(5),F/O]$ is generated by $\xi_1 = 24\omega + 98\omega^2 + \omega^3$ and $\xi_2 = 240\omega^2$, $\mathbb{Z}^2 \subset [\mathbb{C}P(4),F/O]$

is generated by $\xi_1 = 24\omega + 98\omega^2$ and $\xi_2 = 240\omega^2$, and $\mathbb{Z} \subset [\mathbb{C}P(3),F/O]$, $[\mathbb{C}P(2),F/O]$ is generated by $\xi_1 = 24\omega$.

Proof. This follows from Theorem 2.0 and computation of the Adams operations ψ^k in the ring $KO(\mathbb{C}P(6)) = \mathbb{Z}[\omega]/(\omega^4)$, and some arithmetic.

We next recall that there is a cohomology formula for $s\colon [\mathbb{C}P(2n-1),F/O] \to \mathbb{Z}_2$ due to Sullivan. (See Remark 2 following Proposition 3.8.)

Lemma 8.3. There is a class $K = k_2 + k_6 + k_{10} + \cdots \in H^{4*+2}(F/O,\mathbb{Z}_2)$ such that if $f\colon \mathbb{C}P(2n-1) \to F/O$, then

$$\begin{aligned} s(f) &= \langle W(\mathbb{C}P(2n-1)) \cdot f^*(K),[\mathbb{C}P(2n-1)]\rangle \\ &= \langle (1+z)^{2n} f^*(K),[\mathbb{C}P(2n-1)]\rangle \in \mathbb{Z}_2, \end{aligned}$$

where $z \in H^2(\mathbb{C}P(n),\mathbb{Z}_2)$ is the generator.

It is further the case that for the maps $\eta^2\colon S^2 \to F/O$ and $\nu^2\colon S^6 \to F/O$, $(\eta^2)^*(k_2)$ and $(\nu^2)^*(k_6)$ are non-zero. On the other hand, $k_{10} = 0$ in $H^{10}(F/O,\mathbb{Z}_2)$.

Proof. See [24].

Proposition 8.4. $hS(\mathbb{C}P(3)) = \mathrm{kernel}([\mathbb{C}P(3),F/O] \xrightarrow{s} \mathbb{Z}_2) = \mathbb{Z}$. The homotopy $\mathbb{C}P(3)$ corresponding to the integer j, say P^6_j, has trivial normal bundle in the stable vector bundle $\tau_{\mathbb{C}P(3)} + (24j)\omega = (4 + 24j)\omega$ over $\mathbb{C}P(3)$, hence is characterized by its Pontryagin class $p_1(P^6_j) = (4 + 24j)z^2 \in H^4(P^6_j,\mathbb{Z}) = \mathbb{Z}$.

Proof. From Table 8.1, $[\mathbb{C}P(3),F/O] = \mathbb{Z} + \mathbb{Z}_2$. By Lemma 8.3, $s(\nu^2) \neq 0$ and the first statement follows. The second statement is clear from Lemma 8.2.

This result is, of course, well-known [14], [21].

Next, we have from Proposition 3.1 that $s\colon [\mathbb{C}P(4),F/O] \to \mathbb{Z}$ is given by the formula $s(f) = (\frac{1}{8})\langle L(\mathbb{C}P(4))(L(\xi)-1),[\mathbb{C}P(4)]\rangle \in \mathbb{Z}$, where ξ is the composition $\mathbb{C}P(4) \xrightarrow{f} F/O \xrightarrow{i} BSO$. By Lemma 8.2, $\xi = m\xi_1 + n\xi_2$ for some integers m, n, where $\xi_1 = 24\omega + 98\omega^2$ and $\xi_2 = 240\omega^2$. Since $p_1(\omega) = z^2$, $p_2(\omega) = 0$ and $p_1(\omega^2) = 0$, $p_2(\omega^2) = -6z^4$, we have $p_1(\xi) = (24m)z^2$ and $p_2(\xi) = ((12m)(24m-1)-6(98m+240n))z^4$. Thus

$$\begin{aligned} s(f) &= (\tfrac{1}{8})\langle(1 + \frac{p_1(\mathbb{C}P(4))}{3})(\frac{p_1(\xi)}{3} + \frac{7p_2(\xi)-p_1^2(\xi)}{45}),[\mathbb{C}P(4)]\rangle \\ &= (\tfrac{1}{8})\langle(1 + \tfrac{5z}{3})(8mz + (\frac{10\cdot(12)^2m^2-7\cdot 600m-42\cdot 240n}{45})z^2),[\mathbb{C}P(4)]\rangle \\ &= (\tfrac{1}{8})(32m^2 - 80m - 7\cdot 32n) = 4m^2 - 10m - 28n \in \mathbb{Z}. \end{aligned}$$

This computation, together with Proposition 3.2, proves the following, which is also well-known [22].

Proposition 8.5. Let $f \in [\mathbb{C}P(3),F/O]$ correspond to a fibre homotopy trivialization of the bundle $m\xi_1 = (24m)\omega$ over $\mathbb{C}P(3)$. Then $\sigma(f) = s(m\xi_1) = 4m^2 - 10m \in \mathbb{Z} \bmod 28\mathbb{Z}$. In particular, $\partial(P^8_m)_o = (4m^2 - 10m)\cdot\Sigma^7$, where Σ^7 is the Milnor generator of $\Gamma_7 = \mathbb{Z}_{28}$ and $(P^8_m)_o$ is the Hopf bundle over P^6_m. Thus the only homotopy 7-spheres which admit free

S^1 actions are $\{k \cdot \Sigma^7 | \; k \equiv 0,-4,\pm 6,\pm 8,10,14 \bmod 28\}$.

We can also easily describe $hS(\mathbb{C}P(4))$, using the computation of $s\colon [\mathbb{C}P(4),F/O] \to \mathbb{Z}$ above.

<u>Proposition 8.6</u>. $hS(\mathbb{C}P(4)) = \text{kernel}([\mathbb{C}P(4),F/O] \xrightarrow{s} \mathbb{Z})$ $= \{m\xi_1 + n\xi_2 + \tau | \; 4m^2 - 10m - 28n = 0 \text{ and } \tau \in \mathbb{Z}_2^2\} \subset [\mathbb{C}P(4),F/O]$ $= \mathbb{Z}^2 + \mathbb{Z}_2^2$.

Next, we want to compute $s\colon [\mathbb{C}P(5),F/O] \to \mathbb{Z}_2$ and $\sigma\colon [\mathbb{C}P(4),F/O] \to b\,\text{spin}_{10} \subset \Gamma_9$.

<u>Lemma 8.7</u>. The Brown-Kervaire homomorphism $\Psi\colon \Omega_{10}^{\text{spin}} \to \mathbb{Z}_2$ is given by $\Psi = \sum_J \alpha_J \pi^J + \sum_I \beta_I w^I = \pi^2 + \sum_I \beta_I w^I$.

<u>Proof</u>. We have already observed in §3 that $\alpha_{(0)} = 0$. In dimension 10 it only remains to compute $\alpha_{(2)}$ and $\alpha_{(3)}$. But π^3 has filtration 10, hence as a characteristic number of 10-manifolds it coincides with some Stiefel-Whitney number. Thus we may assume that $\alpha_{(3)} = 0$. Finally, if $\mathbb{Q}P(2)$ is the quaternionic projective plane and T^2 is the torous with exotic spin structure, then $\Psi(\mathbb{Q}P(2) \times T^2) = 1$. Since $\mathbb{Q}P(2) \times T^2$ bounds as an ordinary manifold, its Stiefel-Whitney numbers vanish. Thus $\alpha_{(2)} = 1$. (I am grateful to Professor F.P. Peterson for this argument.)

<u>Proposition 8.8</u>. If $f \in [\mathbb{C}P(5),F/O]$ corresponds to a fibre homotopy trivialization of $\xi = m\xi_1 + n\xi_2$ over $\mathbb{C}P(5)$, where ξ_1 and ξ_2 are as in Lemma 8.2, then $s(f) \equiv m \bmod 2$.

Remark. By Lemma 8.3, that is, Sullivan's cohomology formula, $s(f) = \langle(1+z)^6(f^*(k_2) + f^*(k_6)),[\mathbb{C}P(5)]\rangle \in \mathbb{Z}_2$. It is clear from Table 8.1 and Lemma 8.3 that $f^*(k_2) \equiv mz \bmod 2$. Thus $s(f) = \langle(mz)z^4 + (f^*(k_6))z^2,[\mathbb{C}P(5)]\rangle$ and Proposition 8.8 is equivalent to the assertion that $f^*(k_6) = 0$. It may be possible to prove that $f^*(k_6) = 0$ by simply studying the homotopy type of F/O in low dimensions. However, the author did not succeed using that approach, hence we rely instead on the KO-theory formula for $s: [\mathbb{C}P(5),F/O] \to \mathbb{Z}_2$ given in Proposition 3.8.

Proof of 8.8. By Proposition 3.8 and Lemma 8.7 we have $s(f) = c^*\Phi(\pi^2(\nu_{\mathbb{C}P(5)} - \xi)\cdot f^*(\gamma) - \pi^2(\nu_{\mathbb{C}P(5)}))$ where $\gamma \in KO^0(F/O)$ is the element with $ph(\gamma) = \hat{A}$. In particular, since $\hat{A}_1 = -p_1/24$ it follows that $f^*(\gamma) = 1 - m\omega \bmod \omega^2$ in $KO(\mathbb{C}P(5))$.

Now, if we set $\pi_t(\eta) = \sum_{j>0} \pi^j(\eta)t^j$ and $\eta = r(\sum_i L_i)$ is the realification of a sum of complex line bundles, then $\pi_t(\eta) = \prod_i(1 + r(L_i - 1)t)$. Moreover, for any bundles η_1 and η_2, $\pi_t(\eta_1 + \eta_2) = \pi_t(\eta_1)\cdot\pi_t(\eta_2)$ and $\pi_t(n) = 1$ if n is the trivial bundle of dimension n.

We can apply these facts to compute $\pi^2(\nu_{\mathbb{C}P(5)} - \xi)$ $= \pi^2(-((6+24m)\omega + (98m+240n)\omega^2 + m\omega^3))$ and $\pi^2(\nu_{\mathbb{C}P(5)})$ $= \pi^2(-6\omega)$. For $\omega = r(H-1)$, $\omega^2 = r(H^2 - 4H + 3)$ and $\omega^3 = r(H^3 - 6H^2 + 15H - 10)$, hence by an elementary, but long, computation $\pi_t(\nu_{\mathbb{C}P(5)} - \xi) = (1+\omega t)^a(1+(\omega^2-4\omega)t)^b(1+(\omega^3+6\omega^2+9\omega)t)^{-m}$

where $a = 353m + 960n - 6$ and $b = -(92m + 240n)$. By another computation, evaluating the coefficient of t^2 in this product,

$$\pi^2(\nu_{\mathbb{C}P(5)} - \xi) = \left(\frac{a(a-1)}{2} + \frac{81m(m+1)}{2} + \frac{16b(b+1)}{2} - 4ab - 9am + 36mb\right)\omega^2.$$

By an easier computation, $\pi^2(\nu_{\mathbb{C}P(5)}) = \pi^2(-6\omega) = 21\omega^2$.

Let the integer coefficient of ω^2, in the expression for $\pi^2(\nu_{\mathbb{C}P(5)} - \xi)$ above, be denoted by c. From the definitions of a and b in terms of m and n, one can easily show that c is always odd. Then $\pi^2(\nu_{\mathbb{C}P(5)} - \xi)\cdot f^*(\gamma) - \pi^2(\nu_{\mathbb{C}P(5)}) = c\omega^2(1 - m\omega) - 21\omega^2 = (c-21)\omega^2 + m\omega^3$. But $c-21$ is even and from known results about evaluating KO characteristic numbers [3], it follows that $c^*\Phi((c-21)\omega^2 + m\omega^3) \equiv m \bmod 2$. (I am grateful to Professor D.W. Anderson for this argument.)

As our first corollary of Proposition 8.8, we describe $\sigma\colon [\mathbb{C}P(4),F/O] \to b\,\mathrm{spin}_{10} = \mathbb{Z}_2 + \mathbb{Z}_2 = \{\Sigma^9, \nu^3\}$, where Σ^9 is the Kervaire 9-sphere. Then $\{\Sigma^9\} = \mathbb{Z}_2 = bP_{10}$. From Propositions 3.2 and 3.9 and the fact that $s\colon [\mathbb{C}P(5),F/O] \to \mathbb{Z}_2$ is a group homomorphism, it follows that $\sigma\colon [\mathbb{C}P(4),F/O] = \mathbb{Z}^2 + \mathbb{Z}_2^2 \to b\,\mathrm{spin}_{10}$ is a homomorphism. From Table 8.1, the torsion subgroup of $[\mathbb{C}P(4),F/O]$ can be expressed as $\mathbb{Z}_2^2 = \{\nu_4^2, \varepsilon_4\}$, where if $\alpha \in \pi_{2k}(F/O)$ then $\alpha_n \in [\mathbb{C}P(n),F/O]$, $n \geq k$, denotes an extension of the composition $\mathbb{C}P(k) \xrightarrow{j} S^{2k} \xrightarrow{\alpha} F/O$ to $\mathbb{C}P(n) \to F/O$.

<u>Proposition 8.9</u>. $\sigma\colon [\mathbb{C}P(4),F/O] \to b\,\mathrm{spin}_{10}$ is given by $\sigma(m\xi_1 + n\xi_2) = m\Sigma^9$, $\sigma(\nu_4^2) = \nu^3$ and $\sigma(\varepsilon_4) = 0$. Further,

$\sigma(hS(\mathbb{C}P(4))) = \{0, \nu^3\}$, where, of course, $\sigma(hS(\mathbb{C}P(4)))$ is the set of homotopy 9-spheres which admit free S^1 actions.

Proof. The first statement follows from Propositions 3.2, 3.9, and 8.8 and the differential $d_2^{1,2}(\nu^2) = \nu^3$ in Table 8.1. For the second statement, note that $m\xi_1 + n\xi_2 + \tau \in hS(\mathbb{C}P(4))$, $\tau \in \mathbb{Z}_2^2$, implies that m is even. For $s(m\xi_1 + n\xi_2)$ $= 4m^2 - 10m - 28n \not\equiv 0 \mod 4$ if m is odd.

Remark. Consider the composition $\mathbb{R}P(9) \xrightarrow{\pi} \mathbb{C}P(4) \xrightarrow{\xi_1} F/O$, where π is the natural map with fibre S^1. By results of Wall [29], one can obtain by surgery a homotopy equivalence $P^9 \to \mathbb{R}P(9)$ from the element $\xi_1 \circ \pi = f \in [\mathbb{R}P(9), F/O]$. The homotopy real projective space P^9 corresponds to a free $\mathbb{Z}_2$ action on the Kervaire sphere, Σ^9. To see this, note that $\xi_1: \mathbb{C}P(4) \to F/O$ extends to $\xi_1: \mathbb{C}P(5) \to F/O$, hence $\xi_1 \circ \pi: \mathbb{R}P(9) \to F/O$ extends to $\widehat{\xi_1 \circ \pi} = \hat{f}: \mathbb{R}P(10) \to F/O$ (for example, by $\mathbb{R}P(11) \xrightarrow{\pi} \mathbb{C}P(5) \xrightarrow{\xi_1} F/O$). As in Proposition 3.2, the homotopy sphere which covers P^9 is given by $s(\hat{f}) = \langle W(\mathbb{R}P(10)) \cdot \hat{f}^*(K), [\mathbb{R}P(10)]\rangle = \langle (1+x)^{11} \cdot x^2, [\mathbb{R}P(10)]\rangle$ $= 1 \in \mathbb{Z}_2$, where $x \in H^1(\mathbb{R}P(10), \mathbb{Z}_2)$ is the generator. On the other hand, the Browder-Livesay obstruction to constructing a $\mathbb{Z}_2$-invariant $S^8 \subset \Sigma^9$ is given by the formula $s(f|_{\mathbb{R}P(8)}) = \langle W(\mathbb{R}P(8)) \cdot f^*(K), [\mathbb{R}P(8)]\rangle = \langle (1+x)^9 \cdot x^2, [\mathbb{R}P(8)]\rangle$ $= 0$.

This provides a counterexample, suggested to me by Sullivan, to the recent conjecture that the Browder-Livesay invariant of a free $\mathbb{Z}_2$ action on an $8k+1$ sphere in bP_{8k+2} was zero if and only if the sphere was diffeomorphic to S^{8k+1}.

As a second corollary of Proposition 8.8, we have

Proposition 8.10. $hS(\mathbb{C}P(5)) = \text{kernel}([\mathbb{C}P(5),F/O] \xrightarrow{s} \mathbb{Z}_2)$ $= 2\mathbb{Z} + \mathbb{Z} + \mathbb{Z}_2^2 + \mathbb{Z}_3 \subset \mathbb{Z} + \mathbb{Z} + \mathbb{Z}_2^2 - \mathbb{Z}_3 = [\mathbb{C}P(5),F/O]$.

A computation similar to, but longer than, that preceding Proposition 8.5 gives the following

Proposition 8.11. Let $f \in [\mathbb{C}P(6),F/O]$ correspond to a fibre homotopy trivialization of $\xi = m\xi_1 + n\xi_2 + q\xi_3$. Then $s(f) = (\frac{1}{8})\langle L(\mathbb{C}P(6))(L(\xi)-1),[\mathbb{C}P(6)]\rangle = \frac{m(32m^2+301)}{3} - 84m^2 - 224mn + 384n + 496q \in \mathbb{Z}$. Thus $hS(\mathbb{C}P(6)) = \{m\xi_1 + n\xi_2 + q\xi_3 + \tau \mid s(m\xi_1 + n\xi_2 + q\xi_3) = 0 \text{ and } \tau \in \mathbb{Z}_2 + \mathbb{Z}_3\} \subset [\mathbb{C}P(6),F/O]$ $= \mathbb{Z}^3 + \mathbb{Z}_2 + \mathbb{Z}_3$.

Proposition 8.12. $\sigma\colon [\mathbb{C}P(5),F/O] \to \Gamma_{11} = bP_{12} = \mathbb{Z}_{992}$ is given by $\sigma(m\xi_1 + n\xi_2) = (\frac{m(32\cdot m^2+301)}{3} - 84m^2 - 224mn + 384n)\cdot\Sigma^{11}$, $\sigma(\mu_{10}) = 496\cdot\Sigma^{11}$, and $\sigma(\varepsilon_5) = \sigma((\beta_1)_5) = 0$, where Σ^{11} is the Milnor generator of bP_{12}. Thus the set of homotopy 11-spheres which admit free S^1 actions is given by $\sigma(hS(\mathbb{C}P(5)))$ $= \{(\frac{m(32m^2+301)}{3} - 84m^2 - 224mn + 384n + 496q)\cdot\Sigma^{11} \mid m,n,q \in \mathbb{Z}, m \equiv 0 \bmod 2\} \subset \mathbb{Z} \bmod 992\mathbb{Z}$. In particular, such a sphere is an even multiple of Σ^{11}.

Proof. This follows easily from Propositions 3.3, 8.10, and 8.11 and the fact that $\xi_3 \in [\mathbb{C}P(6),F/O]$ may be chosen such that $\xi_3|_{\mathbb{C}P(5)} = \mu_{10} \in [\mathbb{C}P(5),F]$.

Finally, $bP_{14} = 0$ hence $\Gamma_{13} = \pi_{13}^s/\text{im } J = \mathbb{Z}_3 = \{\alpha_1\beta_1\}$. Thus by Proposition 3.2, and because of the differential $d_2^{1,4}(\beta_1) = \alpha_1\beta_1$ in Table 8.1, we have

<u>Proposition 8.13</u>. $\sigma: [\mathbb{C}P(6),F/O] \to \Gamma_{13}$ is given by $\sigma(\xi) = 0$ if $\xi \in \mathbb{Z}^3$, $\sigma(\varepsilon_6) = 0$, and $\sigma((\beta_1)_6) = \alpha_1\beta_1$. In particular, all homotopy 13-spheres admit free S^1 actions.

(2) Spheres in $\pi_{2n+1}^s/\text{im}J \subseteq \Gamma_{2n+1}$, which admit free S^1 actions.

Recall from §3 that there are direct sum decompositions $\Gamma_{2n+1} = bP_{2n+2} \oplus (\pi_{2n+1}^s/\text{im } J)$, at least if $n \neq 4k + 2$, and $[\mathbb{C}P(n),F/O] = \oplus\,\mathbb{Z} + [\mathbb{C}P(n),F]$, with $\sigma(\oplus\,\mathbb{Z}) \subseteq bP_{2n+2}$ and $\sigma([\mathbb{C}P(n),F]) \subseteq \pi_{2n+1}^s/\text{im } J$ ($\sigma([\mathbb{C}P(n),F]) \subseteq \mathbb{Z}_2 \oplus (\pi_{2n+1}^s/\text{im } J)$ if $n = 4k + 1$).

To compute which spheres in bP_{2n+2} admit free S^1 actions involves evaluating certain surgery obstructions and lengthy computations with Pontryagin classes, which is not practical outside the range $n \leq 6$ considered above. On the other hand, if $n = 4k$ or $4k + 1$ then the spheres in $\pi_{2n+1}^s/\text{im } J$ which admit free S^1 actions are given by $\varphi\sigma([\mathbb{C}P(n),F]) = D^{1,q} = \bigcup_s \text{image}(d_s^{1,n-s}) \subseteq \pi_{2n+1}(F/O) = \pi_{2n+1}^s/\text{im } J$, where the $d_s^{1,n-s}$ are differentials in the spectral sequence for $[\mathbb{C}P(n),F/O]$ constructed in §5 and $\varphi: \Gamma_{2n+1} \to \pi_{2n+1}^s/\text{im } J$ is the Kervaire-Milnor map. If $n = 4k + 2$ this still holds on p-torsion components, p odd, since ${}_p\Gamma_{8k+5} \simeq {}_p\pi^s_{8k+5}$. If $n = 4k + 3$, one must restrict σ to $hS(\mathbb{C}P(n)) \cap [\mathbb{C}P(n),F] = \text{kernel}([\mathbb{C}P(n),F] \xrightarrow{s} \mathbb{Z}_2)$. This restriction also involves only the 2-torsion components.

The differentials $d_s^{1,n-s}$ are computed in §7 in a big range. Specifically, for odd p the following is a direct consequence of Proposition 7.2.

<u>Proposition 8.14</u>. For $n < (p^2 + 2p)(p - 1) - 2$, the subgroup of exotic spheres in ${}_p(\pi^s_{2n+1}/\text{im } J) \subseteq \Gamma_{2n+1}$ which admit free S^1 actions is given by

$$\{\alpha_1\beta_1^r\} = \mathbb{Z}_p \subset \Gamma_{\deg(\alpha_1\beta_1^r)}, \qquad r < p$$

$$\{\alpha_1\beta_1^r\beta_s\} = \mathbb{Z}_p \subset \Gamma_{\deg(\alpha_1\beta_1^r\beta_s)}, \qquad r+s \neq p$$

$$\{\alpha_1\varepsilon_i\} = \mathbb{Z}_p \subset \Gamma_{\deg(\alpha_1\varepsilon_i)}, \qquad 1 \leq i < p-2$$

$$\{\beta_1\varepsilon'\} = \mathbb{Z}_p \subset \Gamma_{\deg(\beta_1\varepsilon')}$$

$$\{p\varphi\} = \mathbb{Z}_p \subset \Gamma_{\deg(\varphi)}$$

and is zero in all other dimensions. Thus the lowest dimensional sphere in ${}_p(\pi^s_{2n+1}/\text{im } J) \subseteq \Gamma_{2n+1}$ which does not admit a free S^1 action is $\varepsilon' \in \Gamma_{2(p^2+1)(p-1)-3}$.

Similarly, for $p = 2$ we have

<u>Proposition 8.15</u>. For $n \leq 14$, the following spheres in ${}_2(\pi^s_{2n+1}/\text{im } J) \subseteq \Gamma_{2n+1}$ admit free S^1 actions

$$\{\nu^3\} = \mathbb{Z}_2 \subset \Gamma_9$$

$\{\eta\kappa\} = \mathbb{Z}_2 \subset \Gamma_{15}$

$\{\sigma^3\} = \mathbb{Z}_2 \subset \Gamma_{21}$

$\{\nu\bar{\kappa}\} = \mathbb{Z}_8 \subset \Gamma_{23}$

Proof. This follows from Table 7.5 and Proposition 7.3 and the following remarks. The exact sequence $0 \to \mathbb{Z}_2 = bP_{22} \to \Gamma_{21} \to \pi_{21}^s = \mathbb{Z}_2^2 \to 0$ splits, hence it makes sense to speak of $\pi_{21}^s \subset \Gamma_{21}$ in the third statement of the Proposition. For there are elements $\sigma^2 \in \Gamma_{14} = \pi_{14}(PL/O)$ and $\bar{\kappa} \in \Gamma_{20} = \pi_{20}(PL/O)$, hence $\pi_{21}^s = \mathbb{Z}_2^2 = \{\sigma(\sigma^2), \eta(\bar{\kappa})\} \subset \Gamma_{21} = \pi_{21}(PL/O)$. Also, for a suitable choice of $\bar{\kappa}_{11} \in [\mathbb{C}P(11), F/O]$, we have $s(\bar{\kappa}_{11}) = 0$, hence $\bar{\kappa}_{11} \in hS(\mathbb{C}P(11))$ and the fourth statement of the Proposition is valid. For we may factor $\bar{\kappa}_{11}$ as $\mathbb{C}P(11) \to \mathbb{C}P(11)/\mathbb{C}P(9) = S^{20} \vee S^{22} \xrightarrow{\bar{\kappa} \vee pt} F/O$. Then $s(\bar{\kappa}_{11}) = \langle W(\mathbb{C}P(11)) \cdot (\bar{\kappa}_{11})^*(K), [\mathbb{C}P(11)]\rangle = 0$ (or, since $\bar{\kappa}_{11} \in \text{image}([\mathbb{C}P(11), PL/O] \to [\mathbb{C}P(11), F/O])$, clearly $s(\bar{\kappa}_{11}) = 0$).

The first unsettled question is whether $\langle \sigma^2, 2, \varepsilon\rangle \in \Gamma_{23}$ admits a free S^1 action. The answer would be yes if $d_8^{1,3}(\varepsilon) = \langle\sigma^2, 2, \varepsilon\rangle$ and $s(\varepsilon_{11}) = 0$.

(3) Inertia groups.

Let $I(\mathbb{C}P(n)) \subset \Gamma_{2n}$ be the group of exotic spheres, Σ^{2n}, such that the smoothing $\mathbb{C}P(n) \# \Sigma^{2n}$ is concordant to $\mathbb{C}P(n)$. Note that this coincides with the inertia group of

$\mathbb{C}P(n)$, that is, the group of exotic spheres, Σ^{2n}, such that $\mathbb{C}P(n) \# \Sigma^{2n}$ is diffeomorphic to $\mathbb{C}P(n)$. For given a diffeomorphism $f: \mathbb{C}P(n) \# \Sigma^{2n} \xrightarrow{\simeq} \mathbb{C}P(n)$, we may assume that f is homotopic to Id. For either f or cf, where $c: \mathbb{C}P(n) \xrightarrow{\simeq} \mathbb{C}P(n)$ is conjugation, is homotopic to Id. But as observed in the introduction, $[\mathbb{C}P(n), \Omega(F/PL)] = 0$, hence every PL self-isomorphism of $\mathbb{C}P(n)$ homotopic to Id is PL pseudo-isotopic to Id [24]. Then choosing a PL pseudo-isotopy of $f: \mathbb{C}P(n) \# \Sigma^{2n} \xrightarrow{\simeq} \mathbb{C}P(n)$ to Id induces a concordance between $\mathbb{C}P(n) \# \Sigma^{2n}$ and $\mathbb{C}P(n)$.

Now $\Sigma^{2n} \in I(\mathbb{C}P(n))$ if and only if the composition $\mathbb{C}P(n) \xrightarrow{j} S^{2n} \xrightarrow{\Sigma^{2n}} PL/O$ is null-homotopic. This is equivalent to a factoring of Σ^{2n} through $S^1 \wedge \mathbb{C}P(n-1)$, that is $S^{2n} \to S^1 \wedge \mathbb{C}P(n-1) \to PL/O$, which exists if and only if $\Sigma^{2n} \in D^{2,n-1}(PL/O) = \bigcup_s \text{image}(d_s^{2,n-1-s}) \subseteq \pi_{2n}(PL/O)$, where $d_s^{2,n-1-s}$ are differentials in the spectral sequence of §5 and §6 for $[\mathbb{C}P(n), PL/O]$. We have computed the differentials in the spectral sequence for $[\mathbb{C}P(n), F/O]$ for a range in §7. These results can be used for PL/O because the map $PL/O \to F/O$ induces a map from the spectral sequence for $[\mathbb{C}P(n), PL/O]$ to the spectral sequence for $[\mathbb{C}P(n), F/O]$, and we have

Lemma 8.16. With respect to the inclusion $\pi_{2n}(PL/O) \subset \pi_{2n}(F/O)$, induced by $PL/O \to F/O$, we have $D^{2,n-1}(PL/O) = D^{2,n-1}(F/O)$.

Proof. The inclusion $D^{2,n-1}(PL/O) \subseteq D^{2,n-1}(F/O)$ is obvious by naturality. The inclusion $D^{2,n-1}(F/O) \subseteq D^{2,n-1}(PL/O)$ holds because the fibration $PL/O \to F/O \to F/PL$ and the fact that $[S^1 \wedge \mathbb{C}P(n-1), F/PL] = 0$ implies that $[S^1 \wedge \mathbb{C}P(n-1), PL/O] \to [S^1 \wedge \mathbb{C}P(n-1), F/O]$ is surjective. Thus any composition $S^{2n} \to S^1 \wedge \mathbb{C}P(n-1) \to F/O$ factors through PL/O, that is, $S^{2n} \to S^1 \wedge \mathbb{C}P(n-1) \to PL/O \to F/O$.

From Propositions 7.2 and 7.3 we then have

Proposition 8.17. For $n \leq (p^2 + 2p)(p - 1) - 3$, p odd, ${}_pI(\mathbb{C}P(n))$ is given by

$$\{\beta_1^p\} = \mathbb{Z}_p \subset \Gamma_{2p(p(p-1)-1)}$$

$$\{\beta_1^r \beta_s\} = \mathbb{Z}_p \subset \Gamma_{\deg(\beta_1^r\beta_s)}, \quad r+s = p$$

and in addition

$$\{\beta_1^5\} = \mathbb{Z}_3 \subset \Gamma_{50} \quad \text{if} \quad p = 3$$

and is zero in all other dimensions.

If $p = 2$, $n \leq 13$, then ${}_2I(\mathbb{C}P(n))$ is given by

$$\{\eta\langle\sigma^2, 2, \varepsilon\rangle\} = \mathbb{Z}_2 \subset \Gamma_{24}$$

$$\{\nu^2\bar{\kappa}\} = \mathbb{Z}_2 \subset \Gamma_{26}$$

and is zero in all other dimensions.

(4) Diffeomorphisms of $\mathbb{C}P(n)$.

Given a diffeomorphism $f: \mathbb{C}P(n) \xrightarrow{\sim} \mathbb{C}P(n)$ and a homotopy

f_t of $f = f_o$ to $Id = f_1$, we can associate an element of $[S^1 \wedge \mathbb{C}P(n), F/O]$. Namely, the homotopy f_t induces a homotopy equivalence $h: S^1 \times \mathbb{C}P(n) \to M_f$, where M_f is the mapping torous of f. The associated map $\theta(h): S^1 \times \mathbb{C}P(n) \to F/O$ factors through $S^1 \wedge \mathbb{C}P(n)$ because $h|_{1\times\mathbb{C}P(n)}$ is the identity, hence $\theta(h|_{1\times\mathbb{C}P(n)}): 1 \times \mathbb{C}P(n) \to F/O$ is null-homotopic. Further, by Theorem 1.0 $\theta(h) \in [S^1 \wedge \mathbb{C}P(n), F/O]$ is zero if and only if h is homotopic, rel $1 \times \mathbb{C}P(n)$, to a diffeomorphism $(S^1 \times \mathbb{C}P(n)) \# \Sigma^{2n+1} \xrightarrow{\simeq} M_f$, where $\Sigma^{2n+1} \in bP_{2n+2} \subset \Gamma_{2n+1}$. This is equivalent to deforming the homotopy f_t to a pseudo-isotopy of $f = f_o$ to $\Psi(\gamma) = f_1$, where $\gamma: S^{2n} \xrightarrow{\simeq} S^{2n}$ corresponds to Σ^{2n+1} and $\Psi(\gamma)$ is the diffeomorphism of $\mathbb{C}P(n)$ defined in §6.

Remarks.

1. $S^1 \times \mathbb{C}P(n)$ is not simply connected but for manifolds with $\pi_1 = \mathbb{Z}$ the methods of Sullivan outlined in the introduction go through [8].
2. Given a diffeomorphism $f: \mathbb{C}P(n) \xrightarrow{\simeq} \mathbb{C}P(n)$ and a PL pseudo-isotopy f_t of $f = f_o$ to $Id = f_1$, we can associate an element of $[S^1 \wedge \mathbb{C}P(n), PL/O]$. For the homotopy equivalence $h: S^1 \times \mathbb{C}P(n) \to M_f$ induced by f_t is then a PL isomorphism. This element of $[S^1 \wedge \mathbb{C}P(n), PL/O]$ is zero if and only if f_t deforms through PL pseudo-isotopies to a smooth isotopy of f to Id.

Also, given a PL isomorphism $f\colon \mathbb{C}P(n) \xrightarrow{\simeq} \mathbb{C}P(n)$ and a homotopy f_t of f to Id, we can associate an element of $[S^1 \wedge \mathbb{C}P(n), F/PL]$. Since this last group is zero, every homotopy of f to Id deforms to a PL pseudo-isotopy of f to Id.

Let $\Sigma = \Sigma^{2n+1} \in \Gamma_{2n+1} = \pi_{2n+1}(PL/O)$ correspond to the diffeomorphism σ of S^{2n}. Then the composition $S^1 \wedge \mathbb{C}P(n) \xrightarrow{j} S^{2n+1} \xrightarrow{\Sigma} PL/O$ corresponds to the diffeomorphism $\Psi(\sigma)$ of $\mathbb{C}P(n)$, together with the PL isotopy Ψ_t of $\Psi(\sigma)$ to Id given by the "Alexander trick". That is, we have $D_+^{2n} = \mathbb{C}P(n) - \mathbb{C}P(n)_o$. Embed $\mathrm{cone}(D_+^{2n}) \subset \mathbb{C}P(n) \times I$ as the join of $D_+^{2n} \times 0 \subset \mathbb{C}P(n) \times 0$ with a point $p \in D_+^{2n} \times 1 \subset \mathbb{C}P(n) \times 1$. Then define $\Psi_t\colon \mathbb{C}P(n) \times I \to \mathbb{C}P(n) \times I$ by

$$\begin{aligned} \Psi_t &= \mathrm{Id} && \text{on } \mathbb{C}P(n) \times I - \mathrm{cone}(D_+^{2n}) \\ &= \mathrm{cone}(\sigma) && \text{on } \mathrm{cone}(D_+^{2n}). \end{aligned}$$

The computations of §7 give information on $[S^1 \wedge \mathbb{C}P(n), F/O]$ for a range. For example, we have the following.

Proposition 8.18. For $n \leq 6$, $[S^1 \wedge \mathbb{C}P(n), F/O] = 0$. Thus every diffeomorphism of $\mathbb{C}P(n)$, $n \leq 6$, homotopic to Id is isotopic to $\Psi(\gamma)$ for some $\gamma \in bP_{2n+2}$. $[S^1 \wedge \mathbb{C}P(7), F/O] = \mathbb{Z}_2$ with generator corresponding to the diffeomorphism $\Psi(\eta\kappa)$ of $\mathbb{C}P(7)$.

For p odd, ${}_p[S^1 \wedge \mathbb{C}P(n), F/O] = 0$ for $n \leq p(p-1)^2-2$.

${}_p[S^1 \wedge \mathbb{C}P(p(p-1)^2-1), F/O] = \mathbb{Z}_p$ with generator $\Psi(\alpha_1\beta_1^{p-1})$.

From the discussion above we see that the group $D^{3,n-1} = \bigcup_s \text{image}(d_s^{3,n-1-s}) \subseteq \pi_{2n+1}(PL/O)$ can be characterized as the group of exotic spheres $\Sigma^{2n+1} \in \Gamma_{2n+1}$ such that the homotopy, Ψ_t, of the diffeomorphism $\Psi(\Sigma^{2n+1})$ of $\mathbb{C}P(n)$ to the identity deforms to an isotopy of $\Psi(\Sigma^{2n+1})$ to $\Psi(\gamma)$, $\gamma \in bP_{2n+2}$. From Propositions 7.2 and 7.3 many non-zero elements in $D^{3,n-1}$ can be read off. For instance, for $p = 2$, the diffeomorphisms $\Psi(\nu^3)$ and $\Psi(\mu_9)$ of $\mathbb{C}P(4)$ are isotopic to diffeomorphisms $\Psi(\gamma)$, $\gamma \in bP_{10}$. For p odd, the diffeomorphisms $\Psi(\alpha_1\beta_1^r)$, $1 \leq r < p-1$, of $\mathbb{C}P((rp+1)(p-1)-(r+1))$ are isotopic to $\Psi(\gamma)$, $\gamma \in bP_{2(rp+1)(p-1)-2r}$.

(5) S^1 actions not equivariantly diffeomorphic to their conjugate.

Given a free S^1 action on a homotopy sphere, (Σ^{2n+1}, T), we defined in §4 the conjugate action $((-1)^{n+1}\Sigma^{2n+1}, \overline{T})$, and promised some examples for which (Σ^{2n+1}, T) was not equivariantly diffeomorphic to $((-1)^{n+1}\Sigma^{2n+1}, \overline{T})$. By Proposition 4.1 this is equivalent to constructing elements $f \in [\mathbb{C}P(n), F/O]$ with $s(f) = 0$ such that $c^*(f) \neq f$ where $c\colon \mathbb{C}P(n) \xrightarrow{\sim} \mathbb{C}P(n)$ is conjugation.

If $\Sigma^{2n+1} \in {}_p\Gamma_{2n+1}$ is non-zero, p odd, and n even, then $\Sigma^{2n+1} \neq (-1)^{n+1}\Sigma^{2n+1}$, hence (Σ^{2n+1}, T) cannot be equivariantly diffeomorphic to its conjugate. Such examples can be found in Table 7.4. For instance, $(\beta_1)_6 \in {}_3[\mathbb{C}P(6),F/O]$ corresponds to an S^1 action on $\sigma((\beta_1)_6) = \alpha_1\beta_1 \in \Gamma_{13}$. (Recall $s: {}_p[\mathbb{C}P(n),F/O] \to P_{2n}$ is always zero if p is odd.) Similarly $(\beta_2)_{14} \in {}_3[\mathbb{C}P(14),F/O]$ and $(\varepsilon_1)_{22} \in {}_3[\mathbb{C}P(22),F/O]$ correspond to S^1 actions on $\sigma((\beta_2)_{14}) = \alpha_1\beta_2 \in \Gamma_{29}$ and $\sigma((\varepsilon_1)_{22}) = 3\varphi \in \Gamma_{45}$, respectively. Similar examples for other odd primes can be read off from Proposition 7.2.

From Table 7.5 we can give examples of S^1 actions on the standard sphere S^{2n+1} which are distinct from their conjugate. Consider the element $\nu_9^* \in {}_2[\mathbb{C}P(9),F/O]$. Since the map $\mathbb{Z}_8 + \mathbb{Z}_2 = \{\nu^*\} + \{\mu_{18}\} = {}_2\pi_{18}(F/O) \to {}_2[\mathbb{C}P(9),F/O]$, induced by $j: \mathbb{C}P(9) \to S^{18}$, is injective, we see that $\nu_9^* \in [\mathbb{C}P(9),F/O]$ has order 8. We are interested in the composition $\mathbb{C}P(9) \xrightarrow{c} \mathbb{C}P(9) \xrightarrow{j} S^{18} \xrightarrow{\nu^*} F/O$. Since c has degree -1 on the top cell of $\mathbb{C}P(9)$ it follows that $\nu_9^*c = -\nu_9^* \neq \nu_9^*$. $s(\nu_9^*) = 0$ because $\nu^* \in \text{image}(\pi_{18}(PL/O) \to \pi_{18}(F/O))$ hence $\nu_9^* \in \text{image}([\mathbb{C}P(9),PL/O] \to [\mathbb{C}P(9),F/O])$. (Alternatively, $s(\nu_9^*) = 0$ by Proposition 3.8.) Since $\sigma(\nu_9^*) = \eta\nu^* = 0$, ν_9^* corresponds to an S^1 action on S^{19} distinct from its conjugate. Further, ν_9^* lifts to an element $\nu_{10}^* \in [\mathbb{C}P(10),F/O]$ and an argument similar to the above shows that $\nu_{10}^*c \neq \nu_{10}^*$. Since $\sigma(\nu_{10}^*) = 2\nu\nu^* = 0 \in \pi_{21}(PL/O) = \mathbb{Z}_2^3$, it follows that ν_{10}^* gives an S^1 action on S^{21} not equivariantly diffeomorphic to its conjugate.

We recall that the problem of constructing an equivariant diffeomorphism between an S^1 action, (Σ^{2n+1},T), and its conjugate is equivalent to constructing a diffeomorphism $c: P^{2n} \xrightarrow{\simeq} P^{2n}$ of the orbit space $P^{2n} = \Sigma^{2n+1}/T$ such that $c^*(z) = -z$, $z \in H^2(P^{2n},\mathbb{Z})$. The above examples thus provide examples of homotopy $\mathbb{C}P(n)$ which do not admit such diffeomorphisms. On the other hand, since $c^*: [\mathbb{C}P(n),F/PL] \to [\mathbb{C}P(n),F/PL]$ is the identity [24], there is always a PL isomorphism $c: P^{2n} \xrightarrow{\simeq} P^{2n}$ with $c^*(z) = -z$.

(6) The diagram

$$\begin{array}{ccc} PL & \to & PL/O \\ \downarrow & & \downarrow \\ F & \to & F/O \end{array} .$$

In §2 we showed that mapping $\mathbb{C}P(n)$ into the diagram above induces isomorphisms on p-primary torsion components, p odd. That is,

$$\begin{array}{ccc} {}_p[\mathbb{C}P(n),PL] & \xrightarrow{\simeq} & {}_p[\mathbb{C}P(n),PL/O] \\ \downarrow\wr & & \downarrow\wr \\ {}_p[\mathbb{C}P(n),F] & \xrightarrow{\simeq} & {}_p[\mathbb{C}P(n),F/O]. \end{array}$$

Thus the homotopy projective spaces corresponding to elements of ${}_p[\mathbb{C}P(n),F/O]$ are all given by smoothings of $\mathbb{C}P(n)$. In particular, the orbit spaces of the S^1 actions on spheres $\Sigma^{2n+1} \in {}_p(\pi^s_{2n+1}/\text{im } J) \subseteq \Gamma_{2n+1}$ constructed in (2) of this section (these include the homotopy projective spaces P^{2n} which do not admit diffeomorphisms homotopic to conjugation constructed in (5)) are all PL isomorphic to $\mathbb{C}P(n)$.

For $p = 2$, the maps ${}_2[\mathbb{C}P(n),PL] \xrightarrow{\simeq} {}_2[\mathbb{C}P(n),PL/O]$ and ${}_2[\mathbb{C}P(n),F] \xrightarrow{\simeq} {}_2[\mathbb{C}P(n),F/O]$ are also isomorphisms. The map ${}_2[\mathbb{C}P(n),PL] \to {}_2[\mathbb{C}P(n),F]$ is injective but is definitely not surjective. Using the fact that $\eta^2 \in \pi_2(F)$, $\nu^2 \in \pi_6(F)$, $\sigma^2 \in \pi_{14}(F)$ and $h_{i-1}^2 \in \pi_{2^i-2}(F)$, $i \geq 5$, do not belong to $\text{image}(\pi_*(PL) \to \pi_*(F))$, one can see from Table 7.5 that the elements $\eta_1^2 \in {}_2[\mathbb{C}P(1),F]$, $\nu_j^2 \in [\mathbb{C}P(j),F]$ $j = 3,4$, $\sigma_k^2 \in {}_2[\mathbb{C}P(k),F]$, $k = 7,8,9,10$ and $(h_i^2)_{2^i-1} \in {}_2[\mathbb{C}P(2^i-1),F]$, $i \geq 4$, do not belong to $\text{image}([\mathbb{C}P(n),PL] \to [\mathbb{C}P(n),F])$. Further, $s(\nu_4^2) = s(\sigma_8^2) = s(\sigma_9^2) = s(\sigma_{10}^2) = 0$, hence these elements correspond to manifolds P^8, P^{16}, P^{18}, P^{20} which are tangentially homotopy equivalent to, but not PL isomorphic to $\mathbb{C}P(4)$, $\mathbb{C}P(8)$, $\mathbb{C}P(9)$, and $\mathbb{C}P(10)$, respectively.

Since $\text{cokernel}(\pi_*(PL) \to \pi_*(F))$ is generated by the elements η^2, ν^2, σ^2, h_i^2, $i \geq 4$ [6], it is natural to conjecture that $\text{cokernel}([\mathbb{C}P(n),PL] \to [\mathbb{C}P(n),F])$ is generated by the elements named above not in $\text{image}([\mathbb{C}P(n),PL] \to [\mathbb{C}P(n),F])$. (We point out that if $h_{i-1}^2 \in \pi_{2^i-2}(F)$ is non-zero then $\eta h_{i-1}^2 \in \pi_{2^i-1}(F)$ is also non-zero, $i \geq 5$. Thus $(h_i^2)_{2^i-1} : \mathbb{C}P(2^i-1) \to F$ does not extend to $\mathbb{C}P(2^i) \to F$, $i \geq 4$.)

Since $s(\nu_3^2) = s(\sigma_7^2) = s((h_i^2)_{2^i-1}) = 1 \in \mathbb{Z}_2$, $i \geq 4$, these elements do not correspond to homotopy complex projective spaces. If the conjecture above were true, it would thus follow that every manifold, P^{2n}, tangentially homotopy equivalent to $\mathbb{C}P(n)$, $n \geq 11$, was in fact PL isomorphic to $\mathbb{C}P(n)$.

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