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ON THE HOMOTOPY GROUPS OF *BPL* AND *PL/O* II

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§1. INTRODUCTION

LET BO, BPL, and BF be the classifying spaces for stable vector bundles, PL microbundles, and spherical fibrations, respectively. Let PL/O, F/O, and F/PL be the fibres of the natural maps $BO \to BPL$, $BO \to BF$, and $BPL \to BF$. There is a diagram, with rows and columns fibrations:

$$\begin{array}{ccccc} PL/O & \longrightarrow & F/O & \longrightarrow & F/PL \\ \| & & \downarrow & & \downarrow \\ PL/O & \longrightarrow & BO & \longrightarrow & BPL \\ & & \downarrow & & \downarrow \\ & & BF & = & BF \end{array}$$

Consider the following portions of the homotopy exact sequences of these fibrations.

$$\text{(A)} \qquad \begin{array}{ccccc} & & \pi_k(F/PL) & = & \pi_k(F/PL) \\ & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \Theta} \\ \pi_k(BO) & \xrightarrow{\alpha} & \pi_k(BPL) & \xrightarrow{\beta} & \pi_{k-1}(PL/O) \\ \| & & \downarrow{\scriptstyle J_{PL}} & & \downarrow \\ \pi_k(BO) & \xrightarrow{J} & \pi_k(BF) & \longrightarrow & \pi_{k-1}(F/O) \end{array}$$

In [6], the author investigated the groups and maps in diagram (A) for $k = 4n$. In this paper, using similar ideas, we settle the extensions for $k = 8n + 2$.

Recall that there is an isomorphism $\pi_{k-1}(PL/O) \simeq \Gamma_{k-1}$, where Γ_{k-1} is the group of differentiable structures on the $k - 1$ sphere [8]. The exact sequence in the right column of (A) can be identified with the Kervaire–Milnor exact sequence [9]

$$P_k \xrightarrow{\Theta} \Gamma_{k-1} \longrightarrow \pi^S_{k-1}/\text{im } J$$

where $\pi_k(BF) \simeq \pi^S_{k-1}$, the $k - 1$ stable stem, and $\pi_k(F/PL) = P_k = 0, \mathbf{Z}_2, 0, \mathbf{Z}$ for $k \equiv 1, 2, 3, 4$ mod 4. Also, im $\Theta = bP_k \subseteq \Gamma_{k-1}$ is the subgroup consisting of those exotic spheres which bound π-manifolds [11].

Adams has shown that for $k = 8n + 1, 8n + 2$, the homomorphism $J : \pi_k(BO) = \mathbf{Z}_2 \to \pi_k(BF) = \pi^S_{k-1}$ is injective [1], and Brown and Peterson have shown that for $k = 8n + 2$, $\Theta : \pi_k(F/PL) = \mathbf{Z}_2 \to \pi_k(PL/O) = \Gamma_{k-1}$ is injective [5].

In §2 we consider the homomorphism $d_R: \pi^S_{8n+1} \to \mathbf{Z}_2$ studied by Adams [1]. Let $\lambda: S^{8m+8n+1} \to S^{8m}$ represent $\lambda \in \pi^S_{8n+1}$, $m > n$. Define $d_R(\lambda) = \lambda^* \in \mathrm{Hom}(\widetilde{KO}(S^{8m}), \widetilde{KO}(S^{8m+8n+1})) = \mathrm{Hom}(\mathbf{Z}, \mathbf{Z}_2) = \mathbf{Z}_2$. In [1] it is shown that there exist elements $\mu_n \in \pi^S_{8n+1}$ such that $2\mu_n = 0$ and $d_R(\mu_n) \neq 0$. Hence d_R splits.

Our main result in §2 is that $d_R J_{PL}$ splits. That is,

THEOREM 1.1. *There exist elements $\mu_n \in \pi_{8n+2}(BPL)$ such that $2\mu_n = 0$ and $d_R J_{PL}(\mu_n) \neq 0$.*

It follows from the results of Anderson, Brown, and Peterson on spin cobordism [2] that the image of the natural homomorphism

$$\pi^S_{8n+1} = \Omega^{\text{framed}}_{8n+1} \to \Omega^{\text{spin}}_{8n+1}$$

is $\mathbf{Z}_2$, and that the invariant $d_R: \pi^S_{8n+1} \to \mathbf{Z}_2$ can be identified with this homomorphism. Thus kernel (d_R) consists of framed manifolds which bound spin manifolds. We will use the notation $b\,\mathrm{spin}_{8n+2} = \beta(\mathrm{kernel}(d_R J_{PL})) \subseteq \Gamma_{8n+1}$ for the subgroup of exotic spheres which bound spin manifolds.

Adams has also defined an invariant $e_R: \mathrm{kernel}(d_R) \to \mathbf{Z}_2$ such that the composition $e_R J: \pi_{8n+2}(BO) = \mathbf{Z}_2 \to \mathrm{kernel}(d_R) \to \mathbf{Z}_2$ is the identity. In §3, we define an invariant $f: b\,\mathrm{spin}_{8n+2} \to \mathbf{Z}_2$ such that the composition $f\Theta: \pi_{8n+2}(F/PL) = \mathbf{Z}_2 \to b\,\mathrm{spin}_{8n+2} \to \mathbf{Z}_2$ is the identity. From this and Theorem 1.1 we conclude

THEOREM 1.2. *There is an isomorphism $\Gamma_{8n+1} \simeq bP_{8n+2} \oplus \pi^S_{8n+1}/\mathrm{im}\,J$.*

Theorem 1.2 was first proved for $n = 1, 2$ by D. Sullivan and R. Williamson. Combining all these results, we can write diagram (A):

$$\text{(A)} \qquad \begin{array}{ccccc} & & \mathbf{Z}_2'' & & \mathbf{Z}_2'' \\ & & \downarrow \theta & & \downarrow \Theta \\ \mathbf{Z}_2 & \xrightarrow{\alpha} & \pi_{8n+2}(BPL) & \xrightarrow{\beta} & \Gamma_{8n+1} \\ \| & & \downarrow J_{PL} & & \downarrow \\ \mathbf{Z}_2 & \xrightarrow{J} & \pi^S_{8n+1} & \longrightarrow & \pi^S_{8n+1}/\mathrm{im}\,J \end{array}$$

as

$$\text{(B)} \qquad \begin{array}{ccccc} & & \mathbf{Z}_2'' & = & \mathbf{Z}_2'' \\ & & \downarrow & & \downarrow \\ \mathbf{Z}_2 & \to & \mathbf{Z}_2'' \oplus \pi^S_{8n+1} & \to & \mathbf{Z}_2'' \oplus \pi^S_{8n+1}/\mathbf{Z}_2 \\ \| & & \downarrow & & \downarrow \\ \mathbf{Z}_2 & \longrightarrow & \pi^S_{8n+1} & \to & \pi^S_{8n+1}/\mathbf{Z}_2 \end{array}$$

Here $\mathbf{Z}_2 = \mathrm{im}\,J$ and $\mathbf{Z}_2'' = bP_{8n+2}$. All maps are either inclusions into the indicated summands or projections.

The invariant f is defined by studying KO characteristic numbers of spin manifolds with exotic sphere boundary. The main result in §3 is

THEOREM 1.3. *Let $\Sigma \in b\,\mathrm{spin}_{8n+2} \subseteq \Gamma_{8n+1}$. Then $\Sigma = \partial N^{8n+2}$ where N^{8n+2} is a spin manifold with all Stiefel–Whitney numbers and KO characteristic numbers zero.*

$f(\Sigma) \in \mathbf{Z}_2$ is then defined to be the Brown–Kervaire invariant of such as N^{8n+2} [4].

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§2. THE SPLITTING $0 \to b\,\mathbf{spin}_{8n+2} \to \Gamma_{8n+1} \leftrightarrows \mathbf{Z}_2 \to 0$

The map $BPL \to BF$ is, of course, induced by a map of structure groups $PL \to F$. Adams defines certain elements $\mu_n \in \pi^S_{8n+1}$ in terms of Toda bracket constructions. To define the necessary elements $\mu_n \in \pi_{8n+2}(BPL) = \pi_{8n+1}(PL)$ we need a preliminary lemma relating Toda bracket constructions for spheres to Toda bracket constructions for $F = \lim_{m\to\infty} \Omega^m S^m$. Let $\mathrm{Ad}: \pi_k(F) \xrightarrow{\sim} \pi_{m+k}(S^m)$, $m \gg k$, be the adjoint isomorphism.

LEMMA 2.1. Let $S^{m+k+p+q} \xrightarrow{\gamma'} S^{m+k+p} \xrightarrow{\beta'} S^{m+k} \xrightarrow{\alpha'} S^m$ be maps such that $\alpha'\beta' = \beta'\gamma' = 0$, with $m > k + p + q + 1$ and $k > p + q + 1$. Then there are maps $S^{k+p+q} \xrightarrow{\gamma} S^{k+p} \xrightarrow{\beta} S^k \xrightarrow{\alpha} F$ such that $\Sigma^m\gamma = \gamma'$, $\Sigma^m\beta = \beta'$, $\mathrm{Ad}\,\alpha = \alpha'$, and $\alpha\beta = \beta\gamma = 0$. Moreover, if

$$\langle \alpha, \beta, \gamma \rangle \subseteq \pi_{k+p+q+1}(F)$$

denotes the Today bracket, then

$$\mathrm{Ad}\,\langle \alpha, \beta, \gamma \rangle = \langle \mathrm{Ad}\,\alpha, \Sigma^m\beta, \Sigma^m\gamma \rangle = \langle \alpha', \beta', \gamma' \rangle \subseteq \pi_{m+k+p+q+1}(S^m).$$

Proof. This is an easy consequence of the suspension isomorphism and the geometric definitions of Toda brackets and the adjoint isomorphism. We omit the details.

Next, let $j_{2n} = \mathrm{denom}(B_{2n}/8n)$ where B_{2n} is the $2n$th Bernoulli number. Recall that there is a homomorphism $e_c: \pi^S_{8n-1} \to \mathbf{Z}_{j_{2n}}$ which splits off the cyclic subgroup $\mathrm{im}\,J_{8n-1} \subseteq \mathbf{Z}^S_{8n-1}$ whenever $\mathrm{im}\,J_{8n-1} \simeq \mathbf{Z}_{j_{2n}}$ [1]. It is known that even if $\mathrm{im}\,J_{8n-1} \simeq \mathbf{Z}_{2j_{2n}}$, there is still a splitting $\pi^S_{8n-1} \simeq \mathbf{Z}_{j_{2n}} \oplus \ker(e_c)$ [1, p. 22]. Let $\alpha_n \in \mathbf{Z}_{j_{2n}} \subseteq \pi^S_{8n-1}$ be the element of order 2 defined by such a splitting. Regard e_c as a homomorphism $e_c: \pi^S_{8n-1} \to Q/\mathbf{Z}$. Then $e_c(\alpha_n) = \frac{1}{2}$. Also, since 8 divides j_{2n}, α_n is divisible by 2. Thus $\alpha_n\eta = 0$ where $\eta \in \pi_1{}^S = \mathbf{Z}_2$.

LEMMA 2.2. *Let $\mu_n \in \langle \alpha_n, 2, \eta \rangle \subseteq \pi_{8n+1}(F)$. Then $2\mu_n = 0$ and $d_R(\mu_n) \neq 0$.*

Proof. This is essentially Theorem 12.13 of [1]. Since we need part of the argument below, we indicate the proof. By [12, p. 11 Proposition 1.4] $2\mu_n = \langle \alpha_n, 2, \eta \rangle 2 = \pm\alpha_n\langle 2, \eta, 2\rangle$. But $\langle 2, \eta, 2\rangle = \eta^2$ [12, p. 31 Corollary 3.7]. Thus $2\mu_n = \alpha_n\eta^2 = 0$. Regard d_R as a homomorphism $d_R: \pi^S_{8n+1} \to Q/\mathbf{Z}$. Theorems 11.1 and 7.18 of [1] imply that $d_R\langle \alpha_n, 2, \eta\rangle = \pm 2\, d_R(\eta)e_c(\alpha_n) = 2(\frac{1}{2})(\frac{1}{2}) = 1/2 \in Q/\mathbf{Z}$. Thus $d_R(\mu_n) \neq 0$.

We now prove Theorem 1.1.

Proof of 1.1. By Theorem 4.6 of [6], J_{PL} induces an isomorphism between the 2-torsion subgroups of $\pi_{4k-1}(PL)$ and $\pi_{4k-1}(F) \simeq \pi^S_{4k-1}$ if $k > 2$. $\pi_7(PL) = \mathbf{Z} + \mathbf{Z}_4$ while $\pi_7{}^S \simeq \mathbf{Z}_{240}$ but the $\mathbf{Z}_4$ summand is injected by J_{PL}. Thus let $\alpha_n \in \pi_{8n-1}(PL)$ be the element defined by $J_{PL}(\alpha_n) = \alpha_n \in \pi_{8n-1}(F)$ and $2\alpha_n = 0$. Clearly α_n is divisible by 2. Define $\mu_n \in \pi_{8n+1}(PL)$ by $\mu_n \in \langle \alpha_n, 2, \eta\rangle$. As above, $2\mu_n = \langle \alpha_n, 2, \eta\rangle 2 = \alpha_n\langle 2, \eta, 2\rangle = \alpha_n\eta^2 = 0$. By naturality of Toda brackets, $J_{PL}(\mu_n) \in \langle J_{PL}(\alpha_n), 2, \eta\rangle = \langle \alpha_n, 2, \eta\rangle \subseteq \pi_{8n+1}(F)$. Hence $d_R J_{PL}(\mu_n) \neq 0$. This completes the proof of Theorem 1.1.

Remark 2.3. D. Sullivan has pointed out that a more conceptual proof of Theorem 1.1 can be given when $\operatorname{im} J_{8n-1} \simeq \mathbf{Z}_{j_{2n}}$. Namely, choose the element $\alpha_n \in \pi_{8n-1}(PL)$ above such that $J_{PL}(\alpha_n) \in \pi_{8n-1}(F)$ is the element of order 2 in $\operatorname{im} J_{8n-1}$. Then $\beta\alpha_n \in \pi_{8n-1}(PL/O) = \Gamma_{8n-1}$ is the element of order 2 in bP_{8n}.

Representing homotopy elements by framed manifolds, one can see geometrically that $\langle J_{PL}(\alpha_n), 2, \eta\rangle$ is represented by a framing of the manifold $S^1 \times M^{8n}$ where M^{8n} is an almost parallelizable manifold with $\hat{A}(M^{8n}) = 1$. Moreover, this framed manifold does not bound a spin manifold [10], hence $d_R\langle J_{PL}(\alpha_n), 2, \eta\rangle \neq 0$.

Performing surgery on $S^1 \times M^{8n}$ yields an exotic sphere $\Sigma^{8n+1} \in \Gamma_{8n+1}$ which does not bound a spin manifold. Σ^{8n+1} has order 2 because it belongs to the Toda bracket $\langle \beta\alpha_n, 2, \eta\rangle \subseteq \pi_{8n+1}(PL/O) = \Gamma_{8n+1}$. Thus Σ^{8n+1} splits the sequence

$$0 \to b\ \mathrm{spin}_{8n+2} \to \Gamma_{8n+1} \to \mathbf{Z}_2 \to 0.$$

Besides being geometrically motivated, this proof avoids use of the delicate results of [1].

§3. THE INVARIANT $f: b\ \mathbf{spin}_{8n+2} \to \mathbf{Z}_2$

We will need the results of Anderson, Brown and Peterson relating spin cobordism and *KO*-theory [2]. Their main theorem is that there is a map of spectra $F = \prod f_J \times \prod f_{z_i}$, $J = (j_1 \dots j_k)$ $k \geq 0, j_i > 1$,

3.0 $$F: \mathbf{M}\,\mathrm{spin} \to \prod_{n(J)\ \mathrm{even}} \mathbf{BO}\langle 4n(J)\rangle \times \prod_{n(J)\ \mathrm{odd}} \mathbf{BO}\langle 4n(J) - 2\rangle \times \prod_i \mathbf{K}(\mathbf{Z}_2, \dim z_i)$$

which induces an isomorphism in $\mathbf{Z}_2$-cohomology. The notation is that of [2], which we use throughout. In particular $f_J: \mathbf{M}\,\mathrm{spin} \to BO$ represents the *KO*-theory class

$$\pi^J \,.\, \Phi_{KO}(1) \in KO(\mathbf{M}\,\mathrm{spin}).$$

Let $\Sigma \in b\ \mathrm{spin}_{8n+2} \subseteq \Gamma_{8n+1}$. Choose a smooth, spin manifold M^{8n+2} such that $\Sigma = \partial M^{8n+2}$. Let $\hat{M} = M^{8n+2} \bigcup_\Sigma C\Sigma$.

LEMMA 3.1. *There is an isomorphism of microbundles over* $\hat{M}$

$$\nu_M = \xi + \rho^*\sigma$$

where ν_M *is the PL normal microbundle of* $\hat{M}$, ξ *is an* $8m$*-dimensional spin vector bundle,* $\sigma \in \pi_{8n+2}(BPL)$ *is an* $8m$*-bundle over* S^{8n+2}, *and* $\rho: \hat{M} \to S^{8n+2}$ *is a map of degree one. Moreover,* $\beta(\sigma) = -\Sigma \in \pi_{8n+1}(PL/O) = \Gamma_{8n+1}$. *Finally,* σ *and* ξ *are well-defined by the additional condition* $e_R J_{PL}(\sigma) = 0$.

Proof. The first two statements follow easily from smoothing theory and are proved in [6, Lemma 3.1]. The last statement holds because σ is well-defined modulo vector bundles and $e_R J: \pi_{8n+2}(BO) \to \mathbf{Z}_2$ is non-zero.

We will use Lemma 3.1 to define *KO* characteristic numbers for the almost smooth spin manifold $\hat{M}$. Given a bundle ξ, let $T(\xi)$ denote the Thom space. There is a diagram

$$\begin{array}{ccccccc} S^{16m+8n+2} & \xrightarrow{c} & T(\nu_M) & \xrightarrow{\Delta} & T(\xi) \wedge T(\rho^*\sigma) & \xrightarrow{Id \wedge \rho} & T(\xi) \wedge T(\sigma) \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \hat{M} & \xrightarrow{\Delta} & \hat{M} \times \hat{M} & \xrightarrow{Id \wedge \rho} & \hat{M} \times S^{8n+2} \end{array}$$

where $\nu_M \subset S^{16m+8n+2}$ is an embedding and c is the collapsing map, Δ is the diagonal, and $T(\xi \times \sigma) = T(\xi) \wedge T(\sigma)$ over $\hat{M} \times S^{8n+2}$.

$T(\sigma)$ is the two cell complex $S^{8m} \bigcup_{J_{PL}(\sigma)} e^{8m+8n+2}$. The proof of this for vector bundles given in [1] works for *PL* bundles also. Since $\beta(\sigma)$ bounds a spin manifold, $d_R J_{PL}(\sigma) = 0$. It follows that there is a map $u: T(\sigma) \to BO\langle 8m\rangle$ such that the composition $S^{8m} \xrightarrow{i} T(\sigma) \xrightarrow{u} BO\langle 8m\rangle$ is the generator of $\pi_{8m}(BO\langle 8m\rangle)$. Let $u': T(\xi) \to BO\langle 8m\rangle$ be the canonical *KO*-orientation [3].

We define the *KO*-numbers $\pi^J(M^{8n+2})$ to be

$$\pi^J(M^{8n+2}) = c^*\Delta^*(\pi^J(\xi)u' \cdot \rho^*u) \in KO^{-(16m+8n+2)}(pt) = \mathbf{Z}_2.$$

Here, $\pi^J(\xi) \in KO(\hat{M})$ hence $\pi^J(\xi)u' \in \widetilde{KO}(T(\xi))$ and $\pi^J(\xi)u' \,.\, \rho^*u \in \widetilde{KO}(T(\xi) \wedge T(\rho^*\sigma))$.

From the cofibration sequence $S^{8m+8n+1} \xrightarrow{J_{PL}(\sigma)} S^{8m} \xrightarrow{i} T(\sigma) \xrightarrow{j} S^{8m+8n+2}$, we see that the orientation u may be replaced by $u + j^*v$ where $v \in \pi_{8m+8n+2}(BO\langle 8m\rangle)$ is non-zero.

LEMMA 3.2. *If* $J \neq (0)$ *then* $\pi^J(M^{8n+2})$ *is independent of the choice of* u.

Proof. j^*v has filtration $8m + 8n + 2$ (that is, j^*v is trivial on the $8m + 8n + 1$-skeleton) and u' has filtration $8m$. If $J \neq (0)$ then $\pi^J(\xi)$ has filtration greater than zero, hence $\Delta^*(\pi^J(\xi)u' \,.\, \rho^*j^*v) = 0$. The lemma follows.

We remark that $\pi^{(0)}(M^{8n+2})$ is not well-defined. This is because

$$S^{16m+8n+2} \xrightarrow{\Delta c} T(\xi) \wedge T(\rho^*\sigma) \xrightarrow{Id \wedge j\rho} T(\xi) \wedge S^{8m+8n+2}$$

represents the bottom cell of $T(\xi) \wedge S^{8m+8n+2}$, hence $c^*\Delta^*(u' \,.\, \rho^*j^*v) \neq 0$.

If M is smooth then $\sigma = 0$ and $\nu_M = \xi$. It follows that the *KO*-numbers $\pi^J(M^{8n+2})$ for smooth spin manifolds coincide with those defined in [2].

Also, under the connected sum operation (or, more simply, disjoint union) the collection of almost smooth spin manifolds is a semi-group. The numbers π^J are additive. That is,

$$\pi^J(M^{8n+2} \,\#\, N^{8n+2}) = \pi^J(M^{8n+2}) + \pi^J(N^{8n+2}).$$

From the main theorem of [2], it follows that smooth spin cobordism classes are detected by the *KO* characteristic numbers π^J, $J = (j_1 \cdots j_k)$, $k \geq 0, j_i > 1$, and by Stiefel–Whitney numbers $w^I = w_1{}^{i_1} \cdots w_r{}^{i_r}$. In dimension $8n + 2$, the π^J and w^I span a vector space over $\mathbf{Z}_2$.

LEMMA 3.3. *The relations which hold between the numbers* π^J *and* w^I *for all* $8n + 2$ *smooth, spin manifolds are generated by* (1) *the relations* $w^I = 0$ *if* w^I *involves* w_1 *or* w_2, (2) *the Wu relations* $\Phi^{-1}a_q\Phi(w^{I_{8n+2-q}}) = 0$ *where* $a_q \in A_2{}^q$ *is an element in the* mod 2 *Steenrod*

Algebra, $w^{I_{8n+2-q}}$ *is a monomial of degree* $8n+2-q>0$ *in the Stiefel–Whitney classes, and* $\Phi : H^*(B\,\mathrm{Spin}, \mathbf{Z}_2) \xrightarrow{\approx} H^*(\mathbf{M}\,\mathrm{Spin}, \mathbf{Z}_2)$ *is the Thom isomorphism, and* (3) *the relations* $\pi^J = Y_J$ *between KO-numbers and S-W numbers, where* $4n(J)-2 = 8n+2$ *and* $\Phi(Y_J) = f_J^*(\alpha_{4n(J)-2}) \in H^{8n+2}(\mathbf{M}\,\mathrm{Spin}, \mathbf{Z}_2)$, *where* $\alpha_{4n(J)-2} \in H^{8n+2}(\mathbf{BO}\langle 4n(J)-2\rangle, Z_2)$ *is the generator.*

Proof. From the homotopy equivalence 3.0 it is immediate that an element of $H^{8n+2}(\mathbf{M}\,\mathrm{Spin}, \mathbf{Z}_2)$ vanishes on all spin manifolds if and only if it is decomposable over the Steenrod algebra. Also, the numbers π^J, $J = (j_1 \cdots j_k)$, $j_i > 1$, are independent. Finally, one sees from 3.0 that the only relations between the π^J and w^I are generated by those given in (3).

LEMMA 3.4. *The relations* (1), (2), (3) *of Lemma* 3.3 *hold for almost smooth spin manifolds* $\hat{M}^{8n+2}$.

Proof. Since $w_1(\hat{M}) = w_2(\hat{M}) = 0$, the relations (1) hold. The Wu relations (2) hold for all Poincaré duality spaces. The relations (3) follow by computing the *KO*-theory and $\mathbf{Z}_2$-cohomology maps in the diagram

$$S^{16m+8n+2} \xrightarrow{(Id\wedge\rho)\Delta c} T(\xi)\wedge T(\sigma) \xrightarrow{\xi\wedge u} M\mathrm{Spin}(8m)\wedge BO\langle 8m\rangle \xrightarrow{f_J\wedge Id}$$
$$BO\langle 8m+8n+2\rangle \wedge BO\langle 8m\rangle \xrightarrow{\otimes} BO\langle 16m+8n+2\rangle.$$

For this composition $\pi : S^{16m+8n+2} \to BO\langle 16m+8n+2\rangle$ is by definition the *KO*-number $\pi^J(M^{8n+2})$. It is non-zero if and only if the cohomology map

$$\pi^* : H^{16m+8n+2}(BO\langle 16m+8n+2\rangle) \to H^{16m+8n+2}(S^{16m+8n+2})$$

is non-zero. But in cohomology this clearly gives the Stiefel–Whitney number

$$f_J^*(\alpha_{4n(J)-2})(\xi) = Y_J(\xi) = Y_J(M^{8n+2})$$

since $w_i(v_M) = w_i(\xi)$ by lemma 3.1.

We can now prove Theorem 1.3.

Proof of 1.3. Let $\Sigma = \partial M^{8n+2}$ as above. It follows from Lemma's 3.3 and 3.4 and an easy argument with vector spaces that there is a closed, smooth, spin manifold L^{8n+2} with the same *KO*-numbers and *S-W* numbers as M^{8n+2}. Then $N^{8n+2} = M^{8n+2} \divideontimes L^{8n+2}$ satisfies the requirements of Theorem 1.3.

Actually, the number $\pi^{(0)}(N^{8n+2})$ is not well-defined. This ambiguity in $\pi^{(0)}(N^{8n+2})$ will not matter in our application below since an $8n+2$ spin manifold with only $\pi^{(0)}$ non-zero is cobordant to a framed manifold. [2].

Recall that Brown [4] has defined an invariant $\psi : \Omega^{\mathrm{spin}}_{8n+2} \to Z_2$ which agrees with the Kervaire invariant [9] on $\Omega^{\mathrm{framed}}_{8n+2}$. ψ is defined in terms of secondary cohomology operations and can easily be defined on *PL* manifolds with $w_1 = w_2 = 0$.

Given $\Sigma \in b\ \mathrm{spin}_{8n+2}$, define $f(\Sigma) = \psi(N^{8n+2}) \in Z_2$ where $\Sigma = \partial N^{8n+2}$ and the *KO*-number and S-W numbers of N^{8n+2} vanish. This gives a well defined homomorphism $f : b\,\mathrm{spin}_{8n+2} \to Z_2$. For, first, the ambiguity in $\pi^{(0)}(N^{8n+2})$ is not important since $\Omega^{\mathrm{framed}}_{8n+2} \to \Omega^{\mathrm{spin}}_{8n+2} \xrightarrow{\psi} Z_2$ is zero [5]. Secondly, if we glue two such N^{8n+2} together along their common boundary Σ, we obtain a smooth spin boundary since all the characteristic

numbers vanish. If $\Sigma_0 \in bP_{8n+2}$ is non-zero then we may choose $\Sigma_0 = \partial N_0^{8n+2}$ where N_0^{8n+2} is the $4n$-connected, framed, Kervaire manifold. Since $\psi(N_0^{8n+2}) = 1$, it follows that f splits the exact sequence

$$0 \to bP_{8n+2} = Z_2 \to b\ \mathrm{spin}_{8n+2}$$

Theorem 1.2 is a corollary of this and theorem 1.1.

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