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NORMAL MAPS, COVERING SPACES, AND QUADRATIC FUNCTIONS

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1. Introduction

1.1. *Survey of results*

In this paper we investigate the relations between normal maps, covering spaces, and quadratic functions.

If $\pi : M' \to M$ is an m-fold covering and M', M are closed manifolds, then π can be interpreted as a normal map of degree m. Motivated by the theory of degree 1 normal maps, we are led to study relations between quadratic functions defined on an appropriate cohomology group of M and quadratic functions defined on the corresponding groups for M'. The theory we develop holds in the generality of coverings $\pi : X' \to X$ of Poincaré duality spaces, and using the transfer $\tau : H^*(X', \mathbb{Z}/2) \to H^*(X, \mathbb{Z}/2)$ we obtain formulas relating these associated quadratic functions.

The theory is applied to two types of problems. First we consider coverings of odd degree. Here τ is surjective and K = kernel (τ), which is analogous to a surgery kernel, inherits a canonical quadratic function $\tilde{q} : K \to \mathbb{Z}/2$. If $X' \to X$ is a principal G-bundle, $|G|$ odd, we prove that the Arf invariant of $(K, \tilde{q})$ is $\chi(X)$ if $|G| \equiv 3, 5(8)$ and 0 otherwise, where $\chi(X)$ is the (mod 2) Euler characteristic of X.

Second, if $X' \to X$ is a double cover, we construct canonical quadratic functions $\tilde{q} : H^*(X') \to Q/\mathbb{Z}$. If X is $2n$ dimensional, $H^*(X')$ means $H^n(X', \mathbb{Z}/2)$, while if X is $4n - 1$ dimensional $H^*(X')$ is the torsion subgroup, $T^{2n}(X') \subset H^{2n}(X', \mathbb{Z})$. In the $2n$-dimensional case $A[H^n(X', \mathbb{Z}/2), \tilde{q}]$ is an obstruction to a certain transversality problem for P.D. spaces. In the $4n - 1$ dimensional case our results extend the surgery product formulas of [16], [17] to P.D. spaces.

Since the first version of the present paper appeared, there have been additional applications.

First I. Hambleton and Milgram [22] have constructed free involutions s on spaces homotopy equivalent to $S^2 \times S^2$, $S^3 \times S^3$ with

$$A[H^n(S^n \times S^n, \mathbb{Z}/2, \tilde{q}] \neq 0,$$

and on taking products with CP^{2m} further examples in all even dimensions. In these examples the orbit spaces $X = X'/s$ are thus Poincaré duality spaces such that the map $f : X \to RP^N$ classifying the covering

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$\pi : X' \to X$ cannot be made transversal to RP^{N-1}. In particular X is not the homotopy type of any manifold. Also [22] shows how, in case the map f can be made transversal, the Browder-Livesay invariant [23] comes from our quadratic form.

Secondly, the formulas have applications to the problem of the existence of framed manifolds of Arf invariant one in dimensions $2^i - 2$. J. Jones, at Oxford, has claimed that the dihedral group of order 8, D_4, acts freely on $N^{30} = M^2 \times S^7 \times S^7 \times S^7 \times S^7$, where M^2 is the oriented surface of genus 5, so that N^{30}/D_4 is a π-manifold. Moreover, with respect to the composition series $D_4 \supset K = C_2 \times C_2 \supset C_2 \supset \{1\}$ at least one of the manifolds

$$N^{30}/D_4, \qquad N^{30}/K, \qquad N^{30}/C_2$$

admits a framing with Arf invariant 1. Jones' method is to apply our formula 4.2.1 along with other results, to the sequence of double covers

$$N^{30} \to N^{30}/C_2 \to N^{30}/K \to N^{30}/D_4.$$

Also, Brumfiel has shown that if the Kervaire manifold K^{2^i-2} is smooth, then it admits a fixed point free involution s so that the orbit manifold K/s is also a π manifold which admits framings both of Arf invariant 1 and 0.

1.2. *Constructing quadratic functions*

In 1.4 we summarize the algebraic results on quadratic functions that we need in the paper. In §2, we recall E. H. Brown's method [2] of obtaining quadratic functions on $H^n(X^{2n}, \mathbb{Z}/2)$ from the suspension

$$s : [X, K(\mathbb{Z}/2, n)] \to \{X, K(\mathbb{Z}/2, n)\} = \varinjlim_q\, [\Sigma^q X, \Sigma^q K(\mathbb{Z}/2, n)].$$

Also, we recall from [4] an analogous study of the suspension

$$s : [Y, K(Q/\mathbb{Z}, 2n - 1)] \to \{Y, K(Q/\mathbb{Z}, 2n - 1)\},$$

which leads to a construction of quadratic functions on $T^{2n}(Y)$, Y an oriented $4n - 1$ dimensional P.D. space.

In §3 we study normal maps between P.D. spaces. From the definition it is easy to show how a quadratic function q defined on $H^n(X, \mathbb{Z}/2)$ [resp. $T^{2n}(X)$] induces a quadratic function q' on $H^n(X', \mathbb{Z}/2)$ [resp. $T^{2n}(X')$] for any normal map $\pi : X' \to X$, of any degree.

The quantity which we emphasize is the function $\tilde{q} = q\tau - q'$ defined on $H^n(X', \mathbb{Z}/2)$ [resp. $T^{2n}(X')$], where $\tau : H^*(X') \to H^*(X)$ is the cohomology map induced by $D\pi : \Sigma^q X_+ \to \Sigma^q X'_+$. The function $\tilde{q}$ has the following properties:

(1) $\tilde{q}$ is independent of q.

(2) If $\pi : X' \to X$ is a normal map of odd degree, so that $H^n(X', \mathbb{Z}/2) \cong \pi^* H^n(X, \mathbb{Z}/2) \oplus K^n$, where $K^n = \text{kernel}\,(\tau) \subset H^n(X', \mathbb{Z}/2)$, then $\tilde{q}|_{K^n} = q'|_{K^n}$

is computable as a functional square and $\tilde{q}$ is an analogue of the usual quadratic function defined on a surgery kernel.

(3) If $\pi : X' \to X$ has odd degree, then $\tilde{q}|_{\pi^* H^n(X, \mathbb{Z}/2)} = q - q'\pi^* : H^n(X, \mathbb{Z}/2) \to \mathbb{Z}/2$. Thus formulas for $\tilde{q}$ include the Brown twisting formula [3].

(4) If $\pi : X' \to X$ is a normal map of degree 2, then $\tilde{q} : H^n(X', \mathbb{Z}/2) \to \mathbb{Z}/2$ is actually a *quadratic* function refining the non-singular pairing

$$\tilde{\ell}(a, b) = \langle a \cdot Sb, [X'] \rangle \in \mathbb{Z}/2,$$

where $S = \pi^*\tau - Id : H^n(X', \mathbb{Z}/2) \to H^n(X', \mathbb{Z}/2)$, [resp. $\tilde{q} : T^{2n}(X') \to Q/\mathbb{Z}$ is a quadratic refinement of $\tilde{\ell}(a, b) = \text{linking } (a, Sb) \in Q/\mathbb{Z}$, $S = \pi^*\tau - Id$].

In §4, we show how coverings $\pi : X' \to X$ of degree m can be interpreted as normal maps of degree m in our sense. Then, in some special cases, we give explicit formulae for the function $\tilde{q} : H^n(X', \mathbb{Z}/2) \to \mathbb{Z}/2$ [resp. $\tilde{q} : T^{2n}(X') = Q/\mathbb{Z}$] discussed above.

Consider a double cover $\pi : X' \to X$ of $2n$-dimensional P.D. spaces, and let $S : X' \to X'$ be the involution over X. Then $S^* = \pi^*\tau - Id : H^n(X', \mathbb{Z}/2) \to H^n(X', \mathbb{Z}/2)$. The pairing $\tilde{\ell}(a, b) = \langle a \cdot S^*b, [X'] \rangle \in \mathbb{Z}/2$ is even, that is, $\langle a \cdot S^*a, [X'] \rangle \equiv 0 \pmod 2$. Specifically, $S^*a = \pi^*\tau a - a$ and $\langle a \cdot \pi^*\tau a, [X'] \rangle = \langle \tau a \cdot \tau a, [X] \rangle = \langle Sq^n(\tau a), [X] \rangle = \langle \tau(Sq^n(a)), [X] \rangle = \langle Sq^n(a), [X'] \rangle = \langle a \cdot a, [X'] \rangle$. (We have used certain properties of the transfer τ which will be discussed in greater detail later.)

Construct a classifying diagram for the double cover π.

$$\begin{array}{ccc} X' & \xrightarrow{f'} & S^\infty \\ \pi \downarrow & & \downarrow \\ X & \xrightarrow{f} & RP(\infty) \end{array}$$

There is then a diagram

$$\begin{array}{ccc} X' & \xrightarrow{F'} & S^\infty \times X' \times X' \\ \downarrow & & \downarrow \\ X & \xrightarrow{F} & S^\infty \times_{\mathbb{Z}/2} X' \times X' \end{array}$$

where $F' = f' \times Id \times S$. If $a_\#$ is a cocycle on X', then $1 \otimes a_\# \otimes a_\#$ is a $\mathbb{Z}/2$ equivariant cocycle on $S^\infty \times X' \times X'$, hence represents a class $\alpha \in H^*(S^\infty \times_{\mathbb{Z}/2} X' \times X', \mathbb{Z}/2)$. One of our main formulas (4.2.1) is

$$\tilde{q}(a) = \langle F^*(\alpha), [X] \rangle \in \mathbb{Z}/2$$

where $a \in H^n(X', \mathbb{Z}/2)$, $\tilde{q} = q\tau - q'$ as above. On the cochain level we can write

$$\tilde{q}(a) = \frac{1}{2} \langle a_\# \cdot S^\# a_\#, [X'] \rangle \in \mathbb{Z}/2,$$

where $a_\# \in C^n(X', \mathbb{Z})$ is a cochain representative for $a \in H^n(X', \mathbb{Z}/2)$. ($X'$ need not be orientable. In this chain formula, an *integral chain* $[X']$ is obtained by first lifting the $\mathbb{Z}/2$ fundamental class of X to an integral chain, then using the double cover $\pi : X' \to X$ to obtain two cells of X' for each cell of X.)

A second, but more complicated, formula is also obtained for $\tilde{q} : T^{2n}(X') \to Q/\mathbb{Z}$, where $X' \to X$ is a double cover of oriented $4n - 1$ dimensional P.D. spaces. Let $u_\# \in C^{2n-1}(X', Q)$ be a rational cochain with $\delta u_\# \equiv 0 \pmod{\mathbb{Z}}$. Then $\delta u_\#$ represents a class $a \in T^{2n}(X')$ and (4.2.2) implies

$$\tilde{q}(a) = \frac{1}{2} \langle u_\# \cdot S\delta u_\#, [X'] \rangle \in Q/\mathbb{Z}.$$

There is also a cohomological version obtained from a suitable equivariant class.

Finally, we describe a third example of a formula for $\tilde{q} = q\tau - q'$. Let $\pi : X' \to X$ be a principal G-bundle, $|G| = 2k + 1$. As a covering, π is a normal map, and $H^n(X', \mathbb{Z}/2) = \pi^* H^n(X, \mathbb{Z}/2) \oplus K^n$, since the degree is odd. Now $K^n \subset H^n(X', \mathbb{Z}/2)$ is a module over the quotient ring $\Lambda = \mathbb{Z}/2(G)\Big/\Big(\sum_{g \in G} g\Big)$, where $\mathbb{Z}/2(G)$ is the group ring. This is because $K^n = \text{Kernel}\ (\tau) = \text{Kernel}\ (\pi^*\tau) \subset H^n(X', \mathbb{Z}/2)$, and $\pi^*\tau(a) = \sum_{g \in G} g^* a$ where $g : X' = X'$ are the covering transformations, $g \in G$. Write $G = \{1, g_1 \cdots g_k, g_1^{-1} \cdots g_k^{-1}\}$. Consider

$$\tilde{q}(a) = \langle a \cdot \sum_{i=1}^{k} g_i^* a, [X'] \rangle \in \mathbb{Z}/2.$$

Then $\tilde{q}$: $K^n \to \mathbb{Z}/2$ is quadratic over the cup product pairing in K^n, and is independent of how one represents $G = \{1, g_1 \cdots g_k. g_1^{-1} \cdots g_k^{-1}\}$. Notice that

$$\left\langle a \cdot \sum_{i=1}^{k} g_i^* a, [X'] \right\rangle = \left\langle \sum_{i=1}^{k} (g_i^{-1})^* a \cdot a, [X'] \right\rangle.$$

Thus if $a \in K^n$,

$$a = \sum_{i=1}^{k} g_i^* a + \sum_{i=1}^{k} (g_i^{-1})^* a,$$

and hence

$$\tilde{q}(a) = \left\langle a \cdot \sum_{i=1}^{k} g_i^* a, [X'] \right\rangle$$

is a "coherent" way of dividing $\langle a \cdot a, [X'] \rangle$ by 2. We prove that this formula coincides with the $\tilde{q} = q\tau - q'$ of the general theory in this case also.

1.3. *Some applications*

From the point of view of surgery theory, the important invariant of a quadratic function q: $K \to Q/\mathbb{Z}$ is its Arf invariant, $A(K, q) \in \mathbb{Z}/8$. (See 1.4 for definition and properties. If K is a $\mathbb{Z}/2$ vector space with an even form, then $A(K, q) \in 4\mathbb{Z}/8 = \mathbb{Z}/2$ is the classical Arf invariant.) The main applications of this paper essentially are computations of the Arf invariants of the quadratic functions $\tilde{q} = q\tau - q'$ described above.

For example, consider the case of a principle G-bundle $\pi: X' \to X$, $G = \{1, g_1 \cdots g_k, g_1^{-1} \cdots g_k^{-1}\}$, $K^n \subset H^n(X', \mathbb{Z}/2)$ the kernel of the transfer, and $\tilde{q}: K^n \to \mathbb{Z}/2$ defined by $\tilde{q}(a) = \left\langle a \cdot \sum_{i=1}^{k} g_i^* a, [X'] \right\rangle \in \mathbb{Z}/2$. Then

1.3.1 $$A(K^n, \tilde{q}) = \begin{cases} \chi(X) & \text{if } |G| \equiv 3 \text{ or } 5 \pmod 8 \\ 0 & \text{if } |G| \equiv 1 \text{ or } 7 \pmod 8 \end{cases}$$

where $\chi(X)$ is the (mod 2) Euler characteristic of X.

This result is expected for *manifolds* $\pi: M' \to M$. First it is true for the trivial cover $\pi: M \times G \to M$. Secondly, $A(K^n, q)$ is a bordism invariant of covers $\pi: M' \to M$. Thirdly, if $|G|$ is odd, then $\tilde{H}_*(BG, \mathbb{Z}/2) = 0$, hence $\tilde{\mathfrak{N}}_*(BG) = 0$. However, our computation of $A(K^n, \tilde{q})$ for *P.D. spaces* involved quite a lot of algebra, including the solvability of groups of odd order and the structure of the group rings $\mathbb{Z}/2(\mathbb{Z}/P)$, p an odd prime. Because of their length, these algebraic arguments are not included in the paper.

In general, for covers of odd order, $\pi: X' \to X$, the Arf invariant $A(K^n, \tilde{q})$ seems quite difficult to compute. In the manifold case there are theoretically computable cohomology formulae

1.3.2 $$A(K^n, \tilde{q}) = \langle V^2(M) \cdot f^*(K), [M] \rangle \in \mathbb{Z}/2$$

where M is a manifold and $f: M \to B\,\mathcal{S}_m$ classifies the m-fold cover $\pi: M' \to M$, m odd. Here, $K \in H^*(B\,\mathcal{S}_m, \mathbb{Z}/2)$ is a certain universal characteristic class. In particular, $K_0 = 1 \in H^0(B\,\mathcal{S}_m, \mathbb{Z}/2)$ if $m \equiv 3$ or $5 \pmod 8$, $K_0 = 0$ if $m \equiv 1$ or $7 \pmod 8$, and $K_j = 0$ unless $j = 2^i - 2$, $i \geq 1$. Thus 1.3.1 can be interpreted as a special case of 1.3.2 for P.D. spaces $X \to BG \to B\,\mathcal{S}_m$, $|G| = m$, $G \subset \mathcal{S}_m$ by a regular representation.

Formula 1.3.2 does not hold for all m-fold covers $\pi: X' \to X$, X a P.D. space, m odd. We conjecture that the difference of the two sides of 1.3.2 is a certain Poincaré transversality obstruction. We will discuss this further below.

Next consider a double cover $\pi: X' \to X$ of $2n$ dimensional P.D. spaces. We have the quadratic function $\tilde{q} : H^n(X', \mathbb{Z}/2) \to \mathbb{Z}/2$, $\tilde{q}(a) = (1/2)\langle a_\# \cdot S^* a_\#, [X'] \rangle \in \mathbb{Z}/2$, refining the pairing $\tilde{\ell}(a, b) = \langle a \cdot S^* b, [X'] \rangle \in \mathbb{Z}/2$. If $\pi: M' \to M$ is a double cover of manifolds, then $A(H^n(M', \mathbb{Z}/2), \tilde{q}) = 0$ (see 5.2 for details). Roughly, this is so because if $a_\#$ and $Sa_\#$ have disjoint support, clearly $\tilde{q}(a) = 0$. But if $f : M \to RP(N)$ classifies the double cover $\pi: M' \to M$, then making f transversal to $RP(N-1) \subset RP(N)$ splits M' into two manifolds with a common boundary, interchanged by the involution $S: M' \to M'$ and each carrying half the homology of M'. Thus $\tilde{q}$ vanishes on half of a symplectic basis for the form $\tilde{\ell}$ and hence $A(H^n(M', \mathbb{Z}/2), \tilde{q}) = 0$.

This result, $A(H^n(M', \mathbb{Z}/2), \tilde{q}) = 0$ for *double covers* of *manifolds* $\pi: M' \to M$, can be interpreted positively in at least two ways. First, if M' and

M are manifolds but $\pi: M' \to M$ is only assumed to be a normal map of degree 2, then the quadratic function $\tilde{q} = q\tau - q': H^n(M', \mathbb{Z}/2) \to \mathbb{Z}/2$ is still defined and its Arf invariant $A(H^n(M', \mathbb{Z}/2), \tilde{q}) \in \mathbb{Z}/2$ can be interpreted as an obstruction to cobording $\pi: M' \to M$, through normal maps of degree 2, to a double cover. Secondly, if $\pi: X' \to X$ is a double cover, but X' and X are assumed only to be P.D. spaces, then $A(H^n(X', \mathbb{Z}/2), \tilde{q}) \in \mathbb{Z}/2$ is an obstruction (non-trivial by [22]) to making the classifying map $f: X \to RP(N)$ of the double cover transversal to $RP(N-1) \subset RP(N)$.

We conjecture that double covers and odd covers are related as follows. An m-fold cover $f: X \to B\,\mathcal{S}_m$ determines a double cover $pf: X \to B\,\mathcal{S}_m \to B(\mathcal{S}_m/\mathcal{A}_m) = RP(\infty)$, where $\mathcal{A}_m \subset \mathcal{S}_m$ is the alternating group. We conjecture that if m is odd, then the difference of the two sides of 1.3.2 is exactly the Arf invariant $A(H^n(X', \mathbb{Z}/2), \tilde{q}) \in \mathbb{Z}/2$ of the double cover $\pi: X' \to X$, classified by pf. We prove this, in fact, for some (but not all) non-principle triple covers; note also that 1.3.1 is consistent with this conjecture, since if $|G| = m$ is odd, $G \subset \mathcal{S}_m$, then $G \subset \mathcal{A}_m$, so that associated double cover of a principal G-bundle is trivial.

Next, consider the case $\pi: X' \to X$, a double cover of oriented $4n-1$ dimensional P.D. spaces, and $\tilde{q}: T^{2n}(X') \to Q/\mathbb{Z}$, the resulting quadratic function refining the pairing $\tilde{\ell}(a, b) = \text{linking}\,(a, S^*b) \in Q/\mathbb{Z}$. We use our cochain formula, $\tilde{q}(a) = \frac{1}{2}\langle u_\# S\delta u_\#, [X']\rangle \in Q/\mathbb{Z}$, $u_\# \in C^{2n-1}(X', Q)$, $[\delta u_\#] = a \in T^{2n}(X')$, to prove a certain delicate product formula for surgery obstructions, which we now outline in some detail.

Consider $\pi: X' \to X$ a degree one normal map of $\mathbb{Z}/2$ P.D. spaces. A "$\mathbb{Z}/2$ P.D. space" is constructed as follows (and should not be confused with "Poincaré duality with $\mathbb{Z}/2$ coefficients"). Let X_0 be an oriented P.D. space with boundary, together with an orientation preserving homotopy equivalence from ∂X_0 to two copies of an oriented P.D. space δX. Gluing the two components of ∂X_0 to $\partial(\delta X \times I)$ gives an (unoriented) P.D. space X such that $w_1(X) \in H^1(X, \mathbb{Z}/2)$ is the reduction of a class $z_1(X) \in H^1(X, \mathbb{Z})$. Namely, the Poincaré embedding $\delta X \times I \subset X$ defines a map $z_1: (X, \delta X) \to (S^1, pt)$. δX is called the Bockstein of X. We assume all normal maps $\pi: X' \to X$ between $\mathbb{Z}/2$ P.D. spaces are maps of pairs $\pi: (X', \delta X') \to (X, \delta X)$.

Given such a degree one normal map $\pi: (X', \delta X') \to (X, \delta X)$ of $4n$-dimensional $\mathbb{Z}/2$ P.D. spaces, we define an obstruction $s_2(\pi) \in \mathbb{Z}/2$ to cobording π to a homotopy equivalence of pairs, via a $\mathbb{Z}/2$ P.D. cobordism. Specifically,

$$s_2(\pi) = (1/8)(\text{index}\quad (X_0') - \text{index}\quad (X_0) + 2\theta(\delta X' \xrightarrow{\pi} \delta X)) \in \mathbb{Z}/2$$

where $\theta(\delta X' \xrightarrow{\pi} \delta X) \in \mathbb{Z}/8$ is a difference of Arf invariants. To be more precise, $\pi: \delta X' \to \delta X$ is a degree one normal map of closed, oriented, $4n-1$ dimensional P.D. spaces. Any quadratic function $q: T^{2n}(\delta X) \to Q/\mathbb{Z}$ refining the linking pairing induces a quadratic function $q': T^{2n}(\delta X') \to Q/\mathbb{Z}$. The transfer $\tau: H^{2n}(\delta X', \mathbb{Z}) \to H^{2n}(\delta X, \mathbb{Z})$ induces an orthogonal direct sum splitting

$$T^{2n}(\delta X') \cong K^{2n} \oplus \pi^* T^{2n}(\delta X)$$

where $K^{2n} = \text{kernel } (\tau|_{T^{2n}(\delta X')}) \subset T^{2n}(\delta X')$. Then

$$\theta(\delta X' \xrightarrow{\pi} \delta X) = A(K^{2n}, q') \in \mathbb{Z}/8.$$

Since $q'\pi^*$: $T^{2n}(\delta X) \to Q/\mathbb{Z}$ is also a quadratic function on $T^{2n}(\delta X)$, we have

$$\theta(\delta X' \xrightarrow{\pi} \delta X) = A(T^{2n}(\delta X'), q') - A(T^{2n}(\delta X), q'\pi^*).$$

The obstruction $s_2(\pi)$ for normal maps of *manifolds* π: $M' \to M$ plays a key role in the study of the spaces G/TOP and BTOP, localized at the prime 2, [4], [16], [17]. In particular, s_2 satisfies a certain product formula, which enables one to define *canonical* $\mathbb{Z}_{(2)}$ characteristic classes $k_{4n} \in H^{4n}(G/TOP, \mathbb{Z}_{(2)})$, which are used to split the $\mathbb{Z}_{(2)}$-localization of G/TOP into a product of Eilenberg-MacLane spaces. The most difficult case of this product formula is the following. Let π: $N^{4a-1} \to M^{4a-1}$ be a degree one normal map of $\mathbb{Z}/2$ P.D. spaces of dimension $4a - 1$ and let L^{4b+1} be a $\mathbb{Z}/2$ P.D. space of dimension $4b + 1$. Consider the product

$$\pi \times Id: N \times L \to M \times L.$$

Then $\pi|_{\delta N}: \delta N \to \delta M$ is a normal map of oriented P.D. spaces of dimension $4a - 2$, hence has a Kervaire obstruction $A(\delta N \to \delta M) \in \mathbb{Z}/2$. We claim

1.3.3 $$s_2(\pi \times Id) = A(\delta N \to \delta M) \langle v_{2b}(L) Sq^1 v_{2b}(L), [L] \rangle \in \mathbb{Z}/2.$$

Formula 1.3.3 was proved geometrically for (PL) manifolds in [17] and for differentiable manifolds in [16]. Most of the machinery necessary for a homotopy theoretic proof of 1.3.3 was developed in [16]. We indicate here how the techniques of this paper can be used to complete a homotopy theoretic proof of 1.3.3, which, in particular, extends the surgery results of [16], [17] to P.D. spaces.

It is only necessary to establish 1.3.3 in the following special case. Let K^{4a-2} be the Kervaire manifold, obtained by plumbing together two tangent disc bundles of S^{2a-1}. K^{4a-2} has an orientation reversing involution $t: K^{4a-2} \to K^{4a-2}$, interchanging the two disc bundles. One can easily find an orientation reversing involution t: $S^{4a-2} \to S^{4a-2}$ and a degree one normal map, π: $K^{4a-2} \to S^{4a-2}$ which commutes with t.

We thus obtain a normal map of $\mathbb{Z}/2$ manifolds of dimension $4a - 1$, by mapping the "Kervaire Klein bottle" $\tilde{K}$ to the ordinary Klein bottle $\tilde{S}$. That is, we construct

$$\tilde{K} = K^{4a-2} \times I/(y, 0) \sim (ty, 1)$$
$$\tilde{S} = S^{4a-2} \quad I/(x, 0) \sim (tx, 1)$$

and

$$\tilde{\pi} = \pi \times Id/\sim: \tilde{K} \to \tilde{S}.$$

By the construction $A(\delta\tilde{K} \xrightarrow{\pi} \delta\tilde{S}) = A(K^{4a-2} \xrightarrow{\pi} S^{4a-2}) = 1 \in \mathbb{Z}/2$.

Let L^{4b+1} be any $\mathbb{Z}/2$ P.D. space of dimension $4b + 1$. Consider $\tilde{\pi} \times Id$: $\tilde{K}^{4a-2} \times L^{4b+1} \to \tilde{S}^{4a-1} \times L^{4b+1}$. 1.3.3 reduces to

$$s_2\,(\tilde{\pi} \times Id) = \langle v_{2b}\, Sq^1\, v_{2b}\,(L), [L]\rangle \in \mathbb{Z}/2.$$

Moreover, in the formula for $s_2(\tilde{\pi} \times Id)$, the index terms vanish, that is,

$$0 = \text{index}\,(K^{4a-1} \times I \times L_0^{4b+1}) = \text{index}\,(S^{4a-2} \times I \times L_0^{4b+1}).$$

Thus 1.3.3 reduces to

$$\theta(\delta(\tilde{K} \times L) \xrightarrow{\tilde{\pi} \times Id} \delta(\tilde{S} \times L)) = 4\,\langle v_{2b}Sq^1v_{2b}(L), [L]\rangle \in \mathbb{Z}/8.$$

Now, the Bocksteins $\delta(\tilde{K} \times L)$ and $\delta(\tilde{S} \times L)$ are double covered by $K \times \tilde{L}$ and $S \times \tilde{L}$, respectively, where $\tilde{L}$ is the orientation double cover of L. Specifically, the involution $t \times \hat{t}$ on $K \times \tilde{L}$, where $L = \tilde{L}/\hat{t}$, gives

$$\delta(\tilde{K} \times L) = K \times \tilde{L}/t \times \hat{t},$$

and similarly $\delta(\tilde{S} \times L) = S \times \tilde{L}/t \times \hat{t}$. We thus get a diagram

$$\begin{array}{ccc} K \times \tilde{L} & \to & S \times \tilde{L} \\ \downarrow & & \downarrow \\ \delta(\tilde{K} \times L) & \to & \delta(\tilde{S} \times L) \end{array}$$

where the vertical maps are double covers.

Let $a \in T^*(K \times \tilde{L})$ be in the middle dimensional kernel of $T^*(K \times \tilde{L}) \to T^*(S \times \tilde{L})$. Then $\tau(a) \in T^*(\delta(\tilde{K} \times L))$ is in the middle dimensional kernel of $T^*(\delta(\tilde{K} \times L)) \to T^*(\delta(\tilde{S} \times L))$. A quadratic function q: $T^*(\delta(\tilde{K} \times L)) \to Q/\mathbb{Z}$ induces q': $T^*(K \times \tilde{L}) \to Q/\mathbb{Z}$. But now our chain formula 4.2.2 enables us to compute $\tilde{q}(a) = q\tau(a) - q'(a)$ and the more standard product formulas of [16] compute $q'(a)$ since $T^*(K \times \tilde{L})$ is easy to describe. Combining these results gives a computation of $q\tau(a)$. This provides precisely enough algebraic information to evaluate the difference of Arf invariants $\Theta(\delta(\tilde{K} \times L) \to \delta(\tilde{S} \times L))$ and prove 1.3.3. However, because of the length of the argument, we restrict our discussion in this paper to this outline.

1.4. *Definitions and basic properties of quadratic functions*

In this section we collect the algebraic results we need later in the paper. For the most part we do not give proofs, but refer to the papers [4], [5], [16].

Let K be a finite abelian group, and let ℓ: $K \times K \to Q/\mathbb{Z}$ be a non-singular, symmetric, bilinear pairing. Non-singular means that the associated map $K \to K^* = \text{Hom}\,(K, Q/\mathbb{Z})$ is an isomorphism; equivalently, if $\ell(x, y) = 0$ for all y, then $x = 0$.

Definition 1.4.1. A quadratic function refining ℓ is a function q: $K \to Q/\mathbb{Z}$ with the properties:

(a) $q(x + y) = q(x) + q(y) + \ell(x, y)$, $x, y \in K$

(b) $q(nx) = n^2q(x)$, $n \in \mathbb{Z}$, $x \in K$.

It follows from (a) that $q(0) = 0$. Also, given (a), condition (b) is equivalent to either of the conditions

(b') $2q(x) = \ell(x, x)$, $x \in K$ or

(b'') $q(-x) = q(x)$, $x \in K$.

Suppose q and q' are two quadratic functions refining ℓ. Then (a) implies $q' - q$: $K \to Q/\mathbb{Z}$ is linear, and (b') implies $2(q' - q) = 0$. It can be shown that any non-singular, symmetric pairing, ℓ, on K can indeed be refined to a quadratic function, q. Any other such quadratic function, q', thus has the form

$$q'(x) = q(x) + \ell(x, y)$$

for some $y \in K$ with $2y = 0$. We denote by q_y this perturbation of q by $y \in K$.

Let q: $K \to Q/\mathbb{Z}$ and q': $K' \to Q/\mathbb{Z}$ be two quadratic functions, refining pairings ℓ and ℓ' respectively. Define $q \oplus q' : K \oplus K' \to Q/\mathbb{Z}$ by $q \oplus q'(x, x') = q(x) + q'(x')$ and define $-q$: $K \to Q/\mathbb{Z}$ by $(-q)(x) = -q(x)$, for $x \in K$, $x' \in K'$. Then $q \oplus q'$ refines the direct sum pairing $\ell \oplus \ell'$ on $K \oplus K'$ and $-q$ refines the pairing $-\ell$ on K.

Given a quadratic function q: $K \to Q/\mathbb{Z}$, consider the complex number

$$a(K, q) = \sum_{x \in K} e^{2\pi i q(x)} \in \mathbb{C}.$$

The following is elementary. [4].

PROPOSITION 1.4.2. (a) $a(K, -q) = \overline{a(K, q)}$, ($\overline{}$ *denotes the complex conjugate.*)

(b) $a(K \oplus K', q \oplus q') = a(K, q) \cdot a(K, q')$

(c) $a(K, q) = a(K, q_y) \cdot e^{2\pi i q(y)}$, ($y \in K$ and $2y = 0$).

It is, in fact, a theorem that $\arg(a(K, q))$ is always an eighth root of unity, [16], and (see 1.4.6 below) that $\|a(K, q\| = |K|^{1/2}$, where $|K|$ is the order of the finite abelian group K. We thus define $A(K, q) \in \mathbb{Z}/8$ by the equation

1.4.3 $$a(K, q) = |K|^{1/2} e^{A(K,q)\pi i/4}$$

We call $A(K, q) \in \mathbb{Z}/8$ the Arf invariant of q.

Let $q : K \to Q/\mathbb{Z}$ be a quadratic function refining $\ell : K \times K \to Q/\mathbb{Z}$ and let $i : L \to K$ be a homomorphism of finite abelian groups. Then i has an adjoint $i^* : K \to L^* = \mathrm{Hom}(L, Q/\mathbb{Z})$, defined by $\langle i^*(y), x\rangle = \ell(y, i(x))$ for $x \in L$, $y \in K$. Suppose that in the sequence

1.4.4. $$L \xrightarrow{i} K \xrightarrow{i^*} L^*$$

we have $i^* \circ i = 0$; equivalently, $\ell(i(x), i(x')) = 0$ for all $x, x' \in L$. Then ℓ induces a non-singular, symmetric, bilinear pairing $\bar{\ell} : \bar{K} \times \bar{K} \to Q/\mathbb{Z}$, where $\bar{K} = \mathrm{Kernel}(i^*)/\mathrm{Image}(i)$ is the homology group of the sequence 1.4.4. Namely, if $z, z' \in \mathrm{Kernel}(i^*)$, set

$$\bar{\ell}(z + i(L), z' + i(L)) = \ell(z, z') \in Q/\mathbb{Z}.$$

Moreover, $q \circ i : L \to Q/\mathbb{Z}$ is linear and $2q \circ i = 0$. Thus there is a unique element $y^* \in L^*$ of order 2 defined by $\langle y^*, x\rangle = qi(x)$, $x \in L$. If $x \in \text{Kernel}(i)$ then $\langle y^*, x\rangle = 0$. It follows by considering the dual sequences $0 \to \text{Kernel}(i) \to L \to K$ and $K = K^* \to L^* \to (\text{Kernel}(i))^* \to 0$ that $y^* = i^*(y)$ for some $y \in K$. The following is proved in [5].

PROPOSITION 1.4.5. (a) *$y \in K$ may be chosen such that* $2y \in \text{Image}(L)$

(b) $\bar{q}_y : \bar{K} \to Q/\mathbb{Z}$ *defined by* $\bar{q}_y(z) = q(z) - \ell(z, y) \in Q/\mathbb{Z}$, $z \in \text{Kernel}(i^*)$, *is a quadratic function refining* $\bar{\ell} : \bar{K} \times \bar{K} \to Q/\mathbb{Z}$.

(c) $a(K, q) = \left(\dfrac{|K|}{|\bar{K}|}\right)^{1/2} a(\bar{K}, \bar{q}_y)\, e^{2\pi i q(y)} \in \mathbb{C}$.

Remark 1.4.6. If $L \subset K$, $|L| = |K|^{1/2}$ and $q|_L = 0$ then in the situation above we have $\bar{K} = 0$, $y = 0$, and 1.4.5(c) states $a(K, q) = |K|^{1/2}$. For example, if $q : K \to Q/\mathbb{Z}$ is any quadratic function, form $q \oplus -q : K \oplus K \to Q/\mathbb{Z}$. Let $L = \Delta K = \{(x, x) | x \in K\} \subset K \oplus K$. By 1.4.2 and 1.4.5, $\|a(K, q)\|^2 = a(K \oplus K, q \oplus -q) = |K \oplus K|^{1/2} = |K|$.

There are various simplifications of the theory above if the group K is a vector space over $\mathbb{Z}/2$, that is, if $2x = 0$, all $x \in K$. In fact, in the context of the Arf invariant, this was the case first investigated, particularly by E. H. Brown, Jr., [2], [3].

First, $2\ell(x, y) = 0$ all $x, y \in K$ and, secondly, $4q(x) = 0$, all $x \in K$. Here $\ell : K \times K \to Q/\mathbb{Z}$ is a bilinear pairing and $q : K \to Q/\mathbb{Z}$ is a quadratic refinement. In fact, condition 1.4.1(a), $q(x + y) = q(x) + q(y) + \ell(x, y)$ implies $0 = q(2x) = 2q(x) + \ell(x, x)$. Hence condition 1.4.1(b) is redundant if $2K = 0$.

Brown's notation is somewhat different from ours. He writes $q : K \to \mathbb{Z}/4$ instead of $q : K \to Q/\mathbb{Z}$. (That is, he multiplies by 4). If $\ell : K \times K \to \mathbb{Z}/2$ is the bilinear pairing, the quadratic condition 1.4.1(a) is written $q(x + y) = q(x) + q(y) + 2\ell(x, y) \in \mathbb{Z}/4$.

Let $v \in K$ be the *Wu* class of the pairing ℓ; that is $\ell(v, x) = \ell(x, x)$ for all $x \in K$. Then $2q = 0$ if and only if $v = 0$.

If $v = 0$, we will write $\ell : K \times K \to \mathbb{Z}/2$, $q : K \to \mathbb{Z}/2$, rather than $q : K \to Q/\mathbb{Z}$. (That is, we multiply by 2.) Our notation then agrees with the classical notation. In this case (K, ℓ) has a symplectic basis $\{x_1 \cdots x_n, y_1 \cdots y_n\}$. That is, $\ell(x_i, x_j) = 0$, $\ell(y_i, y_j) = 0$, $\ell(x_i, y_j) = \delta_{ij}$. Thus (K, ℓ) splits as an orthogonal direct sum,

$$K = \bigoplus_{i=1}^{n} \{x_i, y_i\}.$$

If $q : K \to \mathbb{Z}/2$ is a quadratic refinement of ℓ, we have from 1.4.2 and 1.4.3,

1.4.7. $$A(K, q) = 4 \sum_{i=1}^{n} q(x_i)q(y_i) \in \mathbb{Z}/8.$$

2. Quadratic functions on the cohomology of P.D. spaces

2.1. *$\mathbb{Z}/2$ coefficients*

The idea of using results from stable homotopy theory to define quadratic functions on the cohomology of P.D. spaces is due to E. H. Brown, Jr. [2]. We first outline his approach to studying quadratic functions $H^n(X^{2n}, \mathbb{Z}/2) \to \mathbb{Z}/4$ refining the cup product pairing, where X^{2n} is a $2n$ dimensional space satisfying Poincaré duality with $\mathbb{Z}/2$ coefficients.

Let $K(\mathbb{Z}/2, n)$ be the Eilenberg-MacLane space. Then $H^n(X, \mathbb{Z}/2) = [X, K(\mathbb{Z}/2, n)]$. We consider the suspension map

$$s : [X, K(\mathbb{Z}/2, n)] \to \{X, K(\mathbb{Z}/2, n)\} = \lim_{q\to\infty} [\Sigma^q X^{2n}, \Sigma^q K(\mathbb{Z}/2, n)].$$

The following is proved in [2].

PROPOSITION 2.1.1. (a) $\{S^{2n}, K(\mathbb{Z}/2, n)\} = \mathbb{Z}/2$. *A map* $f : S^{q+2n} \to \Sigma^q K(\mathbb{Z}/2, n)$ *is essential if and only if the functional operation* $Sq_f^{n+1}(\Sigma^q\iota_n)$ *is non-zero in the group*

$$\mathbb{Z}/2 = H^{q+2n+1}\left(\Sigma^q K(\mathbb{Z}/2, n) \bigcup_f e^{q+2n+1}, \Sigma^q K(\mathbb{Z}/2, n), \mathbb{Z}/2\right)$$
$$\subset H^{q+2n+1}\left(\Sigma^q K(\mathbb{Z}/2, n) \bigcup_f e^{q+2n+1}, \mathbb{Z}/2\right).$$

(b) *If* X^{2n} *is a connected* $2n$ *dimensional* P.D. *space, there is an exact sequence*

$$0 \to \mathbb{Z}/2 \xrightarrow{i^*} \{X^{2n}, K(\mathbb{Z}/2, n)\} \xrightarrow{ev(\Sigma\iota_n)} H^n(X^{2n}, \mathbb{Z}/2) \to 0,$$

where

$$i^* : \{S^{2n}, K(\mathbb{Z}/2, n)\} = \mathbb{Z}/2 \to \{X^{2n}, K(\mathbb{Z}/2, n)\}$$

is induced by a degree one map $X^{2n} \to S^{2n}$ *and* $ev(\Sigma\iota_n)(f) = f^*(\Sigma^q\iota_n) \in H^{q+n}(\Sigma^q X^{2n}, \mathbb{Z}/2) \simeq H^n(X^{2n}, \mathbb{Z}/2), f : \Sigma^q X^{2n} \to \Sigma^q K(\mathbb{Z}/2, n)$.

(c) *The suspension* $s = H^n(X^{2n}, \mathbb{Z}/2) \to \{X^{2n}, K(\mathbb{Z}/2, n)\}$ *satisfies* $ev(\Sigma\iota_n)\ (s(x)) = x$ *and* $s(x + y) = s(x) + s(y) + i^*\langle x \cdot y, [X^{2n}]\rangle \in \{X^{2n}, K(\mathbb{Z}/2, n)\}$, $x, y \in H^n(X, \mathbb{Z}/2)$.

It follows easily from 2.1.1 that quadratic functions $q : H^n(X^{2n}, \mathbb{Z}/2) \to \mathbb{Z}/4$ refining the cup product pairing correspond bijectively with linear functions $\psi : \{X^{2n}, K(\mathbb{Z}/2, n)\} \to \mathbb{Z}/4$ which satisfy $\psi \cdot i^* = 2 : \mathbb{Z}/2 \to \mathbb{Z}/4$. The correspondence is defined by $q = \psi \circ s$.

2.2. *$Q/\mathbb{Z}$ coefficients*

We next consider oriented P.D. spaces of dimension $4n - 1$, X^{4n-1}. Let $T^{2n}(X)$ denote the torsion subgroup of $H^{2n}(X, \mathbb{Z})$. The coefficient sequence $0 \to \mathbb{Z} \to Q \to Q/\mathbb{Z} \to 0$ yields a natural isomorphism

$$\beta : H^{2n-1}(X, Q/\mathbb{Z})/\text{Image}(H^{2n-1}(X, Q)) \simeq T^{2n}(X) \subset H^{2n}(X, \mathbb{Z}).$$

Moreover, the linking pairing is identified with the pairing

$$\ell(x, y) = \langle x \cdot \beta y, [X]\rangle \in Q/\mathbb{Z}.$$

Here, $x, y \in H^{2n-1}(X, Q/\mathbb{Z})$ and the product $x \cdot \beta y$ corresponds to the pairing $Q/\mathbb{Z} \otimes \mathbb{Z} \to Q/\mathbb{Z}$ of coefficient groups.

We seek quadratic functions $T^{2n}(X) \to Q/\mathbb{Z}$ refining ℓ. Let

$$s : H^{2n-1}(X, Q/\mathbb{Z}) = [X, K(Q/\mathbb{Z}, 2n-1)] \to \{X, K(Q/\mathbb{Z}, 2n-1)\}$$

denote suspension. The following analogue of 2.1.1 is proved in [4].

PROPOSITION 2.2.1. (a) $\{S^{4n-1}, K(Q/\mathbb{Z}, 2n-1)\} = Q/\mathbb{Z}$. *This identification can be chosen such that if* $f: S^{q+4n-1} \to \Sigma^q K(Q/\mathbb{Z}, 2n-1)$ *represents* $\alpha \in Q/\mathbb{Z}$, *then* $2\alpha = \langle f^*(\Sigma^q(\iota \cdot \beta\iota)), [S^{q+4n-1}]\rangle \in Q/\mathbb{Z}$. *Here* $\iota \in H^{2n-1}(K(Q/\mathbb{Z}, 2n-1), Q/\mathbb{Z})$ *is the fundamental class and* $\iota \cdot \beta\iota \in H^{4n-1}(K(Q/\mathbb{Z}, 2n-1), Q/\mathbb{Z})$.

(b) *If* X^{4n-1} *is an oriented, connected,* P.D. *space of dimension* $4n-1$ *then there is an exact sequence*

$$0 \to Q/\mathbb{Z} \xrightarrow{i^*} \{X^{4n-1}, K(Q/\mathbb{Z}, 2n-1)\} \xrightarrow{ev(\Sigma i)} H^{2n-1}(X, Q/\mathbb{Z}) \to 0.$$

Here i^* *is induced by a degree one map* $X^{4n-1} \to S^{4n-1}$ *and*

$$ev(\Sigma\iota)(f) = f^*(\Sigma^q\iota) \in H^{q+2n-1}(\Sigma^q X^{4n-1}, Q/\mathbb{Z}) = H^{2n-1}(X^{4n-1}, Q/\mathbb{Z})$$

where $f: \Sigma^q X^{4n-1} \to \Sigma^q K(Q/\mathbb{Z}, 2n-1)$.

(c) *The suspension* $s : H^{2n-1}(X, Q/\mathbb{Z}) \to \{X, K(Q/\mathbb{Z}, 2n-1)\}$ *satisfies* $ev(\Sigma\iota)(s(y)) = y \in H^{2n-1}(X, Q/\mathbb{Z})$ *and*

$$s(x+y) = s(x) + s(y) + i^*\langle x \cdot \beta y, [X^{4n-1}]\rangle \in \{X, K(Q/\mathbb{Z}, 2n-1)\}, \; x, y \in H^{2n-1}(X, Q/\mathbb{Z}).$$

It follows from 2.2.1 that quadratic functions $q : T^{2n}(X) \to Q/\mathbb{Z}$ refining the linking pairing ℓ, correspond to linear functions $\psi = \{X, K(Q/\mathbb{Z}, 2n-1)\} \to Q/\mathbb{Z}$ which satisfy the three conditions $\psi \circ i^* = Id : Q/\mathbb{Z} \to Q/\mathbb{Z}$, $2\psi = ev(\Sigma(\iota \cdot \beta\iota))$, and $\psi(\mathrm{Image}(\{X, K(Q, 2n-1)\} \to \{X, K(Q/\mathbb{Z}, 2n-1)\})) = 0$. The correspondence is defined by $q = \psi \circ s$, together with the identification of $T^{2n}(X)$ with $H^{2n-1}(X, Q/\mathbb{Z})/\mathrm{Image}(H^{2n-1}(X, Q))$. (It is also proved in [4] that such linear ψ exist.)

3. Normal maps

3.1 *Definitions*

Let $QS^0 = \lim_{q \to \infty} \Omega^q S^q$, and let $\infty \in S^q$ be the basepoint. If X is a space, a map $\varphi: X \to \Omega^q S^q$ has an adjoint $X \times (S^q, \infty) \to X \times (S^q, \infty)$; equivalently, we regard the adjoint of φ as a map $\mathrm{Ad}(\varphi): \Sigma^q X_+ \to \Sigma^q X_+$, where X_+ is the union of X with a disjoint point. In particular, $\pi_0(QS^0) = \mathbb{Z}$, and, if X is connected, a map $\varphi: X \to QS^0$ has a degree, which can be computed as the degree of $\mathrm{Ad}(\varphi)|_{S^q}: S^q \to S^q$, where $S^q \subset \Sigma^q X_+$ is the natural inclusion.

Remark 3.1.1. For later use we point out that there is an obvious suspension isomorphism, (with any coefficients) $\Sigma^q\colon H^i(X) \simeq H^{q+i}(\Sigma^q X_+)$, $i \geq 0$, which is essentially cup product with the generator of $H^q(S^q, \mathbb{Z})$. It follows that the map $(Ad(\varphi))^* : H^*(\Sigma^q X_+) \to H^*(\Sigma^q X_+)$ is multiplication by degree $(\varphi) \in \mathbb{Z}$.

Let X' and X be n-dimensional P.D. spaces. The usual definition of a normal map from X' to X is a diagram

$$\begin{array}{ccc} \nu^q_{X'} & \xrightarrow{\hat{\pi}} & \nu^q_X \\ \downarrow & & \downarrow \\ X' & \xrightarrow{\pi} & X \end{array}$$

where $\nu^q_{X'}$ and ν^q_X are the Spivak normal spherical fibrations of X' and X, and $\hat{\pi}\colon \nu^q_{X'} \to \nu^q_X$ is a bundle map covering $\pi\colon X' \to X$. S-duality Theory yields maps $\Sigma^q X_+ \to \Sigma^q X'_+$, dual to the map of Thom spaces $T\pi\colon T\nu^q_{X'} \to T\nu^q_X$. However, duality depends on choices of degree one maps $S^{q+n} \to T\nu^q_{X'}$ and $S^{q+n} \to T\nu^q_X$; there is no canonical way to obtain $\Sigma^q X_+ \to \Sigma^q X'_+$ from $\hat{\pi}\colon \nu^q_{X'} \to \nu^q_X$.

In the covering space situations studied in this paper, we arrive very naturally at triples of maps $\pi\colon X' \to X$, $D\pi\colon \Sigma^q X_+ \to \Sigma^q X'_+$, $\varphi\colon X \to QS^0$, related in various ways. It thus seems more natural just to define normal maps in terms of this data, and to suppress $\nu^q_{X'} \to \nu^q_X$ altogether.

We thus make the following definition. X' and X are P.D. spaces of dimension n, both oriented or both unoriented. We have fundamental classes $[X'] \in H_n(X')$, $[X] \in H_n(X)$, with $\mathbb{Z}$ or $\mathbb{Z}/2$ coefficients, depending on orientability. We assume X is connected, but not necessarily X'.

Definition 3.1.2. A normal map from X' to X consists of a triple of maps $\pi\colon X' \to X$, $D\pi\colon \Sigma^q X_+ \to \Sigma^q X'_+$, $\varphi\colon X \to QS^0$ such that

(a) $(D\pi)_* \Sigma^q[X] = \Sigma^q[X'] \in H_{q+n}(\Sigma^q X'_+)$,

(b) The composition $\Sigma^q \pi \circ D\pi\colon \Sigma^q X_+ \to \Sigma^q X'_+ \to \Sigma^q X_+$ is homotopic to $Ad(\varphi)\colon \Sigma^q X_+ \to \Sigma^q X_+$,

(c) Let $\tau = (\Sigma^q)^{-1}(D\pi)^*\Sigma^q : H^*(X') \to H^{q+*}(\Sigma^q X'_+) \to H^{q+*}(\Sigma^q X_+) \to H^*(X)$. Then for all $a \in H^*(X')$, $b \in H^*(X)$, we require $\tau(a \cdot \pi^* b) = \tau a \cdot b \in H^*(X)$, whenever the coefficients are such that these products are defined.

We refer to the map $\tau\colon H^*(X') \to H^*(X)$ of (c) as the *transfer map* of the normal map $(\pi, D\pi, \varphi)$.

Remark 3.1.3. It follows from 3.1.1 and 3.1.2(a), (b) that on the bottom cell, the degree of $D\pi\colon \Sigma^q X_+ \to \Sigma^q X'_+$ is equal to degree (φ), and that on the top cell $\pi\colon X' \to X$ has degree equal to degree (φ), that is, $\pi_*[X'] =$ degree $(\varphi)[X]$. (This last statement is interpreted modulo 2 if X' and X are unoriented.)

3.2 *Quadratic functions and normal maps*

We begin this subsection with a reminder that in §2 we outlined a homotopy theoretic construction of quadratic functions $H^n(X^{2n}, \mathbb{Z}/2) \to \mathbb{Z}/4$ or $T^{2n}(X^{4n-1}) \to Q/\mathbb{Z}$ for *connected* P.D. spaces X. The connectedness assumption is no real restriction. If $X = \bigcup_{\text{components}} X_i$, then each X_i is a P.D. space. A function $q\colon H^n(X^{2n}, \mathbb{Z}/2) \to \mathbb{Z}/4$ is quadratic over the cup product pairing if the restrictions to the components $q_i\colon H^n(X_j^{2n}, \mathbb{Z}/2) \to \mathbb{Z}/4$ are quadratic. A similar statement holds in the $4n-1$ dimensional, oriented case.

Suppose X' and X are (unoriented) P.D. spaces of dimension $2n$. Let $\pi\colon X' \to X$, $D\pi\colon \Sigma^q X_+ \to \Sigma^q X'_+$, $\varphi\colon X \to QS^0$ be a normal map. Suppose given a linear homomorphism $\psi\colon \{X^{2n}, K(\mathbb{Z}/2, n)\} \to \mathbb{Z}/4$ such that $\psi i^*(1) = 2 \in \mathbb{Z}/4$, where $i^*\colon \mathbb{Z}/2 = \{S^{2n}, K(\mathbb{Z}/2, n)\} \to \{X^{2n}, K(\mathbb{Z}/2, n)\}$ is as in 2.1.1(b). Then for each component X'_i of X', the composition $\psi' = \psi \circ (D\pi)^*: \{X'_i, K(\mathbb{Z}/2, n)\} \to \{X, K(\mathbb{Z}/2, n)\} \to \mathbb{Z}/4$ is also linear and satisfies $\psi' i^*(1) = 2$, since, by 3.1.2(a), $(D\pi)_* \Sigma^q[X] = \bigcup_{\text{components}} \Sigma^q[X'_i]$.

We thus see that a choice of a quadratic function $q\colon H^n(X, \mathbb{Z}/2) \to \mathbb{Z}/4$ determines a quadratic function $q'\colon H^n(X', \mathbb{Z}/2) \to \mathbb{Z}/4$. Namely, if $q = \psi \circ s\colon [X, K(\mathbb{Z}/2, n)] \to \{X, K(\mathbb{Z}/2, n)\} \to \mathbb{Z}/4$, then $q' = \psi' \circ s\colon [X'_i, K(\mathbb{Z}/2, n)] \to \{X'_i, K(\mathbb{Z}/2, n)\} \to \mathbb{Z}/4$ on each component X'_i of X'.

If X' and X are oriented P.D. spaces of dimension $4n-1$ and $\psi\colon \{X, K(Q/\mathbb{Z}, 2n-1)\} \to Q/\mathbb{Z}$ is a linear function which satisfies the three conditions stated at the end of 2.2, then an argument just like that above gives that for each component X'_i of X', $\psi' = \psi \circ (D\pi)^* : \{X'_i, K(Q/\mathbb{Z}, 2n-1)\} \to \{X, K(Q/\mathbb{Z}, 2n-1)\} \to Q/\mathbb{Z}$, also satisfies these three conditions. Thus, a choice of a quadratic function $q\colon T^{2n}(X) \to Q/\mathbb{Z}$ refining the linking pairing induces a quadratic function $q'\colon T^{2n}(X') \to Q/\mathbb{Z}$.

In the following two subsections we will study these induced quadratic functions, first, for normal maps $X' \to X$ of odd degree with X' and X $2n$ dimensional, unoriented P.D. spaces, and secondly, for normal maps of degree two, in both the unoriented case and the $4n-1$ dimensional, oriented case.

3.3 *Normal maps of odd degree*

Let $\pi\colon X' \to X$, $D\pi\colon \Sigma^q X_+ \to \Sigma^q X'_+$, $\varphi\colon X \to QS^0$ be a normal map of odd degree, where X' and X are $2n$-dimensional P.D. spaces with $\mathbb{Z}/2$ coefficients. Since $\Sigma^q \pi \circ D\pi \sim Ad(\varphi)$, we see from 3.1.1 that the composition

$$H^{q+i}(\Sigma^q X_+, \mathbb{Z}/2) \xrightarrow{(\Sigma^q\pi)^*} H^{q+i}(\Sigma^q X'_+, \mathbb{Z}/2) \xrightarrow{(D\pi)^*} H^{q+i}(\Sigma^q X_+, \mathbb{Z}/2)$$

is the identity. Thus $H^*(X', \mathbb{Z}/2) \simeq K^* \oplus \pi^* H^*(X, \mathbb{Z}/2)$, where $K^* = \text{Kernel}(\tau) \subset H^*(X', \mathbb{Z}/2)$, $\tau : H^*(X', \mathbb{Z}/2) \to H^*(X, \mathbb{Z}/2)$, the transfer. Moreover, if $a \in K^i$, $b \in H^{2n-i}(X, \mathbb{Z}/2)$, then by 3.1.2(a), (c),

$$\begin{aligned} \langle a \cdot \pi^*(b), [X'] \rangle &= \langle \tau(a \cdot \pi^*(b), [X] \rangle \\ &= \langle \tau a \cdot b, [X] \rangle = 0. \end{aligned}$$

Thus the direct sum splitting $H^*(X', \mathbb{Z}/2) = K^* \oplus \pi^* H^*(X, \mathbb{Z}/2)$ is an orthogonal splitting for the cup pairing on $H^*(X', \mathbb{Z}/2)$.

We have seen in 3.2 that a quadratic function $q\colon H^*(X, \mathbb{Z}/2) \to \mathbb{Z}/4$ induces a quadratic function $q'\colon H^n(X', \mathbb{Z}/2) \to \mathbb{Z}/4$. q' will be described if we can describe the restrictions of q' to $K^n \subset H^n(X', \mathbb{Z}/2)$ and to $\pi^*H^n(X, \mathbb{Z}/2) \subset H^n(X', \mathbb{Z}/2)$.

Note that if $a \in K^n$, then in the composition

$$\Sigma^q X_+ \xrightarrow{D\pi} \Sigma^q X'_+ \xrightarrow{s(a)} \Sigma^q K(\mathbb{Z}/2, n)$$

we have $(D\pi)^* s(a)^*(\Sigma^q \iota_n) = 0$. Thus the functional operation $Sq^{n+1}_{s(a)\circ D\pi}(\Sigma^q \iota_n)$ is defined and has its value in the subgroup

$$\mathbb{Z}/2 = H^{q+2n+1}(\Sigma^q K(\mathbb{Z}/2, n) \bigcup_{S(a)\circ D\pi} \text{Cone}(\Sigma^q X_+), \Sigma^q K(\mathbb{Z}/2, n); \mathbb{Z}/2)$$

$$\subset H^{q+2n+1}(\Sigma^q K(\mathbb{Z}/2, n) \bigcup_{S(a)\circ D\pi} \text{Cone}(\Sigma^q X_+), \mathbb{Z}/2),$$

with zero indeterminancy, (since $Sq^{n+1} = 0$: $H^{n-1}(X, \mathbb{Z}/2) \to H^{2n}(X, \mathbb{Z}/2)$).

PROPOSITION 3.3.1. *If $a \in K^n \subset H^n(X', \mathbb{Z}/2)$, then $q'(a) = Sq^{n+1}_{s(a)\circ D\pi}(\Sigma^q \iota_n) \in \mathbb{Z}/2$.*

Proof. This is immediate from the definition of q' in 3.2 together with Prop. 2.1.1.

Next, observe that the degree one component of QS^0 is the space SG. There is the natural map $\Sigma SG \to BSG$, hence there are "suspended *Wu* classes" $\sigma(v_i) \in H^{i-1}(SG, \mathbb{Z}/2)$. We will extend these characteristic classes to arbitrary maps of odd degree, $\varphi: X \to QS^0$.

Begin with $Ad(\varphi): X \times (S^q, \infty) \to X \times (S^q, \infty)$. Form the mapping torous $M_\varphi = X \times S^q \times I/(x, y, 0) \equiv (Ad(\varphi)(x, y), 1)$, which is the analogue of a Hurewicz S^q-fibration with a section, $X \times \infty \times S^1$ over $X \times S^1$. Finally collapse $X \times S^q \times 0 \cup X \times \infty \times S^1 \subset M\varphi$ to a point, obtaining a space $T\varphi$. If degree$(\varphi) = 1$, $T\varphi$ is the Thom space of a spherical fibration over $\Sigma^1 X_+$. Since degree (φ) is odd, $H^q(T\varphi, \mathbb{Z}/2) = \mathbb{Z}/2$, with a generator U, and for $i \geq 1$, there are isomorphisms

$$H^{i-1}(X, \mathbb{Z}/2) \xrightarrow{\Sigma} H^i(\Sigma^1 X_+, \mathbb{Z}/2) \xrightarrow{\Phi} H^{q+i}(T\varphi, \mathbb{Z}/2)$$

where Φ is cup product with U. Define $\varphi^*\sigma(v_i) \in H^{i-1}(X, \mathbb{Z}/2)$ by the formula $\varphi^*\sigma(v_i) = \Sigma^{-1}\Phi^{-1}Sq^i(U)$.

Remark 3.3.2. If degree $(\varphi) = n$, n odd, it is easy to see that there is a map $S^q \bigcup_{n-1} e^{q+1} \to T\varphi$ inducing isomorphisms in $\mathbb{Z}/2$ cohomology in dimensions q and $q + 1$. It follows that

$$\varphi^*\sigma(v_1) = \begin{cases} 1 \in H^0(X, \mathbb{Z}/2) = \mathbb{Z}/2 & \text{if degree } (\varphi) \equiv 3 (\text{mod } 4) \\ 0 \in H^0(X, \mathbb{Z}/2) = \mathbb{Z}/2 & \text{if degree } (\varphi) \equiv 1 (\text{mod } 4). \end{cases}$$

We now return to the computation of the quadratic function q' restricted to $\pi^*H^n(X, \mathbb{Z}/2) \subset H^n(X', \mathbb{Z}/2)$.

PROPOSITION 3.3.3. *If* $b \in H^n(X, \mathbb{Z}/2)$, *then*

$$q'\pi^*(b) = q(b) + 2\langle b \cdot V(X)\varphi^*(\sigma V), [X]\rangle \in \mathbb{Z}/4,$$

where $\varphi^*(\sigma V) = \sum_{i=1}^{\infty} \varphi^*\sigma(v_i) \in H^*(X, \mathbb{Z}/2)$, *and* $V(X) = \sum_{i=0}^{\infty} v_i(X)$ *is the Wu class of the* P.D. *space* X.

This result is proved in [4] for maps $\varphi : X \to QS^0$ of degree one and is due to E. H. Brown, Jr. The general odd degree case is proved by an *identical* argument, if one makes use of the space $T\varphi$ and the definition of $\varphi^*\sigma(v_i)$ given above.

Remark 3.3.4. Note that if $a \in K^n$, $q'(a) \in \mathbb{Z}/2$. An explicit formula is given in 3.3.1. Thus the quadratic function $q' : K^n \to \mathbb{Z}/2$ has an Arf invariant $A(K^n, q') \in 4\mathbb{Z}/8 \subset \mathbb{Z}/8$. $A(K^n, q')$ is called the *Kervaire obstruction* of the normal map $\pi : X' \to X$ of odd degree; we occasionally denote it by $A(X' \xrightarrow{\pi} X) \in \mathbb{Z}/2$.

Suppose now that $\pi_0 : X' \to X$, $D\pi_0 : \Sigma^q X_+ \to \Sigma^q X'_+$, $\varphi_0 : X \to QS^0$ and $\pi_1 : X'' \to X'$, $D\pi_1 : \Sigma^q X'_+ \to \Sigma^q X''_+$, $\varphi_1 : X' \to QS^0$ are normal maps of odd degree n_0 and n_1 respectively. We consider the compositions $\pi = \pi_0\pi_1 : X'' \to X$ and $D\pi = D\pi_1 D\pi_0 : \Sigma^q X_+ \to \Sigma^q X''_+$. One presumes that the maps π, $D\pi$ are part of the structure of a normal map from X'' to X, of degree $n = n_0 \cdot n_1$. In any case, properties 3.1.2(a) and (c) are easily verified, and this is all one needs to establish an orthogonal decomposition with respect to cup product,

$$\begin{aligned} H^*(X'', \mathbb{Z}/2) &= K_1^* \oplus \pi_1^* H^*(X', \mathbb{Z}/2) \\ &= K_1^* \oplus \pi_1^*(K_0^* \oplus \pi_0^* H^*(X, \mathbb{Z}/2)) \\ &= K^* \oplus \pi^* H^*(X, \mathbb{Z}/2), \end{aligned}$$

where $K_1^* \subset H^*(X'', \mathbb{Z}/2)$, $K_0^* \subset H^*(X', \mathbb{Z}/2)$, and $K^* = K_1^* \oplus \pi_1^* K_0^* \subset H^*(X'', \mathbb{Z}/2)$ are the kernels of the transfer maps τ_1, τ_0, and τ associated to $(\pi_1, D\pi_1)$, $(\pi_0, D\pi_0)$ and $(\pi, D\pi)$, respectively.

Moreover, a quadratic function q: $H^n(X, \mathbb{Z}/2) \to \mathbb{Z}/4$ induces a quadratic function q': $H^n(X', \mathbb{Z}/2) \to \mathbb{Z}/4$ which, in turn, induces q'': $H^n(X'', \mathbb{Z}/2) \to \mathbb{Z}/4$. By an elementary naturality property of functional squares and 3.3.1, we have if $a \in K_1^n \subset K^n \subset H^n(X'', \mathbb{Z}/2)$, then

3.3.5. $$q''(a) = Sq^{n+1}_{s(a)D\pi_1}(\Sigma^q\iota_n) = Sq^{n+1}_{s(a)D\pi_1 D\pi_0}(\Sigma^q\iota_n) \in \mathbb{Z}/2.$$

By 3.3.3, if $b \in K_0^n \subset H^n(X', \mathbb{Z}/2)$, then

3.3.6. $$q''(\pi_1^* b) = q'(b) + \langle b \cdot V(X')\varphi_1^*(\sigma V), [X']\rangle \in \mathbb{Z}/2.$$

(Note $q'(b) \in \mathbb{Z}/2 \subset \mathbb{Z}/4$, so we forget the factor 2 on the second term in 3.3.3.)

Now, the element $y = (V(X') \cdot \varphi_1^*(\sigma V))_n \in H^n(X', \mathbb{Z}/2)$ has components in K_0^n and $\pi_0^* H^n(X, \mathbb{Z}/2)$. Specifically, $y = y_{K_0} + y_X \in K_0^n + \pi_0^* H^n(X, \mathbb{Z}/2) = H^n(X', \mathbb{Z}/2)$, where $y_{K_0} = (\pi_0^*\tau_0 - Id)y$ and $y_X = \pi_0^*\tau_0 y$. Since $b \in K_0^n$, we have $\langle by, [X']\rangle = \langle b \cdot (\pi_0^*\tau_0 - Id)y, [X']\rangle$. Also since $V(X') = \pi_0^* V(X)$, 3.1.2(c) implies $y_{K_0} = (\pi_0^*\tau_0 - Id)(V(X') \cdot \varphi_1^*(\sigma V))_n = (V(X') \cdot (\pi_0^*\tau_0 - Id)^*_{\varphi_1} \sigma V))_n$. Finally, since $K^n = K_1^n \oplus \pi_1^* K_0^n \subset H^n(X'', \mathbb{Z}/2)$, we have from 3.3.5, 3.3.6, and 1.4.2(c) a formula for the Arf invariant $A(K^n, q'')$.

3.3.7. $A(X'' \xrightarrow{\pi = \pi_0\pi_1} X) = A(X'' \xrightarrow{\pi_1} X') + A(X' \xrightarrow{\pi_0} X) + 4q'(y_{K_0}) \in \mathbb{Z}/8$

where $y_{K_0} = (V(X') \cdot (\pi_0^*\tau_0 - Id)\varphi_1^*(\sigma V))_n \in K_0^n \subset H^n(X', \mathbb{Z}/2)$ and $q'(y_{K_0}) = Sq^{n+1}_{s(y_{K_0}) \cdot D\pi_0}(\Sigma^q\lambda_n) \in \mathbb{Z}/2$.

Remark 3.3.8. We have assumed $\varphi_1 : X' \to QS^0$ has a degree, n_1. This means each component of X' maps to $Q_{n_1}S^0$ by φ_1, which guarantees that $(\pi_0^*\tau_0 - Id)\varphi_1^*(\sigma V_1) = 0 \in K_1^0 \subset H^0(X', \mathbb{Z}/2)$, even if X' is not connected. As a consequence,

$$(\pi_0^*\tau_0 - Id)\,\varphi_1^*(\sigma V) = (\pi_0^*\tau_0 - Id)\left(\sum_{j=2}^{\infty} \varphi_1^*(\sigma V_j)\right) \in \tilde{H}^*(X', \mathbb{Z}/2).$$

In particular, if $\varphi_1 : X' \to QS^0$ factors through a space with vanishing reduced $\mathbb{Z}/2$ cohomology (for example, $\varphi_1 : X' \to B(\mathbb{Z}/p) \to QS^0$, where $B(\mathbb{Z}/p)$ is the classifying space of $\mathbb{Z}/p$, p an odd prime), then the error term in 3.3.7 drops out. That is, for such a φ_1, $A(X'' \xrightarrow{\pi} X) = A(X'' \xrightarrow{\pi_1} X') + A(X' \xrightarrow{\pi_0} X)$.

Remark 3.3.9. To make $\pi : X'' \to X$, $D\pi : \Sigma^q X_+ \to \Sigma^q X''_+$ into a normal map, where $\pi = \pi_0\pi_1$ and $D\pi = D\pi_1 D\pi_0$, we need an appropriate map $\varphi : X \to QS^0$, constructed from $\varphi_0 : X \to QS^0$ and $\varphi_1 : X' \to QS^0$. The map $D\pi_0 : \Sigma^q X_+ \to \Sigma^q X'_+$ has an adjoint $Ad(D\pi_0) : X \to \Omega^q\Sigma^q X'_+ \subset QX'_+$. Since Q is a functor and since there is a natural map $\alpha : QQS^0 \to QS^0$, we can consider

$$X \xrightarrow{Ad(D\pi_0)} QX'_+ \xrightarrow{Q\varphi_1} QQS^0 \xrightarrow{\alpha} QS^0$$

(α: $QQS^0 \to QS^0$ is $\Omega^q\Sigma^q\Omega^q\Sigma^q(S^0) = \Omega^q[(\Sigma^q\Omega^q)\Sigma^q(S^0)] \to \Omega^q[\Sigma^q(S^0)]$.) We assert $(\pi, D\pi, \varphi)$ is a normal map in the sense of 3.1.2 where $\varphi = \alpha \circ Q\varphi_1 \circ Ad(D\pi_0)$. This is just a diagram chase with Σ and Ω functors and adjoint maps. We never really need this fact, so we don't prove it.

3.4 *Normal maps of degree two*

Let $\pi : X' \to X$, $D\pi : \Sigma^q X_+ \to \Sigma^q X'_+$, $\varphi : X \to QS^0$, be a normal map of degree two. We define $S : H^*(X') \to H^*(X')$ (for any coefficients) by

$$Sa = \pi^*\tau a - a,\ a \in H^*(X'),$$

where $\tau : H^*(X') \to H^*(X)$ is the transfer.

LEMMA 3.4.1. (a) $S^2 a = a$, $a \in H^*(X')$, *hence S is an involution.*
(b) $S\pi^* b = \pi^* b$, $b \in H^*(X)$.

Proof. (a)
$$\begin{aligned} S^2a &= \pi^*\tau\pi^*\tau a - \pi^*\tau a - \pi^*\tau a + a \\ &= \pi^*(2\tau a) - \pi^*(\tau a) - \pi^*(\tau a) + a \\ &= a, \end{aligned}$$

since $\tau\pi^* : H^*(X) \to H^*(X)$ is multiplication by 2, by 3.1.1.

(b) $S\pi^*b = \pi^*\tau\pi^*b - \pi^*b = \pi^*(2b) - \pi^*(b) = \pi^*(b)$.

Remark 3.4.2. As will be apparent in §4, what we have in mind here are *double covers*, $\pi : X' \to X$. If $S : X' \to X'$ is the involution with $X = X'/S$, then it will turn out that the map S defined above is simply $S^* : H^*(X') \to H^*(X')$. This is the case because our transfer $\tau : H^*(X') \to H^*(X)$ will agree with the classical transfer for covering spaces, which satisfies $\pi^*\tau a = a + S^*a$, $a \in H^*(X')$, if $X' \to X = X'/S$, is a double cover.

LEMMA 3.4.2. (a) *Let* X', X *be* $2n$*-dimensional* P.D. *spaces with* $\mathbb{Z}/2$*-coefficients. Then the bilinear pairing* $\tilde{\ell} : H^n(X', \mathbb{Z}/2) \otimes H^n(X', \mathbb{Z}/2) \to \mathbb{Z}/2$

$$\tilde{\ell}(a_1, a_2) = \langle a_1 \cdot Sa_2, [X'] \rangle \in \mathbb{Z}/2$$

is non-singular, symmetric, and even ($\tilde{\ell}(a, a) = 0$).

(b) *Let* X', X *be* $4n - 1$ *dimensional, oriented* P.D. *spaces. Then the bilinear pairing* $\tilde{\ell} : T^{2n}(X') \otimes T^{2n}(X') \to Q/\mathbb{Z}$

$$\tilde{\ell}(a_1, a_2) = \ell'(a_1, Sa_2) \in Q/\mathbb{Z}$$

is non-singular and symmetric, where ℓ' *is the standard linking pairing on* $T^{2n}(X')$.

Proof. (a) $\tilde{\ell}$ is non-singular since S is an isomorphism. $\tilde{\ell}$ is symmetric because

$$\begin{aligned} \tilde{\ell}(a_1, a_2) &= \langle a_1 \cdot Sa_2, [X'] \rangle \\ &= \langle a_1 \cdot (\pi^*\tau a_2 - a_2), [X'] \rangle \\ &= \langle \tau(a_1 \cdot \pi^*\tau a_2), [X] \rangle - \langle a_1 a_2, [X'] \rangle \\ &= \langle \tau a_1 \cdot \tau a_2, [X] \rangle - \langle a_1 \cdot a_2, [X'] \rangle. \end{aligned}$$

$\tilde{\ell}$ is even because $a^2 = a \cdot v_n(X') = a \cdot \pi^*v_n(X) = \tau(a \cdot \pi^*v_n(X)) = \tau a \cdot v_n(X) = (\tau a)^2$. The proof of (b) is almost the same.

We now consider quadratic functions. We have two cases, X', X unoriented, $2n$-dimensional and X', X oriented, $4n - 1$ dimensional. In either case q will denote a quadratic function on an appropriate cohomology group of X, and q' will be the induced quadratic function on the cohomology of X'.

PROPOSITION 3.4.3. (a) *If* X' *and* X *are* $2n$*-dimensional* P.D. *spaces with* $\mathbb{Z}/2$ *coefficients, then the function* $\tilde{q} : H^n(X', \mathbb{Z}/2) \to \mathbb{Z}/4$ *defined by* $\tilde{q}(a) = q(\tau a) - q'(a)$, $a \in H^n(X', \mathbb{Z}/2)$ *is quadratic over the pairing* $\tilde{\ell}$.

(b) *If X' and X are $4n - 1$ dimensional, oriented* P.D. *spaces, then the function $\tilde{q} : T^{2n}(X') \to Q/\mathbb{Z}$ defined by $\tilde{q}(a) = q(\tau a) - q'(a)$, $a \in T^{2n}(X')$ is quadratic over the pairing $\tilde{\ell}$.*

Proof. (b)
$$\begin{aligned} \tilde{q}(a_1 + a_2) &= q(\tau a_1 + \tau a_2) - q'(a_1 + a_2) \\ &= q(\tau a_1) + q(\tau a_2) + \ell(\tau a_1 \cdot \tau a_2) \\ &\qquad - q'(a_1) - q'(a_2) - \ell'(a_1, a_2) \\ &= \tilde{q}(a_1) + \tilde{q}(a_2) + \ell'(a_1, Sa_2) \\ &= \tilde{q}(a_1) + \tilde{q}(a_2) + \tilde{\ell}(a_1, a_2). \end{aligned}$$

Also,
$$\begin{aligned} \tilde{q}(na) &= q(n(\tau a)) - q'(na) \\ &= n^2 q(\tau a) - n^2 q'(a) \\ &= n^2 \tilde{q}(a). \end{aligned}$$

(The proof of (a) is included in this argument.)

Remark 3.4.4. In case (a), since $\tilde{\ell} : H^n(X', \mathbb{Z}/2) \otimes H^n(X', \mathbb{Z}/2) \to \mathbb{Z}/2$ is an even pairing, we have $2\tilde{q} \equiv 0$.

3.5. *The function $\tilde{q}$ associated to a quadratic function*

We continue with our study of a normal map $\pi : X' \to X$ of P.D. spaces. Thus a quadratic function $q : H^*(X) \to Q/\mathbb{Z}$ will induce a quadratic function $q' : H^*(X') \to Q/\mathbb{Z}$, using the results above. (Cohomology will be either $H^n(X, \mathbb{Z}/2)$ or $T^{2n}(X)$, depending on whether X is $2n$ dimensional, unoriented, or $4n - 1$ dimensional, oriented.) There is also the function $\tilde{q} : H^*(X') \to Q/\mathbb{Z}$ defined by $\tilde{q} = q\tau - q'$, $\tau : H^*(X') \to H^*(X)$ the transfer.

Recall that $q : H^*(X) \to Q/\mathbb{Z}$ and $q' : H^*(X') \to Q/\mathbb{Z}$ are defined by $q = \psi \circ s$ and $q' = \psi \circ (D\pi)^* \circ s$, where $\psi : \{X, K\} \to Q/\mathbb{Z}$ is a suitable linear map and s is suspension (K is an Eilenberg-MacLane space $K(\mathbb{Z}/2, n)$ or $K(Q/\mathbb{Z}, 2n - 1)$.) If $a \in H^*(X')$ then $\tilde{q}(a) = q\tau(a) - q'(a)$ is computed in terms of a *non-commutative* diagram

3.5.1.
$$\begin{array}{ccc} & \Sigma^q X_+ & \xrightarrow{D\pi} & \Sigma^q X'_+ \\ s(\tau a) & \downarrow & \swarrow s(a) & \\ & \Sigma^q K & & \end{array}$$

Namely, $\tilde{q}(a) = q\tau(a) - q'(a) = \psi(s(\tau a) - s(a)D\pi) \in Q/\mathbb{Z}$.

Now, the rectangle in the diagram below does commute

3.5.2.
$$\begin{array}{cccc} & \Sigma^q X_+ & \xrightarrow{D\pi} & \Sigma^q X'_+ \\ s(\tau a) & \downarrow & s(a) \swarrow & \downarrow \Sigma^q(a) \\ & \Sigma^q K & \xrightarrow{\Sigma^q \iota} & K(q) \end{array}$$

where $\Omega^q K(q) = K$, that is, $K(q) = K(\mathbb{Z}/2, q + n)$ or $K(Q/\mathbb{Z}, q + 2n - 1)$. Applying the functor Ω^q to 3.5.2 gives

$$\begin{array}{ccccc} & & X_+ & \xrightarrow{Ad(D\pi)} & QX'_+ \\ & \tau a \swarrow & \downarrow & Qa \swarrow & \downarrow \\ K & \xrightarrow{i} & QK & \xrightarrow{\Omega\Sigma\iota} & K \end{array}$$

where the rectangle and left-hand triangle commute, but not the center triangle. (If Y is a space, QY means $\lim_{q \to \infty} \Omega^q \Sigma^q Y$.)

The space QK is homotopy equivalent to $J \times K$, by the map $j \times i : J \times K \to QJ$, where (J, j) is the fibre of $\Omega\Sigma\iota : QK \to K$. Now J is highly connected. Specifically, if $K = K(\mathbb{Z}/2, n)$, $J = J(\mathbb{Z}/2, n)$ is $2n - 1$ connected and $\pi_{2n}(J) = \mathbb{Z}/2$. If $K = K(Q/\mathbb{Z}, 2n - 1)$, $J = J(Q/\mathbb{Z}, 2n - 1)$ is $4n - 2$ connected and $\pi_{4n-1}(J) = Q/\mathbb{Z}$. (See Propositions 2.1.1 and 2.2.1).

If a $\in H^*(X')$, then $s(\tau a) - s(a) \circ D\pi \in \{X, K\}$ corresponds under Ad to $i \circ \tau a - Qa \circ Ad(D\pi) \in [X, QK]$. Under the homotopy equivalence $j \times i : J \times K \to QK$, the elements $i \circ \tau a$ and $Qa \circ Ad(D\pi) \in [X, QK]$ have the same K-component; namely $\tau a \in H^*(X)$. To compute $\tilde{q}(a) = \psi(i \circ \tau a - Qa \circ Ad(D\pi))$, we need to compute the J-component of $i \circ \tau a - Qa \circ Ad(D\pi)$.

Let us choose appropriate "fundamental classes" $\tilde{\kappa} \in H^{2n}(J(\mathbb{Z}/2, n), \mathbb{Z}/2)$ and $\tilde{\kappa} \in H^{4n-1}(J(Q/\mathbb{Z}, 2n - 1), Q/\mathbb{Z})$. In the $Q/\mathbb{Z}$ case, $\tilde{\kappa}$ depends on a choice of isomorphism $\pi_{4n-1}(J(Q/\mathbb{Z}, 2n - 1)) \simeq Q/\mathbb{Z}$. Since $j \times i : J \times K \to QK$ is a homotopy equivalence, there are unique classes $\kappa \in H^{2n}(QK(\mathbb{Z}/2, n), \mathbb{Z}/2)$ and $\kappa \in H^{4n-1}(QK(Q/\mathbb{Z}, 2n - 1), Q/\mathbb{Z})$ with $j^*(\kappa) = \tilde{\kappa}$ and $i^*(\kappa) = 0$. The J-component of any map $f : X \to QK$ is then given by $ev(\kappa)(f) = \langle f^*(\kappa), [X] \rangle \in \mathbb{Z}/2$ or $Q/\mathbb{Z}$.

To obtain more precise formulas, we quote some results from [15], where a specific model of QK is studied. There is the Dyer-Lashof map

$$\mathscr{D} : S^\infty \times_{\mathbb{Z}/2} K \times K \to QK$$

which is trivial on $P^\infty = S^\infty \times_{\mathbb{Z}/2} (* \times *)$.

PROPOSITION 3.5.3. [15] $\mathscr{D} : S^\infty \times_{\mathbb{Z}/2} K \times K/P^\infty \to QK$ *is a homotopy equivalence through (almost) three times the connectivity of* QK.

The significance of 3.5.3 for our purposes is that we can compute a cochain representative for $\mathscr{D}^*(\kappa) \in H^*(S^\infty \times_{\mathbb{Z}/2} K \times K)$, where $\kappa \in H^*(QK)$ is the fundamental class above which detects the first unstable homotopy group, that is, either $\pi_{2n}(QK(\mathbb{Z}/2, n)) = \mathbb{Z}/2$ or $\pi_{4n-1}(QK(Q/\mathbb{Z}, 2n - 1)) = Q/\mathbb{Z}$. Namely, consider the double cover $\pi : S^\infty \times K \times K \to S^\infty \times_{\mathbb{Z}/2} K \times K$. Then also in [15], the homology classes of

$$S^{2n} \to QK(\mathbb{Z}/2, n) \quad \text{or} \quad S^{4n-1} \to QK(Q/\mathbb{Z}, 2n - 1)$$

are characterized in terms of the homology map $\mathcal{D}_*\pi_* : H_*(S^\infty \times K \times K) \to H_*(QK)$. The result is the following.

PROPOSITION 3.5.4. *Consider* $\mathcal{D}\pi : S^\infty \times K \times K \to QK$, $\kappa \in H^*(QK)$. *Then if* $K = K(\mathbb{Z}/2, n)$, $\kappa \in H^{2n}(QK, \mathbb{Z}/2)$ *is the unique class with*

$$\pi^*\mathcal{D}^*(\kappa) = 1 \otimes \iota_n \otimes \iota_n \in H^{2n}(S^\infty \times K \times K, \mathbb{Z}/2).$$

If $K = K(Q/\mathbb{Z}, 2n - 1)$ *then* $\kappa \in H^{4n-1}(QK, Q/\mathbb{Z})$ *is the unique class with*

$$\pi^*\mathcal{D}^*(\kappa) = 1 \otimes \iota_{2n-1} \otimes \beta\iota_{2n-1} \in H^{4n-1}(S^\infty \times K \times K, Q/\mathbb{Z})$$

Remark 3.5.5 (a). The composition $\mathcal{D}\pi : S^\infty \times K \times K \to QK$ is up to homotopy. The multiplication of two copies of $i(K)$ in the H-space QK. That is, $\mathcal{D}\pi = \mu(i \times i) : K \times K \to QK \times QK \to QK$.

(b) It follows that the class $\kappa \in H^*(QK)$ satisfies the following "diagonal formula" for $\mu : QK \times QK \to QK$:

$$\mu^*(\kappa) = \kappa \otimes 1 + 1 \otimes \kappa + \iota_n \otimes \iota_n \quad \text{if} \quad K = K(\mathbb{Z}/2, n)$$

$$\mu^*(\kappa) = \kappa \otimes 1 + 1 \otimes \kappa + \iota_{2n-1} \otimes \beta\iota_{2n-1} \quad \text{if} \quad K = K(Q/\mathbb{Z}, 2n - 1).$$

(In the $Q/\mathbb{Z}$ case, one actually has equality $\iota_{2n-1} \otimes \beta\iota_{2n-1} = \beta\iota_{2n-1} \otimes \iota_{2n-1}$, explaining the seemingly non-symmetric formula.)

(c) These quadratic properties of $\kappa \in H^*(QK)$ enable us to compute $\tilde{q}(a) = \psi(i \circ \tau a - Qa \circ Ad(D\pi)) = ev(\kappa)(i \circ \tau a - Qa \circ Ad(D\pi)) \in \mathbb{Z}/2$ or $Q/\mathbb{Z}$, where a $\in H^*(X')$. Precisely, if $\pi : X' \to X$, $D\pi : \Sigma^q X_+ \to \Sigma^q X'_+$ is a normal map, q is a quadratic function on $H^*(X)$, q' is the induced quadratic function on $H^*(X')$, $\tilde{q} = q\tau - q'$, and $a \in H^*(X')$, we have $\tilde{q}(a) = \langle Ad(B\pi)^* (Qa)^*(\kappa), [X]\rangle \in \mathbb{Z}/2$ or $Q/\mathbb{Z}$ where $Ad(D\pi) : X \to QX'_+$ and $Qa : QX' \to QK$.

We now use Prop. 3.5.3 and 3.5.4 to produce explicit cochain representatives for $\mathcal{D}^*(\kappa) \in H^*(S^\infty \times_{\mathbb{Z}/2} K \times K)$. The cohomology of $S^\infty \times_{\mathbb{Z}/2} K \times K$ can be computed as the cohomology of the complex of $\mathbb{Z}/2$ equivariant cochains on $S^\infty \times K \times K$. As a model for $C_*(S^\infty \times K \times K)$, we take $C_*(S^\infty) \otimes C_*(K) \otimes C_*(K)$, where $C_i(S^\infty)$ has two generators, e_i and Te_i, with $\partial e_i = Te_{i-1} + (-1)^i e_{i-1}$, where $T \in \mathbb{Z}/2$ is the generator. In cohomology, $C^i(S^\infty, \mathbb{Z})$ has two dual generators, e^i and Te^i, with $\delta e^i = Te^{i+1} - (-1)^i e^{i+1}$.

Example 3.5.6. Let $K = K(\mathbb{Z}/2, n)$ and consider $\mathcal{D} : S^\infty \times_{\mathbb{Z}/2} (K(\mathbb{Z}/2, n) \times K(\mathbb{Z}/2, n)) \to QK(\mathbb{Z}/2, n)$. Then $1 \otimes \iota_n \otimes \iota_n \in C^*(S^\infty) \otimes C^*(K(\mathbb{Z}/2, n)) \otimes C^*(K(\mathbb{Z}/2, n))$ is an equivariant $\mathbb{Z}/2$ cocycle. It follows immediately from 3.5.3 and 3.5.4 that

$$\mathcal{D}^*(\kappa) = [1 \otimes \iota_n \otimes \iota_n] \in H^{2n}(S^\infty \times_{\mathbb{Z}/2} (K(\mathbb{Z}/2, n) \times K(\mathbb{Z}/2, n)), \mathbb{Z}/2).$$

Example 3.5.7. Let $K = K(Q/\mathbb{Z}, 2n - 1)$ and consider $\mathcal{D} : S^\infty \times_{\mathbb{Z}/2} K(Q/\mathbb{Z}, 2n - 1) \times K(Q/\mathbb{Z}, 2n - 1) \to QK(Q/\mathbb{Z}, 2n - 1)$. Let $\iota_\# \in$

$C^{2n-1}(K(Q/\mathbb{Z}, 2n-1), Q)$ be a rational cochain which represents the fundamental class $\iota \in H^{2n-1}(K(Q/\mathbb{Z}, 2n-1), Q/\mathbb{Z})$. Thus $\delta\iota_\#$ is a $\mathbb{Z}$-cocycle which represents $\beta\iota \in H^{2n}(K(Q/\mathbb{Z}, 2n-1), \mathbb{Z})$. Consider the rational cochain

$$\mathscr{P}(\iota_\#) = e^0 \otimes \iota_\# \otimes \delta\iota_\# + Te^0 \otimes \delta\iota_\# \otimes \iota_\# + \delta e^0 \otimes \iota_\# \otimes \iota_\#$$
$$\in C^*(S^\infty, \mathbb{Z}) \otimes C^*(K(Q/\mathbb{Z}, 2n-1), Q) \otimes C^*(K(Q/\mathbb{Z}, 2n-1)Q).$$

It is easy to check that $\mathscr{P}(\iota_\#)$ is a $Q/\mathbb{Z}$ cocycle and is $\mathbb{Z}/2$ equivariant. Moreover, $e^0 \otimes \iota_\# \otimes \delta\iota_\# + Te^0 \otimes \delta\iota_\# \otimes \iota_\# + \delta e^0 \otimes \iota_\# \otimes \iota_\# = (e^0 + Te^0) \otimes \iota_\# \otimes \delta\iota_\# + \delta(Te^0 \otimes \iota_\# \otimes \iota_\#) = 1 \otimes \iota_\# \otimes \delta\iota_\# + \delta(Te^0 \otimes \iota_\# \otimes \iota_\#)$. Thus $\pi^*(\mathscr{P}\iota_\#) = 1 \otimes \iota \otimes \beta\iota \in H^*(S^\infty \times K(Q/\mathbb{Z}, 2n-1) \times K(Q/\mathbb{Z}, 2n-1))$. It follows from 3.5.3 and 3.5.4 that

$$\mathscr{D}^*(\kappa) = [\mathscr{P}(\iota_\#)] \in H^{4n-1}(S^\infty \underset{\mathbb{Z}/2}{\times} K(Q/\mathbb{Z}, 2n-1) \times K(Q/\mathbb{Z}, 2n-1), Q/\mathbb{Z}).$$

4. Covering spaces

4.1. *Coverings as normal maps*

Let $\pi : X' \to X$ be an m-fold cover, classified by a map $f : X \to B\mathscr{S}_m$, where $\mathscr{S}_m$ is the symmetric group. If $\rho : E_m \to B\mathscr{S}_m$ is the universal m-fold cover of $B\mathscr{S}_m$, there is a diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & E_m \\ \pi \downarrow & & \downarrow \rho \\ X & \xrightarrow{f} & B\mathscr{S}_m \end{array}$$

Let $E\mathscr{S}_m \to B\mathscr{S}_m$ be the principal $\mathscr{S}_m$ bundle over $B\mathscr{S}_m$. Then $E\mathscr{S}_m$ may be identified with the set of bijections from $\{1,2, \cdots, m\}$ to a fibre of $\rho : E_m \to B\mathscr{S}_m$. We define a map

$$F : X \to E\mathscr{S}_m \underset{\mathscr{S}_m}{\times} (X')^m$$

by $F(x) = (e, x_1', \cdots, x_m')$ where $\{x_1' \cdots x_m'\} = \pi^{-1}(x) \subset X'$ and $e : \{1,2, \cdots, m\} \simeq \rho^{-1}f(x)$ is a bijection with $f'(x_i') = e(i)$. Note that if e is replaced by $e\sigma$, $\sigma \in S_m$, then $(e\sigma, y_{\sigma(1)} \cdots y_{\sigma(m)}) = (e, y_1 \cdots y_m) \in E\mathscr{S}_m \underset{\mathscr{S}_m}{\times} (X')^m$, hence F is well-defined.

Let $i : X_+' \to QX_+'$ be the adjoint of the identity $\Sigma^q X_+' \to \Sigma^q X_+'$. Consider the composition

$$X \xrightarrow{F} E\mathscr{S}_m \underset{\mathscr{S}_m}{\times} (X')^m \xrightarrow{1 \underset{\mathscr{S}_m}{\times} i^m} E\mathscr{S}_m \underset{\mathscr{S}_m}{\times} (QX_+')^m \xrightarrow{\mathscr{D}} Q(X_+'),$$

where $\mathscr{D}$ is the Dyer-Lashof operation [9], [13], [14]. Set

$$D\pi = Ad(\mathscr{D}(1 \underset{\mathscr{S}_m}{\times} i^m)F) : \Sigma^q X_+ \to \Sigma^q X_+'$$

and set

$$\varphi = \mathcal{D}f; X \to B\mathcal{S}_m \to Q_m S^0.$$

Then we have

PROPOSITION 4.1.1. *The triple* $\pi : X' \to X$, $D\pi : \Sigma^q X_+ \to \Sigma^q X'_+$, $\varphi : X \to Q_m S^0$ *is a normal map of degree m, if X and X′ are* P.D. *spaces.*

Proof. We refer to the definition of normal maps, 3.1.2. 3.1.2(b) follows from the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{F} & E\mathcal{S}_m \underset{\mathcal{S}_m}{\times} (X')^m & \xrightarrow{\mathcal{D}} & QX'_+ \\ f \times 1 \downarrow & & \downarrow 1 \times \pi^m & & \downarrow Q\pi \\ B\mathcal{S}_m \times X & \xrightarrow{1 \times \Delta} & E\mathcal{S}_m \underset{\mathcal{S}_m}{\times} (X^m) & \xrightarrow{\mathcal{D}} & QX_+. \end{array}$$

Also, one knows, ([9]), that the homology map $(D\pi)_* : H_*(\Sigma^q X_+) \to H_*(\Sigma^q X'_+)$ is the map associated to the chain map which assigns to each simplex Δ of X the sum $\sum_{i=1}^{m} \Delta_i$ of the m simplexes Δ_i of X' over Δ. Thus $(D\pi)_*$ is the classical homology transfer of a cover. Properties 3.1.2(a), (c) follow.

4.2 *Quadratic functions and covering spaces*

We apply the results of 3.5 to covering spaces $\pi : X' \to X$. In this case, $Ad(D\pi) : X \to QX'_+$ is given by the composition $X \to E\mathcal{S}_m \underset{\mathcal{S}_m}{\times} (X')^m \to QX'_+$. We will derive chain formula for $\tilde{q} = q\tau - q' : H^*(X') \to \mathbb{Z}/2$ or $Q/\mathbb{Z}$ in three cases.

Case 1. X, X', $2n$-dimensional, unoriented and $\pi : X' \to X$ a double cover. Then there is a diagram

$$\begin{array}{ccccc} X' & \xrightarrow{f' \times 1 \times S} & S^\infty \times X' \times X' & \xrightarrow{1 \times a \times a} & S^\infty \times K(\mathbb{Z}/2, n) \times K(\mathbb{Z}/2, n) \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{F} & S^\infty \underset{\mathbb{Z}/2}{\times} X' \times X' & \xrightarrow[\mathbb{Z}/2]{1 \times a \times a} & S^\infty \underset{\mathbb{Z}/2}{\times} K(\mathbb{Z}/2, n) \times K(\mathbb{Z}/2, n), \end{array}$$

where $f' : X' \to S^\infty$ is $\mathbb{Z}/2$ equivariant and $S : X' \to X'$ is the involution on X', and $a \in H^n(X', \mathbb{Z}/2)$. From 3.5.3 and 3.5.5 we deduce

4.2.1.
$$\begin{aligned} \tilde{q}(a) &= q(\tau a) - q'(a) \\ &= \langle F^*(\alpha), [X] \rangle \\ &= (1/2)\langle a_\# \cdot Sa_\#, [X'] \rangle \end{aligned}$$

where $a_\# \in C^n(X', \mathbb{Z})$ represents a and $\alpha \in H^{2n}(S \underset{\mathbb{Z}/2}{\times} X' \times X', \mathbb{Z}/2)$ is represented by the cochain $1 \otimes a_\# \otimes a_\# \in C^*(S^\infty) \otimes C^*(X', \mathbb{Z}) \otimes C^*(X', \mathbb{Z})$.

Case 2. $X, X', 4n-1$ dimensional, oriented and $\pi : X' \to X$ a double cover. Again, consider

$$\begin{array}{ccccc} X' & \xrightarrow{f' \times 1 \times S} & S^\infty \times X' \times X' & \xrightarrow{1 \times a \times a} & S^\infty \times K(Q/\mathbb{Z}, 2n-1) \times K(Q/\mathbb{Z}, 2n-1) \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{F} & S^\infty \underset{\mathbb{Z}/2}{\times} X' \times X' & \xrightarrow[\mathbb{Z}/2]{1 \times a \times a} & S^\infty \underset{\mathbb{Z}/2}{\times} K(Q/\mathbb{Z}, 2n-1) \times K(Q/\mathbb{Z}, 2n-1) \end{array}$$

for any $a \in H^{2n-1}(X', Q/\mathbb{Z})$.

From 3.5.3 and 3.5.6, 3.5.7, we deduce

4.2.2.
$$\begin{aligned} \tilde{q}(a) &= q(\tau a) - q'(a) \\ &= \langle F^*(\alpha), [X] \rangle \\ &= (1/2)\langle a_\# \cdot S\delta a_\#, [X'] \rangle \end{aligned}$$

where $a_\# \in C^{2n-1}(X', Q)$ represents a and $\alpha \in H^{4n-1}(S^\infty \underset{\mathbb{Z}/2}{\times} X' \times X', Q/\mathbb{Z})$ is represented by the cochain $\mathcal{P}(a_\#) = e^0 \otimes a_\# \otimes \delta a_\# + Te^0 \otimes \delta a_\# \otimes a_\# + \delta e^0 \otimes a_\# \otimes a_\# \in C^*(S^\infty) \otimes C^*(X', Q) \otimes C^*(X', Q)$.

Case 3. X, X' $2n$-dimensional, unoriented, $\pi : X' \to X$ a principal G-bundle, $|G| = m$, m odd. Let $\ell : G \to \mathcal{S}_m$ be the representation of G given by left multiplication by elements of G. As an m-fold cover, there is a diagram

$$\begin{array}{ccccc} X' & \xrightarrow{f'} & EG & \xrightarrow{E\ell} & E_m \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{f} & BG & \xrightarrow{B\ell} & B\mathcal{S}_m \end{array}$$

Now, the map $F : X \to E\mathcal{S}_m \underset{\mathcal{S}_m}{\times} (X')^m$ factors through $E\mathcal{S}_m \underset{G}{\times} (X')^m$; that is, we *identify* EG and $E\mathcal{S}_m$, use $BG \overset{\sim}{=} E\mathcal{S}_m/G$, $B\mathcal{S}_m = E\mathcal{S}_m/\mathcal{S}_m$, and construct a diagram

$$\begin{array}{ccccc} X' & \xrightarrow{F'_G} & E\mathcal{S}_m \times (X')^m & = & E\mathcal{S}_m \times (X')^m \\ \pi \downarrow & & \downarrow \pi & & \downarrow \\ X & \xrightarrow{F_G} & E\mathcal{S}_m \times_G (X')^m & \to & E\mathcal{S}_m \underset{\mathcal{S}_m}{\times} (X')^m \end{array}$$

by writing out $G = \{g_1, \cdots, g_m\}$, which explicitly identifies $G = \ell(G) \subset \mathcal{S}_m$. In fact, if $x' \in X'$ is a point, $F'_G(x') = (f'(x'), x'g_1, \cdots, x'g_m) \in EG \times (X')^m = E\mathcal{S}_m \times (X')^m$.

If $a \in H^n(X', \mathbb{Z}/2)$, we get a diagram

$$\begin{array}{ccccc}
X' & \xrightarrow{F'_G} & E\mathcal{S}_m \times (X')^m & & \\
\pi \downarrow & & \downarrow \pi & & \\
X & \xrightarrow{F_G} & E\mathcal{S}_m \underset{G}{\times} (X')^m & \xrightarrow{\rho} & E\mathcal{S}_m \underset{\mathcal{S}_m}{\times} (X')^m \\
\swarrow\ 1 \times a^m & & \swarrow\ 1 \underset{G}{\times} a^m & & \swarrow\ 1 \underset{\mathcal{S}_m}{\times} a^m \\
E\mathcal{S}_m \times K(\mathbb{Z}/2, n)^m & & & & \\
\downarrow \pi & & & & \\
E\mathcal{S}_m \underset{G}{\times} K(\mathbb{Z}/2, n)^m & \xrightarrow{\rho} & E\mathcal{S}_m \underset{\mathcal{S}_m}{\times} K(\mathbb{Z}/2, n)^m & \xrightarrow{\mathcal{D}} & QK(\mathbb{Z}/2, n)
\end{array} \qquad 4.2.3.$$

From 3.5.3, and the fact that $|G| = m$ is odd, we deduce

4.2.4. $$\begin{aligned}\tilde{q}(a) &= \langle F_G^* \rho^*(1 \underset{\mathcal{S}_m}{\times} a^m)^* \mathcal{D}^*(\kappa), [X] \rangle \\ &= \langle (F'_G)^*(1 \times a^m)^* \pi^* \rho^* \mathcal{D}(\kappa), [X'] \rangle.\end{aligned}$$

Now,

$$H^*(E\mathcal{S}_m \times K(\mathbb{Z}/2, n)^m, \mathbb{Z}/2) = \bigotimes_{m\text{-times}} H^*(K(\mathbb{Z}/2, n), \mathbb{Z}/2),$$

with fundamental classes $\iota_1, \cdots, \iota_m$. For each pair $1 \le i < j \le m$, we get a diagram

$$\begin{array}{ccc}
E\mathbb{Z}/2 \times K(\mathbb{Z}/2, n)_i \times K(\mathbb{Z}/2, n)_j & \rightarrow & E\mathcal{S}_m \times K(\mathbb{Z}/2, n)^m \\
\downarrow \pi_2 & & \downarrow \\
E\mathbb{Z}/2 \underset{\mathbb{Z}/2}{\times} K(\mathbb{Z}/2, n)_i \times K(\mathbb{Z}/2, n)_j & \xrightarrow{\sigma} & E\mathcal{S}_m \underset{\mathcal{S}_m}{\times} K(\mathbb{Z}/2, n)^m
\end{array}$$

since we identify $\mathbb{Z}/2$ with the subgroup $\{Id, (ij)\} \subset \mathcal{S}_m$ and identify $E\mathbb{Z}/2$ with $E\mathcal{S}_m$. From 3.5.5, $\pi_2^* \sigma^* \mathcal{D}^*(\kappa) = 1 \otimes \iota_i \otimes \iota_j \in H^{2n}(E(\mathbb{Z}/2) \times K(\mathbb{Z}/2, n)_i \times K(\mathbb{Z}/2, n)_j, \mathbb{Z}/2)$. Finally combining this with diagram 4.2.3 and equation 4.2.4, we have proved

4.2.5. $$\tilde{q}(a) = \sum_{1 \le i < j \le m} \langle g_i^*(a) g_j^*(a), [X'] \rangle$$

where $g_i : X' \to X'$ is the covering transformation associated to $g_i \in G$.

We conclude this chapter with a discussion of the derivation of formula 1.3.1 from 4.2.5. Recall that if $\pi: X' \to X$ is a principal G-bundle, $|G|$ odd, and if $K^n \subset H^n(X')$ is the kernel of the transfer $\tau: H^n(X') \to H^n(X)$, then $\tilde{q} = q\tau - q': K^n \to \mathbb{Z}/2$. Formula 1.3.1 states that

$$A(X' \to X) = A(K^n, \tilde{q}) = \begin{matrix} \chi(X) & \text{if} & |G| \equiv 3, 5 \pmod 8 \\ 0 & \text{if} & |G| \equiv 1, 7 \pmod 8 \end{matrix}$$

First, write $G = \{1, h_1 \cdots h_k, h_1^{-1} \cdots h_k^{-1}\}$. We need to reconcile the seemingly different formulas for $\tilde{q}$ given in 1.3.1 and 4.2.5. Let $m = 2k + 1$, $G = \{g_i\}$, $1 \le i \le m$.

LEMMA 4.2.6. $\sum_{1 \le i < j < m} \langle g_i^* (a) \cdot g_j^*(a), [X'] \rangle = \sum_{i=1}^{k} \langle a \cdot h_i^* (a), [X'] \rangle$ $\in \mathbb{Z}/2$ *for all* $a \in H^n(X', \mathbb{Z}/2)$.

We leave this computation as an exercise.

Secondly, using solvability of groups of odd order, [7], a principal G-bundle $X' \to X$ can be factored $X' = X_r \to X_{r-1} \to \cdots \to X_0 = X$, where each $X_i \to X_{i-1}$ is a principal $\mathbb{Z}/p_i$ bundle, p_i prime. The Arf invariants add in this very special situation by 3.3.8, that is, $A(X' \to X) = \sum_{i=1}^{r} A(X' \to X_{i-1})$. Simple $\mathbb{Z}/8$ arithmetic then shows that if 1.3.1 is true for principal $\mathbb{Z}/p$ bundles, p prime, then 1.3.1 is true for any group G of odd order.

Finally, the proof is completed by a routine but fairly lengthy investigation of the structure of $K^* \subset H^*(X', \mathbb{Z}/2)$ as $\mathbb{Z}/2(\mathbb{Z}/p) \Big/ \left(\sum_{i=1}^{p} T^i \right)$ module, where $\mathbb{Z}/2(\mathbb{Z}/p)$ is the group ring and $T \in \mathbb{Z}/p$ is a generator.

5. Double covers

5.1. *The quadratic function $\tilde{q}$ induced by a double cover*

Let π: $X' \to X$ be a double cover, with classifying diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & S^\infty \\ \pi \downarrow & & \downarrow \\ X & \xrightarrow{f} & RP(\infty) \end{array}$$

In §4, we defined a commutative diagram

5.1.1
$$\begin{array}{ccc} X' & \xrightarrow{f' \times 1 \times S} & S^\infty \times X' \times X' \\ \pi \downarrow & & \downarrow \\ X & \xrightarrow{f} & S^\infty \underset{\mathbb{Z}/2}{\times} X' \times X' \end{array}$$

where $S = X' \to X'$ is the involution over X. We then considered the composition

$$X \xrightarrow{F} S^\infty \underset{\mathbb{Z}/2}{\times} X' \times X' \xrightarrow{i} S^\infty \underset{\mathbb{Z}/2}{\times} (QX'_+ \times QX'_+) \xrightarrow{\mathscr{D}} QX'_+$$

where $\mathscr{D}$ is the Dyer-Lashof map [13], [14], and i is induced by the natural inclusion $X' \subset \Omega^q\Sigma^q X'$.

Moreover, if X' and X are PD spaces and $D\pi = Ad(\mathscr{D} \circ i \circ F)$: $\Sigma^q X_+ \to \Sigma^q X'_+$, then we verified that $(\pi, D\pi, \varphi)$ is a normal map from X' to X, where $\varphi: X \to QS^0$ is the composition

$$X \xrightarrow{f} RP(\infty) = B\mathbb{Z}/2 \xrightarrow{\mathscr{D}} Q_2 S^0,$$

$\mathscr{D}$ the Dyer-Lashof map, [9].

Assume that X' and X are $2n$-dimensional PD spaces with $\mathbb{Z}/2$ coefficients, X connected. A quadratic function $q: H^n(X, \mathbb{Z}/2) \to \mathbb{Z}/4$ induces a quadratic function $q': H^n(X', \mathbb{Z}/2) \to \mathbb{Z}/4$ as in §2. In 3.4.3(a), we proved that the function $\tilde{q}: H^n(X', \mathbb{Z}/2) \to \mathbb{Z}/4$, defined by, $\tilde{q}(a) = q(\tau a) - q'(a)$, is quadratic over the pairing $\tilde{\ell}: H^n(X', \mathbb{Z}/2) \otimes H^n(X', \otimes/2) \to \mathbb{Z}/2$, where $\tilde{\ell}(a_1, a_2) = \langle a_1 \cdot S^* a_2, [X'] \rangle$. In 4.2.1, we established a chain formula for $\tilde{q}(a)$. Namely, if $a_\# \in C^n(X, \mathbb{Z}/2)$ is a cochain representative for $a \in H^n(X', \mathbb{Z}/2)$ then

5.1.2.
$$\begin{aligned}\tilde{q}(a) &= \langle F^\#(1 \otimes a_\# \otimes a_\#), [X] \rangle \\ &= \left(\frac{1}{2}\right)\langle a_\# \cup S^\# a_\#, [X'] \rangle \in \mathbb{Z}/2\end{aligned}$$

where $F: X \to S^\infty \underset{\mathbb{Z}/2}{\times} X' \times X'$ is as in diagram 5.1.1. (By 3.4.4, $2\tilde{q}(a) = 0$, so we take the values of $\tilde{q}$ in $\mathbb{Z}/2$, rather than $\mathbb{Z}/4$.)

It is fairly easy to prove directly that 5.1.2 *defines* a quadratic function $\tilde{q}$ refining $\tilde{\ell}$. The results of this section use nothing more than this, so this section is essentially independent of the rest of the paper. It is interesting, nonetheless, that the $\tilde{q}$ defined by 5.1.2 is a special case of the general theory developed in §§2,3,4.

We first compute $\tilde{q} \circ \pi^*: H^n(X, \mathbb{Z}/2) \to \mathbb{Z}/2$. Since $\tau\pi^*(b) = 2b = 0$, $b \in H^n(X, \mathbb{Z}/2)$, we have $\tilde{q}\pi^*(b) = q'\pi^*(b)$. Also, since $\langle \pi^* b_1 \cdot \pi^* b_2, [X'] \rangle = \langle \pi^*(b_1 b_2), [X'] \rangle = 0$, we have that $\tilde{q}\pi^*: H^n(X, \mathbb{Z}/2) \to \mathbb{Z}/2$ is linear.

PROPOSITION 5.1.3. *If* $b \in H^n(X, \mathbb{Z}/2)$ *then*

$$\tilde{q}\pi^*(b) = \left\langle b \cdot \left(\sum_{j=0}^{\infty} v_{n-2^j+1} \cdot w_1^{2^j-1} \right), [X] \right\rangle \in \mathbb{Z}/2$$

where $w_1 = f^*(e) \in H^1(X, \mathbb{Z}/2)$, $f: X \to \mathbb{R}P(\infty)$, $e \in H^1(\mathbb{R}P(\infty), \mathbb{Z}/2)$ *the generator.*

Proof. This turns out to be a consequence of the definition of the Steenrod squares. From 5.1.1 we construct a diagram

$$\begin{array}{ccc} X & \xrightarrow{F} & S^\infty \underset{\mathbb{Z}/2}{\times} X' \times X' \\ f \times 1 \downarrow & & \downarrow j \times \pi \times \pi \\ \mathbb{R}P(\infty) \times X & \xrightarrow{\Delta} & S^\infty \underset{\mathbb{Z}/2}{\times} X \times X \end{array}$$

From 5.1.2, we obtain ($b_\#$ is a cocycle representative for b)

$$\begin{aligned}\tilde{q}\pi^*(b) &= \langle (f\times 1)^\# \Delta^\#(1\otimes b_\# \otimes b_\#), [X]\rangle \\ &= \left\langle (f\times 1)^*\left(\sum_{i=0}^{\infty} e^i \otimes Sq^{n-i}(b)\right), [X]\right\rangle. \\ &= \left\langle \sum_{i=0}^{\infty} w_1{}^i \otimes Sq^{n-i}(b)), [X]\right\rangle.\end{aligned}$$

The second equality is an application of the definition of Steenrod squares [20]. Since if $Sq = 1 + Sq^1 + Sq^2 + \cdots$,

$$Sq(w_1{}^{2^j-1}) = \sum_{k=0}^{2^j-1} w_1{}^{2^j-1+k},$$

we compute

$$\begin{aligned}\left\langle \sum_{i=0}^{\infty} w_1{}^i \otimes Sq^{n-i}(b), [X]\right\rangle &= \left\langle \sum_{j=0}^{\infty} Sq(w_1{}^{2^j-1}\cdot b), [X]\right\rangle \\ &= \left\langle \sum_{j=0}^{\infty} V(X) w_1{}^{2^j-1}\cdot b, [X]\right\rangle,\end{aligned}$$

which proves 5.1.3.

Remark 5.1.4. Prop. 5.1.3 gives a formula for $q'\pi^*(b)$, $b \in H^n(X, \mathbb{Z}/2)$, since $q'\pi^* = \tilde{q}\pi^*$. This formula is very similar to the formula of Prop. 3.3.3, for $q'\pi^*(b)$, $b \in H^n(X, \mathbb{Z}/2)$, when $\pi: X' \to X$ is a map of *odd* degree. In fact, if $\xi \to \mathbb{R}P(\infty)$ is the canonical line bundle, then the *Wu* class

$$V(\xi) = \sum_{j\geq 0} e^{2^j-1} \in H^*(\mathbb{R}P(\infty), \mathbb{Z}/2).$$

Thus the term

$$\sum_{j\geq 0} V_{n-2^j+1}(X) w_1{}^{2^j-1} = (V(X)\cdot f^*V(\xi))_n$$

in 5.1.3 is provocatively similar to the term $V(X)\varphi^*(\sigma V)$ in 3.3.3. One even expects $\varphi^*(\sigma V)$ to vanish, except in dimensions $2^j - 1$. (This is true for degree *one* maps $\varphi: X \to QS^0$.)

Our proofs of 3.3.3 and 5.1.3 are quite different; one wonders if the similarity can be explained by a uniform proof.

5.2. *A sufficient condition that* $A(H^n(X', \mathbb{Z}/2), \tilde{q}) = 0$
Consider again the diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & S^{2n} \\ \pi \downarrow & & \downarrow \pi \\ X & \xrightarrow{f} & \mathbb{R}P(2n) \end{array}$$

Let $Z^{2n-1} = f^{-1}(\mathbb{R}P(2n-1)) \subset X^{2n}$. Transversality implies that Z^{2n-1} has a regular neighborhood in X homeomorphic to the total space of a line bundle $\xi \to Z^{2n-1}$. Let $V = X - E(\xi)$, where $E(\xi)$ is the open unit interval bundle of ξ. Then $\pi^{-1}(V) \subset X'$ is the disjoint union of two copies of V, say V_0 and V_1. Moreover, $\partial V = \partial E(\xi)$ admits an involution, s, since it double covers Z. X' is constructed from the two copies of V by identifying $p \in \partial V_0$ with $s(p) \in \partial V_1$,

$$X' = V_0 \cup V_1/(p, 0) = (s(p), 1) \quad \text{if} \quad p \in \partial V_0$$

The involution $S: X' \to X'$ over X is defined by $S(p, i) = (p, 1-i)$, all $p \in V_i$, $i = 0, 1$. (If $p \in \partial V_0$, $S(p, 0) = (p, 1) \equiv (s(p), 0) = S(s(p), 1)$, so S is well-defined.)

Definition 5.2.1. $\pi: X' \to X$ is *Poincaré splittable* if after a homotopy of $f: X \to \mathbb{R}P(2n)$, the pair $(V, \partial V)$ is a $2n$-dimensional P.D. space with boundary, and $Z = \partial V/s$ is a $(2n-1)$ dimensional PD space, P.D. embedded in X.

Of course, if $\pi: X' \to X$ is a double cover of *manifolds*, it is splittable by manifold transversality. There are obstructions to Poincaré splittability, in general. For example, our next result shows that the Arf invariant $A(H^n(X', \mathbb{Z}/2), \tilde{q}) \in \mathbb{Z}/2$ is such an obstruction. All we require for this argument is the simple chain formula 5.1.2 for $\tilde{q}$.

PROPOSITION 5.2.2. *If $\pi: X' \to X$ is a Poincaré splittable double cover of $2n$ dimensional PD spaces, then*

$$A(H^n(X', \mathbb{Z}/2), \tilde{q}) = 0.$$

Proof. We consider the cohomology exact sequence of the pair (X', V_0), with $\mathbb{Z}/2$ coefficients.

$$\cdots \to H^{n-1}(V_0) \to H^n(X', V_0) \xrightarrow{p} H^n(X') \xrightarrow{i^*} H^n(V_0) \to H^{n+1}(X', V_0) \to \cdots$$

By excision $j^*: H^*(X', V_0) \xrightarrow{\sim} H^*(V_1, \partial V_1)$, where $j: V_1, \partial V_1 \to X', V_0$ is the inclusion. Thus there is a natural dual pairing

$$H^n(X', V_0) \otimes H^n(V_0) \to \mathbb{Z}/2$$

by identifying $H^n(X', V_0)$ with $H^n(V_1, \partial V_1)$ using j^*, and identifying $H^n(V_0)$ with $H^n(V_1)$ using S. Moreover, if $a \in H^n(X')$, $b \in H^n(X', V_0)$,

$$\begin{aligned} \langle j^*b \cdot Si^*a, [V_1, \partial V_1]\rangle &= \langle S^*p^*b \cdot a, [X']\rangle \\ &= \tilde{\ell}(p^*b, a). \end{aligned}$$

We are thus in the algebraic situation considered in 1.4.4 and 1.4.5, with the simplification that the sequence

$$H^n(X', V_0) \xrightarrow{p^*} H^n(X') \xrightarrow{j^*} H^n(V_0)$$

is exact. Moreover, if $b \in H^n(X', V_0)$ then $\tilde{q}p^*(b) = 0$. Simply choose a cocycle representative $b_\#$ for b with support in the interior of V_1. Then $b_\#$ and

$S^\# b_\#$ have disjoint support and hence

$$\tilde{q}p^*(b) = (1/2)\langle b_\# \cup Sb_\#, [X']\rangle = 0.$$

The proposition now follows, as in Remark 1.4.6.

Conjecture 5.2.3. In general, we conjecture that the Arf invariant $A(H^n(X', \mathbb{Z}/2, \tilde{q}) \in \mathbb{Z}/2$ is the only obstruction to the Poincaré splittability of $\pi: X' \to X$.

5.3. *The Kervaire obstruction of certain non-principal triple covers*

In this section we give an application of Prop. 5.2.2.

We have maps of groups $\mathbb{Z}/2 \xrightarrow{i} \mathcal{S}_3 \xrightarrow{j} \mathbb{Z}/2$ with $ji = Id$, namely i is defined by any two cycle, say (12), in $\mathcal{S}_3$, and j is projection on $\mathcal{S}_3/\mathcal{A}_3 = \mathbb{Z}/2$. Thus there are maps of classifying spaces

$$\mathbb{R}P(\infty) \xrightarrow{B_i} B\mathcal{S}_3 \xrightarrow{B_j} \mathbb{R}P(\infty),$$

and Bi*, Bj^* are isomorphisms on $\mathbb{Z}/2$ cohomology (Bi^* is injective since $i(\mathbb{Z}/2) \subset \mathcal{S}_3$ is a Sylow 2-subgroup and Bi^* is surjective since $Bi^* \circ Bj^* = Id$.)

Consider a map $f: X \to \mathbb{R}P(\infty)$, classifying a double cover $\pi: X' \to X$, X a $2n$-dimensional PD space, and consider the triple cover classified by $Bi \circ f: X \to \mathbb{R}P(\infty) \to B\mathcal{S}_3$. This is, of course, simply $\pi + Id: X' + X \to X$, where + indicates disjoint union.

Let $q: H^n(X, \mathbb{Z}/2) \to \mathbb{Z}/4$ be a quadratic function, $q': H^n(X', \mathbb{Z}/2) \to \mathbb{Z}/4$ the induced quadratic function, and let $\tilde{q} = q' - q\tau$, $\tau: H^n(X', \mathbb{Z}/2) \to H^n(X, \mathbb{Z}/2)$ the transfer for the double cover $\pi: X' \to X$. The triple cover $\pi + Id: X' + X \to X$ has a Kervaire obstruction in $\mathbb{Z}/2$, defined in 3.3.4. Our result is

PROPOSITION 5.3.1. $A\left(X' + X \xrightarrow{\pi + Id} X\right) = A(H^n(X', \mathbb{Z}/2), \tilde{q}) + \left\langle V^2(X)\left(\sum_{j=1}^{\infty} w_1^{2^j-2}\right), [X]\right\rangle$ *where* $w_1 = f^*(e) \in H^1(X, \mathbb{Z}/2), f: X \to \mathbb{R}P(\infty)$.

Proof. The transfer τ_0 associated to the triple cover $X' + X \to X$ is given by

$$\tau_0(a \oplus b) = \tau(a) + b \in H^n(X, \mathbb{Z}/2)$$

if $a \in H^n(X', \mathbb{Z}/2)$, $b \in H^n(X, \mathbb{Z}/2)$. We define an isomorphism

$$H^n(X', \mathbb{Z}/2) \xrightarrow{\sim} K_0^n = \text{Kernel}(\tau_0)$$

by assigning $a + \tau a$ to $a \in H^n(X', \mathbb{Z}/2)$. Let $q_0 : K_0^n \to \mathbb{Z}/2$ be the quadratic function whose Arf invariant is $A(X' + X \to X)$. Then

$$\begin{aligned} q_0(a + \tau a) &= q'(a) + q(\tau a) \\ &= \tilde{q}(a) + 2q(\tau a) \\ &= \tilde{q}(a) + 2\langle \tau a \cdot v_n(X), [X]\rangle \\ &= \tilde{q}(a) + 2\langle a \cdot \pi^* v_n(X), [X']\rangle \in \mathbb{Z}/4 \end{aligned}$$

By 1.4.2(c) and 5.1.3, we compute in $\mathbb{Z}/8$

$$\begin{aligned} A(K_0^n, q_0) &= A(H^n(X', \mathbb{Z}/2), \tilde{q}) - 2\tilde{q}(\pi^* V_n(X)) \\ &= A(H^n(X', \mathbb{Z}/2), \tilde{q}) - 4\left\langle V_n(X) \cdot \left(V(X) \sum_{i=0}^{\infty} w_1^{2^i - 1}\right), [X]\right\rangle \\ &= A(H^n(X', \mathbb{Z}/2), \tilde{q}) - 4\left\langle V^2(X)\left(\sum_{j=1}^{\infty} w_1^{2^j - 2}\right), [X]\right\rangle. \end{aligned}$$

This proves 5.3.1.

Remark 5.3.2. For any odd m, consider the Dyer-Lashof map $\mathscr{D}: B\mathscr{S}_m \to Q_m S^0$. There are surgery obstruction classes $k_{2j} \in H^{2j}(Q_m S^0, \mathbb{Z}/2)$ which measure the Kervaire obstruction of degree m normal maps associated to *manifolds* $\ell: M \to Q_m S_0$. Specifically, if s_k is the Kervaire obstruction

$$s_k(M, \ell) = \left\langle V^2(M)\mathscr{D}^*\left(\sum_{j=0}^{\infty} k_{2j}\right), [M]\right\rangle \in \mathbb{Z}/2.$$

It is not too hard to relate these classes rather closely to the (more familiar) special case of the degree $m = 1$ component of QS^0. In particular, $k_{2j} = 0$ unless $2j = 2^i - 2$, $i \geq 1$. Also (related to the results of §5), in degree 0,

$$k_0 = \begin{cases} 1 & \text{if } m \equiv 3 \text{ or } 5 \pmod 8. \\ 0 & \text{if } m \equiv 1 \text{ or } 7 \pmod 8. \end{cases}$$

If $m = 3$, one can compute $\mathscr{D}^*(k_{2j})$ pretty easily since $Bi: \mathbb{R}P(\infty) \to B\mathscr{S}_3$ is a $\mathbb{Z}/2$ cohomology isomorphism. The result is

$$\mathscr{D}^*(k_{2^j - 2}) = w_1^{2^i - 2} \in H^{2^i - 2}(B\mathscr{S}_3),$$

where

$$w_1 = (Bj)^* e, \; Bj: B\mathscr{S}_3 \to \mathbb{R}P(\infty) = B(\mathscr{S}_3/\mathscr{A}_3).$$

If we combine 5.2.2 and 5.3.1, we have thus proved the following

PROPOSITION 5.3.3. *Let* $f: X \to \mathbb{R}P(\infty)$, $\varphi = \mathscr{D} \circ Bi \circ f: X \to Q_3 S^0$, *$X$ a $2n$-dimensional PD space. Let $\pi: X' \to X$ be the double cover classified by f, and assume it to be Poincaré splittable* (see .2.1). *Let $\pi + Id: X' + X \to X$ be the triple cover $Bi \circ f: X \to \mathbb{R}P(\infty) \to B\mathscr{S}_3$. Then*

$$A(X' + X \xrightarrow{\pi + Id} X) = \left\langle V^2(X) \cdot \varphi^*\left(\sum_{j=1}^{\infty} k_{2^j - 2}\right), [X]\right\rangle$$

In general, for a PD space $f: X^{2n} \to B\mathscr{S}_m$, m odd, one expect the difference

$$A(\hat{X} \xrightarrow{\hat{\pi}} X) - \left\langle V^2(X) f^* \mathscr{D}^*\left(\sum_{j=1}^{\infty} k_{2^j - 2}\right), [X]\right\rangle \in \mathbb{Z}/2$$

to measure some kind of Poincaré transversality obstruction for the m-fold cover $\hat{\pi}: \hat{X} \to X$ classified by f. A plausible conjecture is that this difference is the Poincaré splittability obstruction of the double cover $\pi: X' \to X$ classified by $X \to B\mathcal{S}_m \to B(\mathcal{S}_m/\mathcal{A}_3) = \mathbb{R}P(\infty)$, which, in turn, by Conjecture 5.2.3, should be the Arf invariant $A(H^n(X', \mathbb{Z}/2), \tilde{q})$.

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