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QUADRATIC FUNCTIONS, THE INDEX MODULO 8, AND A $\mathbb{Z}/4$-HIRZEBRUCH FORMULA

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§1. INTRODUCTION

IF M^{4n} is a closed, oriented, smooth manifold then the index of M^{4n}, $I(M) \in \mathbb{Z}$, is given by

$$I(M) = \langle L(\nu_M), [M] \rangle \qquad (*)$$

where $L(\xi) = 1 + L_1(\xi) + L_2(\xi) + \ldots$, $L_i(\xi) \in H^{4i}(B; Q)$, is a characteristic class for vector bundles, $\xi \to B$. $L(\xi)$ is given by the inverse of the Hirzebruch polynomial in the Pontrjagin classes (see [3, page 86]). Thom extended this class to *p.l.* bundles and, in fact, showed that there is a unique characteristic class

$$L(\xi) = 1 + L_1(\xi) + \ldots, \; L_i(\xi) \in H^{4i}(B; \mathbb{Q})$$

for *p.l.* bundles, $\xi \to B$, which is multiplicative, i.e. $L(\xi \oplus \eta) = L(\xi) \cdot L(\eta)$, and such that (*) holds for all *p.l.* manifolds [9]. This extends to topological bundles and manifolds by the triangulation theory in [4].

Thus $I(M)$ is a function of the characteristic classes of the geometric normal bundle of M. If we are only interested in the index modulo 2, then the situation simplifies.

$$I(M) \equiv \operatorname{rank}_{\mathbb{Z}/2}(H^{2n}(M; \mathbb{Z}/2)) \qquad (\text{mod } 2)$$

which is obviously an invariant of the homotopy type of M. Furthermore this rank may be calculated in terms of characteristic classes of the homotopy type of the normal bundle of M. An easy argument shows

$$I(M) \equiv \langle {v_{2n}}^2(\nu_M), [M] \rangle \quad \text{mod } 2 \qquad (**)$$

where $v_{2n}(\nu_M) \in H^{2n}(M, \mathbb{Z}/2)$ is the Wu class of the normal bundle. Formula (**) is valid for any space satisfying Poincaré duality where the role of the normal bundle is played by the Spivak normal fibration [11].

The above formulae may be extended to a larger class of "manifolds", $\mathbb{Z}/2^r$ manifolds. A $\mathbb{Z}/2^r$ manifold is an oriented manifold, M, together with an orientation preserving isomorphism of 2^r copies of a manifold $\delta\tilde{M}$ with ∂M, $\varphi\colon \amalg_{2^r} \delta\tilde{M} \to \partial M$. $\tilde{M}$ denotes the space obtained by collapsing the 2^r copies of $\delta\tilde{M}$ together. Unless $r = 1$, $\tilde{M}$ is not a manifold but does possess a fundamental class $[\tilde{M}] \in H_n(\tilde{M}; \mathbb{Z}/2^r)$, an oriented normal bundle, $\nu_{\tilde{M}}$, and an index, which modulo 2^r is a bordism invariant [10]. A $\mathbb{Z}/2^r$ *P.D.* space is $(X^n, \varphi\colon \amalg_{2^r} \delta\tilde{X} \to \partial X)$ where $(X, \partial X)$ is a *P.D.* pair and φ is a homotopy equivalence. Let $\tilde{X}$ be the associated quotient space. As in the manifold case it has a fundamental class $[\tilde{X}] \in H_n(\tilde{X}; \mathbb{Z}/2^r)$, an oriented normal fibration $\nu_{\tilde{X}}$, and an index which modulo 2^r is a bordism invariant.

Thom's theorem generalizes to the following. There is a unique multiplicative characteristic class

$$L(\xi) = 1 + L_1(\xi) + \dots, L_i(\xi) \in H^{4i}(B; \mathbb{Z}_{(2)}) \quad \text{for} \quad \xi \to B$$

a topological bundle for which (*) is satisfied for closed topological manifolds and (*) is satisfied mod 2^r for closed $\mathbb{Z}/2^r$-manifolds (see [10]).

The above discussion about the index modulo 2 holds more generally, and we have

$$I(\tilde{X}) \equiv \langle v_{2n}{}^2(\nu_{\tilde{X}}), [\tilde{X}] \rangle \mod 2$$

for any $\mathbb{Z}$ or $\mathbb{Z}/2^r$ $P.D.$ space. Since v^2 is multiplicative it follows that $v_{2n}{}^2(\xi) \equiv L_n(\xi) \bmod 2$ for any topological bundle ξ.

One object of this paper is to make an analogous discussion modulo 4. That is, find a characteristic class of spherical fibrations

$$\xi \to B, \quad l(\xi) = 1 + l_1(\xi) + l_2(\xi) + \dots, l_i(\xi) \in H^{4i}(B; \mathbb{Z}/4),$$

so that

$$I(\tilde{X}) \equiv \langle l(\nu_{\tilde{X}}), [\tilde{X}] \rangle \mod 4$$

for any $\mathbb{Z}$ or $\mathbb{Z}/2^r$ manifold or $P.D.$ space, $r \geqq 2$, and so that $l(\xi \oplus \eta) = l(\xi) \cdot l(\eta)$. We first prove (Corollary 6.3) that

$$I(\tilde{X}) \equiv \langle \mathscr{P}(v_{2n}(\nu_{\tilde{X}})), [\tilde{X}] \rangle \mod 4 \qquad (***)$$

where $\mathscr{P}$ is the Pontrjagin square. By adding to $1 + \mathscr{P}(v_2(\xi)) + \mathscr{P}(v_4(\xi)) + \dots$, a polynomial in the Stiefel–Whitney classes, $i_* \sigma_n(w_2(\xi), \dots, w_{4n}(\xi))$, we can produce $l(\xi) \in H^{4*}(B; \mathbb{Z}/4)$ which still gives the index mod 4 for any $\mathbb{Z}$ or $\mathbb{Z}/2^r$ $P.D.$ space ($r \geqq 2$), and which is multiplicative. By uniqueness $l(\xi) \equiv L(\xi) \bmod 4$ for any topological bundle.

To construct l we use only elementary facts from the theory of quadratic functions and cocycle arguments. By using the deeper results of [5] and [6] on Poincaré transversality we define, in another paper, a canonical $\mathbb{Z}/8$ characteristic class for spherical fiber spaces which gives the index mod 8 of any $\mathbb{Z}$ or $\mathbb{Z}/2^r$ $P.D.$ space, $r \geqq 3$, and which agrees for topological bundles with L mod 8. This class restricts mod 4 to l, however it cannot be chosen to be multiplicative.

A key step in the proof of (***) is Theorem 4.3 which is a formula for the index modulo 8 of a $P.D.$ space with boundary, $(W, \partial W)$. This formula is based on a "Gauss sum" formula of Van der Blij [14], for the index modulo 8 of a symmetric matrix. Specifically, we show that if $\hat{v} \in H^{2n}(W, \partial W; \mathbb{Z}/2)$ is a lifting of the Wu class $v_{2n}(\nu_W) \in H^{2n}(W, \mathbb{Z}/2)$, and ψ is a quadratic function on the torsion in $H^{2n}(\partial W, \mathbb{Z})$ which is "compatible" with $\hat{v}$, then $I(W)$ modulo 8 is given by a formula involving $\hat{v}$ and the Arf invariant of ψ. The Arf invariant of ψ is the 8th root of unity $\operatorname{Arg}(\sum e^{2\pi i \psi(x)}) \in S^1$. This $\mathbb{Z}/8$ invariant was first studied in the context of manifold theory by E. H. Brown [1]. More recently it was used by Milgram in [8] and Morgan-Sullivan in [10].

§2 is devoted to developing the requisite theory of quadratic functions and quoting

Van der Blij's theorem. In §3 we study intersection pairings and linking pairings in $P.D.$ spaces. We work in cohomology and use cocycle arguments. If we were only interested in manifolds we could work dually in homology and use embedded submanifolds. It is through this point of view that we first were able to prove Theorem 4.3 for manifolds. It was then necessary to translate to cohomology in order to get proofs that work for $P.D.$ spaces as well.

In §4 we apply the results in §2 and §3 to produce the formula for the index modulo 8 of a $P.D.$ space with boundary. We assume here the existence of compatible liftings of the Wu class and quadratic functions. This is proved in §5. We also prove an equivariant version of compatibility for $\mathbb{Z}/2$-manifolds which is used in the applications.

The applications are given in §6. In addition to the formula for the mod 4 L-class discussed above, we give a formula for the invariant $\sigma(M^{4n-1})$ of a closed, oriented, smooth manifold, M^{4n-1} where $\sigma(M^{4n-1})$ is the number modulo 2 of 2-primary summands in the torsion subgroup of $H^{2n}(M, \mathbb{Z})$. In [7] the analogous invariant for manifolds of dimension $4n+1$ was expressed as a $\mathbb{Z}/2$ cohomology characteristic number, specifically $v_{2n} \cdot Sq^1 v_{2n}$. For manifolds of dimension $4n-1$ such a formula is impossible, since, for example, $\mathbb{R}P(3)$ is parallelizable, but $\sigma(\mathbb{R}P(3)) = 1$. Our formula is a kind of secondary characteristic class formula for $\sigma(M^{4n-1})$. Specifically, let $2M^{4n-1} = \partial W^{4n}$ where W^{4n} is a smooth manifold. Form the resulting $\mathbb{Z}/2$ manifold $\tilde{W}$ and let $\hat{v} \in H^{2n}(\tilde{W}, \delta\tilde{W}, \mathbb{Z}/2) = H^{2n}(W, \partial W, \mathbb{Z}/2)$ lift the Wu class $v_{2n}(v_{\tilde{W}}) \in H^{2n}(\tilde{W}, \mathbb{Z}/2)$. Let $2\colon \mathbb{Z}/2 \to \mathbb{Z}/4$ be the inclusion. Then we prove (Corollary 6.2)

$$2\sigma(M^{4n-1}) = I(W) - \langle \mathscr{P}(\hat{v}), [W, \partial W]\rangle \in \mathbb{Z}/4$$

for any such choice of W and $\hat{v}$.

§2. QUADRATIC FUNCTIONS

Let K be a finite abelian group and let $L\colon K \times K \to Q/\mathbb{Z}$ be a symmetric, bilinear, non-singular pairing. (Here, non-singular means if $L(x, y) = 0$ for all $x \in K$ then $y = 0$.) Such maps L will be called pairings. L identifies K with its dual $K^* = \mathrm{Hom}(K, Q/\mathbb{Z})$. There are obvious notions of an isomorphism between two pairings and orthogonal direct sum of two pairings. The negative of a pairing (K, L), denoted $(K, -L)$, is defined by

$$(-L)(x, y) = -L(x, y) \in Q/\mathbb{Z}.$$

A *quadratic function* over L is a function $\psi\colon K \to Q/\mathbb{Z}$ which satisfies

$$\psi(x+y) - \psi(x) - \psi(y) = L(x, y), \qquad x, y \in K, \quad \text{and} \tag{1}$$

$$\psi(nx) = n^2\psi(x), \qquad n \in \mathbb{Z}, \qquad x \in K. \tag{2}$$

We point out that, given (1), condition (2) is equivalent to

$$2\psi(x) = L(x, x), \qquad x \in K. \tag{2'}$$

In this section we will develop some of the theory of pairings and quadratic functions and relate them to symmetric inner products and quadratic functions on free abelian groups. In our applications, K will generally be the torsion subgroup of $H^{2n}(M, \mathbb{Z})$, where M is a closed, oriented $(4n-1)$ manifold, and L will be the classical linking pairing of torsion cocycles.

Here is one way that pairings arise. Let F be a free abelian group of finite rank and let $B: F \times F \to \mathbb{Z}$ be a symmetric pairing. B is equivalent to a map $B: F \to F^* = \mathrm{Hom}(F, \mathbb{Z})$. We assume that $\det(B) \neq 0$. Let K_B denote the cokernel of $B: F \to F^*$. Then K_B is a finite abelian group, $|K_B| = |\det(B)|$, and we have an exact sequence

$$0 \to F \xrightarrow{B} F^* \xrightarrow{\partial} K_B \to 0.$$

Thus B induces an isomorphism $B: F \otimes Q \xrightarrow{\sim} F^* \otimes Q$. We define $L_B: K_B \times K_B \to Q/\mathbb{Z}$ by $L_B(x + F, y + F) = \langle x, B^{-1}(y)\rangle \in Q/\mathbb{Z}$, where $x, y \in F^*$ and $B^{-1}(y) \in F \otimes Q$. One checks easily that L_B is well-defined and that (K_B, L_B) is a non-singular, symmetric, bilinear pairing.

THEOREM 2.1. *Any pairing (K, L) is isomorphic to (K_B, L_B) for some $B: F \to F^*$.*

Proof. This is proved in Theorem 6 of [15]. (We will not make essential use of Theorem 2.1 below.)

THEOREM 2.2. *Given a bilinear pairing (K, L) there exist quadratic functions over L. Such quadratic functions correspond bijectively with elements $y \in K$ such that $2y = 0$.*

Proof. $K \simeq K_{(\mathrm{odd})} \oplus K_{(2)}$, where $K_{(\mathrm{odd})}$ is the subgroup of elements of odd order and $K_{(2)}$ is the subgroup of elements of order a power of 2. Since we must have $2\psi(x) = L(x, x)$ and $\lambda^2\psi(x) = \psi(\lambda x) = 0$ if $\lambda x = 0$, there is a unique choice for $\psi: K_{(\mathrm{odd})} \to Q/\mathbb{Z}$ and one checks that this ψ is quadratic. If $x_1 \cdots x_r$ is a minimal set of generators for $K_{(2)}$, choose $\alpha_i \in Q/\mathbb{Z}$ with $2\alpha_i = L(x_i, x_i)$ and set $\psi(x_i) = \alpha_i$. Then there is a unique extension to a quadratic function $\psi: K_{(2)} \to Q/\mathbb{Z}$, namely

$$\psi\left(\sum n_i x_i\right) = \sum_i n_i^2\alpha_i + \sum_{i<j} n_i n_j L(x_i, x_j), \qquad n_i \in \mathbb{Z}.$$

If $\psi, \psi': K \to Q/\mathbb{Z}$ are two quadratic functions over L then $\psi - \psi': K \to Q/\mathbb{Z}$ is linear and $2(\psi - \psi') = 0$. Thus all quadratic functions ψ' are obtained by choosing $y \in K$, $2y = 0$ and setting $\psi_y(x) = \psi(x) + L(y, x) \in Q/\mathbb{Z}$, $x \in K$.

The following construction also produces quadratic functions. Let

$$0 \to F \xrightarrow{B} F^* \xrightarrow{\partial} K_B \to 0$$

be as above. By a *Wu class* for B we mean $v \in F$ such that $\langle Bz, v\rangle \equiv \langle Bz, z\rangle \pmod 2$ for all $z \in F$. Given B and a Wu class v we define $\psi_v: K_B \to Q/\mathbb{Z}$ by

$$\psi_v(x + F) = \tfrac{1}{2}(\langle x, B^{-1}(x)\rangle - \langle x, v\rangle) \in Q/\mathbb{Z}, \tag{2.3}$$

where $x \in F^*$. It is not hard to check that ψ_v is well-defined and that ψ_v is quadratic over (K_B, L_B).

A generalization of Theorem 2.1 is that all quadratic functions $\psi: K \to Q/\mathbb{Z}$ arise as $\psi_v: K_B \to Q/\mathbb{Z}$.

THEOREM 2.4. *Let $B: F \to F^*$, (K_B, L_B) be as above. Then the function $v \mapsto \psi_v$ is a bijective correspondence between* mod 2 *reductions of Wu classes for B and quadratic functions over (K_B, L_B).*

Proof. It is clear that the function ψ_v defined in (2.3) depends only on the class of v in

$F \otimes \mathbb{Z}/2$. Conversely, given a quadratic function $\psi\colon K_B \to Q/\mathbb{Z}$, define $v_\psi\colon F^* \to \mathbb{Z}/2$ as follows:

$$v_\psi(x) = \tfrac{1}{2}\langle x, B^{-1}(x)\rangle - \psi(\partial x) \in \tfrac{1}{2}\mathbb{Z}/\mathbb{Z} \subset Q/\mathbb{Z}. \tag{2.5}$$

We have $v_\psi(x_1 + x_2) = v_\psi(x_1) + v_\psi(x_2)$ since ψ is quadratic over L_B. Thus $v_\psi\colon F^* \to \mathbb{Z}/2$ is equivalent to an element $v \in F \otimes \mathbb{Z}/2$ and v is the $\mathbb{Z}/2$ reduction of a Wu class since $\langle Bz, v\rangle = v_\psi(Bz) = \langle Bz, z\rangle$, regarded as elements of $\mathbb{Z}/2$. It is clear that the constructions (2.3) and (2.5) are inverses of each other, and (2.4) follows.

Let $B\colon F \to F^*$ be as above and let $v \in F$ be a Wu class for B. It is easy to see that $\langle Bv, v\rangle$ modulo 2 is independent of v. Our next result relates the index of B, the invariant $\langle Bv, v\rangle \in \mathbb{Z}/2$ and an invariant of K_B.

THEOREM 2.6. index$(B) - \langle Bv, v\rangle \equiv \mathrm{rank}_{\mathbb{Z}/2}(K_B \otimes \mathbb{Z}/2) \pmod 2$.

Proof. There is an exact sequence

$$F \otimes \mathbb{Z}/2 \xrightarrow{B_2} F^* \otimes \mathbb{Z}/2 \xrightarrow{\partial_2} K_B \otimes \mathbb{Z}/2 \to 0.$$

Thus $\mathrm{rank}(F^*) - \mathrm{rank}_{\mathbb{Z}/2}(\mathrm{image}(B_2)) = \mathrm{rank}_{\mathbb{Z}/2}(K_B \otimes \mathbb{Z}/2)$. But $\mathrm{rank}(F^*) \equiv \mathrm{index}(B) \pmod 2$. Also, B_2 defines a non-singular bilinear form on $(F \otimes \mathbb{Z}/2)/(\mathrm{kernel}(B_2))$, which is isomorphic to image (B_2), and $v \in F \otimes \mathbb{Z}/2$ is a Wu class for this non-singular form. Then, by a well-known argument, $\mathrm{rank}_{\mathbb{Z}/2}(\mathrm{image}(B_2)) \equiv \langle B_2(v), v\rangle = \langle Bv, v\rangle \pmod 2$, and (2.6) follows.

Let $\psi\colon K \to Q/\mathbb{Z}$ be a quadratic function over a pairing L. We will study the "Gaussian sum" invariant

$$a(K, \psi) = \sum_{x \in K} e^{2\pi i \psi(x)} \in \mathbb{C}.$$

We first state certain elementary properties of $a(K, \psi)$. For proofs, see [2, Theorem 4.3].

THEOREM 2.7. (i) $a(K, -\psi) = \overline{a(K, \psi)}$ *where $\bar{a}$ is the conjugate of $a \in \mathbb{C}$.*

(ii) $a(K_1 \oplus K_2, \psi_1 \oplus \psi_2) = a(K_1, \psi_1) \cdot a(K_2, \psi_2)$.

(iii) $\|a(K, \psi)\| = |K|^{1/2}$. *In particular,* $a(K, \psi) \neq 0$.

A deeper formula is a theorem of F. van der Blij. See [14] for a proof which uses properties of Fourier series.

THEOREM 2.8. *Let $B\colon F \to F^*$, (K_B, L_B) be as above. Let $v \in F$ be a Wu class for B and let ψ_v be the associated quadratic function on K_B. Then*

$$|K_B|^{1/2} e^{\pi i \cdot \mathrm{index}(B)/4} = e^{\pi i \langle Bv, v\rangle/4}\left(\sum_{x \in K_B} e^{2\pi i \psi_v(x)}\right).$$

If we let $A(K_B, \psi_v) = \mathrm{Arg}(a(K_B, \psi_v)) \in S^1$ then (2.8) implies that $A(K_B, \psi_v)$ is an element of $\tfrac{1}{8}\mathbb{Z}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z} = S^1$. There is the obvious isomorphism $\cdot 8\colon \tfrac{1}{8}\mathbb{Z}/\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/8\mathbb{Z}$ and we will usually regard $A(K_B, \psi_v)$ as an element of $\mathbb{Z}/8\mathbb{Z}$. Thus we have

$$\mathrm{index}(B) = \langle Bv, v\rangle + A(K_B, \psi_v) \in \mathbb{Z}/8. \tag{2.9}$$

We regard (2.9) as a modulo 8 generalization of (2.6). Note that it follows from (2.6) that $A(K_B, \psi_v) \equiv \mathrm{rank}_{\mathbb{Z}/2}(K_B \otimes \mathbb{Z}/2) \pmod 2$.

Since by (2.1) and (2.4) any quadratic function (K, ψ) is isomorphic to (K_B, ψ_v) for some B and some Wu class v, we deduce that $A(K, \psi) = \mathrm{Arg}(a(K, \psi))$ is always in $\mathbb{Z}/8\mathbb{Z} \simeq \tfrac{1}{8}\mathbb{Z}/\mathbb{Z} \subset S^1$ and

$$A(K, \psi) \equiv \mathrm{rank}_{\mathbb{Z}/2}(K \otimes \mathbb{Z}/2) \pmod 2. \tag{2.10}$$

If K is a $\mathbb{Z}/2$ vector space then $A(K, \psi) \in \mathbb{Z}/8$ coincides with the Arf invariant studied by E. H. Brown, Jr. [1]. In general, then, we will refer to $A(K, \psi)$ as the Arf invariant of ψ.

In §3 and §4, we will study the torsion subgroups in the cohomology sequence

$$H^{2n}(W, \mathbb{Z}) \to H^{2n}(\partial W, \mathbb{Z}) \to H^{2n+1}(W, \partial W, \mathbb{Z})$$

where $(W, \partial W)$ is an oriented $4n$-manifold with boundary. The final result in this section isolates the algebraic results we will need then.

Let $\psi: K \to Q/\mathbb{Z}$ be a quadratic function over a pairing L, and let $i: G \to K$ be a homomorphism. Using L, we identify K with $K^* = \mathrm{Hom}(K, Q/\mathbb{Z})$. Form the dual map $i^*: K \to G^*$. Suppose that the sequence

$$G \xrightarrow{\ i\ } K \xrightarrow{\ i^*\ } G^*$$

satisfies $i^* \cdot i = 0$. Let $\bar{K} = \mathrm{kernel}(i^*)/\mathrm{image}(i)$. Then L induces a non-singular pairing $\bar{L}$ on $\bar{K}$, defined by $\bar{L}(x + G, y + G) = L(x, y) \in Q/\mathbb{Z}$ for $x, y \in \mathrm{kernel}(i^*)$. $\bar{L}$ is well-defined since $L(x, i(g)) = \langle i^*(x), g\rangle = 0$ if $x \in \mathrm{kernel}(i^*)$.

Since L vanishes on $G \otimes G$, $\psi \cdot i: G \to Q/\mathbb{Z}$ is linear and $2\psi \cdot i = 0$. Thus there is a unique element $a \in G^*$ such that $2a = 0$ and $\psi i(g) = \langle a, g\rangle$ for all $g \in G$.

THEOREM 2.11. (i) *$a = i^*(b)$ for some $b \in K$ with $2b \in i(G)$.* (ii) *$\psi_b(x + G) = \psi(x) - L(b, x) \in Q/\mathbb{Z}$, $x \in \mathrm{kernel}(i^*)$, defines a quadratic function $\psi_b: \bar{K} \to Q/\mathbb{Z}$ over $\bar{L}$, where b is as in* (i). (iii) *$A(\bar{K}, \psi_b) = A(K, \psi) - \psi(b) \in \frac{1}{8}\mathbb{Z}/\mathbb{Z}$.* (Since $2b \in i(G)$, $0 = 2\psi(2b) = 8\psi(b) \in Q/\mathbb{Z}$, hence $\psi(b) \in \frac{1}{8}\mathbb{Z}/\mathbb{Z} \subset Q/\mathbb{Z}$ and Theorem 2.11(iii) makes sense.)

Proof. In general if $x \in G^*$ then $x \in i^*(K)$ if $\langle x, g\rangle = 0$ for all $g \in G$ with $i(g) = 0$. Since $\langle a, g\rangle = \psi \cdot i(g)$, we see that $a = i^*(b')$ for some $b' \in K$. Since $2a = 0$, $2b' \in \mathrm{kernel}(i^*)$.

For any such choice of b', $\psi_{b'}: \bar{K} \to Q/\mathbb{Z}$, defined in (ii), is well-defined and satisfies $\psi_{b'}(x + y) = \psi_{b'}(x) + \psi_{b'}(y) + \bar{L}(x, y)$, for $x, y \in \bar{K}$. Let $\bar{\psi}: \bar{K} \to Q/\mathbb{Z}$ be a quadratic function over $\bar{L}$. Then $\psi_{b'} - \bar{\psi}: \bar{K} \to Q/\mathbb{Z}$ is linear, hence $\psi_{b'}(x) - \bar{\psi}(x) = \bar{L}(x, z)$ for some $z \in \bar{K}$. Let $z' \in \mathrm{kernel}(i^*)$ represent $z \in \bar{K}$. Then $2b' - 2z' \in \mathrm{image}(i)$ since for all $x \in \mathrm{kernel}(i^*)$, $L(x, 2z') = 2\psi_{b'}(x) - 2\bar{\psi}(x) = 2\psi_{b'}(x) - L(x, x) = L(x, 2b')$. Thus, if we replace b' by $b = b' - z'$, then $a = i^*(b)$ and $2b \in i(G)$. It is easy to check that $2\psi_b(x) = \bar{L}(x, x)$ for $x \in \bar{K}$, hence ψ_b is quadratic over $\bar{L}$. This completes the proof of Theorem 2.11(i) and (ii).

To prove (iii), choose subsets $K_2 \subset K$ such that $-b \in K_2$ and $K_2 \to i^*(K)$ is a bijection and $K_1 \subset \mathrm{kernel}(i^*)$ such that $K_1 \to \bar{K}$ is a bijection. Each element of K has a unique expression $g + x_1 + x_2$, where $g \in i(G)$, $x_1 \in K_1$, $x_2 \in K_2$.

$$\begin{aligned} a(K, \psi) &= \sum_{g, x_1, x_2} e^{2\pi i\psi(g + x_1 + x_2)} \\ &= \sum_{g, x_1, x_2} e^{2\pi i(\psi(g) + \psi(x_1) + \psi(x_2) + L(x_2, g + x_1))} \\ &= \sum_{x_2 \in K_2} e^{2\pi i\psi(x_2)}\left(\sum_{x_1 \in K_1} e^{2\pi i(\psi(x_1) + L(x_2, x_1))}\left(\sum_{g \in i(G)} e^{2\pi i L(b + x_2, g)}\right)\right) \end{aligned}$$

since $\psi(g) = L(b, g)$. Since the sum of the elements in a non-trivial subgroup of S^1 vanishes, the terms above vanish unless $x_2 = -b$. Thus

$$
\begin{aligned}
a(K, \psi) &= e^{2\pi i \psi(-b)} \left(\sum_{x_1 \in K_1} e^{2\pi i (\psi(x_1) - L(b, x_1))} \right) \cdot |i(G)| \\
&= e^{2\pi i \psi(-b)} \cdot a(\bar{K}, \psi_b) \cdot |i(G)|
\end{aligned}
$$

and Theorem 2.11(iii) follows.

§3. CONSEQUENCES OF POINCARÉ DUALITY

In this section we collect some well-known facts implied by Poincaré duality, and prove some technical lemmas needed in later sections. If X is a finite complex, denote by $T^k(X)$ the torsion subgroup of $H^k(X, \mathbb{Z})$ and denote by $F^k(X)$ the quotient $H^k(X, \mathbb{Z})/T^k(X)$. Then $F^k(X)$ is a free abelian group and there is a short exact sequence

$$0 \to T^k(X) \to H^k(X, \mathbb{Z}) \to F^k(X) \to 0.$$

If N is a closed, oriented Poincaré duality space of dimension m, then the cup product pairing $H^i(N, \mathbb{Z}) \otimes H^{m-i}(N, \mathbb{Z}) \to H^m(N, \mathbb{Z}) = \mathbb{Z}$ induces non-singular pairings of free abelian groups $F^i(N) \otimes F^{m-i}(N) \to \mathbb{Z}$. Also, for $i > 0$, there are non-singular pairings $T^i(N) \otimes T^{m+1-i}(N) \to Q/\mathbb{Z}$ defined as follows. If $x \in T^i(N)$, $y \in T^{m+1-i}(N)$ choose cocycles $x_\# \in C^i(N, \mathbb{Z})$ and $y_\# \in C^{m+1-i}(N, \mathbb{Z})$ representing x and y respectively. Since x is a torsion element we have $kx = \delta u_\#$ for some $k \in \mathbb{Z}$ and $u_\# \in C^{i-1}(N, \mathbb{Z})$. Define:

$$L(x, y) = (1/k)\langle u_\# \cup y_\#,\ [N]\rangle \in Q/\mathbb{Z}. \tag{3.1}$$

In particular, if $m = 4n - 1$ then L defines a symmetric, bilinear, non-singular "linking" pairing

$$L\colon T^{2n}(N^{4n-1}) \otimes T^{2n}(N^{4n-1}) \to Q/\mathbb{Z}.$$

Now let $(W, \partial W)$ be an oriented, Poincaré duality pair of dimension m. The cup product pairings $H^i(W, \mathbb{Z}) \otimes H^{m-i}(W, \partial W, \mathbb{Z}) \to H^m(W, \partial W, \mathbb{Z}) = \mathbb{Z}$ define non-singular pairings of free abelian groups $F^i(W) \otimes F^{m-i}(W, \partial W) \to \mathbb{Z}$. Also, there are non-singular pairings $T^i(W) \otimes T^{m+1-i}(W, \partial W) \to Q/\mathbb{Z}$ defined just as in (3.1). If $m = 4n$, the cohomology sequence of the pair $(W, \partial W)$ gives a diagram (which may be regarded as a short exact sequence of chain complexes, with center row acyclic):

$$
\begin{tikzcd}[column sep=small]
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 \arrow[d] \arrow[r] & T^{2n}(W, \partial W) \arrow[r, "j"] \arrow[d] & T^{2n}(W) \arrow[r, "i"] \arrow[d] & T^{2n}(\partial W) \arrow[r, "\delta"] \arrow[d] & T^{2n+1}(W, \partial W) \arrow[r] \arrow[d] & \\
\arrow[r] H^{2n-1}(\partial W, \mathbb{Z}) \arrow[r, "\delta^*"] \arrow[d] & H^{2n}(W, \partial W, \mathbb{Z}) \arrow[r, "j^*"] \arrow[d] & H^{2n}(W, \mathbb{Z}) \arrow[r, "i^*"] \arrow[d] & H^{2n}(\partial W, \mathbb{Z}) \arrow[r, "\delta^*"] \arrow[d] & H^{2n+1}(W, \partial W, \mathbb{Z}) \arrow[r] \arrow[d] & \\
\arrow[r] F^{2n-1}(\partial W) \arrow[r, "\delta"] \arrow[d] & F^{2n}(W, \partial W) \arrow[r, "j"] \arrow[d] & F^{2n}(W) \arrow[r, "i"] \arrow[d] & F^{2n}(\partial W) \arrow[r, "\delta"] \arrow[d] & & \\
0 & 0 & 0 & 0 & &
\end{tikzcd}
\tag{3.2}
$$

The pairing $F^{2n}(W) \otimes F^{2n}(W, \partial W) \to \mathbb{Z}$ identifies $F^{2n}(W)$ with $F^{2n}(W, \partial W)^* = \operatorname{Hom}(F^{2n}(W, \partial W), \mathbb{Z})$. Similarly, the pairing $F^{2n-1}(\partial W) \otimes F^{2n}(\partial W) \to \mathbb{Z}$ identifies $F^{2n}(\partial W)$ with $F^{2n-1}(\partial W)^*$. The well-known formula $\langle i^*(x) \cup y,\ [\partial W]\rangle = \langle x \cup \delta^*(y),\ [W, \partial W]\rangle$, where $x \in H^*(W, \mathbb{Z})$, $y \in H^*(\partial W, \mathbb{Z})$, implies that the maps $\bar{\delta}\colon F^{2n-1}(\partial W) \to F^{2n}(W, \partial W)$ and $\bar{\imath}\colon F^{2n}(W) \to F^{2n}(\partial W)$ are adjoints of each other.

The map $\bar{\jmath}\colon F^{2n}(W, \partial W) \to F^{2n}(W, \partial W)^* = F^{2n}(W)$ corresponds to the cup product form $F^{2n}(W, \partial W) \otimes F^{2n}(W, \partial W) \to \mathbb{Z}$. Thus $\operatorname{index}(\bar{\jmath}) = \operatorname{index}(W)$. The radical of $\bar{\jmath}$ is the subgroup $\operatorname{kernel}(\bar{\jmath}) \subset F^{2n}(W, \partial W)$. Let $F = F^{2n}(W, \partial W)/\operatorname{kernel}(\bar{\jmath}) \xrightarrow{\sim} \operatorname{image}(\bar{\jmath})$. The group $\operatorname{kernel}(\bar{\imath}) \subset F^{2n}(W)$ is easily identified with $F^* = \operatorname{Hom}(F, \mathbb{Z})$, hence $\bar{\jmath}$ induces

$$0 \to F \xrightarrow{B} F^* \to K_B \to 0$$

where $K_B = \operatorname{kernel}(\bar{\imath})/\operatorname{image}(\bar{\jmath})$. The form B induces a non-singular pairing (K_B, L_B) as in §2, and $\operatorname{index}(B) = \operatorname{index}(\bar{\jmath}) = \operatorname{index}(W)$.

The pairings defined above on the torsion subgroups give identifications $T^{2n}(\partial W) \xrightarrow{\sim} \operatorname{Hom}(T^{2n}(\partial W), Q/\mathbb{Z})$ and $T^{2n+1}(W, \partial W) \xrightarrow{\sim} \operatorname{Hom}(T^{2n}(W), Q/\mathbb{Z})$.

LEMMA 3.3. *The maps $i\colon T^{2n}(W) \to T^{2n}(\partial W)$ and $\delta\colon T^{2n}(\partial W) \to T^{2n+1}(W, \partial W)$ are adjoints of each other, that is $L(i(x), y) = \langle x, \delta y\rangle$, where $x \in T^{2n}(W)$ and $y \in T^{2n}(\partial W)$.*

Proof. This is proved using the cocycle definitions of the pairing L and the maps i and δ. Choose cocycles $x_\# \in C^{2n}(W, \mathbb{Z})$ and $y_\# \in C^{2n}(\partial W, \mathbb{Z})$ representing x and y. Choose $\tilde{y}_\# \in C^{2n}(W, \mathbb{Z})$ with $i^\#(\tilde{y}_\#) = y_\#$ and choose $u_\# \in C^{2n-1}(W, \mathbb{Z})$ with $\delta(u_\#) = kx_\#$. Then, by definition (3.1),

$$\begin{aligned} L(i(x), y) &= (1/k)\langle i^*(u_\#) \cup y_\#, [\partial W]\rangle \\ &= (1/k)\langle \delta(u_\# \cup \tilde{y}_\#), [W, \partial W]\rangle \\ &= (1/k)\langle kx_\# \cup \tilde{y}_\#, [W, \partial W]\rangle + (1/k)\langle u_\# \cup \delta\tilde{y}_\#, [W, \partial W]\rangle \\ &\equiv (1/k)\langle u_\# \cup \delta\tilde{y}_\#, [W, \partial W]\rangle \pmod{\mathbb{Z}} \\ &= \langle x, \delta(y)\rangle \in Q/\mathbb{Z}. \end{aligned}$$

Lemma 3.3 implies (see the discussion preceding Theorem 2.11) that the homology group $\bar{K} = \operatorname{kernel}(\delta)/\operatorname{image}(i)$ of the sequence

$$T^{2n}(W) \xrightarrow{i} T^{2n}(\partial W) \xrightarrow{\delta} T^{2n+1}(W, \partial W)$$

inherits a non-singular pairing $\bar{L}\colon \bar{K} \otimes \bar{K} \to Q/\mathbb{Z}$ from the pairing $L\colon T^{2n}(\partial W) \otimes T^{2n}(\partial W) \to Q/\mathbb{Z}$. From Diagram 3.2, regarded as a short exact sequence of chain complexes, there is a natural homology isomorphism

$$K_B = \operatorname{kernel}(\bar{\imath})/\operatorname{image}(\bar{\jmath}) \xrightarrow{\sim} \operatorname{kernel}(\delta)/\operatorname{image}(i) = \bar{K}.$$

The first part of the next lemma shows there is an isomorphism of linking pairings $(\bar{K}, \bar{L}) \xrightarrow{\sim} (K_B, -L_B)$.

LEMMA 3.4. *Let $y, y' \in H^{2n}(W, \mathbb{Z})$ with $i^*(y)$ and $i^*(y')$ in $T^{2n}(\partial W)$.*

(a) $\langle y', (j^*)^{-1}y\rangle + L(i^*y', i^*y) = 0$ *in $Q/\mathbb{Z}$. Here $(j^*)^{-1}y \in H^{2n}(W, \partial W, Q)$.*

(b) *If $y_\# \in C^{2n}(W, \mathbb{Z})$ is a cocycle representative for y and $i^\# y_\# = 2w_\# \in C^{2n}(\partial W, \mathbb{Z})$, and if $w \in H^{2n}(\partial W, \mathbb{Z})$ is the class represented by $w_\#$ and $z \in H^{2n}(W, \partial W; \mathbb{Z}/2)$ is the class represented by $\rho_2(y_\#) \in C^{2n}(W, \partial W; \mathbb{Z}/2)$, then*

$$\tfrac{1}{2}\langle y', z\rangle = \tfrac{1}{2}\langle y', (j^*)^{-1}y\rangle + L(i^*y', w) \quad in \quad Q/\mathbb{Z}.$$

LEMMA 3.5. (a) *If $y, y' \in H^{2n}(W, \mathbb{Z})$ are torsion classes and $y_\#, y_\#' \in C^{2n}(W, \mathbb{Z})$ are cocycle representatives for y and y', with $i^\# y_\# = 2w_\#$ and $i^\# y_\#' = 2w_\#'$ in $C^{2n}(\partial W, \mathbb{Z})$ then*

$$\tfrac{1}{4}\langle y_\# \cup y_\#', [W, \partial W]\rangle = L(w', w) \quad in \quad Q/\mathbb{Z}$$

where w and $w' \in H^{2n}(\partial W, \mathbb{Z})$ are represented by the cocycles $w_\#$ and $w_\#'$ respectively.

(b) *If $y \in H^{2n}(W, \mathbb{Z})$ is a torsion class represented by a cocycle $y_\#$ with $i^\# y_\# = 2w_\# \in C^{2n}(\partial W, \mathbb{Z})$ and if $w \in H^{2n}(\partial W, \mathbb{Z})$ is the class represented by $w_\#$ and $z \in H^{2n}(W, \partial W, \mathbb{Z}/2)$ is the class represented by $\rho_2(y_\#) \in C^{2n}(W, \partial W, \mathbb{Z}/2)$, then*

$$\langle \mathscr{P}(z), [W, \partial W]\rangle = L(w, w) \quad in \quad \mathbb{Z}/4$$

where $\mathscr{P}(z) \in H^{4n}(W, \partial W, \mathbb{Z}/4)$ is the Pontrjagin square of z.

LEMMA 3.6. *If W is a $\mathbb{Z}/2$ P.D. space and $i^*(y) = (w_1, w_2)$ in $T^{2n}(\partial W) = T^{2n}(\delta \tilde{W}) \oplus T^{2n}(\delta \tilde{W})$, with $w_1 + w_2 = 2w$ in $T^{2n}(\delta \tilde{W})$, and if $z \in H^{2n}(\tilde{W}, \mathbb{Z}/2)$ satisfies $\pi^*(z) = \rho_2(y) \in H^{2n}(W, \mathbb{Z}/2)$, where $\pi\colon W \to \tilde{W}$ is the collapsing map, then*

$$\tfrac{1}{4}\langle z^2, [\tilde{W}]\rangle = \tfrac{1}{2}\langle y, (j^*)^{-1} y\rangle + 2L(w, w) - L(w_1, w_2) \quad in \quad Q/\mathbb{Z}.$$

Proofs. Choose $y_\#$, $y_\#'$, cocycle representatives for y and y'. Since i^*y and i^*y' are torsion there are cochains $u_\#$ and $u_\#'$ in $C^{2n-1}(\partial W, \mathbb{Z})$ such that $\delta u_\# = k i^\# y_\#$, $\delta u_\#' = k i^\# y_\#'$. Let $\tilde{u}_\#$ and $\tilde{u}_\#'$ be extensions of these classes to $C^{2n}(W, \mathbb{Z})$. Then

$$\frac{1}{k}(k y_\# - \delta \tilde{u}_\#) \in C^{2n}(W, \partial W; \mathbb{Q})$$

is a cocycle representative for $(j^*)^{-1}y$. Thus

$$\begin{aligned}\langle y' \cdot (j^*)^{-1} y, [W, \partial W]\rangle &= \frac{1}{k}\langle y_\#' \cdot (k y_\# - \delta \tilde{u}_\#), [W, \partial W]\rangle \\ &= \langle y_\#' \cdot y_\#, [W, \partial W]\rangle - \frac{1}{k}\langle \delta(y_\#' \cdot \tilde{u}_\#), [W, \partial W]\rangle \qquad (3.7) \\ &= \langle y_\#' \cdot y_\#, [W, \partial W]\rangle - \frac{1}{k}\langle i^\# y_\#' \cdot u_\#, [\partial W]\rangle \in \mathbb{Q}.\end{aligned}$$

Lemma 3.4(a) follows immediately from this equation since

$$L(i^*y', i^*y) = \frac{1}{k}\langle i^\# y_\#' \cdot u_\#, [\partial W]\rangle \quad \text{in} \quad \mathbb{Q}/\mathbb{Z}.$$

Under the hypothesis of Lemma 3.4(b) we may choose $y_\#$ such that $i^\# y_\# = 2w_\#$ for $w_\#$ a cocycle in $C^{2n}(\partial W, \mathbb{Z})$. Clearly,

$$L(i^*y', w) = \frac{1}{2k}\langle i^\# y_\#' \cdot u_\#, [\partial W]\rangle.$$

$\rho_2(y_\#)$ is a relative cocycle in $C^{2n}(W, \partial W; \mathbb{Z}/2)$ since $y_\# | \partial W \equiv 0 \pmod 2$. Thus $\rho_2(y_\#)$ is a cocycle representative for $z \in H^{2n}(W, \partial W; \mathbb{Z}/2)$ and $\langle y_\#' \cdot y_\#, [W, \partial W]\rangle \equiv \langle y' \cdot z, [W, \partial W]\rangle \pmod 2$. Thus dividing (3.7) by 2 we have $\frac{1}{2}\langle y', (j^*)^{-1} y\rangle = \frac{1}{2}\langle y' \cdot z, [W, \partial W]\rangle - L(i^*y', w)$ in $Q/\mathbb{Z}$. This proves Lemma 3.4(b).

If y and y' are torsion then $\langle y', (j^*)^{-1}y\rangle = 0$. If $i^*y = 2w$ and $i^*y' = 2w'$, then

$$L(w', w) = \frac{1}{4k}\langle i^\# y_\#' \cdot u_\#, [\partial W]\rangle.$$

Thus dividing (3.7) by 4 we see

$$0 = \tfrac{1}{4}\langle y_\#' \cdot y_\#, [W, \partial W]\rangle - L(w', w).$$

This proves Lemma 3.5(a).

To prove Lemma 3.5(b), let $\tilde{w}_\# \in C^{2n}(W, \mathbb{Z})$ be a cochain extending $w_\# \in C^{2n}(\partial W, \mathbb{Z})$. Then $y_\# - 2\tilde{w}_\# \in C^{2n}(W, \partial W, \mathbb{Z})$ is an integral cochain which represents the $\mathbb{Z}/2$ class z. By definition of the Pontrjagin square $\mathscr{P}(z)$ is represented by the cochain (a $\mathbb{Z}/4$ cocycle)

$$\begin{aligned}(y_\# - 2\tilde{w}_\#) \cup (y_\# - 2\tilde{w}_\#) &+ \delta(y_\# - 2\tilde{w}_\#) \cup_1 (y_\# - 2\tilde{w}_\#) \\ &\equiv y_\# \cup y_\# - 2\tilde{w}_\# \cup y_\# - 2y_\# \cup \tilde{w}_\# + 2\delta\tilde{w}_\# \cup_1 y_\# \pmod 4.\end{aligned}$$

From the coboundary formula [12, Theorem 5.1]

$$\delta\left(\tilde{w}_\# \cup_1 y_\#\right) = -\tilde{w}_\# \cup y_\# + y_\# \cup \tilde{w}_\# + \delta\tilde{w}_\# \cup_1 y_\#$$

we deduce that

$$\mathscr{P}(z) + \delta\left(2\tilde{w}_\# \cup_1 y_\#\right) \equiv y_\# \cup y_\# \pmod 4.$$

But $2\tilde{w}_\# \cup_1 y_\#$ is a relative $\mathbb{Z}/4$ cochain, $2\tilde{w}_\# \cup_1 y_\# \in C^{4n-1}(W, \partial W, \mathbb{Z}/4)$, since $i^\#\left(2\tilde{w}_\# \cup_1 y_\#\right) = 2w_\# \cup_1 2w_\#$. Thus

$$\langle \mathscr{P}(z), [W, \partial W]\rangle \equiv \langle y_\# \cup y_\#, [W, \partial W]\rangle \quad \text{in} \quad \mathbb{Z}/4$$

and Lemma 3.5(b) follows immediately from Lemma 3.5(a).

Finally, to prove Lemma 3.6, choose cocycles $(w_1)_\#$, $(w_2)_\#$ and $w_\# \in C^{2n}(\delta\tilde{W}, \mathbb{Z})$ representing w_1, w_2, and w with $2w_\# = (w_1)_\# + (w_2)_\#$. We assume $i^\#(y_\#) = ((w_1)_\#, (w_2)_\#)$. The $\mathbb{Z}/2$ reduction of $y_\#$ is symmetric on $\partial W = 2\delta\tilde{W}$, and hence gives rise to a cocycle $z_\# = \rho_2(y_\#) \in C^{2n}(\tilde{W}, \mathbb{Z}/2)$ which represents a class $z \in H^{2n}(\tilde{W}, \mathbb{Z}/2)$ with $\pi^*(z) = \rho_2(y) \in H^{2n}(W, \mathbb{Z}/2)$. Choose $u_\#$, $(u_i)_\# \in C^{2n-1}(\delta\tilde{W}, \mathbb{Z})$ with $\delta(u_i)_\# = k \cdot w_i$, $i = 1, 2$, $\delta u_\# = kw_\#$, and $(u_1)_\# + (u_2)_\# = 2u_\#$. From (3.7), with $y' = y$, we obtain

$$\begin{aligned}\tfrac{1}{2}\langle y, (j^*)^{-1}y\rangle &= \tfrac{1}{2}\langle z^2, [\tilde{W}]\rangle - \frac{1}{2k}\langle((w_1)_\#, 2w_\# - (w_1)_\#)\cdot((u_1)_\#, 2u_\# - (w_1)_\#), [\partial W]\rangle \\ &= \tfrac{1}{2}\langle z^2, [\tilde{W}]\rangle - \frac{1}{2k}\langle (w_1)_\# \cdot (u_1)_\#, [\delta\tilde{W}]\rangle \\ &\quad - \frac{1}{2k}\langle (2w_\# - (w_1)_\#)\cdot(2u_\# - (u_1)_\#), [\delta\tilde{W}]\rangle \\ &= \tfrac{1}{2}\langle z^2, [\tilde{W}]\rangle - L(w_1, w_1) - L(2w, w) + L(w, w_1) + L(w_1, w) \\ &= \tfrac{1}{2}\langle z^2, [\tilde{W}]\rangle - L(2w, w) + L(w_1, w_2)\end{aligned}$$

as desired.

§4. A FORMULA FOR THE INDEX MODULO 8

Let $(W, \partial W)$ be an oriented, $(4n)$-dimensional Poincaré pair. In this section we prove our main formula, Theorem 4.3, relating the index of W modulo 8 and the invariant $A(T^{2n}(\partial W), \psi)$, where $\psi: T^{2n}(\partial W) \to Q/\mathbb{Z}$ is a quadratic function over the linking pairing L.

Definition 4.1. A lifting $\hat{v} \in H^{2n}(W, \partial W, \mathbb{Z}/2)$ of the Wu class $v_{2n} \in H^{2n}(W, \mathbb{Z}/2)$ is said to be *compatible with* ψ if, for all $x \in T^{2n}(\partial W)$ such that $x = i^*(y)$, where $y \in H^{2n}(W, \mathbb{Z})$, we have

$$\psi(x) = \tfrac{1}{2}\langle y \cdot \hat{v}, [W, \partial W]\rangle - \tfrac{1}{2}\langle y \cdot (j^*)^{-1} y, [W, \partial W]\rangle \in Q/\mathbb{Z}.$$

Definition 4.1 is formally very similar to (2.3). In fact, our plan is to begin with a quadratic function $\psi: T^{2n}(\partial W) \to Q/\mathbb{Z}$ and a compatible lifting of the Wu class $\hat{v} \in H^{2n}(W, \partial W, \mathbb{Z}/2)$ and reduce this to the algebraic situation of (2.3). Then we will apply Theorem 2.8 to obtain a formula for $I(W^{4n})$ (mod 8).

We will assume for now the following theorem. The proof will be given in the next section.

THEOREM 4.2. *Given a quadratic function on $T^{2n}(\partial W)$ there is a compatible lifting of the Wu class, $\hat{v} \in H^{2n}(W, \partial W; \mathbb{Z}/2)$.*

Consider the sequence

$$T^{2n}(W) \xrightarrow{\;i\;} T^{2n}(\partial W) \xrightarrow{\;\delta\;} T^{2n+1}(W, \partial W).$$

By Lemma 3.3 this has the form

$$G \xrightarrow{\;i\;} K \xrightarrow{\;i^*\;} G^*$$

studied in §2. In particular, if $\psi: T^{2n}(\partial W) \to Q/\mathbb{Z}$ is quadratic over L then the composition $\psi i: T^{2n}(W) \to Q/\mathbb{Z}$ is linear and $2\psi i = 0$, hence there is a unique element $a \in T^{2n+1}(W, \partial W)$ such that $2a = 0$ and $\psi i(y) = \langle a, y\rangle$ for all $y \in T^{2n}(W)$. Moreover, by Theorem 2.11(i), there exists $b \in T^{2n}(\partial W)$ such that $\delta(b) = a$ and $2b = i(t)$ for some $t \in T^{2n}(W)$. A choice of b gives a quadratic function $\psi_b: \bar{K} = \text{kernel}(\delta)/\text{image}(i) \to Q/\mathbb{Z}$ over $\bar{L}$ as in Theorem 2.11(ii).

On the other hand, $\bar{K} = K_B$ where

$$0 \longrightarrow F \xrightarrow{\;B\;} F^* \longrightarrow K_B \longrightarrow 0$$

is constructed in §3 from the sequence of free groups

$$F^{2n-1}(\partial W) \xrightarrow{\;\delta\;} F^{2n}(W, \partial W) \xrightarrow{\;j\;} F^{2n}(W) \xrightarrow{\;i\;} F^{2n}(\partial W).$$

We will construct an element $v \in H^{2n}(W, \partial W, \mathbb{Z})$ which represents a Wu class for $B: F \to F^*$. (Recall that $F = F^{2n}(W, \partial W)/\text{kernel}(j)$.) This defines $\psi_v: K_B \to Q/\mathbb{Z}$, quadratic over L_B, as in (2.3). It will turn out (Lemma 4.7) that $\psi_v: K_B \to Q/\mathbb{Z}$ may be identified with $-\psi_b: \bar{K} \to Q/\mathbb{Z}$. But, by (2.9),

$$\begin{aligned} A(K_B, \psi_v) &= \text{index}(B) - \langle Bv, v\rangle \\ &= \text{index}(W) - \langle v^2, [W, \partial W]\rangle \in \mathbb{Z}/8 \end{aligned}$$

and by Theorems 2.11(iii) and 2.7

$$A(\bar{K}, -\psi_b) = \psi(b) - A(T^{2n}(\partial W), \psi) \in \mathbb{Z}/8$$

where we interpret both $\psi(b)$ and the Arf invariants in $\mathbb{Z}/8\mathbb{Z} \simeq \frac{1}{8}\mathbb{Z}/\mathbb{Z}$. We thus deduce our main formula:

THEOREM 4.3. index$(W) = (\langle v^2, [W, \partial W]\rangle + \psi(b)) - A(T^{2n}(\partial W), \psi) \in \mathbb{Z}/8$.

Here are the details.

LEMMA 4.4. *Let $\psi: T^{2n}(\partial W) \to Q/\mathbb{Z}$ be quadratic over L, let $\hat{v} \in H^{2n}(W, \partial W, \mathbb{Z}/2)$ be a compatible lifting of the Wu class and let $b \in T^{2n}(\partial W)$ satisfy $\psi i(y) = L(b, i(y))$ for all $y \in T^{2n}(W)$. Then $\beta\hat{v} = \delta^* b$ where β: $H^{2n}(W, \partial W, \mathbb{Z}/2) \to H^{2n+1}(W, \partial W, \mathbb{Z})$ is the integral Bochstein and $\delta^*: H^{2n}(\partial W, \mathbb{Z}) \to H^{2n+1}(W, \partial W, \mathbb{Z})$ is the coboundary map of the pair $(W, \partial W)$.*

Proof. It suffices to prove that $\langle y, \beta\hat{v}\rangle = \langle y, \delta^* b\rangle$ for all $y \in T^{2n}(W)$. It is not hard to show from the definition of the pairing $T^{2n}(W) \otimes T^{2n+1}(W, \partial W) \to Q/\mathbb{Z}$ that $\langle y, \beta\hat{v}\rangle = \frac{1}{2}\langle y \cdot \hat{v}, [W, \partial W]\rangle \in \frac{1}{2}\mathbb{Z}/\mathbb{Z} \subset Q/\mathbb{Z}$. On the other hand, $\langle y, \delta^* b\rangle = L(i(y), b) = \psi i(y)$. Since y is a torsion element, $(j^*)^{-1}(y) = 0 \in H^{2n}(W, \partial W, Q)$, hence the condition (4.1) that $\hat{v}$ is compatible with ψ reduces to $\psi i(y) = \frac{1}{2}\langle y \cdot \hat{v}, [W, \partial W]\rangle$, as desired.

LEMMA 4.5. *Let $\psi: T^{2n}(\partial W) \to Q/\mathbb{Z}$, $\hat{v} \in H^{2n}(W, \partial W, \mathbb{Z}/2)$ and $b \in T^{2n}(\partial W)$ be as above. Then there is an integral class $v' \in H^{2n}(W, \mathbb{Z})$ such that $\rho_2(v') = v_{2n}(W) \in H^{2n}(W, \mathbb{Z}/2)$ and $i^*(v') = 2b \in H^{2n}(\partial W, \mathbb{Z})$.*

Proof. Lemma 4.4 and an easy diagram chase imply that there exists $v' \in H^{2n}(W, \mathbb{Z})$ with $\rho_2(v') = v_{2n}(W) \in H^{2n}(W, \mathbb{Z}/2)$. An elementary cochain argument, which we leave to the reader, gives that v' may be chosen such that $i^*(v') = 2b \in H^{2n}(\partial W, \mathbb{Z})$.

We now have $i^*(v') = i^*(t) = 2b$, where v', $t \in H^{2n}(W, \mathbb{Z})$. Choose $v \in H^{2n}(W, \partial W, \mathbb{Z})$ such that $j^*(v) = v' - t$.

LEMMA 4.6. *The class $v \in H^{2n}(W, \partial W, \mathbb{Z})$ restricts to a Wu class in $F^{2n}(W, \partial W)$ for the bilinear form $\bar{j}$: $F^{2n}(W, \partial W) \to F^{2n}(W, \partial W)^* = F^{2n}(W)$.*

Proof. We must show that $\langle j^*(z) \cdot z, [W, \partial W]\rangle \equiv \langle j^*(z) \cdot v, [W, \partial W]\rangle \pmod 2$ for all $z \in H^{2n}(W, \partial W, \mathbb{Z})$. But $\langle j^*(z) \cdot z, [W, \partial W]\rangle = \langle z^2, [W, \partial W]\rangle \equiv \langle z \cdot v', [W, \partial W]\rangle \pmod 2$. Since $t \in H^{2n}(W, \mathbb{Z})$ is a torsion element, $\langle z \cdot v', [W, \partial W]\rangle = \langle z \cdot (v' - t), [W, \partial W]\rangle = \langle z \cdot j^*(v), [W, \partial W]\rangle = \langle j^*(z) \cdot v, [W, \partial W]\rangle$, as desired.

It is obvious that v also gives a Wu class for the bilinear form B: $F \to F^*$ where $F =$ image($\bar{j}$: $F^{2n}(W, \partial W) \to F^{2n}(W)$) and $F^* =$ kernel($\bar{i}$: $F^{2n}(W) \to F^{2n}(\partial W)$) as in §3. Thus, as in (2.3), we obtain a quadratic function ψ_v: kernel($\bar{i}$)/image($\bar{j}$) $= K_B \to Q/\mathbb{Z}$ over L_B. We want to compare the quadratic functions (K_B, ψ_v) and $(\bar{K}, \psi_b)$. Recall from Lemma 3.4(a) that $(K_B, L_B) = (\bar{K}, -\bar{L})$.

LEMMA 4.7. $\psi_v = -\psi_b$: $\bar{K} \to Q/\mathbb{Z}$.

Proof. Let $x \in$ kernel(δ) represent $\bar{x} \in \bar{K} =$ kernel(δ)/image(i). Choose $y \in H^{2n}(W, \mathbb{Z})$ with $i^*(y) = x$. Then, by Definition 2.3, $\psi_v(\bar{x}) = \frac{1}{2}\langle y \cdot (j^*)^{-1} y, [W, \partial W]\rangle - \frac{1}{2}\langle y \cdot v, [W, \partial W]\rangle$.

Define $s \in H^{2n}(W, \partial W, \mathbb{Z}/2)$ by $\rho_2(v) = \hat{v} - s$. Clearly, $j^*(s) = \rho_2(t) \in H^{2n}(W, \mathbb{Z}/2)$ since $j^*\rho_2(v) = \rho_2 j^*(v) = \rho_2(v' - t) = v_{2n} - \rho_2(t)$, and by Definition 4.1,

$$\begin{aligned}\psi_v(\bar{x}) &= \tfrac{1}{2}\langle y \cdot (j^*)^{-1} y, [W, \partial W]\rangle - \tfrac{1}{2}\langle y \cdot \hat{v}, [W, \partial W]\rangle + \tfrac{1}{2}\langle y \cdot s, [W, \partial W]\rangle \\ &= -\psi(x) + \tfrac{1}{2}\langle y \cdot s, [W, \partial W]\rangle.\end{aligned}$$

By Lemma 3.4(b). $\frac{1}{2}\langle y \cdot s, [W, \partial W]\rangle = L(b, x)$ in $\mathbb{Q}/\mathbb{Z}$ since $(j^*)^{-1}(t) = 0$ in $H^{2n}(W, \partial W; \mathbb{Q})$. Hence

$$\psi_v(\bar{x}) = -\psi(x) + L(b, x) = -\psi_b(\bar{x})$$

as desired.

Remark 4.8. Our main formula 4.3, merits some discussion. Modulo 2, 4.3 reduces to

$$\operatorname{index}(W) \equiv \langle \hat{v}^2, [W, \partial W]\rangle - \sigma(\partial W) \mod 2.$$

Modulo 4, our formula also simplifies. We will prove in Theorem 6.1 that

$$\operatorname{index}(W) \equiv \langle \mathscr{P}(\hat{v}), [W, \partial W]\rangle - A(T^{2n}(\partial W), \psi) \mod 4.$$

(The first equation follows from the second together with (2.10) which says $A(T^{2n}(\partial W), \psi) \equiv \sigma(\partial W) \bmod 2$.)

§5. COMPATIBILITY

In this section we will prove Theorem 4.2 that for a *P.D.* pair $(W^{4n}, \partial W)$ there are compatible liftings of the Wu class, $\hat{v} \in H^{2n}(W, \partial W; \mathbb{Z}/2)$, and quadratic functions ψ: $T^{2n}(\partial W) \to \mathbb{Q}/\mathbb{Z}$. We shall also prove an equivariant version of this for $\mathbb{Z}/2$ *P.D.* spaces which will be used in §6.

If $(M^{4n}, \partial M)$ is given the structure of a $\mathbb{Z}/2$ manifold, then an equivariant quadratic function ψ: $T^{2n}(\partial M) \to \mathbb{Q}/\mathbb{Z}$ is one with the same values on the two copies of $T^{2n}(\delta\tilde{M})$. An equivariant lifting of the Wu class is a class $\hat{v} \in H^{2n}(M, \partial M; \mathbb{Z}/2)$ such that $\hat{v} \mapsto v_{2n}(\nu_{\tilde{M}})$ under the map $H^{2n}(M, \partial M; \mathbb{Z}/2) \simeq H^{2n}(\tilde{M}, \delta\tilde{M}; \mathbb{Z}/2) \to H^{2n}(\tilde{M}, \mathbb{Z}/2)$. By Theorem 2.2, equivariant quadratic functions exist, and equivariant liftings of the Wu class exist since $v_{2n}(\nu_{\tilde{M}}) | (\delta\tilde{M}) = 0$.

THEOREM 5.1 (Compatibility).

(a) *Given $(M^{4n}, \partial M)$, a P.D. pair, and ψ: $T^{2n}(\partial M) \to \mathbb{Q}/\mathbb{Z}$ a quadratic function, then there is a lifting of the Wu class $v_{2n}(\nu_M)$ to $\hat{v} \in H^{2n}(M, \partial M; \mathbb{Z}/2)$ satisfying*

$$\psi(i^*y) = \tfrac{1}{2}\langle \hat{v}, y\rangle - \tfrac{1}{2}\langle y, (j^*)^{-1}y\rangle \quad \textit{in} \quad \mathbb{Q}/\mathbb{Z} \tag{*}$$

for all $y \in H^{2n}(M, \mathbb{Z})$ with $i^*y \in T^{2n}(\partial M)$.

(b) *If $(M^{4n}, \partial M)$ is a $\mathbb{Z}/2$ P.D. space and ψ as in* (a), *is equivariant then there is an equivariant lifting of the Wu class, $\hat{v}$, which satisfies* (*).

Proof. $\hat{v} \in H^{2n}(M, \partial M; \mathbb{Z}/2)$ is completely determined by its values on $H^{2n}(M; \mathbb{Z}/2)$ and any homomorphism $H^{2n}(M; \mathbb{Z}/2) \to \mathbb{Z}/2$ is realizable as $\langle \hat{v}, \ \rangle$ for some $\hat{v}$. In both cases (a) and (b) there are two subspaces A_1 and A_2 of $H^{2n}(M; \mathbb{Z}/2)$ on which the values of $\hat{v}$ are forced. The values are predetermined on A_1 by the condition that $\hat{v}$ is a lifting of the Wu class (or an equivariant lifting) and on A_2 by the condition that $\hat{v}$ is compatible with ψ. Thus there will be a $\hat{v} \in H^{2n}(M, \partial M; \mathbb{Z}/2)$ meeting both these conditions if and only if they determine the same values on $A_1 \cap A_2$.

Proof of (a). Here $A_1 = \operatorname{im}(j^*\colon H^{2n}(M, \partial M; \mathbb{Z}/2) \to H^{2n}(M; \mathbb{Z}/2))$, and $\langle \hat{v}, j^*(z)\rangle = \langle z, j^*z\rangle$ $(= \langle z^2, [M, \partial M]\rangle)$ is the forced homomorphism $\varphi_1\colon A_1 \to \mathbb{Z}/2$. $A_2 = \operatorname{im}(\rho_2\colon B \to$

$H^{2n}(M;\ \mathbb{Z}/2))$ where $B \subset H^{2n}(M;\ \mathbb{Z})$ is $\{y \mid i^*y \in H^{2n}(\partial M;\ \mathbb{Z})$ is torsion$\}$. Compatibility requires that

$$\tfrac{1}{2}\langle \hat{v}, \rho_2(y)\rangle = \psi(i^*y) + \tfrac{1}{2}\langle y, (j^*)^{-1}y\rangle. \tag{*}$$

The right hand side of this equation is easily seen to define a homomorphism $\varphi_2\colon A_2 \to \mathbb{Z}/2$. We must show that $\varphi_1 | A_1 \cap A_2 = \varphi_2 | A_1 \cap A_2$.

$A_1 \cap A_2$ consists of $\rho_2(y)$ for $y \in B$ with $i^*y \equiv 0 \bmod 2$ (since $i^*y \equiv 0 \bmod 2$ if and only if $\rho_2(y) = j^*z$ for some z in $H^{2n}(M, \partial M;\ \mathbb{Z}/2)$). Thus $i^*y = 2w$ for some $w \in T^{2n}(\partial M)$. By Lemma 3.4(b) $\tfrac{1}{2}\langle z, j^*z\rangle = \tfrac{1}{2}\langle y, (i^*)^{-1}y\rangle + 2L(w, w)$. Since $(\psi i^*y) = \psi(2w) = 4\psi(w) = 2 \cdot 2\psi(w) = 2L(w, w)$, we see that $\varphi_1(\rho_2(y)) = \varphi_2(\rho_2(y))$. This proves $\varphi_1 = \varphi_2$ on $A_1 \cap A_2$ and shows a compatible $\hat{v} \in H^{2n}(M, \partial M;\ \mathbb{Z}/2)$ exists.

Proof of (b). In this case $A_1 = \operatorname{im}(\pi^*\colon H^{2n}(\tilde{M};\ \mathbb{Z}/2) \to H^{2n}(M;\ \mathbb{Z}/2))$ and $\varphi_1\colon A_1 \to \mathbb{Z}/2$ is given by $\varphi_1(\pi^*z) = \langle z^2, [\tilde{M}]\rangle$, since it is shown in [10] that $\langle v_{2n}(\nu_{\tilde{M}}) \cdot z, [\tilde{M}]\rangle = \langle z^2, [\tilde{M}]\rangle$.

$A_2 = \operatorname{im}(\rho_2\colon B \to H^{2n}(M;\ \mathbb{Z}/2))$ as before and $\varphi_2\colon A_2 \to \mathbb{Z}/2$ is again defined by the equation (*).

$A_1 \cap A_2 = \{\rho_2(y) \mid i^*y$ is torsion and $\rho_2(y) = \pi^*z\}$. The condition that $\rho_2(y) = \pi^*(z)$ for some z implies that $i^*(y) = (y_1,\ y_1 + 2w')$ in $T^{2n}(\partial M) = T^{2n}(\delta\tilde{M}) \oplus T^{2n}(\delta\tilde{M})$. We may rewrite this as $i^*(y) = (y_1, 2w - y_1)$. Thus by Lemma 3.6,

$$\tfrac{1}{2}z \cdot z = \tfrac{1}{2}\langle y, (j^*)^{-1}y\rangle + 2L(w, w) - L(y_1, y_2).$$

$$\begin{aligned}\psi(i^*y) &= \psi(y_1) + \psi(y_2) = \psi(y_1) + \psi(2w - y_1) = 2\psi(y_1) + 4\psi(w) - L(2w, y_1)\\ &= L(y_1, y_1) + 2L(w, w) - L(y_1, y_1) - L(y_2, y_1) = 2L(w, w) - L(y_2, y_1).\end{aligned}$$

Thus we see $\varphi_1 = \varphi_2$ on $A_1 \cap A_2$ in this case also. This proves Theorem 5.1.

Note. We could also prove that given a lifting (equivariant lifting) of the Wu class, then there is a compatible quadratic (equivariant quadratic) function. However, there is not a one to one correspondence between liftings of the Wu class and quadratic functions. There is a natural extension of Theorem 5.1 to $\mathbb{Z}/k$ *P.D.* spaces for any k. Once again one may prove that there are compatible equivariant Wu classes and quadratic functions. The proof, however, is much more complicated since if $k \neq 2$, $\tilde{M}$ does not satisfy Poincaré duality with $\mathbb{Z}/2$ coefficients. Thus $v_{2n}(\nu_{\tilde{W}})$ is not determined by its cup product with cohomology classes as in the proof of Theorem 5.1(b).

§6. APPLICATIONS

In this section we will give some applications of our main formula, Theorem 4.3.

THEOREM 6.1. *Let $(W, \partial W)$ be a Poincaré pair of dimension* $4n$ *and let $\psi\colon T^{2n}(\partial W) \to Q/\mathbb{Z}$ be a quadratic function over the linking pairing. Let $\hat{v} \in H^{2n}(W, \partial W, \mathbb{Z}/2)$ be a lifting of the Wu class compatible with ψ. Then*

$$I(W^{4n}) \equiv \langle \mathscr{P}(\hat{v}), [W, \partial W]\rangle - A(T^{2n}(\partial W), \psi) \pmod 4$$

where $\mathscr{P}(\hat{v}) \in H^{4n}(W, \partial W, \mathbb{Z}/4)$ is the Pontrjagin square of $\hat{v}$.

Before proving Theorem 6.1, we give some corollaries.

COROLLARY 6.2. *Let W^{4n} be a $\mathbb{Z}/2$ P.D. space with $\partial W^{4n} = 2M^{4n-1}$. Let $\hat{v} \in H^{2n}(W, \partial W, \mathbb{Z}/2)$ be an equivariant lifting of the Wu class. Then*

$$I(W^{4n}) \equiv \langle \mathscr{P}(\hat{v}), [W, \partial W] \rangle - 2\sigma(M) \in \mathbb{Z}/4$$

where $\sigma(M) \in \mathbb{Z}/2$ is the number modulo 2 of 2-primary summands of $T^{2n}(M)$, and $2: \mathbb{Z}/2 \to \mathbb{Z}/4$ is the inclusion.

COROLLARY 6.3. *Let W be a $\mathbb{Z}/4$ P.D. space and let $\hat{v} \in H^{2n}(W, \partial W, \mathbb{Z}/2)$ be an equivariant lifting of the Wu class. Then*

$$I(W^{4n}) \equiv \langle \mathscr{P}(\hat{v}), [W, \partial W] \rangle \in \mathbb{Z}_4 .$$

Proof of Corollary 6.2. Let $\psi: T^{2n}(\partial W) \to Q/\mathbb{Z}$ be an equivariant quadratic function compatible with $\hat{v} \in H^{2n}(W, \partial W, \mathbb{Z}/2)$, say $\psi = \tilde{\psi} \oplus \tilde{\psi}$, $\tilde{\psi}: T^{2n}(M) \to Q/\mathbb{Z}$. By (2.10) $A(T^{2n}(M), \tilde{\psi}) = \sigma(M) \pmod 2$ hence, by Theorem 2.7(ii), $A(T^{2n}(\partial W), \psi) = 2A(T^{2n}(M), \tilde{\psi}) = 2\sigma(M) \pmod 4$ and Corollary 6.2 follows immediately from Theorem 6.1.

Proof of Corollary 6.3. Since the index modulo 4 of $\mathbb{Z}/4$ *P.D.* spaces is a bordism invariant, it suffices to check Corollary 6.3 for generators of the Poincaré duality bordism group $\Omega_{4n}^{PD}(pt, \mathbb{Z}/4)$. There is an exact sequence

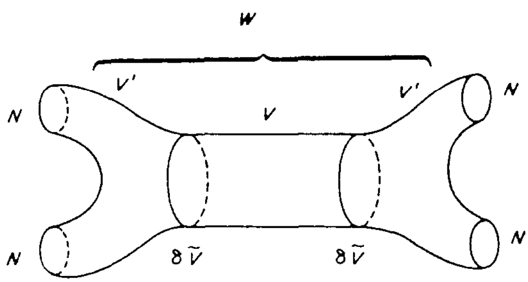
$$\cdots \to \Omega_{4n}^{PD}(pt, \mathbb{Z}/2) \xrightarrow{i} \Omega_{4n}^{PD}(pt, \mathbb{Z}/4) \xrightarrow{j} \Omega_{4n}^{PD}(pt, \mathbb{Z}/2) \xrightarrow{\delta} \Omega_{4n-1}^{PD}(pt, \mathbb{Z}/2) \to \cdots$$

where, if $\tilde{V}$ is a $\mathbb{Z}/2$ *P.D.* space, $i[\tilde{V}] = [2\tilde{V}]$ and $\delta[\tilde{V}] = [\delta\tilde{V}]$. It is clear from Corollary 6.2 or Remark 4.8 that Corollary 6.3 holds for $\mathbb{Z}/4$ *P.D.* spaces $\tilde{W}$, with $[\tilde{W}] = i[\tilde{V}]$, $\tilde{V}$ a $\mathbb{Z}/2$ *P.D.* space.

Suppose $\delta[\tilde{V}] = [\delta\tilde{V}] = 0 \in \Omega_{4n-1}^{PD}(pt, \mathbb{Z}/2)$. Then from the exact sequence

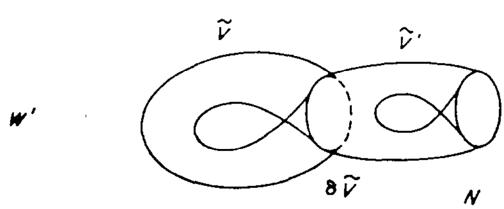
$$\cdots \to \Omega_{4n-1}^{PD}(pt, \mathbb{Z}) \xrightarrow{2} \Omega_{4n-1}^{PD}(pt, \mathbb{Z}) \to \Omega_{4n-1}^{PD}(pt, \mathbb{Z}/2) \to \cdots$$

we can find a *P.D.* cobordism V' from $-\delta\tilde{V}$ to $2N$, for some $(4n-1)$ Poincaré space N. Then $W = V \bigcup_{\partial V} 2V'$ (see diagram) is a $\mathbb{Z}/4$ *P.D.* space with $j[\tilde{W}] = [\tilde{V}] \in \Omega_{4n}^{PD}(pt, \mathbb{Z}/2)$.



It suffices to prove Corollary 6.3 for this $\mathbb{Z}/4$ *P.D.* space W.

First $I(W) = I(V) + 2I(V')$. Secondly, form the space W' by identifying with one another the two copies of V', and then the copies of N.



Then W' has a "normal bundle", and the Wu class of this normal bundle lifts to a class $v' \in H^{2n}(W', \delta\tilde{V} \cup N; \mathbb{Z}/2)$. Define $\hat{v} = \pi^*(v') \in H^{2n}(W, \partial W; \mathbb{Z}/2)$ where $\pi\colon W \to W'$ is the obvious map. Then $\hat{v}$ is an equivariant Wu class for W, $i^*(v') \in H^{2n}(\tilde{V}, \delta\tilde{V}; \mathbb{Z}/2)$ is an equivariant Wu class for the $\mathbb{Z}/2$-manifold $\tilde{V}$, and $j^*(v') \in H^{2n}(V', \partial V'; \mathbb{Z}/2)$ is a lifting of the Wu class for V', where $i\colon \tilde{V} \to W'$ and $j\colon V' \to W'$ are the obvious maps. Moreover, it is not difficult to see that

$$\begin{aligned}\langle \mathscr{P}(\hat{v}), [W, \partial W]\rangle &= \langle \mathscr{P}(i^*(v')), [\tilde{V}, \delta\tilde{V}]\rangle + 2\langle j^*(v')^2, [V', \partial V']\rangle \\ &\equiv I(V) + 2\sigma(\delta\tilde{V}) + 2\langle j^*(v')^2, [V', \partial V']\rangle \pmod 4 \quad \text{by Corollary 6.2} \\ &\equiv I(V) + 2\sigma(\delta\tilde{V}) + 2(I(V') + \sigma(\partial V')) \pmod 4 \quad \text{by Remark 4.8.}\end{aligned}$$

Since $\partial V' = \delta\tilde{V} + 2N$, $2\sigma(\partial V') \equiv 2\sigma(\delta\tilde{V}) \pmod 4$ and hence

$$\langle \mathscr{P}(\hat{v}), [W, \partial W]\rangle \equiv I(V) + 2I(V') = I(W) \pmod 4,$$

which proves Corollary 6.3.

We point out that if we had proved in §5 the existence of compatible equivariant liftings of the Wu class and equivariant quadratic functions for $\mathbb{Z}/4$ *P.D.* spaces, Corollary 6.3 would follow trivially from Theorem 6.1. The above argument turns out to be simpler, however.

Proof of Theorem 6.1. We need to recall some notation from §4. We have the Wu class $v_{2n} \in H^{2n}(W, \mathbb{Z}/2)$ and liftings $\hat{v} \in H^{2n}(W, \partial W, \mathbb{Z}/2)$, $v' \in H^{2n}(W, \mathbb{Z})$ with $j^*(\hat{v}) = \rho_2(v') = v_{2n} \in H^{2n}(W, \mathbb{Z}/2)$. We have a class $b \in T^{2n}(\partial W)$ such that $\psi i(x) = L(b, i(x))$ for $x \in T^{2n}(W)$, and a class $t \in T^{2n}(W)$ such that $i^*(t) = 2b$. Moreover, $i^*(v') = 2b \in H^{2n}(\partial W, \mathbb{Z})$. Choose cocycle representatives $\hat{v}_\#$, $v_\#'$, $b_\#$ and $t_\#$ for $\hat{v}$, v', b and t respectively, such that $j^\#(\hat{v}_\#) = \rho_2(v_\#')$ and $i^\#(v_\#') = i^\#(t_\#) = 2b_\#$. Then $v_\# = v_\#' - t_\# \in C^{2n}(W, \partial W, \mathbb{Z})$ is a relative cocycle, which represents a class $v \in H^{2n}(W, \partial W, \mathbb{Z})$. Let $\rho_2(v_\#) = \hat{v}_\# - s_\# \in C^{2n}(W, \partial W, \mathbb{Z}/2)$. Then $s_\#$ represents a class $s \in H^{2n}(W, \partial W, \mathbb{Z}/2)$ and $j^*(s) = \rho_2(t) \in H^{2n}(W, \mathbb{Z}/2)$.

Formula 4.3 asserts that

$$I(W^{4n}) = \langle v^2, [W, \partial W]\rangle + \psi(b) - A(T^{2n}(\partial W), \psi) \in \mathbb{Z}/8.$$

Since $2\psi(b) = L(b, b) \in \frac{1}{4}\mathbb{Z}/\mathbb{Z} \subset Q/\mathbb{Z}$, we see that if we interpret $L(b, b) \in \mathbb{Z}/4$ we have

$$I(W^{4n}) = \langle v^2, [W, \partial W]\rangle + L(b, b) - A(T^{2n}(\partial W), \psi) \in \mathbb{Z}/4.$$

Since also

$$\begin{aligned}\rho_4(v^2) = \mathscr{P}(v) &= \mathscr{P}(\hat{v} + s) = \mathscr{P}(\hat{v}) + \mathscr{P}(s) + i_*(s^2) \\ &= \mathscr{P}(\hat{v}) - \mathscr{P}(s) \in H^{4n}(W, \partial W; \mathbb{Z}/4),\end{aligned}$$

where $i_*\colon H^*(\ , \mathbb{Z}/2) \to H^*(\ , \mathbb{Z}/4)$ is induced by the inclusion $2\colon \mathbb{Z}/2 \to \mathbb{Z}/4$, Theorem 6.1 follows if we prove that $L(b, b) = \langle \mathscr{P}(s), [W, \partial W]\rangle \in \mathbb{Z}/4$. But this is exactly Lemma 3.5(b), hence Theorem 6.1 is proved.

For our final application we produce a multiplicative characteristic class $l(\xi)$ for spherical fiber spaces, ξ, so that

$$I(\tilde{W}) \equiv \langle l(v_{\tilde{W}}), [\tilde{W}]\rangle \mod 4$$

for any $\mathbb{Z}$ or $\mathbb{Z}/2^r$ $P.D.$ space, $r \geq 2$. It will then follow that $l(\xi) \equiv L(\xi)$ mod 4 for any topological bundle. Here $L(\xi)$ is the characteristic class for topological bundles which is multiplicative, and which gives the index for $\mathbb{Z}$ $P.D.$ spaces and the index mod 2^k for $\mathbb{Z}/2^k$ $P.D.$ spaces. It follows from results of [10] that these two properties characterize $L \in H^{4*}(B_{STOP}; \mathbb{Z}_{(2)})$. A slight generalization of the argument there shows that any $\mathbb{Z}/2^k$ characteristic class of topological bundles which is multiplicative, gives the index mod 2^k for any $\mathbb{Z}$ or $\mathbb{Z}/2^k$ $P.D.$ space, and which agrees with the $\mathbb{Z}/2^k$ reduction of L for vector bundles is equal to the $\mathbb{Z}/2^k$ reduction of L for all topological bundles. We will show l satisfies all these properties.

To construct l, first consider $L \in H^{4*}(B_{SO}, \mathbb{Z}_{(2)})$. Since $\rho_2(L_n) = \rho_2(\mathscr{P}(v_{2n})) = v_{2n}{}^2 \in H^{4n}(B_{SO}, \mathbb{Z}/2)$, it follows that $\rho_4(L_n) = \mathscr{P}(v_{2n}) + i_* \sigma_n(w_2 \ldots w_{4n}) \in H^{4n}(B_{SO}, \mathbb{Z}/4)$, where $\sigma_n(w_2 \ldots w_{4n}) \in H^{4n}(B_{SO}, \mathbb{Z}/2)$ is a polynomial in the Stiefel–Whitney classes, well-defined modulo image(Sq^1). Using the formulas of [13] one can compute the $\sigma_n(w_2 \ldots w_{4n})$. For example

$$\rho_4(L_1) = \mathscr{P}(v_2) + i_*(w_4)$$
$$\rho_4(L_2) = \mathscr{P}(v_4) + i_*(w_8 + w_6 w_2 + w_2{}^4)$$
$$\rho_4(L_3) = \mathscr{P}(v_6) + i_*(w_8 w_2{}^2 + w_6{}^2 + w_6 w_2{}^3 + w_4{}^3 + w_7 w_3 w_2 + w_5 w_4 w_3 + w_3{}^2 w_2{}^3).$$

THEOREM 6.4. *Let* $l_n = \mathscr{P}(v_{2n}) + i_*\sigma_n(w_2 \ldots w_{4n}) \in H^{4n}(B_{SG}, \mathbb{Z}/4)$. *Let* $l = 1 + l_1 + l_2 + \ldots$ *then,*

(i) $l(\xi) \equiv L(\xi)$ mod 4 *for all vector bundles;*
(ii) *l is multiplicative for spherical fiber spaces, and*
(iii) *if $\tilde{W}$ is a $\mathbb{Z}/4$ P.D. space with normal bundle $\nu_{\tilde{W}}$ then*

$$I(\tilde{W}) \equiv \langle l(\nu_{\tilde{W}}), [\tilde{W}]\rangle \bmod 4.$$

As a corollary, we have

COROLLARY 6.5. $\rho_4(L_n) = \mathscr{P}(v_{2n}) + i_*(\sigma(w_2, \ldots, w_{4n})) \in H^{4n}(B_{STOP}; \mathbb{Z}/4)$.

Proof of Theorem 6.4. (i) is obvious from the definition of l.

(ii) Given two oriented spherical fiber spaces ξ, η, we may compute

$$l(\xi \oplus \eta) = \sum_{n \geq 0} (\mathscr{P}(v_{2n}(\xi \oplus \eta)) + i_* \sigma_n(\xi \oplus \eta))$$

mechanically using the formulae for $\mathscr{P}(x + y)$, $\mathscr{P}(x \cdot y)$, $v_k(\xi \oplus \eta)$, and $w_k(\xi \oplus \eta)$. We know that if ξ and η are vector bundles, the result of this computation is $l(\xi \oplus \eta) = l(\xi) \cdot l(\eta)$ since $l = \rho_4(L) \in H^{4*}(B_{SO}, \mathbb{Z}/4)$. But there are natural inclusions $H^*(B_{SO}, \mathbb{Z}/2) \subset H^*(B_{SG}, \mathbb{Z}/2)$ and $H^*(B_{SO}, \mathbb{Z}/4) \subset H^*(B_{SG}, \mathbb{Z}/4)$ (defined by the Stiefel–Whitney classes and their Pontrjagin squares), compatible with all structure involved. Thus $l(\xi \oplus \eta) = l(\xi) \cdot l(\eta)$ for arbitrary spherical fibrations. This proves Theorem 6.4(ii).

(iii) Now let $\tilde{W}$ be a $\mathbb{Z}/4$ Poincaré space. It follows readily from Corollary 6.3 that $I(\tilde{W}) \equiv \langle v^*\mathscr{P}(v_{2n}), [\tilde{W}]\rangle \in \mathbb{Z}/4$. Thus Theorem 6.4(iii) is equivalent to

$$\langle v^*i_* \sigma_n(w_2 \ldots w_{4n}), [\tilde{W}]\rangle = 2\langle v^*\sigma_n(w_2 \ldots w_{4n}), [\tilde{W}]\rangle = 0 \in 2(\mathbb{Z}/2) \subset \mathbb{Z}/4.$$

This is true if $\tilde{W}$ is a closed smooth manifold. Thus the element

$$\sigma_n(w_2 \ldots w_{4n})U \in H^{4n}(MSO, \mathbb{Z}/2)$$

is decomposable over the Steenrod algebra since it vanishes on $\pi_*(MSO)$ and there is a map from MSO to a product of Eilenberg–MacLane spaces, which induces an isomorphism in $\mathbb{Z}/2$ cohomology [16]. It follows that $\sigma_n(w_2 \ldots w_{4n})U \in H^{4n}(MSG, \mathbb{Z}/2)$ is also decomposable over the Steenrod algebra. But if $\tilde{W}$ is a $\mathbb{Z}/4$ Poincaré space we can find a proper embedding of pairs $(\tilde{W}, \delta\tilde{W}) \subset (S_4^{q+4n}, S^{q+4n-1})$ where $S_4^{q+4n} = S^{q+4n-1} \bigcup_4 e^{q+4n}$. Such an embedding defines a collapsing map c: $S_4^{q+4n} \to T(\nu_{\tilde{W}}^q)$, of degree one in the top dimension. Since all Steenrod operations vanish in S_4^{q+4n}, it follows that

$$\langle \nu^*\sigma_n(w_2 \ldots w_{4n}), [\tilde{W}]\rangle = \langle c^*(T\nu)^*\sigma_n(w_2 \ldots w_{4n})U, [S_4^{q+4n}]\rangle = 0,$$

where $T(\nu)$: $T(\nu_{\tilde{W}}^q) \to MSG$. This proves Theorem 6.4(iii).

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