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Splitting manifold approximate fibrations ☆

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Abstract

Suppose M is a topological m-manifold, X is a generalized n-manifold satisfying the disjoint disks property (DDP), $m > n \geqslant 5$, $f: M \to X$ is an approximate fibration, with fiber the shape of a closed topological manifold F, and Y is a closed, 1-LCC, codimension three subset of X. We examine conditions under which f is controlled homeomorphic to an approximate fibration $g: M \to X$ such that $g|g^{-1}(Y): g^{-1}(Y) \to Y$ is, in some sense, an improvement of $f|f^{-1}(Y)$. One of the main results is that if Y is a generalized manifold, and if $f|f^{-1}(Y): f^{-1}(Y) \to Y$ is fiberwise shape equivalent to a manifold approximate fibration $p: E \to Y$, and $Wh(\pi_1(F) \times \mathbb{Z}^k) = 0$, $k = 0, 1, \ldots$, then f is controlled homeomorphic to a manifold approximation $g: M \to X$ such that $g|g^{-1}(Y): g^{-1}(Y) \to Y$ controlled homeomorphic to $p: E \to Y$. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Mapping cylinder neighborhoods have proven to be useful devices in studying geometric properties of ANRs. This has been especially true in working with generalized manifolds (ANR homology manifolds), since there are generalized manifolds that are not locally polyhedral, even stably [4]. If N is the boundary of a mapping cylinder neighborhood W of a generalized manifold X in a topological manifold M, then W is the mapping cylinder of a manifold approximate fibration (MAF) $f: N \to X$ whose fiber has the shape of a k-sphere, $k = \dim M - \dim X - 1$.

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If Y is a topological or generalized submanifold of X, then $f^{-1}(Y)$ may not even be an ANR, so that W, with its given mapping cylinder structure, may not be very useful for investigating properties of the pair (X,Y). It would seem desirable to find an MAF $g:N \to X$, perhaps in some sense equivalent to f, such that the pair $(N,g^{-1}(Y))$ is a manifold pair with $g^{-1}(Y)$ locally flat in N.

The purpose of this paper is to show that under fairly general conditions an MAF $f:N \to X$, with fiber shape equivalent to a closed topological manifold F is controlled equivalent to an MAF $g:N \to X$ that is *split* over a stratified generalized manifold Y in X. As a corollary we will show that if X is a generalized n-manifold having the disjoint disks property, $n \geqslant 5$, tamely embedded in euclidean space $\mathbb{R}^m$, $m-n \geqslant 3$, W is a mapping cylinder neighborhood of X in $\mathbb{R}^m$, and Y is a (topological or generalized) submanifold of X, such that $\dim X - \dim Y \geqslant 3$ and Y is tame in X, then there is a mapping cylinder retraction $\gamma: W \to X$ such that $\gamma^{-1}(Y)$ is a locally flat topological submanifold of W (Theorem 3.3). This result has been applied in [2], for example, to establish transversality theorems for submanifolds of a generalized manifold.

2. Definitions

A *generalized n-manifold* (n-gm) is a locally compact euclidean neighborhood retract (ENR) X such that for each $x \in X$,

$$H_k(X,X-\{x\};\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k=n, \\ 0 & \text{otherwise.} \end{cases}$$

Following Mitchell [12], we say that an ENR X is an n-gm *with boundary* if the condition $H_n(X,X\backslash\{x\};\mathbb{Z}) \cong \mathbb{Z}$ is replaced by $H_n(X,X\backslash\{x\};\mathbb{Z}) \cong \mathbb{Z}$ or 0, and if $\mathrm{bd}\, X = \{x \in X\colon H_n(X,X\backslash\{x\};\mathbb{Z}) \cong 0\}$ is an $(n-1)$-gm embedded in X. (In [12] Mitchell shows that $\mathrm{bd}\, X$ is a homology $(n-1)$-manifold.) An n-gm X, $n \geqslant 5$, has the *disjoint disks property* (DDP) if every pair of maps of the 2-cell B^2 into X can be approximated arbitrarily closely by maps that have disjoint images. A subset A of X is 1-LCC in X if for each $x \in A$ and neighborhood U of x in X, there is a neighborhood V of x in X lying in U such that the inclusion induced homomorphism $\pi_1(V\backslash A) \to \pi_1(U\backslash A)$ is trivial. A closed set A in X of codimension at least three will be called *tame* in X if A is 1-LCC in X.

Given an n-gm X, a *manifold approximate fibration with fiber F* (MAF) over X is an approximate fibration $p:N \to X$, where N is a topological manifold, p is a proper surjection, and the homotopy fiber of p is homotopy equivalent to F. (Equivalently, each $p^{-1}(x)$ has the shape of the space F.) (See [5,8]. In [8] X is also assumed to be a topological manifold.) A *manifold stratified space* is a locally compact ANR Z containing a filtration $Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_k = Z$ such that each Z_i is an ANR and each $Z_i\backslash Z_{i-1}$, $i=0,1,\ldots,k$, is a topological manifold without boundary [15]. We use the convention that $Z_{-1}=\emptyset$. The sets $Z_i\backslash Z_{i-1}$ are the *strata* of Z. A space Z will be called a *generalized manifold stratified space* if it has a filtration with generalized manifolds as strata. (We do not require any of the strata to have the DDP.)

If Z is a (generalized) manifold stratified space, with filtration $Z_0 \subseteq Z_1 \subseteq \cdots \subseteq Z_k$, then a *stratified manifold approximate fibration* is an approximate fibration $p:N \to Z$, where N is a manifold stratified space, with filtration $N_0 \subseteq N_1 \subseteq \cdots \subseteq N_k$, such that $p:(N_i\backslash N_{i-1}) \to$

$(Z_i \setminus Z_{i-1})$ is an MAF (see [7]). For example, Quinn shows in [13] that if K is a polyhedron and $f: E \to K$ is a topological block bundle with manifold fiber F, then f is homotopic (through block bundle maps) to a stratified manifold approximate fibration $f': E \to K$. An MAF $p: N \to X$ is said to be *split over* $Y \subseteq X$ if $p|p^{-1}(Y): p^{-1}(Y) \to Y$ is also an MAF or stratified MAF accordingly as Y is a (topological or generalized) manifold without boundary or (generalized) manifold stratified space, respectively, and $p^{-1}(Y)$ is tame in M. A space F is *K-flat* if $Wh(\pi_1(F) \times \mathbb{Z}^k) = 0$, for all $k = 0, 1, \ldots.$

Suppose that X is a locally compact ANR, A and B are separable metric spaces, and $p: A \to X$ and $q: B \to X$ are maps. Assume B is embedded in Hilbert space, l_2, and that q is extended to a neighborhood V_0 of B in l_2. We generalize the notion of a controlled map described in [8] as follows. A *controlled shape map* F^c from $A \xrightarrow{p} X$ to $B \xrightarrow{q} X$ (or simply $F^c: A \to B$ if the maps are understood) is given by a map $F: A \times [0,1) \to V_0$ such that

1. the map $(q \circ F) \cup (p \times 1): A \times [0,1] \to X$ is continuous, and
2. $\bigcap_{0 \leqslant t < 1} \mathscr{C}l(F(A \times [t,1))) \subseteq B$.

Two controlled shape maps $F^c, G^c: A \to B$ are *controlled shape homotopic*, denoted $F^c \simeq_c G^c$, if there is a third controlled shape map H^c from $A \times I \xrightarrow{p \circ p_1} X$ to $B \times I \xrightarrow{q \circ p_1} X$, where p_1 is the projection to the first factor, defined as follows. Let $V_0, q_0: V_0 \to X$, and $F: A \times [0,1) \to V_0$ be the neighborhood of B, extension of q, and map, respectively, used to define F^c and let $V_1, q_1: V_1 \to X$, and $G: A \times [0,1) \to V_1$ be the neighborhood of B, the extension of q, and the map used to define G^c. Then H^c is defined by a neighborhood V of $B \times I$ in l_2 containing $(V_0 \cup V_1) \times I$, an extension of

$$(q \circ p_1) \cup (q_0) \cup (q_1): (B \times I) \cup (V_0 \times \{0\}) \cup (V_0 \times \{1\}) \to X$$

to V, and a level preserving map $H: A \times [0,1) \times I \to V$ such that $H|A \times [0,1) \times \{0\} = F$ and $H|B \times [0,1) \times \{1\} = G$.

For example, given a controlled shape map $F^c: A \to B$ represented by $F: A \times [0,1) \to V_0$ and a neighborhood V of B in l_2, then a change of parameter (i.e., a homeomorphism $A \times [0,1)$ onto a closed subset of $A \times [0,1)$ that commutes with projection on A) gives a controlled homotopy of F^c to a controlled shape map represented by a map $F': A \times [0,1) \to V$. Thus, we may assume that a representative of F^c maps into any preassigned neighborhood of B. Since X is an ANR, given any two extensions q_0 and q_1 of $q: B \to X$ to neighborhoods V_0 and V_1, there is a neighborhood V_2 of B such that $q_0|V_2$ and $q_1|V_2$ are homotopic rel $q_0|B (= q_1|B)$. Thus, any two controlled shape maps F^c and G^c represented by the same map $F: A \times [0,1) \to l_2$ (but, perhaps different extensions to neighborhoods of B) are controlled shape homotopic. Notice that a controlled shape map from A to B that can be represented by a map which maps $A \times [0,1)$ into B is, in fact, a controlled map as given in [8] (such is the case when B is an ANR).

3. Statement of results

We can now state the main results of this paper. *Throughout the paper X will denote a generalized n-manifold, $n \geqslant 5$, without boundary satisfying the* DDP.

Theorem 3.1. *Suppose that M is a topological m-manifold, $m \geqslant 6$, and $f: M \to X$ is an MAF with fiber shape equivalent to a closed, connected, topological manifold F such that F is K-flat. Suppose Y is generalized manifold stratified space, with filtration $Y_0 \subseteq \cdots \subseteq Y_k$, embedded as a tame, closed subset of X,* $\dim X - \dim Y \geqslant 3$, *such that $f|f^{-1}(Y): f^{-1}(Y) \to Y$ is controlled shape equivalent to a surjective stratified MAF $p: E \to Y$. If E_{i-1} has a mapping cylinder neighborhood in E_i, $i = 1, \ldots, k$, then f is controlled homeomorphic to an MAF $g: M \to X$ such that $g^{-1}(Y) \cong E$, E is tame in M, and $g|g^{-1}(Y) = p$.*

It is perhaps worth emphasizing that the strata of Y are not assumed to have any general position properties such as the DDP.

Next, we state two applications of Theorem 3.1. The first is an immediate corollary.

Corollary 3.2. *Suppose Y is a topological manifold or polyhedron,* $\dim X - \dim Y \geqslant 3$, *tamely embedded as a closed set in X, and $f: M \to X$ ($m \geqslant 6$) is an MAF such that the associated Hurewicz fibration has a topological reduction. Then f is controlled homeomorphic to an MAF $g: M \to X$ that is split over Y.*

Proof. By the classification theorem of [8], the Hurewicz fibration $\hat{f}: \hat{M} \to X$ associated to f is controlled homotopy equivalent to f. Thus, if $\hat{f}$ is fiber homotopy equivalent to a bundle map $p: E \to X$, then the restriction of p to $p^{-1}(Y)$ is controlled shape equivalent to $f|f^{-1}(Y): f^{-1}(Y) \to Y$. □

The next theorem is the main application of Theorem 3.1.

Theorem 3.3. *Suppose that X is compact and is embedded as a tame subset of a topological m-manifold M, $m - n \geqslant 3$ and that Y is a compact topological or generalized manifold,* $\dim X - \dim Y \geqslant 3$, *tamely embedded in X. Then, there is a mapping cylinder neighborhood W of X in M with retraction $\gamma: W \to X$ that splits over Y.*

Proof. We shall only prove the case in which $\partial Y = \emptyset$. Let W be a mapping cylinder neighborhood of X in M as guaranteed by [11,19], with mapping cylinder retraction $\pi: W \to X$. Assume that W is the mapping cylinder of a map $f: \partial W \to X$. In order to get $f|f^{-1}(Y): f^{-1}(Y) \to Y$ controlled shape equivalent to an MAF we shall apply the mapping cylinder neighborhood classification of [3, Theorem 3.3], which we recall now.

Given a compact generalized n-manifold Z (with or without the DDP), let $\mathcal{N}_q(Z)$ denote the collection of germs of codimension q manifold neighborhoods V^{n+q} of Z in which Z is tamely embedded. Two embeddings $\imath_k: Z \to V_k$, $k = 1, 2$, represent the same element of $\mathcal{N}_q(Z)$ if there are neighborhoods N_k of Z in V_k and a homeomorphism $h: N_1 \to N_2$ such that $h \circ \imath_1 = \imath_2$. Let $B\,Top_{q+k,k}$ be the classification space for topological microbundle pairs $\varepsilon^k \subseteq \zeta^{k+q}$, where ε^k denotes the trivial microbundle of rank k, and let $B\,Top_q = \lim_{k\to\infty} B\,Top_{q+k,k}$. Theorem 3.3 of [3] asserts that there is a bijection $\mathcal{N}_q(Z) \to B\,Top_q$ when $q \geqslant 3$. (This requires that Z be compact.) The bijection $\mathcal{N}_q(X) \to B\,Top_q$ ($q = m - n$) associates the neighborhood W of X with a microbundle pair $\varepsilon^k \subseteq \zeta^{k+q}$ over X having the property that the $(q-1)$-spherical fibration

$\phi:\mathscr{E}\to X$ associated to the inclusion $\varepsilon^k \subseteq \zeta^{k+q}$ restricted to X is controlled homotopy equivalent to $f:\partial W\to X$. This implies that $f|f^{-1}(Y)$ is controlled shape equivalent to $\phi|\phi^{-1}(Y)$, which is the fibration associated to $\varepsilon^k|Y \subseteq \zeta^{q+k}|Y$. Applying Theorem 3.3 of [3] again (or just the classification theorem of [17] if Y is a topological manifold), we get $\phi|\phi^{-1}(Y)$ controlled homotopy equivalent to an MAF $q:\partial E\to Y$, whose mapping cylinder E is a topological manifold (see, e.g., [4, Proposition 2.1]).

We may now apply Theorem 3.1 to $f:\partial W\to X$ and $p:\partial E\to Y$ to get a controlled homeomorphism from $\partial W\xrightarrow{f}X$ to $\partial W\xrightarrow{g}X$ such that $g^{-1}(Y)\cong\partial E$ and $g|g^{-1}(Y)=p$. Let $H:\partial W\times[0,1)\to\partial W$ represent the inverse of this controlled homeomorphism. Assume mapping cylinders are parameterized so that their domains are at the 0 level. Then, we can define a homeomorphism h from the mapping cylinder M_g of g to W by

$$h(z,t)=\begin{cases}(H(z,t),t) & \text{if } 0\leqslant t<1,\\ z & \text{if } z\in X.\end{cases}$$

This provides W with the desired mapping cylinder structure.

The proof when $\partial Y\neq\emptyset$ follows from the relative version of the classification theorem of [3]. □

It would be nice to apply Theorem 3.1 to the case in which Y is a polyhedron, but we do not know whether the spherical fibration $\phi:\mathscr{E}\to Y$ is equivalent to a stratified MAF. If W is a mapping cylinder of X in euclidean space, however, then the result does in fact follow.

Corollary 3.4. *Suppose that X is embedded as a tame subset of euclidean space $\mathbb{R}^m$, $m\geqslant 6$ and $m-n\geqslant 3$, and that Y is a closed polyhedron, $\dim X-\dim Y\geqslant 3$, tamely embedded in X. Then there is a mapping cylinder neighborhood W of X in M with retraction $\gamma:W\to X$ that splits over Y.*

Proof. As above, assume W is the mapping cylinder of a map $f:\partial W\to X$ and that $\pi:W\to X$ is the mapping cylinder projection. Then the associated Hurewicz fibration to f is the Spivak normal fibration for X, $v:\mathscr{E}\to X$. By the classification theorem of [8], $v:\mathscr{E}\to X$ is controlled homotopy equivalent to f. On the other hand, by [6] $v:\mathscr{E}\to X$ has a stable topological reduction, hence, an unstable reduction, since we are in codimension at least 3 [9]. Thus, $f|f^{-1}(Y):f^{-1}(Y)\to Y$ is controlled shape equivalent to a bundle map $p:E_0\to Y$. This gives a bundle map to which we may apply Theorem 3.1. The desired mapping cylinder projection $\gamma:W\to X$ is obtained as in Theorem 3.3. □

4. Splitting an MAF over a generalized manifold without boundary

In this section, we shall prove a special case of the splitting theorem in which Y is a tame, generalized submanifold of X without boundary. We suppose the following setting:

1. M is a topological m-manifold, $m\geqslant 6$,

2. $f: M \to X$ is a proper, surjective, MAF with fiber shape equivalent to a closed, connected, topological manifold F such that F is K-flat,
3. $Y \subseteq X$ is a generalized manifold,
4. $\dim X - \dim Y \geqslant 3$,
5. Y is closed and tame in X, and
6. $f|f^{-1}(Y): f^{-1}(Y) \to Y$ is controlled shape equivalent to an MAF $p: E \to Y$.

The idea of the proof is to embed E in M near $f^{-1}(Y)$ and then get a sequence of finer and finer controlled ambient isotopies over Y taking E closer and closer to $f^{-1}(Y)$. By stacking the inverses of these isotopies, we get the desired controlled homeomorphism of M.

Represent the inverse of a controlled shape equivalence from $f|f^{-1}(Y): f^{-1}(Y) \to Y$ to the MAF $p: E \to Y$ by a map

$$\sigma: E \times [0,1) \to M.$$

For any subset $Z \subseteq X$, we will let $\hat{Z} = f^{-1}(Z)$. The first lemma provides "mapping cylinder-like" neighborhoods of $\hat{Y}$.

Lemma 4.1. *$M \setminus \hat{Y}$ has a tame end over Y. Thus, there is a neighborhood N of $\hat{Y}$ in M such that N is a topological manifold with boundary ∂N and $N \setminus \hat{Y} \cong \partial N \times [0,1)$, where the collar structure is controlled over Y.*

Proof. Let U be a neighborhood of Y in X that retracts to Y via a retraction $r: U \to Y$. Then $r \circ f: \hat{U} \to Y$ gives a control map for a neighborhood of the end of $M \setminus \hat{Y}$ over Y. Since Y is locally contractible, small connected open sets V in Y have the property that $f|\hat{V}: \hat{V} \to V$ is controlled shape equivalent to the projection $V \times F \to V$. Thus, since Y is tame in X, the homotopy fiber of the inclusion $\hat{Y} \subseteq M$ is homotopy equivalent to a sphere of dimension $\geqslant 2$, hence, is simply connected. This implies that the end of $M \setminus \hat{Y}$ over Y has locally constant fundamental group isomorphic to $\pi_1(F)$. Quinn's end theorem [13] then applies to give a controlled collar of a neighborhood of the end of $M \setminus \hat{Y}$ over Y. $\square$

Let N be a neighborhood of $\hat{Y}$ as given by the conclusion of Lemma 4.1, and let $c: \partial N \times [0,1) \to N \setminus \hat{Y}$ be a homeomorphism giving a controlled collar structure over Y. For each $t \in [0,1)$ let $N_t = N \setminus c(\partial N \times [0,t))$. Then for any $\varepsilon > 0$ we can find a $t \in [0,1)$ such that N_t isotopically deforms into any preassigned neighborhood of $\hat{Y}$ by an ε-isotopy over Y that is fixed on a smaller neighborhood of $\hat{Y}$. In particular, for every $\varepsilon > 0$ there is a $t \in [0,1)$ such that the inclusion $\hat{Y} \subseteq N_t$ is an ε-shape equivalence over Y. Thus, we have the following lemma, whose proof is immediate. (If Y is not compact, then here, and in the remainder of the section, ε and t may be functions of $y \in Y$.)

Lemma 4.2. *For every $\varepsilon > 0$ there exists $t \in [0,1)$ such that $\sigma(E \times [s,1)) \subseteq N_t$ and $\sigma: E \times \{s_1\} \to N_t$ is an ε-homotopy equivalence over Y for every $s_1 \in [s,1)$.*

We shall now suppose that our parameters have been arranged so that $\sigma(E \times [t,1)) \subseteq \operatorname{int} N_t$ for all $t \in [0,1)$. We will need to use the following version of Ferry-Pedersen's controlled π–π Theorem [6]. Its proof requires only minor adjustments to the argument given for the Bounded π–π Theorem in [6].

Theorem 4.3 (Simply Connected Controlled π–π Theorem). *If B is a finite polyhedron, then there exist $T > 0$ and $\varepsilon_0 > 0$ so that if $(P^n, \partial P)$, $n \geqslant 6$, is an ε-Poincaré duality space over B and $\varepsilon \leqslant \varepsilon_0$, and*

$$\begin{array}{ccc} (M,\partial M) & \xrightarrow{\phi} & (P,\partial P) \\ & & \downarrow p \\ & & B \end{array}$$

is an ε-surgery problem with bundle information assumed as part of the notation so that both $p : P \to B$ and $p| : \partial B \to B$ are UV^1, then we may do surgery to obtain a normal bordism from $(M,\partial M) \to (P,\partial P)$ to $(M',\partial M') \to (P,\partial P)$, where the second map is a $T\varepsilon$-homotopy equivalence of pairs.

The next lemma is a controlled version of Corollary 11.3.4 of [18]. It is a consequence of Theorem 4.3, Lemma 4.2, and Browder's "Top Hat" Theorem [1] (see also, [18, Theorem 11.3]).

Lemma 4.4. *For every $\varepsilon > 0$ there exists $t \in [0,1)$ such that $\sigma|E \times \{s\} : E \times \{s\} \to N_t$ is ε-homotopic over Y to a locally flat embedding for every $s \in [t,1)$.*

For $0 \leqslant t_1 \leqslant t_2 < 1$, let $N[t_1,t_2] = \bigcup_{t_1 \leqslant t \leqslant t_2} N_t \times \{t\} \subseteq M \times [0,1]$, and let $N[t] = N[t,t]$. Analogous to the relative version of Corollary 11.3.4 of [18] we also get a relative version of Lemma 4.4.

Lemma 4.5. *For every $\varepsilon > 0$ there exists $t \in [0,1)$ such that if $t \leqslant t_1 < t_2 < 1$ and if $\sigma|E \times \{t_k\}$ is a locally flat embedding, $k = 1,2$, then $\sigma|E \times [t_1,t_2] : E \times [t_1,t_2] \to M \times [t_1,t_2]$ is ε-homotopic to a locally flat embedding $\sigma_{12} : (E \times [t_1,t_2], E \times \{t_1\}, E \times \{t_2\}) \to (N[t_1,t_2], N \times \{t_1\}, N \times \{t_2\})$ that agrees with $\sigma|E \times \{t_k\}$ for $k = 1,2$.*

Proof of Theorem 3.1 (For a generalized manifold Y without boundary). Let $t_i = 1 - 1/(i+1)$, $i = 0,1,2,\ldots$. Consider the map σ as a map of $E \times [0,1) \to M \times [0,1)$. After a change of parameter, if necessary, we can apply Lemmas 4.4 and 4.5 to assume that a representative $\sigma : E \times [0,1) \to M$ of the controlled equivalence from $E \xrightarrow{p} Y$ to $\hat{Y} \xrightarrow{f|\hat{Y}} Y$, as a map to $M \times [0,1)$, satisfies $\sigma|E \times [t_{i-1},t_i] : (E \times [t_{i-1},t_i], E \times \{t_{i-1}\}, E \times \{t_i\}) \to (N_{t_{i-1}} \times [t_{i-1},t_i], N_{t_{i-1}} \times \{t_{i-1}\}, N_{t_{i-1}} \times \{t_i\})$ is a locally flat embedding. Using the end theorem and thin h-cobordism theorem [13], we see that $N_{t_{i-1}} \times [t_{i-1},t_i]$ has the structure of a mapping cylinder (of triples) over $\sigma(E \times [t_{i-1},t_i])$. Using the approximate lifting property we see that, for any preassigned $\delta > 0$, there is a smaller (mapping cylinder) neighborhood U_i of $\sigma(E \times [t_{i-1},t_i])$, obtained by shrinking down the mapping cylinder structure on $N_{t_{i-1}} \times [t_{i-1},t_i]$, that is a δ-h-cobordism over $\sigma(E \times \{t_i\})$. (Here the control map is the composition of the mapping cylinder retraction with the projection of $\sigma(E\times[t_{i-1},t_i])$ onto $\sigma(E\times\{t_i\})$.) The thin h-cobordism theorem then provides a small product structure on U_i over $\sigma(E\times\{t_i\})$. By the relative thin h-cobordism theorem we can extend this to a small product structure on $N_{t_{i-1}}\times[t_{i-1},t_i]$ over Y that agrees with the natural (vertical) one on $\partial N_{t_{i-1}}\times[t_{i-1},t_i]$. The product structure on U_i gives a proper embedding $\tau: E\times[t_{i-1},t_i]\to U_i$ that is close to $\sigma|E\times[t_{i-1},t_i]$ (over $\sigma(E\times\{t_i\})$) and agrees with σ on $E\times\{t_{i-1}\}$. Applying Miller's isotopy theorem [10], we can get a small ambient isotopy of $N_{t_{i-1}}\times\{t_i\}$ taking $\tau|E\times\{t_i\}$ to $\sigma|E\times\{t_i\}$. As a consequence of these moves, we produce a small pseudoisotopy of $H_i': N_{t_{i-1}}\times[t_{i-1},t_i]\to N_{t_{i-1}}\times[t_{i-1},t_i]$ taking $\sigma(E\times\{t_{i-1}\})$ to $\sigma(E\times\{t_i\})$. By Quinn's pseudoisotopy theorem [16], there is small isotopy of $N_{t_{i-1}}\times[t_{i-1},t_i]$, fixed on $\partial N_{t_{i-1}}\times[t_{i-1},t_i]\cup(N_{t_{i-1}}\times\{t_{i-1},t_i\})$, taking H_i' to an isotopy $H_i: N_{t_{i-1}}\times[t_{i-1},t_i]\to N_{t_{i-1}}\times[t_{i-1},t_i]$. By extending via the identity outside each $N_{t_{i-1}}\times[t_{i-1},t_i]$, we get an isotopy that we shall still call $H_i: M\times[t_{i-1},t_i]\to M\times[t_{i-1},t_i]$. Stacking these isotopies produces a level preserving homeomorphism $H: M\times[0,1)\to M\times[0,1)$. Let $g=\lim_{t\to 1}(f\circ H_t)|(M\times\{t\})$. Then H^{-1} represents a controlled homeomorphism H^{c} from g to f. $\square$

5. Splitting an MAF over a stratified space

As one might naturally expect, the proof of the general case proceeds by induction over the strata. The special case proved in Section 4 gets the induction started. We assume, then, the following setting.

1. $f: M\to X$ is an MAF with fiber shape equivalent to a closed connected topological manifold F such that F is K-flat;
2. $Y\subseteq X$ is a generalized manifold stratified space with filtration $Y_0\subseteq Y_1\subseteq\cdots\subseteq Y_k$;
3. $\dim X-\dim Y\geqslant 3$;
4. Y is closed and tame in X;
5. $f|\hat{Y}:\hat{Y}\to Y$ is controlled shape equivalent to a stratified MAF $p: E\to Y$;
6. for $i>1$, each E_{i-1} has a mapping cylinder neighborhood in E_i;
7. for some fixed $j\geqslant 1$, $\hat{Y}_{j-1}\cong E_{j-1}$, $\hat{Y}_{j-1}$ is tame in M, and $f|\hat{Y}_{j-1}=p|E_{j-1}$.

As before, $\hat{A}=f^{-1}(A)$, for any subset A of X. Set $Z=Y_{j-1}$, $V=Y_j\setminus Y_{j-1}$, $X'=X\setminus Z$, $M'=M\setminus\hat{Z}$, and $E'=E_j\setminus E_{j-1}$. Applying the special case we may assume that $f|M': M'\to X'$ is split over V so that $\hat{V}\cong E'$ and is tame in M, and $f|\hat{V}=p|E'$. This gives a (continuous) map $f|M\to X$ such that $\hat{Z}\cong E_{j-1}$, $\hat{V}\cong E'$, $f|\hat{Z}=p|E_{j-1}$, and $f|\hat{V}=p|E'$. The only problem is that the union $\hat{Y}_j=\hat{Z}\cup\hat{V}$ may not be homeomorphic to E_j. Once we have corrected this defect, the inductive step will be complete.

Let W be a mapping cylinder neighborhood of $\hat{V}$ in M'. Since $\hat{V}$ has a controlled collar at infinity over $\hat{Z}$ and $m\geqslant 6$, we can use the End Theorem [13,14] to see that $(W,\partial W)$ has a collar at infinity, controlled over Z (not $\hat{Z}$). Since E' has a collar at infinity, controlled over E_{j-1}, $\hat{V}$ has a collar at infinity, controlled over Z. It is not difficult, using a uniqueness of collar argument, to get the collar of the end of $(W,\partial W)$ to agree with the collar on $\hat{V}$. Since $\hat{Z}$ is tame in M, we can extend this collaring to a collar of $M'(=M\setminus\hat{Z})$ at the end determined by $\hat{Z}$ that is also controlled over Z.

On the other hand, $\hat{Z}$ is tame in M so that M' has a collar at the end determined by $\hat{Z}$, controlled over $\hat{Z}$ (not just over Z). By the uniqueness of end structures [13,14], there is an isotopy h_t, $t\in[0,1]$, of M', fixed outside a neighborhood of $\hat{Z}$ and controlled over Z taking the collar controlled over Z to the one controlled over $\hat{Z}$. The finishing homeomorphism, $h_1: M' \to M'$, re-embeds W into M' so that its collar at the end determined by $\hat{Z}$ is now controlled over $\hat{Z}$, and we now have $\hat{Z}\cup h_1(\hat{V})\cong E_j$. It is possible to arrange the isotopy h_t of M' so that, for $0\leqslant t<1$, h_t extends to M via the identity on $\hat{Z}$. Thus, we get a controlled homeomorphism of M from f to a map that satisfies property 7 with $j+1$ replacing j.

6. Splitting in codimension one

We conclude the paper with a codimension one version of the splitting theorem, which is an easy application of the end theorem. Although we do not get the theorem for an arbitrary stratified subset Y of X, we are able to drop the assumption that the approximate fibration over Y is equivalent to a known MAF.

Theorem 6.1. *Suppose that M is a topological m-manifold $M, f: M \to X$ is an MAF with fiber homotopy equivalent to a closed topological manifold F such that F is K-flat, and Y is a generalized $(n-1)$-manifold without boundary embedded as a closed, locally two-sided, 1-LCC subset of X. Then f is controlled homeomorphic to an MAF $g: M\to X$ such that $g^{-1}(Y)$ is a locally flat $(m-1)$-dimensional submanifold of M and $g|g^{-1}(Y)$ is an MAF.*

Proof. Let W be a neighborhood of Y in X that is separated by Y. Then $\hat{W}\setminus\hat{Y}$ has two ends over Y, each of which is tame. By the end theorem [13] we can find controlled collars $U_\pm \cong N_\pm\times[0,\infty)$ over Y. Let $U=U_+\cup\hat{Y}\cup U_-$. Then U is a thin h-cobordism over Y, hence, a controlled product: $U\cong N_+\times[-1,1]$. Let us rename $N_+(\cong N_-)$ N and reparameterize the controlled collars (over Y) so that $U\setminus\hat{Y}\cong N\times[-1,0)\cup N\times(0,1]$. For $0<t\leqslant 1$ set $N_{\pm t}=N\times\{\pm t\}$, and let U_t be the closed region in U bounded by $N_{-t}\cup N_t$. Then U_t is a δ_t-thin h-cobordism, where $\delta_t\to 0$ as $t\to 0$, hence, an ε_t product, where $\varepsilon_t\to 0$ as $t\to 0$. (If Y is not compact, we may have to change the t parameter first and let δ_t and ε_t be functions on Y.)

Construct a sequence of ambient isotopies of M as follows. H^1 is fixed on the complement of U, slides $N_{1/2}$ to $N_{1/3}$ along the controlled collar structure on $N\times(0,1]$, and takes $N\times[\frac{1}{3},\frac{1}{2}]$ to $U_{1/3}$, matching up the product structures. This can be done with an (essentially) ε_1-isotopy. In general, H^i is fixed outside $U_{1/i}$, slides $N_{1/(i+1)}$ to $N_{1/(i+2)}$ along the controlled collar structure on $N\times(0,1]$, and takes $N\times[1/(i+2),1/(i+1)]$ to $U_{1/(i+2)}$, matching up the product structures. Stacking these isotopies gives a controlled homeomorphism $H: M\times[0,1)\to M$ whose inverse is a controlled homeomorphism from f to a map $g: M\to X$ such that $g^{-1}(Y)\cong N(=N\times\{\frac{1}{2}\})$. $\square$

If Y is not locally two-sided in X, then there is a double cover $p:(\tilde{X},\tilde{Y}) \to (X,Y)$ such that $\tilde{Y}$ is locally two-sided in $\tilde{X}$. $f:M \to X$ then pulls back to an MAF $\tilde{f}:\tilde{M} \to \tilde{X}$ that splits over $\tilde{Y}$. The question is whether this splitting can be done equivariantly with respect to the $\mathbb{Z}/2$-action on $\tilde{f}:\tilde{M} \to \tilde{X}$. This leads to the following more general question.

Question. Suppose (X,Y) is a generalized manifold pair, where X has the DDP and Y is tame in X, $f:M \to X$ is an MAF. Suppose G is a group acting on M and the pair (X,Y) such that f is G-equivariant and the induced map $\hat{f}:M/G \to X/G$ is a stratified MAF. Given that f splits over Y, under what conditions does $\hat{f}$ split over Y/G?

References

[1] W. Browder, Embedding smooth manifolds, Proceedings of the International Congresses Mathematica Moscow, 1966, pp. 712–719.
[2] J. Bryant, W. Mio, Transversality in generalized manifolds, Topology Appl. 94 (1999) 7–12.
[3] J. Bryant, W. Mio, Embeddings of homology manifolds in codimensions $\geqslant 3$, Topology 38 (1999) 811–821.
[4] J. Bryant, S. Ferry, W. Mio, S. Weinberger, Topology of homology manifolds, Ann. of Math. 143 (2) (1996) 435–467.
[5] D. Coram, P. Duvall, Approximate fibrations, Rocky Mountain J. Math. 7 (1977) 275–288.
[6] S. Ferry, E. Pedersen, Epsilon surgery, in: Novikov Conjectures, Index Theorems and Rigidity, Vol. 2, Oberwolfach, 1993, London Mathematical Society Lecture Series, Vol. 227, Cambridge University Press, Cambridge, 1995, pp. 167–226.
[7] C.B. Hughes, Stratification of mapping cylinders, preprint.
[8] C.B. Hughes, L.R. Taylor, E.B. Williams, Bundle theories for topological manifolds, Trans. Amer. Math. Soc. 319 (1990) 1–65.
[9] R. Kirby, L. Siebenmann, Foundational Essays on Topological Manifolds, Smoothings, and Triangulations, Princeton University Press, Princeton, NJ, 1977.
[10] R. Miller, Close isotopies on piecewise-linear manifolds, Trans. Amer. Math. Soc. 151 (1970) 597–628.
[11] R. Miller, Mapping cylinder neighborhoods of some ANR's, Ann. of Math. 103 (2) (1976) 417–427.
[12] W.J.R. Mitchell, Defining the boundary of a homology manifold, Proc. Amer. Math. Soc. 110 (2) (1990) 509–513.
[13] F. Quinn, Ends of maps I, Ann. of Math. 110 (1979) 275–331.
[14] F. Quinn, Ends of maps III, J. Differential Geom. 17 (1982) 503–521.
[15] F. Quinn, Homotopically stratified sets, J. Amer. Math. Soc. 1 (1988) 441–499.
[16] F. Quinn, Ends of maps IV, Amer. J. Math. 108 (1996) 1139–1161.
[17] C.P. Rourke, B.J. Sanderson, On topological neighborhoods, Compositio Math. 22 (1970) 387–424.
[18] C.T.C. Wall, Surgery on Compact Manifolds, Academic Press, New York, 1970.
[19] J.E. West, Mapping Hilbert cube manifolds to ANR's: a solution to a conjecture of Borsuk, Ann. of Math. 106 (1977) 1–18.