Topology and its Applications 105 (2000) 123–156

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

The cohomology ring of a class of Seifert manifolds

J. Bryden [a,*], C. Hayat-Legrand [b,1], H. Zieschang [c,2], P. Zvengrowski [a,3]

[a] *Department of Mathematics and Statistics, University of Calgary, Calgary, Alberta, Canada T2N 1N4*

[b] *Laboratoire Emile Picard, UMR CNRS 5580, Université Paul Sabatier, UFR MIG, 31062 Toulouse Cedex 4, France*

[c] *Facultät für Mathematik, Ruhr-Universität Bochum, 44780 Bochum, Germany*

Received 23 September 1996; received in revised form 11 February 1998 and 5 February 1999

Abstract

The cohomology groups of the Seifert manifolds are well known. In this article a method is given to compute the cup products in the cohomology ring of any orientable Seifert manifold whose associated orbit surface is S^2, and for any coefficients. In particular the $\mathbb{Z}/2$ cohomology ring is completely determined. This is applied to determine the existence of degree 1 maps from the Seifert manifold to $\mathbb{R}P^3$, and to the Lusternik–Schnirelmann category. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Seifert manifolds; Cohomology ring; Diagonal map; Cup products; Degree 1 maps

AMS classification: 57M25; 20F38; 20J05

1. Introduction

The ring structure of cohomology is a classical invariant associated with any CW-complex and has many applications. For instance, if $\boldsymbol{M}$ is a closed, orientable 3-manifold then the homotopy classes of sections of the bundle of Lorentz metric tensors over the $(3+1)$-dimensional space-time manifold $\boldsymbol{M} \times \mathbb{R}$ turns out to be isomorphic to $[\boldsymbol{M}, \mathbb{R}P^3]$ (cf. Shastri, Williams and Zvengrowski [28], Zvengrowski [32]), which is an Abelian group that depends only on $H^1(\boldsymbol{M}; \mathbb{Z}/2)$ and whether there exists a degree 1 map $\boldsymbol{M} \to \mathbb{R}P^3$ (type $\boldsymbol{M} = 1$) or not (type $\boldsymbol{M} = 2$). It is well known that type $\boldsymbol{M} = 1$ if and only if there is an element $\alpha \in H^1(\boldsymbol{M}; \mathbb{Z}/2)$ with $\alpha^3 \neq 0$ (cf. [28], Shastri and Zvengrowski [27]).

* Corresponding author. Email: bryden@acs.ucalgary.ca. During the preparation of this work the first and fourth named authors were supported in part by NSERC grant OPG0015179.

1 Email: hayat@picard.ups-tlse.fr.

2 Email: marlene.schwarz@rz.ruhr-uni-bochum.de.

3 Email: zvengrow@acs.ucalgary.ca.

PII: S0166-8641(99)00062-0

Another application appears in a theorem by Bredon and Wood [1] which asserts the equivalence between the existence of a non-zero cup cube in $H^3(\boldsymbol{M}; \mathbb{Z}/2)$ and the existence of an embedded, closed surface of odd Euler characteristic in the orientable 3-manifold $\boldsymbol{M}$ (cf. [10]). As an example, this equivalence is used in a theorem by Kirby and Melvin [13] relating the existence of two spin structures with different mod 4 Rochlin invariant and having one non-zero Witten–Reshetikhin–Turaev invariant. Furthermore, the existence of a non-orientable surface in an orientable 3-manifold is closely related to the existence of one-sided Heegaard splittings (cf. Rubinstein [23]). Other applications of the existence of a non-zero cup cube include the calculation of the invariants of 3-manifolds introduced by Murakami, Ohtsuki and Okada [17, §3], and in the construction of normal bordisms from $\boldsymbol{M}$ to $\mathbb{R}P^3$ (cf. Taylor [31]).

The explicit calculation of the cup products is often quite complicated. There are several methods available for determining the cup products of $H^*(X; A)$ for any commutative ring A. One such method, for a manifold $\boldsymbol{M}$, uses the intersection theory of chains and Poincaré duality. On the other hand, to obtain the cup products from their definition (cf. [29]), it is necessary to find a chain approximation to the diagonal $\boldsymbol{M} \to \boldsymbol{M} \times \boldsymbol{M}$. This algebraic method will be used herein. The theory behind the construction of a chain approximation to the diagonal is elementary (cf. Steenrod and Epstein [30, Chapter 5]), but in practice the algebraic calculations involved can be daunting. One may well ask if the construction of the diagonal approximation could be carried out on a computer. To the best of the authors' knowledge this is not feasible. This is because at each stage of the construction there are infinitely many choices which must be correlated to produce a single formula which is applicable in every case. On the other hand, a computer verification of the diagonal formula, once constructed, may very well be possible.

If the fundamental group, Π, of an irreducible, orientable 3-manifold $\boldsymbol{M}$ is infinite, which is always the case apart from a few well known exceptions (cf. Orlik [19]), then $\boldsymbol{M}$ is an Eilenberg–MacLane space $K(\Pi, 1)$ (cf. Epstein [8] or [20, Satz 5]). The fact that $\boldsymbol{M}$ is an Eilenberg–MacLane space is crucial since it then follows from MacLane [15, Theorem 11.5] that $H^*(\boldsymbol{M}; A) = H^*(\Pi; A)$. Thus the computation of the cup products in $H^*(\boldsymbol{M}; A)$ can be transformed into a purely algebraic calculation in group cohomology, so it is necessary to obtain a projective $\mathbb{Z}\Pi$-resolution of $\mathbb{Z}$. One such resolution is the equivariant chain complex, that is, the chain complex of the universal cover, M, of $\boldsymbol{M}$. This method was developed by Reidemeister (cf. [22]) in order to classify lens spaces and to calculate the Alexander polynomial in knot theory. The boundary maps are determined with the help of the Fox calculus (cf. Fox [9], Burde and Zieschang [4]). A similar construction has been used by Chevalley [5] in the theory of Lie groups.

In this paper the problem will be restricted to the case of the orientable Seifert manifolds with orbit surface S^2. The results that are described here have (apart from Theorem 1.4) been announced in [2], and it is the purpose of this work to give full details of the methods and proofs involved, as well as to lay the foundation for a forthcoming generalization of these results to the cohomology ring of any orientable Seifert manifold with $\mathbb{Z}/p$ coefficients (cf. [3]), for any prime p (which will then solve the problem of the existence of degree one maps onto any lens space $L(p, q)$, by applying Theorem 2.1 [10]).

The family of Seifert fibred 3-manifolds has been studied extensively since they were first introduced by Seifert in 1932. Any such 3-manifold is the union of disjoint circles, called fibres, in a particular way. The orbit space, obtained by identifying each fibre to a point, is a surface called the orbit surface of the manifold. Seifert developed a complete set of invariants that characterize this family up to fibre preserving homeomorphism. Aside from some well known exceptions, they fall into the category of 3-manifolds which are completely determined up to homeomorphism by their fundamental group. The basic definition and properties of Seifert manifolds are discussed in Seifert's original paper [25] (which was translated into English by Heil and can be found in the Appendix of [26]) and in such works as Hempel [12], Montesinos [16], Orlik, Vogt and Zieschang [20], and Scott [24].

The notation $\boldsymbol{M} = (O, o, 0 \mid e : (a_1, b_1), \ldots, (a_m, b_m))$ denotes an orientable Seifert manifold with m singular fibres having invariants $(a_1, b_1), \ldots, (a_m, b_m)$, Euler number e and orbit surface S^2. For each i, (a_i, b_i) is a pair of relatively prime integers, with $a_i \geqslant 2$. The fundamental group of $\boldsymbol{M}$ is

$$\Pi = \pi_1(\boldsymbol{M}) = \left\langle s_1, \ldots, s_m, h \mid [s_j, h], s_j^{a_j} h^{b_j}, \text{ for } 1 \leqslant j \leqslant m,\ s_1 \ldots s_m h^{-e} \right\rangle$$

and $R = \mathbb{Z}\Pi$ is the integral group ring of Π.

More specifically, the additive structure of the cohomology ring of the Seifert manifold $\boldsymbol{M} = (O, o, 0 \mid e : (a_1, b_1), \ldots, (a_m, b_m))$ is well known and can be determined by first calculating its homology by using a CW-decomposition (or by abelianizing Π) and then applying Poincaré duality. Assume that $a_1, \ldots, a_n$ are even and $a_{n+1}, \ldots, a_m$ are odd.

(i) If $n > 0$ then $H^1(\boldsymbol{M}; \mathbb{Z}/2) \approx H^2(\boldsymbol{M}; \mathbb{Z}/2) \approx (\mathbb{Z}/2)^{n-1}$.

(ii) If $n = 0$, let $b_1, \ldots, b_p \equiv 2 \pmod 4$, $b_{p+1}, \ldots, b_r \equiv 0 \pmod 4$, and $b_{r+1}, \ldots, b_m \equiv 1 \pmod 2$, then

(a) $H^1(\boldsymbol{M}; \mathbb{Z}/2) \approx H^2(\boldsymbol{M}; \mathbb{Z}/2) = 0$, if $m - r + e \equiv 1 \pmod 2$ and

(b) $H^1(\boldsymbol{M}; \mathbb{Z}/2) \approx H^2(\boldsymbol{M}; \mathbb{Z}/2) \approx \mathbb{Z}/2$, if $m - r + e \equiv 0 \pmod 2$.

Our main results are described in the following three theorems (cf. Section 3).

Theorem 1.1. *Let $\boldsymbol{M}$ be the Seifert manifold $(O, o, 0 \mid e : (a_1, b_1), \ldots, (a_m, b_m))$ and let δ_{jk} denote the Kronecker delta.*

(i) *If $n > 0$ (so $a_1, \ldots, a_n$ are even and $a_{n+1}, \ldots, a_m$ are odd, as above), then there are generators α_i, β_i, γ in dimensions 1, 2, and 3, respectively, such that as a vector space $H^*(\boldsymbol{M}; \mathbb{Z}/2) = \mathbb{Z}/2\{1, \alpha_i, \beta_i, \gamma \mid 2 \leqslant i \leqslant n\}$. For $2 \leqslant j, k \leqslant n$, the cup products in $H^*(\boldsymbol{M}; \mathbb{Z}/2)$ are given by*:

$$\alpha_j \cdot \alpha_k = \binom{a_1}{2}\beta_1 + \delta_{jk}\binom{a_j}{2}\beta_j \quad \text{and} \quad \alpha_j \cdot \beta_k = \delta_{jk}\gamma;$$

where $\beta_1 = \beta_2 + \cdots + \beta_n$. Moreover, if $2 \leqslant i \leqslant n$ as well, then

$$\alpha_i \cdot \alpha_j \cdot \alpha_k = \binom{a_1}{2}\gamma \quad \text{if } i \neq j \text{ or } j \neq k,$$
$$\alpha_i^3 = \left[\binom{a_1}{2} + \binom{a_i}{2}\right]\gamma.$$

(ii) *If* $n=0$ *and* α *denotes the generator of* $H^1(\boldsymbol{M};\mathbb{Z}/2)$ *when* $m-r+e\equiv 0 \pmod 2$ *then,*

$$\alpha^3\neq 0 \quad \text{if and only if} \quad e-2p+\sum_{j=r+1}^{m} a_j b_j \equiv 2 \pmod 4.$$

Furthermore, if $\alpha^3=0$ *then* $\alpha^2=0$.

Theorem 1.2. *With the above notation, if* $n\geqslant 2$, *then* type $\boldsymbol{M}=1$ *exactly when*

$$\binom{a_i}{2}+\binom{a_j}{2}\equiv 1 \pmod 2 \quad \text{for some } i,\ j,\ 1\leqslant i,\ j\leqslant n.$$

If $n=1$, *then* type $\boldsymbol{M}=2$. *Finally, if* $n=0$, *then* type $\boldsymbol{M}=1$ *if and only if*

$$m-r+e\equiv 0 \pmod 2 \quad \text{and} \quad e-2p+\sum_{j=r+1}^{m} a_j b_j\equiv 2 \pmod 4.$$

Let $\operatorname{cat} X$ denote the normalized Lusternik–Schnirelmann category of the space X, that is, $\operatorname{cat} X+1$ is the least number of open sets, each contractible in X, that cover X. The Ganea conjecture for Lusternik–Schnirelmann category states that, for $m>0$,

$$\operatorname{cat}\bigl(X\times S^m\bigr)=\operatorname{cat}(X)+1$$

(cf. [(3), §4]). This conjecture is resolved for Seifert manifolds in a number of different cases.

Theorem 1.3. *With the above notation, suppose that*
(1) $n\geqslant 2$ *and there exists* $\binom{a_i}{2}\not\equiv\binom{a_j}{2} \pmod 2$ *for some* $1\leqslant i,\ j\leqslant n$, *or*
(2) $n=0$, $m-r+e\equiv 0 \pmod 2$, *and* $e-2p+\sum_{j=r+1}^{m} a_j b_j\equiv 2 \pmod 4$.
Then $\operatorname{cat}\boldsymbol{M}=3$ *and furthermore for any integers* $n_1,\dots,n_k\geqslant 1$,

$$\operatorname{cat}\bigl(\boldsymbol{M}\times S^{n_1}\times\cdots\times S^{n_k}\bigr)=\operatorname{cat}\boldsymbol{M}+k=3+k.$$

In Section 2 we describe the equivariant chain complex $\mathcal{C}=C_*(M)\ (=C_*(M;\mathbb{Z}))$ for the universal cover M of $\boldsymbol{M}$, and state the "Diagonal Approximation Theorem" giving the necessary chain approximation to the diagonal $\Delta:\mathcal{C}\to\mathcal{C}\otimes\mathcal{C}$. The calculation of the cohomology ring $H^*(\boldsymbol{M};\mathbb{Z}/2)$ and the application to type is made in Section 3. In Remark 3.11 we show that the results on type, which follow from Theorem 1.2, generalize the previous calculations on type given in [27,32]. The final application given in Theorem 3.12, resolves the Ganea conjecture for the Seifert manifolds in numerous cases. The construction of the equivariant chain complex $\mathcal{C}$ is carried out in Section 4. Finally, Section 5 contains the details of the verification of the Diagonal Approximation Theorem. The paper has been arranged so that all of the main results are contained in Sections 1, 2, and 3 (which can be read independently of the other sections), while Sections 4 and 5 contain the somewhat arduous technical details of our method.

2. The equivariant chain complex and the diagonal approximation

The equivariant chains for the universal cover M of the Seifert manifold $\boldsymbol{M}$ are the free R-modules in dimensions $0, 1, 2, 3$ with generators (which are described completely in Section 4)

(G_0) 0: $\sigma_0^0, \dots, \sigma_m^0$;

(G_1) 1: $\sigma_1^1, \dots, \sigma_m^1; \rho_0^1, \dots, \rho_m^1; \eta_0^1, \dots, \eta_m^1$;

(G_2) 2: $\sigma_1^2, \dots, \sigma_m^2; \rho_0^2, \dots, \rho_m^2; \mu_0^2, \dots, \mu_m^2; \delta^2$;

(G_3) 3: $\sigma_0^3, \dots, \sigma_m^3; \delta^3$.

The definition of the boundary map, ∂, of the chain complex C requires the following conventions and definitions in the group ring R. First of all, in addition to the list of generators given in (G_1), (G_2) adopt the notation $\sigma_0^1 = 0$, $\sigma_0^2 = 0$. Next, let $r_j = s_0 s_1 \dots s_j$, $r_{-1} = 1$ and observe that $r_m = 1$. Given relatively prime integers $a_j \geqslant 2$, b_j, choose integers $c_j > 0$, $d_j > 0$ so that

$$\begin{vmatrix} a_j & b_j \\ c_j & d_j \end{vmatrix} = 1$$

and let $t_j = s_j^{c_j} h^{d_j}$. Then $s_j = t_j^{-b_j}$ and $h = t_j^{a_j}$. When $j = 0$ set $a_0 = 1$, $b_0 = e$, so that $s_0 = h^{-e}$. Now define the Laurent polynomials

$$f_{l,j} = 1 + t_j + \dots + t_j^{l-1}, \quad l \geqslant 1; \qquad f_{a_j,j} = F_j = \frac{t_j^{a_j} - 1}{t_j - 1},$$

$$g_{l,j} = t_j^{-1} + t_j^{-2} + \dots + t_j^{-l}, \quad l \geqslant 1; \qquad g_{b_j,j} = G_j = \frac{1 - t_j^{-b_j}}{t_j - 1},$$

$$P_j = 1 + t_j^{-b_j} + \dots + t_j^{-b_j(c_j-1)}, \qquad Q_j = 1 + t_j^{a_j} + \dots + t_j^{a_j(d_j-1)}.$$

In particular,

$$F_0 = 1 \quad \text{and} \quad G_0 = (1 - h^{-e})/(h - 1).$$

The free R-resolution $\mathcal{C}$ is given by the exact sequence

$$\mathcal{C}: 0 \to C_3(M) \xrightarrow{\partial_3} C_2(M) \xrightarrow{\partial_2} C_1(M) \xrightarrow{\partial_1} C_0(M) \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

where $\varepsilon(\sigma_j^0) = 1$. The differentials are given by

$(R_{1,1})$ $\partial \sigma_j^1 = \sigma_j^0 - \sigma_0^0$, $1 \leqslant j \leqslant m$;

$(R_{1,2})$ $\partial \rho_j^1 = (s_j - 1)\sigma_j^0$, $0 \leqslant j \leqslant m$;

$(R_{1,3})$ $\partial \eta_j^1 = (h - 1)\sigma_j^0$, $0 \leqslant j \leqslant m$;

$(R_{2,1})\qquad \partial\sigma_j^2 = \eta_0^1 - \eta_j^1 + (h-1)\sigma_j^1, \qquad 1 \leqslant j \leqslant m;$

$(R_{2,2})\qquad \partial\rho_j^2 = (1-s_j)\eta_j^1 + (h-1)\rho_j^1, \quad 0 \leqslant j \leqslant m;$

$(R_{2,3})\qquad \partial\delta^2 = \sum_{j=0}^{m} \pi_j^1,$

where $\pi_j^1 = r_{j-1}(\sigma_j^1 + \rho_j^1) - r_j\sigma_j^1$ (which implies $\pi_0^1 = \rho_0^1$);

$(R_{2,4})\qquad \partial\mu_j^2 = F_j \cdot \rho_j^1 + G_j \cdot \eta_j^1, \quad 0 \leqslant j \leqslant m;$

$(R_{3,1})\qquad \partial\sigma_j^3 = \rho_j^2 + (1-t_j)\mu_j^2, \quad 0 \leqslant j \leqslant m;$

$(R_{3,2})\qquad \partial\delta^3 = (1-h)\delta^2 - \sum_{j=0}^{m} \pi_j^2,$

where $\pi_j^2 = -r_{j-1}(\sigma_j^2 + \rho_j^2) + r_j\sigma_j^2$ (and this implies $\pi_0^2 = -\rho_0^2$).

Remark 2.1. Observe that (a) $\partial\pi_j^1 = (r_j - r_{j-1})\sigma_0^0$, (b) $\partial\pi_j^2 = (r_j - r_{j-1})\eta_0^1 + (1-h)\pi_j^1$.

The free resolution $\mathcal{C}$ suffices to find the additive structure of $H^*(\boldsymbol{M}; A)$. However, to find the ring structure (i.e., the cup products), make $\mathcal{C}\otimes\mathcal{C}$ into a R-chain complex by setting $\partial(x\otimes y) = \partial x \otimes y + (-1)^{\deg(x)} x \otimes \partial y$, and $(nu + mv)(x\otimes y) = n(ux\otimes uy) + m(vx \otimes vy)$ for $m, n \in \mathbb{Z}$, $u, v \in \Pi$, $x, y \in \mathcal{C}$. Then seek a diagonal approximation $\Delta : \mathcal{C} \to \mathcal{C}\otimes\mathcal{C}$, such that

(a) Δ is a R-chain map,

(b) Δ preserves augmentation, i.e., there is a commutative diagram:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Delta} & \mathcal{C}\otimes\mathcal{C} \\ \Big\downarrow{\scriptstyle\varepsilon} & & \Big\downarrow{\scriptstyle\varepsilon\otimes\varepsilon} \\ \mathbb{Z} & \xrightarrow{\approx} & \mathbb{Z}\otimes\mathbb{Z} \end{array}$$

Such a diagonal map Δ exists by acyclic models (cf. [30, Chapter 5]), but it must be found explicitly (it is not unique). Of course, by (a), it suffices to know Δ on the (free) generators of the complex $\mathcal{C}$.

Before stating the central result, we introduce the 1-chain $\tau_j^1 = P_j\rho_j^1 + t_j^{-b_jc_j} Q_j\eta_j^1$.

Diagonal Approximation Theorem 2.2. *A diagonal approximation of the equivariant chain complex is defined on the generators of the chain complex* $\mathcal{C}$ *as follows*:

$\Delta(\sigma_j^0) = \sigma_j^0 \otimes \sigma_j^0; \qquad \Delta(\sigma_j^1) = \sigma_j^1 \otimes \sigma_j^0 + \sigma_0^0 \otimes \sigma_j^1;$

$\Delta(\rho_j^1) = s_j\sigma_j^0 \otimes \rho_j^1 + \rho_j^1 \otimes \sigma_j^0; \qquad \Delta(\eta_j^1) = h\sigma_j^0 \otimes \eta_j^1 + \eta_j^1 \otimes \sigma_j^0;$

$\Delta(\sigma_j^2) = h\sigma_0^0 \otimes \sigma_j^2 - h\sigma_j^1 \otimes \eta_j^1 + \sigma_j^2 \otimes \sigma_j^0 + \eta_0^1 \otimes \sigma_j^1;$

$\Delta(\rho_j^2) = \rho_j^2 \otimes \sigma_j^0 + s_j\eta_j^1 \otimes \rho_j^1 - h\rho_j^1 \otimes \eta_j^1 + hs_j\sigma_j^0 \otimes \rho_j^2;$

$$
\begin{aligned}
\Delta\big(\mu_j^2\big) = {} & \mu_j^2 \otimes t_j^{-b_j}\sigma_j^0 + t_j^{a_j-b_j}\sigma_j^0 \otimes \mu_j^2 \\
& - \sum_{k=0}^{a_j-1}\sum_{l=-b_j}^{k-1} t_j^k\rho_j^1 \otimes t_j^l\tau_j^1 - \sum_{l=0}^{a_j-1}\sum_{k=l-b_j}^{a_j-b_j-1} t_j^k\tau_j^1 \otimes t_j^l\rho_j^1 \\
& - \sum_{k=1-b_j}^{-1}\sum_{l=-b_j}^{k-1} t_j^k\eta_j^1 \otimes t_j^l\tau_j^1 + \sum_{l=1-b_j}^{-1}\sum_{k=a_j-b_j}^{a_j+l-1} t_j^k\tau_j^1 \otimes t_j^l\eta_j^1 \\
& - G_j\sum_{r=1}^{a_j-1} t_j^r\tau_j^1 \otimes f_{r,j}\tau_j^1 - F_j\sum_{r=1}^{b_j} t_j^{-r}\tau_j^1 \otimes g_{r,j}\tau_j^1;
\end{aligned}
$$

$$
\begin{aligned}
\Delta(\delta^2) = {} & \delta^2 \otimes s_0\sigma_0^0 + r_{m-1}\sigma_0^0 \otimes \delta^2 + \pi_m^1 \otimes \pi_m^1 + \rho_0^1 \otimes \rho_0^1 + \pi_m^1 \otimes \rho_0^1 \\
& - \sum_{j=2}^{m-1}\sum_{i=1}^{j-1}\pi_j^1 \otimes \pi_i^1 + \sum_{j=1}^{m} r_j\sigma_j^1 \otimes r_{j-1}\rho_j^1 - \sum_{j=1}^{m}\pi_j^1 \otimes r_{j-1}\sigma_j^1;
\end{aligned}
$$

$$
\begin{aligned}
\Delta\big(\sigma_j^3\big) = {} & \sigma_j^3 \otimes \sigma_j^0 + t_j^{a_j-b_j}\sigma_j^0 \otimes \sigma_j^3 - t_j\mu^2 \otimes t_j^{-b_j}\tau_j^1 - t_j^{a_j-b_j}\tau_j^1 \otimes t_j\mu_j^2 \\
& + t_j\mu_j^2 \otimes G_j\tau_j^1 - \mu_j^2 \otimes G_j\tau_j^1 \\
& - t_j^{-b_j}P_j\mu_j^2 \otimes \big(\rho_j^1 + G_j\tau_j^1\big) - t_j^{a_j}\big(\rho_j^1 + G_j\tau_j^1\big) \otimes P_j\mu_j^2;
\end{aligned}
$$

$$
\begin{aligned}
\Delta(\delta^3) = {} & \delta^3 \otimes s_0\sigma_0^0 + r_{m-1}h\sigma_0^0 \otimes \delta^3 - h\delta^2 \otimes s_0\eta_0^1 \\
& - r_{m-1}\eta_0^1 \otimes \delta^2 - \rho_0^2 \otimes \rho_0^1 + \pi_m^2 \otimes \pi_m^1 \\
& - h\pi_m^1 \otimes \pi_m^2 + \pi_m^2 \otimes \rho_0^1 + h\pi_m^1 \otimes \rho_0^2 + h\rho_0^1 \otimes \rho_0^2 \\
& - \sum_{j=2}^{m-1}\sum_{i=1}^{j-1}\pi_j^2 \otimes \pi_i^1 + \sum_{j=2}^{m-1}\sum_{i=1}^{j-1} h\pi_j^1 \otimes \pi_i^2 \\
& - \sum_{j=1}^{m}\pi_j^2 \otimes r_{j-1}\sigma_j^1 - \sum_{j=1}^{m} h\pi_j^1 \otimes r_{j-1}\sigma_j^2 \\
& - \sum_{j=1}^{m} r_j\sigma_j^2 \otimes r_{j-1}\rho_j^1 + \sum_{j=1}^{m} r_jh\sigma_j^1 \otimes r_{j-1}\rho_j^2.
\end{aligned}
$$

3. The cohomology ring $H^*(\boldsymbol{M}; \mathbb{Z}/2)$

For the Seifert fibred manifold $\boldsymbol{M} = (O, o, 0 \mid e : (a_1, b_1), \ldots, (a_m, b_m))$ with infinite fundamental group, first assume that $a_1, \ldots, a_n \equiv 0 \pmod 2$, $1 \leqslant n \leqslant m$, and $a_{n+1}, \ldots, a_m \equiv 1 \pmod 2$ (this implies $b_1, c_1, \ldots, b_n, c_n \equiv 1 \pmod 2$). The $\mathbb{Z}/2$-cohomology is determined by the cochain complex:

$$
\mathrm{Hom}_R(C_0; \mathbb{Z}/2) \xrightarrow{\partial^0} \mathrm{Hom}_R(C_1; \mathbb{Z}/2) \xrightarrow{\partial^1} \mathrm{Hom}_R(C_2; \mathbb{Z}/2) \xrightarrow{\partial^2} \mathrm{Hom}_R(C_3; \mathbb{Z}/2),
$$

where $\mathbb{Z}/2$ is regarded as a trivial left R-module. For any generator α of C_i, let $\hat{\alpha}$ denote the dual generator of $\mathrm{Hom}_R(C_i;\mathbb{Z}/2)$; that is, $\hat{\alpha}(\alpha)=1$, $\hat{\alpha}(\beta)=0$ for any other generator β of C_i, for $i=0,\ldots,3$. For the case $n=0$, i.e., when all a_j are odd, see Lemma 3.7.

Theorem 3.1. *If $\boldsymbol{M}$ is an orientable Seifert fibred 3-manifold with $1\leqslant n\leqslant m$ as above, then*

$$H^i(\boldsymbol{M};\mathbb{Z}/2)=\begin{cases}\mathbb{Z}/2, & i=0,3,\\ (\mathbb{Z}/2)^{n-1}, & i=1,2,\\ 0, & i>3.\end{cases}$$

Moreover, the generators are

$$\begin{cases}1=[\sum_{j=0}^m\hat{\sigma}_j^0], & \textit{dimension } 0,\\ \alpha_j:=[\hat{\rho}_j^1+\hat{\rho}_1^1],\ 2\leqslant j\leqslant n, & \textit{dimension } 1,\\ \beta_j:=[\hat{\mu}_j^2]=[\hat{\sigma}_j^2],\ 2\leqslant j\leqslant n, & \textit{dimension } 2,\\ \gamma:=[\hat{\delta}^3]=[\hat{\sigma}_0^3]=\cdots=[\hat{\sigma}_m^3], & \textit{dimension } 3.\end{cases}$$

Proof. The mod 2 cohomology groups of $\boldsymbol{M}$ can easily be computed by Poincaré duality and the fact that $H_1(\boldsymbol{M};\mathbb{Z})=(\pi_1(\boldsymbol{M}))_{\mathrm{ab}}$. However, in order to compute the cohomology ring structure, it is also necessary to determine the generators of the cohomology groups and so the cohomology of $\boldsymbol{M}$ must be computed directly.

Computation of H^0. First of all observe that $\partial^0\hat{\sigma}_0^0=\sum_{j=1}^m\hat{\sigma}_j^1$ and that $\partial^0\hat{\sigma}_j^0=\hat{\sigma}_j^1$. It follows that $\partial^0\sum_{j=0}^m\hat{\sigma}_j^0=\partial^0\hat{\sigma}_0^0+\partial^0\sum_{j=1}^m\hat{\sigma}_j^0=0$ and hence $\ker\partial^0=\langle\sum_{j=0}^m\hat{\sigma}_j^0\rangle$. Thus $H^0(\boldsymbol{M};\mathbb{Z}/2)=\langle\sum_{j=0}^m\hat{\sigma}_j^0\rangle=\mathbb{Z}/2$.

Computation of H^1. A straightforward calculation reveals

$$\begin{aligned}&\partial^1\hat{\sigma}_j^1=0,\\ &\partial^1\hat{\eta}_0^1=\sum_{j=1}^m\hat{\sigma}_j^2+e\cdot\hat{\mu}_0^2, && \partial^1\hat{\rho}_0^1=\hat{\delta}^2+\hat{\mu}_0^2,\\ &\partial^1\hat{\eta}_j^1=\begin{cases}\hat{\sigma}_j^2+\hat{\mu}_j^2, & 1\leqslant j\leqslant n,\\ \hat{\sigma}_j^2+b_j\hat{\mu}_j^2, & n+1\leqslant j\leqslant m,\end{cases} && \partial^1\hat{\rho}_j^1=\begin{cases}\hat{\delta}^2, & 1\leqslant j\leqslant n,\\ \hat{\delta}^2+\hat{\mu}_j^2, & n+1\leqslant j\leqslant m.\end{cases}\end{aligned}\tag{3.1}$$

Thus $\ker(\partial^1)=\langle\hat{\sigma}_i^1,\hat{\rho}_j^1+\hat{\rho}_1^1\mid 1\leqslant i\leqslant m,\ 2\leqslant j\leqslant n\rangle$ and since $\mathrm{im}(\partial^0)=\langle\hat{\sigma}_j^1\mid 1\leqslant j\leqslant m\rangle$, it follows that $H^1(\boldsymbol{M};\mathbb{Z}/2)=\langle\alpha_j:=[\hat{\rho}_j^1+\hat{\rho}_1^1]\mid 2\leqslant j\leqslant n\rangle=(\mathbb{Z}/2)^{n-1}$.

Computation of H^2. Since

$$\begin{cases}\partial^2\hat{\sigma}_j^2=0,\\ \partial^2\hat{\rho}_j^2=\hat{\sigma}_j^3+\hat{\delta}^3,\\ \partial^2\hat{\delta}^2=0,\\ \partial^2\hat{\mu}_j^2=0,\end{cases}\tag{3.2}$$

the 2-cocycles are generated by $\hat{\sigma}_j^2$, $\hat{\delta}^2$, $\hat{\mu}_j^2$. The relations in (3.1) imply that $\hat{\mu}_j^2 \sim \hat{\sigma}_j^2$ for $1 \leqslant j \leqslant n$, and $\hat{\delta}^2 \sim \hat{\mu}_0^2 \sim \hat{\mu}_{n+1}^2 \sim \cdots \sim \hat{\mu}_m^2$. Since $\partial^1 \hat{\rho}_1^1 = \hat{\delta}^2$ for $1 \leqslant j \leqslant n$, $\hat{\delta}^2 \sim \hat{\mu}_0^2 \sim \hat{\mu}_{n+1}^2 \sim \cdots \sim \hat{\mu}_m^2 \sim 0$. Furthermore, $b_j \hat{\mu}_j^2 + \hat{\sigma}_j^2 \sim 0$ for $n+1 \leqslant j \leqslant m$, and hence $\hat{\sigma}_j^2 \sim 0$ for $n+1 \leqslant j \leqslant m$. This eliminates all but the elements $\hat{\sigma}_j^2$, $1 \leqslant j \leqslant n$, from consideration. Finally observe that $\sum_{j=1}^m \hat{\sigma}_j^2 + e \cdot 0 \sim \sum_{j=1}^n \hat{\sigma}_j^2 \sim 0$ is the only relation amongst these elements. This shows that $[\hat{\sigma}_1^2] = \sum_{j=2}^n [\hat{\sigma}_j^2]$. Thus

$$H^2(\boldsymbol{M}; \mathbb{Z}/2) = \big\langle [\hat{\sigma}_j^2] \mid 2 \leqslant j \leqslant n \big\rangle = \big\langle [\hat{\mu}_j^2] \mid 2 \leqslant j \leqslant n \big\rangle = (\mathbb{Z}/2)^{n-1}.$$

Define $\beta_j := [\hat{\sigma}_j^2] = [\hat{\mu}_j^2]$, $1 \leqslant j \leqslant n$, and observe that $\beta_1 = \sum_{j=2}^n \beta_j$.

Computation of H^3. It follows from (3.2) that $\hat{\delta}^3 \sim \hat{\sigma}_j^3$. This means that $[\hat{\delta}^3] = [\hat{\sigma}_j^3]$ for $1 \leqslant j \leqslant m$, and hence $H^3(\boldsymbol{M}; \mathbb{Z}/2) = \langle [\hat{\delta}^3] \rangle = \langle [\hat{\sigma}_j^3] \rangle = \mathbb{Z}/2$. Let γ denote the generator of $H^3(\boldsymbol{M}; \mathbb{Z}/2)$. □

The goal now is to turn to the computation of the ring structure in $H^*(\boldsymbol{M}; \mathbb{Z}/2)$, that is, to find the cup products $\alpha_j \cdot \alpha_k$, $\alpha_j \cdot \beta_k$. Since $H^1(\boldsymbol{M}; \mathbb{Z}/2) \approx H^2(\boldsymbol{M}; \mathbb{Z}/2) = 0$ when $n = 1$, assume that $n \geqslant 2$.

By definition $\alpha_j = [\hat{\rho}_j^1 + \hat{\rho}_1^1]$. It follows that $\alpha_j \cdot \alpha_k = [(\hat{\rho}_j^1 + \hat{\rho}_1^1) \smile (\hat{\rho}_k^1 + \hat{\rho}_1^1)]$, where for any $z \in C_2$:

$$\big((\hat{\rho}_j^1 + \hat{\rho}_1^1) \smile (\hat{\rho}_k^1 + \hat{\rho}_1^1)\big)(z) = \times\big((\hat{\rho}_j^1 \otimes \hat{\rho}_k^1 + \hat{\rho}_j^1 \otimes \hat{\rho}_1^1 + \hat{\rho}_1^1 \otimes \hat{\rho}_k^1 + \hat{\rho}_1^1 \otimes \hat{\rho}_1^1)(\Delta z)\big)$$

and $\times : \mathbb{Z}/2 \otimes \mathbb{Z}/2 \to \mathbb{Z}/2$ is the multiplication map. Then for some $\kappa, \kappa_i^\sigma, \kappa_i^\rho, \kappa_i^\mu \in \mathbb{Z}/2$,

$$\big(\hat{\rho}_j^1 + \hat{\rho}_1^1\big) \smile \big(\hat{\rho}_k^1 + \hat{\rho}_1^1\big) = \kappa \hat{\delta}^2 + \sum_{i=0}^m \kappa_i^\sigma \hat{\sigma}_i^2 + \sum_{i=0}^m \kappa_i^\rho \hat{\rho}_i^2 + \sum_{i=0}^m \kappa_i^\mu \hat{\mu}_i^2$$

and the coefficients $\kappa, \kappa_i^\sigma, \kappa_i^\rho, \kappa_i^\mu$ are given by:

$$\begin{aligned}
\kappa &= \times\big((\hat{\rho}_j^1 \otimes \hat{\rho}_k^1 + \hat{\rho}_j^1 \otimes \hat{\rho}_1^1 + \hat{\rho}_1^1 \otimes \hat{\rho}_k^1 + \hat{\rho}_1^1 \otimes \hat{\rho}_1^1)(\Delta \delta^2)\big), \\
\kappa_i^\sigma &= \times\big((\hat{\rho}_j^1 \otimes \hat{\rho}_k^1 + \hat{\rho}_j^1 \otimes \hat{\rho}_1^1 + \hat{\rho}_1^1 \otimes \hat{\rho}_k^1 + \hat{\rho}_1^1 \otimes \hat{\rho}_1^1)(\Delta \sigma_i^2)\big), \\
\kappa_i^\rho &= \times\big((\hat{\rho}_j^1 \otimes \hat{\rho}_k^1 + \hat{\rho}_j^1 \otimes \hat{\rho}_1^1 + \hat{\rho}_1^1 \otimes \hat{\rho}_k^1 + \hat{\rho}_1^1 \otimes \hat{\rho}_1^1)(\Delta \rho_i^2)\big), \\
\kappa_i^\mu &= \times\big((\hat{\rho}_j^1 \otimes \hat{\rho}_k^1 + \hat{\rho}_j^1 \otimes \hat{\rho}_1^1 + \hat{\rho}_1^1 \otimes \hat{\rho}_k^1 + \hat{\rho}_1^1 \otimes \hat{\rho}_1^1)(\Delta \mu_i^2)\big).
\end{aligned} \tag{3.3}$$

Next recall that $\beta_k = [\hat{\mu}_k^2] = [\hat{\sigma}_k^2]$ and observe that $\alpha_j \cdot \beta_k = \lambda \gamma$, for some $\lambda \in \mathbb{Z}/2$. Moreover, since $\alpha_j \cdot \beta_k = [(\hat{\rho}_j^1 + \hat{\rho}_1^1) \smile \hat{\mu}_k^2] = [\hat{\rho}_j^1 \smile \hat{\mu}_k^2 + \hat{\rho}_1^1 \smile \hat{\mu}_k^2]$ it follows that there exist $\zeta, \zeta_i^\sigma \in \mathbb{Z}/2$ such that $\hat{\rho}_j^1 \smile \hat{\mu}_k^2 + \hat{\rho}_1^1 \smile \hat{\mu}_k^2 = \zeta \hat{\delta}^3 + \sum_{i=0}^m \zeta_i^\sigma \hat{\sigma}_i^3$, where

$$\begin{aligned}
\zeta &= \times\big((\hat{\rho}_j^1 \otimes \hat{\mu}_k^2 + \hat{\rho}_1^1 \otimes \hat{\mu}_k^2)(\Delta \delta^3)\big), \\
\zeta_i^\sigma &= \times\big((\hat{\rho}_j^1 \otimes \hat{\mu}_k^2 + \hat{\rho}_1^1 \otimes \hat{\mu}_k^2)(\Delta \sigma_i^3)\big), \\
\lambda &= \zeta + \sum_{i=0}^m \zeta_i^\sigma.
\end{aligned} \tag{3.4}$$

Lemma 3.2. *Let δ_{jk} denote the Kronecker delta. Then $\alpha_j \cdot \alpha_k = \binom{a_1}{2}\beta_1 + \delta_{jk}\binom{a_j}{2}\beta_j$, for $2 \leqslant j, k \leqslant n$.*

Proof. From the discussion above, it suffices to compute the coefficients κ, κ_i^σ, κ_i^ρ, $\kappa_i^\mu \in \mathbb{Z}/2$.

It is clear that the coefficients

$$\kappa_i^\sigma = \times\big((\hat\rho_j^1 \otimes \hat\rho_k^1 + \hat\rho_j^1 \otimes \hat\rho_1^1 + \hat\rho_1^1 \otimes \hat\rho_k^1 + \hat\rho_1^1 \otimes \hat\rho_1^1)(\Delta\sigma_i^2)\big),$$
$$\kappa_i^\rho = \times\big((\hat\rho_j^1 \otimes \hat\rho_k^1 + \hat\rho_j^1 \otimes \hat\rho_1^1 + \hat\rho_1^1 \otimes \hat\rho_k^1 + \hat\rho_1^1 \otimes \hat\rho_1^1)(\Delta\rho_i^2)\big)$$

are zero, since the expressions for $\Delta\sigma_i^2$ and $\Delta\rho_i^2$, given in the Diagonal Approximation Theorem, do not involve any terms of the form $\rho_j^1 \otimes \rho_k^1$, $\rho_j^1 \otimes \rho_1^1$, $\rho_1^1 \otimes \rho_k^1$, or $\rho_1^1 \otimes \rho_1^1$.

Furthermore, since $\delta^2 \sim 0 \in H^2(\boldsymbol{M}; \mathbb{Z}/2)$, the coefficient, κ, of δ^2 is immaterial.

Finally, consider $\kappa_l^\mu = \times(\hat\rho_j^1 \otimes \hat\rho_k^1 + \hat\rho_j^1 \otimes \hat\rho_1^1 + \hat\rho_1^1 \otimes \hat\rho_k^1 + \hat\rho_1^1 \otimes \hat\rho_1^1)(\Delta\mu_l^2)$. Observe that the only terms of $\Delta\mu_l^2$ which contribute to κ_i^μ are of the form $\rho_l^1 \otimes \rho_l^1$ (for $j = k = l$) and $\rho_1^1 \otimes \rho_1^1$ (for all j, k). To complete the calculation of $\alpha_j \cdot \alpha_k$, it suffices to count the number of terms of the form $\rho_l^1 \otimes \rho_l^1$ in $\Delta\mu_l^2$ for $1 \leqslant j = k = l \leqslant n$, modulo 2.

Thus, when $1 \leqslant l \leqslant n$, the number of terms of the form $\rho_l^1 \otimes \rho_l^1$ in $\Delta\mu_l^2$ is

$$\sum_{i=0}^{a_l-1}\sum_{j=-b_l}^{i-1}(c_l) + \sum_{j=0}^{a_l-1}\sum_{i=j-b_l}^{a_l-b_l-1}(c_l) + b_l\sum_{r=1}^{a_l-1}(rc_l^2) + a_l\sum_{r=1}^{b_l}(rc_l^2)$$
$$= c_l\left[\sum_{i=0}^{a_l-1}(i+b_l) + \sum_{j=0}^{a_l-1}(a_l - j) + b_l c_l\binom{a_l}{2} + a_l c_l\binom{b_l+1}{2}\right]. \tag{3.5}$$

Since $b_l \equiv c_l \equiv 1 \pmod 2$, and $a_l \equiv 0 \pmod 2$ it follows that the expression in (3.5) is equal to

$$c_l\left[\binom{a_l}{2} + a_l b_l + a_l^2 - \binom{a_l}{2} + b_l c_l\binom{a_l}{2} + a_l c_l\binom{b_l+1}{2}\right] \equiv \binom{a_l}{2} \pmod 2. \qquad \square$$

Lemma 3.3. *Let $2 \leqslant j, k \leqslant n$. Then $\alpha_j \cdot \beta_k = \delta_{jk}\gamma$.*

Proof. Since $\Delta\delta^3$ does not involve μ_i^2 at all, $\zeta = 0$ in either case.

For $j \neq k$, it is clear that $\zeta_i^\sigma = 0$, since $\Delta\sigma_i^3$ only involves cells with subscript i. Finally, for $j = k$, the same reasoning implies that $\zeta_i^\sigma = 0$ for $i \neq k$, while

$$\zeta_k^\sigma = \times\big((\hat\rho_k^1 \otimes \hat\mu_k^2)(\Delta\sigma_k^3)\big)$$
$$= \times\big((\hat\rho_k^1 \otimes \hat\mu_k^2)\big(t_k^{a_k-b_k}\tau_k^1 \otimes t_k\mu_k^2 + t_k^{a_k}(\rho_k^1 + G_k\tau_k^1) \otimes P_k\mu_k^2\big)\big).$$

Recalling $G_k = t_k^{-1} + \cdots + t_k^{-b_k}$, $\tau_k^1 = P_k\rho_k^1 + t^{-b_k c_k}Q_k\eta_k^1$ with $P_k = 1 + t_k^{-b_k} + \cdots + t_k^{-b_k(c_k-1)}$, and by also recalling that b_k, c_k are both odd here, it follows that

$$\zeta_k^\sigma = c_k \cdot 1 + (1 + b_k c_k) \cdot c_k = 1 \in \mathbb{Z}/2. \qquad \square$$

Corollary 3.4. *If $i \neq j$ or $j \neq k$, $2 \leqslant i, j, k \leqslant n$, then $\alpha_i \cdot \alpha_j \cdot \alpha_k = \binom{a_1}{2}\gamma$.*

The seemingly exceptional role of the invariant a_1 is due to the choice of the generators $\{\alpha_i \mid 2 \leqslant i \leqslant n\}$ made in Theorem 3.1.

Corollary 3.5. *If $2 \leqslant k \leqslant n$, then*

$$\alpha_k^3 = \begin{cases} 0, & \binom{a_1}{2} + \binom{a_k}{2} \equiv 0 \pmod 2, \\ \gamma, & \binom{a_1}{2} + \binom{a_k}{2} \equiv 1 \pmod 2. \end{cases}$$

This corollary can be rewritten in a more perspicacious form. Observe that $\binom{a_1}{2} + \binom{a_k}{2} \equiv 1 \pmod 2$ for some k, $2 \leqslant k \leqslant n$, if and only if the set $\{\binom{a_i}{2} \mid 1 \leqslant i \leqslant n\}$ contains two numbers of opposite parity. The next theorem follows from this observation.

Theorem 3.6. *One has $\alpha_k^3 = \gamma$ for some k, $2 \leqslant k \leqslant n$, if and only if*

$$\binom{a_i}{2} + \binom{a_j}{2} \equiv 1 \pmod 2, \quad \textit{for some } i, j, 1 \leqslant i, j \leqslant n.$$

This concludes the case $n > 0$ (i.e., when at least one a_j is even).

Lemma 3.7. *Suppose that $a_i \equiv 1 \pmod 2$ for all $i = 1, \ldots, m$, $b_1, \ldots, b_p \equiv 2 \pmod 4$, $b_{p+1}, \ldots, b_r \equiv 0 \pmod 4$, $b_{r+1}, \ldots, b_m \equiv 1 \pmod 2$, and set $s = m - r$.*

(1) *If $s + e \equiv 1 \pmod 2$ then $H^1(\boldsymbol{M}; \mathbb{Z}/2) = H^2(\boldsymbol{M}; \mathbb{Z}/2) = 0$.*

(2) *If $s + e \equiv 0 \pmod 2$, then $H^1(\boldsymbol{M}; \mathbb{Z}/2) = H^2(\boldsymbol{M}; \mathbb{Z}/2) = \mathbb{Z}/2$.*

Furthermore, if α is the generator of $H^1(\boldsymbol{M}; \mathbb{Z}/2)$, then $\alpha^3 \neq 0$ if and only if

$$e - 2p + \sum_{j=r+1}^{m} a_j b_j \equiv 2 \pmod 4.$$

Proof. This relatively simple case can be settled using [27, Theorem 2.3]. To apply this theorem it is necessary to compute $H_1(\boldsymbol{M}; \mathbb{Z}/2)$ and $H_1(\boldsymbol{M}; \mathbb{Z}/4)$. Abelianizing $\pi_1(\boldsymbol{M})$ modulo 2 to obtain $H_1(\boldsymbol{M}; \mathbb{Z}/2)$ gives the abelian group generated by $h, s_1, \ldots, s_m$ with relations

$$\sum_{j=1}^{m} s_j = eh, \quad s_1 = \cdots = s_r = 0, \quad s_{r+1} = \cdots = s_m = h.$$

It follows that $(m - r)h = eh$, or $(m - r - e)h = (s - e)h = 0$. In case $s - e \equiv 1 \pmod 2$, this implies $h = 0$ and consequently $H_1(\boldsymbol{M}; \mathbb{Z}/2) = 0$, which completes (1) (using Poincaré duality), while if $s - e \equiv 0 \pmod 2$ one obtains $H_1(\boldsymbol{M}; \mathbb{Z}/2) = \mathbb{Z}/2$ (generated by $\alpha = [h] = [s_{r+1}] = \cdots = [s_m]$), which similarly completes (2) as far as the additive structure is concerned.

To determine the multiplicative structure, assuming (2) henceforth, in $H^*(\boldsymbol{M}; \mathbb{Z}/2)$ it suffices to know whether or not $\alpha^3 = 0$. Using [27], and the universal coefficient theorem, we see from the above that $H_1(\boldsymbol{M}; \mathbb{Z}/4)$ is either $\mathbb{Z}/4$ or $\mathbb{Z}/2$, corresponding respectively to

$\alpha^3 = 0$, $\alpha^3 \neq 0$. As in the above computation of $H_1(\boldsymbol{M};\mathbb{Z}/2)$, it is found that $H_1(\boldsymbol{M};\mathbb{Z}/4)$ is generated by the same classes with relations

$$\sum_{j=1}^{m} s_j = eh, \quad s_1 = \cdots = s_p = 2h, \quad s_{p+1} = \cdots = s_r = 0,$$
$$a_j s_j + b_j h = 0, \quad \text{for } r+1 \leqslant j \leqslant m.$$

Since $a_j^2 \equiv 1 \pmod 4$, one obtains $s_j = -a_j b_j h$ and hence

$$0 = eh - \sum_{j=1}^{m} s_j = eh - 2ph + \left(\sum_{j=r+1}^{m} a_j b_j\right) h = Ah,$$

where $A = e - 2p + \sum_{j=r+1}^{m} a_j b_j$. Thus $H_1(\boldsymbol{M};\mathbb{Z}/4) = \mathbb{Z}/4$ when $A \equiv 0 \pmod 4$, and $H_1(\boldsymbol{M};\mathbb{Z}/4) = \mathbb{Z}/2$ when $A \equiv 2 \pmod 4$, completing the proof for the multiplicative structure (A cannot be odd since only case (2) is under consideration). □

Remark 3.8. In Lemma 3.7(2), it is apparent that in fact $\alpha^2 = 0$ when $e - 2p + \sum_{j=r+1}^{m} a_j b_j \equiv 0 \pmod 4$, since $\alpha^2 = Sq^1\alpha$, which is just the Bockstein homomorphism applied to α.

Returning to the question of determining the type of $\boldsymbol{M}$, recall that $\boldsymbol{M}$ has type 1 if and only if there exists $\alpha \in H^*(\boldsymbol{M};\mathbb{Z}/2)$ with $\alpha^3 \neq 0$, and otherwise has type 2. Thus, the type of $\boldsymbol{M}$ can now be calculated from Theorem 3.6, when $n \geqslant 1$, or from Lemma 3.7 when $n = 0$. The following theorem is immediate.

Theorem 3.9. *If $a_1, \ldots, a_n$ are even, $2 \leqslant n \leqslant m$, then* type$(\boldsymbol{M}) = 1$ *exactly when* $\binom{a_i}{2} + \binom{a_j}{2} \equiv 1 \pmod 2$ *for some i, j, $1 \leqslant i, j \leqslant n$. If exactly one a_i is even, then* type$(\boldsymbol{M}) = 2$. *Finally, if all the a_i are odd, then using the notation of Lemma* 3.7, type$(\boldsymbol{M}) = 1$ *if and only if* $m - r + e \equiv 0 \pmod 2$ *and* $e - 2p + \sum_{j=r+1}^{m} a_j b_j \equiv 2 \pmod 4$.

Remark 3.10. For $i = 1, \ldots, k$ let $\boldsymbol{M}_i$ be from the class of Seifert manifolds under consideration. Suppose that $\boldsymbol{M}$ is the connected sum $\boldsymbol{M}_1 \# \cdots \# \boldsymbol{M}_k$. The type of $\boldsymbol{M}$ can be determined by Theorem 3.9 and [27, Corollary 2.5] which states that type$(\boldsymbol{M}) = \min\{\text{type}(\boldsymbol{M}_1), \ldots, \text{type}(\boldsymbol{M}_k)\}$.

Remark 3.11. Given $a_1, \ldots, a_n \equiv 0 \pmod 2$, suppose that $a_1, \ldots, a_r \equiv 0 \pmod 4$ and $a_{r+1}, \ldots, a_n \equiv 2 \pmod 4$. Theorem 2.3 [27] states that if there are an odd number of $\mathbb{Z}/2$'s in the cyclic decomposition of $H_1(\boldsymbol{M};\mathbb{Z})$ then $\boldsymbol{M}$ has type 1. This result can be compared to Theorem 3.9 by computing $H_1(\boldsymbol{M};\mathbb{Z}/4)$ from the chain complex $\mathcal{C} \otimes \mathbb{Z}/4$, where $\mathbb{Z}/4$ is considered as the trivial R-module. The following cases cover all possibilities (the details are omitted).

(1) $r \geqslant 1$: in this case [27, Theorem 2.3] shows that $\boldsymbol{M}$ has type 1 if $n - r$ is odd, whereas Theorem 3.9 shows that $\boldsymbol{M}$ has type 1 if $n - r \geqslant 1$,

(2) $r = 0$, n odd: here [27, Theorem 2.3] gives no conclusion, while Theorem 3.9 shows that $\boldsymbol{M}$ has type 2,
(3) $r = 0$, $n \geqslant 2$ even: both theorems show that the type of $\boldsymbol{M}$ is 2,
(4) $n = 0$: in this final case, both theorems can be used to determine the type of $\boldsymbol{M}$ and the results agree.

This demonstrates that Theorem 3.9 not only agrees with all previous results but generalizes these results as well.

There is a final application of the preceding results to Lusternik–Schnirelmann category and the Ganea conjecture. This will be stated here, but for complete details see [3].

Theorem 3.12. *With the above notation, suppose that*
(1) $n \geqslant 2$ *and there exists* $\binom{a_i}{2} \not\equiv \binom{a_j}{2} \pmod 2$ *for some* $1 \leqslant i, j \leqslant n$, *or*
(2) $n = 0$, $m - r + e \equiv 0 \pmod 2$, *and* $e - 2p + \sum_{j=r+1}^{m} a_j b_j \equiv 2 \pmod 4$.

Then $\operatorname{cat} \boldsymbol{M} = 3$ *and furthermore for any integers* $n_1, \ldots, n_k \geqslant 1$,

$$\operatorname{cat}(\boldsymbol{M} \times S^{n_1} \times \cdots \times S^{n_k}) = \operatorname{cat} \boldsymbol{M} + k = 3 + k.$$

Although the proof of Theorem 3.12 is not given here the hypotheses of the theorem are equivalent to the existence of a non-trivial 3-fold cup product of one-dimensional classes in $H^1(M; \mathbb{Z}/2)$. The result follows from this fact.

4. The equivariant chain complex of a Seifert manifold

We will start with an introduction to the theory of equivariant chain complexes (which were called 'Homotopieketten' by Reidemeister [22]).

4.1. The general theory of equivariant chain complexes

Let $\boldsymbol{W}$ be a CW-complex, $\pi : W \to \boldsymbol{W}$ a regular covering with automorphism group G (cf. Spanier [28]). Then there is a regular CW-structure on W such that
(i) every open cell of W is mapped bijectively to a cell of $\boldsymbol{W}$, and
(ii) to any two cells $\sigma_1^k, \sigma_2^k \subset W$ which are mapped to the same cell $\boldsymbol{\sigma}^k \subset \boldsymbol{W}$ there is a uniquely determined element $g \in G$ such that $g(\sigma_1^k) = \sigma_2^k$.

The CW-structure on W (cf. Dubrovin, Fomenko, and Novikov [7, Chapter 1, §11]) is obtained by first lifting the 0-cells of $\boldsymbol{W}$, then the 1-cells, etc., to W. This is possible because the closed cells are continuous images of the contractible disks D^k.

For $g \in G$ and a k-cell $\tau^k \in W$ define $g \cdot \tau^k := g(\tau^k)$. This is a G-action on the cells of W, and is free by (ii) above. Define the augmentation $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ by $\sum_g n_g g \mapsto (\sum_g n_g g)^\varepsilon = \sum_g n_g$. Let R denote the group ring $\mathbb{Z}G$. The G-action on the cells of W induces an R-action on the k-chains $C_k(W)$. The elements $\{\sigma^k \mid \boldsymbol{\sigma}^k \in \boldsymbol{W}\}$ form an R-basis of $C_k(W)$ and hence $C_k(W)$ is free as an R-module.

To complete the description of the augmented R-complex

$$\mathcal{C} : \cdots \to C_{k+1}(W) \xrightarrow{\partial_{k+1}} C_k(W) \xrightarrow{\partial_k} \cdots \xrightarrow{\partial_1} C_0(W) \xrightarrow{\varepsilon} \mathbb{Z} \to 0,$$

it remains to describe the boundary maps ∂_k. Since these are R-maps and since $C_k(W)$ is a free R-module, it suffices to define $\partial(e\cdot\tau^k)$, for each k-cell τ^k of W, $k\geqslant 1$ (here e denotes the identity element of G). As a slight abuse of notation, write τ^k for $e\cdot\tau^k$. Since W is a regular CW-complex, its boundary is determined (cf. [6, Chapter 2]) by an incidence function $[\tau^k:\sigma^{k-1}]$, for each k-cell τ^k and each $(k-1)$-cell σ^{k-1}, satisfying:

(1) $[\tau^k:\sigma^{k-1}]=0$ if $\bar\tau^k\cap\sigma^{k-1}=\emptyset$,
(2) $[\tau^k:\sigma^{k-1}]=\pm 1$ if $\sigma^{k-1}\subset\bar\tau^k$,
(3) set $D_k(\tau^k)=\sum[\tau^k:\sigma^{k-1}]\sigma^{k-1}$, then $D_{k-1}D_k(\tau^k)=0$, for all k and τ^k.

(Note that the sum in (3) is necessarily finite and that the conditions given in (1), (2) exhaust all possibilities for a k-cell τ^k and a $(k-1)$-cell σ^{k-1}.) It is then possible to choose an orientation on the cells of W so that the boundary map of the complex equals D_k, thus $\partial_k=D_k$.

The formulae for ∂_1 are self evident because the boundary of a 1-cell in W is just the initial and final point of a path lifted from a closed path in $\boldsymbol{W}$. Since the boundary of any 2-cell τ^2 represents a relation in G, the Fox (or free) calculus can be used to determine ∂_2. Its use is described in more detail in Section 4.3. In general, for $k\geqslant 3$, ∂_k is determined by properties (1)–(3) above.

The covering $\pi:W\to\boldsymbol{W}$ will henceforth be assumed to be the universal cover. Since π is regular $G=\pi_1(\boldsymbol{W})$. In the next two sections these techniques will be applied to the case where $\boldsymbol{W}=\boldsymbol{M}$. Since $\boldsymbol{M}$ is a 3-manifold, it follows that $C_k(\boldsymbol{M})=C_k(M)=0$ for $k>3$.

4.2. *The* CW-*structure of the Seifert manifolds*

Before describing the CW-structure of the Seifert manifold itself, we consider the CW-structure of a twisted solid torus. To deal with a Seifert fibration it is necessary to have a CW-structure which uses a "twisted" meridian and longitude. So first let $\boldsymbol{V}:=D^2\times S^1$. Let $\boldsymbol{\mu}_*^1=\partial D^2\times\{1\}$ and $\boldsymbol{\lambda}_*^1=\{1\}\times S^1$ be the respective standard meridian and longitude of $\partial\boldsymbol{V}$. Now let $\boldsymbol{\rho}_*^1$, $\boldsymbol{\eta}_*^1$ be a pair of simple closed curves on $\partial\boldsymbol{V}$ which cut the torus into a disk with base point $\boldsymbol{\sigma}^0=\boldsymbol{\rho}_*^1\cap\boldsymbol{\eta}_*^1=\boldsymbol{\mu}_*^1\cap\boldsymbol{\lambda}_*^1$. Then, for suitable integers a,b,c,d with $ad-bc=\pm 1$, there are homologies

$$\boldsymbol{\mu}_*^1\sim a\boldsymbol{\rho}_*^1+b\boldsymbol{\eta}_*^1,\qquad \boldsymbol{\lambda}_*^1\sim c\boldsymbol{\rho}_*^1+d\boldsymbol{\eta}_*^1\quad \text{on } \partial\boldsymbol{V}.$$

Assume that $ad-bc=1$, then there exists a map $\varphi:D^2\to\boldsymbol{V}$ such that $\varphi|_{\mathring{D}^2}$ is an embedding of the interior of the disk into $\mathring{\boldsymbol{V}}$. Furthermore, $\varphi(\partial D^2)\subset\boldsymbol{\rho}_*^1\cup\boldsymbol{\eta}_*^1$ and the cycle $\varphi|_{(\partial D^2)}\sim a\boldsymbol{\rho}_*^1+b\boldsymbol{\eta}_*^1$ on $\partial\boldsymbol{V}$. This defines a 2-cell $\boldsymbol{\mu}^2:=\operatorname{Im}(\varphi)\subset\boldsymbol{V}$ and the complement $\mathring{\boldsymbol{V}}\setminus\boldsymbol{\mu}^2$ is an open 3-cell, denoted by $\boldsymbol{\sigma}^3$. Thus, the solid torus $\boldsymbol{V}$ has the following CW-structure:

$$\boldsymbol{\sigma}^0=\boldsymbol{\rho}_*^1\cap\boldsymbol{\eta}_*^1;\quad \boldsymbol{\rho}^1=\boldsymbol{\rho}_*^1\setminus\boldsymbol{\sigma}^0,\quad \boldsymbol{\eta}^1=\boldsymbol{\eta}_*^1\setminus\boldsymbol{\sigma}^0;$$
$$\boldsymbol{\rho}^2=\partial\boldsymbol{V}\setminus(\boldsymbol{\rho}_*^1\cup\boldsymbol{\eta}_*^1),\quad \boldsymbol{\mu}^2;\quad \boldsymbol{\sigma}^3.$$

In this construction the mapping $\varphi|_{\partial D^2}:\partial D^2\to\partial\boldsymbol{V}$ of the boundary of the disk is not unique. By deforming φ over the 2-cell of $\partial\boldsymbol{V}$ and within the graph $\boldsymbol{\rho}_*^1\cup\boldsymbol{\eta}_*^1$, it is clear that any two choices of the map φ are homotopic on $\partial\boldsymbol{V}$.

Next consider an orientable Seifert manifold $\boldsymbol{M}$ having orbit surface S^2. Such a manifold is characterized by the Seifert invariants: $\boldsymbol{M} = (O, o, 0 \mid e : (a_1, b_1), \ldots, (a_m, b_m))$ (cf. Seifert [24]). The manifold will be decomposed into $m+1$ solid tori $\boldsymbol{V}_1, \ldots, \boldsymbol{V}_m, \boldsymbol{V}_0$ and its *central part* $\boldsymbol{M}' \cong \boldsymbol{B}(m+1) \times S^1$, where $\boldsymbol{B}(m+1) = \overline{S^2 \setminus (D_0^2 \cup \cdots \cup D_m^2)}$, is the closure of the sphere minus $m+1$ disks. The solid tori $\boldsymbol{V}_j$, for $1 \leqslant j \leqslant m$, are regular neighborhoods (cf. [12, p. 7]) of the m exceptional fibres with characteristic numbers (a_j, b_j) and $\boldsymbol{V}_0$ is a regular neighborhood of a normal fibre with characteristic numbers $(1, e)$. Now, $\boldsymbol{B}(m+1)$ has the following cell structure: on the jth boundary component there is a 0-cell $\boldsymbol{\sigma}_j^0$ and the 1-cell $\boldsymbol{\rho}_j^1$, for $j = 0, 1, \ldots, m$. Furthermore there are m 1-cells $\boldsymbol{\sigma}_j^1$ from $\boldsymbol{\sigma}_0^0$ to $\boldsymbol{\sigma}_j^0$, $1 \leqslant j \leqslant m$, and an open 2-cell $\boldsymbol{\delta}^2$ obtained from the sphere S^2 by removing the closed disks bounded by the 1-cells ρ_j^1 along with the 1-cells given by the paths σ_j^1. The attaching map for $\boldsymbol{\delta}^2$ is given by

$$(\boldsymbol{\rho}_0^1) \cdot \big(\boldsymbol{\sigma}_1^1 \boldsymbol{\rho}_1^1 (\boldsymbol{\sigma}_1^1)^{-1}\big) \cdot \cdots \cdot \big(\boldsymbol{\sigma}_m^1 \boldsymbol{\rho}_m^1 (\boldsymbol{\sigma}_m^1)^{-1}\big).$$

Moreover $\boldsymbol{M}'$ has $(m+1)$ 1-cells $\boldsymbol{\eta}_j^1$ and 2-cells $\boldsymbol{\rho}_j^2$, for $0 \leqslant j \leqslant m$, on the boundary components $\partial \boldsymbol{V}_j$. The 1-cells $\boldsymbol{\eta}_j^1$ correspond to the twisted longitude $\boldsymbol{\eta}_{*j}^1$ (defined above) of $\partial \boldsymbol{V}_j$ with the 0-cell $\boldsymbol{\sigma}_j^0$ deleted, while the 2-cells are defined to be $\boldsymbol{\rho}_j^2 := \partial \boldsymbol{V}_j \setminus (\boldsymbol{\rho}_{*j}^1 \cup \boldsymbol{\eta}_{*j}^1)$ and have attaching maps given by:

$$(\boldsymbol{\eta}_j^1) \cdot (\boldsymbol{\rho}_j^1) \cdot (\boldsymbol{\eta}_j^1)^{-1} \cdot (\boldsymbol{\rho}_j^1)^{-1}$$

In addition there are m 'interior' 2-cells $\boldsymbol{\sigma}_j^2$: that sit over the 1-cells $\boldsymbol{\sigma}_j^1$ with attaching maps

$$(\boldsymbol{\eta}_0^1) \cdot \big(\boldsymbol{\sigma}_j^1 (\boldsymbol{\eta}_j^1)^{-1} (\boldsymbol{\sigma}_j^1)^{-1}\big).$$

Finally, there is a 3-cell $\boldsymbol{\delta}^3$ which sits over $\boldsymbol{\delta}^2$ in $\boldsymbol{M}'$, defined by $\boldsymbol{\delta}^3 := \boldsymbol{\delta}^2 \times \boldsymbol{\eta}_0^1$.

Now each of the solid tori $\boldsymbol{V}_j$, $j = 0, \ldots, m$, has the CW-structure described above and is compatible with the cell structure of $\boldsymbol{M}'$. This completely describes the CW-structure of the Seifert manifold $\boldsymbol{M}$.

4.3. *The boundary map*

Having established the CW-structure of $\boldsymbol{M}$ in Section 4.2, the boundary maps ∂_1, ∂_2 ∂_3 in $\mathcal{C} = C_*(M)$ can now be determined. The formulae $(R_{1,1})$, $(R_{1,2})$, and $(R_{1,3})$ (which are stated in Section 2) giving ∂_1 are obvious consequences of the respective facts that $\boldsymbol{\sigma}_j^1$ is a path in $\boldsymbol{M}$ from the base point $\boldsymbol{\sigma}_0^0$ to $\boldsymbol{\sigma}_j^0$, while $\boldsymbol{\rho}_j^1$ represents the element $s_j \in \Pi$ and $\boldsymbol{\eta}_j^1$ represents the generic fibre $h \in \Pi$ (based at $\boldsymbol{\sigma}_j^0$).

The Fox calculus is used to derive the formulae given in $(R_{2,1})$–$(R_{2,4})$ for ∂_2. This procedure will now be outlined. Consider the general situation of the covering space $W \xrightarrow{\pi} \boldsymbol{W}$ described in Section 4.1, and for the time being suppose that $\boldsymbol{W}$ has a single 0-cell $\boldsymbol{e}^0$. Also suppose that a finite presentation of $G := \pi(\boldsymbol{W}, \boldsymbol{e}^0)$ is specified. Then each 1-cell $\boldsymbol{\sigma}_i^1$ represents a generator $x_i \in G$ and the boundary of each 2-cell $\boldsymbol{\tau}_j^2$ represents a word ω_j in the presentation of G. Given any such $\boldsymbol{\tau}_j^2$, suppose that $x_1, \ldots, x_k$ are the elements of G which correspond to the elements occurring in ω_j. Then

$$\text{(F1)} \qquad \partial\tau_j^2 = \sum_{i=1}^{k} \frac{\partial\omega_j}{\partial x_i}\sigma_i^1,$$

where $\partial\omega_j/\partial x_i$ denotes the Fox derivative of the word ω_j. The Fox derivatives are characterized by the following formulae:

$$\text{(F2)} \qquad \frac{\partial x_j}{\partial x_i} = \delta_{i,j}, \qquad \frac{\partial(uv)}{\partial x} = \frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}.$$

It follows from these two formulae that:

$$\text{(F3)} \qquad \begin{aligned} &\frac{\partial x^n}{\partial x} = 1 + x + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}, \\ &\frac{\partial x^{-n}}{\partial x} = -(x^{-1} + \cdots + x^{-n}) = \frac{x^{-n} - 1}{x - 1}, \qquad n > 0. \end{aligned}$$

This method can be generalized to the case where $\boldsymbol{W}$ may have several 0-cells, $\boldsymbol{e}_i^0$, and where some of the $\boldsymbol{\sigma}_i^1$ may be paths rather than loops (thus, in this case, $x_i \in \mathcal{G}(\boldsymbol{W})$, the fundamental groupoid of $\boldsymbol{W}$). When the union of the proper paths (i.e., the non-loops) in $\boldsymbol{W}$ is a tree, as is the case for the CW-structure of $\boldsymbol{M}$, these paths can simply be identified with $e \in \pi_1(\boldsymbol{W})$ when computing the Fox derivatives. (For related references see Lustig, Thiele, and Zieschang [14], Nielsen [18], Osborne and Zieschang [21].)

To apply this to $\boldsymbol{M}$, let $\boldsymbol{\sigma}_j^1$, the path from $\boldsymbol{\sigma}_0^0$ to $\boldsymbol{\sigma}_j^0$, for all $1 \leqslant j \leqslant m$, represent $q_j \in \mathcal{G}(\boldsymbol{M})$. These are the only proper paths among the 1-cells of $\boldsymbol{M}$ and their union is a tree. Let h_j denote the element of $\mathcal{G}(\boldsymbol{M})$ corresponding to $\boldsymbol{\eta}_j^1$. Now, when calculating the Fox derivatives, identify q_j with e in $\pi_1(\boldsymbol{M}, \boldsymbol{\sigma}_0^0)$ (as mentioned above) and h_j with h. The formulae given in $(R_{2,1})$, $(R_{2,2})$ and $(R_{2,3})$ follow immediately from the above procedure applied to the words in $\mathcal{G}(\boldsymbol{M})$ represented by the boundaries of $\boldsymbol{\sigma}_j^2$, $\boldsymbol{\rho}_j^2$ and $\boldsymbol{\delta}^2$ (respectively $h_0q_jh_j^{-1}q_j^{-1}$, $h_js_jh_j^{-1}s_j^{-1}$, $s_0q_1s_1q_1^{-1}q_2s_2q_2^{-1}\ldots q_ms_mq_m^{-1}$).

For example, to obtain the expression for $\partial_2\sigma_j^2$ given in $(R_{2,1})$, let $\omega = h_0q_jh_j^{-1}q_j^{-1}$. The basic formulae of Fox calculus (F2), (F3) show that:

$$\frac{\partial\omega}{\partial q_j} = h_0 - h_0q_jh_j^{-1}q_j^{-1}, \qquad \frac{\partial\omega}{\partial h_0} = 1, \qquad \frac{\partial\omega}{\partial h_j} = -h_0q_jh_j^{-1},$$

which reduce to:

$$\frac{\partial\omega}{\partial q_j} = h - 1, \qquad \frac{\partial\omega}{\partial h_0} = 1, \qquad \frac{\partial\omega}{\partial h_j} = -1,$$

in $\Pi = \pi_1(\boldsymbol{M}, \boldsymbol{\sigma}_0^0)$. The expression

$$(R_{2,1}) \qquad \partial\sigma_j^2 = (h-1)\sigma_j^1 + \eta_0^1 - \eta_j^1$$

now follows directly from Eq. (F1). (Recall that q_j, h_0, h_j are represented by the respective 1-cells $\boldsymbol{\sigma}_j^1$, $\boldsymbol{\eta}_0^1$, $\boldsymbol{\eta}_j^1$ in $\boldsymbol{M}$.) The derivation of $(R_{2,2})$ and $(R_{2,3})$ are similar and are omitted.

To obtain $(R_{2,4})$, start with the cell $\hat{\mu}_j^2$ representing the remaining relation $s_j^{a_j} h^{b_j}$ and apply the formulae given in (F1)–(F3), to obtain

$$\partial(\hat{\mu}_j^2) = \frac{s^a - 1}{s - 1}\rho_j^1 + s^a \frac{h^b - 1}{h - 1}\eta_j^1 = \frac{t^{-ab} - 1}{t^{-b} - 1}\rho_j^1 + t^{-ab}\frac{t^{ab} - 1}{t^a - 1}\eta_j^1,$$

where the subscript j is now dropped from s_j, t_j, a_j and b_j. A more convenient choice for this generator (with a simpler boundary) is given by $\mu_j^2 = \hat{\mu}_j^2 + x\rho_j^2$, for a suitable choice of $x \in R$. In fact $x \in \mathbb{Z}[t, t^{-1}] \subset R$ (that is, x is a Laurent polynomial in t) and is specified in the following lemma.

Lemma 4.1. *For relatively prime integers a, b,*

$$x := \frac{1}{1 - t} - \frac{t^{-ab} - 1}{(t^a - 1)(t^{-b} - 1)} \in \mathbb{Z}[t, t^{-1}].$$

Proof. First of all rewrite x in the following manner:

$$\begin{aligned} x &= \frac{1}{1 - t} - \frac{1 - t^{ab}}{t^{(a-1)b}(t^a - 1)(1 - t^b)} = \frac{1}{1 - t} - \frac{1 + t^b + t^{2b} + \cdots + t^{(a-1)b}}{t^{(a-1)b}(t^a - 1)} \\ &= \frac{p(t)}{t^{(a-1)b}(t^a - 1)} \end{aligned}$$

where

$$\begin{aligned} p(t) &:= t^{(a-1)b+1} + t^{(a-1)b+2} + \cdots + t^{(a-1)(b+1)} - \left(1 + t^b + t^{2b} + \cdots + t^{(a-1)b}\right) \\ &= t^{(a-1)b+1}\left(1 + t + t^2 + \cdots + t^{a-1}\right) - \left(1 + t^b + t^{2b} + \cdots + t^{(a-1)b}\right). \end{aligned}$$

To complete the proof of the lemma, it suffices to show that $p(t)$ is divisible by $t^a - 1$, as polynomials in $\mathbb{Z}[t]$. Consider $p(t)$ and $t^a - 1$ as polynomials in $\mathbb{C}[t]$. Clearly, $p(1) = 0$. For any ath root of unity $\omega \in \mathbb{C}$, $\omega \neq 1$, recall that $1 + \omega + \omega^2 + \cdots + \omega^{a-1} = 0$. Also, since a, b are relatively prime, ω^b is another ath root of unity with $\omega^b \neq 1$. It is then clear that $p(\omega) = 0$. Thus, $p(t)$ is divisible by all $t - \omega$, for all ath roots of unity (including $\omega = 1$), whence $(t^a - 1) | p(t)$ in $\mathbb{C}[t]$. But $t^a - 1$ is monic and $p(t) \in \mathbb{Z}[t]$, so the quotient must in fact be in $\mathbb{Z}[t]$. □

Now take x as given in Lemma 4.1 and $\mu_j^2 = \hat{\mu}_j^2 + x\rho_j^2$. Then $\partial_2\mu_j^2 = \partial_2\hat{\mu}_j^2 + x\partial_2\rho_j^2$ and a straightforward calculation gives the expression in $(R_{2,4})$.

Finally, consider the 3-cells σ_j^3, δ^3, which are somewhat analogous to the 2-cells μ_j^2 and δ^2. To verify that the formulae $(R_{3,1})$ and $(R_{3,2})$ are correct, one need only check (cf. Section 4.1), say for σ_j^3, that the geometric boundary of $\partial\boldsymbol{\sigma}_j^3$ is contained in $\boldsymbol{\rho}_j^2$ and $\boldsymbol{\mu}_j^2$, that $\partial_2\partial_3\sigma_j^3 = 0$, and that the coefficients in the formula for $\partial_3\sigma_j^3$ are ± 1. The first condition is clear from Section 4.2, the second is an easy calculation, and the third is obvious since $(R_{3,1})$ may be written $\partial_3\sigma_j^3 = 1 \cdot \rho_j^2 + 1 \cdot \mu_j^2 + (-1) \cdot t_j\mu_j^2$. The proof for $(R_{3,2})$ is similar.

5. Proof of the Diagonal Approximation Theorem

This section contains the details of the verification of the fact that $\partial\Delta = \Delta\partial$, for Δ given in the Diagonal Approximation Theorem 2.2. This will complete the proof.

There are several routine cases to be considered first. Before proceeding, recall that the boundary of the tensor product of two complexes is given by: $\partial(x \otimes y) = \partial x \otimes y + (-1)^p x \otimes \partial y$ if $p = \dim(x)$.

The diagonal on σ_j^0. Clearly $\partial(\Delta\sigma_j^0) = 0 = \Delta(\partial\sigma_j^0)$, and $(\varepsilon \otimes \varepsilon)(\Delta\sigma_j^0) = 1 = \varepsilon(\sigma_j^0)$.

The diagonal on σ_j^1. Observe that

$$\begin{aligned}
&\partial\big(\sigma_j^1 \otimes \sigma_j^0 + \sigma_0^0 \otimes \sigma_j^1\big) \\
&\quad = \sigma_j^0 \otimes \sigma_j^0 - \sigma_0^0 \otimes \sigma_j^0 + \sigma_0^0 \otimes \sigma_j^0 - \sigma_0^0 \otimes \sigma_0^0 \\
&\quad = \sigma_j^0 \otimes \sigma_j^0 - \sigma_0^0 \otimes \sigma_0^0 = \Delta\big(\sigma_j^0 - \sigma_0^0\big) = \Delta\big(\partial\sigma_j^1\big).
\end{aligned}$$

Thus, taking $\Delta\sigma_j^1 = \sigma_j^1 \otimes \sigma_j^0 + \sigma_0^0 \otimes \sigma_j^1$ (as in Theorem 2.2), we have $\partial(\Delta\sigma_j^1) = \Delta(\partial\sigma_j^1)$ as required. The same reasoning applies to ρ_j^1, η_j^1, σ_j^2 and ρ_j^2. The routine verifications of these cases is omitted.

The rest of the proof is divided into two parts. In (A) the verification is done for μ_j^2, σ_j^3, and in (B) for δ^2 and δ^3. The proofs given in parts (A) and (B) are simple in principle, as in the above cases they involve nothing more than showing that $\partial\Delta = \Delta\partial$, i.e., that Δ is a chain map. However these calculations are combinatorially complex.

Part A. μ_j^2 and σ_j^3

The diagonal on μ_j^2. Here a, b are relatively prime positive integers and $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$ (the subscript j is dropped uniformly throughout Part A. Using the formulae for $\Delta\rho^1$, $\Delta\eta^1$ it follows that

$$\begin{aligned}
\Delta(\partial\mu^2) &= F\Delta\rho^1 + G\Delta\eta^1 \\
&= F(t^{-b}\sigma^0 \otimes \rho^1 + \rho^1 \otimes \sigma^0) + G(t^a\sigma^0 \otimes \eta^1 + \eta^1 \otimes \sigma^0).
\end{aligned} \tag{5.1}$$

Thus

$$\begin{aligned}
\Delta(\partial\mu^2) = {} & \underbrace{\sum_{j=0}^{a-1} t^{j-b}\sigma^0 \otimes t^j\rho^1}_{(5)} + \underbrace{\sum_{i=0}^{a-1} t^i\rho^1 \otimes t^i\sigma^0}_{(3)} + \underbrace{\sum_{j=1-b}^{-1} t^{a+j}\sigma^0 \otimes t^j\eta^1}_{(2)} \\
& + \underbrace{t^{a-b}\sigma^0 \otimes t^{-b}\eta^1}_{(4)} + \underbrace{\sum_{i=1-b}^{-1} t^i\eta^1 \otimes t^i\sigma^0}_{(6)} + \underbrace{t^{-b}\eta^1 \otimes t^{-b}\sigma^0}_{(1)}.
\end{aligned} \tag{5.2}$$

It remains to show that $\partial(\Delta\mu^2)$ equals the expression in (5.2) above.

First recall (from Section 2) that $\tau^1 = P\rho^1 + t^{-bc}Q\eta^1$, and notice that $\partial\tau^1 = (t-1)\sigma^0$. Also observe that there is the following telescoping series (this identity is used frequently but not mentioned):

$$\partial\left(\sum_{k=u}^{v} t^k\tau^1\right) = (t^{v+1} - t^u)\sigma^0. \tag{5.3}$$

It follows from (5.3) that

$$\partial(f_j\tau^1) = (t^j - 1)\sigma^0, \qquad \partial(g_j\tau^1) = (1 - t^{-j})\sigma^0.$$

Also notice that $\partial\rho^1 = (t^{-b} - 1)\sigma^0$, $\partial\eta^1 = (t^a - 1)\sigma^0$. For clarity each term in (5.2) and in the following expression (5.4) is labeled, such that the corresponding terms in the two expressions receive the same label.

$$\begin{aligned}
&\partial(\Delta\mu^2)\\
&\quad= \underbrace{\left(\sum_{i=0}^{a-1} t^i\rho^1\right)\otimes t^{-b}\sigma^0}_{(c4)} + \underbrace{\left(\sum_{i=1-b}^{-1} t^i\eta^1\right)\otimes t^{-b}\sigma^0}_{(c1)} + \underbrace{t^{-b}\eta^1\otimes t^{-b}\sigma^0}_{(1)}\\
&\qquad + \underbrace{t^{a-b}\sigma^0\otimes\left(\sum_{i=0}^{a-1} t^i\rho^1\right)}_{(c3)} + \underbrace{t^{a-b}\sigma^0\otimes\left(\sum_{j=1-b}^{-1} t^i\eta^1\right)}_{(c2)} + \underbrace{t^{a-b}\sigma^0\otimes t^{-b}\eta^1}_{(4)}\\
&\qquad + \underbrace{\sum_{i=0}^{a-1}\sum_{j=-b}^{i-1}(t^i - t^{i-b})\sigma^0\otimes t^j\tau^1}_{(A)} + \underbrace{\sum_{i=0}^{a-1} t^i\rho^1\otimes t^i\sigma^0}_{(3)} - \underbrace{\sum_{i=0}^{a-1} t^i\rho^1\otimes t^{-b}\sigma^0}_{(c4)}\\
&\qquad - \sum_{j=0}^{a-1}\big(\underbrace{t^{a-b}\sigma^0\otimes t^j\rho^1}_{(c3)} - \underbrace{t^{j-b}\sigma^0\otimes t^j\rho^1}_{(5)}\big) + \underbrace{\sum_{j=0}^{a-1}\sum_{i=j-b}^{a-b-1} t^i\tau^1\otimes(t^{j-b} - t^j)\sigma^0}_{(B)}\\
&\qquad + \underbrace{\sum_{i=1-b}^{-1}\sum_{j=-b}^{i-1}(t^i - t^{a+i})\sigma^0\otimes t^j\tau^1}_{(A')} + \sum_{i=1-b}^{-1}\big(\underbrace{t^i\eta\otimes t^i\sigma^0}_{(6)} - \underbrace{t^i\eta\otimes t^{-b}\sigma^0}_{(c1)}\big)\\
&\qquad + \sum_{j=1-b}^{-1}\big(\underbrace{t^{a+j}\sigma^0\otimes t^j\eta^1}_{(2)} - \underbrace{t^{a-b}\sigma^0\otimes t^j\eta^1}_{(c2)}\big)\\
&\qquad + \underbrace{\sum_{j=1-b}^{-1}\sum_{i=a-b}^{a+j-1} t^i\tau^1\otimes(t^j - t^{a+j})\sigma^0}_{(B')}
\end{aligned}$$

$$+\underbrace{G\left(\sum_{j=1}^{a-1}(t^j - t^{j+1})\sigma^0 \otimes f_j\tau^1\right)}_{(X)} + \underbrace{G\left(\sum_{j=1}^{a-1} t^j\tau^1 \otimes (t^j - 1)\sigma^0\right)}_{(Y)}$$

$$+\underbrace{F\left(\sum_{j=1}^{b}(t^{-j} - t^{1-j})\sigma^0 \otimes g_j\tau^1\right)}_{(X')} + \underbrace{F\left(\sum_{j=1}^{b} t^{-j}\tau^1 \otimes (1 - t^{-j})\sigma^0\right)}_{(Y')}, \tag{5.4}$$

where two terms that are underscored with the same "c" cancel each other. By comparing terms in expressions (5.2) and (5.4) it follows that the proof can be completed by showing that

$$A + A' + X + X' = 0, \;\; B + B' + Y + Y' = 0. \tag{5.5}$$

For this it will suffice to establish the following identities.

$$A + A' = G(t^a\sigma^0 \otimes F\tau^1) - F(t^{-b}\sigma^0 \otimes G\tau^1) = -(X + X'), \tag{5.6}$$

$$B + B' = G(F\tau^1 \otimes \sigma^0) - F(G\tau^1 \otimes \sigma^0) = -(Y + Y'). \tag{5.7}$$

Since

$$X + X' = G\left(\sum_{j=1}^{a-1}(t^j - t^{j+1})\sigma^0 \otimes f_j\tau^1\right) + F\left(\sum_{j=1}^{b}(t^{-j} - t^{1-j})\sigma^0 \otimes g_j\tau^1\right),$$

where

$$\begin{aligned}
&\sum_{j=1}^{a-1}(t^j - t^{j+1})\sigma^0 \otimes f_j\tau^1 \\
&\quad = \sum_{j=1}^{a-1} t^j\sigma^0 \otimes \sum_{i=0}^{j-1} t^i\tau^1 - \sum_{j=2}^{a} t^j\sigma^0 \otimes \sum_{i=0}^{j-2} t^i\tau^1 \\
&\quad = t^1\sigma^0 \otimes \tau^1 + \sum_{j=2}^{a-1} t^j\sigma^0 \otimes \left(\sum_{i=0}^{j-1} t^i\tau^1 - \sum_{i=0}^{j-2} t^i\tau^1\right) - t^a\sigma^0 \otimes \sum_{i=0}^{a-2} t^i\tau^1 \\
&\quad = \sum_{j=1}^{a-1} t^j\sigma^0 \otimes t^{j-1}\tau^1 - t^a\sigma^0 \otimes \sum_{i=0}^{a-2} t^i\tau^1 \\
&\quad = \sum_{j=0}^{a-2} t^j(t\sigma^0 \otimes \tau^1) + t^{a-1}(t\sigma^0 \otimes \tau^1) - t^a\sigma^0 \otimes \sum_{i=0}^{a-1} t^i\tau^1 \\
&\quad = F(t\sigma^0 \otimes \tau^1) - t^a\sigma^0 \otimes F\tau^1
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=1}^{b}(t^{-j} - t^{1-j})\sigma^0 \otimes \sum_{k=-j}^{-1} t^k\tau^1 \\
&\quad = \sum_{j=1}^{b} t^{-j}\sigma^0 \otimes \sum_{k=-j}^{-1} t^k\tau^1 - \sum_{j=0}^{b-1} t^{-j}\sigma^0 \otimes \sum_{k=-(j+1)}^{-1} t^k\tau^1
\end{aligned}$$

$$= \sum_{j=1}^{b-1} t^{-j}\sigma^0 \otimes \left(\sum_{k=-j}^{-1} t^k\tau^1 - \sum_{k=-(j+1)}^{-1} t^k\tau^1 \right)$$

$$+ t^{-b}\sigma^0 \otimes \sum_{k=-b}^{-1} t^k\tau^1 - \sigma^0 \otimes t^{-1}\tau^1$$

$$= -\sum_{j=1}^{b-1} t^{-j}\sigma^0 \otimes t^{-j-1}\tau^1 - \sigma^0 \otimes t^{-1}\tau^1 + t^{-b}\sigma^0 \otimes G\tau^1$$

$$= -G(t\sigma \otimes \tau^1) + t^{-b}\sigma^0 \otimes G\tau^1,$$

it follows that

$$\begin{aligned} X + X' &= G\big[F(t\sigma^0 \otimes \tau^1) - t^a\sigma^0 \otimes F\tau^1\big] + F\big[-G(t\sigma^0 \otimes \tau^1) + t^{-b}\sigma^0 \otimes G\tau^1\big] \\ &= FG(t\sigma^0 \otimes \tau^1) - G(t^a\sigma^0 \otimes F\tau^1) - FG(t\sigma^0 \otimes \tau^1) + F(t^{-b}\sigma^0 \otimes G\tau^1) \\ &= -G(t^a\sigma^0 \otimes F\tau^1) + F(t^{-b}\sigma^0 \otimes G\tau^1), \end{aligned}$$

as required. A somewhat more straightforward calculation yields

$$\begin{aligned} Y + Y' &= G\left(\sum_{j=0}^{a-1} (t^j\tau^1 \otimes t^j\sigma^0 - t^j\tau^1 \otimes \sigma^0)\right) \\ &\quad + F\left(\sum_{j=1}^{b} (t^{-j}\tau^1 \otimes \sigma^0 - t^{-j}\tau^1 \otimes t^{-j}\sigma^0)\right) \\ &= G\big(F(\tau^1 \otimes \sigma^0) - F\tau^1 \otimes \sigma^0\big) + F\big(G\tau^1 \otimes \sigma^0 - G(\tau^1 \otimes \sigma^0)\big) \\ &= GF(\tau^1 \otimes \sigma^0) - G(F\tau^1 \otimes \sigma^0) - FG(\tau^1 \otimes \sigma^0) + F(G\tau^1 \otimes \sigma^0) \\ &= -G(F\tau^1 \otimes \sigma^0) + F(G\tau^1 \otimes \sigma^0), \end{aligned}$$

as required.

It remains to show that $A + A'$, $B + B'$ are as given in (5.6), (5.7). For $A + A'$, first note that expanding $G(t^a\sigma^0 \otimes F\tau^1) - F(t^{-b}\sigma^0 \otimes G\tau^1)$ gives

$$\sum_{k=0}^{a-1} \sum_{l=-b}^{-1} (t^{a+l}\sigma^0 \otimes t^{k+l}\tau^1 - t^{-b+k}\sigma^0 \otimes t^{k+l}\tau^1).$$

The fact that $A + A'$ equals this expression is immediate from the following identity where $u(i, j)$ can be any elements of an Abelian group, defined for $i, j \in \mathbb{Z}$.

$$\begin{aligned} &\sum_{i=0}^{a-1} \sum_{j=-b}^{i-1} \big[u(i, j) - u(i - b, j)\big] + \sum_{i=1-b}^{-1} \sum_{j=-b}^{i-1} \big[u(i, j) - u(a + i, j)\big] \\ &\quad = \sum_{k=0}^{a-1} \sum_{l=-b}^{-1} \big[u(a + l, k + l) - u(-b + k, k + l)\big], \end{aligned} \tag{5.8}$$

where of course for our purpose we substitute $u(i, j) = t^i\sigma^0 \otimes t^j\tau^1$. The proof of (5.7) for $B + B'$ is quite similar, and is based on the identity:

$$\sum_{j=0}^{a-1}\sum_{i=j-b}^{a-b-1}\bigl[u(i,j-b)-u(i,j)\bigr]+\sum_{j=1-b}^{-1}\sum_{i=a-b}^{a+j-1}\bigl[u(i,j)-u(i,a+j)\bigr]$$
$$=\sum_{k=0}^{a-1}\sum_{l=-b}^{-1}\bigl[u(k+l,l)-u(k+l,k)\bigr], \tag{5.9}$$

where in this case $u(i,j)=t^i\tau^1\otimes t^j\sigma^0$.

Identities (5.8) and (5.9) (as well as (5.10) below) can all be established by first converting all summands to $u(i,j)$, then making suitable cancellations in the double summations. The proof for (5.8) is now indicated, the others are quite similar and can safely be omitted. First, the left hand side of (5.8) is equal to

$$\left(\sum_{i=0}^{a-1}\sum_{j=-b}^{i-1}-\sum_{i=-b}^{a-b-1}\sum_{j=-b}^{i+b-1}+\sum_{i=1-b}^{-1}\sum_{j=-b}^{i-1}-\sum_{i=1+a-b}^{a-1}\sum_{j=-b}^{i-a-1}\right)u(i,j).$$

Combining the first and fourth sums, the second and third, and also omitting $u(i,j)$ henceforth, one obtains

$$\sum_{i=0}^{a-b}\sum_{j=-b}^{i-1}+\sum_{i=a+1-b}^{a-1}\left(\sum_{j=-b}^{i-1}-\sum_{j=-b}^{i-a-1}\right)$$
$$-\sum_{i=-b}^{-b}\sum_{j=-b}^{-1}-\sum_{i=1-b}^{-1}\left(\sum_{j=-b}^{i+b-1}-\sum_{j=-b}^{i-1}\right)-\sum_{i=0}^{a-b-1}\sum_{j=-b}^{i+b-1}$$
$$=\sum_{i=0}^{a-b}\sum_{j=-b}^{i-1}+\sum_{i=a+1-b}^{a-1}\sum_{j=i-a}^{i-1}-\sum_{i=-b}^{-b}\sum_{j=-b}^{-1}-\sum_{i=1-b}^{-1}\sum_{j=i}^{i+b-1}-\sum_{i=0}^{a-b-1}\sum_{j=-b}^{i+b-1}$$
$$=\sum_{i=0}^{a-b-1}\sum_{j=-b}^{i-1}+\sum_{i=a-b}^{a-1}\sum_{j=i-a}^{i-1}-\sum_{i=-b}^{-1}\sum_{j=i}^{i+b-1}-\sum_{i=0}^{a-b-1}\sum_{j=-b}^{i+b-1}$$
$$=\sum_{i=a-b}^{a-1}\sum_{j=i-a}^{i-1}-\sum_{i=-b}^{-1}\sum_{j=i}^{i+b-1}-\sum_{i=0}^{a-b-1}\sum_{j=i}^{i+b-1}=\sum_{i=a-b}^{a-1}\sum_{j=i-a}^{i-1}-\sum_{i=-b}^{a-b-1}\sum_{j=i}^{i+b-1}.$$

This last expression is precisely the right hand side of (5.8) when converted to the summand $u(i,j)$.

The diagonal on σ_j^3.

Lemma 5.1.
(a) $F\tau^1-\eta^1=\partial[(1+t^{-b}+\cdots+t^{-b(c-1)})\mu^2]=\partial(P\mu^2)$,
(b) $G\tau^1+\rho^1=\partial[t^{-bc}(1+t^a+\cdots+t^{a(d-1)})\mu^2]=\partial(t^{-bc}Q\mu^2)$.

Proof. Use the formula

$$PG+1=Ft^{-bc}Q$$

which follows from the fact that on the left-hand side there appear with coefficient 1 all powers of t between t^{-bc} and t^0, and that FQ consists of all powers from t^0 to t^{ad-1}. Therefore

$$\begin{aligned}\partial(P\mu^2) &= P(F\rho^1 + G\eta^1) = FP\rho^1 + (PG+1)\eta^1 - \eta^1\\ &= FP\rho^1 + Ft^{-bc}Q\eta^1 - \eta^1 = F\tau^1 - \eta^1\end{aligned}$$

and

$$\begin{aligned}\partial(t^{-bc}Q\mu^2) &= t^{-bc}Q\big(F\rho^1 + G\eta^1\big) = t^{-bc}QF\eta^1 + t^{-bc}QF\rho^1\\ &= t^{-bc}QG\eta^1 + (PG+1)\rho^1\\ &= G\big(P\rho^1 + t^{-bc}Q\eta^1\big) + \rho^1 = G\tau^1 + \rho^1. \qquad \square\end{aligned}$$

The rest of the proof is divided into two steps. The first part, given in the next lemma, is a simplification of the expression for $\Delta(\partial\sigma^3)$. Once this is carried out, the formula for $\Delta\sigma^3$ is quite easy to verify. It will first be useful to have another identity for double summations over Abelian groups similar to (5.8) and (5.9) (and given without proof).

$$\begin{aligned}&\sum_{i=1}^{a-1}\sum_{j=0}^{i-1}\big[u(i,j) - u(i-b, j-b)\big] + \sum_{i=1}^{b}\sum_{j=1}^{i}\big[u(a-i, a-j) - u(-i,-j)\big]\\ &\quad= \sum_{i=1}^{b}\sum_{j=0}^{a-1}\big[u(a-i, j) - u(j-b, -i)\big]. \qquad (5.10)\end{aligned}$$

Lemma 5.2.

$$\begin{aligned}\Delta(\partial\sigma^3) = {} & \underbrace{\rho^2\otimes\sigma^0}_{(1)} + \underbrace{t^{-b}\eta^1\otimes\rho^1}_{(2)} - \underbrace{t^a\rho^1\otimes\eta^1}_{(3)} + \underbrace{t^{a-b}\sigma^0\otimes\rho^2}_{(4)} + \underbrace{\mu^2\otimes t^{-b}\sigma^0}_{(5)}\\ & + \underbrace{t^{a-b}\sigma^0\otimes\mu^2}_{(6)} - \underbrace{t\mu^2\otimes t^{1-b}\sigma^0}_{(7)} - \underbrace{t^{1+a-b}\sigma^0\otimes t\mu^2}_{(8)} - \underbrace{\rho^1\otimes G\tau^1}_{(9)}\\ & + \underbrace{t^a\rho^1\otimes F\tau^1}_{(10)} + \underbrace{t^a\rho^1\otimes G\tau^1}_{(11)} - \underbrace{t^{-b}F\tau^1\otimes\rho^1}_{(12)} + \underbrace{\eta^1\otimes G\tau^1}_{(13)}\\ & - \underbrace{t^aG\tau^1\otimes\eta^1}_{(14)} + \underbrace{t^aG\tau^1\otimes F\tau^1}_{(15)} - \underbrace{t^{-b}F\tau^1\otimes G\tau^1}_{(16)}\\ & - \underbrace{t(F\rho^1 + G\eta^1)\otimes t^{-b}\tau^1}_{(17)} + \underbrace{t^{a-b}\tau^1\otimes t(F\rho^1 + G\eta^1)}_{(18)}.\end{aligned}$$

Proof. From the calculation of Δ on ρ^2 and μ^2, it follows that

$$\begin{aligned}\Delta\partial\sigma^3 &= \Delta\rho^2 + \Delta\mu^2 - t\Delta\mu^2\\ &= \rho^2\otimes\sigma^0 + s\eta^1\otimes\rho^1 - h\rho^1\otimes\eta^1 + hs\sigma^0\otimes\rho^2 + \mu^2\otimes t^{-b}\sigma^0\\ &\quad + t^{a-b}\sigma^0\otimes\mu^2 - t\mu^2\otimes t^{-b+1}\sigma^0 - t^{a-b+1}\sigma^0\otimes t\mu^2\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=0}^{a-1}\sum_{j=-b}^{i-1}\bigl(-t^i\rho^1\otimes t^j\tau^1+t^{i+1}\rho^1\otimes t^{j+1}\tau^1\bigr)\\
&+\sum_{j=0}^{a-1}\sum_{i=j-b}^{a-b-1}\bigl(-t^i\tau^1\otimes t^j\rho^1+t^{i+1}\tau^1\otimes t^{j+1}\rho^1\bigr)\\
&+\sum_{i=1-b}^{-1}\sum_{j=-b}^{i-1}\bigl(-t^i\eta^1\otimes t^j\tau^1+t^{i+1}\eta^1\otimes t^{j+1}\tau^1\bigr)\\
&+\sum_{j=1-b}^{-1}\sum_{i=a-b}^{a-1+j}\bigl(t^i\tau^1\otimes t^j\eta^1-t^{i+1}\tau^1\otimes t^{j+1}\eta^1\bigr)\\
&-G\sum_{i=1}^{a-1}\bigl(t^i\tau^1\otimes f_i\tau^1-t^{i+1}\tau^1\otimes tf_i\tau^1\bigr)\\
&-F\sum_{i=1}^{b}\bigl(t^{-i}\tau^1\otimes g_i\tau^1-t^{-i+1}\tau^1\otimes tg_i\tau^1\bigr).
\end{aligned}
$$

To simplify this observe that the four double sums "telescope" and reduce to nine single sums, while the second to last sum simplifies by writing $t^i\tau^1\otimes f_i\tau^1-t^{i+1}\tau^1\otimes tf_i\tau^1=(1-t)(t^i\tau^1\otimes f_i\tau^1)$ and using the fact that $-G(1-t)=1-t^{-b}$. A similar remark holds for the last sum, since $t^a-1=F(t-1)$. Thus

$$
\begin{aligned}
\Delta(\partial\sigma^3)&=\rho^2\otimes\sigma^0+t^{-b}\eta^1\otimes\rho^1-t^a\rho^1\otimes\eta^1\\
&\quad+t^{a-b}\sigma^0\otimes\rho^2+\mu^2\otimes t^{-b}\sigma^0+t^{a-b}\sigma^0\otimes\mu^2\\
&\quad-t\mu^2\otimes t^{-b+1}\sigma^0-t^{a-b+1}\sigma^0\otimes t\mu^2-\sum_{j=-b}^{-1}\rho^1\otimes t^j\tau^1\\
&\quad-\sum_{i=1}^{a-1}t^i\rho^1\otimes t^{-b}\tau^1+\sum_{j=-b+1}^{a-1}t^a\rho^1\otimes t^j\tau^1-\sum_{i=-b}^{a-b-1}t^i\tau^1\otimes\rho^1\\
&\quad+\sum_{j=1}^{a}t^{a-b}\tau^1\otimes t^j\rho^1-\sum_{i=1-b}^{-1}t^i\eta^1\otimes t^{-b}\tau^1+\sum_{j=1-b}^{-1}\eta^1\otimes t^j\tau^1\\
&\quad+\sum_{j=1-b}^{-1}t^{a-b}\tau^1\otimes t^j\eta^1-\sum_{i=a-b+1}^{a-1}t^i\tau^1\otimes\eta^1\\
&\quad+(1-t^{-b})\sum_{i=1}^{a-1}t^i\tau^1\otimes f_i\tau^1+(t^a-1)\sum_{i=1}^{b}t^{-i}\tau^1\otimes g_i\tau^1\\
&=\rho^2\otimes\sigma^0+t^{-b}\eta^1\otimes\rho^1-t^a\rho^1\otimes\eta^1\\
&\quad+t^{a-b}\sigma^0\otimes\rho^2+\mu^2\otimes t^{-b}\sigma^0+t^{a-b}\sigma^0\otimes\mu^2\\
&\quad-t\mu^2\otimes t^{-b+1}\sigma^0-t^{a-b+1}\sigma^0\otimes t\mu^2-\rho^1\otimes G\tau^1-(F-1)\rho^1\otimes t^{-b}\tau^1
\end{aligned}
$$

$$+\,t^a\rho^1\otimes(F+G-t^{-b})\tau^1 - t^{-b}F\tau^1\otimes\rho^1 + t^{a-b}\tau^1\otimes tF\rho^1$$
$$-\,(G-t^{-b})\eta^1\otimes t^{-b}\tau^1 + \eta^1\otimes(G-t^{-b})\tau^1 + t^{a-b}\tau^1\otimes(G-t^{-b})\eta^1$$
$$-\,t^a(G-t^{-b})\tau^1\otimes\eta^1 + t^aG\tau^1\otimes F\tau^1 - t^{-b}F\tau^1\otimes G\tau^1,$$

where the final two terms of the second last expression simplify to the final two terms of the last expression by substituting $f_i=\sum_{j=0}^{i-1}t^j$, $g_i=\sum_{j=1}^{i}t^{-j}$, and then using (5.10). The last expression now quickly simplifies to that given in the lemma by grouping

(1) $-(F-1)\rho^1\otimes t^{-b}\tau^1 + t^a\rho^1\otimes(-t^{-b}\tau^1) = -tF\rho^1\otimes t^{-b}\tau^1$,
(2) $-(G-t^{-b})\eta^1\otimes t^{-b}\tau^1 + \eta^1\otimes(-t^{-b}\tau^1) = -tG\eta^1\otimes t^{-b}\tau^1$, and
(3) $t^{a-b}\tau^1\otimes(G-t^{-b})\eta^1 + t^{a-b}\tau^1\otimes\eta^1 = t^{a-b}\tau^1\otimes tG\eta^1$. □

It is now easy to prove the formula for $\Delta\sigma^3$. Indeed, it follows from Lemma 5.1 that

$$\begin{aligned}
\partial(\Delta\sigma^3) = {} & \underbrace{\rho^2\otimes\sigma^0}_{(1)} + \mu^2\otimes\sigma^0 - t\mu^2\otimes\sigma^0 \\
& + \underbrace{t^{a-b}\sigma^0\otimes\rho^2}_{(4)} + \underbrace{t^{a-b}\sigma^0\otimes\mu^2}_{(6)} - t^{a-b}\sigma^0\otimes t\mu^2 \\
& - \underbrace{t(F\rho^1+G\eta^1)\otimes t^{-b}\tau^1}_{(17)} - \underbrace{t\mu^2\otimes t^{1-b}\sigma^0}_{(7)} + t\mu^2\otimes t^{-b}\sigma^0 \\
& - \underbrace{t^{1+a-b}\sigma^0\otimes t\mu^2}_{(8)} + t^{a-b}\sigma^0\otimes t\mu^2 \\
& + \underbrace{t^{a-b}\tau^1\otimes t(F\rho^1+G\eta^1)}_{(18)} + \underbrace{t^a\rho^1\otimes G\tau^1}_{(11)} - \underbrace{\rho^1\otimes G\tau^1}_{(9)} \\
& + \underbrace{\eta^1\otimes G\tau^1}_{(13)} - t^{-b}\eta^1\otimes G\tau^1 + t\mu^2\otimes(\sigma^0 - t^{-b}\sigma^0) \\
& - \mu^2\otimes\sigma^0 + \underbrace{\mu^2\otimes t^{-b}\sigma^0}_{(5)} - \underbrace{t^{-b}F\tau^1\otimes\rho^1}_{(12)} \\
& - \underbrace{t^{-b}F\tau^1\otimes G\tau^1}_{(16)} + \underbrace{t^{-b}\eta^1\otimes\rho^1}_{(2)} + t^{-b}\eta^1\otimes G\tau^1 \\
& + \underbrace{t^a\rho^1\otimes F\tau^1}_{(10)} + \underbrace{t^aG\tau^1\otimes F\tau^1}_{(15)} - \underbrace{t^a\rho^1\otimes\eta^1}_{(3)} - \underbrace{t^aG\tau^1\otimes\eta^1}_{(14)}
\end{aligned}$$

which matches exactly with the result in Lemma 5.2 apart from the extra terms which cancel in (five) pairs.

Part B. δ^2 and δ^3

Throughout this section recall the following notation introduced in Section 2; $\sigma_0^1=\sigma_0^2=0$, $r_0=s_0$, $r_{-1}=1$.

Lemma 5.3. *Let $u(j,i)$ be any elements of an Abelian group, defined for $i, j \in \mathbb{Z}$. Then*

$$\sum_{j=2}^{m-1}\sum_{i=1}^{j-1}\big[u(j,i)-u(j-1,i)\big]=\sum_{j=1}^{m-1}\big[u(m-1,j)-u(j,j)\big].$$

Proof.

$$\begin{aligned}
&\sum_{j=2}^{m-1}\sum_{i=1}^{j-1}\big[u(j,i)-u(j-1,i)\big]\\
&\quad=\sum_{j=1}^{m-1}\sum_{i=1}^{j-1}\big[u(j,i)-u(j-1,i)\big]\\
&\quad=\sum_{j=1}^{m-1}\sum_{i=1}^{j-1}u(j,i)-\sum_{j=1}^{m-2}\sum_{i=1}^{j}u(j,i)\\
&\quad=\sum_{i=1}^{m-2}u(m-1,i)+\sum_{j=1}^{m-2}\left(\sum_{i=1}^{j-1}u(j,i)-\sum_{i=1}^{j}u(j,i)\right)\\
&\quad=\sum_{j=1}^{m-2}\big[u(m-1,j)-u(j,j)\big]=\sum_{j=1}^{m-1}\big[u(m-1,j)-u(j,j)\big]. \qquad \square
\end{aligned}$$

The next corollary follows from this lemma and Remark 2.1. It will be used to compute $\partial(\Delta\delta^2)$ and $\partial(\Delta\delta^3)$.

Corollary 5.4.

(a) $$\partial\left(\sum_{j=2}^{m-1}\sum_{i=1}^{j-1}\pi_j^1\otimes\pi_i^1\right)$$
$$=\sum_{j=1}^{m-1}r_{m-1}\sigma_0^0\otimes\pi_j^1-\sum_{j=1}^{m-1}r_j\sigma_0^0\otimes\pi_j^1+\sum_{j=1}^{m-1}\pi_j^1\otimes s_0\sigma_0^0-\sum_{j=1}^{m-1}\pi_j^1\otimes r_{j-1}\sigma_0^0,$$

(b) $$\partial\left(\sum_{j=2}^{m-1}\sum_{i=1}^{j-1}\pi_j^2\otimes\pi_i^1\right)$$
$$\begin{aligned}
&=-\sum_{j=2}^{m-1}\sum_{i=1}^{j-1}h\pi_j^1\otimes\pi_i^1+\sum_{j=2}^{m-1}\sum_{i=1}^{j-1}\pi_j^1\otimes\pi_i^1+\sum_{j=1}^{m-1}r_{m-1}\eta_0^1\otimes\pi_j^1\\
&\quad-\sum_{j=1}^{m-1}r_j\eta_0^1\otimes\pi_j^1-\sum_{j=1}^{m-1}\pi_j^2\otimes s_0\sigma_0^0+\sum_{j=1}^{m-1}\pi_j^2\otimes r_{j-1}\sigma_0^0,
\end{aligned}$$

(c) $\partial\left(\sum_{j=2}^{m-1}\sum_{i=1}^{j-1} h\pi_j^1 \otimes \pi_i^2\right)$

$$= \sum_{j=2}^{m-1}\sum_{i=1}^{j-1} h\pi_j^1 \otimes h\pi_i^1 - \sum_{j=2}^{m-1}\sum_{i=1}^{j-1} h\pi_j^1 \otimes \pi_i^1 + \sum_{j=1}^{m-1} r_{m-1}h\sigma_0^0 \otimes \pi_j^2 - \sum_{j=1}^{m-1} r_j h\sigma_0^0 \otimes \pi_j^2 - \sum_{j=1}^{m-1} h\pi_j^1 \otimes r_{j-1}\eta_0^1 + \sum_{j=1}^{m-1} h\pi_j^1 \otimes s_0\eta_0^1.$$

Finally, take note of the following two formulae which will be used to simplify the computations of both $\Delta(\partial\delta^2)$ and $\Delta(\partial\delta^3)$. These formulae follow directly from the definitions of π_j^1, π_j^2 in Section 2 and the formulae for $\Delta(\sigma_j^t)$, $\Delta(\rho_j^t)$, $t = 1, 2$, given in Theorem 2.2.

$$\Delta\pi_j^1 = \pi_j^1 \otimes r_{j-1}\sigma_j^0 + r_{j-1}\sigma_0^0 \otimes r_{j-1}\sigma_j^1 + r_j\sigma_j^1 \otimes (r_{j-1} - r_j)\sigma_j^0 - r_j\sigma_0^0 \otimes r_j\sigma_j^1 + r_j\sigma_j^0 \otimes r_{j-1}\rho_j^1,$$

$$\begin{aligned}\Delta\pi_j^2 = {} & \pi_j^2 \otimes r_{j-1}\sigma_j^0 + h\pi_j^1 \otimes r_{j-1}\eta_j^1 - r_j\eta_j^1 \otimes r_{j-1}\rho_j^1 - r_j h\sigma_j^0 \otimes r_{j-1}\rho_j^2 \\ & - r_{j-1}h\sigma_0^0 \otimes r_{j-1}\sigma_j^2 - r_j\sigma_j^2 \otimes r_{j-1}\sigma_j^0 \\ & - r_{j-1}\eta_0^1 \otimes r_{j-1}\sigma_j^1 + r_j h\sigma_0^0 \otimes r_j\sigma_j^2 \\ & + r_j\sigma_j^2 \otimes r_j\sigma_j^0 + r_j\eta_0^1 \otimes r_j\sigma_j^1 + r_j h\sigma_j^1 \otimes r_{j-1}\eta_j^1 - r_j h\sigma_j^1 \otimes r_j\eta_j^1. \end{aligned} \tag{5.11}$$

The diagonal on δ^2. Since $\partial\delta^2 = \sum_{j=0}^m \pi_j^1$, it follows from (5.11) that

$$\begin{aligned}\Delta(\partial\delta^2) = {} & \underbrace{\sum_{j=0}^m \pi_j^1 \otimes r_{j-1}\sigma_j^0}_{(A)} + \underbrace{\sum_{j=0}^m r_{j-1}\sigma_0^0 \otimes r_{j-1}\sigma_j^1}_{(B)} + \underbrace{\sum_{j=0}^m r_j\sigma_j^1 \otimes (r_{j-1} - r_j)\sigma_j^0}_{(C)} \\ & - \underbrace{\sum_{j=0}^m r_j\sigma_0^0 \otimes r_j\sigma_j^1}_{(D)} + \underbrace{\sum_{j=0}^m r_j\sigma_j^0 \otimes r_{j-1}\rho_j^1}_{(E)}. \end{aligned} \tag{5.12}$$

The expression for $\Delta\delta^2$ in Theorem 2.2 and Corollary 5.4(a) now give

$$\begin{aligned}\partial(\Delta\delta^2) = {} & \underbrace{\sum_{j=0}^m \pi_j^1 \otimes s_0\sigma_0^0}_{(c1)} + \underbrace{r_{m-1}\sigma_0^0 \otimes \sum_{j=0}^m \pi_j^1}_{(c2)} \\ & + \underbrace{\sigma_0^0 \otimes \pi_m^1}_{(s3)} - \underbrace{r_{m-1}\sigma_0^0 \otimes \pi_m^1}_{(c4)} - \underbrace{\pi_m^1 \otimes \sigma_0^0}_{(c5)} \\ & + \underbrace{\pi_m^1 \otimes r_{m-1}\sigma_0^0}_{(c6)} + \underbrace{s_0\sigma_0^0 \otimes \rho_0^1}_{(s7)} - \underbrace{\sigma_0^0 \otimes \rho_0^1}_{(c8)} \end{aligned}$$

$$
\begin{aligned}
&-\underbrace{\rho_0^1\otimes s_0\sigma_0^0}_{(c9)}+\underbrace{\rho_0^1\otimes\sigma_0^0}_{(s10)}+\underbrace{\sigma_0^0\otimes\rho_0^1}_{(c8)}-\underbrace{r_{m-1}\sigma_0^0\otimes\rho_0^1}_{(c11)}\\
&-\underbrace{\pi_m^1\otimes s_0\sigma_0^0}_{(c12)}+\underbrace{\pi_m^1\otimes\sigma_0^0}_{(c5)}\\
&-\underbrace{\sum_{j=1}^{m-1} r_{m-1}\sigma_0^0\otimes\pi_j^1}_{(c13)}+\underbrace{\sum_{j=1}^{m-1} r_j\sigma_0^0\otimes\pi_j^1}_{(s14)}-\underbrace{\sum_{j=1}^{m-1}\pi_j^1\otimes s_0\sigma_0^0}_{(c15)}\\
&+\underbrace{\sum_{j=1}^{m-1}\pi_j^1\otimes r_{j-1}\sigma_0^0}_{(c16)}+\underbrace{\sum_{j=1}^{m} r_j\sigma_j^0\otimes r_{j-1}\rho_j^1}_{(s17)}\\
&-\underbrace{\sum_{j=1}^{m} r_j\sigma_0^0\otimes r_{j-1}\rho_j^1}_{(s18)}-\underbrace{\sum_{j=1}^{m} r_j\sigma_j^1\otimes r_j\sigma_j^0}_{(s19)}\\
&+\underbrace{\sum_{j=1}^{m} r_j\sigma_j^1\otimes r_{j-1}\sigma_j^0}_{(s20)}-\underbrace{\sum_{j=1}^{m} r_j\sigma_0^0\otimes r_{j-1}\sigma_j^1}_{(s21)}\\
&+\underbrace{\sum_{j=1}^{m} r_{j-1}\sigma_0^0\otimes r_{j-1}\sigma_j^1}_{(s22)}+\underbrace{\sum_{j=1}^{m}\pi_j^1\otimes r_{j-1}\sigma_j^0}_{(s23)}-\underbrace{\sum_{j=1}^{m}\pi_j^1\otimes r_{j-1}\sigma_0^0}_{(c24)}.
\end{aligned}
\tag{5.13}
$$

As a first step in the verification that (5.12) equals (5.13), observe that the terms in the expression (5.13) labelled c5 and c8 cancel in two pairs. Furthermore, a direct calculation shows that the following expressions from (5.13), displayed in (5.14), equal 0.

$$
\begin{aligned}
&\mathrm{c1}+\mathrm{c9}+\mathrm{c15}+\mathrm{c12}=0,\\
&\mathrm{c2}+\mathrm{c4}+\mathrm{c11}+\mathrm{c13}=0,\quad \mathrm{c6}+\mathrm{c16}+\mathrm{c24}=0.
\end{aligned}
\tag{5.14}
$$

Finally, observe that the sum of the remaining terms of the expression given in (5.13) is equal to the expression for $\Delta(\partial\delta^2)$, given in (5.12), as indicated below in display (5.15).

$$
\begin{aligned}
&A=\mathrm{s10}+\mathrm{s23}, \qquad B=\mathrm{s22}, \qquad C=\mathrm{s20}+\mathrm{s19},\\
&D=\mathrm{s3}+\mathrm{s14}+\mathrm{s18}+\mathrm{s21}, \quad E=\mathrm{s7}+\mathrm{s17}.
\end{aligned}
\tag{5.15}
$$

The diagonal on δ^3. Since

$$
\partial\delta^3=(1-h)\delta^2-\sum_{j=0}^{m}\pi_j^2
$$

it follows from the expression for $\Delta\pi_j^2$ in (5.11) that

$$
\begin{aligned}
\Delta(\partial\delta^3) = {} & \underbrace{\delta^2 \otimes s_0\sigma_0^0}_{(A)} + \underbrace{r_{m-1}\sigma_0^0 \otimes \delta^2}_{(B)} + \underbrace{\pi_m^1 \otimes \pi_m^1}_{(C)} + \underbrace{\rho_0^1 \otimes \rho_0^1}_{(D)} + \underbrace{\pi_m^1 \otimes \rho_0^1}_{(E)} \\
& - \underbrace{h\delta^2 \otimes s_0 h\sigma_0^0}_{(F)} - \underbrace{r_{m-1}h\sigma_0^0 \otimes h\delta^2}_{(G)} \\
& - \underbrace{h\pi_m^1 \otimes h\pi_m^1}_{(H)} - \underbrace{h\rho_0^1 \otimes h\rho_0^1}_{(I)} - \underbrace{h\pi_m^1 \otimes h\rho_0^1}_{(J)} \\
& - \underbrace{\sum_{j=2}^{m-1}\sum_{i=1}^{j-1} \pi_j^1 \otimes \pi_i^1}_{(K)} + \underbrace{\sum_{j=1}^{m} r_i\sigma_j^1 \otimes r_{j-1}\rho_j^1}_{(L)} \\
& - \underbrace{\sum_{j=1}^{m} \pi_j^1 \otimes r_{j-1}\sigma_j^1}_{(M)} + \underbrace{\sum_{j=2}^{m-1}\sum_{i=1}^{j-1} h\pi_j^1 \otimes h\pi_i^1}_{(N)} \\
& - \underbrace{\sum_{j=1}^{m} r_j h\sigma_j^1 \otimes r_{j-1}h\rho_j^1}_{(O)} + \underbrace{\sum_{j=1}^{m} h\pi_j^1 \otimes r_{j-1}h\sigma_j^1}_{(P)} \\
& - \underbrace{\sum_{j=0}^{m} \pi_j^2 \otimes r_{j-1}\sigma_j^0}_{(Q)} - \underbrace{\sum_{j=0}^{m} h\pi_j^1 \otimes r_{j-1}\eta_j^1}_{(R)} \\
& + \underbrace{\sum_{j=0}^{m} r_j\eta_j^1 \otimes r_{j-1}\rho_j^1}_{(S)} + \underbrace{\sum_{j=0}^{m} r_j h\sigma_j^0 \otimes r_{j-1}\rho_j^2}_{(T)} \\
& + \underbrace{\sum_{j=0}^{m} r_{j-1}h\sigma_0^0 \otimes r_{j-1}\sigma_j^2}_{(U)} + \underbrace{\sum_{j=0}^{m} r_j\sigma_j^2 \otimes r_{j-1}\sigma_j^0}_{(V)} \\
& + \underbrace{\sum_{j=0}^{m} r_{j-1}\eta_0^1 \otimes r_{j-1}\sigma_j^1}_{(W)} - \underbrace{\sum_{j=0}^{m} r_j h\sigma_0^0 \otimes r_j\sigma_j^2}_{(X)} \\
& - \underbrace{\sum_{j=0}^{m} r_j\sigma_j^2 \otimes r_j\sigma_j^0}_{(Y)} - \underbrace{\sum_{j=0}^{m} r_j\eta_0^1 \otimes r_j\sigma_j^1}_{(Z)} \\
& - \underbrace{\sum_{j=0}^{m} r_j h\sigma_j^1 \otimes r_{j-1}\eta_j^1}_{(A')} + \underbrace{\sum_{j=0}^{m} r_j h\sigma_j^1 \otimes r_j\eta_j^1}_{(B')}. \qquad (5.16)
\end{aligned}
$$

The expression for $\Delta\delta^3$ in Theorem 2.2 and Corollary 5.4(b), (c) now give

$$\begin{aligned}
\partial(\Delta\delta^3) = {} & \underbrace{\delta^2 \otimes s_0\sigma_0^0}_{(s1)} - \underbrace{h\delta^2 \otimes s_0\sigma_0^0}_{(c2)} \\
& - \underbrace{\sum_{j=0}^{m} \pi_j^2 \otimes s_0\sigma_0^0}_{(c3)} + \underbrace{r_{m-1}h\sigma_0^0 \otimes \delta^2}_{(c4)} - \underbrace{r_{m-1}h\sigma_0^0 \otimes h\delta^2}_{(s5)} \\
& - \underbrace{r_{m-1}h\sigma_0^0 \otimes \sum_{j=0}^{m} \pi_j^2}_{(c6)} - \underbrace{\sum_{j=0}^{m} h\pi_j^1 \otimes s_0\eta_0^1}_{(c7)} \\
& - \underbrace{h\delta^2 \otimes s_0h\sigma_0^0}_{(s8)} + \underbrace{h\delta^2 \otimes s_0\sigma_0^0}_{(c2)} - \underbrace{r_{m-1}h\sigma_0^0 \otimes \delta^2}_{(c4)} \\
& + \underbrace{r_{m-1}\sigma_0^0 \otimes \delta^2}_{(s9)} + \underbrace{r_{m-1}\eta_0^1 \otimes \sum_{j=0}^{m} \pi_j^1}_{(c10)} \\
& - \underbrace{\eta_0^1 \otimes \rho_0^1}_{(c11)} + \underbrace{s_0\eta_0^1 \otimes \rho_0^1}_{(s12)} - \underbrace{h\rho_0^1 \otimes \rho_0^1}_{(c13)} + \underbrace{\rho_0^1 \otimes \rho_0^1}_{(s14)} \\
& - \underbrace{\rho_0^2 \otimes s_0\sigma_0^0}_{(c15)} + \underbrace{\rho_0^2 \otimes \sigma_0^0}_{(s16)} + \underbrace{\eta_0^1 \otimes \pi_m^1}_{(s17)} - \underbrace{r_{m-1}\eta_0^1 \otimes \pi_m^1}_{(c18)} \\
& - \underbrace{h\pi_m^1 \otimes \pi_m^1}_{(c19)} + \underbrace{\pi_m^1 \otimes \pi_m^1}_{(s20)} + \underbrace{\pi_m^2 \otimes \sigma_0^0}_{(c21)} \\
& - \underbrace{\pi_m^2 \otimes r_{m-1}\sigma_0^0}_{(c22)} - \underbrace{h\sigma_0^0 \otimes \pi_m^2}_{(s23)} + \underbrace{hr_{m-1}\sigma_0^0 \otimes \pi_m^2}_{(c24)} \\
& + \underbrace{h\pi_m^1 \otimes \eta_0^1}_{(c25)} - \underbrace{h\pi_m^1 \otimes r_{m-1}\eta_0^1}_{(c26)} - \underbrace{h\pi_m^1 \otimes h\pi_m^1}_{(s27)} \\
& + \underbrace{h\pi_m^1 \otimes \pi_m^1}_{(c19)} + \underbrace{\eta_0^1 \otimes \rho_0^1}_{(c11)} - \underbrace{r_{m-1}\eta_0^1 \otimes \rho_0^1}_{(c28)} - \underbrace{h\pi_m^1 \otimes \rho_0^1}_{(c29)} \\
& + \underbrace{\pi_m^1 \otimes \rho_0^1}_{(s30)} + \underbrace{\pi_m^2 \otimes s_0\sigma_0^0}_{(c31)} - \underbrace{\pi_m^2 \otimes \sigma_0^0}_{(c21)} \\
& + \underbrace{h\sigma_0^0 \otimes \rho_0^2}_{(c32)} - \underbrace{hr_{m-1}\sigma_0^0 \otimes \rho_0^2}_{(c33)} - \underbrace{h\pi_m^1 \otimes \eta_0^1}_{(c25)} \\
& + \underbrace{h\pi_m^1 \otimes s_0\eta_0^1}_{(c34)} - \underbrace{h\pi_m^1 \otimes h\rho_0^1}_{(s35)} + \underbrace{h\pi_m^1 \otimes \rho_0^1}_{(c29)}
\end{aligned}$$

$$+\underbrace{s_0h\sigma_0^0\otimes\rho_0^2}_{(s36)}-\underbrace{h\sigma_0^0\otimes\rho_0^2}_{(c32)}-\underbrace{h\rho_0^1\otimes\eta_0^1}_{(s37)}+\underbrace{h\rho_0^1\otimes s_0\eta_0^1}_{(c38)}$$

$$-\underbrace{h\rho_0^1\otimes h\rho_0^1}_{(s39)}+\underbrace{h\rho_0^1\otimes\rho_0^1}_{(c13)}+\underbrace{\sum_{j=2}^{m-1}\sum_{i=1}^{j-1}h\pi_j^1\otimes\pi_i^1}_{(c40)}$$

$$-\underbrace{\sum_{j=2}^{m-1}\sum_{i=1}^{j-1}\pi_j^1\otimes\pi_i^1}_{(s41)}-\underbrace{\sum_{j=1}^{m-1}r_{m-1}\eta_0^1\otimes\pi_j^1}_{(c42)}+\underbrace{\sum_{j=1}^{m-1}r_j\eta_0^1\otimes\pi_j^1}_{(s43)}$$

$$+\underbrace{\sum_{j=1}^{m-1}\pi_j^2\otimes s_0\sigma_0^0}_{(c44)}-\underbrace{\sum_{j=1}^{m-1}\pi_j^2\otimes r_{j-1}\sigma_0^0}_{(c45)}$$

$$+\underbrace{\sum_{j=2}^{m-1}\sum_{i=1}^{j-1}h\pi_j^1\otimes h\pi_i^1}_{(s46)}-\underbrace{\sum_{j=2}^{m-1}\sum_{i=1}^{j-1}h\pi_j^1\otimes\pi_i^1}_{(c40)}+\underbrace{\sum_{j=1}^{m-1}r_{m-1}h\sigma_0^0\otimes\pi_j^2}_{(c47)}$$

$$-\underbrace{\sum_{j=1}^{m-1}r_jh\sigma_0^0\otimes\pi_j^2}_{(s48)}-\underbrace{\sum_{j=1}^{m-1}h\pi_j^1\otimes r_{j-1}\eta_0^1}_{(c49)}$$

$$+\underbrace{\sum_{j=1}^{m-1}h\pi_j^1\otimes s_0\eta_0^1}_{(c50)}-\underbrace{\sum_{j=1}^{m}r_j\eta_0^1\otimes r_{j-1}\sigma_j^1}_{(s51)}+\underbrace{\sum_{j=1}^{m}r_{j-1}\eta_0^1\otimes r_{j-1}\sigma_j^1}_{(s52)}$$

$$+\underbrace{\sum_{j=1}^{m}h\pi_j^1\otimes r_{j-1}\sigma_j^1}_{(c53)}-\underbrace{\sum_{j=1}^{m}\pi_j^1\otimes r_{j-1}\sigma_j^1}_{(s54)}$$

$$-\underbrace{\sum_{j=1}^{m}\pi_j^2\otimes r_{j-1}\sigma_j^0}_{(s55)}+\underbrace{\sum_{j=1}^{m}\pi_j^2\otimes r_{j-1}\sigma_0^0}_{(c56)}-\underbrace{\sum_{j=1}^{m}r_jh\sigma_0^0\otimes r_{j-1}\sigma_j^2}_{(s57)}$$

$$+\underbrace{\sum_{j=1}^{m}r_{j-1}h\sigma_0^0\otimes r_{j-1}\sigma_j^2}_{(s58)}+\underbrace{\sum_{j=1}^{m}h\pi_j^1\otimes r_{j-1}\eta_0^1}_{(c59)}$$

$$-\underbrace{\sum_{j=1}^{m}h\pi_j^1\otimes r_{j-1}\eta_j^1}_{(s60)}+\underbrace{\sum_{j=1}^{m}h\pi_j^1\otimes hr_{j-1}\sigma_j^1}_{(s61)}-\underbrace{\sum_{j=1}^{m}h\pi_j^1\otimes r_{j-1}\sigma_j^1}_{(c53)}$$

$$
\begin{aligned}
&\underbrace{-\sum_{j=1}^{m} r_j\eta_0^1\otimes r_{j-1}\rho_j^1}_{\text{(s62)}}\underbrace{+\sum_{j=1}^{m} r_j\eta_j^1\otimes r_{j-1}\rho_j^1}_{\text{(s63)}}\\
&\underbrace{-\sum_{j=1}^{m} r_jh\sigma_j^1\otimes r_{j-1}\rho_j^1}_{\text{(c64)}}\underbrace{+\sum_{j=1}^{m} r_j\sigma_j^1\otimes r_{j-1}\rho_j^1}_{\text{(s65)}}\underbrace{-\sum_{j=1}^{m} r_j\sigma_j^2\otimes r_j\sigma_j^0}_{\text{(s66)}}\\
&\underbrace{+\sum_{j=1}^{m} r_j\sigma_j^2\otimes r_{j-1}\sigma_j^0}_{\text{(s67)}}\underbrace{+\sum_{j=1}^{m} r_jh\sigma_j^0\otimes r_{j-1}\rho_j^2}_{\text{(s68)}}\\
&\underbrace{-\sum_{j=1}^{m} r_jh\sigma_0^0\otimes r_{j-1}\rho_j^2}_{\text{(s69)}}\underbrace{-\sum_{j=1}^{m} r_jh\sigma_j^1\otimes r_{j-1}\eta_j^1}_{\text{(s70)}}\underbrace{+\sum_{j=1}^{m} r_jh\sigma_j^1\otimes r_j\eta_j^1}_{\text{(s71)}}\\
&\underbrace{-\sum_{j=1}^{m} r_jh\sigma_j^1\otimes r_{j-1}h\rho_j^1}_{\text{(s72)}}\underbrace{+\sum_{j=1}^{m} r_jh\sigma_j^1\otimes r_{j-1}\rho_j^1}_{\text{(c64)}}.
\end{aligned}
\tag{5.17}
$$

As a first step in the verification that (5.16) equals (5.17), observe that the terms in the expression (5.17) which have the same labels cancel in twelve pairs (namely, the pairs labelled c2, c4, c11, c13, c19, c21, c25, c29, c32, c40, c53 and c63). Furthermore, a straightforward calculation shows that the following expressions from (5.17), given in display (5.18), equal 0.

$$
\begin{aligned}
&\text{c44}+\text{c31}+\text{c15}+\text{c3}=0, &&\text{c47}+\text{c33}+\text{c24}+\text{c6}=0,\\
&\text{c50}+\text{c38}+\text{c34}+\text{c7}=0, &&\text{c10}+\text{c42}+\text{c28}+\text{c18}=0,\\
&\text{c56}+\text{c45}+\text{c22}=0, &&\text{c59}+\text{c49}+\text{c26}=0.
\end{aligned}
\tag{5.18}
$$

Finally, observe that the sum of the remaining terms of the expression given in (5.17) is equal to the expression for $\Delta(\partial\delta^3)$, given in (5.16), as indicated below in (5.19).

$$
\begin{aligned}
&A=\text{s1}, && H=\text{s27}, && O=\text{s72}, && V=\text{s67},\\
&B=\text{s9}, && I=\text{s39}, && P=\text{s61}, && W=\text{s52},\\
&C=\text{s20}, && J=\text{s35}, && Q=\text{s55}+\text{s16}, && X=\text{s48}+\text{s57}+\text{s69}+\text{s23},\\
&D=\text{s14}, && K=\text{s41}, && R=\text{s60}+\text{s37}, && Y=\text{s66},\\
&E=\text{s30}, && L=\text{s65}, && S=\text{s63}+\text{s12}, && Z=\text{s43}+\text{s51}+\text{s62}+\text{s17},\\
&F=\text{s8}, && M=\text{s54}, && T=\text{s68}+\text{s36}, && A'=\text{s70},\\
&G=\text{s5}, && N=\text{s46}, && U=\text{s58}, && B'=\text{s71}. \qquad \square
\end{aligned}
\tag{5.19}
$$

Acknowledgement

The authors wish to thank Kerstin Aaslepp for her valuable commentary in the preparation of this paper.

References

[1] G.E. Bredon, J.H. Wood, Non-orientable surfaces in orientable 3-manifolds, Invent. Math. 7 (1969) 83–110.
[2] J. Bryden, C. Hayat-Legrand, H. Zieschang, P. Zvengrowski, L'anneau de cohomologie d'une variété de Seifert, C. R. Acad. Sci. Paris Sér. 1 324 (1997) 323–326.
[3] J. Bryden, P. Zvengrowski, The cohomology ring of the orientable Seifert manifolds and applications to Lusternik–Schnirelmann category, in: Proc. of the Workshop on Geometry and Homotopy, Warsaw, 1997.
[4] G. Burde, H. Zieschang, Knots, de Gruyter, Berlin, 1995.
[5] C. Chevalley, Théorie de Groupes de Lie II: Groupes Algébriques, Hermann and Cie, Paris, 1951.
[6] G. Cooke, R. Finney, Homology of Cell Complexes, Princeton Univ. Press, Princeton, NJ, 1967.
[7] B.A. Dubrovin, A.T. Fomenko, S.P. Novikov, Modern Geometry—Methods and Applications. Part III. Introduction ot Homology Theory, GTM 124, Springer, Berlin, 1990.
[8] D.B.A. Epstein, Projective planes in 3-manifolds, Proc. London Math. Soc. 11 (1961) 469–484.
[9] R.H. Fox, Free differential calculus. I. Derivations in the free group ring, Ann. of Math. 57 (1953) 547–560.
[10] C. Hayat-Legrand, S. Wang, H. Zieschang, Degree-one maps onto lens spaces, Pacific J. Math. 176 (1996) 19–32.
[11] C. Hayat-Legrand, S. Wang, H. Zieschang, Minimal Seifert manifolds, Math. Ann. 308 (1997) 673–700.
[12] J. Hempel, 3-Manifolds, Vol. 86, Annals of Math. Studies, Princeton Univ. Press, Princeton, NJ, 1976, 115–135.
[13] R. Kirby, P. Melvin, On the 3-manifold invariants of Reshetikhin–Turaev for $SL(2,\mathbb{C})$, Invent. Math. 105 (1991) 474–545.
[14] M. Lustig, E.-M. Thiele, H. Zieschang, Primitive elements in the free product of two finite cyclic groups, Abh. Math. Sem. Univ. Hamburg 68 (1995) 277–281.
[15] S. MacLane, Homology, Springer, Berlin, 1963.
[16] J.M. Montesinos, Classical Tesselations amd Three-Manifolds, Springer, Berlin, 1987.
[17] H. Murakami, T. Ohtsuki, M. Okada, Invariants of three-manifolds derived from linking matrices of framed links, Osaka J. Math. 29 (1992) 545–572.
[18] J. Nielsen, Die Isomorphismen der allgemeinen, unendlichen Gruppe mit zwei Erzeugenden, Math. Ann. 78 (1918) 385–397.
[19] P. Orlik, Seifert Manifolds, Lecture Notes in Math. 291, Springer, Berlin, 1972.
[20] P. Orlik, E. Vogt, H. Zieschang, Zur Topologie gefaserter dreidimensionaler Mannigfaltigkeiten, Topology 6 (1967) 49–64.
[21] R.P. Osborne, H. Zieschang, Primitives in the free group on two generators, Invent. Math. 63 (1981) 17–24.
[22] K. Reidemeister, Homotopieringe und Linsenräume, Abh. Math. Sem. Univ. Hamburg 11 (1935) 102–109.
[23] J.H. Rubinstein, One-sided Heegaard splittings of 3-manifolds, Pacific J. Math. 76 (1970) 185–200.
[24] P. Scott, The geometries of 3-manifolds, Bull. London Math. Soc. 15 (56) (1983) 401–487.

[25] H. Seifert, Topologie dreidimensionaler gefaserter Räume, Acta Math. 60 (1932) 147–238.
[26] H. Seifert, W. Threlfall, A Textbook of Topology, Academic Press, New York, 1980.
[27] A.R. Shastri, P. Zvengrowski, Type of 3-manifolds and addition of relativistic kinks, Rev. Math. Phys. 3 (4) (1991) 467–478.
[28] A.R. Shastri, J.G. Williams, P. Zvengrowski, Kinks in general relativity, Internat. J. Theoret. Phys. 19 (1) (1980) 1–23.
[29] E.H. Spanier, Algebraic Topology, McGraw-Hill, New York, 1966.
[30] N. Steenrod, D.B.A. Epstein, Cohomology Operations, Princeton Univ. Press, Princeton, NJ, 1962.
[31] L.R. Taylor, Relative Rochlin invariants, Topology Appl. 18 (1984) 259–280.
[32] P. Zvengrowski, 3-manifolds and relativistic kinks, in: Proceedings of the Winter School Geometry and Physics, Srní, Supplemento ai Rendiconti de Circolo Matematico di Palermo Ser. II 30, 1993, pp. 157–162.