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ON THE IMAGE OF THE LAWRENCE-KRAMMER REPRESENTATION

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ABSTRACT

A non-singular sesquilinear form is constructed that is preserved by the Lawrence-Krammer representation. It is shown that if the polynomial variables q and t of the Lawrence-Krammer representation are chosen to be appropriate algebraically independent unit complex numbers, then the form is negative-definite Hermitian. Using the fact that non-invertible knots exist this result implies that there are matrices in the image of the Lawrence-Krammer representation that are conjugate in the unitary group, yet the braids that they correspond to are not conjugate as braids. The two primary tools involved in constructing the sesquilinear form are Bigelow's interpretation of the Lawrence-Krammer representation, together with the Morse theory of functions on manifolds with corners.

1. Introduction

This paper takes a Morse-theoretic approach to the Lawrence-Krammer representation. The Lawrence-Krammer representation is an injective homomorphism $B_n \to \mathrm{GL}_{\binom{n}{2}}\mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$ [4] [18]. The representation has a natural description as the action of the braid group on the middle-dimensional homology of a certain four-dimensional manifold, where the homology is thought of as a module over a Laurent polynomial ring, which enters the picture as the group ring of a free abelian group of covering transformations similar to the Burau representation. The middle dimensional homology of any even-dimensional manifold has an intersection product pairing, and this is used to construct a sesquilinear form that is preserved by the Lawrence-Krammer representation, analogously to the work of Long [21] and similar work for the Gassner and Burau representations by Abdulrahim [1] and Squier [26]. Using the Morse theory of functions on manifolds with corners developed by Handron [12] [13], this sesquilinear pairing is explicitly computed in Section 4.

The embeddings of $\mathbf{Z}[q^{\pm 1}, t^{\pm 1}] \to \mathbf{C}$ are parametrized by algebraically independant $q, t \in \mathbf{C}$, so we can think of the Lawrence-Krammer representation as a map $\mathbf{C}^2 \to \mathrm{Hom}(\mathrm{B_n}, \mathrm{GL}_{\binom{n}{2}}\mathbf{C})$. Provided $|q| = |t| = 1$ the sesquilinear form that is preserved by the Lawrence-Krammer representation is Hermitian. It is shown in Theorem 2 that for certain values of t and q this Hermitian form is negative definite, and thus the image of the Lawrence-Krammer representation $B_n \to GL_{\binom{n}{2}}\mathbf{C}$ has compact closure. Using this result, we address the question of conjugacy in the image of the Lawrence-Krammer representation in Section 5. This result may be of interest to Braid Cryptographers [2] as it gives insight into the difference between the conjugacy problem in braid groups versus the conjugacy problem in the target matrix group, $GL_{\binom{n}{2}}\mathbf{C}$.

2. Generalities on the Lawrence-Krammer representation

This section begins with the definition of the Lawrence-Krammer representation and more generally the Lawrence representations and the sesquilinear forms that they preserve.

Definition 1. The **configuration space** $\mathcal{C}_n X$ of n points in a topological space X is the space $(X^n - \Delta_n X)/S_n$. Here $\Delta_n X = \{(x_1, \cdots, x_n) \in X^n : x_i = x_j \text{ for some } i \neq j\}$. S_n is the symmetric group, acting by permuting the factors of the product X^n.

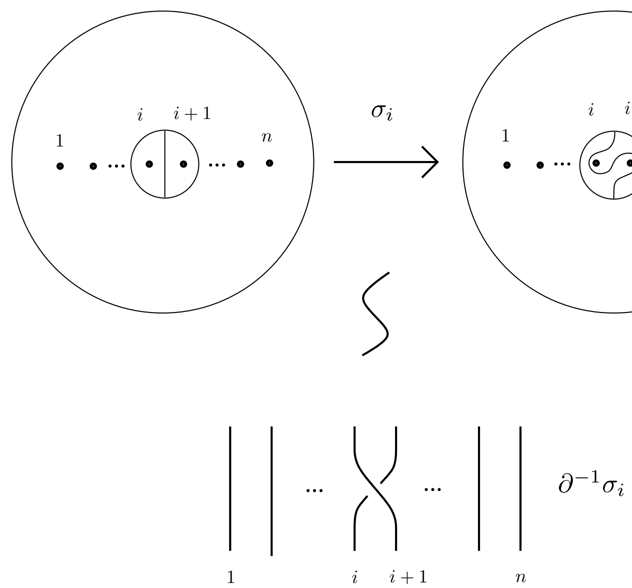
A convention on the fundamental group that is used throughout this paper is that if $f, g : \mathbf{I} = [\mathbf{0}, \mathbf{1}] \to \mathbf{X}$ are loops, then the concatenation fg denotes the loop such that $(fg)(t) = g(2t)$ for $0 \leq t \leq \frac{1}{2}$ and $(fg)(t) = f(2t-1)$ for $\frac{1}{2} \leq t \leq 1$.

The **braid group** $B_n := \pi_0 \mathrm{Diff}(\mathrm{D}^2, \mathrm{n})$ is the mapping class group of a disc with n marked points in the interior, where the diffeomorphisms restrict to the identity on the boundary. An equivalent definition of the braid group is the fundamental group of the configuration space of n points in a disc, $B_n = \pi_1 \mathcal{C}_n D^2$. The fact that the two definitions are equivalent is an easy consequence of the homotopy long exact sequence of the fibration $\mathrm{Diff}(\mathrm{D}^2) \to \mathcal{C}_\mathrm{n}\mathrm{D}^2$, together with Smale's theorem that $\mathrm{Diff}(\mathrm{D}^2)$ is contractible [25]. The map $\mathrm{Diff}(\mathrm{D}^2) \to \mathcal{C}_\mathrm{n}\mathrm{D}^2$ is given by fixing a configuration in $\mathcal{C}_n D^2$ and evaluating it on a diffeomorphisms of $\mathrm{Diff}(\mathrm{D}^2)$. The fact that evaluation maps are fibrations is due to Palais [23]. The boundary map $\pi_1 \mathcal{C}_n D^2 \to \pi_0 \mathrm{Diff}(\mathrm{D}^2, \mathrm{n})$ is a homomorphism with the above concatenation convention in $\pi_1 \mathcal{C}_n D^2$.

$\mathcal{P}_n$ will denote the closed unit disc with n interior points removed. Let $ab : B_i \to \mathbf{Z}$ for $i \in \{1, 2, 3, \cdots\}$ be the abelianization maps. A convention in this paper is that $ab(\sigma_i) = 1$, where σ_i is the half Dehn twist in Figure 1. Let $T : \pi_1 \mathcal{C}_k \mathcal{P}_n \to \mathbf{Z}$ be the composite of the forgetful map $\pi_1 \mathcal{C}_k \mathcal{P}_n \to \pi_1 \mathcal{C}_k D^2$ with the abelianization map, and similarly let $R : \pi_1 \mathcal{C}_k \mathcal{P}_n \to \mathbf{Z}$ be the composite of the inclusion map $\pi_1 \mathcal{C}_k \mathcal{P}_n \to \pi_1 \mathcal{C}_{k+n} D^2$ with the abelianization map, and define $Q : \pi_1 \mathcal{C}_k \mathcal{P}_n \to \mathbf{Z}$ by the identity $Q(f) := \frac{R(f) - T(f)}{2}$. Let $\mathcal{L}\mathcal{C}_k \mathcal{P}_n$ be the abelian Galois covering space of $\mathcal{C}_k \mathcal{P}_n$ such that the image of the map $\pi_1 \mathcal{L}\mathcal{C}_k \mathcal{P}_n \to \pi_1 \mathcal{C}_k \mathcal{P}_n$ is $ker(Q) \cap ker(T)$. The k-th Lawrence representation of B_n, as described by Bigelow in [6] is the action of

B_n on $\mathrm{H}_k(\mathcal{LC}_k\mathcal{P}_n)$.

Figure 1



An example of the isomorphism $\partial : \pi_1\mathcal{C}_nD^2 \to \pi_0\mathrm{Diff}(\mathrm{D}^2, \mathrm{n})$

$\mathcal{LC}_k\mathcal{P}_n \to \mathcal{C}_k\mathcal{P}_n$ is a 'normal' or Galois cover. For $k \geq 2$ its group of covering transformations is precisely $\mathbf{Z}\times\mathbf{Z}$ and can be identified with the image of $Q \times T$. q and t will denote the covering transformations corresponding to 1×0 and 0×1 respectively in the image of $Q \times T$. With these definitions, $\mathrm{H}_2(\mathcal{LC}_2\mathcal{P}_n)$ is a module over the Laurent polynomial ring $\mathbf{Z}[q^{\pm1}, t^{\pm1}]$, which is the group ring of the group of covering transformations $\langle q, t\rangle = \mathbf{Z}\times\mathbf{Z}$.

In [4], the Lawrence-Krammer representation was defined as the action of the braid group B_n on $\mathrm{H}_2(\mathcal{LC}_2\mathcal{P}_n) \otimes_{\mathbf{Z}[q^{\pm1},t^{\pm1}]} \mathcal{F}$ where $\mathcal{F}$ is some field containing $\mathbf{Z}[q^{\pm1}, t^{\pm1}]$ such as $\mathbf{C}$. It has since been shown that $\mathrm{H}_2(\mathcal{LC}_2\mathcal{P}_n)$ is free over the Laurent polynomial ring $\mathbf{Z}[q^{\pm1}, t^{\pm1}]$ [24]. Unfortunately, those free generators are not convenient to use, so in this paper we will restrict the Lawrence-Krammer representation to a certain full-rank, free, B_n-invariant submodule of $\mathrm{H}_2(\mathcal{LC}_2\mathcal{P}_n)$ which will be described precisely in Section 3.

Definition 2. The **total intersection product** is a sesquilinear form

$$\langle\cdot,\cdot\rangle : \mathrm{H}_2(\mathcal{LC}_2\mathcal{P}_n) \oplus \mathrm{H}_2(\mathcal{LC}_2\mathcal{P}_n) \to \mathbf{Z}[q^{\pm1}, t^{\pm1}]$$

defined by

$$\langle v, w\rangle := \sum_{(i,j)\in\mathbf{Z}\times\mathbf{Z}} \mu(v, t^iq^jw)t^iq^j$$

where $\mu : \mathrm{H}_2(\mathcal{LC}_2\mathcal{P}_n) \oplus \mathrm{H}_2(\mathcal{LC}_2\mathcal{P}_n) \to \mathbf{Z}$ is the intersection product. See [27] or [8] for definitions of the intersection product on manifolds, and [20] for basics on sesquilinear forms. Sometimes this pairing is called a Blanchfield form [16].

To compute the intersection product, one could take the CW-decomposition given in Bigelow's paper [4] and notice that all homology classes are realizable by compact surfaces. Unfortunately, this is potentially very difficult, as Bigelow's generators $v_{i,j}$ are genus two surfaces and rather difficult to visualize, moreover they are not transverse. To bypass this difficulty, we compute the intersection product using two easy to visualize transverse CW-decompositions of $\mathcal{C}_2\mathcal{P}_n$ that come from a Morse function on $\mathcal{C}_2\mathcal{P}_n$.

3. A little Morse theory

In section 3.1 we review Morse theory on manifolds with corners. In section 3.2 the Morse theory is applied to get dual CW-decompositions of the configuration space of two points in a planar surface, and in 3.3 we apply these results to study the Lawrence-Krammer representation.

3.1. *A survey of Morse theory on manifolds with corners*

Morse theory on manifolds with corners has been studied for some time, although it is not a commonly known branch of Morse theory. A good general reference for Morse theory is Milnor's book [22], and for Morse theory on manifolds with corners, Handron's papers [12] [13]. A summary of the relevant theory is given below.

Definition 3. A smooth n-dimensional **manifold with corners** is a 2^{nd}-countable Hausdorff topological space X together with a family of maps $\mathbf{A}$ where if $\xi \in \mathbf{A}$ then $\xi : U \to \mathbf{H}^n_j$ is a homeomorphism between an open subset U of X and $\mathbf{H}^n_j$ for some $j \in \{0, 1, \cdots, n\}$.

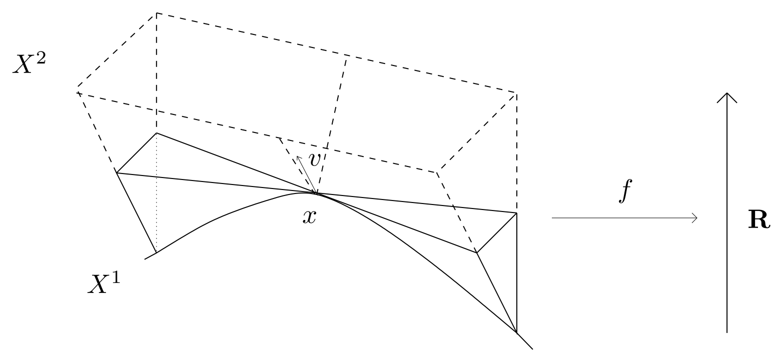
$$\mathbf{H}^n_j = \{w \in \mathbf{R}^n : w \cdot e_i \geq 0 \ \forall\ 1 \leq i \leq j\}$$

$\xi \in \mathbf{A}$ is called a **chart**. We demand that the union of the domains of the charts in $\mathbf{A}$ is X, and if any two charts ξ, ψ have overlapping domains $\xi : U \to \mathbf{H}^n_j$ and $\psi : V \to \mathbf{H}^n_k$ then they must be smoothly compatible in the sense that $\xi \circ \psi^{-1}_{|\psi(U\cap V)}$ must be a smooth diffeomorphism from $\psi(U \cap V)$ to $\xi(U \cap V)$ [11]. This allows us to define smooth functions between manifolds with corners and derivatives of such functions analogously to [15].

The **i-dimensional strata** $X^{(i)}$ of X is the set of points $x \in X$ such that there exists a chart $\xi : U \to \mathbf{H}^n_{n-i}$, $x \in U$ with $\xi(x) \cdot e_j = 0 \ \forall\ 1 \leq j \leq n - i$. A **critical point** of a function $f : X \to \mathbf{R}$ is a point $x \in X$ such that if $x \in X^{(i)}$ then $Df_{|X^{(i)}}(x) = 0$. A critical point x of f is **non-degenerate** if the Hessian matrix $D^2 f_{|X^{(i)}}(x)$ is non-singular and if for all $v \in T_xX$ that point into the strata $X^{(i+1)}$, $Df_x(v) \neq 0$. A function $f : X \to \mathbf{R}$ is a **Morse function** if all of its critical points are non-degenerate. A non-degenerate critical point is **essential** if for all $v \in T_xX$ that point into $X^{(i+1)}$, $Df_x(v) > 0$.

Theorem 1. [10] [29] [12] [13] Given a Morse function $f : X \to \mathbf{R}$ the homotopy type of $f^{-1}(-\infty, c]$ changes only at $c = f(x)$ for x an essential critical point of f. Provided $f^{-1}(c)$ contains only one essential critical point, $f^{-1}[c-\epsilon, c+\epsilon]$ is homotopy equivalent to $f^{-1}(c-\epsilon)$ union a cell, the dimension of the cell is given by the index of $D^2 f_{|X^{(i)}}(x)$.

Figure 2



Essential critical point on a 1-dimensional strata resulting in a 1-cell attachment.

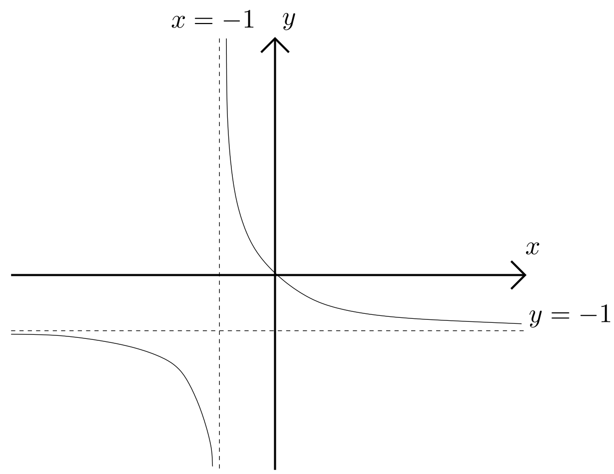
Given M a smooth manifold with boundary, M^n is naturally a smooth manifold with corners. Thus, $\mathcal{C}_n M$ is a smooth manifold with corners.

Lemma 1. Let R be a sub-manifold of $\mathbf{R}^2$, and consider the distance function $d : \mathcal{C}_2 R \to \mathbf{R}$, $d([(z_1, z_2)]) = |z_1 - z_2|$ where $|\cdot|$ is the standard Euclidean norm on $\mathbf{R}^2$. A point $[z_1, z_2] \in \mathcal{C}_2 R$ is a critical point of d if and only if $[z_1, z_2]$ sits on the 2-dimensional strata of $\mathcal{C}_2 R$ and $z_1 - z_2$ is perpendicular to both $T_{z_1}\partial R$ and $T_{z_2}\partial R$. The Hessian is non-singular at $[z_1, z_2]$ provided the polynomial $xy + x + y$ does not have a root at $(x, y) = (\langle k_1, z_1 - z_2\rangle, \langle z_2 - z_1, k_2\rangle)$ where k_1 and k_2 are the curvature vectors of ∂R at z_1 and z_2 respectively. The index of the Hessian is the number of roots $t \in (0, \infty)$ of the polynomial $xy + x + y$ where $(x, y) = (\langle k_1, z_1 - z_2\rangle + t, \langle z_2 - z_1, k_2\rangle + t)$.

Proof. d has critical points only on the 2-dimensional strata of $\mathcal{P}_n$ since if $[z_1, z_2]$ is not on the 2-dimensional strata, at least one of z_1 or z_2 must be in the interior of R and so the derivative of d must be non-zero. Let $[z_1, z_2] \in (\mathcal{C}_2 R)^{(2)}$, thus both z_1 and $z_2 \in \partial R$. Let $f : (-\epsilon, \epsilon) \to \partial R$ and $g : (-\epsilon, \epsilon) \to \partial R$ be arclength-preserving parametrizations of neighborhoods of z_1 and z_2 in ∂R respectively with $f(0) = z_1$ $g(0) = z_2$. The critical points of $d \circ [f(x), g(y)]$ are precisely the same as the critical points of the square of the distance function, which is a polynomial function $d^2 \circ [f(x), g(y)] = \langle f(x) - g(y), f(x) - g(y)\rangle$. A quick computation shows that the derivative of the above polynomimal is the 1×2 matrix $[\langle f'(x), f(x) - g(y)\rangle, \langle g(y) - f(x), g'(y)\rangle]$. To prove the statement about non-singularity and index of the Hessian, we compute $D^2(d^2 \circ [f(x), g(y)])_{(0,0)}$. This is the matrix $\begin{bmatrix} c_1 + 1 & -1 \\ -1 & c_2 + 1 \end{bmatrix}$ where

$c_1 = \langle f''(0), f(0) - g(0)\rangle$ and $c_2 = \langle g(0) - f(0), g''(0)\rangle$. The result follows. □

Figure 3



Roots of the polynomial $xy + x + y$

3.2. *A CW-decomposition*

For the purposes of the remainder of this paper, consider the punctured disc $\mathcal{P}_n$ to be the genus zero compact connected 2-manifold with $n+1$ boundary components. Consider an embedding of $\mathcal{P}_n$ in $\mathbf{R}^2$ where the boundary consists entirely of ellipses, as in Figure 4.

Lemma 2. The distance function $d : \mathcal{C}_2\mathcal{P}_n \to \mathbf{R}$ is Morse. The CW decomposition given by the Morse function d has one 0-cell, $2n+1$ 1-cells, labeled $x_i, y_i, i \in \{1, \cdots, n\}$ and b. There are also $\binom{n}{2} + 2n$ 2-cells, with attaching maps:

$$\begin{array}{lll} Z_{i,j} & [bx_ib^{-1}, x_j^{-1}] & 1 \leq i < j \leq n \\ Z_i & y_i^{-1}b^{-1}x_ibx_i & 1 \leq i \leq n \\ Z'_i & [b, y_i] & 1 \leq i \leq n \end{array}$$

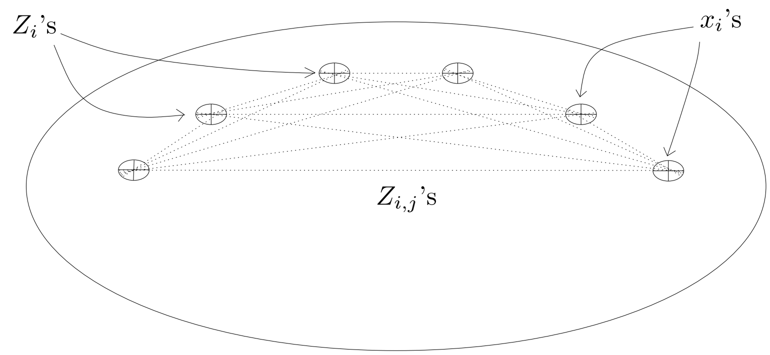
We use the convention that $[a,b] = aba^{-1}b^{-1}$.

Proof. $d^{-1}(-\infty, x]$ for $x > 0$ small is has the homotopy type of the projectivised unit tangent bundle of $\mathcal{P}_n$, which is diffeomorphic to $S^1 \times \vee_{i=1}^n S^1$. We give $S^1 \times \vee_{i=1}^n S^1$ the product CW-decomposition, with counter-clockwise orientations given to the 1-cells. The cell corresponding to the S^1 factor will be denoted b and the n wedge summands will be denoted y_i for $i \in \{1, \cdots, n\}$. The product structure gives us the 2-cells Z'_i with attaching maps $[b, y_i]$.

There are $4\binom{n}{2}+10n+2$ critical points, but only $\binom{n}{2}+2n$ of them are essential. If one restricts d to each connected 2-dimensional stratum, the essential critical points are the maxima on each stratum, and if both points of the configuration lay on the same ellipse, then there is an additional essential critical point corresponding to the minor axis of the ellipse.

Of the $\binom{n}{2}+2n$ essential critical points, $\binom{n}{2}+n$ are 2-cell attachments (the maxima of d on each connected 2-dimensional stratum), and the remaining n essential critical points are 1-cell attachments. This is true because of Lemma 1 tells us that a critical point corresponding to the minor axis of an ellipse is a saddle point.

Figure 4.



Essential critical points of d, $n = 6$.

The 1-cells which correspond to the minor axis of an ellipse will be labeled x_i for $i \in \{1, \cdots, n\}$. x_i can be thought of as a loop of configurations (starting at the base-point) where one point of the configuration is stationary and the other traverses an embedded circle which bounds a disc that contains only the i-th puncture, in particular this disc does not contain the stationary point of the configuration.

The cells corresponding to the major axis of an ellipse will be labeled Z_i for $i \in \{1, \cdots, n\}$. Z_i can be thought of as the homotopy between y_i and $b^{-1}x_ibx_i$. The cells corresponding to essential critical points $[x, y]$ where x is on the i-th ellipse and y is on the j-th ellipse will be labeled $Z_{i,j}$ for $1 \leq i < j \leq n$. The attaching map for these critical points is clearly a commutator, we choose the attaching map to be $[bx_ib^{-1}, x_j^{-1}]$ for technical convenience in section 3.3. □

We can reduce the CW-complex Z, using n handle slides described by the attaching maps for Z_i for $i \in \{1, \cdots, n\}$ to get the following CW-decomposition.

Corollary 1. $\mathcal{C}_2\mathcal{P}_n$ deformation retracts to a subspace with CW-decomposition denoted simply by Y, with one 0-cell, $n+1$ 1-cells, labeled $x_i : i \in \{1, \cdots, n\}$ and b, and $\binom{n}{2}+n$ 2-cells, with attaching maps given by:

$$\begin{array}{lll} Y_{i,j} & [bx_ib^{-1}, x_j^{-1}] & 1 \leq i < j \leq n \\ Y_i & [b, x_ibx_i] & 1 \leq i \leq n \end{array}$$

3.3. *The dual CW-decomposition*

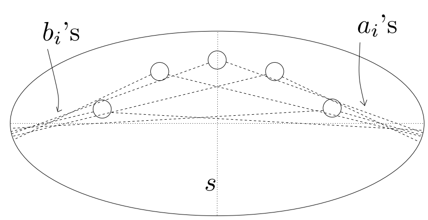
Lemma 3. The negative of the distance function $-d$ gives a CW decomposition of $\mathcal{C}_2\mathcal{P}_n$ with 1 0-cell, $2n+1$ 1-cells labeled $s, a_i, b_i : i \in \{1, \cdots, n\}$, and $\binom{n}{2}+2n$ 2-cells, with attaching maps given by:

$X_{i,j}$	$[a_i, b_j]$	$1 \leq i < j \leq n$
X_i	$s^{-1}b_i^{-1}sa_i$	$1 \leq i \leq n$
X_i'	$b_i((b_{i+1}\cdots b_n s^{-1}a_1\cdots a_{i-1}) \to a_i^{-1})$	$1 \leq i \leq n$

where $(x \to y) = xyx^{-1}$

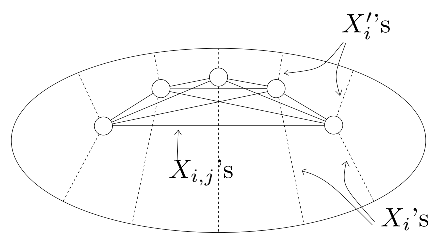
Proof. This proof differs very little from the proof of Lemma 2. d has all the same critical points as $-d$, they only differ in their essential critical points. There are $\binom{n}{2} + 2n$ 2-cell attachments, $2n+1$ 1-cell attachments and a 0-cell. The 0-cell is the configuration corresponding to the major axis of the big ellipse. The 1-cell attachments correspond to the minor axis of the big ellipse, labeled s, and the maxima of $-d$ on the 2-dimensional strata where one configuration is on the big ellipse. Label these cells a_i and b_i for $i \in \{1, \cdots, n\}$.

Figure 5



Index 0 and 1 critical points of $-d$, and cell labels.

Figure 6



Index 2 critical points of $-d$ and cell labels.

The 2-cells $X_{i,j}$ are as in Corollary 1. They are described as the attaching map for the single 2-cell of the torus that consists of configurations of two points, one point that lays on a circle that bounds the i-th puncture and one point that lays on a circle that bounds the j-th puncture. X_i and X_i' express the a_i's in terms of the b_j's in the two ways: conjugating by s and also conjugating by its inverse.□

A CW-decomposition of the base space of a covering space lifts to a CW-decomposition of the cover. Choose a base-point in $\mathcal{LC}_2\mathcal{P}_n$ that is above the 0-cell for our CW-decomposition Y. We give $\mathcal{LC}_2\mathcal{P}_n$ a CW-structure by lifting the cells of Y to $\mathcal{LC}_2\mathcal{P}_n$. Introducing a mild notational ambiguity, we interpret $Y_{i,j}$ to be the lift of $Y_{i,j}$ to $\mathcal{LC}_2\mathcal{P}_n$ so that the attaching map starts at the base-point. This is a slight abuse of notation because $Y_{i,j}$ is also a 2-cell in $\mathcal{C}_2\mathcal{P}_n$. We do the same for all the remaining cells of Y. With these conventions, $\partial Y_{i,j} = q^{-1}t^{-1}(q-1)((-x_i+tx_j)-(q-1)b)$ and $\partial Y_i = q^{-1}t^{-1}(q^{-1}t^{-1}+1)((1-t)x_i+(q-1)b)$.

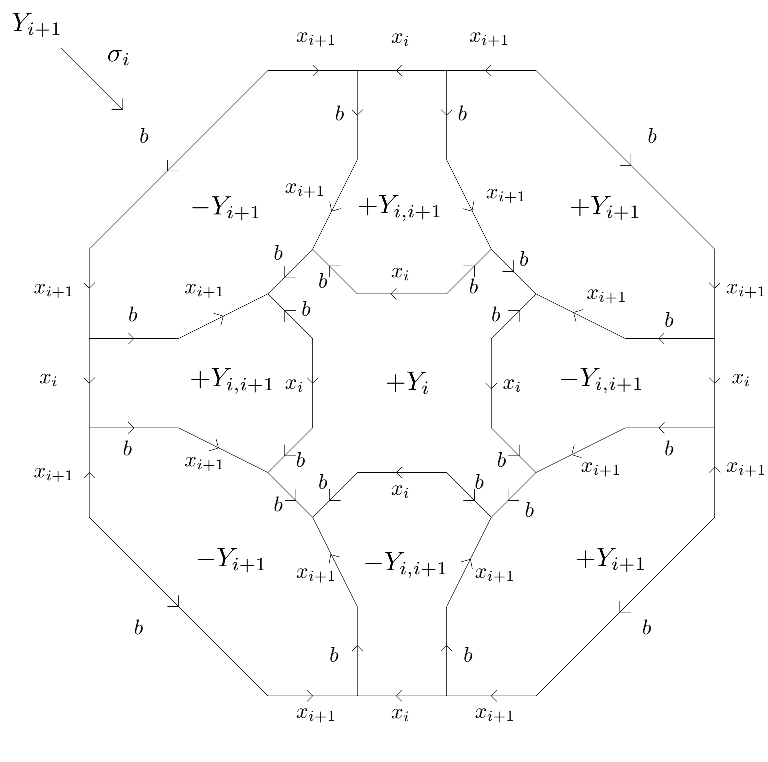
Proposition 1. The homology $\mathrm{H}_2(\mathcal{LC}_2\mathcal{P}_n)$ contains a free rank $\binom{n}{2}$ $\mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$-module, which is B_n-invariant, and spanned by $v_{i,j} = qt(q-1)(Y_i - tY_j) + (1-t)(1+qt)Y_{i,j}$. This submodule is the Lawrence-Krammer module and denoted by $\mathcal{L}_n$. The corresponding representation of B_n is called the Lawrence-Krammer representation.

Proof. Homotope $\sigma_i : \mathcal{LC}_2\mathcal{P}_n \to \mathcal{LC}_2\mathcal{P}_n$ to a cellular map. Since $\pi_2 Y = *$, we can ask how σ_i acts on the elements $Y_j, Y_{j,k} \in \pi_2(Y, Y^1)$, where Y is the CW-structure from Corollary 1, lifted to $\mathcal{LC}_2\mathcal{P}_n$. Choose the cellular approximation so that $\sigma_i \cdot b = b$ and $\sigma_i \cdot x_j = x_j$ unless $i = j-1$ in which case $\sigma_i \cdot x_j = x_j^{-1} x_{j-1} x_j$ or if $i = j$ then $\sigma_i \cdot x_j = x_{j+1}$. We can now compute $\sigma_i \cdot Y_{j,k}$ and $\sigma_i Y_j$.

$$\sigma_i \cdot Y_{j,k} = \begin{cases} Y_{j+1,k} & j = i, k > i+1 \\ Y_{j,k+1} & j < i, k = i \\ -qtY_{j,k} + qt(1-q)Y_k & j = i, k = i+1 \\ qY_{j-1,k} + (1-q)Y_{j,k} & j = i+1 \\ (1-q)Y_{j,k} + qY_{j,k-1} & j < i, k = i+1 \\ Y_{j,k} & \text{otherwise} \end{cases}$$

$$\sigma_i \cdot Y_j = \begin{cases} Y_{j+1} & j = i \\ (1+qt)(1-q)Y_j + q^2 Y_{j-1} + t^{-1}(1+qt)(1-t)Y_{j-1,j} & j = i+1 \\ Y_j & \text{otherwise} \end{cases}$$

Figure 7



Filling $\partial(\sigma_i \cdot Y_{i+1})$

The most involved of these computations is for $\sigma_i \cdot Y_{i+1}$ so it will be done in some detail. The remaining computations are simpler. $\partial Y_{i+1} = [b, x_{i+1} b x_{i+1}]$ so $\partial(\sigma_i \cdot Y_{i+1}) = [b, x_{i+1}^{-1} x_i x_{i+1} b x_{i+1}^{-1} x_i x_{i+1}]$. Since $\pi_2 Y = *$ there is only one element of $\pi_2(Y, Y^1)$ with this boundary. The problem of finding this element is much like solving a jigsaw-puzzle, one needs to find the 2-cells of Y that fit together so that they have boundary $[b, x_{i+1}^{-1} x_i x_{i+1} b x_{i+1}^{-1} x_i x_{i+1}]$. The solution is given in Figure 7.

The above computation proves more than the fact that subspace spanned by the $v_{i,j}$'s is invariant. It gives us the matrices for the Lawrence-Krammer representation. They are:

$$\sigma_i \cdot v_{j,k} = \begin{cases} v_{j,k} & i \notin \{j-1, j, k-1, k\}, \\ qv_{i,k} + (q^2-q)v_{i,j} + (1-q)v_{j,k} & i = j-1 \\ v_{j+1,k} & i = j \neq k-1, \\ qv_{j,i} + (1-q)v_{j,k} - (q^2-q)tv_{i,k} & i = k-1 \neq j, \\ v_{j,k+1} & i = k, \\ -tq^2 v_{j,k} & i = j = k-1. \end{cases}$$

□

The above computation gives an independent verification that the matrices in [3] are correct.

Even though the $v_{i,j}$'s form a basis for $\mathrm{H}_2(\mathcal{L}\mathcal{C}_2\mathcal{P}_n) \otimes_{\mathbf{Z}[q^{\pm 1}, t^{\pm 1}]} \mathcal{F}$ where $\mathcal{F}$ is a field containing $\mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$, the $v_{i,j}$'s do not span $\mathrm{H}_2(\mathcal{L}\mathcal{C}_2\mathcal{P}_n)$. For example, the homology classes given by Bigelow in [6] are not in the $\mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$-span of the $v_{i,j}$'s. This is similar to, although much less complete, than the results of Paoluzzi and Paris [24], where they prove that $\mathrm{H}_2(\mathcal{L}\mathcal{C}_2\mathcal{P}_n)$ is a free rank $\binom{n}{2}$ $\mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$-module.

4. On the image of the Lawrence-Krammer representation

It is not uncommon for a general manifold to have a singular intersection product pairing. For example, with the cylinder $S^1 \times \mathbf{I}$ the intersection product is zero. There is a Poincaré duality theorem for abelian covers of compact manifolds (See for example [16] Appendix E), but like Poincaré duality for compact manifolds, Poincaré duality does not directly give information about the intersection product provided the manifolds have non-empty boundary.

To compute the intersection product, one could take the CW-decomposition given in [4] and notice that all homology classes are realizable by compact, genus two surfaces. General position is sufficient to compute the pairings. Unfortunately, Bigelow's $v_{i,j}$'s are not transverse. Instead, we compute the intersection product on $\mathrm{H}_2(\mathcal{L}\mathcal{C}_2\mathcal{P}_n)$ using the two transverse CW-decompositions, X and Y.

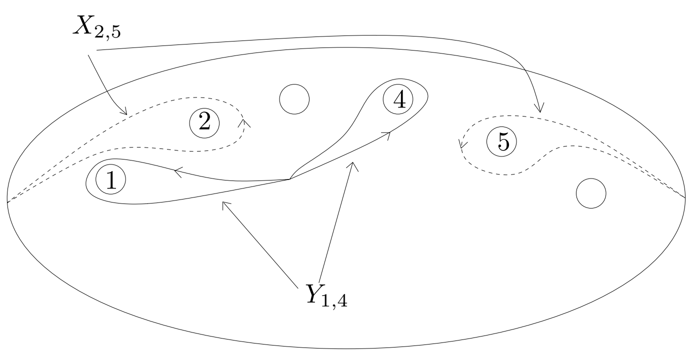
We lift the cells of X to $\mathcal{L}\mathcal{C}_2\mathcal{P}_n$ as we did for Y, that is, fix some lift of X^0 to $\mathcal{L}\mathcal{C}_2\mathcal{P}_n$ and call it the base-point. For every cell $X_{i,j}$, X_i and X_i' of X, we use the same notation to denote the lift of the cell in $\mathcal{L}\mathcal{C}_2\mathcal{P}_n$ whose attaching map starts at X^0.

Theorem 2. The intersection product $\langle v_{i,j}, v_{k,l}\rangle$ is given by the formula

$$-(1-t)(1+qt)(q-1)^2t^{-2}q^{-3}\begin{cases} -q^2t^2(q-1) & i=k<j<l \text{ or } i<k<j=l \\ -(q-1) & k=i<l<j \text{ or } k<i<j=l \\ t(q-1) & i<j=k<l \\ q^2t(q-1) & k<l=i<j \\ -t(q-1)^2(1+qt) & i<k<j<l \\ (q-1)^2(1+qt) & k<i<l<j \\ (1-qt)(1+q^2t) & k=i, j=l \\ 0 & \text{otherwise} \end{cases}$$

Proof. To see that the above formula is correct, notice that the two CW-decompositions given for $\mathcal{C}_2\mathcal{P}_n$ in Corollary 1 and Lemma 3 are transverse. In fact, the only cells of X that intersect cells of Y are the $\binom{n}{2}$ pairs, $X_{i,j}\cap Y_{i,j}$, which intersect in precisely four points (before lifting to the cover). This can be seen easily because the 2-cells X_i and X_i' are all contained in the 3-dimensional stratum of $\mathcal{C}_2\mathcal{P}_n$, and $X_{i,j}$ is disjoint from $Y_{k,l}$ unless $k=i$ and $j=l$ as in Figure 8. The disjointness observation comes from the Morse Theory of [13] – any cell in a CW-decomposition for a Morse function f can be realized in a very simple way: a Morse function near a critical point z has a local coordinate system where $f(x_1,\cdots,x_n) = -(x_1^2+\cdots+x_k^2)+(x_{k+1}^2+\cdots+x_n^2)+f(z)$, where k is the index of the critical point. Let D be a compact k-dimensional disc that corresponds to a neighborhood of 0 inside the subspace $x_{k+1}=\cdots=x_n=0$ in the above coordinate system. Then the cell corresponding to the critical point z consists of D union the forward orbit of ∂D under the flow of the negative gradient of the Morse function.

Figure 8



2-cells $Y_{1,4}$ and $X_{2,5}$, $n=6$.

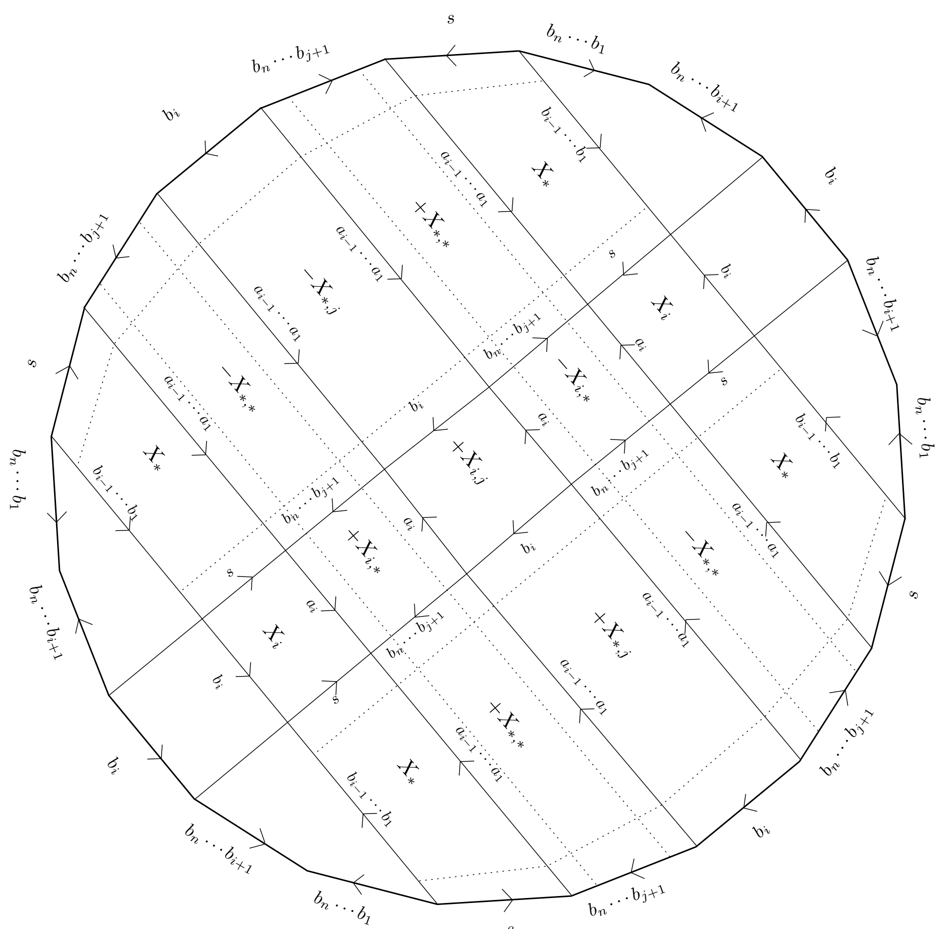
An easy computation gives $\langle X_{i,j}, Y_{i,j}\rangle = q^{n+3-(i+j)}(q-1)^2$. To compute $\langle v_{i,j}, v_{k,l}\rangle$ we represent $v_{i,j}$ in the X cellular homology and use the previous formula for $\langle X_{i,j}, Y_{k,l}\rangle$. Since both X and Y are CW-decompositions of deformation retractions of $\mathcal{C}_2\mathcal{P}_n$, they are canonically homotopy equivalent. The homotopy equivalence $Y\to X$ can be homotoped to a cellular map where $x_i\to(b_n\cdots b_{i+1})b_i^{-1}(b_n\cdots b_{i+1})^{-1}$, and $b\to s(b_n\cdots b_1)^{-1}$. Using Lemma 4, we can compute the homotopy equivalence on the 2-cells of Y as in the proof of Proposition 1.

$$Y_{i,j} = q^{-4}q^{i+j-n}X_{i,j} - \sum_{\substack{0<a<i\\ j<b<n+1}} q^{-5}(q-1)^2 q^{a+b-n}X_{a,b} + \sum_a l_a X_a +$$

$$\sum_{j<b<n+1} q^{-5}(q-1)q^{i+b-n}X_{i,b} + \sum_{0<a<i} -q^{-4}(q-1)q^{a+j-n}X_{a,j}$$

See Figure 9 for details on the above computation for $Y_{i,j}$.

Figure 9



Filling $\partial Y_{i,j} = [s(b_n \cdots b_1)^{-1} \to (b_n \cdots b_{i+1} \to b_i^{-1}), b_n \cdots b_{j+1} \to b_j]$

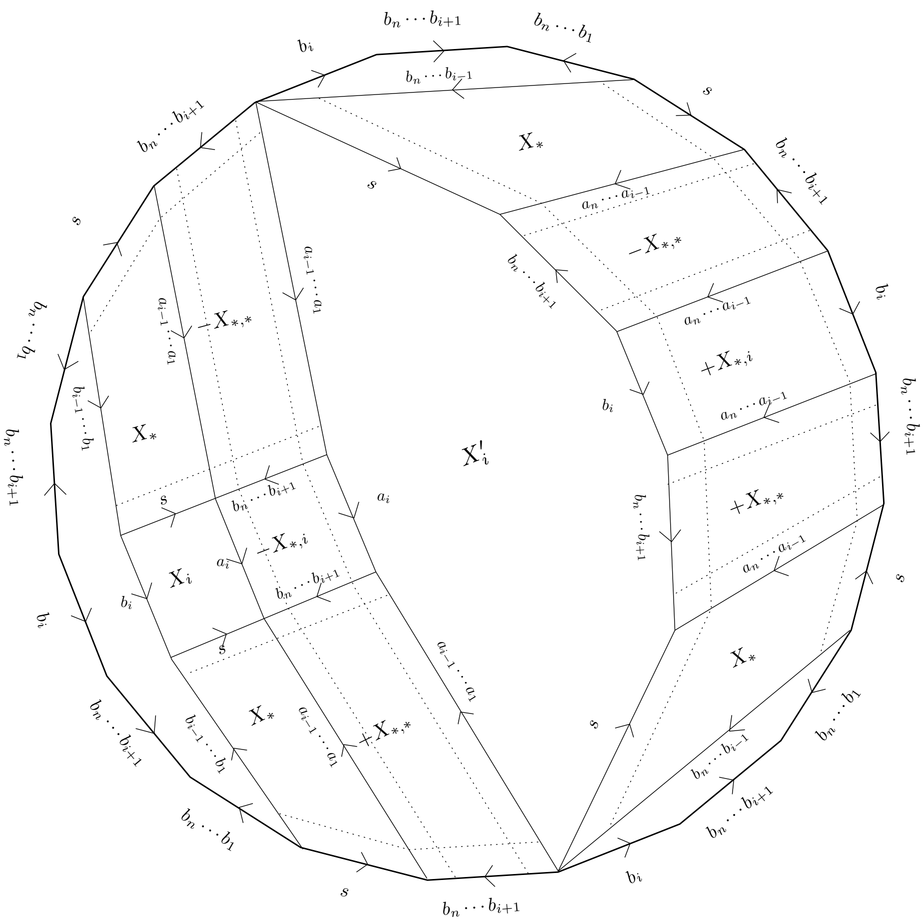
$$Y_i = -\sum_{\substack{0<a<i\\ i<b<n+1}} q^{-6}t^{-1}(q-1)(1+qt)q^{a+b-n}X_{a,b} + \sum_a p_a X_a +$$

$$\sum_{0<a<i} q^{-4}q^{a+i-n}X_{a,i} - \sum_{i<b<n+1} q^{-6}t^{-1}q^{i+b-n}X_{i,b}$$

See Figure 10 for details on the computation for Y_i.

There is no need to compute the coefficients l_a and p_a as they do not contribute to the intersection product.

Figure 10



$$\partial Y_i = [s(b_n \cdots b_1)^{-1}, (b_n \cdots b_{i+1} \to b_i^{-1})s(b_n \cdots b_1)^{-1}(b_n \cdots b_{i+1} \to b_i^{-1})]$$

□

Proposition 2. The intersection product is non-singular, or equivalently, the dual map $v \to \langle \cdot, v\rangle$ is injective.

Proof. We need to prove that the $\binom{n}{2} \times \binom{n}{2}$ matrix of coefficients $\langle v_{i,j}, v_{k,l}\rangle$ has rank $\binom{n}{2}$, thus it suffices to show that the determinant of this matrix is nonzero. Since $\mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$ is an integral domain (moreover it is a unique factorization domain [16]), we really only need to show that the determinant of the matrix

$M_{(i,j),(k,l)} = \frac{\langle v_{i,j}, v_{k,l}\rangle}{c_{p,q}}$ is non-zero where $c_{p,q} = -(1-t)(1+qt)(q-1)^2t^{-2}q^{-3}$. Notice the term in the determinant of M corresponding to the diagonal entries is precisely $\left((1-qt)(1+q^2t)\right)^{\binom{n}{2}}$. Similarly notice that every other term in the determinant of M is divisible by $(q-1)$. Since neither $(1-qt)$ or $(1+q^2t)$ are divisible by $(q-1)$, the determinant must be non-zero. □

Theorem 3. For appropriate choices of q and $t \in \mathbf{C}$ the intersection product is a negative-definite Hermitian form.

Proof. Take an arbitrary $v \in \mathcal{L}_n$ and compute $\langle v, v\rangle$. Let $v = \sum_{i,j} \lambda_{i,j} v_{i,j}$ for $\lambda_{i,j} \in \mathbf{C}$, and notice that for $|q| = |t| = 1$, $-(1-t)(1-q^2t^2)(q-1)^2(1+q^2t)t^{-2}q^{-3} \in \mathbf{R}$. By Theorem 2

$$\frac{\langle v, v\rangle}{-(1-t)(1-q^2t^2)(q-1)^2(1+q^2t)t^{-2}q^{-3}} = \sum_{i,j} \lambda_{i,j}\bar{\lambda}_{i,j} + 2\mathrm{Re}\left(\frac{q-1}{(1-qt)(1+q^2t)}k\right)$$

where

$$k = \sum_{\substack{a=c,b>d \\ or \\ b=d,c<a}} \lambda_{a,b}\bar{\lambda}_{c,d} + \sum_{a=d} q^2t\lambda_{a,b}\bar{\lambda}_{c,d} + \sum_{c<a<d<b} (q-1)(1+qt)\lambda_{a,b}\bar{\lambda}_{c,d}$$

therefore, for $|q-1| < \frac{1}{2n^4+6n^3}$ and $|t-i| < \frac{1}{2n^4+6n^3}$ the Lawrence-Krammer representation is definite. Note that $-(1-t)(1-q^2t^2)(q-1)^2(1+q^2t)t^{-2}q^{-3}$ is negative in this case.□

5. Conjugacy in the image

Since the Lawrence-Krammer representation is faithful, one may ask if it gives insight into the conjugacy problem for braid groups. One way to approach this would be via canonical forms of matrices. Given a braid $f \in B_n$ let f_* denote the action of f on $\mathcal{L}_n$. Unitary matrices can be diagonalized, and diagonal matrices are conjugate if an only if they have the same characteristic polynomial. If two matrices in the image of the Lawrence-Krammer representation are conjugate in $U_{\binom{n}{2}}$, are they conjugate by a matrix in the image of the Lawrence-Krammer representation? If the answer is yes, this would be an exceptionally fast solution to the conjugacy problem in braid groups. It turns out the answer is no, and this will be proved in Corollary 2.

The fact that the Lawrence-Krammer representation is unitary, together with its simple topological definition allows the proof of certain symmetry relations among the eigenvalues of f_* and the eigenvalues of the matrices of related braids. These symmetries, together with the existence of non-invertible knots [28] will be used to show that the characteristic polynomial does not separate conjugacy classes.

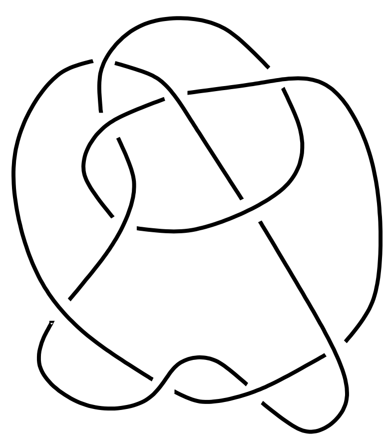
Given a braid $f \in B_n$ there is an associated braid cfc, where $c : \mathcal{P}_n \to \mathcal{P}_n$ is any orientation-reversing diffeomorphism of the punctured disc that fixes the n puncture points. c can be chosen to have order 2. If one thinks of $\mathcal{P}_n$ as the unit disc in the complex plane with puncture points along the real axis, then c can be taken to be complex conjugation. The map from $B_n \to B_n$ given by $f \to cfc$ is the only outer automorphism of the braid groups B_n [9].

Proposition 3. The matrices f_* and $(cf^{-1}c)_*$ are conjugate in $U_{\binom{n}{2}}$.

Proof. Notice that since c can be realized as complex conjugation on the punctured disc, c defines an involution of $\mathcal{C}_2\mathcal{P}_n$ which lifts to an involution of $\mathcal{L}\mathcal{C}_2\mathcal{P}_n$. Notice that the induced map c_* on $\mathcal{L}_n$ is not linear, in fact $c_*(t^a q^b v) = t^{-a}q^{-b}c_*(v)$. Therefore, $(cfc)_* = c_* f_* c_* = (c_*\tilde{I})\circ(\tilde{I}f_*\tilde{I})\circ(\tilde{I}c_*)$ is a composite of three $\mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$-linear maps, where $\tilde{I} : \mathcal{L}_n \to \mathcal{L}_n$ is the unique $\mathbf{Z}$-linear map such that $\tilde{I}(v_{i,j}) = v_{i,j}$ and $\tilde{I}(t^a q^b v) = t^{-a}q^{-b}\tilde{I}(v)$. Notice $\tilde{I}f_*\tilde{I} = \overline{f_*}$. This proves that $\overline{f_*}$ is conjugate to $(cfc)_*$. Since the Lawrence-Krammer representation is unitary, $\overline{f_*}$ is conjugate to $(f_*^{-1})^\tau$, where τ denotes the transpose operation. The matrix that conjugates the one to the other is the matrix of products $\langle v_{i,j}, v_{k,l}\rangle$. Therefore, $(cf^{-1}c)_*$ is conjugate to f_*^τ, but f_*^τ and f_* have the same characteristic polynomials and are therefore conjugate.□

If we think of the closed braids associated to the four braids f, cfc, $cf^{-1}c$ and f^{-1}, then the links associated to f and f^{-1} are mirror reflections of each other, and the links associated to cfc and f are also mirror reflections of each other. The mirror reflection $f \to f^{-1}$ changes the orientation of the knot, while $f \to cfc$ preserves the orientation. The oriented knots associated to f and $cf^{-1}c$ are inverses of each other.

Figure 11



The knot 10_{82}

Corollary 2. There exists matrices in the image of the Lawrence-Krammer representation that are conjugate, yet the braids they are associated to are not conjugate.

Trotter has shown that non-invertible knots exist [28]. Trotter's example is a rather complicated pretzel knot. With the advent of sophisticated computer algorithms such as Jeff Week's Snappea, simpler non-invertible knots have been found [14]. For example the hyperbolic knot that is denoted 10_{82} in Rolfsen's knot tables.

More generally, provided all of the Lawrence representations [19] [6] are unitary the above proof would go through to prove that f_* is conjugate to $(cf^{-1}c)_*$ for all braids f and all Lawrence representations. As mentioned earlier, implicit in the work of Long [21], the Lawrence representations all preserve a non-singular sesquilinear form and therefore the characteristic polynomials of f_* and $(cf^{-1}c)_*$ are the same for all Lawrence representations by the proof of Proposition 3.

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