# On the topological contents of $\eta$-invariants

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**Abstract**

We discuss an universal bordism invariant obtained from the Atiyah-Patodi-Singer $\eta$-invariant from the analytic and homotopy theoretic point of view. Classical invariants like the Adams $e$-invariant, $\rho$-invariants and $String$-bordism invariants are derived as special cases. The main results are a secondary index theorem about the coincidence of the analytic and topological constructions and intrinsic expressions for the bordism invariants.

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# 1 Introduction

This work is a mixture of a research article and a survey. Its purpose is to understand which topological information is encoded in the $\eta$-invariant, a spectral-geometric invariant introduced by Atiyah-Patodi-Singer [6] in the context of index theory for boundary value problems for Dirac-operators. By now there are many examples in the literature where the $\eta$-invariant has been used to define bordism invariants ([8],[12], [11], [21], [29], [42], [27], ...[1]). In the present paper we consider the universal structure behind these examples. We define a bordism invariant which we call the universal $\eta$-invariant. We use Section 5 in order to review some of the known $\eta$-invariant based bordism invariants. We put the emphasis on the demonstration how they can be interpreted as special cases of our universal construction. Though we have not checked it in detail our construction may also subsume (by constructions similar to the one in Subsection 5.5) other invariants of Kreck-Stolz type or Eeells-Kuiper type, [45], [30], or the generalized Rochin invariants of [48].
The universal $\eta$-invariant comes in two incarnations. The analytic version $\eta^{an}$ (Def. 3.5) is the bordism invariant which is easily derived from the appearance of the reduced $\eta$-invariant in the local index theorem for the Atiyah-Patodi-Singer boundary value problem by canceling out the dependence on geometric data. The ideas for this construction are more or less standard and have been used previously in many special situations.
The topological counterpart $\eta^{top}$ (Def. 2.2) is constructed by a simple homotopy-theoretic consideration using the interplay of $\mathbb{Q}/\mathbb{Z}$-bordism and $K$-theory. By their universal character the precise definitions of $\eta^{an}$ and $\eta^{top}$ are somewhat technical and will therefore not be reproduced in this introduction.
While it is not so complicated to see that $\eta^{an}$ is a bordism invariant, to understand its homotopy-theoretic meaning is slightly deeper. The bridge between analysis and topology

[1] A complete list would be too long to be given here!

is provided by our first main Theorem 3.6 stating that

$$\eta^{an} = \eta^{top} \ .$$

Its proof uses standard methods in index theory like the analytic picture of $K$-homology [14], [43], the Atiyah-Patodi-Singer index theorem [6], and some ideas from $\mathbb{Z}/l\mathbb{Z}$-index theory [33].
Bordism classes can be represented geometrically by manifolds with additional structures, called cycles (see Subsection 3.2 for details). It is then an interesting question how one can calculate the universal $\eta$-invariant or its specializations in terms of the cycle. The definition of both, the topological or the analytical version of the universal $\eta$-invariant involves the choice of a zero bordism of some multiple of the cycle. In applications it is often complicated to find such a zero bordism. It is a striking advantage of the analytic picture that it can be reorganized to an expression which only involves structures on the cycle itself. In special cases this has been previously exploited in [54], [29] (the case of the Adams $e$-invariant, see Subsection 5.1), and [21] (to calculate $String$-bordism invariants, see Subsection 5.4). We consider the intrinsic formula for $\eta^{an}$ given in Theorem 4.12 as one of the main original contributions of the present paper. This formula is based on a new object which we call a geometrization. We will sketch the main idea here and refer to Definition 4.2 for a precise description. We consider a map from a manifold to a space $f : M \to B$ and let $I : \hat{K}^0(M) \to K^0(M_+)$ denote the map which associates to a differential $K$-theory class (see Subsection 4.2 for a review) the underlying topological $K$-theory class. A geometrization of $(M, f)$ is a lift $\mathcal{G}$

$$\begin{tikzcd} & \hat{K}^0(M) \arrow[d, "I"] \\ K^0(B_+) \arrow[r, "f^*"] \arrow[ur, dotted, "\mathcal{G}"] & K^0(M_+) \end{tikzcd}$$

satisfying some additional properties explained in detail in Definition 4.2. If $\Gamma$ is a compact Lie group, then a map $f : M \to B\Gamma$ classifies a $\Gamma$-principal bundle on the manifold $M$. A connection on this principal bundle allows to define for every representation $\rho$ of $\Gamma$ a vector bundle $\mathbf{V}_\rho = (V_\rho, h^{V_\rho}, \nabla^{V_\rho})$ on $M$ with hermitean metric and connection by the associated bundle construction. Using the completion theorem [9] this construction extends to a geometrization

$$\mathcal{G} : K^0(B\Gamma_+) \to \hat{K}^0(M) \ , \quad [\rho] \mapsto [\mathbf{V}_\rho] \ .$$

The notion of a geometrization thus partially generalizes the notion of a connection on the in general non-existent principal bundle classified by the map $f : M \to B$. The details are slightly more complicated since we will take structures on the normal bundle into account.
In this paper we generally decided to work with complex $K$-theory. We think that there is a real version of the whole theory which can be obtained by replacing complex $K$-theory by

real $KO$-theory, $BSpin^c$ by $BSpin$, and taking the real structures on the spinor bundles into account properly on the analytic side. The real version of the universal $\eta$-invariant would be slightly stronger than its complex counterpart which loses some two-torsion classes. In order to recover the Adams $e$-invariant or the string bordism invariant [21] completely as special cases of the universal $\eta$-invariant we would need the real version.

Let us now describe the contents of the paper. In Section 2 we introduce the topological version $\eta^{top}$ of the universal $\eta$-invariant and study its properties. Most interesting is probably the relation with the Adams spectral sequence Proposition 2.9 and its consequence Corollary 2.10 which asserts (under the simplifying Assumption 2.6) that the universal $\eta$-invariant exactly detects the first non-trivial subquotient of the bordism theory with respect to the $K$-theory based Adams filtration.

In Section 3 we introduce the analytic version $\eta^{an}$ of the universal $\eta$-invariant and prove the secondary index Theorem 3.6 stating that $\eta^{an} = \eta^{top}$. Before we can define $\eta^{an}$ we have to recall in Subsections 3.2 and 3.3 some preliminary technical details concerning the relation of structures on the stable normal bundle as they come out of the Pontrjagin-Thom construction, and structures on the tangent bundle which will be used to do geometry and analysis.

Section 4 is devoted to geometrizations (Def. 4.2) and the intrinsic formula for $\eta^{an}$ (Thm. 4.12).

In the last Section 5 we discuss in detail various specializations of the universal $\eta$-invariant. It contains mainly a review of known constructions and results with slight improvements or generalizations at some points (e.g. Corollary 5.11). In the Propositions 5.12 and 5.13 we show how the usual geometric structures of *Spin*- and *String*-geometry (see [59] for the latter) give rise to geometrizations which lead to the known intrinsic formulas for the corresponding bordism invariants. It has been the initial motiviation for this work to understand the general principles behind the *String*-bordism invariants introduced in [21]. The arguments used here for the $String = MO\langle 8\rangle$-bordism case should easily be adaptable to bordism theories $MO\langle n\rangle$ associated to higher connected covers $BO\langle n\rangle$ of $BO$.

This paper is written for a reader with basic background in homotopy theory without being an expert. We will assume some knowledge on spectra and how they represent homology and cohomology theories. We decided, however, to explain some of the homotopy theoretic constructions and their translation to the geometric picture with more details as one would usually do in a paper written for topologists.

On the analytic side, in particular in the proof of Theorem 3.6, we assume a sound familiarity with $K$-theoretic arguments in index theory.

The theory of geometrizations and some of the examples in Section 5 require a certain experience with differential cohomology theories and elements of Chern-Weyl theory like transgressions etc.

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# 2 The topological construction

## 2.1 A rough description of $\eta^{top}$

In this section we introduce the homotopy theoretic version of the universal $\eta$-invariant of torsion classes in bordism theory. Before we come to technical details we will give a rough description of the idea. Any torsion class in the respective bordism group vanishes after rationalization. Therefore we can choose a preimage under the Bockstein operator assoziated to the sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0 \ .$$

This preimage is thus a $\mathbb{Q}/\mathbb{Z}$-bordism class of one degree higher. To this bordism class we apply the unit of complex $K$-theory in order to obtain a $\mathbb{Q}/\mathbb{Z}$-$K$-homology class of the corresponding bordism theory. It can be paired with $K$-cohomology classes in order to obtain values in $\mathbb{Q}/\mathbb{Z}$. In this way our original bordism class gives rise to a $\mathbb{Q}/\mathbb{Z}$-valued homomorphism from the $K$-cohomology of the considered bordism theory. The preimage of an element under the Bockstein operator is not uniquely determined, in general. In order to obtain a well-defined bordism invariant we therefore take the equivalence class of our homomorphism in a quotient of the group of $\mathbb{Q}/\mathbb{Z}$-valued homomorphisms from $K$-theory which is defined exactly such the ambiguity of the choice of that preimage does not matter any more.

Let us now describe the contents of the subsequent Subsections. In Subsection 2.2 we collect some basic facts from the homotopy theory of Thom spaces. An experienced reader may skip this section and immediately proceed to the construction of $\eta^{top}$ in Subsection 2.3. In Subsection 2.4 we analyse the target group of the universal $\eta$-invariant in greater detail. Subsection 2.5 is devoted to the relation between the universal $\eta$-invariant and the Adams spectral sequence. Finally, in Subsection 2.6 we study the functorial properties of the universal $\eta$-invariant with respect to transformations of the data.

## 2.2 Thom spaces and the Thom isomorphism

On the homotopy theoretic side bordism theories are represented by Thom spectra. In the following we briefly review the basic construction of Thom spectra. More details can be found e.g. in [53, Ch.IV].

If $V \to Y$ is a $k$-dimensional real vector bundle over a $CW$-complex $Y$ we define its Thom space $\mathtt{Th}(V)$ as the homotopy quotient

$$\mathtt{Th}(V) := V/_h(V \setminus Y) \ ,$$

where we identify $Y$ with the zero section of $V$. The associated Thom spectrum is then defined by

$$Y^V := \Sigma^{\infty-k}\mathtt{Th}(V) \ ,$$

where $\Sigma^\infty$ is the suspension spectrum functor, the left-adjoint of the adjoint pair

$$\Sigma^\infty : \mathtt{Top}_* \leftrightarrows \mathtt{Sp} : \Omega^\infty \qquad (1)$$

of functors between pointed topological spaces and spectra, and $\Sigma^{\infty-k}$ stands for the composition of $\Sigma^\infty$ with the $k$-fold down shift.
The following functoriality will be important. If $Y' \subseteq Y$ is an inclusion of a subcomplex carrying a vector bundle $V' \to Y'$ of dimension $k' \le k$, and we have fixed an identification

$$V_{|Y'} \cong V' \oplus (Y' \times \mathbb{R}^{k-k'}) \ ,$$

then we get an induced map $(Y')^{V'} \to Y^V$ of Thom spectra. This allows us to define the Thom spectra associated to the sequences of classical groups $(G(n))_{n\ge 1}$, where $G$ belongs to the list

$$\{U, SU, O, SO, Spin, Spin^c, Sp\} \ .$$

These groups come with families of stabilization maps

$$G(n) \to G(n+1) \ , \quad n \ge 1 \ .$$

We consider the homotopy colimit of the corresponding sequence of classifying spaces

$$BG := \mathtt{hocolim}_n\, BG(n) \ .$$

For any map $B \to BO$ from a $CW$-complex $B$ we define a Thom spectrum $MB$ as follows. The restriction of the map to a finite subcomplex $A$ has a factorization $A \to BO(n)$. It classifies an $\mathbb{R}^n$-bundle $V_A \to A$ which gives rise to the Thom spectrum $A^{V_A}$. The functoriality of the construction explained above provides natural maps $(A')^{V_{A'}} \to A^{V_A}$ for inclusions $A' \subseteq A \subseteq B$ of finite subcomplexes. The Thom spectrum associated to the map $B \to BO$ is then defined as

$$MB := \mathtt{hocolim}_A\, A^{V_A} \ ,$$

where $A$ runs over the finite subcomplexes of $B$.
For a map of $CW$-complexes over $BO$ with a given homotopy making the diagram

$$\begin{tikzcd} B' \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & BO & \end{tikzcd} \tag{2}$$

we get a homotopy class of maps of Thom spectra $MB' \to MB$. In this way the Thom spectrum construction provides a functor from the homotopy category of $\mathtt{CW}/_h BO$ of $CW$-complexes over $BO$ to the homotopy catgeory of spectra, where the subscript $h$ indicates that a morphism in this over-category is a triangle (2) filled by a specified homotopy.
For our theory it will be important that the bordism theory is $K$-oriented. The natural $K$-oriented bordism theory is $Spin^c$-bordism [5], [53, Ch. VI].
We take the flexibility of considering bordism theories which are $K$-oriented through a map to $Spin^c$-bordism.
We now come back to the description of our set-up. The compatible family of maps

$$Spin^c(n) \to O(n) \ , \quad n \ge 0$$

induces a map

$$BSpin^c \to BO \tag{3}$$

and therefore a Thom spectrum $MBSpin^c$. In order to keep a compatible notation in our paper we use this symbol for the Thom spectrum which would usually denoted with $MSpin^c$. We fix a map of spaces

$$\sigma : B \to BSpin^c \ . \tag{4}$$

It gives rise to a triangle

$$\begin{tikzcd} B \arrow[rr] \arrow[dr] & & BSpin^c \arrow[dl] \\ & BO & \end{tikzcd}$$

filled by the constant homotopy. It therefore induces a map of Thom spectra

$$M\sigma : MB \to MBSpin^c \ . \tag{5}$$

The homology theory represented by the spectrum $MB$ will be called $B$-bordism theory. If $Y$ is a pointed space or a spectrum, then the $B$-bordism groups of $Y$ are defined as $\pi_n(MB \wedge Y)$.
Let us note the following important special case where $B = *$ is the one-point space. In this case the Thom spectrum $S := MB$ is the sphere spectrum.
We let $K$ denote the complex $K$-theory spectrum. It is a commutative ring spectrum with unit and multiplication

$$\epsilon_K : S \to K \ , \quad mult \ : \ K \wedge K \to K \ . \tag{6}$$

By [5] the $Spin^c$-bordism theory is $K$-oriented. This orientation is given by a Thom class

$$\alpha : MBSpin^c \to K \ . \tag{7}$$

Its composition with (5) induces the $K$-orientation

$$\beta := M\sigma \circ \alpha : MB \to K \tag{8}$$

of $B$-bordism theory. It leads to Thom isomorphisms in homology and cohomology. We need some details of the construction of the Thom isomorphisms later in the proof of Theorem 3.6.
We start with the cohomological version. Let $Y$ be a pointed space or spectrum. If $E$ is another spectrum, then we let $E^*(Y)$ denote the $E$-cohomology of $Y$. Note that we use this notation for the reduced cohomology. If $Y$ is a space, then the unreduced $E$-cohomology of $Y$ will be given by $E^*(Y_+)$, where $Y_+$ is the disjoint union of $Y$ with an additional base point.
The Thom spectrum $MB$ is $K$-oriented by (8). We therefore have for any spectrum or pointed space $Y$ a Thom isomorphism

$$\mathtt{Thom}^K : K^*(B_+ \wedge Y) \xrightarrow{\sim} K^*(MB \wedge Y) \ . \tag{9}$$

Let us describe this map in detail. We consider the product $B \wedge B_+$ as a space over $BO$ via the projection onto the first factor. The diagonal $B \to B \wedge B_+$ then becomes a map over $BO$ and thus induces a map of Thom spectra

$$\Delta : MB \to M(B \wedge B_+) \cong MB \wedge B_+ \tag{10}$$

which is usually called the Thom diagonal. Let $\phi \in K^*(B_+ \wedge Y)$ be represented by

$$\phi : B_+ \wedge Y \to \Sigma^{-*}K \ .$$

Then the image of $\phi$ under the Thom isomorphism is represented by the composition

$$MB \wedge Y \stackrel{\Delta}{\to} MB \wedge B_+ \wedge Y \stackrel{\beta\wedge\phi}{\to} K \wedge \Sigma^{-*}K \stackrel{mult}{\to} \Sigma^{-*}K \ . \tag{11}$$

In the present paper we denote the $E$-homology of a pointed space or spectrum $Y$ by $\pi_*(E\wedge Y)$ (we do not use the notation $E_*(Y)$ since the swaps like $E\mathbb{Q}/\mathbb{Z}_*(Y) = E_*(Y\mathbb{Q}/\mathbb{Z})$ some times become notationally confusing). We must understand the Thom isomorphism

$$\mathtt{Thom}_K : \pi_*(K \wedge MB \wedge Y) \stackrel{\sim}{\to} \pi_*(K \wedge B_+ \wedge Y)$$

in $K$-homology in a similarly explicit way. It is induced by the composition

$$K \wedge MB \wedge Y \stackrel{\Delta}{\to} K \wedge MB \wedge B_+ \wedge Y \stackrel{\beta}{\to} K \wedge K \wedge B_+ \wedge Y \stackrel{mult}{\to} K \wedge B_+ \wedge Y \ . \tag{12}$$

Its precomposition with the unit $MB \wedge Y \stackrel{\epsilon_K}{\to} K \wedge MB \wedge Y$ coincides with

$$MB \wedge Y \stackrel{\Delta}{\to} MB \wedge B_+ \wedge Y \stackrel{\beta}{\to} K \wedge B_+ \wedge Y \ . \tag{13}$$

This gives the relation

$$\mathtt{Thom}_K(\epsilon_K(z)) = \beta(\Delta(z)) \in \pi_*(K \wedge B_+ \wedge Y) \tag{14}$$

for $z \in \pi_*(MB \wedge Y)$.

More generally, we have Thom isomorphisms for cohomology theories which are $K$-module theories. These isomorphisms will be denoted by the same symbols $\mathtt{Thom}^K$ and $\mathtt{Thom}_K$. They are defined by natural modifications of (12) and (11). We will apply this e.g. in the case of $K\mathbb{Q}/\mathbb{Z}$ or $K[[q]]$ which will be introduced later.

We now describe the pairing between homology and cohomology and derive some formulas clarifying its compatibility with the Thom isomorphisms. For spectra or pointed spaces $Y, Z$ we have a pairing between the $K$-theory $K^*(Y)$ and the $K$-homology $\pi_*(K \wedge Y \wedge Z)$ with values in $\pi_*(K \wedge Z)$. If the homology and cohomology classes are represented by maps

$$\phi : Y \to \Sigma^{-i}K \ , \quad y : \Sigma^j S \to K \wedge Y \wedge Z \ ,$$

then the pairing is given by the composition

$$\langle \phi, y \rangle : \Sigma^{i+j} S \stackrel{y}{\to} \Sigma^i K \wedge Y \wedge Z \stackrel{\phi}{\to} K \wedge K \wedge Z \stackrel{mult}{\to} K \wedge Z \ . \tag{15}$$

It turns out to be useful to consider the cohomology groups of a space or spectrum as topological groups. The topology on a cohomology group is determined by the system of neighbourhoods of the identity given by the system of kernels of restrictions to finite complexes or spectra. This is the profinite[2] topology defined in [18, Def. 4.9]. By $\mathtt{Hom}^{cont}$ we mean continuous homomorphisms into the target which in this paper is always a discrete group. We will interpret the evaluation pairing described above as a map

$$\pi_j(K \wedge Y \wedge Z) \to \mathtt{Hom}^{cont}(K^*(Y), \pi_{*+j}(K \wedge Z)) \ . \tag{16}$$

For later use let us consider the situation where $y \in \pi_j(K \wedge MB \wedge Y \wedge Z)$ and $\phi \in K^i(B_+ \wedge Y)$. Then we have the relation

$$\langle \mathtt{Thom}^K(\phi), y\rangle = \langle \phi, \mathtt{Thom}_K(y)\rangle \in \pi_{j+i}(K \wedge Z) \ . \tag{17}$$

Let us further assume that $y = \epsilon_K(z)$ for $z \in \pi_j(MB \wedge Y \wedge Z)$. Then we can apply the orientation $\beta : MB \to K$ to $\Delta(z) \in \pi_j(MB \wedge B_+ \wedge Y \wedge Z)$. Combining (14) and (17) we get the following equality

$$\langle \mathtt{Thom}^K(\phi), \epsilon_K(z)\rangle = \langle \phi, \beta(\Delta(z))\rangle \in \pi_{j+i}(K \wedge Z) \ . \tag{18}$$

## 2.3 The definition of $\eta^{top}$

On the level of spectra we define rationalizations and $\mathbb{Q}/\mathbb{Z}$-versions of a homology theory using Moore spectra. For an abelian group $A$ we let $\mathtt{M}A$[3] denote the Moore spectrum of $A$. It is characterized by the property that

$$\pi_n(H\mathbb{Z} \wedge \mathtt{M}A) \cong \begin{cases} A & n = 0 \\ 0 & n \neq 0 \end{cases} \ .$$

More generally, for a spectrum $Y$ we have exact sequences

$$0 \to \pi_n(Y) \otimes A \to \pi_n(Y \wedge \mathtt{M}A) \to \mathtt{Tor}(\pi_{n-1}(Y), A) \to 0 \tag{19}$$

([19, (2.1)]) for all $n \in \mathbb{Z}$. For cyclic groups $A$ a construction of $\mathtt{M}A$ is indicated in Subsection 3.5. Note that $\mathtt{M}\mathbb{Z}$ and $\mathtt{M}\mathbb{Q}$ are equivalent to the sphere spectrum $S$ and the Eilenberg-MacLane spectrum $H\mathbb{Q}$, respectively. For any spectrum $E$ we abbreviate $EA := E \wedge \mathtt{M}A$. For example, we have an equivalence $E\mathbb{Z} \cong E$. We let $M\mathbb{Z} \to M\mathbb{Q}$ be the map of Moore spectra induced by the inclusion $\mathbb{Z} \to \mathbb{Q}$. It extends to a fibre sequence

$$\mathtt{M}\mathbb{Z} \to \mathtt{M}\mathbb{Q} \to \mathtt{M}\mathbb{Q}/\mathbb{Z} \to \Sigma\mathtt{M}\mathbb{Z} \tag{20}$$

of Moore spectra.

Our invariant will be defined on the torsion elements of the $B$-bordism theory of a space $X$ (or a spectrum $\mathcal{X}$ as explained later in Subsection 2.6).

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[2] Note that this topology is in general not the profinite topology in the sense of group theory!

[3] Please do not confuse the letter $\mathtt{M}$ used for Moore spectra and $M$ used to denote Thom spectra.

Smashing the fibre sequence of Moore spectra (20) with $MB \wedge X_+$ and $K \wedge MB \wedge X_+$ gives the vertical sequences in the following diagram:

$$\begin{tikzcd}
\Sigma^{-1} MB\mathbb{Q} \wedge X_+ \arrow[r] \arrow[d] & \Sigma^{-1} K \wedge MB\mathbb{Q} \wedge X_+ \arrow[d] \\
{}^{\hat{x}}\ \Sigma^{-1} MB\mathbb{Q}/\mathbb{Z} \wedge X_+ \arrow[r] \arrow[d] & {}^{\tilde{x}}\ \Sigma^{-1} K \wedge MB\mathbb{Q}/\mathbb{Z} \wedge X_+ \arrow[d] \\
{}^{x}\ MB \wedge X_+ \arrow[r] \arrow[d] & K \wedge MB \wedge X_+ \arrow[d] \\
{}^{0}\ MB\mathbb{Q} \wedge X_+ \arrow[r] & K \wedge MB\mathbb{Q} \wedge X_+
\end{tikzcd} \quad . \tag{21}$$

The horizontal arrows are all induced by the unit (6) of $K$-theory.
The little letters in diagram (21) denote elements which will be chased through this diagram. We consider a torsion class $x \in \pi_n(MB \wedge X_+)_{tors}$. It has a lift $\hat{x} \in \pi_{n+1}(MB\mathbb{Q}/\mathbb{Z} \wedge X_+)$ which is well-defined up to classes in the image of $\pi_{n+1}(MB\mathbb{Q} \wedge X_+)$. This lift $\hat{x}$ maps to a class $\tilde{x} \in \pi_{n+1}(K \wedge MB\mathbb{Q}/\mathbb{Z} \wedge X_+)$. This element is well-defined up to the image of $\pi_{n+1}(MB\mathbb{Q} \wedge X_+)$. We thus have defined a homomorphism $x \mapsto \tilde{x}$

$$\pi_n(MB \wedge X_+)_{tors} \to \frac{\pi_{n+1}(K \wedge MB\mathbb{Q}/\mathbb{Z} \wedge X_+)}{\mathtt{image}(\pi_{n+1}(MB\mathbb{Q} \wedge X_+))} \ . \tag{22}$$

We now specialize the evaluation pairing (16) to our situation. We consider the case where $Z = M\mathbb{Q}/\mathbb{Z}$ and $Y = MB \wedge X_+$. We furthermore precompose with the Thom isomorphism (9) in $K$-theory and get a homomorphism

$$\pi_{n+1}(K \wedge MB\mathbb{Q}/\mathbb{Z} \wedge X_+) \to \mathtt{Hom}^{cont}(K^0(B_+ \wedge X_+), \pi_{n+1}(K\mathbb{Q}/\mathbb{Z})) \ . \tag{23}$$

We let

$$U \subseteq \mathtt{Hom}^{cont}(K^0(B_+ \wedge X_+), \pi_{n+1}(K\mathbb{Q}/\mathbb{Z})) \tag{24}$$

denote the subgroup given by the pairings with the elements in $\mathtt{image}(\pi_{n+1}(MB\mathbb{Q} \wedge X_+))$. The following group plays a central role in the present paper as the target of our universal $\eta$-invariant.

**Definition 2.1** *We define*

$$Q_n(B, X) := \frac{\mathtt{Hom}^{cont}(K^0(B_+ \wedge X_+), \pi_{n+1}(K\mathbb{Q}/\mathbb{Z}))}{U} \ . \tag{25}$$

Note that $Q_n(B, X)$ also depends on the choice of the map $\sigma$ in (4), but this dependence will be suppressed from the notation. Note that $Q_n(B, X)$ is a torsion group. Indeed, by continuity every element of $\mathtt{Hom}^{cont}(K^0(B_+ \wedge X_+), \pi_{n+1}(K\mathbb{Q}/\mathbb{Z}))$ factors over a finitely generated quotient of $K^0(B_+ \wedge X_+)$. Since it has values in a torsion group it is of finite order.

Composing (22) with (23) and with the projection to the quotient by $U$ we obtain a homomorphism

$$\eta^{top} : \pi_n(MB \wedge X_+)_{tors} \to Q_n(B, X) \ . \tag{26}$$

Let us collect the essentials of this construction in the following definition.

**Definition 2.2** *The homotopy theoretic version of the universal $\eta$-invariant is the homomorphism*

$$\eta^{top} : \pi_n(MB \wedge X_+)_{tors} \to Q_n(B, X)$$

*defined by the following prescription: If $x \in \pi_n(MB \wedge X_+)_{tors}$, then we choose a lift $\hat{x} \in \pi_{n+1}(MB\mathbb{Q}/\mathbb{Z} \wedge X_+)$ whose pairing with $K^0(B_+ \wedge X_+)$ represents the class $\eta^{top}(x) \in Q_n(B, X)$.*

## 2.4 Simplification of $Q_n(B, X)$

The target of our universal $\eta$-invariant is the group $Q_n(B, X)$ defined in (25). The universal $\eta$-invariant is designed to detect torsion elements in the bordism group $\pi_n(MB \wedge X_+)$. For an effective use of the universal $\eta$-invariant we have to know the group $Q_n(B, X)$. To this end first of all we must know the $K$-theory $K^0(B_+ \wedge X_+)$. In many examples this $K$-theory is a lot easier than the bordism theory. So take for example $B = *$ and $X = *$. Then $\pi_n(MB \wedge X_+) = \pi_n^S$ are the stable homotopy groups of the sphere, a central and very complicated object of stable homotopy theory which has not been calculated completely so far. On the other hand $K^0(B_+ \wedge X_+) \cong \mathbb{Z}$ is known. Another interesting example is the case where $B = BG$ for $G \in \{U, SU, Spin, Spin^c\}$ and $X = B\Gamma$ for another compact Lie group $\Gamma$. In this case we can calculate $K^0(B_+ \wedge X_+)$ using the completion theorem [9] in terms of the representation rings

$$K^0(B_+ \wedge X_+) \cong \mathtt{lim}_n \, R(G(n) \times \Gamma)\hat{}_{I_n} \ , \tag{27}$$

where $I_n \subset R(G(n) \times \Gamma)$ is the augmentation ideal.
In addition to the $K$-theory group $K^0(B_+ \wedge X_+)$ the other ingredient of the construction of $Q_n(B, X)$ is the subgroup $U$ given in (24). Its calculation requires the knowledge of the rationalization of $\pi_{n+1}(MB \wedge X_+)$ which is easy in some cases, but may be complicated in others.
One goal of the present section is to give a simplified picture of the group $Q_n(X, B)$. This picture turns out to be quite useful when we try to construct maps out of this group to simpler targets. We make a simplifying assumption.

**Assumption 2.3** *We assume that $B_+ \wedge X_+$ is rationally of finite type, i.e.* $\dim H^i(B_+ \wedge X_+; \mathbb{Q})$ *is finite for all* $\in \mathbb{N}$.

This assumption is satisfied for eample, if $B$ and $X$ are of finite type. This means that they are homotopy equivalent to $CW$-complexes whose skeleta are are build with finitely many cells.

Let

$$HP\mathbb{Q} := \bigvee_{i\in\mathbb{Z}} \Sigma^{2i} H\mathbb{Q} \tag{28}$$

be the spectrum which represents 2-periodic rational homology. For each $i \in \mathbb{Z}$ we have a projection $p_{2i} : HP\mathbb{Q} \to \Sigma^{2i}H\mathbb{Q}$ to the corresponding component. We set $p_n := 0$ for odd $n$. It is useful to write $HP\mathbb{Q} \cong H\mathbb{Q}[b, b^{-1}]$, where $\deg(b) = 2$.
The Chern character is an equivalence

$$\mathbf{ch} : K\mathbb{Q} \xrightarrow{\sim} HP\mathbb{Q} \ .$$

We furthermore need the Todd class $\mathbf{Td} \in HP\mathbb{Q}^0(BSpin^c_+)$ which we describe in the following passage. The splitting principle gives a way to define multiplicative characteristic classes for real vector bundle from even formal power series (see [9] for a detailed account). For a complex line bundle $L \to X$ we let $c_1(L) \in H^2(X;\mathbb{Z})$ denote its first Chern class. We consider a formal power series

$$K(x) = 1 + x^2a_1 + x^4a_2 + \cdots \in \mathbb{Q}[b, b^{-1}][[x^2]]$$

of total degree zero where $\deg(x) = 2$ and $a_i \in \mathbb{Q}[b, b^{-1}]$ has degree $-2i$. Then there exists a unique multiplicative characteristic class which associates to the real vector bundle $V \to X$ the class $K(V) \in HP\mathbb{Q}^0(X_+)$ such that $K(L_\mathbb{R}) = K(ic_1(L))$, where $L_\mathbb{R}$ is $L$ considered as a real bundle. Multiplicativity here refers to the property that

$$K(V \oplus V') = K(V) \cup K(V') \ .$$

The characteristic class associated to the formal power series

$$\frac{\frac{x}{2b}}{\sinh(\frac{x}{2b})} \in \mathbb{Q}[b, b^{-1}][[x^2]]$$

is usually denoted by $\hat{\mathbf{A}}(V)$. By the multiplicativity of $\hat{\mathbf{A}}$ the family of classes $(\hat{\mathbf{A}}(\xi_n) \in HP\mathbb{Q}^0(BO(n)_+))_{n\geq 1}$ of the universal bundles $\xi_n \to BO(n)$ is compatible with restriction along the maps $BO(n) \to BO(n+1)$ and therefore gives rise to a class $\hat{\mathbf{A}} \in HP\mathbb{Q}^0(BO_+)$ which restricts to the classes $\hat{\mathbf{A}}(\xi_n)$ for all $n$. In order to simplify the notation we use the same symbol $\hat{\mathbf{A}}$ in order to denote the pull-back of this class along a map $e : B \to BO$, i.e. we will write $\hat{\mathbf{A}} \in HP\mathbb{Q}^0(B_+)$ for instead of $e^*\hat{\mathbf{A}}$. In particular we have a class

$$\hat{\mathbf{A}} \in HP\mathbb{Q}^0(BSpin^c_+) \ .$$

The group $Spin^c(n)$ has a natural character $\chi_n : Spin^c(n) \to U(1)$ whose restriction to the center $U(1) \cong Z(Spin^c(n))$ is the two-fold covering. This character determines a line bundle $L_n \to BSpin^c(n)$ and therefore a class $c_1(L_n) \in H^2(BSpin^c(n);\mathbb{Z})$. The collection of classes $(c_1(L_n))_{n\geq 1}$ is compatible with respect to the restrictions along the family of maps $BSpin^c(n) \to BSpin^c(n+1)$ and therefore determines a class

$$c_1 \in H^2(BSpin^c;\mathbb{Z})$$

which restricts to the classes $c_1(L_n)$ for all $n \geq 1$. The Todd class is defined as the class

$$\mathbf{Td} := \hat{\mathbf{A}} \cup \exp(\frac{c_1}{2b}) \in HP\mathbb{Q}^0(BSpin^c_+) \ .$$

The Todd class is a unit in the cohomology ring so that $\mathbf{Td}^{-1} \in HP\mathbb{Q}^0(BSpin^c)$ is defined as well.
The importance of the Todd class comes from its role in the compatibility of the Thom isomorphism for $Spin^c$-vector bundles and the Chern character. Let $\mathtt{Thom}^K$denote the Thom isomorphism (9). There is an ordinary orientation

$$MB \to MBSpin^c \to MBSO \to HP\mathbb{Q}$$

of $B$ bordism theory which induces a Thom isomorphism

$$\mathtt{Thom}^{HP\mathbb{Q}} : HP\mathbb{Q}^*(B_+ \wedge X_+) \stackrel{\sim}{\to} HP\mathbb{Q}^*(MB \wedge X_+)$$

in periodic rational cohomology. Then we have the relation in $HP\mathbb{Q}^*(MB \wedge X_+)$

$$\mathbf{ch} \circ \mathtt{Thom}^K(\phi) = \mathtt{Thom}^{HP\mathbb{Q}}(\mathbf{ch}(\phi) \cup \mathbf{Td}^{-1}) \tag{29}$$

for all $\phi \in K^*(B_+ \wedge X_+)$.
The composition of the Chern character with multiplication by the inverse of the universal Todd class and the projection $p_{n+1}$ gives a map

$$p_{n+1}(\mathbf{Td}^{-1} \cup \mathbf{ch}(\dots)) : K^0(B_+ \wedge X_+) \to H\mathbb{Q}^{n+1}(B_+ \wedge X_+) \ .$$

We consider the kernel

$$V := \mathtt{ker} \left( K^0(B_+ \wedge X_+) \stackrel{p_{n+1}(\mathbf{Td}^{-1} \cup \mathbf{ch}(\dots))}{\longrightarrow} H\mathbb{Q}^{n+1}(B_+ \wedge X_+) \right) \ . \tag{30}$$

We now formulate the desired simplification of the description of the group $Q_n(B, X)$.

**Lemma 2.4** *The restriction to $V \subseteq K^0(B_+ \wedge X_+)$ induces a well-defined map*

$$Q_n(B, X) \to \mathtt{Hom}^{cont}(V, \pi_{n+1}(K\mathbb{Q}/\mathbb{Z})) \tag{31}$$

*which is an isomorphism if we assume 2.3.*

*Proof.* First we show that the restriction is well-defined. We must show that if $\phi \in V$, then the pairing of $\phi$ with an element of $\pi_{n+1}(K \wedge MB\mathbb{Q}/\mathbb{Z} \wedge X_+)$ coming from $y \in \pi_{n+1}(MB\mathbb{Q} \wedge X_+)$ vanishes. This pairing can be calculated as the image in $\pi_{n+1}(K\mathbb{Q}/\mathbb{Z})$ of the evaluation pairing (15) (with $Y = MB \wedge X_+$ and $Z = M\mathbb{Q}$)

$$\langle \mathtt{Thom}^K(\phi), \epsilon_K(y) \rangle \in \pi_{n+1}(K\mathbb{Q}) \cong \mathbb{Q} \ ,$$

where $\epsilon_K$ is the map induced by the unit of $K$-theory. The identification of $\pi_{n+1}(K\mathbb{Q}) \cong \mathbb{Q}$ can be viewed as given by the application of $p_{n+1} \circ \mathbf{ch} : \pi_{n+1}(K\mathbb{Q}) \to \pi_0(H\mathbb{Q}) \cong \mathbb{Q}$. We therefore can write

$$\langle \mathtt{Thom}^K(\phi), \epsilon_K(y) \rangle = p_{n+1}(\langle \mathbf{ch} \circ \mathtt{Thom}^K(\phi), \mathbf{ch}(\epsilon_K(y)) \rangle) \ .$$

Note that

$$p_k(\mathbf{ch}(\epsilon_K(y))) = p_k(\epsilon_{HP\mathbb{Q}}(y)) = \begin{cases} 0 & k \neq 0 \\ \epsilon_{H\mathbb{Q}}(y) & k = 0 \end{cases} \ ,$$

where $\epsilon_{HP\mathbb{Q}}$ and $\epsilon_{H\mathbb{Q}}$ is induced by the unit of periodic and non-periodic rational homology. Therefore by equation (29) and the fact that $p_{n+1} \circ \mathtt{Thom}^{HP\mathbb{Q}} = \mathtt{Thom}^{H\mathbb{Q}} \circ p_{n+1}$ we obtain

$$\langle \mathtt{Thom}^K(\phi), \epsilon_K(y) \rangle = \langle \mathtt{Thom}^{H\mathbb{Q}}(p_{n+1}(\mathbf{Td}^{-1} \cup \mathbf{ch}(\phi))), \epsilon_{H\mathbb{Q}}(y) \rangle \ . \tag{32}$$

If $\phi \in V$, then the right-hand side of this equality vanishes. This shows that the restriction map (31) is well-defined.

We now show that it is an isomorphism under Assumption 2.3. We use the general fact that if $f : A \to V$ is a homomorphism of an abelian group into a $\mathbb{Q}$-vector space such that its image is finitely generated as an abelian group, then there exists a splitting $A \cong \mathtt{ker}(f) \oplus A'$. Indeed, in this case the image is free and hence projective.

Note that there exists an integer $N$ (only depending on $n$) such that the image of $p_{n+1}(\mathbf{Td}^{-1} \cup \mathbf{ch}(\dots))$ is contained in $\frac{1}{N} H^{n+1}(B_+ \wedge X_+; \mathbb{Z}) \subseteq H^{n+1}(B_+ \wedge X_+; \mathbb{Q})$ and is therefore finitely generated as an abelian group since we assume that $B_+ \wedge X_+$ is rationally of finite type. We conclude that

$$K^0(B_+ \wedge X_+) \cong V \oplus V^c \ ,$$

where $V^c \cong \mathtt{image}(p_{n+1}(\mathbf{Td}^{-1} \cup \mathbf{ch}(\dots))$ is a free abelian group. This immediately implies that (31) is surjective.

Any homomorphism $\phi \in \mathtt{Hom}^{cont}(K^0(B_+ \wedge X_+), \pi_{n+1}(K\mathbb{Q}/\mathbb{Z}))$ can uniquely be decomposed as a sum of its restrictions to $V$ and $V^c$. We claim that $U \cong \mathtt{Hom}(V^c, \pi_{n+1}(K\mathbb{Q}/\mathbb{Z}))$, where $U$ is as in (24). The claim implies that (31) is injective.

A homomorphism $f : V^c \to \pi_{n+1}(K\mathbb{Q}/\mathbb{Z})$ can be lifted to a homomorphism $\hat{f} : V^c \to \pi_{n+1}(K\mathbb{Q})$ since $V^c$ is free. This lift further extends uniquely to an homomorphism of $\mathbb{Q}$-vector spaces $\hat{f}_{\mathbb{Q}} : V^c \otimes \mathbb{Q} \to \pi_{n+1}(K\mathbb{Q})$. Via the Chern character we can view $V^c \otimes \mathbb{Q}$ as a subspace of $H\mathbb{Q}^{n+1}(B_+ \wedge X_+)$. Hence there exists a homomorphism $\tilde{f} : H\mathbb{Q}^{n+1}(B_+ \wedge X_+) \to \mathbb{Q}$ which restricts to $\hat{f}_{\mathbb{Q}}$. We interpret $\tilde{f}$ as a homology class $\tilde{f} \in \pi_{n+1}(H\mathbb{Q} \wedge B_+ \wedge X_+)$. We now use the identification $\mathtt{M}\mathbb{Q} \cong H\mathbb{Q}$ and the Thom isomorphism for rational homology in order to get an isomorphism $\pi_{n+1}(MB\mathbb{Q} \wedge X_+) \cong \pi_{n+1}(H\mathbb{Q} \wedge B_+ \wedge X_+)$. It follows that any homomorphism $V^c \to \pi_{n+1}(K\mathbb{Q}/\mathbb{Z})$ can be obtained as evaluation against a class in $\pi_{n+1}(MB\mathbb{Q} \wedge X_+)$. □

The definition of $\eta^{top}$ is based on first lifting the torsion element in the bordism group to an $\mathbb{Q}/\mathbb{Z}$-bordism element which is then paired with elements of $K$-theory. The pairing with torsion $K$-theory elements can be expressed in a dual way as a pairing of the original bordism class with $\mathbb{Q}/\mathbb{Z}$-lifts of the $K$-theory elements. We now explain the details. Assume that $\phi \in K^0(B_+ \wedge X_+)$ satisfies $\mathbf{ch}(\phi) = 0$. Then $\phi \in V$ and we get an evaluation

$$\mathtt{ev}_\phi : Q_n(B, X) \to \pi_{n+1}(K\mathbb{Q}/\mathbb{Z}) \ .$$

In view of the exact sequence

$$K\mathbb{Q}/\mathbb{Z}^{-1}(B_+ \wedge X_+) \xrightarrow{\partial} K^0(B_+ \wedge X_+) \xrightarrow{\mathbf{ch}} HP\mathbb{Q}(B_+ \wedge X_+)$$

we can choose $\hat{\phi} \in K\mathbb{Q}/\mathbb{Z}^{-1}(B_+ \wedge X_+)$ such that $\partial\hat{\phi} = \phi$. If we want to calculate $\mathtt{ev}_\phi(\eta^{top}(x))$, then instead of lifting the class $x$ to a $\mathbb{Q}/\mathbb{Z}$ class we can instead evaluate the class $\epsilon_K(x) \in \pi_n(K \wedge MB \wedge X_+)$ against the lift $\hat{\phi}$. Indeed, using that $K\mathbb{Q}/\mathbb{Z}$ is a $K$-module spectrum we get a Thom isomorphism

$$\mathtt{Thom}^K : K\mathbb{Q}/\mathbb{Z}^*(B_+ \wedge X_+) \xrightarrow{\sim} K\mathbb{Q}/\mathbb{Z}^*(MB \wedge X_+) \ .$$

The following assertion follows easily from the definition of $\eta^{top}$ and commutativity of the diagram

$$\begin{tikzcd} B_+ \wedge X_+ \wedge \mathtt{M}\mathbb{Q}/\mathbb{Z} \arrow[r, "\phi\wedge\mathtt{id}_{\mathtt{M}\mathbb{Q}/\mathbb{Z}}"] \arrow[d, "\partial"] & K\mathbb{Q}/\mathbb{Z} \arrow[d, equal] \\ B_+ \wedge X_+ \arrow[r, "\hat{\phi}"] & K\mathbb{Q}/\mathbb{Z} \end{tikzcd} \ .$$

**Lemma 2.5** *For $x \in \pi_n(MB \wedge X_+)_{tors}$ we have*

$$\mathtt{ev}_\phi(\eta^{top}(x)) = \langle \mathtt{Thom}^K(\hat{\phi}), \epsilon_K(x) \rangle \ .$$

This Lemma will play a role when we compare the universal $\eta$-invariant of the present paper with other classical secondary invariants, e.g. in Subsection 5.2.

## 2.5 Relation with the Adams spectral sequence

A classical approach to a calculation of the homotopy groups $\pi_*(MB \wedge X_+)$ uses the Adams spectral sequence. We refer to [2] and [52] for a detailed description of this method. For our purpose the Adams spectral sequence $(E_r, d_r)$ based on complex $K$-theory is of particular importance. It does not really calculate the $B$-bordism groups $\pi_*(MB \wedge X_+)$ of $X$, but rather the homotopy groups $\pi_*((MB \wedge X_+)_K)$ of the Bousfield localization of $MB \wedge X_+$ with respect to $K$-theory [19].
In order to understand the strength of the universal $\eta$-invariant it is interesting to understand its relation with the $K$-theory based Adams spectral sequence. In this subsection we will show that the universal $\eta$-invariant factorizes over the first line of this spectral sequence and exactly detects the first subquotient of $\pi_n(MB \wedge X_+)$ with respect to the associated Adams filtration. The main result of the present Subsection will be formulated

as Proposition 2.9 and Corollary 2.10. It is relevant if one wants to calculate $\eta^{top}(x)$ for an element $x \in \pi_n(MB \wedge X_+)_{tors}$ about which one knows the element in $[x] \in E_2^{1,n+1}$ which detects $x$ (also called the symbol of $x$). This approach has been used to calculate invariants of *String*-bordism classes in [21], see Subsection 5.4.
This relation between $\eta^{top}$ and the $K$-theory based Adams spectral sequence is particularly clean under some simplifying assumtions which we will adopt here.

**Assumption 2.6** *1. We assume that $\pi_*(K \wedge MB \wedge X_+)$ is torsion free.*

*2. We further assume that the groups $E_2^{s,t}(MB \wedge X_+)$ are finite for all $t \in \mathbb{Z}$ and $s \geq 1$.*

These assumptions hold true e.g. if $B \in \{*, BU, BSU, BSpin, BSpin^c\}$ and $X = B\Gamma$ for some compact Lie group. In order to see 1. in this case note that by the Thom isomorphism

$$\pi_*(K \wedge MB \wedge X_+) \cong \pi_*(K \wedge B_+ \wedge X_+) \ .$$

The right-hand side vanishes in odd degree and is given by (27) for even degree.
Let us now describe the Adams filtration. Let $Y$ be a spectrum or pointed space. The unit $\epsilon_K$ of $K$-theory extends to a fibre sequence

$$\Sigma^{-1}\bar{K} \xrightarrow{\delta} S \xrightarrow{\epsilon_K} K \to \bar{K}$$

of spectra which in particular determines the spectrum $\bar{K}$ and the map $\delta$. The latter induces the maps in the following sequence

$$\Sigma^{-k}\bar{K}^{\wedge^k} \wedge Y \to \Sigma^{-(k-1)}\bar{K}^{\wedge^{k-1}} \wedge Y \to \cdots \to \Sigma^{-1}\bar{K} \wedge Y \to Y$$

which is called the Adams tower. The Adams filtration measures how far elements in the homotopy group of $Y$ can be lifted in the Adams tower. More precisely, an element $x \in \pi_n(Y)$ belongs to the step $F^k\pi_n(Y)$ if it can be lifted to an element in $\pi_{n+k}(\bar{K}^{\wedge^k} \wedge Y)$ in the Adams tower. In this way we obtain the decreasing Adams filtration

$$F^k\pi_n(Y) \subseteq F^{k-1}\pi_n(Y) \subseteq \cdots \subseteq F^1\pi_n(Y) \subseteq \pi_n(Y)$$

on $\pi_n(Y)$.
We want to apply the geometric boundary theorem [52, Thm. 2.3.4] and the discussion of [21, Sec. 5.3]. The Assumption 2.6 implies the assumption for the geometric boundary theorem, namely that

$$\pi_{*+1}(K \wedge MB\mathbb{Q}/\mathbb{Z} \wedge X_+) \to \pi_*(K \wedge MB \wedge X_+) \tag{33}$$

vanishes. Note that Assumption 2.6,1. implies that

$$\pi_n(MB \wedge X_+)_{tors} \subseteq F^1\pi_n(MB \wedge X_+) \ . \tag{34}$$

**Lemma 2.7** *If we assume 2.6, then the restriction of $\eta^{top}$ to $F^2\pi_n(MB \wedge X_+)_{tors}$ is trivial.*

*Proof.* Assume that $x \in F^2\pi_n(MB \wedge X_+)_{tors}$. If we can show that we can choose $\hat{x} \in F^1\pi_{n+1}(MB\mathbb{Q}/\mathbb{Z} \wedge X_+)$, then $\tilde{x} = 0$ and therefore $\eta^{top}(x) = 0$.
Let $[\hat{x}] \in E_2^{0,n+1}(MGB\mathbb{Q}/\mathbb{Z} \wedge X_+)$ be the symbol of $\hat{x}$. The geometric boundary theorem gives a map

$$\delta_2 : E_2^{0,n+1}(MB\mathbb{Q}/\mathbb{Z} \wedge X_+) \to E_2^{1,n+1}(MB \wedge X_+) \tag{35}$$

so that $\delta_2[\hat{x}] = [x]$ is the symbol of $x$. By our assumtion on $x$ we have $\delta_2[\hat{x}] = 0$. By [21, Sec. 5.3] we have a long exact sequence

$$0 \to E_2^{0,n+1}(MB \wedge X_+) \otimes \mathbb{Q}/\mathbb{Z} \xrightarrow{j} E_2^{0,n+1}(MB\mathbb{Q}/\mathbb{Z} \wedge X_+) \xrightarrow{\delta_2} E_2^{1,n+1}(MB \wedge X_+)_{tors} \to 0 \ . \tag{36}$$

Therefore $[\hat{x}] = j(\alpha)$ for some class $\alpha \in E_2^{0,n+1}(MB \wedge X_+) \otimes \mathbb{Q}/\mathbb{Z}$. We further have surjections

$$E_2^{*,*}(MB\mathbb{Q} \wedge X_+) \cong E_2^{*,*}(MB \wedge X_+) \otimes \mathbb{Q} \xrightarrow{q} E_2^{*,*}(MG \wedge X_+) \otimes \mathbb{Q}/\mathbb{Z} \to 0 \ . \tag{37}$$

Our assumption 2.6.2 implies that $E_2^{*,s}(MB\mathbb{Q} \wedge X_+) = 0$ for $s \geq 1$. Therefore the Adams spectral sequence for $MB\mathbb{Q} \wedge X_+$ degenerates at the $E_2$-term.
We can write $\alpha = q(\beta)$ for some $\beta \in E_2^{0,n+1}(MB\mathbb{Q} \wedge X_+)$ which is necessarily a permanent cycle. We choose $b \in \pi_{n+1}(MB\mathbb{Q} \wedge X_+)$ with symbol $[b] = \beta$. The image $\hat{y} \in \pi_{n+1}(MB\mathbb{Q}/\mathbb{Z} \wedge X_+)$ of the element $b$ has the property that $\hat{x} - \hat{y} \in F^1\pi_{n+1}(MB\mathbb{Q}/\mathbb{Z} \wedge X_+)$. We can replace our choice of $\hat{x}$ by this difference. □

We define a map

$$\kappa : E_2^{1,n+1}(MB \wedge X_+) \to Q_n(B, X) \tag{38}$$

as follows. By Assumption 2.6, 2. and (36) the map $\delta_2$ in (35) is surjective. For $\gamma \in E_2^{1,n+1}(MB \wedge X_+)$ we thus can choose $\beta \in E_2^{0,n+1}(MB\mathbb{Q}/\mathbb{Z} \wedge X_+)$ such that $\delta_2(\beta) = \gamma$. The element $\beta$ is well-defined up to the image of $j$. We have a natural inclusion map

$$a : E_2^{0,n+1}(MB\mathbb{Q}/\mathbb{Z} \wedge X_+) \to \pi_{n+1}(K \wedge MB\mathbb{Q}/\mathbb{Z} \wedge X_+) \ .$$

We define $\kappa(\gamma) \in Q_n(B, X)$ as the element represented by the homomorphism $K^0(B_+ \wedge X_+) \to \pi_{n+1}(K\mathbb{Q}/\mathbb{Z})$ given by the pairing with $a(\beta)$. In the proof of Lemma 2.7 we see that pairings with elements in the image of $j$ give the zero element in $Q_n(B, X)$. Therefore $\kappa(\gamma)$ is well-defined independ of the choice of $\beta$.

**Lemma 2.8** *If we assume 2.6 and that $n$ is odd, then the homomorphism*

$$\kappa : E_2^{1,n+1}(MB \wedge X_+) \to Q_n(B, X)$$

*is injective.*

*Proof.* Let $\gamma \in E_2^{1,n+1}(MB \wedge X_+)$. Assume that $\kappa(\gamma) = 0$. We write $\gamma = \delta_2(\beta)$ for some $\beta \in E_2^{0,n+1}(MB\mathbb{Q}/\mathbb{Z} \wedge X_+)$. We have $\kappa(\gamma) = 0$ if an only if there exists $y \in \pi_{n+1}(MB\mathbb{Q} \wedge X_+)$

which induces the same pairing with $K^0(B_+ \wedge X_+)$ as $a(\beta) \in \pi_{n+1}(K \wedge MB\mathbb{Q}/\mathbb{Z} \wedge X_+)$. By Pontrjagin duality and the Thom isomorphism we have

$$\pi_{n+1}(K \wedge MB\mathbb{Q}/\mathbb{Z} \wedge X_+) \cong \mathtt{Hom}^{cont}(K^0(MB \wedge X_+), \mathbb{Q}/\mathbb{Z}) \cong \mathtt{Hom}^{cont}(K^0(B_+ \wedge X_+), \mathbb{Q}/\mathbb{Z}) \ .$$

We conclude that $a(\beta) = a(j(q([y])))$, where $[y] \in E_2^{0,n+1}(MB\mathbb{Q} \wedge X_+)$ and $q$ and $j$ are as in (37) and 36). Since $a$ is injective this implies $\gamma = \delta_2(j(q([y]))) = 0$. □

By (34) we have a natural map

$$s : \pi_{n+1}(MB \wedge X_+)_{tors} \to E_2^{1,n+1}(MB \wedge X_+)$$

mapping $x$ to its symbol $s(x) = [x]$.

**Proposition 2.9** *If we assume 2.6, then there is a factorization*

$$\eta^{top} = \kappa \circ s : \pi_n(MB \wedge X_+)_{tors} \to E_2^{1,n+1}(MB \wedge X_+) \to Q_n(B, X) \ .$$

*Proof.* Let $x \in \pi_n(MB \wedge X_+)_{tors}$ and $\hat{x} \in \pi_{n+1}(MB\mathbb{Q}/\mathbb{Z} \wedge X_+)$ be its lift. Then $\eta^{top}(x)$ is represented by the homomorphism given by the pairing with $\epsilon_K(\hat{x}) \in \pi_{n+1}(K \wedge MB \wedge X_+)$. On the other hand if, in the construction of $\kappa$, we set $\gamma := s(x)$, then we can take $\beta$ such that $a(\beta) = [\epsilon_K(\hat{x})]$. It follows that $\kappa(s(x))$ is also represented by the homomorphism given by the pairing with $\epsilon_K(\hat{x}) \in \pi_{n+1}(K \wedge MB \wedge X_+)$. □

If we combine Lemma 2.8 with Proposition 2.9 and use the inclusion

$$F^1\pi_n(MB \wedge X_+)_{tors}/F^2\pi_n(MB \wedge X_+)_{tors} \subseteq E_2^{1,n+1}(MB \wedge X_+)$$

we get the following corollary.

**Corollary 2.10** *If we assume 2.6 and that $n$ is odd, then we have an injection*

$$\eta^{top} : F^1\pi_n(MB \wedge X_+)_{tors}/F^2\pi_n(MB \wedge X_+)_{tors} \to Q_n(B, X) \ .$$

## 2.6 Functorial properties and a stable version

We let $\mathtt{CW}/_h BSpin^c$ be the categeory of $CW$-complexes over $BSpin^c$ (see the text after (2) where the meaning of the subscript $h$ is explained). Furthermore, we let $\mathtt{Top}$ denote the category of spaces. The association $(B, X) \mapsto Q_n(B, X)$ becomes a functor from the product

$$\mathtt{CW}/_h BSpin^c \times \mathtt{Top} \to \mathtt{Ab}$$

as follows.
Let us consider a pair of morphisms

$$\begin{tikzcd} B \arrow[rr, "\phi"] \arrow[dr, "\sigma"'] & & B' \arrow[dl, "\sigma'"] \\ & BSpin^c & \end{tikzcd} \quad , \quad \psi : X \to X' \tag{39}$$

in $\mathtt{CW}/_h BSpin^c$ and $\mathtt{Top}$. Pull-back along $(\phi,\psi)^*: K^0(B'_+\wedge X'_+)\to K^0(B_+\wedge X_+)$ induces a map

$$\mathtt{Hom}^{cont}(K^0(B_+\wedge X_+),\pi_{n+1}(K\mathbb{Q}/\mathbb{Z}))\to \mathtt{Hom}^{cont}(K^0(B'_+\wedge X'_+),\pi_{n+1}(K\mathbb{Q}/\mathbb{Z}))$$

whose restriction to the subgroup $U$ has values in the corresponding subgroup $U'$. Hence we get a functorial map

$$(\phi,\psi)_*: Q_n(B,X)\to Q_n(B',X')\ .$$

We also have an induced map of spectra

$$M\phi\wedge\psi: MB\wedge X_+\to MB'\wedge X_+\ .$$

**Lemma 2.11** *The following diagram commutes:*

$$\begin{tikzcd}
\pi_n(MB\wedge X_+)_{tors} \arrow[r, "\eta^{top}"] \arrow[d, "M\phi\wedge\psi"] & Q_n(B,X) \arrow[d, "(\phi,\psi)_*"] \\
\pi_n(MB'\wedge X'_+)_{tors} \arrow[r, "\eta^{top}"] & Q_n(B',X')
\end{tikzcd}\ .$$

*Proof.* This is an easy consequence of the definitions. □

Observe that the group $Q_n(B,X)$ depends on $X$ only via the space $X_+$ obtained by attaching a disjoint base point. There is an immediate extension of the definitions to arbitrary pointed spaces $Z$ in place of $X_+$. By a slight abuse of notation we write $Q_n(B,Z)$ for the resulting groups. Our motivation for using spaces of the form $X_+$ is that the geometric picture of the bordism group $\pi_n(MB\wedge X_+)$ is simpler than in the general case $\pi_n(MB\wedge Z)$. Note that the homotopy theoretic construction of $\eta^{top}$ immediately extends to general pointed spaces so that we get by adapting Definition 2.2 a transformation

$$\eta^{top}:\pi_n(MB\wedge Z)_{tors}\to Q_n(B,Z)\ .$$

More generally, let $\mathcal{X}$ be a spectrum and $\Omega^\infty\mathcal{X}$ be the associated infinite loop space. We can redo the construction of $\eta^{top}$ in 2.3 with the spectrum $\mathcal{X}$ in place of $X_+$. Then in the definition of $Q_n(B,\mathcal{X})$ we have to intepret $K^0(B_+\wedge\mathcal{X})$ as spectrum cohomology. The counit of the adjunction $(\Sigma^\infty,\Omega^\infty)$ (see (1)) is a natural map

$$u:\Sigma^\infty\Omega^\infty\mathcal{X}\to\mathcal{X} \tag{40}$$

which induces a map

$$Q_n(B,\Sigma^\infty\Omega^\infty\mathcal{X})\to Q_n(B,\mathcal{X})$$

such that the following diagram commutes.

**Lemma 2.12**

$$\begin{array}{ccc}\pi_n(MB\wedge\Omega^\infty\mathcal{X})_{tors} & \xrightarrow{\eta^{top}} & Q_n(B,\Omega^\infty\mathcal{X}) \\ \downarrow{\scriptstyle \mathtt{id}\wedge u} & & \downarrow{\scriptstyle (\mathtt{id},u)} \\ \pi_n(MB\wedge\mathcal{X})_{tors} & \xrightarrow{\eta^{top}} & Q_n(B,\mathcal{X})\end{array} .$$

*Proof.* This is again an easy consequence of the definitions. □

These generalizations will play a role in the application to algebraic $K$-theory developed in Subsection 5.3.

# 3 The spectral geometric construction

## 3.1 Motivation

In this section we define an analytic invariant of torsion elements in the $B$-bordism theory of a space $X$. The analytic invariant will be derived from geometric and spectral geometric quantities associated to geometric cycles for bordism classes. The relation between the geometric and homotopy theoretic picture of the bordism group is given by Thom-Pontrjagin construction, see [53, Ch IV.7]. In Subsection 3.2 we give the details of the geometric picture of the $B$-bordism theory. Subsection 3.3 is devoted to some technical details on the transfer of $Spin^c$-structures from the normal bundle to the tangent bundle. A reader with some experience with the Thom-Pontrjagin construction and $Spin^c$-structures may immediately proceed to the construction of $\eta^{an}$ in Subsection 3.4. The final Subsection 3.5 of this part contains the proof the main theorem about the equality of the analytic and topological universal $\eta$-invariant.

## 3.2 Geometric cycles for $B$-bordism theory

Cycles for elements of the $B$-bordism group $\pi_n(MB\wedge X_+)$ of $X$ are triples $(M,f,g)$ which we now describe in detail. The first entry is a closed $n$-dimensional manifold $M$. The remaining two entries are maps $f:M\to B$ and $g:M\to X$, where the map $f$ classifies a $B$-structure on the stable normal bundle of $M$. To say that $f$ classifies a $B$-structure on the stable normal bundle of $M$ involves an additional structure which we drop in the notation for simplicity. This structure fixes the relation between the map $f$ and the tangent bundle of $M$. Since $M$ is compact, for suitable $k\geq 1$ there exists a factorization

$$\begin{tikzcd} & & BO(k) \arrow[d] \\ M \arrow[r, "f"] \arrow[urr, dotted, "\hat f"] & B \arrow[r] & BO \end{tikzcd}$$

up to homotopy. Let $\xi_k\to BO(k)$ denote the $k$-dimensional universal real euclidean vector bundle. To say that $f$ represents a $B$-structure on the stable normal bundle of $f$

means that we are given a trivialization of the sum

$$TM \oplus \hat{f}^*\xi_k \cong M \times \mathbb{R}^{n+k} \ . \tag{41}$$

**Definition 3.1** *We call the choice of $\hat{f}$ together with such an isomorphism a representative of the normal $B$-structure on $M$.*

The normal $B$-structure itself is given by the equivalence class of the representative under the equivalence relation generated by stabilization (which allows to change $k$) and joint homotopy of $\hat{f}$ together with isomorphism (41).
The bordism group $\pi_n(MB \wedge X_+)$ is the set of equivalence classes of cycles, where the equivalence relation is given by bordism, and the group structure is induced by the disjoint sum. A zero bordism of $(M, f, g)$ is given by a triple $(W, F, G)$ of similar data, where $W$ is a compact $n+1$-dimensional with boundary $\partial W \cong M$, and $F : W \to B$ and $G : W \to X$ are maps which extend $f$ and $g$. We require that the $B$-structure on the stable normal bundle of $W$ represented by $F$ restricts to the $B$-structure on the stable normal bundle of $M$ represented by $f$. In detail this means the following. The representative of the normal $B$-structure on $W$ by $F$ involves an isomorphism

$$TW \oplus \hat{F}^*\xi_k \cong W \times \mathbb{R}^{n+1+k} \ . \tag{42}$$

An outgoing normal field of $TW_{|\partial W}$ provides a decomposition

$$TW_{|M} \cong TM \oplus (M \times \mathbb{R}) \ .$$

Let $\hat{f}^s : M \to BO(k+1)$ be the composition of $\hat{f}$ with the map $BO(k) \to BO(k+1)$ so that

$$\hat{f}^{s*}\xi_{k+1} \cong (M \times \mathbb{R}) \oplus \hat{f}^*\xi_k \ .$$

The restriction of the isomorphism (42) induces an isomorphism

$$TM \oplus \hat{f}^{s*}\xi_{k+1} \cong TM \oplus (M \times \mathbb{R}) \oplus \hat{f}^*\xi_k \cong TW_{|M} \oplus f_k^*\xi_k \stackrel{(42)}{\cong} M \times \mathbb{R}^{n+1+k} \ .$$

The requirement about the restriction of the $B$-structure from $W$ to $M$ thus is that isomorphism represents the given $B$-structure on $M$.

## 3.3 Normal and tangential $Spin^c$-structures

Because of the factorization $B \to BSpin^c \to BO$ a normal $B$-structure induces a normal $Spin^c$-structure. As we will do geometry on the tangent bundle we must transfer normal $Spin^c$-structures to tangential $Spin^c$-structures. The homotopy theoretic picture of this transition is explained in [21, Sec. 8] in the example of *String*-structures. In the following we describe its geometric counterpart.

Let $V \to M$ be an $m$-dimensional real vector bundle. Then a geometric $Spin^c$-structure on $V$ is pair $(P, \kappa)$, where $P \to M$ is a $Spin^c(m)$-principal bundle and $\kappa$ is an isomorphism of real vector bundles

$$\kappa : P \times_{Spin^c(n)} \mathbb{R}^m \cong V \ .$$

With this definition a $Spin^c$-structure induces a euclidean metric and an orientation on $V$ so that the oriented orthonormal frame bundle is $SO(V) := P \times_{Spin^c(m)} SO(m)$.
The collection of all $Spin^c$-structures on the vector bundle $V$ naturally forms a groupoid. For glueing and certain functorial constructions we have to be very careful with identifications. In these cases it is not sufficient to work with the set of isomorphism classes of $Spin^c$-structures.
For an oriented euclidean vector bundle $V$ we let $Spin^c(V)$ denote the groupoid of $Spin^c$-structures which induce the given metric and orientation. The objects of the groupoid $Spin^c(V)$ are the $Spin^c$-structures $(P, \kappa)$, and the morphisms $(P, \kappa) \to (P', \kappa')$ are isomorphisms of $Spin^c(m)$-principal bundles $P \to P'$ which are compatible with the isomorphisms $\kappa$ and $\kappa'$. This in particular implies that automorphisms of $(P, \kappa)$ are given by the central action of $C^\infty(M, U(1))$ on $P$.
If we associate to any open subset $A \subseteq M$ the groupoid $Spin^c(V_{|A})$ of $Spin^c$-structures on the restriction of $V$ to $A$, then we obtain a sheaf of groupoids $\underline{Spin^c(V)}$ which actually is an $U(1)$-banded gerbe. We refer to [20],[49], or [36] for an introduction to gerbes. Isomorphism classes of $U(1)$-banded gerbes $\mathcal{G}$ are classified by their Dixmier-Douady classes $\mathtt{DD}(\mathcal{G}) \in H^3(M; \mathbb{Z})$. In particular, the Dixmier-Douday class of the $Spin^c$-gerbe $\underline{Spin^c(V)}$ is the class

$$\mathtt{DD}(\underline{Spin^c(V)}) = W_3(V) = \beta w_2(V) \in H^3(M; \mathbb{Z}) \ ,$$

where $w_2(V) \in H^2(M; \mathbb{Z}/2\mathbb{Z})$ is the second Stiefel-Whitney class and $\beta : H^2(M; \mathbb{Z}/2\mathbb{Z}) \to H^3(M; \mathbb{Z})$ is the Bockstein operator [46, Thm. D2]. The groupoid $Spin^c(V)$ is non-empty exactly if $W_3(V) = 0$, i.e. the class $W_3(V)$ is the obstruction against the existence of a $Spin^c$-structure on $V$. In the following we will simplify the notation and write $P$ for the $Spin^c$-structure $(P, \kappa)$.
Let $\mathcal{B}U(1)(M)$ denote the Picard groupoid (see [28]) of $U(1)$-principal bundles on $M$. Given an $U(1)$-principal bundle $E \in \mathcal{B}U(1)(M)$ and a $Spin^c$-structure $P \in Spin^c(V)$, we can define a new $Spin^c$-structure $E \otimes P \in Spin^c(V)$. This is most easily described in terms of cocycles. If $(\phi_{\alpha\beta})$ is the cocycle for $E$ and $(\lambda_{\alpha\beta})$ is the cocycle for $P$ with respect to some open covering of $M$, then $(\phi_{\alpha\beta}\lambda_{\alpha\beta})$ is the cocycle for $E \otimes P$, where the action uses the central embedding $U(1) \subset Spin^c(m)$. The structure map $\kappa_{E\otimes P}$ is induced by $\kappa_P$ in the canonical manner. Alternatively, a global formula is given by (45) specialized to the case $n = 0$, see below. This construction defines bifunctor

$$\mathcal{B}U(1)(M) \times Spin^c(V) \to Spin^c(V) \ . \tag{43}$$

If $Spin^c(V)$ is not empty, then the set of isomorphism classes of $Spin^c$-structures on $V$ is a torsor over the group of isomorphism classes in $\mathcal{B}U(1)(M)$. Since the latter is canonically isomorphic to $H^2(M;\mathbb{Z})$ we get a simply transitive action of $H^2(M;\mathbb{Z})$ on $Spin^c(V)/iso$. Furthermore, the tensor product with the respective identities induces isomorphisms

$$C^\infty(M,U(1)) \cong \mathtt{Aut}_{\mathcal{B}U(1)(M)}(E) \cong \mathtt{Aut}_{Spin^c(V)}(E\otimes P) \cong \mathtt{Aut}_{Spin^c(V)}(P) \ .$$

The sum of two vector bundles with $Spin^c$-structures has a naturally induced $Spin^c$-structure. This is formalized with the natural bifunctor

$$Spin^c(V)\times Spin^c(U)\to Spin^c(V\oplus U) \ . \tag{44}$$

On the level of objects this bifunctor is given by

$$(P,Q)\mapsto P\otimes Q \ ,$$

where the $Spin^c(n+m)$-principal bundle

$$P\otimes Q := (P\times_M Q)\times_{(Spin^c(n)\times Spin^c(m))} Spin^c(n+m) \tag{45}$$

is obtained from the $Spin^c(n)\times Spin^c(m)$-principal bundle $P\times_M Q$ by extension of structure groups along the natural map

$$\begin{tikzcd} Spin^c(n)\times Spin^c(m) \arrow[r] \arrow[d] & Spin^c(n+m) \arrow[d] \\ SO(n)\times SO(m) \arrow[r] & SO(n+m) \end{tikzcd} \ .$$

Here $n=\dim(V)$ and $m=\dim(U)$, and the compatibility with the lower part of this diagram is used to define the structure map $\kappa_{P\otimes Q}$ from $\kappa_P$ and $\kappa_Q$. The bifunctor comes equipped with natural associativity constraints. We omit the details of the latter two aspects.

We set $Spin^c(0):=U(1)$ and let $0_M$ denote the zero dimensional vector bundle on $M$. Then we get an identification of $Spin^c(0_M)\cong \mathcal{B}U(1)(M)$, and for $n=0$ the bifunctor (44) specializes to (43). As a consequence of associativity the bifunctor (44) is compatible with the action (43) of $\mathcal{B}U(1)(M)$ in the sense that for $E\in\mathcal{B}U(1)(M)$ have natural isomorphisms

$$(E\otimes P)\otimes Q\cong E\otimes(P\otimes Q)\cong P\otimes(E\otimes Q) \ . \tag{46}$$

A trivialized vector bundle $M\times\mathbb{R}^n$ has a preferred trivial $Spin^c$-structure $Q(n):=M\times Spin^c(n)$. We can use this to produce a canonical equivalence of groupoids

$$Spin^c(V)\cong Spin^c(V\oplus(M\times\mathbb{R}^n)) \ , \quad P\mapsto P\otimes Q(n) \ .$$

On the level of $Spin^c$-structures we speak of stabilizations.

Let us now consider a pair $(M,f)$ of a compact oriented $n$-dimensional Riemannian manifold and a map $f:M\to B$ which represents a normal $B$-structure. Then we can assume that $f$ has a factorization over $BSpin^c(k)$ as in the diagram

$$\begin{tikzcd} & & \xi_k^{Spin^c} \arrow[r] \arrow[d] & \xi_k^{SO} \arrow[r] \arrow[d] & \xi_k \arrow[d] \\ & & BSpin^c(k) \arrow[r] \arrow[d] & BSO(k) \arrow[r] \arrow[d] & BO(k) \arrow[d] \\ M \arrow[r, "f"] \arrow[urr, dotted, "\tilde{f}"] & B \arrow[r] & BSpin^c \arrow[r] & BSO \arrow[r] & BO \end{tikzcd} .$$

The map $\tilde{f}$ classifies a $Spin^c(k)$-principal bundle $\tilde{f}^*Q_k \to M$, where $Q_k \to BSpin^c(k)$ denotes the universal $Spin^c(k)$-bundle. Note that we have $\tilde{f}^*Q_k \in Spin^c(\tilde{f}^*\xi_k^{Spin^c})$.
We let $\hat{f} : M \to BO(k)$ be induced by $\tilde{f}$ so that $\hat{f}^*\xi_k \cong \tilde{f}^*\xi_k^{Spin^c}$. With these identifications the trivialization (41) induces a bifunctor (44)

$$Spin^c(TM) \times Spin^c(\tilde{f}^*\xi_k^{Spin^c}) \cong Spin^c(M \times \mathbb{R}^{n+k}) .$$

Since $\mathcal{B}U(1)(M)$ acts simply transitively on isomorphisms classes we conclude using (46) that there is a unique isomorphism class of geometric $Spin^c$-structures $P \in Spin^c(TM)$ such that

$$P \otimes \tilde{f}^*Q_k \cong Q(n+k) . \tag{47}$$

One can further check that this isomorphism class does only depend on the normal $B$-structure represented by $f$ and not on its representative. This is the tangential $Spin^c$-structure determined by the normal $Spin^c$-atructure.
For constructions which involve glueing or in the notion of a $Spin^c$-map we need a rigidified notion of a tangential $Spin^c$-structure.

**Definition 3.2** *Assume that we have fixed a representative of a normal $B$-structure in terms of the factorization $\tilde{f}$ and the isomorphism (41). Then we define a tangential representative of the normal $Spin^c$-structure as a pair of a $Spin^c$-structure $P \in Spin^c(TM)$ together with a choice of an isomorphism in (47).*

There are many tangential representatives of the normal $Spin^c$-structure, but the main point is that two of them are isomorphic by a unique isomorphism.
Let $h : M \to W$ be a smooth map and assume that we are given oriented euclidean vector bundles $V_M \to M$ and $V_W \to W$ together with an isomorphism

$$V_M \oplus (M \times \mathbb{R}^k) \cong h^*V_W \oplus (M \times \mathbb{R}^l) . \tag{48}$$

Assume further that we are given $Spin^c$-structures $P_M \in Spin^c(V_M)$ and $P_W \in Spin^c(V_W)$.

**Definition 3.3** *A representative of a refinement of $h$ to a $Spin^c$-map is a choice of an isomorphism*

$$h^*P_W \otimes Q(l) \cong P_M \otimes Q(k) \tag{49}$$

*in $Spin^c(V_M \oplus (M \times \mathbb{R}^k))$ (this uses (48)). These representative are subject to the equivalence relation generated by stabilization und homotopy of both isomorphisms (48) and (49). A $Spin^c$-map is a map $h$ together with an equivalence class of representatives of a refinement of $h$ to a $Spin^c$-map.*

Being a $Spin^c$-map is an additional datum, not just a property of the map. Observe that we can compose $Spin^c$-maps in a natural way.
We now assume that $(W, F)$ is a zero bordism of $(M, f)$. Recall that this involves an identification of the restriction of the $B$-structure on the stable normal bundle of $W$ classified by $F$ to $M$ with the normal $B$-structure on the stable normal bundle of $M$ represented by $f$. This has be explained in detail in Subsection 3.2. We choose a representative of the normal $B$-structure on $W$ involving the factorization $\tilde{F} : W \to BSpin^c(k)$. On $M$ we take the induced factorization $\tilde{f} := \tilde{F}_{|M}$.
We choose a Riemannian metric on $M$ and extend it to a Riemannian metric on $W$ with a product structure close to the boundary. Then we have a natural decomposition of oriented euclidean vector bundles

$$TW_{|M} \cong TM \oplus (M \times \mathbb{R}) \ , \tag{50}$$

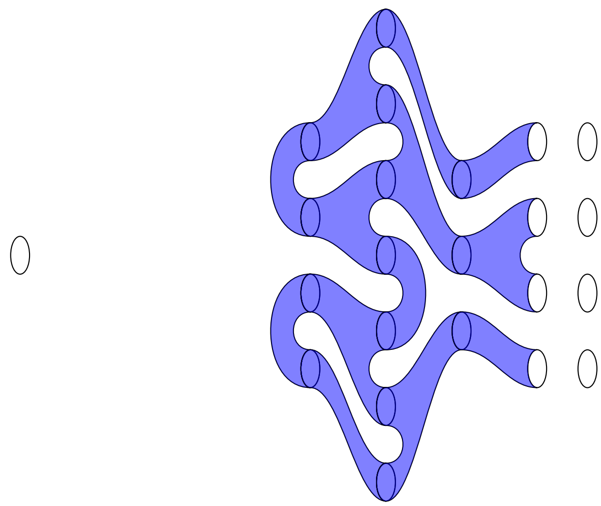
where we trivialize the normal bundle by the outgoing unit normal vector field. Assume now that we have chosen tangential representatives $P(TM)$ and $P(TW)$ of the normal $Spin^c$-structures on $M$ and $W$. Note that this involves the choices of isomorphisms of the type (47) which we dropped from the notation. We claim that in this situation we get a natural refinement of the inclusion $M \to W$ to a $Spin^c$-map. This refinement is distinguished by the condition that the following diagram in $Spin^c(M \times \mathbb{R}^{n+1+k})$

$$\begin{tikzcd} P(TM) \otimes Q(1) \otimes \tilde{f}^*Q_k \arrow[r, "\cong"] \arrow[d, "\cong"] & P(TW)_{|M} \otimes \tilde{f}^*Q_k \arrow[d, "\cong"] \\ Q(n+1+k) \arrow[r, equal] & Q(n+1+k) \end{tikzcd}$$

commutes up to homotopy. Here the upper corners are interpreted in $Spin^c(M \times \mathbb{R}^{n+1+k})$ using the representative of the normal $B$-structure on $M$ or $W$, respectively. The vertical morphisms are given by the tangential representative of the normal $Spin^c$-structures. Finally, the upper horizontal isomorphism uses (50) and fixes the refinement of the inclusion $M \to W$ to a $Spin^c$-map.

### 3.4 The definition of $\eta^{an}$

We now assume that the class $x = [M, f, g] \in \pi_n(MB \wedge X_+)$ is torsion. Then there exists a suitable integer $l \in \mathbb{N}$ such that $lx = 0$. We can thus find a zero bordism $(W, F, G)$ of the disjoint union $l(M, f, g)$ of $l$ copies of $(M, f, g)$.



A picture of $M$ and the zero bordism $W$ of $4M$

We will define $\eta^{an}(x) \in Q_n(B, X)$ in terms of a collection of indices of associated $\mathbb{Z}/l\mathbb{Z}$-index problems [33]. In order to formulate these index problems and to express the indices in terms of geometric and spectral invariants we must choose appropriate geometric structures.

We choose a Riemannian metric on $M$ and observe that $TM$ is oriented. We can now choose a tangential representative $(P, \kappa) \in Spin^c(TM)$ of the normal $Spin^c$-structure determined by $f$. A connection $\tilde{\nabla}^{TM}$ on $P$ induces via $\kappa$ a connection on $TM$. We say that $\tilde{\nabla}^{TM}$ is a $Spin^c$-extension of the Levi-Civita connection on $M$ if it induces the Levi-Civita connection $\nabla^{TM,LC}$ on $TM$.

The group $Spin^c(n)$ has a distinguished unitary representation called the spinor representation $\Delta^n$. For even $n$ its dimension is $2^{n/2}$, and it has a decomposition $\Delta^n \cong \Delta^{n,+} \oplus \Delta^{n,-}$. It is related with the odd-dimensional case by $\Delta^{n,+}_{|Spin^c(n-1)} \cong \Delta^{n-1}$.

The bundle $S(TM) := P \times_{Spin^c(n)} \Delta^n \to M$ is called the Spinor bundle of $M$. If we have chosen a $Spin^c$-extension $\tilde{\nabla}^{TM}$ of the Levi-Civita connection on $M$, then the spinor bundle carries the structure of a Dirac bundle. We thus obtain the $Spin^c$-Dirac operator $\not{D}_M$ which acts on sections of $S(TM)$. Standard references for these constructions are [15, Ch. 3], [46, App. D].

We have a map $(f, g) : M \to B_+ \wedge X_+$. If we are given a class $\phi \in K^0(B_+ \wedge X_+)$, then we can choose a $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle $V \to M$ whose $K$-theory class satisfies $[V] = (f, g)^*\phi \in K^0(M_+)$. We choose a hermitean metric $h^V$ and a metric connection $\nabla^V$ which preserve the grading. The triple $\mathbf{V} := (V, h^V, \nabla^V)$ will then be called a geometric vector bundle. We let $\not{D}_M \otimes \mathbf{V}$ be the Dirac operator twisted by $\mathbf{V}$. It acts on sections of $S(TM) \otimes V$.

We now assume that $n = \dim(M)$ is odd. The $\eta$-invariant [6] of the twisted Dirac operator

$$\eta(\not{D}_M \otimes \mathbf{V}) \in \mathbb{R}$$

is defined as the value at $s = 0$ of the meromorphic continuation of the $\eta$-function function

$$\eta(\not{D}_M \otimes \mathbf{V})(s) := \mathtt{Tr}_s\, |D_M \otimes \mathbf{V}|^{-s} \mathtt{sign}(D_M \otimes \mathbf{V})\ ,$$

where $\mathtt{Tr}_s$ is the super trace with respect to the grading of $V$. Note that the trace exists if $\Re(s) > n$, and that the meromorphic continuation of the $\eta$-function is regular at $s = 0$ by the results of [6]. The $\eta$-invariant depends on the geometry of $M$ and $V$ in a possibly discontinuous way with jumps when eigenvalues of $\not{D}_M \otimes \mathbf{V}$ cross zero. In order to get a quantity which depends continuously on the geometry one usually considers the reduced $\eta$-invariant for which we will use the symbol $\xi$ in the present paper:

$$\xi(\not{D}_M \otimes \mathbf{V}) := [\frac{\eta(\not{D}_M \otimes \mathbf{V}) + \dim \mathtt{ker}(\not{D}_M \otimes \mathbf{V}))}{2}] \in \mathbb{R}/\mathbb{Z}. \tag{51}$$

We now take into account that we have a zero-bordism $(W, F, G)$ of $l$ copies of $(M, f, g)$. In an appropriate model of $\pi_n(MB\mathbb{Q}/\mathbb{Z} \wedge X_+)$ the object $(W, F, G)$ geometrically represents the lift of $x$ to a class

$$\hat{x} = [W, F, H] \in \pi_{n+1}(MB\mathbb{Q}/\mathbb{Z} \wedge X_+)\ ,$$

using the notation of the diagram (21). We refer to Lemma 3.7 for more details.
We choose a Riemannian metric on $W$ which extends the Riemannian metric on $\partial W$ induced by the previously chosen metric on $M$ with a product structure. Furthermore, we extend the associated Levi-Civita connection to a $Spin^c$-connection $\tilde{\nabla}^{TW}$ which extends the connection on $\partial W$ induced by $\tilde{\nabla}^{TM}$. The class $(F, G)^*\phi \in K^0(W_+)$ extends the class $(F, G)^*_{|\partial W}\phi \in K^0(\partial W_+)$ which restricts to $(f, g)^*\phi \in K^0(M_+)$ on the copies of $M$ in the boundary of $W$. Hence we can assume, after adding some trivial bundles to the even and odd parts of $V$, that the bundle on $\partial W$ induced by $V$ has an extension $U$ to $W$. We choose a hermitean metric $h^U$ and a metric connection $\nabla^U$ on $U$ which extend the corresponding already given data on the boundary. In this way we get a geometric bundle $\mathbf{U} := (U, h^U, \nabla^U)$.
We can now form the Atiyah-Patodi-Singer boundary value problem for $\not{D}_W \otimes \mathbf{U}$. The analytic details of that boundary value problem are not important for our present purpose so that we refer to [6] for a precise description. We only have to know that it produces a Fredholm operator $(\not{D}_W \otimes \mathbf{U})_{APS}$ which has a well-defined index

$$\mathtt{index}((\not{D}_W \otimes \mathbf{U})_{APS}) \in \mathbb{Z}\ ,$$

and that the following index formula proved in [6] holds true:

$$\mathtt{index}((\not{D}_W \otimes \mathbf{U})_{APS}) = \int_W p_{n+1}(\mathbf{Td}(\tilde{\nabla}^{TW}) \wedge \mathbf{ch}(\nabla^U)) - l\frac{\eta(\not{D}_M \otimes \mathbf{V}) + \dim \mathtt{ker}(\not{D}_M \otimes \mathbf{V}))}{2}\ . \tag{52}$$

In this formula the closed form $\mathbf{Td}(\tilde{\nabla}^{TW}) \in \Omega^0(W)[b, b^{-1}]$ is the Chern-Weyl representative determined the universal class $\mathbf{Td} \in HP\mathbb{Q}^0(BSpin^c_+)$ and the connection $\tilde{\nabla}^{TM}$. Similarly, the form $\mathbf{ch}(\nabla^U) \in \Omega^0(W)[b, b^{-1}]$ is the Chern-Weyl representative determined

by the class $\mathbf{ch} \in HP\mathbb{Q}^0(BU_+)$ and the connection $\nabla^U$. Note that we use powers of $b$ to shift the higher form-degree components to total degree zero.
We consider the element

$$e := [\frac{\mathtt{index}(\not{D}_W \otimes U)_{APS}}{l}] \in \mathbb{Q}/\mathbb{Z} \ . \tag{53}$$

Equivalently, by the index theorem (52) and (51) we can write

$$e = [\frac{1}{l}\int_W p_{n+1}(\mathbf{Td}(\tilde{\nabla}^{TW}) \wedge \mathbf{ch}(\nabla^U))] - \xi(\not{D}_M \otimes \mathbf{V}) \tag{54}$$

if we interpret this equality in $\mathbb{R}/\mathbb{Z}$. The quantity $e$ can be interpreted as a $\mathbb{Z}/l\mathbb{Z}$-index in the sense of [33]. In the following proposition we formulate how the number $e$ depends on the data.

**Proposition 3.4** *1. The value of $e$ does not depend on the choices of geometric structures on $M$ and $W$.*

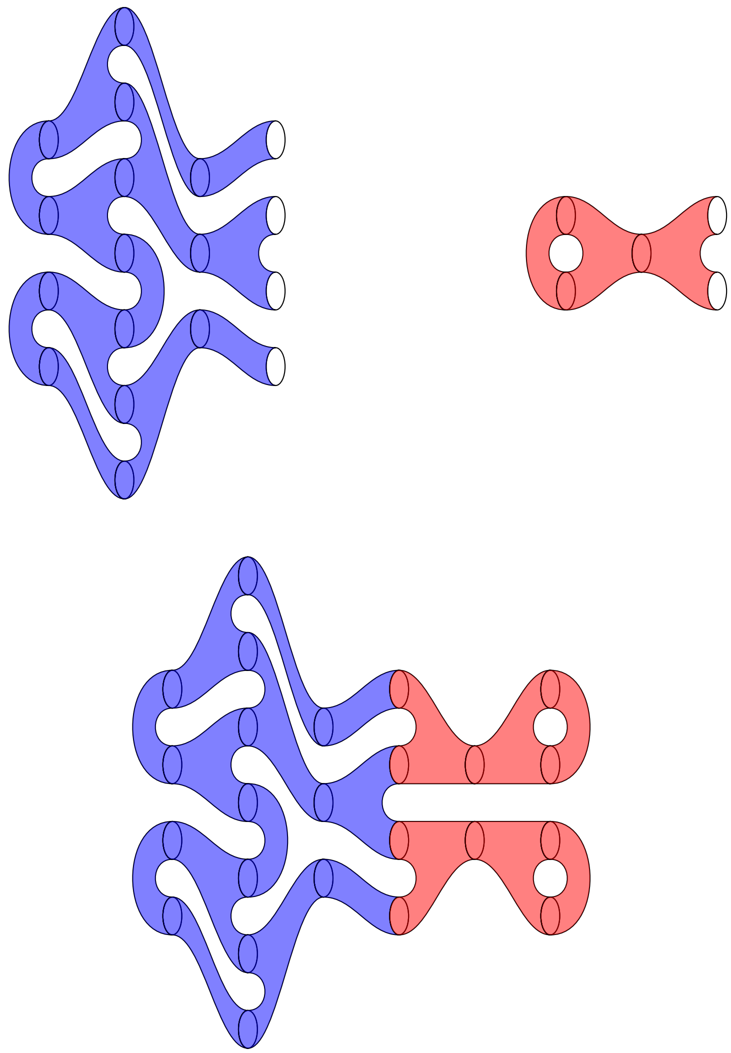
*2. The value of $e$ only depends on the $K$-theory class $\phi$. This dependence is additive and determines an element $\tilde{e} \in \mathtt{Hom}^{cont}(K^0(B_+ \wedge X_+), \mathbb{Q}/\mathbb{Z})$.*

*3. The class $[\tilde{e}] \in Q_n(B, X)$ of this homomorphism does not depend on $l$ or the choice of the zero bordism of $(W, F, G)$.*

*4. The element $[\tilde{e}] \in Q_n(B, X)$ described in 3. only depends on the bordism class $x$. This dependence is additive so that we obtain a well-defined homomorphism*

$$\eta^{an} : \pi_n(MB \wedge X_+)_{tors} \to Q_n(B, X) \ .$$

*Proof.* On the one hand, we have $e \in \frac{1}{l}\mathbb{Z}/\mathbb{Z} \subseteq \mathbb{R}/\mathbb{Z}$. On the other hand, we know that the right-hand side of (54) depends continuously on the geometric data. This shows that $e$ does not depend on the geometric structures at all since two choices of geometric structures can be connected by a family. This proves 1.
The element $e$ is additive in the bundle $U$. It therefore descends uniquely to a homomorphism $\tilde{e} \in \mathtt{Hom}(K^0(B_+ \wedge X_+), \mathbb{Q}/\mathbb{Z})$. Since it factors over the restriction along the map $(f, g) : M \to B_+ \wedge X_+$ and $M$ is compact this homomorphism is continuous. This shows 2.
Assume that we have a second zero bordism $(W', F', G')$ of $l'(M, f, g)$ yielding $\tilde{e}' \in \mathtt{Hom}^{cont}(K^0(B_+ \wedge X_+), \mathbb{Q}/\mathbb{Z})$. Then by glueing along boundary components we can form the closed $n+1$-dimensional $B$-manifold $\tilde{W} := l'W \cup_{ll'M} lW'$ which comes with maps $\tilde{F} : \tilde{W} \to B$ and $\tilde{G} : \tilde{W} \to X$.



Pictures of $W$, $W'$, and $\frac{1}{2}\tilde{W}$ with $l=4$ and $l'=2$

Note that the tangential representative of the normal $Spin^c$-structures $(P,\kappa)$ and $(P',\kappa')$ come with isomorphisms of the type (47). Compatibility with these fixes the morphism which we have to use order to glue $P$ with $P'$. In this way we get a tangential representative of the normal $Spin^c$-structure on $\tilde{W}$. The triple $(\tilde{W},\tilde{F},\tilde{G})$ is thus cycle for a class

$$y:=[\tilde{W},\tilde{F},\tilde{G}]\in\pi_{n+1}(MB\wedge X_+)\ .$$

Then for $\phi\in K^0(B_+\wedge X_+)$ we get from the right-hand side of (54) that

$$\tilde{e}(\phi)-\tilde{e}'(\phi)=[\frac{1}{ll'}\langle\mathbf{Td}(T\tilde{W})\cup(\tilde{F},\tilde{G})^*\mathbf{ch}(\phi),[\tilde{W}]\rangle]\ .$$

Since $(\tilde{F},\tilde{G})^*\mathbf{Td}^{-1}=\mathbf{Td}(T\tilde{W})$ this is exactly the formula (32) for the evaluation of $\epsilon_K(\frac{1}{ll'}y)\in\pi_{n+1}(K\wedge MB\mathbb{Q}\wedge X_+)$ against $\mathtt{Thom}^K(\phi)\in K^0(MB\wedge X_+)$. Therefore the class

$[\tilde{e}] \in Q_n(X,B)$ is independent of the choice of $l$ and the zero bordism $(W,F,G)$. This finishes the verification of 3.
We observe that the map which associates to $(M,f,g)$ the class $[\tilde{e}] \in Q_n(B,X)$ is additive under disjoint unions. Moreover, if $(M,f,g)$ itself is zero bordant, i.e. we can find $(W,F,G)$ as above with $l=1$, then $[\tilde{e}]=0$. It follows that the construction above uniquely descends to a homomorphism

$$\eta^{an} : \pi_n(MG \wedge X_+)_{tors} \to Q_n(B,X) \ . \tag{55}$$

$\square$

Let us collect the essentials of this construction in the following definition.

**Definition 3.5** *We define $\eta^{an} := 0$ for even $n$. For odd $n$ we define the homomorphism*

$$\eta^{an} : \pi_n(MB \wedge X_+)_{tors} \to Q_n(B,X)$$

*by the following prescription: If $x \in \pi_n(MB \wedge X_+)$ is represented by $(M,f,g)$, then we choose a zero bordism $(W,F,G)$ of $l(M,f,g)$ for a suitable $l \in \mathbb{N}$. Let $\phi \in K^0(B_+ \wedge X_+)$ then we choose a bundle $U \to W$ which represents $(F,G)^*\phi$ in $K$-theory and whose restrictions to the $l$ copies of $M$ in the boundary are pairwise isomorphic.*
*We choose a $Spin^c$-geometry for $W$ and a geometric refinement $\mathbf{U}$ for $U$ whose restrictions to the $l$ copies of $M$ in the boundary of $W$ are again pairwise isomorphic. Then $\eta^{an}(x) \in Q_n(B,X)$ is represented by the homomorphism*

$$K^0(B_+ \wedge X_+) \ni \phi \mapsto [\frac{1}{l}\mathtt{index}((\not{D}_W \otimes \mathbf{U})_{APS})] \in \mathbb{Q}/\mathbb{Z} \cong \pi_{n+1}(K\mathbb{Q}/\mathbb{Z}) \ . \tag{56}$$

## 3.5 The secondary index theorem

In the Definitions 2.2 and 3.5 we have described homomorphisms

$$\eta^{top} : \pi_n(MB \wedge X_+)_{tors} \to Q_n(B,X) \ , \quad \eta^{an} : \pi_n(MB \wedge X_+)_{tors} \to Q_n(B,X) \ .$$

Both constructions follow a common idea. Given a torsion element $x \in \pi_n(MB \wedge X)_{tors}$ in a first step a lift $\hat{x} \in \pi_n(MB\mathbb{Q}/\mathbb{Z} \wedge X_+)$, respectively a geometric representative of such a lift, is chosen. The homotopy theoretic invariant $\eta^{top}(x)$ is the represented by the homomorphism $K^0(B_+ \wedge X_+) \to \mathbb{Q}/\mathbb{Z}$ induced by the lift in a homotopy theoretic way. The analytic variant $\eta^{an}(x)$ is represented by a homomorphisms, which this time is obtained by constructing a suitable family of Atiyah-Patodi-Singer index problems on the geometric representative of the lift $\hat{x}$. Because of these coincidences it is very natural to expect that the following theorem holds true.

**Theorem 3.6**

$$\eta^{an} = \eta^{top} \ .$$

*Proof.* The main task in the proof is to find a bridge where one can translate the analytic constructions going into the definition of $\eta^{an}$ to homotopy theory. What remains then is the identification of the resulting homotopy theoretic picture with $\eta^{top}$. An obvious option is to apply the $\mathbb{Z}/l\mathbb{Z}$-index theorem [33] directly to $\eta^{an}$ in order to express it in homotopy theoretic terms.

In this paper we decided to go a different path. It is interesting since it explains in greater detail in which sense the homotopy theoretic construction of $\eta^{top}$ and the geometric or analytic constructions involved in $\eta^{an}$ correspond to each other. Our bride between analysis and topology will be the identification of homotopy theoretic $K$-homology with the analytic picture [14] and the ordinary Atiyah-Singer index theorem for elliptic operators [10], respectively its local form described in [15, Ch. IV].

Some ideas of our proof of Theorem 3.6, in particular about the usage of Moore spaces, are taken from [33] and the proof of the $\mathbb{R}/\mathbb{Z}$-index theorem [8, Thm 5.3].

We start with a description of Moore spaces for cyclic groups. Moore spaces are related with Moore spectra as discussed in 2.3 via the suspension spectrum construction. Let $S^1 \to S^1$ be the $l$-fold covering of the pointed circle. Its mapping cylinder $Z_l$ and mapping cone $C_l$ fit into the cofibre sequence of spaces

$$S^1 \to Z_l \to C_l \xrightarrow{\partial} \Sigma S^1 \to \dots \ . \tag{57}$$

Note that the shifted suspension spectrum $\Sigma^{\infty-1}C_l$ is then a model for the Moore spectrum $\mathtt{M}\mathbb{Z}/l\mathbb{Z}$. Further note that the inclusion of the cylinder basis $S^1 \to Z_l$ is a homotopy equivalence. Hence we have equivalences $\Sigma^{\infty-1}S^1 \cong S \cong \mathtt{M}\mathbb{Z}$. Applying the functor $\Sigma^{\infty-1}$ to the sequence (57) and using these identifications we get the fibre sequence

$$\mathtt{M}\mathbb{Z} \xrightarrow{l} \mathtt{M}\mathbb{Z} \to \mathtt{M}\mathbb{Z}/l\mathbb{Z} \xrightarrow{\partial} \Sigma\mathtt{M}\mathbb{Z} \tag{58}$$

of Moore spectra. We use the Moore spectra $M\mathtt{M}\mathbb{Z}/l\mathbb{Z}$ and the sequence (58) as approximations for $\mathtt{M}\mathbb{Q}/\mathbb{Z}$ and (20) in the sense that

$$\mathtt{M}\mathbb{Q}/\mathbb{Z} \cong \mathtt{hocolim}_l\, \mathtt{M}\mathbb{Z}/l\mathbb{Z} \ .$$

The connecting maps are fixed by their compatibility with the inclusions

$$\mathbb{Z}/l\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \ , \quad [n] \mapsto [\frac{n}{l}] \ . \tag{59}$$

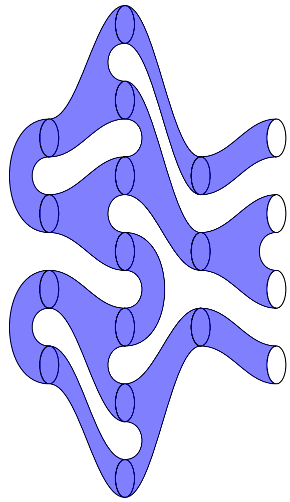
Smashing the sequence (58) with $MB \wedge X_+$ and taking homotopy groups we get a long exact sequence of abelian groups

$$\dots \to \pi_{n+1}(MB\mathbb{Z}/l\mathbb{Z} \wedge X_+) \xrightarrow{\partial} \pi_n(MB \wedge X_+) \xrightarrow{l} \pi_n(MB \wedge X_+) \to \dots \ . \tag{60}$$

Let $x = [M, f, g] \in \pi_n(MB \wedge X_+)$ be an $l$-torsion element. In the following we construct a geometric representative $(\tilde{W}, \tilde{F}, \tilde{G})$ of a lift $\hat{x} \in \pi_{n+1}(MB\mathbb{Z}/l\mathbb{Z} \wedge X_+)$, where $\tilde{F} : \tilde{W} \to B$ represents a $B$-structure and $\tilde{G} = (\tilde{G}_1, \tilde{G}_2) : \tilde{W} \to C_l \wedge X_+$. Here we implicitly use the identification of spectra

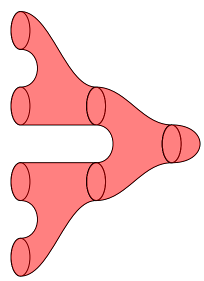
$$\Sigma MB\mathbb{Z}/l\mathbb{Z} \wedge X_+ \cong MB \wedge (C_l \wedge X_+) \ . \tag{61}$$

We will obtain $(\tilde{W}, \tilde{F}, \tilde{G})$ by closing up the boundary of the triple $(W, F, G)$ found in 3.4.



A picture of $S^1 \times W$

The details follow. We consider a two-sphere $S^2_l$ with $l$ holes.



A picture of $S^2_4 \times M$

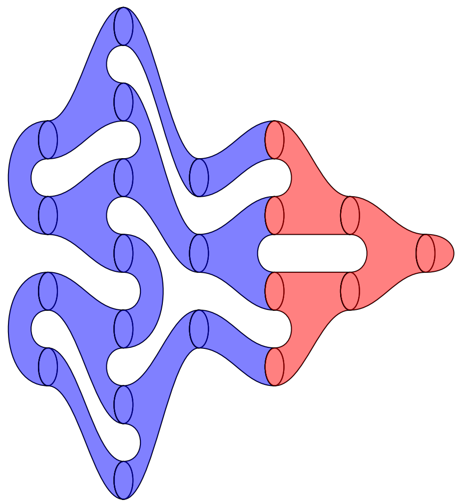
More precisely we let $S^2_l \subset S^2$ be the closed submanifold with boundary $\partial S^2_l \cong \bigsqcup_{i=1}^l S^1$ obtained by deleting the interiors of $l$ disjoint discs from $S^2$. The identification of the boundary with the $l$ copies of $S^1$ is fixed such that it preserves the natural orientations. We now have an identification

$$\partial(S^1 \times W) \cong l(S^1 \times M) \cong \partial(S^2_l \times M) \ .$$

We let

$$\tilde{W} := (S^1 \times W) \cup_{l(S^1 \times M)} (S^2_l \times M) \tag{62}$$

be the manifold obtained by glueing along the boundary.



A picture of $\tilde{W}$

We define $\tilde{F} : \tilde{W} \to B$ such that it restricts to

$$S^1 \times W \overset{\mathtt{pr}_W}{\to} W \overset{F}{\to} B \ , \quad S^2_l \times M \overset{\mathtt{pr}_M}{\to} M \overset{f}{\to} B \ .$$

Since $S^2_l$ and $S^1$ have stable normal framings the composition $\tilde{W} \overset{\tilde{F}}{\to} B \overset{(4)}{\to} BSpin^c$ represents a stable normal $Spin^c$-structure of $\tilde{W}$.
In a similar manner we define $\tilde{G}_2 : \tilde{W} \to X$ such that it restricts to

$$S^1 \times W \overset{\mathtt{pr}_W}{\to} W \overset{G}{\to} X \ , \quad S^2_l \times M \overset{\mathtt{pr}_M}{\to} M \overset{g}{\to} X \ .$$

We now consider the map

$$S^1 \times W \overset{\mathtt{pr}_{S^1}}{\to} S^1 \overset{i}{\to} C_l \ , \tag{63}$$

where $i : S^1 \to C_l$ is the identification of $S^1$ with the basis of the mapping cone. Note that the map

$$\sqcup_{j=1}^l i : \partial S^2_l \cong \bigsqcup_{j=1}^{l} S^1 \to C_l$$

can be extended to a map

$$g_1 : S^2_l \to C_l \ . \tag{64}$$

We can and will restrict the choice $g_1$ such that it is smooth on the preimage of a neighbourhood $\partial C_l \subset U \subset C_l$ of the cone basis $\partial C_l$, and regular values of $g_1$ in the interior $U \setminus \partial C_l$ have exactly one preimage. We advise the reader to make his own picture of this situation. The restriction of the map (63) to $\partial(S^1 \times W) \cong l(S^1 \times M)$ thus has an extension across the other part $S^2_l \times M$ of $\tilde{W}$ given by

$$S^2_l \times M \overset{\mathtt{pr}_{S^2_l}}{\to} S^2_l \overset{g_1}{\to} C_l \ ,$$

Alltogether we obtain the first component $\tilde{G}_1 : \tilde{W} \to C_l$ of the map $\tilde{G} = (\tilde{G}_1, \tilde{G}_2) : \tilde{W} \to C_l \wedge X_+$. The cycle $(\tilde{W}, \tilde{F}, \tilde{G})$ represents a class in

$$\hat{x} := [\tilde{W}, \tilde{F}, \tilde{G}] \in \pi_{n+2}(MB \wedge C_l \wedge X_+) \stackrel{(61)}{\cong} \pi_{n+1}(MB\mathbb{Z}/l\mathbb{Z} \wedge X_+) \ .$$

First of all we have an unreduced class $[\tilde{W}, \tilde{F}, \tilde{G}] \in \pi_{n+2}(MB \wedge (C_l \wedge X_+)_+)$. But it belongs to the reduced bordism group since $[\tilde{W}, \tilde{F}]$ is zero-bordant. To see this fill $S^1 \times W$ by $D^2 \times W$ and $S^2_l \times M$ by $D^3 \times M$ compatibly with the glueing (62).
Let $\partial : \pi_{n+1}(MB\mathbb{Z}/l\mathbb{Z} \wedge X_+) \to \pi_n(MB \wedge X_+)$ be the boundary as in (60).

**Lemma 3.7** *We have $\partial \hat{x} = x$*

*Proof.* The boundary operator $\partial$ in the Lemma is induced by the map denoted by the same symbol in (57)

$$\partial : C_l \stackrel{p}{\to} \Sigma S^1 \cong S^2 \ ,$$

where $p$ is the projection which contracts the cone basis to a point. Therefore $\partial \hat{x} \in \pi_{n+2}(MB \wedge S^2 \wedge X)$ is represented by $(\tilde{W}, \tilde{F}, (p \circ \tilde{G}_1, \tilde{G}_2))$. We must show that it corresponds to $x$ under the suspension isomorphism

$$\pi_n(MB \wedge X_+) \cong \pi_{n+2}(MB \wedge S^2 \wedge X_+) \ .$$

To this end we invert the suspension isomorphism in the geometric picture. This inverse is of course given by taking the inverse image of a regular point in $S^2$ of the corresponding component $p \circ \tilde{G}_1$ of the structure map. If we take the inverse image of a point in the neighbourhood $U \setminus \partial C_l$ mentioned above we exactly recover the representative $(M, f, g)$ of $x$. □

The construction of $\eta^{top}$ involves the $K$-homology of a based space $Y$ defined homotopy theoretically as $\pi_*(K \wedge Y)$. It is equivalent to the analytic picture introduced in [14]. The analytic $K$-homology is subsumed in the more general bivariant $KK$-theory (see [43] and the text book [17]) which allows to treat $K$-homology and cohomology on equal footing. Of particular importance for our purpose is that the product in $KK$-theory provides a description of the $\cap$-product between $K$-homology and $K$-theory which easily compares with the operation of twisting Dirac operators.
The unit of $K$-theory induces the map

$$\epsilon_K : \pi_{n+2}(MB \wedge C_l \wedge X_+) \to \pi_{n+2}(K \wedge MB \wedge C_l \wedge X_+) \ . \tag{65}$$

We use the Thom isomorphism for $MB$ in $K$-homology in order to identify

$$\mathtt{Thom}_K : \pi_{n+2}(K \wedge MB \wedge C_l \wedge X_+) \stackrel{\sim}{\to} \pi_{n+2}(K \wedge B_+ \wedge C_l \wedge X_+) \ . \tag{66}$$

Finally we use $KK$-theory in order represent this $K$-homology of a pointed space analytically. For the moment we assume that $X$ and $B$ are compact. This is no real restriction since we are calculating with a finite number of cycles at a time and their structure maps can only hit compact parts of the spaces $B$ and $X$. For a compact based space $Y$ we let $C(Y)$ denote the $C^*$-algebra of functions which vanish on the base point. Then by the equivalence between homotopy theoretic and analytic $K$-homology [14] we have an isomorphism

$$\pi_{n+2}(K \wedge B_+ \wedge C_l \wedge X_+) \cong KK_{n+2}(C(B_+ \wedge C_l \wedge X_+), \mathbb{C}) \ . \tag{67}$$

Recall that in Subsection 3.4 we have already chosen a Riemannian metric and a $Spin^c$-extension of the Levi-Civita connection $W$. We choose a Riemanian metric and a $Spin^c$-extension of the Levi-Civita connection on $S^1$. Then we get a corresponding geometric product structure on $S^1 \times W$. The choice of the geometry on $S^1$ also induces a geometric structure on the boundary $\partial S^2_l \cong lS^1$ which we extend to $S^2_l$, again with a product structure. We get a corresponding product metric and $Spin^c$-extension of the Levi-Civita connection on $S^2_l \times M$. These geometric structures glue nicely and give a Riemannian metric and a $Spin^c$-extension of the Levi-Civita connection on $\tilde{W}$. We let $\not{D}_{\tilde{W}}$ denote the corresponding Dirac operator. It acts on the complex spinor bundle $S(T\tilde{W})$. The Hilbert space $L^2(\tilde{W}, S(\tilde{W}))$ of square integrable sections of this bundle carries an action $\rho$ of the $C^*$-algebra $C(\tilde{W}_+)$ of continuous functions on $\tilde{W}$ by multiplication. The triple

$$(\not{D}_{\tilde{W}}) := (L^2(\tilde{W}, S(\tilde{W})), \not{D}_{\tilde{W}}, \rho)$$

is an unbounded Kasparov module for the pair of $C^*$-algebras $(C(\tilde{W}_+), \mathbb{C})$ and represents a class

$$[\not{D}_{\tilde{W}}] \in KK_{n+2}(C(\tilde{W}_+), \mathbb{C}) \ .$$

The map $(\tilde{F}, \tilde{G})$ induces a homomorphism of $C^*$-algebras

$$(\tilde{F}, \tilde{G})^* : C(B_+ \wedge C_l \wedge X_+) \to C(\tilde{W}_+)$$

which in turn induces the push-forward in analytic $K$-homology in the statement of the following Lemma.

**Lemma 3.8** *The image of the class $\hat{x} \in \pi_{n+2}(MB \wedge C_l \wedge X_+)$ under the composition of the unit (65), Thom isomorphism, (66) and the identification (67) is given by*

$$(\tilde{F}, \tilde{G})_*[\not{D}_{\tilde{W}}] \in KK_{n+2}(C(B_+ \wedge C_l \times X_+), \mathbb{C}) \ .$$

*Proof.* The image of $\hat{x}$ under the unit and Thom isomorphism is given by (14) as

$$\mathtt{Thom}_K(\epsilon_K(\hat{x}) = \beta(\Delta(\hat{x})) \in \pi_{n+2}(K \wedge B_+ \wedge C_l \wedge X_+) \ .$$

The triple $(\tilde{W}, \tilde{F}, \tilde{G})$ of a $B$-manifold with a map $\tilde{G} : \tilde{W} \to C_l \wedge X_+$ represents the class $\hat{x} \in \pi_{n+2}(MB \wedge C_l \wedge X_+)$. Then $\Delta(\hat{x}) = [\tilde{W}, \tilde{F}, (\tilde{F}, \tilde{G})]$. Formally we can view this as the push-forward of the $B$-bordism fundamental class of $\tilde{W}$ along the map $(\tilde{F}, \tilde{G})$. Its image under the $K$-orientation $\beta : MB \to K$ of $B$-bordism theory is then the push-forward of the $K$-theory fundamental class of $\tilde{W}$ associated to the $Spin^c$-structure along this map. In the analytic picture of $K$-homology the $K$-theory fundamental class of $\tilde{W}$ is represented by the $Spin^c$-Dirac operator. Hence it is equal to $[\not{D}_{\tilde{W}}]$. We thus get

$$\beta([\tilde{W},\tilde{F},(\tilde{F},\tilde{G})] = (\tilde{F},\tilde{G})_*[\not{D}_{\tilde{W}}] \ .$$

□

We let $\phi \in K^0(B_+ \wedge X_+)$. The pairing on the right-hand side in the following calculation in $\pi_{n+2}(K \wedge C_l) \cong \mathbb{Z}/l\mathbb{Z}$ is reminiscent to the evaluation occuring in the definition of $\eta^{top}$:

$$\begin{aligned}
\langle \mathtt{Thom}^K(\phi), \epsilon_K(\hat{x})\rangle &\overset{(17)}{=} \langle \phi, \mathtt{Thom}_K(\epsilon_K(\hat{x}))\rangle \\
&\overset{Lemma\ 3.8)}{=} \langle \phi, (\tilde{F},\tilde{G})_*[\not{D}_{\tilde{W}}]\rangle \\
&= \tilde{G}_{1*}([\not{D}_{\tilde{W}}] \cap (\tilde{F},\tilde{G}_2)^*\phi) \ .
\end{aligned}$$

We choose a geometric bundle $\tilde{\mathbf{V}}$ whose underlying $K$-theory class is equal to $(\tilde{F},\tilde{G}_2)^*\phi$. The restriction of the maps $\tilde{F}$ and $\tilde{G}_2$ to the part $S^1 \times W \subset \tilde{W}$ factor over the projection to $W$ and $(F,G) : W \to B_+ \wedge X_+$. Hence we can assume that the restriction of $\tilde{\mathbf{V}}$ to $S^1 \times W \subset \tilde{W}$ is isomorphic to the pull-back of the bundle $\mathbf{U}$ on $W$, if we allow some stablization of $\tilde{\mathbf{V}}$ and $\mathbf{U}$.
In the $KK$-picture the $\cap$-product

$$[\not{D}_{\tilde{W}}] \cap (\tilde{F},\tilde{G}_2)^*\phi \in KK_{n+2}(C(\tilde{W}_+),\mathbb{C})$$

is realized by the unbounded Kasparov module $(L^2(\tilde{W}, S(\tilde{W}) \otimes \tilde{\mathbf{V}}), \not{D}_{\tilde{W}} \otimes \tilde{\mathbf{V}}, \rho)$ associated to the twisted Dirac operator $\not{D}_{\tilde{W}} \otimes \tilde{\mathbf{V}}$, where $\rho$ again denotes the action of $C(\tilde{W}_+)$ on $L^2(\tilde{W}, S(\tilde{W}) \otimes \tilde{\mathbf{V}})$ by multiplication. Hence we have

$$[\not{D}_{\tilde{W}} \otimes \tilde{\mathbf{V}}] = [\not{D}_{\tilde{W}}] \cap (\tilde{F},\tilde{G}_2)^*\phi \ .$$

We conclude that $\eta^{top}(x) \in Q_n(X,B)$ is represented by the map

$$K^0(B_+ \wedge X_+) \ni \phi \mapsto \tilde{G}_{1*}[\not{D}_{\tilde{W}} \otimes \tilde{\mathbf{V}}] \in KK_{n+2}(C(C_l),\mathbb{C}) \cong \pi_{n+1}(K\mathbb{Z}/l\mathbb{Z}) \subset \pi_{n+1}(K\mathbb{Q}/\mathbb{Z}) \ , \tag{68}$$

where the last inclusion is induced by (59).
Next we want to calculate the element in $\mathbb{Z}/l\mathbb{Z}$ given by $\tilde{G}_{1*}([\not{D}_{\tilde{W}} \otimes \tilde{\mathbf{V}}])$. Since the usual index theorem [10] calculates integral indizes we have to construct and calculate an integral representative of this $\mathbb{Z}/l\mathbb{Z}$-valued index. The inclusion of the cone base $i : S^1 \to C_l$ induces a surjective map

$$\mathbb{Z} \cong \pi_{n+2}(K \wedge S^1) \to \pi_{n+2}(K \wedge C_l) \cong \mathbb{Z}/l\mathbb{Z} \ .$$

We try to construct a lift of $\tilde{G}_{1*}([\not{D}_{\tilde{W}} \otimes \tilde{\mathbf{V}}])$ to $\pi_{n+2}(K \wedge S^1)$ by providing a factorization $\gamma$ as in the diagram

$$\begin{tikzcd} & S^1 \arrow[d, "i"] \\ \tilde{W} \arrow[ur, dotted, "\gamma"] \arrow[r, "\tilde{G}_1"] & C_i \end{tikzcd} \ .$$

For our given representative such a factorization does not exist in general. The idea is to modify the representative without changing its $\mathbb{Z}/l\mathbb{Z}$-valued index such that this lift exists for the modified cycle.
Note that $M$ is a closed odd-dimensional manifold. The Dirac operator $\not{D}_M \otimes \mathbf{V}$ is selfadjoint. We can find a selfadjoint smoothing operator $Q$ on $L^2(M, S(M) \otimes V)$ such that $\not{D}_M \otimes \mathbf{V} + Q$ is invertible. In [23] such a perturbation was called a taming. As described in this reference such a taming can be lifted to the product $S^2_l \times M$ and also to collar neighbourhood $l(S^1 \times (-1, 0] \times M) \cong Z \subset S^1 \times W$ of $\partial(S^1 \times W)$. This lift is a selfadjoint operator $\bar{Q}$ on $L^2(Z \cup_{l(S^1\times M)} S^2_l \times M, S(\tilde{W}) \otimes \tilde{V})$ which is an integral operator along $M$ and local in the remaining directions. Let $\chi : \tilde{W} \to [0, 1]$ be a cut-off function which is supported on $Z \cup_{l(S^1\times M)} S^2_l \times M$, is equal to one in a neighbourhood of the subset $S^2_l \times M$, and only depends on the normal variable near $\partial(S^1 \times W)$. We define the extension $\tilde{Q} := \chi\hat{Q}\chi$ of $\bar{Q}$ to all of $\tilde{W}$. Note that $\tilde{Q}$ commutes with the image of $\tilde{G}^*_1(C(C_l))$. Adding $\tilde{Q}$ to $\not{D}_W \otimes \tilde{\mathbf{V}}$ gives a relatively compact perturbation. Therefore

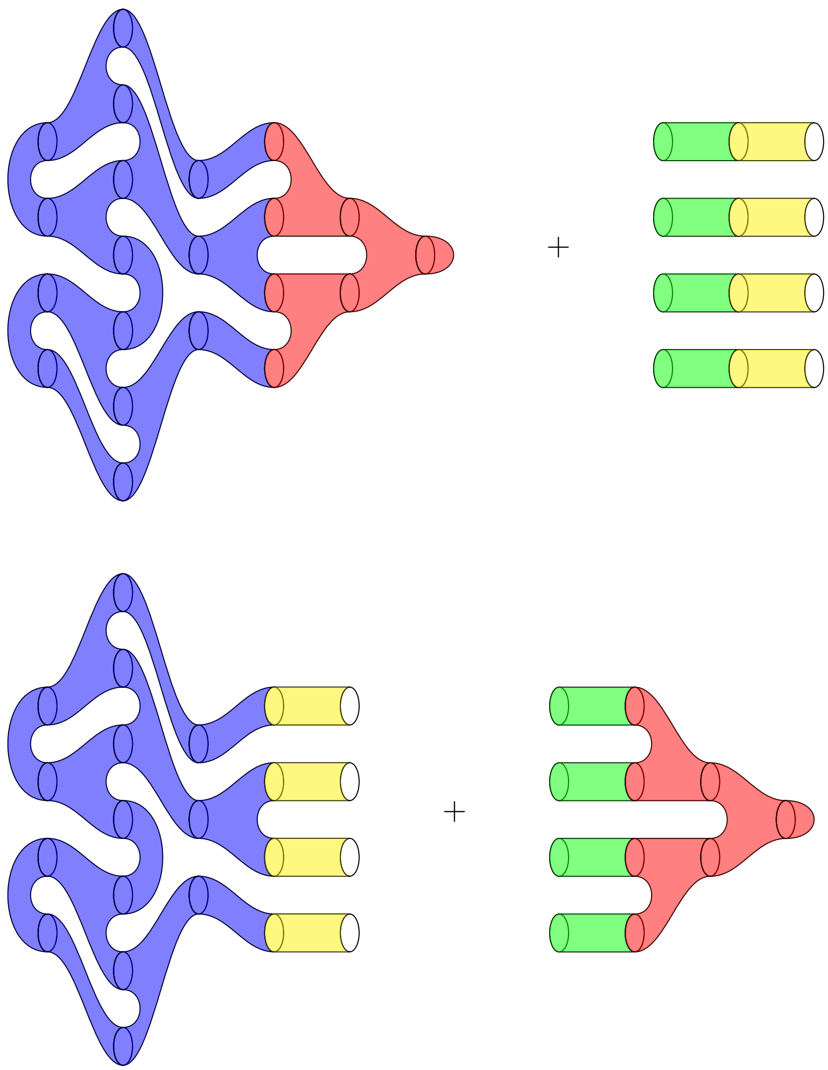
$$\tilde{G}_{1*}[\not{D}_{\tilde{W}} \otimes \tilde{\mathbf{V}}] = \tilde{G}_{1*}[\not{D}_{\tilde{W}} \otimes \tilde{\mathbf{V}} + \tilde{Q}] \ .$$

On the part $S^2_l \times M \subset \tilde{W}$ the perturbed operator $\not{D}_W \otimes \tilde{\mathbf{V}} + \tilde{Q}$ is invertible along the fibres of the projection to $S^2_l$. We define the manifold

$$\tilde{\tilde{W}} := S^1 \times W \cup_{l(S^1\times M)} l(S^1 \times [0, \infty) \times M) \ .$$

Its geometry is the cylindrical extension of the geometry of the piece $S^1 \times W$. In a similar manner we define the geometric bundle $\tilde{\tilde{\mathbf{V}}}$ on $\tilde{\tilde{W}}$ by a cylindrical extension of $\tilde{\mathbf{V}}_{|S^1\times W}$. We define an operator $\tilde{\tilde{Q}}$ similarly to $\tilde{Q}$ by lifting $Q$ to the cylinder $S^1 \times [0, \infty) \times M$ and cutting off in the interior of $S^1 \times W$. Finally we let $\tilde{\tilde{G}}_1 : \tilde{\tilde{W}} \to C_l$ be given by $G_1$ on $S^1 \times W$ and the radially constant extension to of $(\tilde{G}_1)_{|\partial(S^1\times W)}$ to the cylinder $l([0, \infty) \times S^1 \times M)$. The operator $\not{D}_{\tilde{\tilde{W}}} \otimes \tilde{\tilde{\mathbf{V}}} + \tilde{\tilde{Q}}$ is invertible along the fibre $M$ of the projection from the cylindrical end of $\tilde{\tilde{W}}$ to $S^1 \times [0, \infty)$. Therefore $(L^2(\tilde{\tilde{W}}, S(T\tilde{\tilde{W}}) \otimes \tilde{\tilde{V}}), \not{D}_{\tilde{\tilde{W}}} \otimes \tilde{\tilde{\mathbf{V}}} + \tilde{\tilde{Q}}, \tilde{\tilde{\rho}})$ is a Kasparov module over the $C^*$-algebra $C(\tilde{\tilde{W}})$ of bounded continuous functions.
The operators $\not{D}_W \otimes \tilde{\mathbf{V}} + \tilde{Q}$ and $\not{D}_{\tilde{\tilde{W}}} \otimes \tilde{\tilde{\mathbf{V}}} + \tilde{\tilde{Q}}$ coincide on $S^1 \times W$ and are invertible along the fibres $M$ outside of this submanifold of $\tilde{W}$ and $\tilde{\tilde{W}}$. In this situation we can apply a relative index theorem (the version [22]) in order to get

$$\tilde{G}_{1*}[\not{D}_{\tilde{W}} \otimes \tilde{\mathbf{V}}] = \tilde{\tilde{G}}_{1*}[\not{D}_{\tilde{\tilde{W}}} \otimes \tilde{\tilde{\mathbf{V}}} + \tilde{\tilde{Q}}] \ .$$



*A picture of the relative index theorem. The operator is invertible on the parts which are not blue. The index of the operator associated to the upper picture is the index of its left part $\tilde{W}$. The index is preserved under cut-and paste as indicated. The index of the operator associated to the lower picture is again the index of the left part $\tilde{\tilde{W}}$.*

Note the factorization $\tilde{\tilde{G}}_1 : \tilde{\tilde{W}} \stackrel{\mathtt{pr}_{S^1}}{\to} S^1 \stackrel{i}{\to} C_l$, where the last map is the embedding of the cone basis. Therefore

$$\mathtt{pr}_{S^1 *}[\not{D}_{\tilde{\tilde{W}}} \otimes \tilde{\tilde{\mathbf{V}}} + \tilde{\tilde{Q}}] \in \pi_{n+2}(K \wedge S^1) \cong \mathbb{Z}$$

represents the desired integral lift.

Note that $\tilde{\tilde{W}} \cong S^1 \times \hat{W}$, where $\hat{W} = W \cup_{\partial W} l([0,\infty) \times M)$. We equip $\hat{W}$ with the cylindrical extension of the geometry of $W$. Similarly we let $\hat{\mathbf{U}}$ be the geometric bundle

on $\hat{W}$ obtained by the cylindrical extension of $\mathbf{U}$. The taming $Q$ has a unique lift $\hat{Q}$ to $\hat{W}$ such that its lift to the product $\tilde{\hat{W}} = S^1 \times \hat{W}$ coincides with $\tilde{\hat{Q}}$. Then the whole operator $\not{D}_{\tilde{\hat{W}}} \otimes \tilde{\hat{\mathbf{V}}} + \tilde{\hat{Q}}$ is the lift of $\not{D}_{\hat{W}} \otimes \hat{\mathbf{U}} + \hat{Q}$ to this product. Therefore $\mathtt{pr}_{S^1 *}[\not{D}_{\tilde{\hat{W}}} \otimes \tilde{\hat{\mathbf{V}}} + \tilde{\hat{Q}}] \in \pi_{n+2}(K \wedge S^1)$ represents the suspension of $[\not{D}_{\hat{W}} \otimes \hat{\mathbf{U}} + \hat{Q}] \in \pi_{n+1}(K) \cong \mathbb{Z}$.
We now calculate this index. The index theory for these kinds of perturbations of Dirac operators has been developed in [23]. In the language of this reference the operator $Q$ defines taming $(M \otimes \mathbf{V})_t$ of the geometric manifold $M \otimes \mathbf{V}$ and a boundary taming $(W \otimes \mathbf{U})_{bt}$ of the geometric manifold $W \otimes \mathbf{U}$. The integer $[\not{D}_{\hat{W}} \otimes \hat{\mathbf{U}} + \hat{Q}]$ is the index of the boundary tamed geometric manifold $\mathtt{index}((W \otimes \mathbf{U})_{bt})$. The index theorem [23, Thm. 4.18] gives

$$\mathtt{index}((W \otimes \mathbf{U})_{bt}) = \int_W p_{n+1}(\mathbf{Td}(\tilde{\nabla}^{TW}) \wedge \mathbf{ch}(\nabla^U)) - l\eta((M \otimes \mathbf{V})_t) \ .$$

In $\mathbb{R}/\mathbb{Z}$ we have $[\eta((M \otimes \mathbf{U})_t)] = \xi(D_M \otimes \mathbf{V})$. Hence by comparison with (52) we get the equality in $\mathbb{Q}/\mathbb{Z}$

$$[\frac{1}{l}\mathtt{index}((W \otimes \mathbf{U})_{bt})] = [\frac{1}{l}\mathtt{index}((\not{D}_W \otimes \mathbf{U})_{APS})] \ .$$

In view of the construction of $\eta^{an}$, in particular of (56), we see that the map (68) also represents $\eta^{an}(x)$. This finishes the proof of Theorem 3.6. □

# 4 An intrinsic formula

## 4.1 Motivation

In a typical situation for the theory of the present paper one is given a geometric representative $(M, f, g)$ for a torsion class $x = [M, f, g] \in \pi_n(MB \wedge X_+)$ and wants to calculate the universal $\eta$-invariants $\eta^{top}(x) = \eta^{an}(x) \in Q_n(B, X)$. The expressions for the universal $\eta$-invariant that we have at our disposal at the moment share the disadvantage that one has to find a lift $\hat{x} \in \pi_{n+1}(MB\mathbb{Q}/\mathbb{Z} \wedge X_+)$ or a geometric zero bordism $(W, F, G)$ of $l$ copies of $(M, f, g)$ explicitly. It is at this point where differential and spectral geometry helps. Often the cycle $(M, f, g)$ already comes with geometric structures, e.g. connections on appropriate bundles. In the present section we develop a generalization of Chern-Weil theory and Cheeger-Simons [26]characteristic classes which is designed to finally obtain formulas for the universal $\eta$-invariant which are intrinsic in the cycle $(M, f, g)$.
The main new object is the notion of a geometrization of $(M, f, g, \tilde{\nabla})$ which is defined in Definition 4.2. The notion of a geometrization involves differential $K$-theory which is reviewed in Subsection 4.2. In the Subsection 4.3 further we show existence of geometrizations and study their functorial properties. In Subsection 4.4 we introduce a special class of geometrizations which we called good. In contrast to general geometrizations they have the property that they extend over zero bordisms. The main result is the intrinsic formula for the universal $\eta$-invariant formulated in Theorem 4.12.

## 4.2 Review of Differential $K$-theory

The main technical tool in the definition of a geometrization is the notion of a differential extension of $K$-theory, or shorter, differential $K$-theory. Recall that Chern classes of complex vector bundles take values in integral cohomology. Complex vector bundles with hermitean connections have refined Chern classes in differential integral cohomology. This theory has been introduced in [26] where differential integral cohomology classes are called differential characters. The relation of differential $K$-theory to topological $K$-theory is very similar to the relation of differential integral cohomology with integral cohomology. In particular, a complex vector bundle represents a $K$-theory class, and the datum of a hermitean connection refines this class to an element differential $K$-theory.
Differential $K$-theory by now also has some history [32], [38], [24]. Nevertheless we think that it is less standard so that we will recall the relevant structures in the following. A complete model has been constructed in [24], and by [25] we know that the properties listed below uniquely characterize differential $K$-theory so that one can take and use these properties as axioms.
Differential $K$-theory is a five-tuple

$$(\hat{K}, I, R, a, \int)$$

of the following objects. The first entry is a contravariant functor

$$\hat{K} : \mathtt{smooth\ manifolds} \longrightarrow \mathbb{Z}/2\mathbb{Z} - \mathtt{graded\ commutative\ rings}\ .$$

The remaining entries are natural transformations between functors. The domains and ranges of the first three are given by

$$I : \hat{K} \to K\ , \quad R : \hat{K} \to \Omega P_{cl}\ .$$

$$a : \Omega P/\mathtt{im}(d)[1] \to \hat{K}$$

Here $\Omega P(M) := \Omega(M)[b, b^{-1}]$ is denotes the space of two-periodic smooth differential forms on $M$. By $\Omega P_{cl}(M) \subseteq \Omega P(M)$ we denote its subspace of closed forms. The transformations $R$ and $I$ preserve the ring structures while $a$ is just additive. These transformations are compatible in the sense that for every manifold $M$ the following diagram commutes

$$\begin{tikzcd}
 & \Omega P^{*-1}(M)/\mathtt{im}(d) \arrow[rr, "d"] \arrow[dr, "a"] & & \Omega P^*_{cl}(M) \arrow[dr, "Rham"] & \\
HP\mathbb{R}^{*-1}(M_+) \arrow[ur] \arrow[dr] & & \hat{K}^*(M) \arrow[ur, "R"] \arrow[dr, "I"] & & HP\mathbb{R}^*(M_+) \\
 & K\mathbb{R}/\mathbb{Z}^{*-1}(M_+) \arrow[rr, "Bockstein"] \arrow[ur] & & K^*(M_+) \arrow[ur, "\mathbf{ch}"] &
\end{tikzcd} .$$

Here we define $HP\mathbb{R}$ similarly as $HP\mathbb{Q}$ in (28). The flat part of differential $K$-theory is defined as the kernel of the curvature transformation $R$. It is canonically isomorphic to $\mathbb{R}/\mathbb{Z}$-$K$-theory (with a shift), i.e. we have

$$\hat{K}^*_{flat}(M) := \mathtt{ker}(R : \hat{K}^*(M) \to \Omega P^*_{cl}(M)) \cong K\mathbb{R}/\mathbb{Z}^{*-1}(M_+) \ . \tag{69}$$

Furthermore, we have the equalities

$$a(x) \cup y = a(x \wedge R(y)) \ , \quad \text{for all} \quad x \in \Omega P^{*-1}(M)/\mathtt{im}(d) \ , \quad y \in \hat{K}^*(M) \ ,$$

and the sequence

$$K^{*-1}(M_+) \stackrel{\mathbf{ch}}{\to} \Omega P^{*-1}(M)/\mathtt{im}(d) \stackrel{a}{\to} \hat{K}^*(M) \stackrel{I}{\to} K^*(M_+) \to 0 \tag{70}$$

is exact. The integration is a natural transformation

$$\int : \hat{K}^*(S^1 \times M) \to \hat{K}^{*-1}(M)$$

whose existence and compatibility with the other structures fixes the odd part of the differential extension uniquely up to unique isomorphism as discussed in [25]. Since we do not need the integration in the present paper we will not write out the long list of these compatibilities explicitly.
Let $\mathbf{V} = (V, h^V, \nabla^V)$ be a geometric bundle on the manifold $M$, where $h^V$ is a hermitean metric which is preserved by the connection $\nabla^V$. Then we have a natural class

$$[\mathbf{V}] \in \hat{K}^0(M) \ . \tag{71}$$

This class is in fact tautological in the model [24] in view of [24, 2.1.4]. It satisfies

$$I([\mathbf{V}]) = [V] \in K^0(M_+) \ , \quad R([\mathbf{V}]) = \mathbf{ch}(\nabla^V) \in \Omega P^0_{cl}(M) \ .$$

## 4.3 Geometrizations

Let $M$ be a compact manifold equipped with maps $f : M \to B$ and $g : M \to X$. At the moment we do not require any connection of $f$ with the tangent bundle. Nevertheless we must imitate this situation. We can assume that $f$ has a factorization over $\tilde{f} : M \to BSpin^c(k)$ which classifies a $Spin^c(k)$-bundle $\tilde{f}^*Q_k \in Spin^c(\tilde{f}^*\xi_k^{Spin^c})$ on $M$. The role of the tangent bundle is taken by the choice of a complementary $Spin^c$-bundle. In detail, we choose an $l$-dimensional oriented euclidean vector bundle $\eta \to M$ for some $l \geq 0$ together with an orientation preserving isomorphism of euclidean vector bundles.

$$\eta \oplus \tilde{f}^*\xi_k^{Spin^c} \cong M \times \mathbb{R}^{l+k} \ . \tag{72}$$

Then we choose a $Spin^c$-structure $P \in Spin^c(\eta)$ together with an isomorphism

$$P \otimes \tilde{f}_k^* Q_k \cong Q(l+k) \ ,$$

where we use the isomorphism (72) in order view the left- and right-hand sides in the same groupoid $Spin^c(M\times\mathbb{R}^{l+k})$ (see Subsection 3.3 for details).
We choose a connection $\tilde{\nabla}$ on $P$ and get an induced Todd form $\mathbf{Td}(\tilde{\nabla})\in\Omega P^0_{cl}(M)$ which represents the class $f^*\mathbf{Td}^{-1}\in HP\mathbb{Q}^0(M_+)$.
We now consider a continuous homomorpism

$$\mathcal{G}:K^0(B_+\wedge X_+)\to\hat{K}^0(M)\ ,$$

where the domain has the profinite topology (see Subsection 2.2) and the target is discrete. Since $\Omega P^0_{cl}(M)$ is a rational vector space and

$$\mathbf{Td}^{-1}\cup\mathbf{ch}(\dots)\otimes\mathbb{Q}:K^0(B_+\wedge X_+)\otimes\mathbb{Q}\to HP\mathbb{Q}^0(B_+\wedge X_+)$$

is an isomorphism onto a dense subspace, there exists a unique continuous factorization $c_{\mathcal{G}}$ in the following diagram

$$\begin{tikzcd} K^0(B_+\wedge X_+) \arrow[r, "\mathcal{G}"] \arrow[d, "\mathbf{Td}^{-1}\cup\mathbf{ch}(\dots)"] & \hat{K}^0(M) \arrow[d, "\mathbf{Td}(\tilde{\nabla})\wedge R(\dots)"] \\ HP\mathbb{Q}^0(B_+\wedge X_+) \arrow[r, "c_{\mathcal{G}}"] & \Omega P^0_{cl}(M) \end{tikzcd} \quad .$$

**Definition 4.1** *The map $c_{\mathcal{G}}$ is called the cohomological character of $\mathcal{G}$.*

We say that the cohomological character $c_{\mathcal{G}}$ preserves degree if it preserves the decompositions

$$HP\mathbb{Q}^0(B_+\wedge X_+)\cong\prod_{k\in\mathbb{Z}}b^{-k}\tilde{H}^{2k}(B_+\wedge X_+;\mathbb{Q})\ ,\quad \Omega P^0_{cl}(M)\cong\prod_{k\in\mathbb{Z}}b^{-k}\Omega^{2k}_{cl}(M)\ .$$

**Definition 4.2** *A geometrization of $(M,f,g,\tilde{\nabla})$ is a continuous map*

$$\mathcal{G}:K^0(B_+\wedge X_+)\to\hat{K}^0(M)$$

*such that the following diagram*

$$\begin{tikzcd} & \hat{K}^0(M) \arrow[d, "I"] \\ K^0(B_+\wedge X_+) \arrow[ur, dotted, "\mathcal{G}"] \arrow[r, "(f,g)^*"] & K^0(M) \end{tikzcd}$$

*commutes and the associated cohomological character $c_{\mathcal{G}}$ preserves degree.*

As already indicated in the Introduction 1 the notion of a geometrization generalizes the notion of a connection. This is demonstrated in Lemma 5.12 for the case $B=BSpin$ and $X=*$. At this place we will discuss another example where we put $B$ in the background and consider connections on a bundle classfied by the auxiliary map $g:M\to X$. Let us assume that we already have a geometrization $\mathcal{G}^0$ of $(M,f,\tilde{\nabla})$. Its existence is garanteed by Proposition 4.4. We now consider a compact Lie group $\Gamma$ and set $X=B\Gamma$. The map $g:M\to B\Gamma$ classifies a $\Gamma$-principal bundle $R\to M$. We let $\nabla^R$ be a connection on $R$.

**Lemma 4.3** *There exists a natural geometrization $\mathcal{G}$ of $(M, f, g, \tilde{\nabla})$ associated to this data.*

*Proof.* The completion theorem [9] gives an isomorphism $K^0(B\Gamma_+) \cong R(\Gamma)\hat{}_{I_\Gamma}$ of topological groups, where $I_\Gamma \subseteq R(\Gamma)$ is the dimension-ideal of the integral representation ring. We consider a representation $\sigma : \Gamma \to U(m_\sigma)$ which represents an element $[\sigma] \in K^0(B\Gamma)$. The associated complex vector bundle $V_\sigma := R \times_{\Gamma,\sigma} \mathbb{C}^{m_\sigma}$ on $M$ then represents the element $[V_\sigma] = f^*[\sigma] \in K^0(M_+)$. This bundle comes with a hermitean metric $h^{V_\sigma}$ and a metric connection $\nabla^{V_\sigma}$ induced by $\nabla^R$. We therefore get a geometric bundle $\mathbf{V}_\sigma := (V_\sigma, \nabla^{V_\sigma}, \nabla^{V_\sigma})$. It represents the class $[\mathbf{V}_\sigma] \in \hat{K}^0(M)$ refining $[V_\sigma] \in K^0(M_+)$, i.e we have $I([\mathbf{V}_\sigma]) = [V_\sigma]$, see (71). Let $\phi \in K^0(B_+)$. Then we get the element $\phi \times [\sigma] \in K^0(B_+ \wedge B\Gamma_+)$. We define

$$\mathcal{G}(\phi \times [\sigma]) := \mathcal{G}^0(\phi) \cup [\mathbf{V}_\sigma] \ .$$

This construction defines $\mathcal{G}$ by linear extension on a dense subgroup of $K^0(B_+ \wedge B\Gamma_+)$. We now show that the map $\mathcal{G}$ extends by continuity to all of $K^0(B_+ \wedge B\Gamma_+)$ and defines a geometrization of $(M, f, g, \tilde{\nabla})$. Indeed, the map $R(\Gamma) \to \hat{K}^0(M)$ induced by $\sigma \mapsto [\mathbf{V}_\sigma]$ is multiplicative and annihilates $I_\Gamma^{2n+1}$. Since $\mathcal{G}^0$ is continuous, this implies that $\mathcal{G}$ is continuous. We let $c_\Gamma : HP\mathbb{Q}^0(B\Gamma_+) \to \Omega P^0_{cl}(M)$ be the unique continuous map such that $\mathbf{ch}(\nabla^{V_\sigma}) = c_\Gamma(\mathbf{ch}([\sigma]))$. Note that $c_\Gamma$ preserves degree. Since the cohomological character $c_{\mathcal{G}^0}$ preserves degree, the cohomological character $c_{\mathcal{G}} = c_{\mathcal{G}^0} \times c_\Gamma$ of $\mathcal{G}$ preserves degree, too. $\square$

The geometrization $\mathcal{G}$ allows to recover the Chern character form of $\nabla^{V_\sigma}$ by

$$\mathbf{ch}(\nabla^{V_\sigma}) = \mathbf{Td}(\tilde{\nabla})^{-1} \wedge R(\mathcal{G}(1 \otimes [\sigma])) \ .$$

It also allows to partially recover transgressions. If $\nabla^{R\prime}$ is a second connection on $R$ and $\mathcal{G}'$ is the associated geometrization, then

$$\mathcal{G}'(1 \otimes [\sigma]) - \mathcal{G}'_M(1 \otimes [\sigma]) = a(\mathbf{Td}(\tilde{\nabla}) \wedge \tilde{\mathbf{ch}}(\nabla^{V_\sigma\prime}, \nabla^{V_\sigma})) \ .$$

Here $\tilde{\mathbf{ch}}(\nabla^{V_\sigma\prime}, \nabla^{V_\sigma}) \in \Omega P^{-1}(M)$ denotes the transgression form which satisfies

$$d\tilde{\mathbf{ch}}(\nabla^{V_\sigma\prime}, \nabla^{V_\sigma}) = \mathbf{ch}(\nabla^{V_\sigma\prime}) - \mathbf{ch}(\nabla^{V_\sigma}) \ .$$

The following Proposition 4.4 asserts that geometrizations exist. Its proof uses the functoriality of geometrizations in the pair $(B, X)$. Consider a pair of maps $\phi$ and $\psi$ as in (39). Given a geometrization $\mathcal{G}$ of $(M, \phi \circ f, \psi \circ g, \tilde{\nabla})$ we get a geometrization

$$(\phi, \psi)_* \mathcal{G} := \mathcal{G} \circ (\phi, \psi)^* \tag{73}$$

of $(M, f, g, \tilde{\nabla})$.

Note that our standing assumption is that $M$ is compact.

**Proposition 4.4** *Given $(M, f, g, \tilde{\nabla})$ there exists a geometrization.*

*Proof.* Since $M$ is compact the maps $f$ and $g$ factor over compact subspaces of $B$ or $X$ respectively. In view of the functoriality of the geometrization (73) we can assume that $B$ and $X$ are compact. Then $K^0(B_+ \wedge X_+)$ is a finitely generated abelian group. We choose a (non-canonical) decomposition

$$K^0(B_+ \wedge X_+) \cong A_{tors} \oplus A_{free}$$

into a torsion and a free part. We write

$$A_{tors} := \bigoplus_{y \in I} y\mathbb{Z}/\mathtt{ord}(y)\mathbb{Z}$$

for some set of generators $I \subset A_{tors}$. For all $y \in I$, using the exactess at the right end of (70), we choose $\tilde{y}_0 \in \hat{K}^0(M)$ such that $I(\tilde{y}_0) = (f,g)^* y$. Then $\mathtt{ord}(y)\tilde{y}_0 = a(\omega_y)$ for some $\omega_y \in \Omega P^{-1}(M)/\mathtt{im}(d)$, again by (70). We define

$$\tilde{y} := \tilde{y}_0 - a(\frac{1}{\mathtt{ord}(y)}\omega_y) \ .$$

Then $\mathtt{ord}(y)\tilde{y} = 0$ and we can define $\mathcal{G}_{|A_{tors}} : A_{tors} \to \hat{K}^0(M)$ such that $\mathcal{G}(y) = \tilde{y}$ for all $y \in I$. Since $\mathbf{Td}^{-1} \wedge \mathbf{ch}$ vanishes on $A_{tors}$ and $\mathcal{G}_{|A_{tors}}$ maps to flat classes it is clear that the cohomological character of this part of $\mathcal{G}$ preserves degree.
We now come to the free part. We choose a basis $J \subset A_{free}$ and classes $\tilde{z}_0 \in \hat{K}^0(M)$ such that $I(\tilde{z}_0) = (f,g)^* z$ for all $z \in J$. We further choose a basis $J' \subset A_{free} \otimes \mathbb{Q}$ such that $\{\mathbf{Td}^{-1} \wedge \mathbf{ch}(z')\}_{z' \in J'}$ is a homogeneous basis with respect to the decomposition

$$HP\mathbb{Q}^0(B_+ \wedge X_+) \cong \bigoplus_{m \in \mathbb{Z}} b^{-m} H^{2m}(B_+ \wedge X_+; \mathbb{Q}) \ .$$

We define the even integers $n_{z'} := \deg(\mathbf{Td}^{-1} \wedge \mathbf{ch}(z'))$ for all $z' \in J'$. Then there exists an invertible rational $(J, J')$-indexed matrix $A$ such that $z = \sum_{z' \in J'} A_{zz'} z'$ for all $z \in J$. We now can choose a collection of forms $\alpha_{z'} \in \Omega P^{-1}(M)/\mathtt{im}(d)$ for $z' \in J'$ such that

$$\sum_{z \in J} A^{-1}_{z'z} \ \mathbf{Td}(\tilde{\nabla}) \wedge R(\tilde{z}_0) - d\alpha_{z'} \in b^{-\frac{n_{z'}}{2}} \Omega^{n_{z'}}_{cl}(M) \subseteq \Omega P^0_{cl}(M)$$

for all $z' \in J'$. We define

$$\mathcal{G}_{|A_{free}} : A_{free} \to \hat{K}^0(M)$$

by linear extension such that

$$\mathcal{G}(z) = \tilde{z}_0 - a(\mathbf{Td}(\tilde{\nabla})^{-1} \wedge \sum_{z' \in J'} A_{zz'} \alpha_{z'}) \ .$$

By construction its cohomological character preserves degree.

□

Geometrizations can be pulled back along $Spin^c$-maps over $X$. Let $(M', f', g')$ be a manifold with maps $f' : M \to B$ and $g' : M' \to X$. We consider a smooth map $h : M' \to M$ that $f \circ h$ is homotopic to $f'$. This implies that we can choose a stable isomorphism of complementary bundles

$$\eta' \oplus (M' \times \mathbb{R}^s) \cong \eta \oplus (M \times \mathbb{R}^t) \ ,$$

This is exactly the situation where we can talk about a refinement of $h$ to a $Spin^c$-map by choosing an isomorphism

$$P' \otimes Q(s) \cong h^* P \otimes Q(t) \ . \tag{74}$$

The refinement of $h$ to a $Spin^c$-map is given by the joint homotopy class of this pair isomorphisms. In order to define a pull-back of geometrizations we need of course also a compatibility of the maps to $X$, i.e. we require that $g'$ and $g \circ h$ are homotopic. In this case we speak of a $Spin^c$-map over $X$.
Assume now that we have connections $\tilde{\nabla}$ on $P$ and $\tilde{\nabla}'$ on $P'$. They induce connections on the stabilizations $P \otimes Q(t)$ and $P' \otimes Q(s)$. We thus can define the transgression

$$\tilde{\mathbf{Td}}(h^*\tilde{\nabla}, \tilde{\nabla}') \in \Omega P^{-1}(M')/\mathtt{im}(d)$$

where we use the isomorphism (74) in order to compare the stabilization of $h^*\tilde{\nabla}$ with that of $\tilde{\nabla}'$ on the same bundle. For this transgression to be well-defined it is important that the isomorphism (74) is fixed up to homotopy by the structure of a $Spin^c$-map on $h$. The transgression satisfies

$$d\tilde{\mathbf{Td}}(h^*\tilde{\nabla}, \tilde{\nabla}') = h^*\mathbf{Td}(\tilde{\nabla}) - \mathbf{Td}(\tilde{\nabla}') \ .$$

Let $\mathcal{G}$ be a geometrization of $(M, f, g, \tilde{\nabla})$.

**Lemma 4.5** *If $h : M' \to M$ is a $Spin^c$-map over $X$, then there exists a functorial construction of a pulled-back geometrization $\mathcal{G}' := h^*\mathcal{G}$ of $(M', f', g', \tilde{\nabla}')$.*

*Proof.* By our assumptions the equivalence class

$$\beta := \tilde{\mathbf{Td}}(h^*\tilde{\nabla}, \tilde{\nabla}')\mathbf{Td}(\tilde{\nabla}')^{-1} \in \Omega P^{-1}(M')/\mathtt{im}(d) \tag{75}$$

of forms is defined. It satisfies

$$d\beta = h^*\mathbf{Td}(\tilde{\nabla})\mathbf{Td}(\tilde{\nabla}')^{-1} - 1 \ .$$

We define the pull-back $\mathcal{G}' := h^*\mathcal{G}$ by

$$\mathcal{G}'(y) := h^*\mathcal{G}(y) + a(\beta \wedge h^*R(\mathcal{G}(y))) \ , \quad y \in K^0(B_+ \wedge X_+) \ , \tag{76}$$

where $a$ and $R$ belong to the structure maps of differential $K$-theory. We have by construction

$$\mathbf{Td}(\tilde{\nabla}') \wedge R(\mathcal{G}'(y)) = h^*(\mathbf{Td}(\tilde{\nabla}) \wedge R(\mathcal{G}(y)))$$

and hence the equality $c_{\mathcal{G}'} = h^* c_{\mathcal{G}}$ of cohomological characters. Since the cohomological character $c_{\mathcal{G}}$ preserves degree, so does the cohomological character of $\mathcal{G}'$.
We show that the pull-back is functorial. We consider a second tuple $(M'', f'', g'', \tilde{\nabla}'')$ with a $Spin^c$-map $h' : M'' \to M'$ over $X$ and the associated transgression form $\beta'$. Then we have for the iterated pull-back

$$\begin{aligned}
\mathcal{G}''(y) &= h'^*(h^*(\mathcal{G}(y))) + h'^*(a(\beta \wedge h^* R(\mathcal{G}(y)))) + a(\beta' \wedge h'^* R(\mathcal{G}'(y))) \\
&= (h \circ h'^*)^*(\mathcal{G}(y)) \\
&+ a(h'^*\beta \wedge h'^*(h^*(R(\mathcal{G}(y))) + \beta' \wedge h'^*(h^* R(\mathcal{G}(y))) + \beta' \wedge h'^* d\beta \wedge h'^*(h^*(R(\mathcal{G}(y)))))
\end{aligned}$$

Let $\tilde{\beta}$ be the transgression form for the composition $h \circ h'$ of $Spin^c$-maps over $X$. Then we must show that

$$\tilde{\beta} - h'^*\beta + \beta' + h'^*\beta' \wedge d\beta \in \mathtt{im}(d)$$

This follows from

$$d(h'^*\beta + \beta' + h'^*\beta' \wedge d\beta) = h'^* h^* \mathbf{Td}(\tilde{\nabla})^{-1} \mathbf{Td}(\tilde{\nabla}'') - 1 = d\tilde{\beta}$$

and the fact that all these forms are defined by transgressions. □

The identity of $M$ refines to a $Spin^c$-map over $X$ in a natural way by choosing the identity in (74). The pull-back of geometrizations for the identity of $M$ can be used to transfer a geometrization defined for one choice of the connection $\tilde{\nabla}$ to a second choice. This allows to define a notion of geometrization which is independent of the choice of the connection. This could play a role of one wants to classify geometrizations. We will not pursue that goal in the present paper.

## 4.4 Good geometrizations

Assume that $(W, F, G)$ is a zero-bordism of the $n$-dimensional cycle $(M, f, g)$. We choose a Riemannian metric on $M$ and extend it to $W$ with product sructures. We fix tangential representatives $P(TW) \in Spin^c(TW)$ and $P(TM) \in Spin^c(TM)$ of the normal $Spin^c$-structures on $W$ and $M$, see Definition 3.2. As explained in Subsection 3.3 there is a natural homotopy class of isomorphisms of $Spin^c$-structures

$$P(TM) \otimes Q(1) \cong P(TW)_{|M} \tag{77}$$

which turns the inclusion

$$i_{M \to W} : M \to W$$

into a $Spin^c$-map. We fix a choice of such an isomorphism in this class.
We choose a $Spin^c$-extension of the Levi-Civita connection $\tilde{\nabla}^{TW}$ on $W$ with product structure and a $Spin^c$-extension of the Levi-Civita connection $\tilde{\nabla}^{TM}$ on $M$ such that the isomorphism (77) preserves the connections. This implies that the forms (75) are trivial

in this situation. Assume now that we have a geometrization of $(W, F, G, \tilde{\nabla}^{TW})$. Then we can define the restriction $\mathcal{G}_{\partial W} := (\mathcal{G}_W)_{|\partial W}$ as in Lemma 4.5. It is given by

$$\mathcal{G}_{\partial W}(\phi) = \mathcal{G}_W(\phi)_{|\partial W}, \quad \phi \in K^0(B_+ \wedge X_+) \ . \tag{78}$$

This restriction can be compared with a given geometrization $\mathcal{G}_M$ of $(M, f, g, \tilde{\nabla}^{TM})$. In general we do not expect that a given $\mathcal{G}_M$ extends to a geometrization $\mathcal{G}_W$ of $(W, F, G, \tilde{\nabla}^{TW})$. In this respect geometrizations seem to be more rigid than connections.
Here is a very simple example of a geometrization which does not extend. We consider the case $B = X = *$ and $M = S^3$ with its standard metric. We choose a normal framing of $S^3$ which extends over $D^4$ so that framed bordism class $[S^3]$ is trivial. We let $\mathcal{G}_0$ be the good geometrization of $(S^3, \tilde{\nabla}^{TS^3})$ defined in Subsection 5.1. We have $K^0(B_+ \wedge X_+) \cong K^0(*_+) \cong \mathbb{Z}$ so that a geometrization is fixed by the image of 1 in $\hat{K}^0(S^3)$. Let $\omega \in \Omega^3(S^3)$ be some form. Then we can define a new geometrization $\mathcal{G}_\omega$ of $(S^3, \tilde{\nabla}^{TS^3})$ by

$$\mathcal{G}_\omega(1) := \mathcal{G}_0(1) + a(\omega) \ .$$

It is easy to check, using the fact that $\mathcal{G}_0$ does extend by Lemma 4.9, that $\mathcal{G}_\omega$ extends to $D^4$ if and only if $\int_{S^3} \omega \in \mathbb{Z}$.
In order to deal with the problem of non-extendability of geometrizations appropriately we introduce the notion of a good geometrization. If $\mathcal{G}_M$ is good, then it will extend to zero bordisms. We define the notion of good geometrizations constructively. The rough idea is as follows. We choose a sufficiently high connected approximation $(f_u, g_u) : M_u \to B_+ \wedge X_+$ of $B_+ \wedge X_+$ by a smooth manifold $M_u$ and a geometrization $\mathcal{G}_u$. The map $(f, g) : M \to B_+ \wedge X_+$ then has a unique factorization up to homotopy through $h : M \to M_u$. The main observation is that there exists a natural refinement of $h$ to a $Spin^c$-map. We declare a geometrization $\mathcal{G}_M$ obtained as $\mathcal{G}_M := h^*\mathcal{G}_u$ as good. For a zero bordism $(W, F, G)$ of $(M, f, g)$ we can extend the factorization $h$ to $H : W \to M_u$ which again naturally refines to a $Spin^c$-map. The crucial point then is that

$$H \circ i_{M \to W} = h \ . \tag{79}$$

holds true as an equality of $Spin^c$-maps. Then $\mathcal{G}_W := H^*\mathcal{G}_u$ is the desired extension of $\mathcal{G}_M$ over $W$.
In order to be able to approximate $B_+ \wedge X_+$ by compact manifolds we now make the following assumption.

**Assumption 4.6** *The spaces $X$ and $B$ have the homotopy type of CW-complexes with finite skeleta.*

By the assumption we can find a compact manifold $M_u$ with a pair of maps $f_u : M \to B$ and $g_u : M_u \to X$ such that $f_u \times g_u : M_u \to B \times X$ is an $n+1$-equivalence. We choose a complement $\eta_u \to M_u$ of the bundle $\hat{f}_u^*\xi_k \to M_u$ and a complementary $Spin^c$-structure $P_u \in Spin^c(\eta_u)$ in the sense explained in Subsection 4.3. We further choose a connection $\tilde{\nabla}^u$ on $P_u$. By Proposition 4.4 there exists a geometrization $\mathcal{G}_u$ of the triple $(M_u, f_u, g_u, \tilde{\nabla}^u)$. Given $(M, f, g)$ with $\dim(M) = n$ there exists a unique factorization

$$\begin{tikzcd} & M_u \arrow[d, "{(f_u,g_u)}"] \\ M \arrow[ur, dotted, "h"] \arrow[r, "{(f,g)}"'] & X \times B \end{tikzcd} \tag{80}$$

of the map $(f, g)$ over the $(n + 1)$-equivalence $(f_u, g_u)$ up to homotopy.
Note that $f_u$ has no relation with the tangent bundle of $M_u$, but the choice of the isomorphisms

$$\eta_u \oplus \hat{f}_u^* \xi_k \cong M_u \times \mathbb{R}^{m+k}$$

and

$$P_u \otimes \tilde{f}_u^* Q_k \cong Q(m + k) \tag{81}$$

models the relation between normal and tangential structures. First of all we can choose a stable isomorphism

$$TM \oplus (M \times \mathbb{R}^s) \cong h^* \eta_u \oplus (M \times \mathbb{R}^t) \tag{82}$$

such that the induced trivialization (using $\hat{f}^* \xi_k \cong h^* \hat{f}_u^* \xi_k$ )

$$TM \oplus (M \times \mathbb{R}^s) \oplus \hat{f}^* \xi_k \cong M \times \mathbb{R}^{n+s+t}$$

is stably homotopic to the trivialization given by the representative of the normal $B$-structure on $M$. Then we choose the isomorphism

$$P(TM) \otimes Q(s) \cong h^* P_u \otimes Q(t) \ . \tag{83}$$

such that (81) induces the homotopy class of the isomorphism given by the tangential representative of the normal $Spin^c$-structure on $M$. This turns $h$ into a $Spin^c$-map so that the pull-back $h^* \mathcal{G}_u$ is defined.

**Definition 4.7** *A geometrization $\mathcal{G}_M$ of $(M, f, g, \tilde{\nabla}^{TM})$ obtained in this way is called good.*

Note that in this definition we do not fix the choices of $M_u, f_u, g_u, \mathcal{G}_u$ or the $Spin^c$-refinement of the map $h$. By varying these choices we define the subset of good among all geometrizations of $(M, f, g, \tilde{\nabla}^{TM})$. At the moment we do not have a nice intrinsic characterization of this subset. But by Lemma 4.9 non-extendability to zero bordisms produces obstructions against goodness.
As a consequence of the above discussion we get the following Lemma.

**Lemma 4.8** *If we assume 4.6, then for every tuple $(M, f, g, \tilde{\nabla}^{TM})$ there exists a good geometrization.*

**Lemma 4.9** *Let $\mathcal{G}_M$ be a good geometrization of $(M, f, g, \tilde{\nabla}^{TM})$. If $(W, F, G)$ is a zero bordism of $(M, f, g)$ with connection $\tilde{\nabla}^{TW}$, then there exists a geometrization $\mathcal{G}_W$ of $(W, F, G, \tilde{\nabla}^{TW})$ which restricts to $\mathcal{G}_M$.*

*Proof.* Since $(f_u, g_u) : M_u \to B \times X$ is an $n+1$-equivalence and $\dim(W) = n+1$ we can extend the factorization (80) to a factorization

$$\begin{tikzcd} & M_u \arrow[d, "(f_u,g_u)"] \\ W \arrow[r, "(F,G)"'] \arrow[ur, dotted, "H"] & X \times B \end{tikzcd} \tag{84}$$

up to homotopy such that $H$ coincides with $h$ on the boundary of $W$. The stable isomorphism (82) is chosen such that it extends to a stable isomorphism between $TW$ and $H^*\eta_u$ satisfying a similar condition as (82). Similarly, if we define the stable isomorphism between $P(TW)$ and $H^*P_u$ by similar conditions as for (83), then we obtain a refinement of $H$ to a $Spin^c$ map which has the crucial property that the composition of the inclusion $i_{M\to W} : M \to W$ with $H$ coincides with $h$ as $Spin^c$-maps.
Then we can define the pull-back $\mathcal{G}_W := H^*\mathcal{G}_u$. We get by (79) and the functoriality of the pull-back shown in Lemma 4.5 that the restriction of $\mathcal{G}_W$ to $M$ is $\mathcal{G}_M$. □

## 4.5 An intrinsic formula for $\eta^{an}$

The main goal of the present Subsection is to give an intrinsic formula for $\eta^{an}(x)$ which only involves structures on the cycle $(M, f, g)$ for $x \in \pi_n(MB \wedge X_+)_{tors}$.
The geometric and analytic terms in the formula (54) for $\eta^{an}(x)$ separately have values in $\mathbb{R}/\mathbb{Z}$; only their sum belongs to $\mathbb{Q}/\mathbb{Z}$. In order to deal with these terms separately it is useful to use a real version $Q_n^{\mathbb{R}}(B,X)$ of the group $Q_n(B,X)$. We start with introducing this group. We further show that there is no loss information when going over the this real version. We let (compare with 24)

$$U^{\mathbb{R}} \subseteq \mathtt{Hom}^{cont}(K^0(B_+ \wedge X_+), \pi_{n+1}(K\mathbb{R}/\mathbb{Z})) \tag{85}$$

be the subgroup given by evaluations against elements in $\pi_{n+1}(MB\mathbb{R} \wedge X_+)$ and define

$$Q_n^{\mathbb{R}}(B,X) := \frac{\mathtt{Hom}^{cont}(K^0(B_+ \wedge X_+), \pi_{n+1}(K\mathbb{R}/\mathbb{Z}))}{U^{\mathbb{R}}} . \tag{86}$$

The inclusion $\pi_{n+1}(K\mathbb{Q}/\mathbb{Z}) \to \pi_{n+1}(K\mathbb{R}/\mathbb{Z})$ induces a map

$$i_{\mathbb{R}} : Q_n(B,X) \to Q_n^{\mathbb{R}}(B,X) .$$

**Lemma 4.10** *The map $i_{\mathbb{R}} : Q_n(B,X) \to Q_n^{\mathbb{R}}(B,X)$ is injective.*

*Proof.* Let $\kappa \in Q_n(B,X)$ be represented by $\hat{\kappa} \in \mathtt{Hom}^{cont}(K^0(B_+ \wedge X_+), \pi_{n+1}(K\mathbb{Q}/\mathbb{Z}))$. Since $\kappa$ is continuous it factors over a finitely generated quotient of $K^0(B_+ \wedge X_+)$. Hence there exists $N \in \mathbb{N}$ such that $N\hat{\kappa}$ vanishes. Assume now that $i_{\mathbb{R}}(\kappa) = 0$. Then there exists

$w \in \pi_{n+1}(MB\mathbb{R} \wedge X_+)$ such that $\hat{\kappa}(\phi) = [\langle w, \phi\rangle] \in \pi_{n+1}(K\mathbb{R}/\mathbb{Z})$. Since $\pi_{n+1}(MB\mathbb{R} \wedge X_+) \cong \pi_{n+1}(MB \wedge X_+) \otimes \mathbb{R}$ (see (19)) there exists a finite subset $I \subset \pi_{n+1}(MB \wedge X_+)$ and a map $\lambda : I \to \mathbb{R}$ such that $w = \sum_{v \in I} \lambda(v)v$. We have $\hat{\kappa}(\phi) = \sum_{v\in I}[\lambda(v)\langle\phi, v\rangle]$, where here $\langle\phi, v\rangle \in \pi_{n+1}(K)$. For $v \in I$ we define $\hat{v} \in \mathtt{Hom}^{cont}(K^0(B_+ \wedge X_+), \pi_{n+1}(K))$ by $\hat{v}(\phi) := \langle\phi, v\rangle$. The set $\{\hat{v}|v \in I\}$ generates a free abelian subgroup $A \subseteq \mathtt{Hom}^{cont}(K^0(B_+ \wedge X_+), \pi_{n+1}(K))$. We can choose a minimal subset $J \subseteq I$ which generates a subgroup of $A$ of full rank. Then we can write $\hat{\kappa}(\phi) = \sum_{v\in J}[\mu(v)\hat{v}(\phi)]$ for a suitable map $\mu : J \to \mathbb{R}$. The image of $K^0(B_+ \wedge X_+) \to \mathtt{Hom}(A, \pi_{n+1}(K\mathbb{Z}))$ has full rank. Hence for every $v \in J$ there exists $\phi_v \in K^0(B_+ \wedge X_+)$ such that $\hat{v}(\phi_v) \neq 0$ and $\hat{v}'(\phi_v) = 0$ for all $J \ni v' \neq v$. It follows that $\hat{\kappa}(\phi_v) = [\mu(v)\hat{v}(\phi_v)]$. Since $0 = N\hat{\kappa}(\phi_v) = [N\mu(v)\hat{v}(\phi_v)]$ it follows that $\mu(v) \in \mathbb{Q}$. We set $w_\mathbb{Q} := \sum_{v\in J}\mu(v)v \in \pi_{n+1}(MB \wedge X_+) \otimes \mathbb{Q} \cong \pi_{n+1}(MB\mathbb{Q} \wedge X_+)$. Then we have $\hat{\kappa}(\phi) = [\langle\phi, w_\mathbb{Q}\rangle]$ for all $\phi \in K^0(B_+ \wedge X_+)$. This shows that $\hat{\kappa} \in U$ and $\kappa = 0$. $\square$

The standing assumption for the following is 4.6 which ensures the existence of good geometrizations by Lemma 4.8. Let $x \in \pi_n(MB \wedge X_+)_{tors}$ be an $l$-torsion element in the $B$-bordism group of $X$ and $(M, f, g)$ be a cycle for $x$. We choose a Riemannian metric and a $Spin^c$-extension $\tilde{\nabla}^{TM}$ of the Levi-Civita connection on $M$. We further choose a good geometrization $\mathcal{G}_M$ of $(M, f, g, \tilde{\nabla}^{TM})$ (see Definition 4.7).
For every $\phi$ in $K^0(B_+\wedge X_+)$ we choose a $\mathbb{Z}/2\mathbb{Z}$-graded vector bundle $V_\phi \to M$ such that we have the equality of $K$-theory classes $[V_\phi] = (f, g)^*\phi \in K^0(M_+)$. We furthermore choose a hermitean metric $h^{V_\phi}$ and a metric connection $\nabla^{V_\phi}$ so that we get the geometric bundle $\mathbf{V}_\phi = (V_\phi, h^{V_\phi}, \nabla^{V_\phi})$. It represents a differential $K$-theory class $[\mathbf{V}_\phi] \in \hat{K}^0(M)$ such that $I([\mathbf{V}_\phi]) = [V_\phi] = (f, g)^*\phi$. This class can be compared with the differential refinement $\mathcal{G}_M(\phi) \in \hat{K}^0(M)$ of $(f, g)^*\phi$ given by the good geometrization. By the exactness of (70) the difference of these two classes uniquely determines an element

$$\gamma_\phi \in \Omega P^{-1}(M)/\mathtt{im}(\mathbf{ch})$$

such that

$$\mathcal{G}_M(\phi) - [\mathbf{V}_\phi] = a(\gamma_\phi) \ . \tag{87}$$

**Definition 4.11** *We will refer to $\gamma_\phi$ as the correction form associated to $\phi$.*

**Theorem 4.12** *The element $i_\mathbb{R}(\eta^{an}([M, f, g])) \in Q^\mathbb{R}_n(B, X)$ is represented by the homomorphism*

$$K^0(B_+ \wedge X_+) \ni \phi \mapsto [-\int_M \mathbf{Td}(\tilde{\nabla}^{TM}) \wedge \gamma_\phi] - \xi(\not{D}_M \otimes \mathbf{V}_\phi) \in \mathbb{R}/\mathbb{Z} \ . \tag{88}$$

*Proof.* The integral in formula(88) belongs to $\mathbb{R}[b, b^{-1}]^{-n-1}$ which will be identified with $\mathbb{R}$ using the generator $b^{-\frac{n}{2}}$. First note that, despite of the fact that $\gamma_\phi$ is only defined up the image of $\mathbf{ch} : K^{-1}(M_+) \to HP\mathbb{Q}^{-1}(M_+)$, the class

$$[\int_M \mathbf{Td}(\tilde{\nabla}^{TM}) \wedge \gamma_\phi] \in \mathbb{R}/\mathbb{Z}$$

is well-defined. Indeed, we have $\langle [M], \mathbf{Td}(TM) \cup \mathbf{ch}(\psi)\rangle \in \mathbb{Z}$ for all $\psi \in K^{-1}(M_+)$ by the odd version of Atiyah-Singer index theorem. We use (54) in order to express the right-hand side of (56) as

$$[\frac{1}{l}\int_W \mathbf{Td}(\tilde{\nabla}^{TW}) \wedge \mathbf{ch}(\nabla^{U_\phi})] - \xi(\not{D}_M \otimes \mathbf{V}_\phi) \ .$$

The whole idea of is now to turn the integral over $W$ into an integral over $M$. To this end we assume by Lemma 4.9 that the good geometrization $\mathcal{G}_M$ has an extension $\mathcal{G}_W$ to $W$. The $K$-theory class $(f,g)^*\phi$ extends across $W$ as $(F,G)^*\phi$. We can thus assume, after adding a bundle of the form $\mathbf{W} \oplus \mathbf{W}^{op}$, that the bundle $\mathbf{V}_\phi$ has an extension $\mathbf{U}_\phi$ as a geometric bundle to $W$. Note that this sort of stabilization does not effect the correction form $\gamma_\phi$ and the reduced $\eta$-invariant $\xi(\not{D}_M \otimes \mathbf{V}_\phi)$. From now on we assume that $\mathbf{V}_\phi$ extends. We let $\gamma^W_\phi \in \Omega P^{-1}(W)/\mathtt{im}(\mathbf{ch})$ be the correction form defined by

$$\mathcal{G}_W(\phi) - [\mathbf{U}_\phi] = a(\gamma^W_\phi) \ .$$

By (78) we conclude that $(\gamma^W_\phi)_{|\partial W}$ coincides with $\gamma_\phi$ on all copies of $M$. We now use Stokes' theorem in order to rewrite

$$\begin{aligned}
[\frac{1}{l}\int_W \mathbf{Td}(\tilde{\nabla}^{TW}) \wedge \mathbf{ch}(\nabla^{U_\phi})] &= [\frac{1}{l}\int_W \mathbf{Td}(\tilde{\nabla}^{TW}) \wedge R(\mathcal{G}_W(\phi)) - \frac{1}{l}\int_W \mathbf{Td}(\tilde{\nabla}^{TW}) \wedge d\gamma^W_\phi] \\
&= [\frac{1}{l}\int_W \mathbf{Td}(\tilde{\nabla}^{TW}) \wedge R(\mathcal{G}_W(\phi))] - [\int_M \mathbf{Td}(\tilde{\nabla}^{TM}) \wedge \gamma_\phi]
\end{aligned}$$

The integrand of the integral over $W$ is exactly the cohomological character of $\mathcal{G}_W$ applied to $\phi$. Since it preserves degree it follows that the homomorphism

$$\kappa : \phi \mapsto [\frac{1}{l}\int_W \mathbf{Td}(\tilde{\nabla}^{TW}) \wedge R(\mathcal{G}_W(\phi))] \in \mathbb{R}/\mathbb{Z}$$

factors over

$$K^0(B_+ \wedge X_+) \stackrel{\frac{1}{l}p_{n+1}(\mathbf{Td}^{-1}\cup\mathbf{ch})}{\longrightarrow} H\mathbb{Q}^{n+1}(B_+ \wedge X_+)$$

(see 30). Assumption 4.6 implies that $B_+ \wedge X_+$ is rationally of finite type. In the proof of Lemma 2.4 we have seen that then

$$\pi_{n+1}(MB\mathbb{Q} \wedge X_+) \to \mathtt{Hom}(H\mathbb{Q}^{n+1}(B_+ \wedge X_+), \mathbb{Q})$$

is surjective. Since $\dim(H\mathbb{Q}^{n+1}(B_+ \wedge X_+)) < \infty$ the tensor product of this map with $\mathbb{R}$

$$\pi_{n+1}(MB\mathbb{R} \wedge X_+) \to \mathtt{Hom}(H\mathbb{Q}^{n+1}(B_+ \wedge X_+), \mathbb{R})$$

is still surjective. This implies that $\kappa \in U^{\mathbb{R}}$. Therefore $i_{\mathbb{R}}(\eta^{an}([M,f,g]))$ is also represented by the map (88). $\square$

Let us mention the following aspect of the intrinsic formula (88) which is not yet completely understood at the moment. For the intrinsic formula to make sense we do not need to assume that $[M, f, g] \in \pi_n(MB \wedge X_+)$ is a torsion element. Hence it provides an extension of the universal $\eta$-invariant to the whole bordism group, i.e. we get a homomorphism

$$\eta^{intrinsic} : \pi_n(MB \wedge X_+) \to Q_n^{\mathbb{R}}(B, X)$$

which restricts to $i_{\mathbb{R}} \circ \eta^{top} = i_{\mathbb{R}} \circ \eta^{an}$ on $\pi_n(MB \wedge X_+)_{tors}$. In general we do not know the topological contents of this extension. For an example see the text after Corollary 5.20.

# 5 Examples

## 5.1 Adams' $e$-invariant

In this Subsection we explain the relation between our universal $\eta$-invariant and the Adams $e$-invariant. We consider the case of framed bordism, i.e. we set $B := *$ and $X := *$. With this choice the product $MB \wedge X_+$ is the sphere spectrum $S$ whose homotopy groups are called the stable homotopy groups (of the sphere) and will be denoted by

$$\pi_n^S := \pi_n(S) \ .$$

The stable homotopy groups form a basic object of interest in stable homotopy theory. Though they turn out to be quite resistive against a complete calculation a lot about their of structure is known [52].
For $n \geq 1$ the stable homotopy group $\pi_n^S$ is finite by Serre's theorem [55]. Therefore the universal $\eta$-invariant is defined on all of $\pi_n^S$.
Concerning the target group $Q_n := Q_n(*, *)$ of the universal $\eta$-invariant defined in 2.1 note that with the choices of $B$ and $X$ as above we have an identification

$$K^0(B_+ \wedge X_+) = K^0(S^0) \cong \mathbb{Z} \ .$$

The finiteness of the stable homotopy groups in higher degree implies that their rationalizations are trivial. In the present situation the subgroup (24) thus vanishes. After identifying $\pi_{2m+2}(K\mathbb{Q}/\mathbb{Z})$ with $\mathbb{Q}/\mathbb{Z}$ for all $m \geq 0$ we obtain the identification

$$Q_{2m+1} \cong \mathtt{Hom}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$$

given by evaluation against $1 \in \mathbb{Z}$. Our universal $\eta$-invariant is thus interpreted as a homomorphism

$$e_{\mathbb{C}}^{Adams} = \eta : \pi_{2m+1}^S \to \mathbb{Q}/\mathbb{Z} \ , \tag{89}$$

where $\eta$ stands for $\eta^{top}$, or equivalently $\eta^{an}$ by Theorem 3.6. The notation indicates that this is the complex version of Adams' $e$-invariant which was introduced in order to detect the image of the $J$-homomorphism

$$J : KO_{2m+2} \to \pi_{2m+1}^S \tag{90}$$

(see the series of papers by Adams culminating in [1]). Note that the relation between Adams $e$-invariant and its complex version on $\pi^S_{4m-1}$ is given simply by

$$e^{Adams}_{\mathbb{C}} = \begin{cases} e^{Adams} & m\ even \\ 2e^{Adams} & m\ odd \end{cases} .$$

Our goal in this Subsection is to explain that the results of the present paper specialize to classically well-known facts about the $e$-invariant. Let us first discuss the relation with the Adams filtration and spectral sequence. We observe that Assumption 2.6 is satisfied. Let

$$\dots \subseteq F^2\pi^S_* \subseteq F^1\pi^S_* \subseteq F^0\pi^S_* = \pi^S_*$$

be the filtration of the stable homotopy groups associated to the $K$-theory based Adams spectral sequence (see e.g. [2]). Then it follows from Corollary 2.10 that the complex version of Adams $e$-invariant (89) induces an injection

$$F^1\pi^S_{2m+1}/F^2\pi^S_{2m+1} \hookrightarrow \mathbb{Q}/\mathbb{Z} .$$

We now come to the second classical fact about the Adams $e$-invariant, namely that it is related to spectral geometry. This has first been observed in [8]. The spectral geometric calculation of Adams $e$-invariant is based on the Pontrjagin-Thom isomorphism which allows to represent elements in the stable homotopy group by stably normally framed manifolds. It has the favorable property that it provides an intrinsic formula, a fact which has been successfully exploited e.g. in [29], [54]. The goal of the following discussion is to derive, as an illustration, the well-known intrinsic formula [7, Thm 4.14] for $e^{Adams}_{\mathbb{C}}$ from Theorem 4.12. This is finally our argument that (89) is really the complex version of Adams' $e$-invariant.

We equip $M$ with a Riemannian metric. As a normally stably framed manifold $M$ it comes with a trivialization $TM \oplus (M \times \mathbb{R}^k) \cong M \times \mathbb{R}^{n+k}$ for some sufficiently large $k$. A tangential representative of the associated normal $Spin^c$-structure is now given by a stable trivialization

$$P(TM) \otimes Q(k) \cong M \times Spin^c(n+k) . \tag{91}$$

We can in fact assume that $P(TM)$ comes from a $Spin$-structure. In this case the Levi-Civita connection induces a canonical $Spin^c$-connection $\tilde{\nabla}^{TM}$.

In order to apply Theorem 4.12 we must first choose a good geometrization of $(M, \tilde{\nabla}^{TM})$. In the notation of Subsection 4.4 we can choose the manifold $M_u$ to be a point. It then has a unique geometrization $\mathcal{G}_u$. Let $h : M \to M_u$ be the constant map. Note that $h^*P_u$ is trivial. Hence the given trivialization (91) refines $h$ to a $Spin^c$-map. Using this refinement we define the good geometrization $\mathcal{G} := h^*\mathcal{G}_u$. In view of the identification $K^0(B_+ \wedge X_+) \cong \mathbb{Z}$ the geometrization $\mathcal{G}_u$ associates to $1 \in \mathbb{Z}$ the class $\mathcal{G}_u(1) = 1 \in \hat{K}^0(M_u)$. We now use Lemma 4.5 in order to calculate $\mathcal{G}(1) \in \hat{K}^0(M)$. By equation (76) we have

$$\mathcal{G}(1) = 1 + a\left(\frac{\tilde{\mathbf{Td}}(\tilde{\nabla}^{triv}, \tilde{\nabla}^{TM})}{\mathbf{Td}(\tilde{\nabla}^{TM})}\right) ,$$

where $\nabla^{triv}$ is the connection on $P(TM) \otimes Q(k)$ induced by the trivialization (91). Let $\mathbf{V}_1$ be the trivial one-dimensional geometric bundle on $M$. Then $[\mathbf{V}_1] = 1 \in \hat{K}^0(M)$ and in view of Equation (87) we must take the correction form

$$\gamma_1 := \frac{\tilde{\mathbf{Td}}(\tilde{\nabla}^{triv}, \tilde{\nabla}^{TM})}{\mathbf{Td}(\tilde{\nabla}^{TM})} .$$

We now specialize Theorem 4.12 to the present situation. Using the above formula for the correction form and invoking the notational simplification coming from the fact that we twist by a one-dimensional trivial bundle we obtain the intrinsic formula

$$i_{\mathbb{R}}(\eta^{an}([M]))(1) = [-\int_M \tilde{\mathbf{Td}}(\tilde{\nabla}^{triv}, \tilde{\nabla}^{TM})] - \xi(\not{D}_M) \in \mathbb{R}/\mathbb{Z}$$

This formula directly compares with the formula for $i_{\mathbb{R}}(e^{Adams}_{\mathbb{C}}([M])) \in \mathbb{R}/\mathbb{Z}$ derived by specializing [7, Thm 4.14].

## 5.2 $\rho$-invariants and the index theorem for flat bundles

The reduced $\eta$-invariant $\xi(\not{D}_M \otimes \mathbf{V})$ of the $Spin^c$-Dirac operator on a closed odd-dimensional manifold $M$ twisted by a geometric bundle $\mathbf{V}$ depends on the geometric structures of the manifold and the bundle. A usual way to describe this dependence is in terms of variation formulas. Their main feature is that the variation of $\xi(\not{D}_M \otimes \mathbf{V})$ with respect to the geometric structures can be obtained by integrating a density over the manifold which admits a local expression in terms of the variation of the geometry.
Let us rigidify the geometric bundle $\mathbf{V}$ by assuming that it is a flat hermitean vector bundle of dimension $k$. Then locally $\not{D}_M \otimes \mathbf{V}$ is isomorphic to a direct sum of $k$ copies of $\not{D}_M$. This implies that the variation of the difference of reduced $\eta$-invariants

$$\rho(\not{D}_M, \mathbf{V}) := \xi(\not{D}_M \otimes \mathbf{V}) - k\xi(\not{D}_M) \tag{92}$$

is invariant under variations of the geometry of $M$. The $\rho$-invariant is thus a differential topological invariant of the $Spin^c$-manifold with a flat hermitean bundle. This is the classical case where the topological contents of the $\eta$-invariant has beed studied. The $\rho$-invariants have successfully been applied for example in order to detect elements in $Spin^c$-bordism groups of classifying spaces of finite cyclic groups [12], [11]. We refer to these reference for examples of explicit calculations of $\rho$-invariants.
As explained above it follows from an variation argument that $\rho$-invariants are of topological nature. The precise homotopy theoretic description of $\rho$-invariants is given by the index theorem for flat bundles [8, Thm. 5.3]. The goal of the following discussion is to explain the relation of our theory, in particular of the relation $\eta^{an} = \eta^{top}$ shown in Theorem 3.6, with the index theorem for flat bundles. Roughly speaking, this goes as follows. The index theorem for flat bundles is about the pairing of the $K$-homology class represented by the $Spin^c$-Dirac operator with the torsion $K$-cohomology classes obtained from the flat bundle, while our index theorem considers the pairing of a torsion $K$-homology class

with $K$-theory classes. Clearly the case of intersection is when both classes are torsion. In this case the relation between both pairings is clarified in Lemma 2.5.
We first translate the data of the $Spin^c$-manifold $M$ of dimension $n = 2m + 1$ with a flat hermitean bundle $\mathbf{V}$ into the bordism picture. Let $U(k)^\delta$ denote the unitary group equipped with the discrete topology. Its classifying space $BU(k)^\delta$ is universal for flat hermitean vector bundles of dimension $k$. We let $f : M \to BSpin^c$ be a classifying map of the stable normal bundle of the $Spin^c$-manifold $M$ as explained in Subsection 3.4. Furthermore, we let $g : M \to BU(k)^\delta$ be a classifying map for $\mathbf{V}$. The triple $(M, f, g)$ then represents a class

$$[M, f, g] \in \pi_n(MBSpin^c \wedge BU(k)^\delta_+) \ .$$

We are thus forced to apply our theory in the case where $B := BSpin^c$ and $X := BU(k)^\delta$. In order to be in the domain of the universal $\eta$-invariant we must assume that $[M, f, g]$ is a torsion class. Due to this restriction we can not rederive [8, Thm. 5.3] in full generality. We do not try to control $\pi_{n+1}(MBSpin^c \wedge BU(k)^\delta_+) \otimes \mathbb{Q}$ or to calculate the group $Q_n(BSpin^c, BU(k)^\delta)$. We rather observe that the $K$-theory class $\lambda_k \in K^0(BU(k)^\delta_+)$ of the universal $\mathbb{C}^k$-bundle on $BU(k)^\delta$ can be pulled back along the projection

$$q : BSpin^c_+ \wedge BU(k)^\delta_+ \to BU(k)^\delta_+ \tag{93}$$

to a class

$$q^*\lambda_k \in K^0(BSpin^c_+ \wedge BU(k)^\delta_+) \ ,$$

and that the evaluation against the difference $q^*\lambda_k - k$ provides a well-defined homomorphism

$$\mathtt{ev}_{q^*\lambda_k - k} : Q_n(BSpin^c, BU(k)^\delta) \to \mathbb{Q}/\mathbb{Z} \ .$$

It indeed follows from $\mathbf{ch}(\lambda_k - k) = 0$ that the evaluation against $q^*\lambda_k - k$ vanishes on the subgroup (24).
In the following discussion we identify the representatives of the elements

$$\mathtt{ev}_{q^*\lambda_k - k}(\eta^{an}([M, f, g])), \quad \mathtt{ev}_{q^*\lambda_k - k}(\eta^{top}([M, f, g])) \quad \in \mathbb{Q}/\mathbb{Z}$$

explicitly in terms of the $\rho$-invariant, and homotopy theoretic data, respectively.
The equality $\eta^{an} = \eta^{top}$ of Theorem 3.6 has the form of an index theorem, here formulated as Corollary 5.1, which is the announced special case of [8, Thm. 5.3].
We first explain that $\mathtt{ev}_{q^*\lambda_k - k}(\eta^{an}([M, f, g]))$ is exactly given by $\rho(\not{D}_M, \mathbf{V})$ defined in (92). We are going to use the notation introduced in Subsection 3.4. As an intermediate step we choose, for a suitable non-vanishing integer $l$, a zero bordism $(W, F, G)$ of the union of $l$ copies of the cycle $(M, f, g)$ with $Spin^c$-geometry. The geometric bundle $\mathbf{U}$ is then the flat hermitean bundle classified by $F$, and we have by Definition 3.5

$$\mathtt{ev}_{q^*\lambda_k - k}(\eta^{an}([M, f, g])) = [\frac{1}{l}\mathtt{index}(\not{D}_W \otimes \mathbf{U})] - [\frac{k}{l}\mathtt{index}(\not{D}_W)] \ .$$

If we use (54) instead, then we express this evaluation in terms of an integral over local data on $W$ and the reduced $\eta$-invariants. Because of $\mathbf{ch}(\nabla^U) = k$ the local contributions cancel out and we remain with

$$\mathtt{ev}_{q^*\lambda_k - k}(\eta^{an}([M,f,g])) = \rho(\not{D}_M, \mathbf{V})$$

as desired. This is also the analytic side of the index theorem for flat bundles in [8, Thm. 5.3].

We now discuss the topological side and calculate $\mathtt{ev}_{q^*\lambda_k - k}(\eta^{top}([M,f,g]))$ following the prescription of Subsection 2.3. In order to shorten the notation we write $x := [M,f,g]$. According to this prescription we first must choose a lift $\hat{x} \in \pi_{n+1}(MBSpin^c\mathbb{Q}/\mathbb{Z} \wedge BU(k)^\delta_+)$ such that $\partial\hat{x} = x$, where

$$\partial : \pi_{n+1}(MBSpin^c\mathbb{Q}/\mathbb{Z} \wedge BU(k)^\delta_+) \to \pi_n(MBSpin^c \wedge BU(k)^\delta_+)$$

is a Bockstein operator. Such a lift exists since we assume that $x$ is a torsion class. We apply to this element the unit of $K$-theory and the Thom isomorphism

$$\pi_{n+1}(MBSpin^c\mathbb{Q}/\mathbb{Z}\dots) \stackrel{unit}{\to} \pi_{n+1}(K \wedge MBSpin^c\mathbb{Q}/\mathbb{Z}\dots) \stackrel{Thom}{\cong} \pi_{n+1}(K\mathbb{Q}/\mathbb{Z} \wedge BSpin^c_+ \dots) .$$

In this way we obtain a $K\mathbb{Q}/\mathbb{Z}$-homology class $\tilde{x} \in \pi_{n+1}(K\mathbb{Q}/\mathbb{Z} \wedge BSpin^c_+ \wedge BU(k)^\delta_+)$. The element $\mathtt{ev}_{q^*\lambda_k - k}(\eta^{top}([M,f,g])) \in \mathbb{Q}/\mathbb{Z}$ is now given by the pairing between this and the $K$-theory class $q^*\lambda_k - k \in K^0(BSpin^c_+ \wedge BU(k)^\delta_+)$:

$$\mathtt{ev}_{q^*\lambda_k - k}(\eta^{top}([M,f,g])) = \langle \mathtt{Thom}^K(q^*\lambda_k - k), \tilde{x}\rangle \in \pi_{n+1}(K\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z} . \qquad (94)$$

As already mentioned above, the topological side of the index theorem for flat bundles [8, Thm. 5.3] is not given as the pairing of a $K\mathbb{Q}/\mathbb{Z}$-homology class with a $K$-theory class, but rather by a pairing between a $K$-homology class and a $K\mathbb{R}/\mathbb{Z}$-cohomology class. In the following we rewrite the right-hand side of (94) in this way.

We use flat hermitean bundles give naturally rise to $K\mathbb{R}/\mathbb{Z}$-theory classes. This has already been observed in [8]. For our present purpose we (see also [42, Sec. 2] for a related construction) describe the corresponding universal class

$$\Lambda_k \in K\mathbb{R}/\mathbb{Z}^{-1}(BU(k)^\delta_+) \qquad (95)$$

such that $\partial\Lambda_k = \lambda_k - k$, where $\partial$ is again a Bockstein

$$\partial : K\mathbb{R}/\mathbb{Z}^{-1}(BU(k)^\delta_+) \to K^0(BU(k)^\delta_+) .$$

For every map from a compact manifold $h : N \to BU(k)^\delta$ the class $h^*(\lambda_k)$ is represented by a unitary flat $k$-dimensional bundle $\mathbf{U}$ obtained by pull-back of the universal bundle. The differential $K$-theory class $[\mathbf{U}]-k \in \hat{K}^0(N)$ is flat in therefore belongs to the subgroup $K\mathbb{R}/\mathbb{Z}^{-1}(N_+) \cong \hat{K}^0_{flat}(N)$, compare with (69). It further satisfies $\partial([\mathbf{U}]-k) = h^*(\lambda_k - k)$. In this way we have constructed a homotopy invariant section of the restriction of the functor $K\mathbb{R}/\mathbb{Z}^{-1}(\dots)$ to the category of compact manifolds over $BU(k)^\delta$. This section is

represented by a universal class $\Lambda_k \in K\mathbb{R}/\mathbb{Z}^{-1}(BU(k)^\delta_+)$ which satisfies $\partial\Lambda_k = \lambda_k - k$ as required.
Recall that $q$ denotes the projection (93). We now apply Lemma 2.5 to the right-hand side of (94) in order to get the equality

$$\mathtt{ev}_{q^*\lambda_k - k}(\eta^{top}(x)) = \langle \mathtt{Thom}^K(q^*\Lambda_k), \epsilon_K(x)\rangle \ .$$

Using further a variant of (17)

$$\langle \mathtt{Thom}^K(q^*\Lambda_k), \epsilon_K(x)\rangle = \langle q^*\Lambda_k, \mathtt{Thom}_K(\epsilon_K(x))\rangle = \langle g^*\Lambda_k, [M_K]\rangle$$

we get the formula which expresses the topological side in terms of the data given by the cycle:

$$\mathtt{ev}_{q^*\lambda_k - k}(\eta^{top}(x)) = \langle g^*\Lambda_k, [M_K]\rangle \ .$$

Finally note that by the construction of $\Lambda_k$ the pull-back $g^*\Lambda_k$ has a geometric description in terms of differential $K$-theory on $M$, namely that we have $g^*\Lambda_k = [\mathbf{V}] - k \in \hat{K}^0_{flat}(M) \cong K\mathbb{R}/\mathbb{Z}^{-1}(M_+)$. The formula

$$\mathtt{ev}_{q^*\lambda_k - k}(\eta^{top}(x)) = \langle [\mathbf{V}] - k, [M]_K\rangle$$

now identifies our topological side with the topological side of the index theorem for flat bundles of [8, Thm. 5.3]. The following Corollary now immediately follows from the equality $\eta^{top} = \eta^{an}$.

**Corollary 5.1** *Let $M$ be a closed $n = 2m-1$-dimensional $Spin^c$-manifold with a flat hermitean $k$-dimensional vector bundle $\mathbf{V}$. We assume in addition that the class $[M, f, g] \in \pi_n(MBSpin^c \wedge BU(k)^\delta_+)$ is torsion, where $f$ classifies the normal $Spin^c$-structure and $g$ is a classifying map of $\mathbf{V}$. Then we have the following equality in $\mathbb{R}/\mathbb{Z}$:*

$$\rho(\not{D}_M, \mathbf{V}) = \langle [\mathbf{V}] - k, [M]_K\rangle \tag{96}$$

In this way Theorem 4.12 implies a special case of [8, Thm. 5.3]. Let us again remark, that by [8, Thm. 5.3] the equality (96) holds true without the additional assumption that $[M, f, g]$ is a torsion class.

## 5.3 Algebraic $K$-theory

It is well-known that a $k$-dimensional flat complex vector bundle $\mathbf{V}$ on a manifold $M$ represents an algebraic $K$-theory class $[\mathbf{V}]_{alg} \in K(\mathbb{C})^0(M_+)$. Let us describe this in greater detail. There exists algebraic $K$-theory machines which associate to a ring $R$ an algebraic $K$-theory spectrum $K(R)$ (see the two foundational papers [51], [58]). The symbol $K(\mathbb{C})^0(M_+)$ stands for the degree-zero group of the cohomology theory represented by the spectrum $K(\mathbb{C})$ and evaluated on $M$. In order to understand the class $[\mathbf{V}]_{alg}$ we need an explicit identification of the homotopy type of the underlying infinite loop space $\Omega^\infty K(\mathbb{C})$ of the $K$-theory spectrum of $\mathbb{C}$. We let $\mathbb{C}^\delta$ be the complex numbers equipped with the discrete topology. Then we consider the homotopy colimit of classifying spaces of discrete general linear groups

$$BGL(\mathbb{C}^\delta) = \mathtt{hocolim}_k\, BGL(k, \mathbb{C}^\delta) \ .$$

We let

$$p : BGL(\mathbb{C}^\delta) \to BGL(\mathbb{C}^\delta)^+ \tag{97}$$

be Quillens +-construction (see [3, Ch. 3] for a detailed description). This map induces the abelization on the level of fundamental groups and isomorphisms in arbitrary homology and cohomology theories. We then have an equivalence

$$\Omega^\infty K(\mathbb{C}) \cong \mathbb{Z} \times BGL(\mathbb{C}^\delta)^+ \ .$$

This gives an identification

$$K(\mathbb{C})^0(M_+) \cong [M_+, \mathbb{Z} \times BGL(\mathbb{C}^\delta)^+] \ .$$

A flat complex vector bundle $\mathbf{V} := (V, \nabla^V)$ of dimension $k$ is classified by a homotopy class of maps $h : M \to BGL(k, \mathbb{C}^\delta)$. The bundle $V$ carries in addition a parallel hermitean metric if and only if this map has a factorization up to homotopy over $BU(k)^\delta$. The class $[\mathbf{V}]_{alg} \in K(\mathbb{C})^0(M_+)$ is then represented by the homotopy class of the composition

$$g : M \xrightarrow{h} BGL(k, \mathbb{C}^\delta) \xrightarrow{i} BGL(\mathbb{C}^\delta) \xrightarrow{p} BGL(\mathbb{C}^\delta)^+ \xrightarrow{x \mapsto (k,x)} \mathbb{Z} \times BGL(\mathbb{C}^\delta)^+ \ . \tag{98}$$

We let $g_0 := p \circ i \circ h$ denote the composition of the first three of these maps.
In Subsection 5.2 we have seen that, at least in the case where $\mathbf{V}$ is hermitean and $M$ is closed and normally $Spin^c$, we can define the analytical invariant $\rho(\not{D}_M, \mathbf{V}) \in \mathbb{Q}/\mathbb{Z}$ and understand it in terms of homotopy theory. The main objective of the present Subsection is to explain in which way this construction provides an invariant of the algebraic $K$-theory class represented by $\mathbf{V}$.
As a first step we translate our geometric situation into the bordism theoretic language. We will consider some bordism theory determined by a map $\sigma : B \to BSpin^c$, but already the case where $B$ is a point (the case of stable homotopy) is interesting. We assume that the manifold $M$ is closed and has a normal $B$-structure classified by a map $f : M \to B$, see Subsection 3.4 for details. Then we can consider the triple $(M, f, g_0)$ as a cycle for a class

$$[M, f, g_0] \in \pi_n(MB \wedge BGL(\mathbb{C}^\delta)^+_+) \ .$$

**Lemma 5.2** *Every class in $x \in \pi_n(MB \wedge BGL(\mathbb{C}^\delta)^+_+)$ can be obtained in this way.*

*Proof.* By the universal properties of the +-construction the map $p$ in (97) induces an isomorphism in $B$-bordism homology theory. Hence there exists a triple $(M, f, \tilde{g}_0)$, where $M$ is a closed $n$-dimensional manifold with a $B$-structure classified by $f$ and

$\tilde{g}_0 : M \to BGL(\mathbb{C}^\delta)$ such that $x = p_*([M, f, \tilde{g}_0])$. Since $M$ is compact we can assume that $\tilde{g}_0$ has a factorization $g_0$ over $BGL(k, \mathbb{C}^\delta)$ for some $k$. $\square$

If we replace $g_0$ by $g$, then we get a bordism class

$$[M, f, g] \in \pi_n(MB \wedge \Omega^\infty K(\mathbb{C})_+) \tag{99}$$

of the infinite loop space of algebraic $K$-theory. We will use this notation also in the more general case where $M$ has several connected components and the bundle $V$ may have different dimensions on these components. With this convention we have the following generalization of Lemma 5.2.

**Lemma 5.3** *Every class in $\pi_n(MB \wedge \Omega^\infty K(\mathbb{C})_+)$ can be obtained in this way.*

*Proof.* This is essentially the same proof as in Lemma 5.2 $\square$
If a space $Y$ has already a base point, then we have a natural map $Y_+ \to Y$ of based spaces which sends the new external base point to the base point in $Y$. For any spectrum $E$ it induces a map from the unreduced to the reduced $E$-homology theory

$$\pi_*(E \wedge Y_+) \to \pi_*(E \wedge Y) \ .$$

Just to explain a further notation, we can identify

$$\pi_*(E \wedge Y) \cong \pi_*(E \wedge \Sigma^\infty Y)$$

by reinterpreting the meaning of the smash product. This will be applied to the based space $\Omega^\infty K(\mathbb{C})$. When we write $[M, f, g] \in \pi_n(MB \wedge \Sigma^\infty\Omega^\infty K(\mathbb{C}))$ we mean the image of (99) under this projection and identification. We can use the counit (40) $u : \Sigma^\infty\Omega^\infty K(\mathbb{C}) \to K(\mathbb{C})$ in order to obtain the bordism class

$$u_*([M, f, g]) \in \pi_n(MB \wedge K(\mathbb{C}))$$

of the algebraic $K$-theory spectrum.
In the case of stable homotopy (i.e. where $B$ is a point) we have the following assertion.

**Lemma 5.4** *Every class in $x \in \pi_n(K(\mathbb{C}))$ can be written in the form $x = u_*([M, g])$ for a suitable normally stably framed closed $n$-manifold and a map $g : M \to \Omega^\infty K(\mathbb{C})_+$ arising from a flat complex vector bundle on $M$. If $x$ is a torsion class, then we can in addition assume that $[M, g] \in \pi_n(S \wedge \Omega^\infty K(\mathbb{C})_+)$ is a torsion class.*

*Proof.* Let $x \in \pi_n(K(\mathbb{C}))$. This map is represented by a map $\tilde{g} : S^n \to \Omega^\infty K(\mathbb{C})$. On $S^n$ we choose the stable normal framing which extends to the disc $D^{n+1}$. In this way we get a stable homotopy class $[S^n, \tilde{g}] \in \pi_n(S \wedge \Omega^\infty K(\mathbb{C})_+)$. If $x$ was a torsion class, then $[S^n, \tilde{g}]$ is a torsion class, too. We now apply Lemma 5.3 which asserts that we can write $[S, \tilde{g}] = [M, g]$, where $g$ has a factorization as in (98) with possibly different dimensions $k$ on different components of $M$. Then $u_*([M, g]) = x$. If $x$ was torsion, then the class

$[M, g]$ is torsion, too. $\square$

We start with an unstable version

$$\eta^{top} : \pi_n(MB \wedge BGL(\mathbb{C}^\delta)^+_+) \to Q_n(B, BGL(\mathbb{C}^\delta)^+) \ .$$

A calculation of the target of this map seems intractable, but we will construct an interesting map out of this group with values in $\mathbb{Q}/\mathbb{Z}$. The identity map $\mathbb{C}^\delta \to \mathbb{C}$ (where $\mathbb{C}$ in the target has the usual euclidean topology) induces a map of topological groups $GL(\mathbb{C}^\delta) \to GL(\mathbb{C})$. It induces the first map in the following chain

$$\tilde{\Theta}_0 : BGL(\mathbb{C}^\delta) \to BGL(\mathbb{C}) \stackrel{x \mapsto (0,x)}{\to} \mathbb{Z} \times BGL(\mathbb{C}) \cong \Omega^\infty K \ .$$

We can interpret this map as topological $K$-theory class $\tilde{\Theta}_0 \in K^0(BGL(\mathbb{C}^\delta)_+)$. Again, since $p$ in (97) induces an isomorphism in complex $K$-theory there exists a unique class $\Theta_0 \in K^0(BGL(\mathbb{C}^\delta)^+_+)$ such that $p^*\Theta_0 = \tilde{\Theta}_0$. Let $\lambda_k$ denote $K$-theory class of the universal flat $\mathbb{C}^k$ -bundle on $BGL(k, \mathbb{C}^\delta)$. Then the restriction of $\Theta_0$ to $BGL(k, \mathbb{C}^\delta)$ is equal to $\lambda_k - k$.
Let $q : B_+ \wedge BGL(\mathbb{C}^\delta)^+_+ \to BGL(\mathbb{C}^\delta)^+_+$ denote the projection. We have $\mathbf{ch}(\Theta_0) = 0$ so that by Lemma 2.4 evaluation against $q^*\Theta_0$ defines a well-defined map

$$\mathtt{ev}_{q^*\Theta_0} : Q_n(B, BGL(\mathbb{C}^\delta)^+) \to \mathbb{Q}/\mathbb{Z} \ .$$

We can now consider the composition

$$\varepsilon_0 : \pi_n(MB \wedge BGL(\mathbb{C}^\delta)^+_+) \stackrel{\eta^{top}}{\to} Q_n(B, BGL(\mathbb{C}^\delta)^+) \stackrel{\mathtt{ev}_{q^*\Theta_0}}{\to} \mathbb{Q}/\mathbb{Z} \ .$$

Our first goal is to give a more or less analytic description of the element

$$\varepsilon_0([M, f, g_0]) \in \mathbb{Q}/\mathbb{Z} \ .$$

Our result is stated as Theorem 5.5. In order to formulate its statement we must generalize the construction of the universal $K\mathbb{R}/\mathbb{Z}^{-1}$-class of the space $BU(k)^\delta$ given in Subsection 5.2 to the stable and non-unitary case. We will obtain an universal element

$$\Lambda_0 \in K\mathbb{R}/\mathbb{Z}^{-1}(BGL(\mathbb{C}^\delta)^+_+)$$

such that $\partial\Lambda_0 = \Theta_0$, where $\partial$ denotes the Bockstein. Its pull-back along the projection $\Omega^\infty K(\mathbb{C})_+ \cong (\mathbb{Z} \times BGL(\mathbb{C}^\delta)^+)_+ \to BGL(\mathbb{C}^\delta)^+_+$ is a class

$$\Lambda \in K\mathbb{R}/\mathbb{Z}^{-1}(\Omega^\infty K(\mathbb{C})_+) \ .$$

By construction it will represent the real part of the natural transformation of functors

$$e : K(\mathbb{C})^0(\dots) \to K\mathbb{C}/\mathbb{Z}^{-1}(\dots) \qquad (100)$$

defined in [42, Sec. 2].
For every map from a compact manifold $N \to BGL(\mathbb{C}^\delta)$ there exists a factorization $h : N \to BGL(k, \mathbb{C}^\delta)$ for some $k$. This map classifies a flat complex vector bundle $\mathbf{V}^{flat} = (V, \nabla^{V,flat})$. We choose some not necessarily flat metric $h^V$ and form the unitary connection

$$\nabla^V := \frac{1}{2}\left(\nabla^{V,flat} + (\nabla^{V,flat})^{*_{h^V}}\right) .$$

The Chern form of a connection on a complex vector bundle is a closed complex form which is real if the connection is unitary. We thus have a complex transgression

$$\tilde{\mathbf{ch}}(\nabla^V, \nabla^{V,flat}) \in \Omega P^{-1}_{\mathbb{C}}(M)/\mathtt{im}(d)$$

such that

$$d\tilde{\mathbf{ch}}(\nabla^V, \nabla^{V,flat}) = \mathbf{ch}(\nabla^V) - k .$$

Since the connection $\nabla^V$ is unitary its Chern form is real and we also have the equality

$$d\Re(\tilde{\mathbf{ch}}(\nabla^V, \nabla^{V,flat})) = \mathbf{ch}(\nabla^V) - k . \tag{101}$$

As a side remark, the imaginary part $\Im(\tilde{\mathbf{ch}}(\nabla^V, \nabla^{V,flat}))$ is a closed form and represents a characteristic class of the flat bundle $(V, \nabla^{V,flat})$ which played e.g. a prominent role in [16]. We let $\mathbf{V} := (V, h^V, \nabla^V)$ be the induced geometric bundle over $N$. We get a flat class

$$[\mathbf{V}] - k - a(\Re(\tilde{\mathbf{ch}}(\nabla^V, \nabla^{V,flat}))) \in \hat{K}^0_{flat}(N) \cong K\mathbb{R}/\mathbb{Z}^{-1}(N_+) .$$

Since any two choice of metrics on $V$ can be connected by a path it follows from the homotopy invariance of the functor $K\mathbb{R}/\mathbb{Z}^{-1}(\dots)$ that this class does not depend on the choice of the metric $h^V$. By similar reasoning we see that we have constructed a homotopy invariant section of the restriction of the functor $K\mathbb{R}/\mathbb{Z}^{-1}(\dots)$ to the category of compact smooth manifolds over $BGL(k, \mathbb{C}^\delta)$. This section is thus represented by an universal class $\tilde{\Lambda}_k \in K\mathbb{R}/\mathbb{Z}^{-1}(BGL(k, \mathbb{C}^\delta)_+)$. The restriction of $\tilde{\Lambda}_k$ to $BU(k)^\delta_+ \subset BGL(k, \mathbb{C}^\delta)_+$ is equal to the class $\Lambda_k$ in (95). We further observe that the collection of these classes is compatible with stabilization in $k$. We therefore get a universal class $\tilde{\Lambda} \in K\mathbb{R}/\mathbb{Z}^{-1}(BGL(\mathbb{C}^\delta)_+)$. Finally we use that the $+$-construction map (97) induces an isomorphism in $K\mathbb{R}/\mathbb{Z}$-cohomology theory so that we get a class $\Lambda_0 \in K\mathbb{R}/\mathbb{Z}^{-1}(BGL(\mathbb{C}^\delta)^+_+)$ such that $p^*\Lambda_0 = \tilde{\Lambda}$. It satisfies $\partial\Lambda_0 = \Theta_0$.

**Theorem 5.5** *Let $(M, f)$ be an $n$-dimensional closed $B$-manifold and $g_0 : M \to BGL(\mathbb{C}^\delta)^+$ be a map induced by a flat bundle $\mathbf{V}$ as in (98). We assume that $[M, f, g_0] \in \pi_n(MB \wedge BGL(\mathbb{C})^+_+)_{tors}$. If $\mathbf{V}$ carries a flat hermitean metric, then*

$$\varepsilon_0([M, f, g_0]) = \rho(\not{D}_M, \mathbf{V}) . \tag{102}$$

*In general we have*

$$\varepsilon_0([M, f, g_0])) = \langle [M]_K, g_0^*\Lambda_0 \rangle . \tag{103}$$

*where $[M]_K \in \pi_n(K \wedge M_+)$ denotes the $K$-theory fundamental class of $M$.*

*Proof.* We first prove the general case (103) by the following chain of equalities

$$\begin{aligned}
\varepsilon_0([M,f,g_0]) &\overset{Lemma\ 2.5}{=} \mathtt{ev}_{q^*\Theta_0}(\eta^{top}([M,f,g_0])) \\
&= \langle \mathtt{Thom}^{K\mathbb{Q}/\mathbb{Z}}(q^*\Lambda_0), \epsilon_K([M,f,g_0])\rangle \\
&\overset{variant\ of\ (18)}{=} \langle g_0^*\Lambda_0, [M]_K\rangle\ .
\end{aligned}$$

In the unitary case we observe that $g_0^*\Lambda_0 = [\mathbf{V}] - k$. The equality (102) now follows from (96) and the chain

$$\varepsilon_0([M,f,g_0]) \overset{(103)}{=} \langle g_0^*\Lambda_0, [M]_K\rangle = \langle [\mathbf{V}] - k, [M]_K\rangle \overset{(96)}{=} \rho(\not{D}_M, \mathbf{V})\ .$$

Note that in the present paper we have shown (96) under the assumption that $\sigma_*[M,f,g_0^u]$ is a torsion class in $\pi_n(MBSpin^c_+ \wedge BU(k)^\delta_+)$, where $g_0^u : M \to BU(k)^\delta$ denotes an unitary factorization of $g_0$ and $\sigma_*$ is induced by the map $M\sigma : MB \to MBSpin^c$. Since this might not be the case in general we have to appeal to the proof of this formula (96) without such assumption given in [8, Thm. 5.3]. □

We now come to the stable version of $\eta^{top}$. We must choose a stable class $\Theta \in K^0(K(\mathbb{C}))$ in order to define an evaluation on $Q_n(MB, K(\mathbb{C}))$. The class $\Theta_0 \in K^0(\Omega^\infty K(\mathbb{C})_+)$ used in the unstable version does not come from a stable class. We rather use the natural map

$$\Theta : K(\mathbb{C}) \to K$$

of spectra. The Chern character of this class does not vanish but is concentrated in degree zero. Hence for a general bordism theory the class $q^*\Theta \in K^0(B_+ \wedge K(\mathbb{C}))$ may evaluate non-trivially against elements coming from $\pi_{n+1}(MB\mathbb{Q} \wedge K(\mathbb{C}))$. In order to have a well-defined evaluation we restrict to the case of stable homotopy, i.e. to the case $B = *$.

The domain of the stable version of $\eta^{top}$ is the torsion subgroup $\pi_n(K(\mathbb{C}))_{tors}$ which has been determined by Suslin [57, Thm 4.9] as

$$\pi_n(K(\mathbb{C}))_{tors} \cong \left\{ \begin{array}{cc} \mathbb{Q}/\mathbb{Z} & n\ odd \\ 0 & n\ even \end{array} \right. \tag{104}$$

Indeed [57, Thm 4.9] states a stronger assertion. In positive degrees the algebraic $K$-theory of the complex numbers is given by its torsion part up to a uniquely divisible group. In the stable homotopy case we have $\mathbf{Td} = 1$ so $\mathbf{Td} \cup \mathbf{ch}(q^*\Theta)$ vanishes in positive degrees. It follows from Lemma 2.4 that the evaluation

$$\mathtt{ev}_{q^*\Theta} : Q_n(K(\mathbb{C})) \to \mathbb{Q}/\mathbb{Z}$$

is well-defined. We now consider the composition

$$\varepsilon : \pi_n(K(\mathbb{C}))_{tors} \overset{\eta^{top}}{\to} Q_n(K(\mathbb{C})) \overset{\mathtt{ev}_{q^*\Theta}}{\to} \mathbb{Q}/\mathbb{Z}\ .$$

Let $(M,g)$ be an $n$-dimensional closed normally stably framed manifold with a map $g : M \to \Omega^\infty K(\mathbb{C})$ induced by a flat bundle $\mathbf{V}$ as in (98). It gives rise to the classes $[M,g] \in \pi_n(\Omega^\infty K(\mathbb{C})_+)$ and hence $u_*([M,g]) \in \pi_n(K(\mathbb{C}))$. By Lemma 5.4 all classes in $\pi_n(K\mathbb{C})$ can be represented in this way. This gives many examples to which the Theorem 5.6 applies. Note that $g$ factors over the inclusion

$$i_k : BGL(\mathbb{C}^\delta)^+ \cong \{k\} \times BGL(\mathbb{C}^\delta)^+ \hookrightarrow \Omega^\infty K(\mathbb{C}) \ .$$

Therefore we can consider $[M,g]$ as the image under this inclusion of a class $[M,g_0] \in \pi_n(S \wedge BGL(\mathbb{C}^\delta)^+_+)$. The closed normally stably framed $n$-manifold $M$ also represents the stable homotopy class $[M] \in \pi_n^S$ which has the Adams' invariant $e_{\mathbb{C}}^{Adams}([M]) \in \mathbb{Q}/\mathbb{Z}$ defined in (89).
We can now state the stable version of Theorem 5.5.

**Theorem 5.6** *Let $(M,g)$ be an $n$-dimensional closed normally stably framed manifold with a map $g : M \to \Omega^\infty K(\mathbb{C})$ induced by a flat bundle $\mathbf{V}$ as in (98). We assume that $[M,g] \in \pi_n(S \wedge \Omega^\infty K(\mathbb{C})_+)_{tors}$. If $\mathbf{V}$ carries a flat hermitean metric, then*

$$\varepsilon(u_*([M,g])) = \rho(\not{D}_M, \mathbf{V}) + k e_{\mathbb{C}}^{Adams}([M]) \ . \tag{105}$$

*In general we have*

$$\varepsilon(u_*([M,g])) = \langle g_0^* \Lambda_0, [M]_K \rangle + k e_{\mathbb{C}}^{Adams}([M]) \ . \tag{106}$$

*where $[M]_K \in \pi_n(K \wedge M_+)$ denotes the $K$-theory fundamental class of $M$.*

*Proof.* We again start with the general case (106). Using Lemma 2.12 we calculate

$$\begin{aligned} \varepsilon(u_*([M,g])) &= \mathtt{ev}_\Theta(\eta^{top}(u_*([M,g]))) = \mathtt{ev}_\Theta(u^*(\eta^{top}([M,g]))) \\ &= \mathtt{ev}_{u^*\Theta}(\eta^{top}([M,g])) = \mathtt{ev}_{i_k^* u^* \Theta}(\eta^{top}([M,g_0])) \\ &= \mathtt{ev}_{\Theta_0 + k}(\eta^{top}([M,g_0])) \ . \end{aligned}$$

Note that $k \in K^0(BGL(\mathbb{C}^\delta)^+_+)$ is pulled-back from a point. We can now use the unstable case (103) and (89) in order to deduce

$$\varepsilon(u_*([M,g])) = \langle g_0^*, [M]_K \Lambda_0 \rangle + k e_{\mathbb{C}}^{Adams}([M]) \ .$$

The unitary case (105) now follows from (106) and (102). □

We now show how one can deduce a special case of [42, Thm A] from (106). In [42] an algebraic $K$-theory class is constructed from a homology sphere $M$ of dimension $n$ and a representation $\alpha : \pi_1(M) \to GL(k, \mathbb{C}^\delta)$. One gets an induced map $\tilde{g} : M \xrightarrow{\alpha} BGL(k, \mathbb{C}^\delta) \to BGL(\mathbb{C}^\delta)$ to which Quillens +-construction is applied. The fundamental group of a homology sphere is perfect so that the +-construction $M^+$ of $M$ is homotopy equivalent to a simply-connected homology sphere, hence to $S^n$. Thus we get a map

$g^+ : S^n \simeq M^+ \stackrel{\tilde{g}^+}{\to} BGL(\mathbb{C}^\delta)^+$ which represents a class $[S^n, g^+] \in \pi_n(K(\mathbb{C}))$. The homology sphere $M$ admits a stable normal framing (see e.g. [44] or [31, Lem. 1]) and the composition $g : M \to M^+ \stackrel{\tilde{g}^+}{\to} BGL(\mathbb{C}^\delta)^+$ gives the class $[M, g] \in \pi_n(S \wedge \Omega^\infty K(\mathbb{C})_+)$ such that $u_*([M, g]) = [S^n, g^+]$.

Let $\mathbf{V}^{flat}$ denote the flat vector bundle determined by the representation $\alpha$. The statement of [42, Thm A] is the equality

$$e([S^n, g^+]) = \rho_{\mathbb{C}}(\not{D}_M, \mathbf{V}^{flat}) \tag{107}$$

in $\mathbb{C}/\mathbb{Z}$, where on the left-hand side we consider $[S^n, g^+] \in K^0(S^n)$, the map $e$ is the transformation (100), and we have identified $K\mathbb{C}/\mathbb{Z}^{-1}(S^n) \cong \mathbb{C}/\mathbb{Z}$. The subscript $\mathbb{C}$ on the right-hand side indicates a complex version of the $\rho$-invariant defined for flat vector bundles without requiring a flat hermitean metric. In the language of the present paper, using Lemma 2.5, the left-hand side of (107) can be identified with

$$\varepsilon([S^n, g^+]) - ke_{\mathbb{C}}^{Adams}([M]) \in \mathbb{R}/\mathbb{Z}$$

if $[S^n, g^+]$ is a torsion class.

The real part of the complex $\rho$-invariant for a (not necessarily unitarily) flat vector bundle $\mathbf{V}^{flat}$ can be expressed as

$$\langle g^*\Lambda_0, [M]_K\rangle \ .$$

Therefore, if $[S^n, g^+] \in \pi_n(K(\mathbb{C}))$ is torsion, then (106) is equivalent to the real part of (107).

Let us comment on the fact that Adams' $e$-invariant appears on the right-hand sides of (105) and (106). Note that $K(\mathbb{C})$ is a ring spectrum with unit $\epsilon_{K(\mathbb{C})} : S \to K(\mathbb{C})$. The unit induces a homomorphism $\pi_*^S \to \pi_*(K(\mathbb{C}))$. Since the image $\mathtt{im}(J)$ of the $J$-homomorphism (90) is a well-known summand of $\pi_*^S$ it was an interesting question to determine its image under the unit $\epsilon_{K(\mathbb{C})}$. Let us consider the case

$$\frac{\mathbb{Z}}{\frac{B_m}{4m}\mathbb{Z}} \cong \mathtt{im}(J)_{4m-1} \subseteq \pi_{4m-1}^S \ .$$

In [50] it has been shown that this piece goes injectively to algebraic $K$-theory. This was deduced from the following two facts:

1. $\mathtt{im}(J)_{4m-1}$ is detected by (the real version of) Adams' $e$-invariant $e : \pi_{4m-1}^S \to \mathbb{Q}/\mathbb{Z}$.
2. The $e$-invariant has a factorization over the analog
$$K(\mathbb{R})^0(\dots) \to KO\mathbb{C}/\mathbb{Z}^{-1}(\dots)$$
of the homomorphism (100).

The complex case of this factorization is easily seen from Theorem 5.6. In fact the elements in $\mathtt{im}(J)_{4m-1}$ can be represented by cycles of the form $u_*([S^{4m-1}_{fr}, g])$, where $fr$ is a normal framing obtained from the standard framing by twisting with an element of $\pi_{4m-1}(O)$. For $\mathbf{V}$ we take the trivial bundle of dimension 1. Then we get from Theorem 5.6

$$\varepsilon(u_*([S^{4m-1}_{fr}, g])) = e_{\mathbb{C}}^{Adams}([S^{4m-1}_{fr}]) \ .$$

## 5.4 *String*-bordism

In this Subsection we describe the connection of the present paper with constructions in [21]. In this reference we constructed an invariant $b^{an}$ of elements of the *String*-bordism group in dimension $4m-1$ using a formula which shares a lot of similarities with the intrinsic formula (88) for $\eta^{an}$. For the evaluation of $b^{an}$ on the subset of string bordism classes in the kernel of the natural map to *Spin*-bordism we in addition gave extrinsic formulas involving the *Spin* zero bordisms explicitly. One of the interesting features of the restriction of $b^{an}$ to this subset is that it has a factorization over the String orientation of `tmf`, the spectrum of topological modular forms. Since $b^{an}$ is calculable in interesting cases it can be used to detect the `tmf`-class represented by a closed *String*-manifold. This will be the topic of another publication.

Our goal here is to give the precise relation between $b^{an}$ and the universal $\eta$-invariant of the present paper. The result is formulated in Theorem 5.8. In the course of this discussion we show one of the conjectures stated [21] asserting that the factorization of $b^{an}$ over topological modular forms holds true on the whole *String*-bordism group, i.e. we get rid of the restriction to the kernel to the *Spin*-bordism. This is formulated as Corollary 5.11. Formally our proof is complete in dimensions $8m-1$, while in dimensions $8m-5$ we lose some two-torsion since in the present paper we work with complex $K$-theory instead of real $K$-theory. We strongly believe that the relevant part of the theory has a real version which does prove the case in dimension $8m-5$ completely, too.

We designed the notion of a geometrization such that it allows to produce an intrinsic formula for the universal $\eta$-invariant which specializes to the previously known intrinsic formula for $b^{an}$. In Proposition 5.13 we show how the Riemannian geometry on a *String*-manifold together with a geometric string structure give rise to a geometrization, and we derive the corresponding intrinsic formula.

We start with recalling the definition of *String*-bordism. The space $BString$ is defined as a stage in the Postnikov tower of $BO$:

$$\begin{tikzcd}
BString \arrow[d, "p"] & \\
BSpin \arrow[r, "\frac{p_1}{2}"] \arrow[d] & K(\mathbb{Z},4) \\
BSO \arrow[r, "w_2"] \arrow[d] & K(\mathbb{Z}/2\mathbb{Z},2) \\
BO \arrow[r, "w_1"] & K(\mathbb{Z}/2\mathbb{Z},1)
\end{tikzcd} .$$

The space $BString = BO\langle 8\rangle$ is just a low instance of a whole tower of higher connected coverings of the classifying space $BO$. Starting with $BString$ these higher spaces are no longer associated to classical families of compact Lie groups.

In the *String*-case the search for appropriate geometric models (see e.g. [34]) plays an important role in the developments around the Stolz-Teichner program [56]. The

naming $BString$ is related to the fact that string structures on a manifold are related with $Spin$-structures on its loop space. The principal idea of $String$-geometry is to translate geometric structures on the infinite-dimensional $Spin$-principal bundle of the loop space to finite-dimensional geometric objects on the manifold itself by a sort of transgression. Geometrizations of $String$-manifolds can be considered as one aspect of $String$-geometry though in the present paper we will not consider the connection with the loop space.
In Lemma 5.12 we demonstrate that a connection on a $Spin$-principal bundle gives rise to a geometrization. While the connection on the principal bundle allows to define connections on all associated vector bundles, the geometrization partially keeps this information in terms of the differential $K$-theory classes represented by these vector bundles with connections. The geometrization associated to a geometric $String$ structure in this sense replaces the theory of connections on the non-existing principal bundle with structure group $String$. We think that the methods used in the case of $BString = BO\langle 8\rangle$ can easily be adapted to the higher stages $BO\langle n\rangle$.
In order to comply with the notation of the present paper we write $MBString$ for the Thom spectrum which would usually be denoted by $MString$ or $MO\langle 8\rangle$.
We apply our theory in the case where $X = *$ and $B = BString$ with the map

$$\sigma : BString \to BSpin \to BSpin^c \ .$$

The $String$-bordism spectrum $MBString$ is rationally even (see [39], [41], [40] for more calculations), so that for $n = 4m - 1$ we have

$$\pi_n(MBString)_{tors} = \pi_n(MBString) \ .$$

We write $Q_n(BString) := Q_n(BString, \{*\})$ and consider the analytic and topological aspect of the universal $\eta$-invariant

$$\eta^{top} = \eta^{an} : \pi_n(MBString) \to Q_n(BString) \ .$$

We will show that we can obtain $b^{an}$ from the universal $\eta$-invariant by defining an interesting homomorphism out of $Q_n(BString)$. It involves evaluations against a collection of elements $R_k(\xi_n^{String}) \in K^0(BString_+)$ for all $k \geq 0$. It is useful to organize this collection in a formal power series

$$R(\xi_n^{String}) := \sum_{k\geq 0} q^k R_k(\xi_n^{String}) \in K[[q]]^0(BString_+)$$

which we will describe in the following. By $K[[q]]$ we denote the multiplicative cohomology theory (resp. the corresponding spectrum) which associates to a space $Y$ the ring

$$K[[q]]^*(Y_+) := K^*(Y_+)[[q]]$$

of formal power series with coefficients in $K^*(Y_+)$. The following constructions with real vector bundles are standard in the theory of the Witten genus (111), compare e.g. with [35], [21]. Given a real vector bundle $V \to Y$ we consider the element

$$R(V) \in K[[q]]^0(Y_+)$$

defined by

$$R(V) := \sum_{k=0}^{\infty} R_k(V) q^k \ , \tag{108}$$

where $R_k(V)$ is the $K$-theory class of the virtual bundle given by the coefficient in front of $q^k$ in the expansion of

$$\prod_{k\geq 1}(1-q^k)^{2\dim(V)} \bigotimes_{k\geq 1} \mathtt{Sym}_{q^k}(V \otimes_{\mathbb{R}} \mathbb{C}) \ ,$$

where

$$\mathtt{Sym}_p(V) := \bigoplus_{k\geq 0} p^k \mathtt{Sym}^k(V) \ .$$

The transformation $V \mapsto R(V)$ is exponential, i.e. it satisfies $R(V \oplus W) = R(V) \cup R(W)$. Moreover, it has values in the group of multiplicative units $K[[q]]^0(Y_+)^\times$ because the power series starts with 1, i.e. we have $R(V) = 1 + O(q)$. In view of the universal property of $KO^0$ it therefore extends to a natural transformation

$$R : KO^0(Y_+) \to K[[q]]^0(Y_+)^\times \ .$$

The composition

$$BString \to BO \stackrel{x \mapsto (0,x)}{\longrightarrow} \mathbb{Z} \times BO \cong \Omega^\infty KO$$

classifies the universal class $\theta^{String} \in KO^0(BString_+)$. We fix $n = 4m-1$ and let

$$\lambda_n^{String} := n - \theta^{String} \in KO^0(BString_+) \ . \tag{109}$$

If $(M, f)$ is a cycle for $\pi_n(MBString)$, where the map $f : M \to BString$ classifies the $String$-structure on the stable normal bundle of $M$, then we obviously have the equality

$$[TM] + 1 = f^* \lambda_{n+1}^{String} \in KO^0(M_+) \tag{110}$$

We have well-defined classes $R_k(\lambda_{n+1}^{String}) \in K^0(BString_+)$ for all $k \geq 0$ and therefore $R(\lambda_{n+1}^{String}) \in K[[q]]^0(BString_+)$. With this notation the Witten genus

$$\sigma_{Witten}^{\mathbb{C}} : \pi_{n+1}(MBString) \to \pi_{n+1}(K[[q]])$$

is given by

$$\sigma_{Witten}^{\mathbb{C}}(y) = \langle \mathtt{Thom}^K(R(\lambda_{n+1}^{String})), \epsilon_K(x) \rangle \ . \tag{111}$$

We use the superscript $\mathbb{C}$ in order to indicate that we work with the image of the $KO[[q]]$-valued Witten genus in complex $K$-theory $K[[q]]$.
Motivated by the above constructions we organize the evaluations of $Q_n(BString)$ against the family of classes $R_k(\lambda_{n+1}^{String})$ into a formal power series and define a homomorphism

$$W : \mathtt{Hom}(K^0(BString_+), \pi_{n+1}(K\mathbb{Q}/\mathbb{Z})) \to \mathbb{Q}/\mathbb{Z}[[q]] := \prod_{k\geq 0} \mathbb{Q}/\mathbb{Z}\, q^k \tag{112}$$

by

$$W(\phi) := \sum_{k\geq 0} \mathtt{ev}_{R_k(\lambda^{String}_{n+1})}(\phi) q^k \ .$$

Here we identify $\pi_{n+1}(K\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$. The homomorphism (112) does not yet factorize over the quotient $Q_n(BString)$ since it does not vanish on the subgroup $U$ defined in (24). In order to get such a factorization we must replace the target $\mathbb{Q}/\mathbb{Z}[[q]]$ of $W$ by the quotient by a subgroup which contains $W(U)$. This subgroup will be defined using modular forms. We let $\mathcal{M}^R_{2m}$ denote the space of modular forms for $SL(2,\mathbb{Z})$ of weight $2m$ whose $q$-expansion has coefficients in the subring $R \subseteq \mathbb{C}$ (see [35] for an introduction). In particular, we let

$$\mathcal{M}^{\mathbb{Q}}_{2m}[[q]] \subseteq \mathbb{Q}[[q]]$$

be the finite-dimensional vector space of $q$-expansions of rational modular forms $\mathcal{M}^{\mathbb{Q}}_{2m}$ of weight $2m$. Its image in $\mathbb{Q}/\mathbb{Z}[[q]]$ will be denoted by $\overline{\mathcal{M}^{\mathbb{Q}}_{2m}[[q]]}$. We define

$$T_{2m} := \frac{\mathbb{Q}/\mathbb{Z}[[q]]}{\overline{\mathcal{M}^{\mathbb{Q}}_{2m}[[q]]}} \ . \tag{113}$$

Up to the replacement of $\mathbb{Q}$ by $\mathbb{R}$ this is exactly the group defined in [21, Def. 1.1].

**Lemma 5.7** *The composition of (112) with the projection to the quotient (113) induces a well-defined map*

$$\bar{W} : Q_{4m-1}(BString) \to T_{2m} \ .$$

*Proof.* We must show that under this composition the subgroup $U$ defined in (24) is mapped to $\overline{\mathcal{M}^{\mathbb{Q}}_{2m}[[q]]}$. By (111) we have for $y \in \pi_{n+1}(MBString)$ that

$$\langle \mathtt{Thom}^K(R(\lambda^{String}_{n+1})), \epsilon_K(y)\rangle = \sigma^{\mathbb{C}}_{Witten}(y) \in \pi_{n+1}(K[[q]]) \ .$$

If we identify $\pi_{n+1}(K[[q]]) \cong \mathbb{Z}[[q]]$, then the Witten genus has values in $\mathcal{M}^{\mathbb{Z}}_{2m}[[q]] \subset \mathbb{Z}[[q]]$. More generally, for $y \in \pi_{n+1}(MBString\mathbb{Q})$ we get

$$\langle \mathtt{Thom}^K(R(\lambda^{String}_{n+1})), \epsilon_K(y)\rangle \in \mathcal{M}^{\mathbb{Q}}_{2m}[[q]] \ .$$

This shows that $\bar{W}(U) \subseteq \overline{\mathcal{M}^{\mathbb{Q}}_{2m}[[q]]}$. □

In [21, Sec 3.3] we have constructed homomorphisms

$$b^{an} : \pi_{4m-1}(MBString) \to T_{2m} \ , \quad b^{top} : A_{4m-1} \to T_{2m}$$

where

$$A_{4m-1} = \mathtt{ker}(\pi_{4m-1}(MBString) \to \pi_{4m-1}(MBSpin)) \ .$$

Since in the present paper we work we complex $K$-theory as opposed to real $K$-theory in [21, Sec 3.3] we define

$$b^*_{\mathbb{C}} := \begin{cases} b^* & m\ even \\ 2b^* & m\ odd \end{cases} \ , \quad * \in \{an, top, \mathtt{tmf}\} \ . \tag{114}$$

The following theorem clarifies the relation between $b^{an}_{\mathbb{C}}, b^{top}_{\mathbb{C}}$ and the universal $\eta$-invariant of the present paper.

**Theorem 5.8** *We have the equalities*

$$\bar{W} \circ \eta^{top}_{|A_{4m-1}} = b^{top}_{\mathbb{C}}$$

*and*

$$\bar{W} \circ \eta^{an} = b^{an}_{\mathbb{C}}$$

*Proof.* We extend the map $MBString \to MBSpin$ to a fibre sequence

$$\Sigma^{-1}MBSpin \to \mathcal{A} \to MBString \to MBSpin$$

which defines the spectrum $\mathcal{A}$. The smash product of the fibre sequence with the Bockstein sequence

$$\Sigma^{-1}\mathtt{M}\mathbb{Q}/\mathbb{Z} \to \mathtt{M}\mathbb{Z} \to \mathtt{M}\mathbb{Q} \to \mathtt{M}\mathbb{Q}/\mathbb{Z}$$

yields the following quadratic diagram

$$\begin{array}{ccccccc}
\Sigma^{-2}MBSpin\mathbb{Q} & \longrightarrow & \Sigma^{-1}\mathcal{A}\mathbb{Q} & \longrightarrow & \Sigma^{-1}MBString\mathbb{Q} & \longrightarrow & {}^{\hat{w}}\ \Sigma^{-1}MBSpin\mathbb{Q} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{-2}MBSpin\mathbb{Q}/\mathbb{Z} & \longrightarrow & \Sigma^{-1}\mathcal{A}\mathbb{Q}/\mathbb{Z} & \longrightarrow & {}^{\hat{x}}\ \Sigma^{-1}MBString\mathbb{Q}/\mathbb{Z} & \longrightarrow & {}^{\hat{y}}\ \Sigma^{-1}MBSpin\mathbb{Q}/\mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1}MBSpin & \longrightarrow & {}^{\hat{z}}\ \mathcal{A} & \longrightarrow & {}^{x}\ MBString & \longrightarrow & {}^{0}\ MBSpin \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
{}^{-\hat{w}}\ \Sigma^{-1}MBSpin\mathbb{Q} & \longrightarrow & {}^{\tilde{z}}\mathcal{A}\mathbb{Q} & \longrightarrow & {}^{0}\ MBString\mathbb{Q} & \longrightarrow & MBSpin\mathbb{Q}
\end{array}$$

We start with $x \in A_{4m-1} \subseteq \pi_{4m-1}(MBString)$. This element goes to zero if it is mapped to the right or down. The class $\bar{W}(\eta^{top}(x))$ is represented by the power series

$$\sum_{k\ge 0} \langle \mathtt{Thom}^K(R_k(\xi^{String}_{4m})), \epsilon_K(\hat{x})\rangle q^k \in \mathbb{Q}/\mathbb{Z}[[q]] \ . \tag{115}$$

Note that we can define classes $\theta^{Spin}$ and $\lambda^{Spin}_n := n - \theta^{Spin} \in KO^0(BSpin_+)$ analogously to (109). Then we have equalities of the evaluations

$$\begin{aligned}
\langle \mathtt{Thom}^K(R_k(\lambda^{String}_{4m})), \epsilon_K(\hat{x})\rangle &= \langle \mathtt{Thom}^K(R_k(\lambda^{Spin}_{4m})), \epsilon_K(\hat{y})\rangle \\
&= [\langle \mathtt{Thom}^K(R_k(\lambda^{Spin}_{4m})), \epsilon_K(\hat{w})\rangle] \in \mathbb{Q}/\mathbb{Z} \ ,
\end{aligned} \tag{116}$$

where the elements $\hat{y}$ and $\hat{w}$ are images and lifts as indicated in the above diagram, and where we use the compatibility of the $K$-theory Thom isomorphisms for $MBString$ and $MBSpin$. In the construction of $b^{top}$ in [21, Sec 4.1] we go the other way. We first lift $x$ to $\hat{z}$ which maps to $\tilde{z}$ which is then again lifted to $\Sigma^{-1}MBSpin$. Modulo the obvious ambiguities this element in the lower left corner is the negative of $\hat{w}$ from the upper right corner. By the definition of $b^{top}$ in [21, Sec 4.1] we see that $b^{top}_{\mathbb{C}}(x)$ is represented by

$$\sum_{k\geq 0}[\langle \mathtt{Thom}^K(R_k(\lambda^{Spin}_{4m})), \epsilon_K(\hat{w})\rangle]q^k \in \mathbb{Q}/\mathbb{Z}[[q]] \ .$$

Combining this with (115) and (116) we see that

$$\bar{W}\circ \eta^{top}_{|A_{4m-1}} = b^{top}_{\mathbb{C}} \ .$$

This proves the first assertion of Theorem 5.8.
We now show the second. Let $x = [M, f] \in \pi_n(MBString)$ be an $l$-torsion element represented by the cycle $(M, f)$ and $(W, F)$ be a zero-bordism of the union of $l$ copies of $(M, f)$. We choose the geometry on $M$ and $W$ as in Subsection 3.4. The $Spin^c$-structures come from $Spin$-structures so hat the Levi-Civita connections have canonical $Spin^c$-extensions $\tilde{\nabla}^{TM}$ and $\tilde{\nabla}^{TW}$. In view of Equation (110) the $K[[q]]$-theory class $R(\lambda^{String}_{4m})$ can be represented by a formal power series of $\mathbb{Z}/2\mathbb{Z}$-graded bundles $R(TM\oplus 1)$ associated to the tangent bundle. The Riemannian metric and the Levi-Civita connection turn $TM$ into a geometric bundle. The construction of $R(TM\oplus 1)$ therefore produces a formal power series of geometric bundles $\mathbf{R}(TM\oplus 1)$.
The construction of $b^{an}$ involves the choice of a geometric $String$-structure $\alpha$ on $M$. This notion has been introduced in [59]. As a main feature it produces a form $H_\alpha \in \Omega^3(M)$ with the property that

$$2dH_\alpha = p_1(\nabla^{TM,LC}) \ . \tag{117}$$

In the following we use characteristic forms associated to certain power series

$$\tilde{\Phi}\ ,\ \Phi\ ,\ \Theta\ \in \mathbb{Q}[[q]][b, b^{-1}][[p_1, p_2, \ldots]] \ .$$

We refer to [21, Sec 3.3] for an explicit definition. In the present paper distribute the powers of $b$ such that $\Phi$ and $\Theta$ have total degree zero, and $\tilde{\Phi}$ has total degree $-4$. In the the notation of (126) we have

$$\Phi = \Phi_{R(\lambda^{Spin}_{4m})}\ ,\quad \tilde{\Phi} = \tilde{\Phi}_{R(\lambda^{Spin}_{4m})}\ ,\quad \Theta = \Phi - p_1\tilde{\Phi} \ .$$

The notation $\tilde{\Phi}(\nabla^{TM})$ is as in (127). We start with the representative of $b^{an}_{\mathbb{C}}(x)$ given in [21, Def 4.1]

$$[2\int_M H_\alpha \wedge \tilde{\Phi}(\nabla^{TM})] - \xi(\not{D}_M \otimes \mathbf{R}(TM\oplus 1))] \in \mathbb{R}/\mathbb{Z}[[q]] \ , \tag{118}$$

where here and below we ignore the power $b^{-2m}$. We use the APS index formula (52) in order to express the reduced $\eta$-invariants (51). Using the equality

$$\Phi(\nabla^{TW}) = \mathbf{Td}(\tilde{\nabla}^{TM}) \wedge \mathbf{ch}(\nabla^{\mathbf{R}(TM\oplus 1)})$$

we get

$$(118) = [2\int_M H_\alpha \wedge \tilde{\Phi}(\nabla^{TM}) - \frac{1}{l}\int_W \Phi(\nabla^{TW})] + [\frac{1}{l}\mathtt{index}(\not{D}_W \otimes \mathbf{R}(TW))_{APS}] \ .$$

We now use Stokes' theorem and the relation (117) in order to calculate

$$\begin{aligned} 2\int_M & H_\alpha \wedge \tilde{\Phi}(\nabla^{TM}) - \frac{1}{l}\int_W \Phi(\nabla^{TW}) \\ &= \frac{1}{l}\int_W \Big(p_1(\tilde{\nabla}^{TW}) \wedge \tilde{\Phi}(\nabla^{TM}) - \Phi(\tilde{\nabla}^{TW})\Big) \\ &= \frac{1}{l}\int_W \Theta(\tilde{\nabla}^{TW}) \in \mathcal{M}^{\mathbb{R}}_{2m}[[q]] \ . \end{aligned}$$

For the last inclusion we use the crucial fact that

$$p_{4m}(\Theta) \in \mathcal{M}^{\mathbb{Q}}_{2m}[[q]][p_1, p_2, \dots] \subset \mathbb{Q}[[q]][[p_1, p_2, \dots]] \ ,$$

see [21, Sec. 3.3]. Therefore

$$[\frac{1}{l}\mathtt{index}(\not{D}_W \otimes \mathbf{R}(TW))_{APS}] \in \mathbb{R}/\mathbb{Z}[[q]]$$

is a representative of $b^{an}_{\mathbb{C}}(x) \in T_{2m}$, too. But in view of Definition 3.5 and the construction of $\bar{W}$ this is also a representative of $\bar{W}(\eta^{an}(x)) \in T_{2m}$. This shows

$$\bar{W} \circ \eta^{an} = b^{an}_{\mathbb{C}} \ .$$

□

As a consequence of the equality $\eta^{an} = \eta^{top}$ shown in Theorem 3.6 we get another proof of [21, Thm. 2.2].

**Corollary 5.9** *We have the equality*

$$b^{an}_{\mathbb{C}|A_{4m-1}} = b^{top}_{\mathbb{C}} \ .$$

The spectrum of topological modular forms $\mathtt{tmf}$ has been constructed by Miller, Goerss and Hopkins, and in an alternative way by Lurie, see the survey [47]. It is related to $K$-theory and $String$-bordism by a factorization of the Witten genus

$$\begin{tikzcd} MBString \arrow[r, "\sigma_{AHR}"] \arrow[rr, bend left, dotted, "\sigma_{Witten}"] \arrow[rrr, bend left=50, dotted, "\sigma^{\mathbb{C}}_{Witten}"] & \mathtt{tmf} \arrow[r, "W"] \arrow[rr, bend right, dotted, "W_{\mathbb{C}}"'] & KO[[q]] \arrow[r] & K[[q]] \end{tikzcd} \ ,$$

where $\sigma_{AHR}$ is the String-orientation of $\mathtt{tmf}$ constructed by Ando, Hopkins and Rezk. We now recall from [21, Sec.4.3] the construction of the homomorphism

$$b^{\mathtt{tmf}} : \pi_{4m-1}(\mathtt{tmf}) \to T_{2m}$$

which is very similar to that of $\eta^{top}$. Note that $\pi_{4m-1}(\mathtt{tmf})$ is a torsion group (see [37], [13] for more calculations of $\pi_*(\mathtt{tmf})$). Therefore an element $y \in \pi_{4m-1}(\mathtt{tmf})$ can be lifted to an element $\hat{y} \in \pi_{4m}(\mathtt{tmf}\mathbb{Q}/\mathbb{Z})$. Then

$$b^{\mathtt{tmf}}(y) := [W(\hat{y})] \in T_{2m} \ ,$$

where $[W(\hat{y})]$ denotes the class in $T_{2m}$ of the element $W(\hat{y}) \in \pi_{4m}(KO[[q]]\mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}[[q]]$. The complex version $b^{\mathtt{tmf}}_{\mathbb{C}}$ of $b^{\mathtt{tmf}}$ is defined similarly by

$$b^{\mathtt{tmf}}_{\mathbb{C}}(y) := [W_{\mathbb{C}}(\hat{y})] \in T_{2m} \ ,$$

or alternatively, by (114).

**Proposition 5.10** *We have the equality*

$$b^{\mathtt{tmf}}_{\mathbb{C}} \circ \sigma_{AHR} = \bar{W} \circ \eta^{top} : \pi_{4m-1}(MString) \to T_{2m} \ .$$

*Proof.* If $x \in \pi_{4m-1}(MString)$ and $\hat{x} \in: \pi_{4m}(MString\mathbb{Q}/\mathbb{Z})$ is a lift, then

$$\langle \mathtt{Thom}^K(R(\lambda^{String}_{4m})), \epsilon_K(\hat{x})\rangle \in \mathbb{Q}/\mathbb{Z}[[q]]$$

represents $\bar{W}\circ\eta^{top}(x)$. We have already seen in the proof of Lemma 5.7 that this expression is equal to the Witten genus (extended to $\mathbb{Q}/\mathbb{Z}$-theory)

$$\langle \mathtt{Thom}^K(R(\lambda^{String}_{4m})), \epsilon_K(\hat{x})\rangle = \sigma^{\mathbb{C}}_{Witten}(\hat{x}) \ .$$

The Witten genus can now be decomposed as

$$\sigma^{\mathbb{C}}_{Witten}(\hat{x}) = W_{\mathbb{C}}(\sigma_{AHR}(\hat{x})) \ .$$

We can take $\sigma_{AHR}(\hat{x}) \in \pi_{4m}(\mathtt{tmf}\mathbb{Q}/\mathbb{Z})$ as the lift of $\sigma_{AHR}(x) \in \pi_{4m}(\mathtt{tmf})$ so that $W_{\mathbb{C}}(\sigma_{AHR}(\hat{x}))$ represents $b^{\mathtt{tmf}}_{\mathbb{C}}(\sigma_{AHR}(x))$. Hence we can conclude that

$$b^{\mathtt{tmf}}_{\mathbb{C}} \circ \sigma_{AHR}(x) = \bar{W} \circ \eta^{top}(x) \ .$$

$\square$

Using $\eta^{an} = \eta^{top}$ and $\bar{W} \circ \eta^{an} = b^{an}_{\mathbb{C}}$ (Theorem 5.8) we get

**Corollary 5.11** *We have*

$$b^{an}_{\mathbb{C}} = b^{\mathtt{tmf}}_{\mathbb{C}} \circ \sigma_{AHR} \ .$$

This proves the complex version of the conjecture 3 in [21, Sec 1.5]. In fact, for even $m$ there is no difference between the real and complex case, but in the case of odd $m$ the complex version implies the real version up to two-torsion which was known before. We believe that a real version of the present theory would prove the conjecture completely.
The formula for $b^{an}$ given in [21, Sec 3.3] and reproduced here as (118) is an intrinsic formula which uses the notion of a geometric $String$-structure [59]. In the following we show that a geometric $String$-structure gives rise to a good geometrization $\mathcal{G}^{String}$ such that the intrinsic formula 4.12 specializes to the one for $b^{an}$. Since $String$-structures refine $Spin$-structures we start with the construction of a geometrization for a $Spin$-structure.
Let $(M,f)$ with $f:M\to BSpin$ be an $n$-dimensional manifold with a normal $Spin$-structure. We are going to use a version of Subsection 3.3 for $Spin$-structures. If $V\to M$ is a real euclidean oriented vector bundle, then the $Spin$-gerbe $\underline{Spin(V)}$ of $V$ associates to each open subset $A\subseteq M$ the groupoid $Spin(V_{|A})$ of $Spin$-structures on the restriction of $V$ to $A$. This gerbe has the band $\mathbb{Z}/2\mathbb{Z}$, and its isomorphism class is classified by the Dixmier-Douady class

$$\mathtt{DD}(\underline{Spin(V)}) = w_2(V)\in H^2(M;\mathbb{Z}/2\mathbb{Z})\ ,$$

the second Stiefel-Whitney class of $V$.
We choose a Riemannian metric on $M$ and a tangential representative of the normal $Spin$-structure on $TM$. It is given by a geometric $Spin$-structure $P\in Spin(TM)$ together with a trivialization

$$P\otimes \tilde f^*Q_k^{Spin}\cong Q(n+k)\ , \qquad (119)$$

where $\tilde f: M\to BSpin(k)$ is some factorization of $f$, and $Q_k^{Spin}\to BSpin$ this time denotes the universal $Spin$-bundle. It naturally induces a tangential representative of the induced normal $Spin^c$-structure by extension of structure groups along $Spin(l)\to Spin^c(l)$ (see [46, Example D.5]).
The Levi-Civita connection gives rise to a connection $\nabla^{TM}$ on $P$ which in turn has a natural $Spin^c$-extension $\tilde\nabla^{TM}$. We furthermore choose a connection $\nabla^k:=\nabla^{\tilde f^*Q_k^{Spin}}$ on $\tilde f^*Q_k^{Spin}$.

**Proposition 5.12** *There exists a good geometrization $\mathcal{G}^{Spin}$ of $(M,f,\tilde\nabla^{TM})$.*

*Proof.* The connections $\nabla^{TM}$ and $\nabla^k$ together induce a connection $\nabla^{TM}\otimes\nabla^k$ on $P\otimes \tilde f^*Q_k^{Spin}$ which can be compared with a trivial connection using the isomorphism (119). Therefore the transgression form $\tilde{\mathbf{Td}}(\nabla^{TM}\otimes\nabla^k,\nabla^{triv})\in \Omega P^{-1}(M)$ is defined and satisfies

$$d\tilde{\mathbf{Td}}(\nabla^{TM}\otimes\nabla^k,\nabla^{triv}) = \mathbf{Td}(\nabla^{TM})\wedge\mathbf{Td}(\nabla^k)-1\ .$$

We let

$$\mu := \mathbf{Td}(\nabla^k)^{-1}\wedge\tilde{\mathbf{Td}}(\nabla^{TM}\otimes\nabla^k,\nabla^{triv})\ . \qquad (120)$$

For any pointed space or spectrum $Y$ we let $\bar K^*(Y)$ denote the completion the topological group $K^*(Y)$ equipped with the profinite topology (see [18, Def. 4.9] and Subsection 2.2). We have $BSpin_+ := \mathtt{hocolim}_l\, BSpin(l)_+$ and therefore

$$\bar K^*(BSpin_+)\cong \mathtt{lim}_l\,\bar K^*(BSpin(l)_+)\ .$$

The completion theorem [9] gives

$$K^*(BSpin(k)_+) = \bar{K}^*(BSpin(k)_+) \cong R(Spin(k))\hat{}_{I_{Spin(k)}} \ .$$

We therefore get the following description of the completion of the $K$-theory of $BSpin$:

$$\bar{K}^*(BSpin_+) \cong \texttt{lim}_l\, K^*(BSpin(k+l)_+) \cong \texttt{lim}_l\, R(Spin(k+l))\hat{}_{I_{Spin(k+l)}} \ .$$

Given $l \geq 0$ and a representation $\rho$ of $Spin(k+l)$ we obtain a geometric bundle $\mathbf{V}_\rho$ associated to the stabilization $\tilde{f}^*Q_k^{Spin} \otimes Q(l)$ of $\tilde{f}^*Q_k^{Spin}$ with the connection $\nabla^k \otimes \nabla^{Q(l)}$ induced by $\nabla^k$. We define

$$G(\rho) := [\mathbf{V}_\rho] - a(\mu \wedge \mathbf{ch}(\nabla^{V_\rho})) \in \hat{K}^0(M) \ .$$

We have chosen the form $\mu$ in (120) such that the following equality holds true in $\Omega P^0_{cl}(M)$:

$$\mathbf{Td}(\tilde{\nabla}^{TM}) \wedge R(G(\rho)) = \mathbf{Td}(\nabla^k)^{-1} \wedge \mathbf{ch}(\nabla^{V_\rho}) \ . \tag{121}$$

The map $\rho \mapsto G(\rho)$ extends to a map $G : R(Spin(k+l)) \to \hat{K}^0(M)$ by linearity. This extension annihilates the power $I^{2n+1}_{Spin(k+l)} \subseteq R(Spin(k+l))$ of the dimension ideal. In order to see this note that if $\rho \in I^p_{Spin(k+l)}$ and $2p > n$, then we have $\mathbf{ch}(\nabla^{V_\rho}) = 0$. For those $\rho$ we have $G(\rho) = [\mathbf{V}_\rho]$, and this class is flat. If $p > n$, then we have $[V_\rho] = 0$ so that $G(\rho) = a(\omega)$ for some $\omega \in HP\mathbb{R}^{-1}(M_+)$. The product of a flat class with a class of this form vanishes. Hence $G(\rho) = 0$ if e.g. $p > 2n$ (this is not optimal). The map $G$ thus further extends be continuity to a map

$$G : K^0(BSpin(k+l)_+) \to \hat{K}^0(M) \ .$$

One now checks that for $l \geq 1$ we have

$$G(\rho) = G(\rho_{|Spin(k+l-1)}) \ .$$

In this way the maps $G$ for the various $l$ are compatible and define a continuous map

$$\mathcal{G}^{Spin} : \bar{K}^0(BSpin_+) \to \hat{K}^0(M) \ .$$

It follows from (121) that $\mathbf{Td}(\tilde{\nabla}^{TM}) \wedge R(G(\rho))$ is the Chern-Weyl representative of the class $\mathbf{Td}^{-1} \cup \mathbf{ch}([\rho]) \in HP\mathbb{Q}^0(BSpin(k+l)_+)$ associated to the connection $\nabla^k \otimes \nabla^{Q(l)}$, where $[\rho] \in K^0(BSpin(k+l)_+)$ is the class represented by $\rho$. Note that

$$H^*(BSpin; \mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \dots] \tag{122}$$

is the polynomial ring generated by the universal Pontrjagin classes. The cohomological character

$$c_{\mathcal{G}^{Spin}} : HP\mathbb{Q}^0(BSpin_+) \to \Omega P^0_{cl}(M)$$

of $\mathcal{G}^{Spin}$ maps the class $b^{-2i}p_i \in HP\mathbb{Q}^0(BSpin_+)$ to the corresponding characteristic form $b^{-2i}p_i(\nabla^k) \in \Omega P^0_{cl}(M)$. This map clearly preserves degrees.
We now show that the geometrization $\mathcal{G}^{Spin}$ is good. By an inspection of the construction of $\mathcal{G}^{Spin}$ we observe that the connection of the map $f : M \to B$ with the normal bundle has not been used. This map can be arbitrary if we replace $TM$ by some complement $\eta \to M$ of $\hat{f}^*\xi_k$ as in Subsection 4.3 and choose some connection $\nabla^\eta$ of the associated complementary $Spin$-principal bundle $P^{Spin} \in Spin(\eta)$ in the place of $\nabla^{TM}$. We obtain the $Spin^c$-bundle $P$ with connection $\tilde{\nabla}^\eta$ which replaces $\tilde{\nabla}^{TM}$ by extension of the structure group.
We choose an $n+1$-connected approximation $f_u : M_u \to BSpin$ such that there is a factorization of $f$ over a closed embedding $h : M \to M_u$. As in Subsection 4.4 we obtain a natural refinement of $h$ to a $Spin^c$-map. Since $h$ is a closed embedding we can choose the connections $\tilde{\nabla}^u$ on $P_u$ and $\nabla^{k,u}$ on $\tilde{f}_u^*Q_k^{Spin}$ such that $h^*\tilde{\nabla}^u = \tilde{\nabla}^{TM}$ stably and $h^*\nabla^{k,u} = \nabla^k$. We now define $\mathcal{G}_u^{Spin}$ as above. Then by construction $\mathcal{G}^{Spin} = h^*\mathcal{G}_u^{Spin}$ since the correction forms (75) vanishes. Hence $\mathcal{G}^{Spin}$ is good. □

Let $p : BString \to BSpin$ be the natural map. We now consider a manifold $(M, f)$ where $f : M \to BString$ represents a normal $String$-structure. We have an induced normal $Spin$-structure represented by $p \circ f$, and we adopt the geometric choices made before the statement of Proposition 5.12.

**Proposition 5.13** *A choice of a geometric string structure $\alpha$ on $(\tilde{f}^*Q_k^{Spin}, \nabla^k)$ naturally determines a good geometrization $\mathcal{G}^{String}$ of $(M, f, \tilde{\nabla}^{TM})$. For $\phi \in \bar{K}^0(BSpin_+)$ it is given by*

$$\mathcal{G}^{String}(p^*\phi) = \mathcal{G}^{Spin}(\phi) - a(\nu_\phi)$$

*with*

$$\nu_\phi := \mathbf{Td}(\tilde{\nabla}^{TM})^{-1} \wedge 2H_\alpha \wedge \tilde{\Phi}_\phi(\nabla^k) \in \Omega P^{-1}(M) \ .$$

*Here $\mathcal{G}^{Spin}$ is as in Lemma 5.12, $\tilde{\Phi}_\phi(\nabla^k)$ is defined below in (127), and $H_\alpha \in \Omega^3(M)$ is the three-form given by the geometric string structure.*

*Proof.* We have a fibre sequence

$$K(\mathbb{Z}, 3) \to BString \xrightarrow{p} BSpin \to K(\mathbb{Z}, 4) \ .$$

By [4] the group $K^*(K(\mathbb{Z}, 3))$ (note that this is reduced $K$-theory) is divisible and consists of phantom classes, i.e. classes which vanish when pulled-back to finite $CW$-complexes. This suggests the following proposition which is probably well-known, but we could not find a reference for it.

**Proposition 5.14** *The projection $p : BString \to BSpin$ induces a continuous injective map*

$$p^* : \bar{K}^*(BSpin_+) \to \bar{K}^*(BString_+) \ . \qquad (123)$$

*with a dense image.*

*Proof.* It follows immediately from the definition of the profinite topology that maps between spaces or spectra induce continuous maps on the cohomology groups. In order to show the remaining two assertions we need some preparations about divisible groups. Let $A$ be some abelian group. Then we define its subgroup

$$A_{div} := \{a \in A \,|\, \forall n \in \mathbb{N}\, \exists a' \in A \text{ s.t. } a = na'\}$$

of divisible elements. We consider the exact sequence

$$0 \to A_{div} \to A \to \bar{A} \to 0 \ .$$

Since a divisible group is injective this sequence is split. Hence

$$A \cong A_{div} \oplus \bar{A} \ .$$

This implies that $\bar{A}_{div} = 0$. We now consider a short exact sequence of groups

$$0 \to A \to B \to C \to 0$$

together with a map $B \to X$, where $X$ is finitely generated.

**Lemma 5.15** *If $c \in C_{div}$, then we can find a lift $b \in B$ of $c$ whose image in $X$ vanishes.*

*Proof.* We consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
 & A_{div} \arrow[r] \arrow[d] & B_{div} \arrow[r] \arrow[d] & C_{div} \arrow[d] \\
 & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\
 & \bar{A} \arrow[r] \arrow[d] & \bar{B} \arrow[r] \arrow[d] & \bar{C} \arrow[d] \\
 & 0 & 0 & 0
\end{tikzcd}$$

with vertical exact sequences. The Snake Lemma gives an isomorphism

$$\frac{C_{div}}{\mathtt{im}(B_{div} \to C_{div})} \cong \frac{\mathtt{ker}(\bar{B} \to \bar{C})}{\mathtt{im}(\bar{A} \to \bar{B})} \ .$$

This shows that the group on the right-hand side is divisible. Since any map from a divisible group to a finitely generated group is trivial we have a factorization $\bar{B} \to X$ of the map $B \to X$. We thus get a homomorphism

$$\frac{\mathtt{ker}(\bar{B} \to \bar{C})}{\mathtt{im}(\bar{A} \to \bar{B})} \to \frac{X}{Y} \ ,$$

where $Y \subseteq X$ is the image of $\bar{A} \to \bar{B} \to X$. The quotient $X/Y$ is still finitely generated, and this implies that this map is trivial since its domain is divisible.
We now choose a preimage $b_0 \in B$. Its image in $\bar{b}_0 \in \bar{B}$ then vanishes when mapped to $\bar{C}$ so that it represents a class in $\frac{\mathtt{ker}(\bar{B}\to\bar{C})}{\mathtt{im}(\bar{A}\to\bar{B})}$. The image of this class in $X/Y$ vanishes so that there exists $\bar{a} \in \bar{A}$ such that the image of $\bar{b}_0 - \bar{a}$ in $X$ vanishes. We choose some lift $a \in A$ of $\bar{a}$. Then the image of $b := b_0 - a$ in $X$ vanishes. Moreover, $b$ can be taken as a lift of $c$, too. □

We now turn to the actual proof of the Proposition 5.14. We consider the map of maps

$$\begin{tikzcd} BSpin \arrow[r] \arrow[d, "\mathtt{id}"] & BString \arrow[d, "p"] \\ BSpin \arrow[r] & BSpin \end{tikzcd}$$

which induces a morphism of Serre spectral sequences

$$p^* : E(id) \to E(p) \ .$$

The Serre spectral sequence $E(\mathtt{id})$ is of course the Atiyah-Hirzebruch spectral sequence of $BSpin$ which converges to $\bar{K}^*(BSpin_+)$. After choosing a cellular structure on $BSpin$ the first page of $E(p)$ is given by the cellular cochain complex

$$E_1^{s,t}(p) = C^s(BSpin, K^t(K(\mathbb{Z},3)_+))$$

of $BSpin$ with coefficients in $K^*(K(\mathbb{Z},3)_+)$. We have an exact sequence

$$0 \to E_1^{s,t}(p)_{div} \to E_1^{s,t}(p) \to \overline{E}_1^{s,t}(p) \to 0 \ .$$

Since $K^*(K(\mathbb{Z},3))$ is divisible by [4] the composition

$$E_1^{s,t}(\mathtt{id}) \xrightarrow{p^*} E_1^{s,t}(p) \to \overline{E}_1^{s,t}(p)$$

is an isomorphism. Moreover, since $p^*$ is a chain map and there are no non-trivial maps from a divisible group to a finitely generated group we have a decomposition of chain complexes

$$E_1^{*,*}(p) \cong E_1^{*,*}(p)_{div} \oplus p^* E_1^{*,*}(\mathtt{id}) \ .$$

The same reasoning applies to the higher pages and we get a decomposition of the whole spectral sequence as

$$E(p) \cong E(p)_{div} \oplus p^* E(\mathtt{id}) \ .$$

We conclude that (123) is injective.
We now show that it has a dense range. Let $\phi \in K^*(BString_+)$. We must approximate $\phi$ by elements in the image of $p^*$. Let $t : T \to BString$ be a map from a finite $CW$-complex

so that $\mathtt{ker}(t^*) \subseteq K^*(BString_+)$ is some neighborhood of zero. We must show that we can find $\psi \in K^*(BSpin_+)$ such that

$$\phi - p^*\psi \in \mathtt{ker}(t^*) \ . \tag{124}$$

The cellular structure of $BSpin$ induces a filtration

$$\cdots \subseteq BSpin^k \subseteq BSpin^{k+1} \subseteq \dots \ .$$

We let

$$\cdots \subseteq BString^k \subseteq BString^{k+1} \subseteq \dots$$

be the induced filtration by preimages under $p$. We let

$$F^kK^*(BString_+) := \mathtt{ker}(K^*(BString_+) \to K^*(BString^{k-1})_+) \ .$$

Then we have

$$E_\infty^{k,*}(p) \cong F^kK^*(BString_+)/F^{k+1}K^*(BString_+) \ .$$

Since $T$ is compact there exists a number $k_\infty \in \mathbb{N}$ such that there is a factorization $t : T \to BString^{k_\infty - 1} \to BString$. We then have $F^{k_\infty}K^*(BString_+) \subseteq \mathtt{ker}(t^*)$. Assume that $k$ is maximal such that $\phi \in F^kK^*(BString_+)$. It suffices to find $\hat{\psi} \in K^*(BSpin_+)$ and $\rho \in \mathtt{ker}(t^*)$ such that $\phi - p^*\hat{\psi} - \rho \in F^{k+1}K^*(BString_+)$. Then a finite iteration produces the required $\psi \in K^*(BSpin_+)$ such that (124) holds true.
The element $\phi$ gives rise to an element in $u \in E_\infty^{k,*}(p)$ which we can decompose as $u = v \oplus p^*u_0$ with $v \in E_\infty^{k,*}(p)_{div}$. We let $\hat{\psi} \in K^*(BSpin_+)$ be an element with $\hat{\psi}_{|BSpin^{k-1}} = 0$, and which is represented by $u_0 \in E_\infty^{k,*}(\mathtt{id})$. We apply Lemma 5.15 to the exact sequence

$$0 \to F^{k+1}K^*(BString_+) \to F^kK^*(BString_+) \to E_\infty^{k,*}(p) \to 0$$

and the map $t^* : F^kK^*(BString_+) \to K^*(X_+)$. By Lemma 5.15 we can find an element $\rho \in F^kK^*(BString_+) \cap \mathtt{ker}(t^*)$ which represents $v$. Then we have

$$\phi - p^*\hat{\psi} - \rho \in F^{k+1}K^*(BString_+) \ .$$

□

We now come to the construction of the good geometrization $\mathcal{G}^{String}$. We choose an $n+1$-connected approximation $f_u : M_u \to BString$ such that we can factorize $f : M \to BString$ over the closed embedding $h : M \to M_u$. As in Subsection 4.4 we obtain a natural refinement of $h$ to a $Spin^c$-map. Since $h$ is a closed embedding we can choose the connections $\tilde{\nabla}^u$ on $P_u$ and $\nabla^{k,u}$ on $f_u^*Q_k^{Spin}$ such that $h^*\tilde{\nabla}^u = \tilde{\nabla}^{TM}$ stably and $h^*\nabla^{k,u} = \nabla^k$. Geometric string structure behave as flexible as connections and metrics [59]. We can therefore assume that there is a geometric string structure $\alpha_u$ on $(f_u^*Q_k^{Spin}, \nabla^{k,u})$ which restricts to the geometric string structure $\alpha$ on $M$. Let $\mathcal{G}_u^{Spin}$ denote the geometrization of $(M_u, p \circ f_u, \tilde{\nabla}^u)$ constructed in Lemma 5.12. As a first approximation we define

$$\mathcal{G}_{u,0}^{String} : \mathtt{im}\left(p^* : \bar{K}^0(BSpin_+) \to \bar{K}^0(BString_+)\right) \to \hat{K}^0(M_u)$$

by
$$\mathcal{G}^{String}_{u,0}(p^*(\phi)) := \mathcal{G}^{Spin}_u(\phi)\ , \quad \phi \in \bar{K}^0(BSpin_+)\ .$$
We can do this since $p^*$ is injective by Proposition 5.14. The map $\mathcal{G}^{String}_{u,0}$ is defined on a dense subset of $\bar{K}^0(BString_+)$, but is not continuous in general. The idea is now to add a correction term in order to improve the continuity, and then extend by continuity.
Note that via $p^*$ we can identify
$$H^*(BString;\mathbb{Q}) \cong \mathbb{Q}[p_2,\dots]$$
with the quotient ring of $H^*(BSpin;\mathbb{Q})$ given in (122) by setting $p_1 = 0$. The problem with continuity comes from the contribution of $p_1(\tilde{\nabla}^u)$ to the curvature of $\mathcal{G}^{String}_{u,0}$. The idea is now to kill this contribution by a correction term given by a geometric string structure $\alpha_u$ on $(\tilde{f}^*_u Q^{Spin}_k, \nabla^{k,u})$. The geometric string structure provides the form $H_{\alpha_u} \in \Omega^3(M_u)$ with the property that $2dH_{\alpha_u} = p_1(\nabla^{k,u})$ (see 117).
For a formal power series
$$\Lambda \in \mathbb{Q}[b,b^{-1}][[p_1,p_2,\dots]]$$
we define a new formal power series
$$\tilde{\Lambda} := \frac{\Lambda - i_{p_1=0}\Lambda}{p_1} \in \mathbb{Q}[b,b^{-1}][[p_1,p_2,\dots]]\ . \tag{125}$$
In other words, the power series $\tilde{\Lambda}$ is $p_1^{-1}$ times the sum of those monomials of $\Lambda$ which contain $p_1$. Since the periodic rational cohomology of any pointed space $Y$ is complete, i.e. we have $HP\mathbb{Q}^*(Y) \cong \overline{HP\mathbb{Q}}^*(Y)$, the Chern character factorizes over the completion of $K$-theory as $\mathbf{ch} : \bar{K}^0(BSpin_+) \to HP\mathbb{Q}^0(BSpin_+)$. Let $\phi \in \bar{K}^0(BSpin_+)$. Then we define
$$\Phi_\phi := \mathbf{Td}^{-1} \cup \mathbf{ch}(\phi) \in HP\mathbb{Q}^0(BSpin_+) \cong \mathbb{Q}[b,b^{-1}][[p_1,p_2,\dots]] \tag{126}$$
and obtain $\Phi'_\phi$ as described above. We define the form
$$\nu_{u,\phi} := \mathbf{Td}(\tilde{\nabla}^u)^{-1} \wedge 2H_{\alpha_u} \wedge \tilde{\Phi}_\phi(\nabla^{k,u}) \in \Omega P^{-1}(M_u)\ ,$$
where we use the abbreviation
$$\tilde{\Phi}_\phi(\nabla^{k,u}) := \tilde{\Phi}_\phi(p_1(\nabla^{k,u}), p_2(\nabla^{k,u}),\dots)\ . \tag{127}$$
We can now define the map
$$\mathcal{G}^{String}_u : \mathtt{im}(p^*) \to \hat{K}^0(M)$$
by the following prescription:
$$\mathcal{G}^{String}_u(p^*(\phi)) := \mathcal{G}^{String}_{u,0}(p^*(\phi)) - a(\nu_{u,\phi})\ .$$
We further define
$$\mathcal{G}^{String} := h^*\mathcal{G}^{String}_u\ .$$
Unfortunately we can not verify directly that $\mathcal{G}^{String}_u$ is continuous, but we have the following Lemma.

**Lemma 5.16** *The map*

$$\mathcal{G}^{String} : \mathtt{im}\left(p^* : \bar{K}^0(BSpin_+) \to \bar{K}^0(BString_+)\right) \to \hat{K}^0(M)$$

*extends by continuity to all of $\bar{K}^0(BString_+)$ with cohomological character given by $p_i \mapsto p_i(\nabla^k)$ for all $i \geq 2$. The continuous extension (for which we use the same symbol $\mathcal{G}^{String}$) is therefore a geometrization of $(M, f, \tilde{\nabla}^{TM})$.*

*Proof.* Let us consider a sequence $(\phi_k)$ in $\bar{K}^0(BSpin_+)$ such that $p^*\phi_k \to 0$ in the profinite topology of $\bar{K}^0(BString_+)$ as $k \to \infty$. We must show that there exists a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have $\mathcal{G}^{String}(p^*\phi_k) = 0$. Since $M_u$ is compact we can choose a $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have $f_u^* p^*\phi_k = 0$ and $f_u^* p^*(\mathbf{Td}^{-1} \wedge \mathbf{ch}(\phi_k)) = 0$. Since the pull-back $f_u^* : H^*(BString_+; \mathbb{Q}) \to H^*(M_u; \mathbb{Q})$ is injective in degrees $\leq n$ we conclude that $i_{p_1=0}\Phi_{\phi_k}$ is a polynomial in the generators $p_l$ with $4l \geq n+1$. We now calculate using (121) and (125) that for $\phi \in \bar{K}^0(BSpin_+)$

$$\begin{aligned}
\mathbf{Td}(\tilde{\nabla}^u) \wedge R(\mathcal{G}_u^{String}(p^*(\phi))) &= \mathbf{Td}(\tilde{\nabla}^u) \wedge R(\mathcal{G}_u^{Spin}(\phi)) - p_1(\tilde{\nabla}^{k,u}) \wedge \tilde{\Phi}_\phi(\nabla^{k,u}) \\
&= \Phi_\phi(\nabla^{k,u}) - p_1(\tilde{\nabla}^{k,u}) \wedge \tilde{\Phi}_\phi(\nabla^{k,u}) \\
&= (i_{p_1=0}\Phi_\phi)(\nabla^{k,u}) \ . \qquad (128)
\end{aligned}$$

We conclude that $\mathbf{Td}(\tilde{\nabla}^u) \wedge R(\mathcal{G}_u^{String}(p^*(\phi_k))) = 0$ for $k \geq k_0$. It follows for those $k \geq k_0$ that the class $\mathcal{G}_u^{String}(p^*(\phi_k))$ is flat and in the kernel of $I : \hat{K}^0(M_u) \to K^0(M_{u,+})$. We conclude that

$$\mathcal{G}_u^{String}(p^*(\phi_k)) \in HP\mathbb{R}^{-1}(M_{u,+})/\mathtt{im}(\mathbf{ch}) \ .$$

An $n+1$-connected map induces an isomorphism in ordinary cohomology in degrees $\leq n$. Since $BString_+$ is rationally even the odd-dimensional real cohomology of the $n+1$-connected approximation $M_u$ is concentrated in degrees $\geq n+1$. Since $M$ is $n$-dimensional the restriction $h^* : HP\mathbb{R}^{-1}(M_{u,+}) \to HP\mathbb{R}^{-1}(M_+)$ is trivial. This implies that $\mathcal{G}^{String}(p^*(\phi_k)) = h^*\mathcal{G}_u^{String}(p^*(\phi_k)) = 0$ for all $k \geq k_0$. The assertion about the cohomological character follows from the relation

$$\mathbf{Td}(\tilde{\nabla}^{TM}) \wedge R(\mathcal{G}^{String}(\phi)) = (i_{p_1=0}\Phi_{(p^*)^{-1}(\phi)})(\nabla^k)$$

derived from (128). This finishes the proof of Lemma 5.16. □

In order to show that the geometrization $\mathcal{G}^{String}$ constructed in Lemma 5.16 is good we must show that $\mathcal{G}_u^{String}$ is continuous itself. To this end we argue similarly by representing this geometrization as a pull-back from a $\dim(M_u)+1$-connected approximation of $BString_+$. □

We now specialize Theorem 4.12 in order to derive an intrinsic formula for

$$b^{an}([M, f]) = \bar{W} \circ \eta^{an}([M, f]) \in T_{2m} \ .$$

The connection $\nabla^k$ on the $Spin(k)$-principal bundle $\tilde{f}^*Q_k^{Spin} \to M$ turns the real vector bundle $\tilde{f}^*\xi_k^{String}$ into a geometric bundle $\mathbf{N}_k$. It is a geometric representative of the stable normal bundle of $M$, hence the notation. We have

$$R([TM]+1) = R(n+1+k-[\tilde{f}^*\xi_k^{String}]) \in K[[q]]^0(M_+) .$$

Therefore we get an interpretation of $\mathbf{R}(n+1+k-\mathbf{N}_k)$ as a virtual geometric representative of $R([TM]+1)$ (which differs from $\mathbf{R}(TM+1)$ used in (118) since we work with the geometry on the normal bundle). By construction we have

$$\mathcal{G}^{String}(\lambda_{n+1}^{String}) = [\mathbf{R}(n+1+k-\mathbf{N}_k)] + a(\nu_{R(\lambda_{n+1}^{Spin})}) \in \hat{K}^0(M)[[q]] .$$

In other words, the correction form for $[\lambda_{n+1}^{String}] \in K[[q]]^0(BString_+)$ is given by

$$\gamma_{R(\lambda_{n+1}^{String})} = \nu_{R(\lambda_{n+1}^{Spin})} = \mathbf{Td}(\tilde{\nabla}^{TM})^{-1} \wedge 2H_\alpha \wedge \tilde{\Phi}_{R(\lambda_{n+1}^{Spin})}(\nabla^k) \in \Omega P^{-1}(M)[[q]] .$$

By Theorem 4.12 the composition $\bar{W}\circ\eta^{top}([M,f]) \in T_{2m}$ is now represented by the formal power series

$$[-\int_M 2H_\alpha \wedge \tilde{\Phi}_{R(\lambda_{n+1}^{Spin})}(\nabla^k)] - \xi(\not{D}_M \otimes \mathbf{R}(n+1+k-\mathbf{N}_k)) \in \mathbb{R}/\mathbb{Z}[[q]] .$$

This is the version of (118) using the normal bundle geometry on the twisting bundles.

## 5.5 The Crowley-Goette invariants

In this subsection we show how one can derive the Crowley-Goette invariant introduced in [27] for $S^3$-principal bundles on certain $n = 4m-1$-dimensional manifolds as a special case of our universal $\eta$-invariant. We start with recalling the definitions from [27]. Since in the present paper we decided to work with $Spin^c$-bordism and complex Dirac operators we will define the variant $t_M^{\mathbb{C}}$ which coincides with the Crowley-Goette invariant for odd $m$ and is its double for even $m$. Let $S^3$ be the group of unit quaternions and $BS^3$ be its classifying space. The set of homotopy casses $[M, BS^3]$ is the set of isomorphism classes of $S^3$-principal bundles on $M$ denoted in [27] by $\mathtt{Bun}(M)$.
Let $M$ be a closed $n$-dimensional $Spin$-manifold such that $H^3(M;\mathbb{Q}) = 0$ and $H^4(M;\mathbb{Q}) = 0$. Then the Crowley-Goette invariant is defined as a certain function

$$t_M : \mathtt{Bun}(M) \to \mathbb{Q}/\mathbb{Z} .$$

In the following we recall the intrinsic formula [27, (1.9)]. We identify the quaternions with $\mathbb{C}^2$ using the right-multiplication by $I$. The left multiplication of $S^3$ on the quaternions gives a representation $\rho$ on $\mathbb{C}^2$. Note that

$$HP\mathbb{Q}^*(BS^3_+) \cong \mathbb{Q}[b, b^{-1}][[c_2]] ,$$

and by the completion theorem [9] we have the isomorphism

$$K^0(BS^3_+) \cong R(S^3)\hat{}_{I_{S^3}} \ .$$

The representation $\rho$ gives rise to a class $[\rho] \in K^0(BS^3_+)$ and a power series $\mathbf{ch}([\rho]) \in \mathbb{Q}[b,b^{-1}][[c_2]]^0$ of total degree zero. There exists a unique power series $\tilde{\Phi} \in \mathbb{Q}[b,b^{-1}][[c_2]]^{-4}$ of total degree $-4$ such that

$$2 - \mathbf{ch}([\rho]) = c_2\, \tilde{\Phi} \ .$$

Let $\tilde{g} \in \mathtt{Bun}(M)$ and $R \to M$ be an $S^3$-bundle classified by $\tilde{g}$. We choose a connection $\nabla^R$ on $R$. For every unitary representation $\lambda$ of $S^3$ we let $E_\lambda := P \times_{S^3,\lambda} \mathbb{C}^2$ be the vector bundle associated to $R$ and $\lambda$. It comes with a natural hermitean metric $h^{E_\lambda}$. The connection $\nabla^R$ induces a connection $\nabla^{E_\lambda}$ which preserves $h^{E_\lambda}$. In this way we get a geometric bundle $\mathbf{E}_\lambda = (E_\lambda, h^{E_\lambda}, \nabla^{E_\lambda})$. By our assumptions on the rational cohomology the Chern-Weyl representative $c_2(\nabla^R)$ of $c_2$ is exact, and there exists a unique element $\hat{c}_2(\nabla^R) \in \Omega^3(M)/\mathtt{im}(d)$ such that $d\hat{c}_2(\nabla^R) = c_2(\nabla^R)$. We define $\tilde{\Phi}_\rho(\nabla^R) \in \Omega P^{-4}_{cl}(M)$ by replacing $c_2$ by $c_2(\nabla^R)$ in the power series $\tilde{\Phi}$.
We choose a Riemannian metric on $M$ which induces the Levi-Civita connection. Furthermore we choose the *Spin*$^c$-structure induced by the *Spin*-structure. We then get a natural *Spin*$^c$-extension $\tilde{\nabla}^{TM}$ of the Levi-Civita connection. The complex version

$$t^{\mathbb{C}}_M : \mathtt{Bun}(M) \to \mathbb{R}/\mathbb{Z}$$

of $t_M$ is now given by [27, (1.9)]

$$t^{\mathbb{C}}_M(\tilde{g}) := [\int_M \mathbf{Td}(\tilde{\nabla}^{TM}) \wedge \hat{c}_2(\nabla^R) \wedge \tilde{\Phi}(\nabla^R)] - 2\xi(\not{D}_M) + \xi(\not{D}_M \otimes \mathbf{E}_\rho) \in \mathbb{R}/\mathbb{Z} \ . \qquad (129)$$

To be precise, the value of the integral belongs to $\mathbb{R}[b,b^{-1}]^{-4}$ which will be identified with $\mathbb{R}$ using the generator $b^{-2}$. In order to relate the Crowley-Goette invariant with our universal $\eta$-invariant we are led to consider a bordism theory of *Spin*$^c$-manifolds with $S^3$-bundles with rationally trivial second Chern class. This bordism theory is set up as follows. We have a fibration $\partial$

$$\begin{tikzcd} K(\mathbb{Q},3) \arrow[r] & K(\mathbb{Q}/\mathbb{Z},3) \arrow[d, "\partial"] & \\ & K(\mathbb{Z},4) \arrow[r] & K(\mathbb{Q},4) \end{tikzcd}$$

of Eilenberg-MacLane spaces. We define the space $X$ by the following homotopy pull-back

$$\begin{tikzcd} X \arrow[r] \arrow[d, "q"] & K(\mathbb{Q}/\mathbb{Z},3) \arrow[d, "\partial"] \\ BS^3 \arrow[r, "c_2"] & K(\mathbb{Z},4) \end{tikzcd} \ . \qquad (130)$$

Since $c_2$ is a rational isomorphism and $K(\mathbb{Q}/\mathbb{Z},3)$ is rationally trivial we see that the space $X$ is rationally contractible. We conclude that for $n=4m-1$

$$\pi_n(MSpin^c \wedge X_+)\otimes\mathbb{Q} \cong \pi_n(MSpin^c \wedge X\mathbb{Q}_+) \cong \pi_n(MSpin^c\mathbb{Q}) \cong 0\ .$$

It follows that

$$\pi_n(MSpin^c \wedge X_+)_{tors} = \pi_n(MSpin^c \wedge X_+)$$

so that the universal $\eta$-invariant is defined on the whole $Spin^c$-bordism group of $X$:

$$\eta^{top} = \eta^{an} : \pi_n(MSpin^c \wedge X_+) \to Q_n(BSpin^c, X)\ .$$

Next we calculate the $K$-theory $K^*(BSpin^c_+ \wedge X_+)$. We have a fibration

$$BSpin^c_+ \wedge X_+ \stackrel{(\mathtt{id},q)}{\to} BSpin^c_+ \wedge BS^3_+$$

with fibre $K(\mathbb{Q},3)$. Since $K^*(K(\mathbb{Q},3))$ is divisible and consists of phantoms by [4] the proof of Proposition 5.14 applies and shows that

$$(\mathtt{id},q)^* : \bar{K}^*(BSpin^c_+ \wedge BS^3_+) \to \bar{K}^*(BSpin^c_+ \wedge X_+) \tag{131}$$

is injective with dense image. The domain of this map can be calculated the using the completion theorem [9]. At the moment we will only use this map in order to name specific elements in its target. The element $2-\rho$ of the representation ring $R(S^3)$ generates the dimension ideal $I_{S^3}$. If we let $A := 2-[\rho] \in \bar{K}^0(BS^3_+)$, then we have $\bar{K}^*(BS^3_+) \cong \mathbb{Z}[[A]]$ and

$$\bar{K}^*(BSpin^c_+ \wedge BS^3_+) \cong \bar{K}^*(BSpin^c_+)[[A]]\ .$$

Since $X$ is rationally contractible we have

$$\mathbf{ch}((\mathtt{id},q)^*A) = p^*q^*\mathbf{ch}(A) = 0\ ,$$

where $p : BSpin^c_+ \wedge X_+ \to X_+$ is the projection. Hence by Lemma 2.4 the evaluation

$$\mathtt{ev}_{(\mathtt{id},q)^*A} : Q_n(BSpin^c, X) \to \mathbb{Q}/\mathbb{Z}$$

is well-defined. We define

$$\varepsilon := \mathtt{ev}_{(\mathtt{id},q)^*A} \circ \eta^{top} : \pi_n(MSpin^c \wedge X_+) \to \mathbb{Q}/\mathbb{Z}\ .$$

In order to be able to relate the universal $\eta$-invariant with the Crowley-Goette invariant the following simple observation is crucial.

**Lemma 5.17** *A pair $(M,\tilde{g})$ of a compact oriented $n$-dimensional $Spin^c$-manifold $M$ which satisfies the assumptions of Crowley-Goette that $H^3(M;\mathbb{Q})=0$ and $H^4(M;\mathbb{Q})=0$, and a map $\tilde{g} \in \mathtt{Bun}(M)$ gives naturally rise to a class $[M,f,g] \in \pi_n(MBSpin^c \wedge X_+)$.*

*Proof.* The main point is to show that $\tilde{g}: M \to BS^3$ has a natural lift to $g: M \to X$ in the diagram (130). The rationalization of $\tilde{g}^* c_2$ vanishes so that there exists a class $\hat{c}_2 \in H^3(M;\mathbb{Q}/\mathbb{Z})$ such that $\partial \hat{c}_2 = \tilde{g}^* c_2$. This lift is unique up to the image of a rational class of degree 3, hence unique by our assumption. The map $\tilde{g}$ and the lift $\hat{c}_2 : M \to K(\mathbb{Q}/\mathbb{Z}, 3)$ together determine the lift $g: M \to X$. The map $f: M \to BSpin^c$ of course classifies the normal $Spin^c$-structure on $M$. $\square$

If we fix the $Spin^c$-manifold $M$, then we have defined a map

$$s_M : \mathtt{Bun}(M) \to \pi_n(MBSpin^c \wedge X_+) \ , \quad \tilde{g} \mapsto [M, f, g] \ .$$

The following proposition clarifies the relation between $t^{\mathbb{C}}_M$ and the universal $\eta$-invariant.

**Proposition 5.18** *Let $M$ be a compact oriented $n$-dimensional $Spin^c$-manifold $M$ which satisfies $H^3(M;\mathbb{Q}) = 0$ and $H^4(M;\mathbb{Q}) = 0$. Then we have the relation*

$$t^{\mathbb{C}}_M = \varepsilon \circ s_M : \mathtt{Bun}(M) \to \mathbb{Q}/\mathbb{Z} \ .$$

*Proof.* It is an instructive exercise in the use of geometrizations to derive an intrinsic formula for the composition $\varepsilon \circ s_M$ which compares with the formula (129) for $t^{\mathbb{C}}_M$. In a first step we must approximate the space $X$ by spaces with finite skeleta. Note that we can write (compare with (59) for the connecting maps)

$$K(\mathbb{Q}/\mathbb{Z}, 3) := \mathtt{hocolim}_l \, K(\mathbb{Z}/l\mathbb{Z}, 3) \ .$$

If we define $X_l$ by the pull-back

$$\begin{array}{ccccc} X_l & \longrightarrow & K(\mathbb{Z}/l\mathbb{Z},3) & & \\ \downarrow{\scriptstyle q_l} & & \downarrow{\scriptstyle \partial_l} & & \\ BS^3 & \xrightarrow{c_2} & K(\mathbb{Z},4) & \xrightarrow{l} & K(\mathbb{Z},4) \end{array} \quad , \qquad (132)$$

then we get connecting maps $X_l \to X_{l'}$ if $l|l'$ and

$$X \cong \mathtt{hocolim}_l \, X_l \ , \quad \pi_n(MSpin^c \wedge X_+) = \mathtt{colim}_l \, \pi_n(MBSpin^c \wedge X_{l,+}) \ .$$

The main advantage of $X_l$ is that it has finite skeleta. We consider a closed $n$-dimensional manifold with a normal $Spin^c$-structure classified by a map $f: M \to BSpin^c$ and an auxiliary map $g: M \to X$. We can assume that $g$ has a factorization

$$g : M \stackrel{g_l}{\to} X_l \to X$$

for some $l$. We choose a Riemannian metric on $M$ and a $Spin^c$-extension $\tilde{\nabla}^{TM}$ of the Levi-Civita connection. We are going to construct a good geometrization for $(M, f, g_l, \tilde{\nabla}^{TM})$ using similar ideas as in the *String*-bordism case Proposition 5.13. We choose a compact $\max(n+1,4)$-connected approximation $(f_u, g_u) : M_u \to BSpin^c \times X_l$ such that the map

$(f, g_l)$ factorizes over a closed embedding $h : M \to M_u$. We choose the $Spin^c$-connection $\tilde{\nabla}^u$ as in Subsection 4.4. The map $h$ has a refinement to a $Spin^c$-map and we can assume that $h^*\tilde{\nabla}^u = \tilde{\nabla}^{TM}$ stably.
The composition $q_l \circ g_u : M_u \to BS^3$ classifies an $S^3$-principal bundle $R_u \to M_u$ on which we choose a connection $\nabla^{R_u}$. We can assume that $R \cong h^*R_u$ with connection $\nabla^R = h^*\nabla^{R_u}$.
We let

$$\tilde{\mathcal{G}}_u : K^0(BSpin^c_+ \wedge BS^3_+) \to \hat{K}^0(M_u)$$

denote the geometrization of $(M_u, f_u, q_l \circ g, \tilde{\nabla}^u)$ constructed in Lemma 4.3.
We have a fibration

$$BSpin^c_+ \wedge X_{l,+} \stackrel{(\mathtt{id},q_l)}{\to} BSpin^c_+ \wedge BS^3_+ \tag{133}$$

with fibre $K(\mathbb{Z}, 3)$. Since $K^*(K(\mathbb{Z}, 3))$ is divisible and consists of phantoms by [4] the proof of Proposition 5.14 applies again and shows that

$$(\mathtt{id}, q_l)^* : \bar{K}^*(BSpin^c_+ \wedge BS^3_+) \to \bar{K}^*(BSpin^c_+ \wedge X_{l,+}) \tag{134}$$

is injective with dense image. We define

$$\mathcal{G}_{u,0} : \mathtt{im}((\mathtt{id}, q_l)^*) \to \hat{K}^0(M_u)$$

by

$$\mathcal{G}_{u,0}((\mathtt{id}, q_l)^*\phi) := \tilde{\mathcal{G}}_u(\phi) \in \hat{K}^0(M_u) \ .$$

This densely defined map again needs a correction in order to be continuous. We must kill the contribution of $c_2(\nabla^{R_u})$ to the curvature of $\mathcal{G}_{u,0}((\mathtt{id}, q_l)^*\phi)$. Note that $q_l^*c_2 \in H^4(X_l; \mathbb{Z})$ is $l$-torsion. Hence we can choose a form $\alpha_u \in \Omega^3(M_u)/\mathtt{im}(d)$ such that $d\alpha_u = c_2(\nabla^{R_u})$. By an easy application of Serres spectral sequence to the fibration (133) we see that

$$p^* : H^*(BSpin^c_+; \mathbb{Q}) \to H^*(BSpin^c_+ \wedge X_{l,+}; \mathbb{Q})$$

is an isomorphism. Since $H^*(BSpin^c_+; \mathbb{Q})$ is concentrated in even degrees the odd-dimensional cohomology of $M_u$ is concentrated in degrees $\geq n+1$. In particular we see that $\alpha_u$ uniquely determined. Moreover, the restriction $h^* : HP\mathbb{R}^{-1}(M_{u,+}) \to HP\mathbb{R}^{-1}(M_+)$ is trivial.
We have

$$\begin{aligned} HP\mathbb{Q}^*(BSpin^c_+ \wedge BS^3) &\cong \mathbb{Q}[b, b^{-1}][[c_1, p_1, p_2 \dots, c_2]] \ , \\ HP\mathbb{Q}^*(BSpin^c_+ \wedge X_{l,+}) &\cong \mathbb{Q}[b, b^{-1}][[c_1, p_1, p_2 \dots]] \ , \end{aligned}$$

where $c_1$ and the Pontrjagin classes come from $BSpin^c$, and $c_2$ is pulled back from $BS^3$. The pull-back $(\mathtt{id}, q_l)^*$ is the quotient map defined by setting $c_2 = 0$. For $\phi \in K^0(BSpin^c_+ \wedge BS^3)$ we define the formal power series

$$\Phi_\phi := \mathbf{Td}^{-1} \cup \mathbf{ch}(\phi) \in \mathbb{Q}[b, b^{-1}][[c_1, p_1, p_2 \dots, c_2]]^0$$

and set

$$\tilde{\Phi}_\phi := \frac{\Phi_\phi - i_{c_2=0}\Phi_\phi}{c_2} \in \mathbb{Q}[b,b^{-1}][[c_1,p_1,p_2\dots,c_2]]^{-4} \ .$$

For $\phi \in \bar{K}^0(BSpin^c_+ \wedge BS^3_+)$ now define

$$\mathcal{G}_u((\mathtt{id},q_l)^*(\phi)) := \mathcal{G}_{u,0}((\mathtt{id},q_l)^*(\phi)) - a(\alpha_u \wedge \mathbf{Td}(\tilde{\nabla}^u)^{-1} \wedge \tilde{\Phi}_\phi(\nabla^u,\nabla^{R_u})) \ ,$$

where $\tilde{\Phi}_\phi(\nabla^u,\nabla^{R_u}) \in \Omega P^{-4}(M)$ is obtained from $\tilde{\Phi}_\phi$ by replacing the generators $c_1, p_i$ and $c_2$ by their corresponding Chern-Weyl representatives $c_1(\tilde{\nabla}^u)$, $p_i(\tilde{\nabla}^u)$, and $c_2(\nabla^{R_u})$. We calculate similarly as in (128) that

$$\mathbf{Td}(\tilde{\nabla}^u) \wedge R(\mathcal{G}_u(\phi)) = i_{c_2=0}\Phi_\phi(\nabla^u,\nabla^{R_u}) \ . \tag{135}$$

We now define

$$\mathcal{G} : \mathtt{im}((\mathtt{id},q)^*) \to \hat{K}^0(M)$$

by

$$\mathcal{G}((\mathtt{id},q_l)^*(\phi)) := h^*\mathcal{G}_u((\mathtt{id},q_l)^*(\phi)) \ .$$

We claim that $\mathcal{G}$ extends by continuity to a good geometrization of $(M,f,g_l,\tilde{\nabla}^{TM})$. The argument is very similar to that of Lemma 5.16. We first show continuity. If $(\phi_k)$ is a sequence in $\bar{K}^0(BSpin^c_+ \wedge BS^3_+)$ with $(\mathtt{id},q_l)^*\phi_k \to 0$ as $k\to\infty$ in the profinite topology of $\bar{K}^0(BSpin^c_+ \wedge X_{l,+})$, then we can find a $k_0 \in \mathbb{N}$ such that for all $k \ge k_0$ have $(f_u,g_u)^*(\mathtt{id},q_l)^*\phi_k = 0$ and $(f_u,g_u)^*(\mathtt{id},q_l)^*(\mathbf{Td}^{-1}\cup \mathbf{ch}(\phi_k)) = 0$. This implies that

$$\mathcal{G}_u(\phi_k) \in HP\mathbb{R}^{-1}(M_{u,+})/\mathtt{im}(\mathbf{ch})$$

for all $k \ge k_0$. It follows $h^*\mathcal{G}_u(\phi_k) = 0$.
Because of (135) cohomological character of $\mathcal{G}$ is given by

$$c_1 \mapsto c_1(\tilde{\nabla}^{TM}) \ , \quad p_i \mapsto p_i(\tilde{\nabla}^{TM}) \ .$$

Hence it preserves degree.
It follows that $\mathcal{G}$ is a geometrization. In order to see that it is good we show that $\mathcal{G}_u$ itself is continuous using a similar argument based on a $\dim(M_u)+1$-connected approximation of $BSpin^c \times X_l$.
We can now apply Theorem 4.12 in order to derive a formula for $\varepsilon([M,f,g_l]) \in \mathbb{R}/\mathbb{Z}$. We can take $\hat{c}_2(\nabla^R) := h^*\alpha_u$ and have $\tilde{\Phi}_A = \mathbf{Td}^{-1}\tilde{\Phi}$. We have by construction

$$\mathcal{G}((\mathtt{id},q_l)^*A) = [2-\mathbf{E}_\rho] - a(\hat{c}_2(\nabla^R)\wedge\tilde{\Phi}(\nabla^R)) \ ,$$

hence the correction form (Definition 4.11) is given by

$$\gamma_{(\mathtt{id},q_l)^*A} = -\hat{c}_2 \wedge \tilde{\Phi}(\nabla^R) \ .$$

It follows that

$$\mathtt{ev}_{(\mathtt{id},q_l)^*A}(\eta^{an}([M,f,g_l])) = [\int_M \mathbf{Td}(\tilde{\nabla}^{TM}) \wedge \hat{c}_2(\nabla^R)\wedge\tilde{\Phi}(\nabla^R)] - 2\xi(\not{D}_M) + \xi(\not{D}_M \otimes \mathbf{E}) \ .$$

This is exactly the formula for $t^{\mathbb{C}}_M(\tilde{g}) \in \mathbb{R}/\mathbb{Z}$. The Proposition 5.18 now follows from Lemma 2.11 which gives the first equality in the chain

$$\mathtt{ev}_{(\mathtt{id},q_l)^*A}(\eta^{an}([M,f,g_l])) = \mathtt{ev}_{(\mathtt{id},q)^*A}(\eta^{an}([M,f,g])) = \varepsilon(s_M(\tilde{g})) \ .$$

□

The paper [27] provides a lot of interesting explicit calculations. Our general point of view is probably not of much help here. But it is useful to understand structural results like the relation with the Adams $e$-invariant [27, Prop 1.11]. This is what we are going to explain now. We define the space $Y$ by extending the diagram (130) by another cartesian square as follows

$$\begin{tikzcd} Y \arrow[r, "H"] \arrow[d, "r"] & X \arrow[r] \arrow[d, "q"] & K(\mathbb{Q}/\mathbb{Z},3) \arrow[d, "\partial"] \\ S^4 \arrow[r, "h"] & BS^3 \arrow[r, "c_2"] & K(\mathbb{Z},4) \end{tikzcd} \ , \qquad (136)$$

where $h$ generates $\pi_4(BS^3)$ such that $h^*c_2 \in H^4(S^4;\mathbb{Z})$ is the positive orientation class. We use the Serre spectral sequence in order to calculate the rational cohomology of $Y$:

$$H^k(Y;\mathbb{Q}) = \left\{ \begin{array}{cc} \mathbb{Q} & k=0,7 \\ 0 & k \notin \{0,7\} \end{array} \right. \ .$$

This implies

$$\pi_{4m-1}(S \wedge Y_+)_{tors} = \pi_{4m-1}(S \wedge Y_+) \qquad (137)$$

for $m \geq 3$.

From now on we assume that $m \geq 2$. By Lemma 2.11 we get the commutativity of the squares (except of the lower right which will be explained below) of the following commutative diagram:

$$\begin{tikzcd}
\pi_{4m-1}(MBSpin^c \wedge X_+) \arrow[r, "\eta^{top}"] \arrow[rr, dotted, bend left, "\varepsilon"] & Q_{4m-1}(BSpin^c, X) \arrow[r, "\mathtt{ev}_{p^*q^*A}"] & \mathbb{Q}/\mathbb{Z} \\
\pi_{4m-1}(S \wedge Y_+)_{tors} \arrow[u, "(\epsilon_{MBSpin^c} \wedge H)_*"] \arrow[r, "\eta^{top}"] \arrow[d, "(\mathtt{id} \wedge r)_*"'] & Q_{4m-1}(*, Y) \arrow[u, "(i_*, H)"'] \arrow[r, "ev_{r^*h_+^*A}"] \arrow[d, "(\mathtt{id}, r)"'] & \mathbb{Q}/\mathbb{Z} \arrow[u, equal] \\
\pi_{4m-1}(S \wedge S^4_+) \arrow[r, "\eta^{top}"] \arrow[d, "w_*"'] & Q_{4m-1}(*, S^4) \arrow[r, "\mathtt{ev}_{h_+^*A}"] \arrow[d, "w_*"'] & \mathbb{Q}/\mathbb{Z} \arrow[u, equal] \\
\pi_{4m-1}(S \wedge S^4) \arrow[r, "\eta^{top}"] \arrow[d, "\cong"'] & Q_{4m-1}(*, \bar{S}^4) \arrow[r, "\mathtt{ev}_{h^*A}"] \arrow[d, "\cong"'] & \mathbb{Q}/\mathbb{Z} \arrow[u, equal] \\
\pi_{4m-5}(S) \arrow[r, "\eta^{top}"] \arrow[rr, dotted, bend right, "e^{Adams}_{\mathbb{C}}"'] & Q_{4m-5}(*, *) \arrow[r, "\mathtt{ev}_1"] & \mathbb{Q}/\mathbb{Z} \arrow[u, equal]
\end{tikzcd} \quad . \tag{138}$$

We need the condition $m \geq 2$ in order to have well-defined evaluations $\mathtt{ev}_{h_+^*A}$, $\mathtt{ev}_{h^*A}$ and $\mathtt{ev}_1$. The map $w_*$ is induced by the map $w : S^4_+ \to S^4$ which is the identity on $S^4$ and maps the extra base point to the base point of $S^4$. We have

$$K^0(*_+ \wedge S^4_+) \cong K^0(S^4_+) \cong K^0(S^4) \oplus \mathbb{Z} ,$$

where the first summand (which is of course another copy of $\mathbb{Z}$) is the image of $w^*$. This map induces

$$w_* : Q_{4m-1}(*, S^4) \to Q_{4m-1}(*, \bar{S}^4)$$

Sorry for the notational inconvenience caused by the abuse of notation for $Q_n$ of pointed spaces adopted in Subsection 2.6. Here by $\bar{S}^4$ we denote $S^4$ with the internal base point. We use the symbol $h_+ : S^4_+ \to BS^3$ for the map induced by $h$ which maps the extra base point to a base point of $BS^3$. In order to see that the $(2,3)$-square commutes we also use that $\dim(A) = 0$. The lower left vertical map is the suspension isomorphism. The lower middle vertical isomorphism is again induced by suspension and the Bott isomorphism

$$K^0(S^4) \cong K^{-4}(S^0) \xrightarrow{b^2} K^0(S^0) .$$

In order to see that the lower left square commutes note that this isomorphism maps $h^*A$ to 1. This follows from

$$\mathbf{ch}(2 - A) = b^{-2}c_2 + O(b^{-3})$$

and the fact that $c_2 \in H^4(S^4; \mathbb{Z})$ is the suspension of $1 \in H^0(*; \mathbb{Z})$. The composition of the lower two arrows is the definition (89) of the complex version of the Adams $e$-invariant.

We conclude that

$$\varepsilon \circ (\epsilon_{MBSpin^c} \wedge H) = e^{Adams}_{\mathbb{C}} \circ w_* \ . \tag{139}$$

The same argument as for Lemma 5.17 gives

**Lemma 5.19** *A pair $(M, \tilde{g})$ of a compact oriented $4m-1$-dimensional normally framed manifold $M$ which satisfies $H^3(M;\mathbb{Q}) = 0$ and $H^4(M;\mathbb{Q}) = 0$, and a map $\tilde{g} \in [M, S^4]$ gives naturally rise to a class $[M, g] \in \pi_{4m-1}(S \wedge Y_+)$.*

If $M$ satisfies the assumption of the Lemma, then we have a map

$$\tilde{s}_M : [M, S^4] \to \pi_{4m-1}(S \wedge Y_+)$$

and conclude from Proposition 5.18, (139) and (137) that for $m \geq 3$ (or $m = 2$ and $[M, g] \in \pi_7(Y_+)$ is a torsion class)

$$t^{\mathbb{C}}_M = e^{Adams}_{\mathbb{C}} \circ w_* \circ \tilde{s}_M : [M, S^4] \to \mathbb{Q}/\mathbb{Z} \ .$$

This is [27, Prop 1.11] if one takes the following geometric description of the composition $w_* \circ \tilde{s}_M(\tilde{g})$ into account. First of all we have $\tilde{s}_M(\tilde{g}) = [M, g]$, where $g : M \to Y$ is the lift of $\tilde{g}$. Then $w_*([M, g]) = [M, \tilde{g}] - [M, const] \in \pi_{4m-1}(S \wedge S^4)$. The geometric representative of the 4-fold desuspension of this class is the stably normally framed manifold obtained by taking the preimage $Y := \tilde{g}^{-1}(s)$ of a regular point $s \in S^4$ of $\tilde{g}$.

**Corollary 5.20** *[27, Prop 1.11][4] We assume that $m \geq 2$. Let $(M, \tilde{g})$ be a pair of a compact oriented $4m-1$-dimensional normally framed manifold $M$ which satisfies $H^3(M;\mathbb{Q}) = 0$ and $H^4(M;\mathbb{Q}) = 0$, and a map $\tilde{g} \in [M, S^4]$. If $m = 2$, then in addition we assume that $[M, g] \in \pi_7(Y_+)$ is a torsion class. Then we have*

$$t^{\mathbb{C}}_M(h \circ \tilde{g}) = e^{Adams}_{\mathbb{C}}(Y) \ ,$$

*where $Y$ is the preimage $Y := \tilde{g}^{-1}(s)$ of a regular point $s \in S^4$ of $\tilde{g}$ with its induced normal framing.*

In the following we discuss an example which shows that the intrinsic extension of the universal $\eta$-invariant mentioned at the end of Subsection 4.5 behaves quite unexpectedly. We consider the Hopf fibration $\tilde{g} : S^7 \to S^4$. By Lemma 5.19 we get an element $[S^7, g] \in \pi_7(S \wedge Y_+)$. We claim that this element is not torsion. If it would be a torsion element, then by Corollary 5.20 we would have $t^{\mathbb{C}}_{S^7}(h \circ \tilde{g}) = e^{Adams}_{\mathbb{C}}(Y)$, where $Y$ is a Hopf fibre with the induced framing. It has been shown in [27, Example 3.5] that $t^{\mathbb{C}}_{S^7}(h \circ \tilde{g}) = 0$. On the other hand, since the Hopf fibration generates the stable homotopy group $\pi^S_3 \cong \mathbb{Z}/24\mathbb{Z}$ which is detected completely by $e^{Adams}$ we know that $e^{Adams}_{\mathbb{C}}(Y) \in \mathbb{Q}/\mathbb{Z}$ has order 12, in particular is non-trivial. Now $(\mathtt{id} \wedge r)_*([S^7, g])$ is a torsion class. We conclude that

$$\eta^{top}((\mathtt{id} \wedge r)_*([S^7, g])) \neq (i_*, H)(\eta^{intrinsic}([S^7, g])) \ .$$

In other words, the $(2,1)$-square in (138) does not commute if one deletes the subscript $(\dots)_{tors}$ in the second line and replaces the corresponding $\eta^{top}$ by $\eta^{intrinsic}$. We find this surprising.

[4] The $e$-invariant used in the present paper is the negative of the $e$-invariant in the conventions of [27, Prop 1.11]. This accounts for the different sign.

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