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The η-form and a generalized Maslov index

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Abstract. Given a family $\{L_0(b), L_1(b)\}_{b\in B}$ of pairs of transverse Lagrangian subspaces of a hermitean symplectic vector space we define a family of Dirac operators on the unit interval and consider its η-form $\eta(L_0, L_1) \in \Omega^*(B)$. To a family $\{L_0(b), L_1(b), L_2(b)\}_{b\in B}$ of pairwise transverse Lagrangian subspaces we associate the cocycle $\eta(L_0, L_1) + \eta(L_1, L_2) + \eta(L_2, L_1)$ which is a closed form. We identify its cohomology class with a generalization to families of the triple Maslov index.

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1. Introduction

In this note we consider the η-form of a family of Dirac operators $\mathcal{D}(b)$, $b \in B$, on the interval $[0, 1]$ over a base space B. The η-form was introduced by Bismut-Cheeger [3] as the boundary contribution to the local index theorem for families of Dirac operators. It also appeared in the study of adiabatic limits of η-invariants [2]. In our case the operator $\mathcal{D}(b)$ depends on $b \in B$ only through the boundary conditions. If B is a point, then the η-form reduces to the usual η-invariant of $\mathcal{D}$ which was explicitly calculated by Lesch-Wojciechowski [9]. In [5] we found a relation between the η-invariant and the Maslov index. The Maslov index was first introduced in Wall [12] as a measure of the non-additivity under gluing of the signature of manifolds with boundary. This non-additivity was generalized to arbitrary Dirac operators in [5].

In the present note we relate the η-form with a family version of the Maslov index. The family version of the Maslov index conjecturally plays the same role in the non-additivity of the family index of families of Dirac operators on manifolds with cylindrical ends (or APS-boundary conditions) as the usual Maslov index does for the usual index.

Let V be a finite-dimensional Hilbert space equipped with a hermitean symplectic structure Ω (this just means that $\imath\Omega$ is a non-degenerate hermitean form of index (l,l), $\dim_{\mathbf{C}}(V) = 2l$). If $\{L_0(b), L_1(b)\}_{b\in B}$ is a smooth family of pairs of transverse Lagrangian subspaces of V, then we define the η-form $\eta(L_0, L_1) \in C^\infty(B, \Lambda^{ev}T^*B)$. Our main result is

Theorem 1.1. *If $\{L_0(b), L_1(b), L_2(b)\}_{b\in B}$ is a smooth family of triples of pairwise transverse Lagrangian subspaces of V, then*
(1) $d(\eta(L_0, L_1) + \eta(L_1, L_2) + \eta(L_2, L_0)) = 0$,
(2) *and if we define the cohomology class $\tau(L_0, L_1, L_2)$ by*

$$\tau(L_0, L_1, L_2) := [\eta(L_0, L_1) + \eta(L_1, L_2) + \eta(L_2, L_0)] \in H^{even}(B, \mathbf{R}) ,$$

then $\tau(L_0, L_1, L_2) = \mathbf{ch}(L_0^+) - \mathbf{ch}(L_0^-)$, where $L_0 = L_0^+ \oplus L_0^-$ is the splitting of the bundle of Lagrangian subspaces $L_0 \subset B\times V$ into the positive and negative eigenspaces of the quadratic form $Q(x_0) := \Omega(x_1, x_2)$, where $x_i \in L_i$, $x_0 = x_1 + x_2$.

The proof of the theorem is based on a local index theorem for families of Dirac operators on manifolds with cylindrical ends and boundaries with local boundary condition. Instead of saying how the existing proofs Bismut-Cheeger [3], Melrose-Piazza [11] should be modified in order to include the additional boundaries we prefer to work out again the essential arguments. Our approach is modelled on the b-calculus proof of [11], but is more direct and might be of independent interest.

2. Definition of $\eta(L_0, L_1)$

Our definition of the η-form is modelled on the case of even-dimensional base space of [2]. Since the total spaces of the fibrations considered in [2] are odd-dimensional, this corresponds to the case of odd-dimensional fibres.

Let V be a finite-dimensional complex Hilbert space with scalar product $(.,.)$. Let $I \in \mathrm{End}(V)$ be a complex structure, i.e. $I^2 = -1$, $I^* = -I$. We assume that $\mathrm{tr}\, I = 0$. Then $\Omega(x,y) := (Ix, y)$ is a hermitean symplectic structure on V. We consider the formally selfadjoint differential operator $D := I\frac{d}{dt}$ acting on $C^\infty([0,1], V)$.

A complex subspace $L \subset V$ is called Lagrangian, if $L \perp IL$ and $L \oplus IL = V$. We want to consider D as an unbounded operator on the Hilbert space $L^2([0,1], V)$. In order to define selfadjoint extensions $\mathcal{D}$ of D

we choose two Lagrangian subspaces $L_i \subset V, i = 0, 1$. Then we define the domain $\mathrm{dom}(\mathcal{D})$ of $\mathcal{D}$ by

$$\mathrm{dom}(\mathcal{D}) := \{f \in C^\infty([0,1], V) \mid f(i) \in L_i, \ i = 0, 1\} .$$

Then $\mathcal{D}$ is essentially selfadjoint, and we denote its unique selfadjoint extension by $\mathcal{D}$, too. We have $\ker(\mathcal{D}) \cong L_0 \cap L_1$. In particular $\mathcal{D}$ is invertible iff $L_0 \cap L_1 = 0$.

We now turn to families. Let B be some manifold. We consider a pair of smooth families of Lagrangian subspaces $B \ni b \mapsto L_i(b)$, $i = 0, 1$, and we assume that $L_0(b) \cap L_1(b) = 0$, $\forall b \in B$. We obtain a corresponding family $\{\mathcal{D}(b)\}_{b \in B}$ of invertible operators. We want to apply the superconnection formalism in order to define the η-form of that family. Since this formalism involves derivatives of the family with respect to b we prefer to work with an unitary equivalent family $\{\tilde{\mathcal{D}}(b)\}_{b\in B}$ which has the advantage that its domain is independent of $b \in B$.

The η-form is a local object with respect to the base space. In order to define it we only consider a germ of the family near a point $b_0 \in B$. Let $U_I(V)$ denote the group of unitary operators on V which commute with I. The group $U_I(V)$ acts transitively on the space Λ of all Lagrangian subspaces of V. Thus we can find germs of smooth families of unitaries $b \mapsto U_i(b) \in U_I(V)$, $i = 0, 1$, with $U_i(b_0) = 1$ and $U_i(b)L_i(b) = L_i(b_0)$. We can define germs of smooth families $A_i(b) := \log(U_i(b))$ of anti-hermitean matrices using the standard branch of the logarithm. Let $\chi_i \in C^\infty([0,1])$ be cut-off functions with $\chi_0(t) = 1$ for $t < 1/5$, $\chi_0(t) = 0$ for $t > 2/5$, $\chi_1(t) = 1$ for $t > 4/5$, and $\chi_1(t) = 0$ for $t < 3/5$. We set $W(t, b) := \exp(\chi_0(t)A_0(b) + \chi_1(t)A_1(b))$. Then $b \mapsto W(., b)$ can be considered as a germ of a family of unitary multiplication operators on $L^2([0,1], V)$. We set

$$\begin{aligned}\tilde{D}(b) &:= W(., b) D W^*(., b) \\ &= D - \chi_0' I A_0(b) - \chi_1' I A_1(b) ,\end{aligned}$$

where "$'$" denotes the derivative with respect to t. We define the selfadjoint extension of $\tilde{\mathcal{D}}(b)$ using the Lagrangian subspaces $L_0(b_0)$, $L_1(b_0)$. Then $\tilde{\mathcal{D}}(b)$ is unitary equivalent to $\mathcal{D}(b)$ and its domain is independent of b.

We now turn to the definition of the η-form of the family $\{\tilde{\mathcal{D}(b)}\}_{b\in B}$ following [2], Thm. 2.43. Let C_1 denote the graded algebra over $\mathbf{C}$ generated by σ satisfying $\sigma^2 = 1$, $\sigma^* = \sigma$, and $\deg(\sigma) = 1$. Let $\mathcal{H}$ denote the germ at b_0 of the trivial Hilbert space bundle with fibre $L^2([0,1], V) \otimes C_1$ over B. We define the superconnection A_s, $s > 0$, on $\mathcal{H}$ associated to $\tilde{\mathcal{D}}$ by

$$A_s = \nabla - d(\chi_0 A_0 + \chi_1 A_1) + \sqrt{s}\sigma\tilde{\mathcal{D}} ,$$

where d differentiates along B. Here ∇ is the canonical connection of $\mathcal{H}$ and $\nabla - d(\chi_0 A_0 + \chi_1 A_1) = W\nabla W^*$. For $\mathrm{Re}(u) > 1$ we can define the holomorphic family of germs of smooth, even differential forms

$$\eta(u) := \frac{1}{2\sqrt{\pi}} \int_0^\infty \mathrm{tr}_\sigma^{even}(\sigma \tilde{\mathcal{D}} \mathrm{e}^{-A_s^2}) s^{u-1/2} ds \,. \tag{1}$$

Here $\mathrm{tr}_\sigma^{even}(\dots)$ stands for the even form part of $\mathrm{tr}(\sigma \dots)$. As usual the asymptotic expansion of the heat kernels for small times implies that $\eta(u)$ has a meromorphic continuation with respect to u to all of $\mathbf{C}$ having at most first order poles. Following [7] we define

Definition 2.1.

$$\eta(\tilde{\mathcal{D}}) := P.F.\, \eta(0) \,,$$

where "$P.F.$" stands for the finite part of the Laurent expansion of $\eta(u)$ at $u = 0$. As defined above the form $\eta(\tilde{\mathcal{D}})$ may depend on the choices made for the definition of $\tilde{\mathcal{D}}$. But the following lemma justifies the notation $\eta(L_0, L_1) := \eta(\tilde{\mathcal{D}})$.

Lemma 2.2. *$\eta(\tilde{\mathcal{D}})$ does not depend on the choices of the families U_i and the cut-off functions χ_i.*

Proof. Let $\hat{U}_i$, $\hat{\chi}_i$ be another choice and define $\hat{W}$ as above. Let $\hat{\tilde{\mathcal{D}}}$ denote the corresponding family of operators. We set $V = W\hat{W}^*$. Then $\hat{\tilde{\mathcal{D}}} = V^* \tilde{\mathcal{D}} V$. If $\hat{A}_s$ is the superconnection associated to $\hat{\tilde{\mathcal{D}}}$, then $\hat{A}_s = V^* A_s V$. It now follows from the cyclicity of the trace that $\hat{\eta}(u) = \eta(u)$, where $\hat{\eta}(u)$ corresponds to $\hat{A}_s$. $\square$

3. The Maslov cocycle

Now we turn to the generalized Maslov index. Let $\{L_i(b)\}_{b \in B}$, $i = 0, 1, 2$, be smooth families of Lagrangian subspaces of V such that $L_i(b) \cap L_j(b) = \{0\}, \forall b \in B, i \neq j$. Let $\mathbf{ch} : K^0(B) \to H^{ev}(B, \mathbf{R})$ be the Chern character. In the present section we prove

Proposition 3.1. *The form $\eta(L_0, L_1) + \eta(L_1, L_2) + \eta(L_2, L_0)$ is closed. Moreover*

$$\begin{aligned} \tau(L_0, L_1, L_2) &:= [\eta(L_0, L_1) + \eta(L_1, L_2) + \eta(L_2, L_0)] \\ &\in \mathbf{ch}(K^0(B)) \subset H^{even}(B, \mathbf{R}) \,. \end{aligned}$$

Remark. The zero component $\tau(L_0, L_1, L_2)^0 \in \mathbf{Z}$ is the Maslov index (in its hermitean symplectic generalization) of the triple (L_0, L_1, L_2) (see [5]). For an exposition of the usual Maslov index we refer to [10]). In Proposition 4.1 below we explicitly compute the class $\tau(L_0, L_1, L_2) \in H^{even}(B, \mathbf{R})$ in terms of the hermitean symplectic geometry of the family $\{L_0(b), L_1(b), L_2(b)\}_{b\in B}$.

Proof. The idea of the proof is to formulate an index problem for a family of Dirac operators $\partial\!\!\!/^+$ such that the form $\eta(L_0, L_1)+\eta(L_1, L_2)+\eta(L_2, L_0)$ represents the Chern character of the index bundle of $\partial\!\!\!/^+$.

We consider a compact oriented surface M_c with boundary and corners which is diffeomorphic to a rectangular convex 12-gon in the hyperbolic plane. We label the boundary pices $\partial_i M_c$, $i = 0, \dots, 11$, according to their cyclic order.

We choose a Riemannian metric on M_c such that $\partial_i M_c$ are isometric to the interval $[0, 1]$, and such that the boundary pieces intersect in twelve rectangular corners. We assume that the metric is product in a neighbourhood of the interior of the pieces $\partial_i M$, and that neighbourhoods of the corners are isometric to a neighbourhood of the vertex of the euclidean quadrant $\mathbf{R}^+ \times \mathbf{R}^+$.

Let M be the oriented, non-compact Riemann surface with six boundary components $\partial_i M, i = 0, \dots, 5$, isomorphic to $\mathbf{R}$ which is obtained by gluing infinite cylinders $[0, \infty) \times [0, 1]$ along the boundary pieces of M_c with odd label. The boundary components of M are again labelled according to their cyclic order.

We consider the spinor bundle $S = S^+ \oplus S^-$ of M and fix a finite-dimensional Hilbert space W of dimension $\dim(V)/2$. We consider the graded vector bundle $E = S \otimes (W \oplus W^{op})$. By $\partial\!\!\!/$ we denote the corresponding twisted Dirac operator. Let $\partial\!\!\!/^\pm$ be the parts mapping sections of $E^\pm$ to those of $E^\mp$. We formulate an index problem for $\partial\!\!\!/^+$ by putting boundary conditions at $\partial_j M$ depending on $L_i(b)$, $i = 0, 1, 2$.

The metric of M is flat near infinity and the boundaries (the flat region). We claim that the holonomy of the parallel transport in S along ∂M_c is trivial, where we consider M_c as a submanifold of M. Note that M is topologically a disc. Thus we can choose a trivialization of the tangent bundle TM. Measured with respect to the trivialization the parallel transport in TM along ∂M_c gives a rotation by -4π. The trivialization of TM induces one of the $Spin(2)$-principal bundle of M. The parallel transport along ∂M_c in this bundle corresponds to a rotation by -2π in the structure group. This implies the claim. We fix some point in ∂M_c and identify the fibres of the bundle S near infinity and ∂M with the fibre over this point using the parallel transport inside the flat region. We denote this fibre by S, too. Analogously we denote the fibre of E over this point by E.

Let (s,t) be oriented euclidean orthonormal coordinates near a point in the flat region. Then we have $\partial\!\!\!/ = \sigma_s\partial_s + \sigma_t\partial_t$, where $\sigma_s, \sigma_t \in \mathrm{Hom}^{odd}(E,E)$ depend on the choice of coordinates. Again we consider the components $\sigma_s^\pm, \sigma_t^\pm \in \mathrm{Hom}^{odd}(E^\pm, E^\mp)$. The operator $I := \sigma_s^-\sigma_t^+ \in \mathrm{Aut}(E^+)$ is invariantly defined. It satisfies $I^* = -1$, $I^2 = -1$, $\mathrm{tr} I = 0$, and it defines a hermitean symplectic structure on E^+. We fix an isometry $V \cong E^+$ which is compatible with the complex structures I on V and E^+.

Now we introduce the family of boundary conditions defining the family $\{\partial\!\!\!/^+(b)\}_{b\in B}$. We let $\partial\!\!\!/^+(b)$ be the differential operator $\partial\!\!\!/^+$ mapping

$$\{\phi \in C_c^\infty(M,E^+) \mid \phi(x) \in L_i(b) \forall x \in \partial_i M \text{ or } x \in \partial_{i+3}M\}$$

to $L^2(M,E^-)$.

First we show that $\partial\!\!\!/^+(b)$ gives rise to a smooth family of Fredholm operators such that the index bundle is well-defined. Then we apply the superconnection formalism in order to compute the Chern character of the index bundle.

First we conjugate the family $\{\partial\!\!\!/^+(b)\}_{b\in B}$ to a family $\{\tilde{\partial\!\!\!/}^+(b)\}_{b\in B}$ with constant domain. This will be done again on the level of germs at a point $b_0 \in B$. In the remainder of the present section we replace B by a sufficiently small neighbourhood of b_0. We define a germ of a family of smooth $U(V)$-valued functions $W^+(b,m)$, $m \in M$, such that near $\partial_i M$, $\partial_{i+3}M$, $i = 0,1,2$,

$$W^+(b,(s,t)) = \exp(\chi_0(s)A_i(b)) \;, \tag{2}$$

where A_i, χ_0 were defined above. Here (s,t) are orthonormal euclidean coordinates, s being normal to the boundary given by $s=0$. W^+ is determined by (2) near ∂M and we continue W^+ to the interior of M by the constant $1 \in U(V)$. Similarly we set $W^- = -\sigma_s^+ W^+ \sigma_s^-$ near ∂M and continue W^- to the interior of M by $1 \in U(E^-)$.

Let $\tilde{\partial\!\!\!/}^+ := W^-\partial\!\!\!/(W^+)^*$. Then $\tilde{\partial\!\!\!/}^+(b)$ is a germ of a family of (now b-dependent) Dirac operators with domain (now independent of b) given as above by the Lagrangian subspaces $L_i(b_0)$ at $\partial_i M$, $\partial_{i+3}M$, $i = 0,1,2$.

Recall that M has 6 cylindrical ends isomorphic to $[0,\infty)\times[0,1]$. We can consider $[0,\infty)\times[0,1]$ as a subset of the cylinder $\mathbf{R}\times[0,1]$. For $i = 0,\dots,5$ the family $\tilde{\partial\!\!\!/}^+(b)$ induces families of translation invariant operators $\tilde{\partial\!\!\!/}_i^+(b)$ on the infinite cylinder $\mathbf{R}\times[0,1]$ together with boundary conditions. Let $\tilde{\partial\!\!\!/}_i^-(b)$ be the formal adjoint (with the adjoint boundary condition) of $\tilde{\partial\!\!\!/}_i^+(b)$ and let

$$\tilde{\partial\!\!\!/}_i(b) := \begin{pmatrix} 0 & \tilde{\partial\!\!\!/}_i^-(b) \\ \tilde{\partial\!\!\!/}_i^+(b) & 0 \end{pmatrix} .$$

Then $\tilde{\partial\!\!\!/}_i(b)$ acts on sections of a bundle which we also denote by E.

Lemma 3.2.
(1) *The symmetric operator $\tilde{\not{\partial}}_i(b)$ defined on the space of smooth sections with compact support satisfying the boundary conditions is essentially selfadjoint.*
(2) *It has a bounded inverse $\tilde{\not{\partial}}_i(b)^{-1}$.*

Proof. To prove the Lemma it is better to consider the original non gauge-transformed operator. We fix the point $b \in B$ and consider without loss of generality the case $i = 0$. Let $(t, s) \in \mathbf{R} \times [0, 1]$ be the coordinates of the cylinder. Then we put $\not{\partial} := \sigma_t \partial_t + \sigma_s \partial_s$ acting on $C^\infty(\mathbf{R} \times [0, 1], E)$. We let $\mathrm{dom}(\not{\partial})$ be the space of all $f \in C_c^\infty(\mathbf{R} \times [0, 1], E)$ such that $f(t, k) \in (L_k(b) + \sigma_t^+ L_k(b))$, $k = 0, 1$. Then $\not{\partial}$ becomes an symmetric operator on $L^2(\mathbf{R} \times [0, 1], E)$ which is unitary equivalent with $\tilde{\not{\partial}}_0(b)$. We show that $\not{\partial}$ is essentially selfadjoint.

Let $\mathcal{D} := \sigma_t^- \sigma_s^+ \partial_s$ be defined on $\{f \in C^\infty([0, 1], E^+) \mid f(k) \in L_k(b),\ k = 0, 1\}$. Then $\mathcal{D}$ is essentially selfadjoint on $L^2([0, 1], E^+)$. Let $\sigma(\mathcal{D})$ be the spectrum of $\mathcal{D}$ with multiplicity. Moreover let ϕ_λ, $\lambda \in \sigma(\mathcal{D})$, be an orthonormal base of $L^2([0, 1], E^+)$ of eigenvectors of $\mathcal{D}$. Then we let $\mathcal{H}_\lambda \subset L^2([0, 1], E)$ be the subspace spanned by $\phi_\lambda, \sigma_t^+ \phi_\lambda$. We can write

$$L^2(\mathbf{R} \times [0, 1], E) = \bigoplus_{\lambda \in \sigma(\mathcal{D})} \mathcal{H}_\lambda \otimes L^2(\mathbf{R}) = \bigoplus_{\lambda \in \sigma(\mathcal{D})} H_\lambda \,.$$

Note that $E = E^+ \oplus E^-$ induces a splitting $H_\lambda = H_\lambda^+ \oplus H_\lambda^-$. We define

$$\not{\partial}_\lambda := \sigma_t \partial_t + \lambda \begin{pmatrix} 0 & \sigma_t^- \\ -\sigma_t^+ & 0 \end{pmatrix} \,.$$

Let $\check{\not{\partial}} := \oplus_{\lambda \in \sigma(\mathcal{D})} \not{\partial}_\lambda$ be defined on

$$\mathrm{dom}(\check{\not{\partial}}) := \{f \in C_c^\infty(\mathbf{R} \times [0, 1], E) \mid f = \sum_{\lambda \in \sigma(\mathcal{D})}^{finite} f_\lambda, f_\lambda \in H_\lambda\} \,.$$

Then $\check{\not{\partial}}$ is symmetric and $\check{\not{\partial}} \subset \not{\partial}$. If we show that $\check{\not{\partial}}$ is essentially selfadjoint, then we are done since $\bar{\check{\not{\partial}}} \subset \bar{\not{\partial}} \subset \not{\partial}^* \subset \check{\not{\partial}}^*$ implies $\bar{\not{\partial}} = \not{\partial}^*$, where $\bar{\not{\partial}}$ denotes the closure of $\not{\partial}$. Now $\not{\partial}_\lambda$ is essentially selfadjoint on the domain $\mathrm{dom}(\not{\partial}_\lambda) := C_c^\infty(\mathbf{R}, \mathcal{H}_\lambda)$. Assume that $f \in \mathrm{dom}(\check{\not{\partial}}^*)$. Then $|\langle f, \check{\not{\partial}}\psi\rangle| \leq C(f)\|\psi\|$, $\forall \psi \in \mathrm{dom}(\check{\not{\partial}})$. We write $f = \sum_{\lambda \in \sigma(\mathcal{D})} f_\lambda$. Then we can conclude that $|\langle f_\lambda, \not{\partial}_\lambda u\rangle| \leq C(f_\lambda)\|u\|$, $\forall u \in \mathrm{dom}(\not{\partial}_\lambda)$. Thus $f_\lambda \in \mathrm{dom}(\not{\partial}_\lambda^*) = \mathrm{dom}(\bar{\not{\partial}}_\lambda)$. Thus any finite sum of f_λ is in $\mathrm{dom}(\bar{\check{\not{\partial}}})$. We have $\lim_{N \to \infty} \sum_{\lambda \in \sigma(\mathcal{D}), |\lambda| \leq N} f_\lambda = f$, and $\lim_{N \to \infty} \bar{\check{\not{\partial}}} \sum_{\lambda \in \sigma(\mathcal{D}), |\lambda| \leq N} f_\lambda = \check{\not{\partial}}^* f$

exists. Hence $f \in \mathrm{dom}(\bar{\check{\not\partial}})$. This proves that $\check{\not\partial}$ is essentially selfadjoint and thus **(1)**.

We now show **(2)**. Note that $\not\partial_\lambda^2 \geq \lambda^2$. Since $L_0(b) \cap L_1(b) = \{0\}$ we have $\inf_{\lambda \in \sigma(\mathcal{D})} \lambda^2 = c > 0$. It follows that $\not\partial_\lambda^{-1}$ is bounded and $\|\not\partial_\lambda^{-1}\| \leq c^{-1}$. Thus $\not\partial^{-1} = \oplus_{\lambda \in \sigma(\mathcal{D})} \not\partial_\lambda^{-1}$ exists and is bounded by c^{-1}. □

The distribution kernel of $\tilde{\not\partial}_i(b)^{-1}$ gives parametrices for $\tilde{\not\partial}(b)$ at infinity, in the interior, and also near the boundary of M. By patching we build a global parametrix $Q(b)$ such that the smoothing remainders $R(b) := \tilde{\not\partial}(b)Q(b) - 1$, $R_1(b) := Q(b)\tilde{\not\partial}(b) - 1$ have compact support.

Lemma 3.3. (1) *The operator $\tilde{\not\partial}(b)$ is essentially selfadjoint.*
(2) *The domain $\mathcal{H}$ of $\bar{\tilde{\not\partial}}(b)$ is independent of b.*
(3) *$\{\bar{\tilde{\not\partial}}^+(b)\}$ viewed as a family of bounded operators from $\mathcal{H}^+$ to $L^2(M, E^-)$ is a smooth family of Fredholm operators.*

Proof. We start with **(1)**. Let $h \in L^2(M, E)$ be arbitrary. We claim that $Q(b)h \in \mathrm{dom}(\bar{\tilde{\not\partial}}(b))$. We approximate h in $L^2(M, E)$ by a sequence $h_\alpha \in C_c^\infty(M, E)$. Then $Q(b)h_\alpha \in C^\infty(M, E)$ satisfies the boundary conditions. Let $\cdots \subset K_N \subset K_{N+1} \subset \cdots \subset M$ be an exaustion of M by compact subsets admitting a sequence of smooth functions $\{\chi_N\}_{N \in \mathbf{N}}$ such that $\mathrm{supp}(\chi_N) \subset K_{N+1}$, $\chi_{|K_N} = 1$, and such that $\sup_{m \in M} \|d\chi_N(m)\| \to 0$ as $N \to \infty$. We have $h_\alpha = \tilde{\not\partial}(b)Q(b)h_\alpha - R(b)h_\alpha$. Note that $\chi_N Q(b)h_\alpha \in \mathrm{dom}(\tilde{\not\partial})$. The existence of

$$\begin{aligned} &\lim_{\alpha \to \infty} \lim_{N \to \infty} \tilde{\not\partial}_i(b)\chi_N Q(b)h_\alpha \\ &= \lim_{\alpha \to \infty} \lim_{N \to \infty} [\chi_N(R(b) + 1)h_\alpha + \mathrm{grad}(\chi_N)Q(b)h_\alpha] \\ &= (R(b) + 1)h \end{aligned}$$

implies the claim.

Let now $f \in \mathrm{dom}(\tilde{\not\partial}(b)^*)$. We must show that f belongs to the domain of $\bar{\tilde{\not\partial}}(b)$. We write $f = Q(b)\tilde{\not\partial}(b)^* f - R_1(b)f$. By the above $Q\tilde{\not\partial}_i(b)^* f$ belongs to the domain of $\bar{\tilde{\not\partial}}(b)$. Moreover $R_1(b)f$ is smooth and has compact support. Since $\tilde{\not\partial}(b)^* f \in L^2(M, E)$ the trace $f_{|\partial M} \in L^2_{loc}(\partial M, E)$ is well-defined. It follows from

$$|\langle f, \tilde{\not\partial}(b)\psi \rangle| \leq C(f)\|\psi\|, \quad \forall \psi \in \mathrm{dom}(\tilde{\not\partial}(b))$$

that $f_{|\partial M}$ satisfies the same boundary conditions used to define $\tilde{\not\partial}(b)$. We conclude that $R_1(b)f \in \mathrm{dom}(\tilde{\not\partial}(b))$ and hence $f \in \bar{\tilde{\not\partial}}(b)$. This finishes the proof of **(1)**.

Since

$$\tilde{\not{\partial}}(b) = \tilde{\not{\partial}}(b_0) + V(b)\,, \tag{3}$$

where $V(b)$ is a smooth family of bounded operators the domain $\mathcal{H}$ of $\tilde{\not{\partial}}(b)$ does not depend on $b \in B$. This proves **(2)**.

For **(3)** we employ the spectral comparison theorem for manifolds which coincide at infinity stating that $\sigma_{ess}(\tilde{\not{\partial}}(b)) = \cup_i \sigma_{ess}(\tilde{\not{\partial}}_i(b))$, hence $\inf \sigma_{ess}(\tilde{\not{\partial}}(b)) > 0$ for all $b \in B$. From (3) it follows by perturbation theory that the family $\{\tilde{\not{\partial}}^+(b)\}$ is a smooth family of Fredholm operators when it is viewed as a family of bounded operators from $\mathcal{H}^+$ to $L^2(M, E^-)$. □

We now apply the superconnection formalism in order to obtain a formula for the Chern character $\mathbf{ch}(\mathrm{index}(\tilde{\not{\partial}}^+))$ of the index bundle of $\{\tilde{\not{\partial}}^+(b)\}_{b\in B}$.

As a first step we make $\tilde{\not{\partial}}^+(b)$ surjective following [1], Ch.9.5. Let $N \in \mathbf{N}$ be given. Then we consider the manifold $M' := M \cup *$. We extend the bundle E^+ to a bundle $(E')^+$ over M' such that the fibre over the point $*$ is $\mathbf{C}^N$. If we are given a map $\psi : \mathbf{C}^N \to C_c^\infty(M, E^-)$, then we define the operator $\tilde{\not{\partial}}^+_\psi(b) : \mathcal{H}^+ \oplus C^N \to L^2(M, E^-)$ by $\tilde{\not{\partial}}^+_\psi(b)(f \oplus u) := \tilde{\not{\partial}}^+(b)f + \psi(u)$.

Assume that B is compact. Then as in [1], Lemma 9.30, there exists an integer N and a linear map $\psi : \mathbf{C}^N \to C_c^\infty(M, E^-)$ such that $\tilde{\not{\partial}}^+_\psi(b)$ is surjective for all $b \in B$. In this case we have a bundle $\ker(\tilde{\not{\partial}}^+_\psi) := \{\ker(\tilde{\not{\partial}}^+_\psi(b))\}_{b\in B}$. The index bundle $\mathrm{index}(\tilde{\not{\partial}}^+) \in K^0(B)$ is represented by $\ker(\tilde{\not{\partial}}^+_\psi) - B \times \mathbf{C}^N$. We fix ψ as above, and in addition we can assume that $\mathrm{supp}(\psi(u)) \in M_c$, $\forall u \in \mathbf{C}^N$.

In the remainder of the present section we replace M by M', E^+ by $(E')^+$, and $\tilde{\not{\partial}}^+$ by $\tilde{\not{\partial}}^+_{\epsilon\psi}$ such that $\tilde{\not{\partial}}^+_\psi$ is surjective.

First we take $\epsilon = 1$. Let ∇ denote the trivial connection on the bundle $B \times L^2(M, E)$. We set $W := W^+ \oplus W^-$. Let $\tilde{\nabla} = W \nabla W^*$. Then we define the superconnection

$$B_t = \tilde{\nabla} + \sqrt{t}\tilde{\not{\partial}}\,.$$

The curvature B_t^2 has the form

$$B_t^2 = t\tilde{\not{\partial}}^2 + \sqrt{t}R\,,$$

where R is a one-form with values in the odd endomorphisms of E. The heat operator $\mathrm{e}^{-B_t^2}$ can be constructed using the Volterra series [1], Prop.9.46.

Let $P_t(x,y)$ denote the smooth integral kernel of $\mathrm{e}^{-B_t^2}$. In order to simplify the notation we omit the smooth dependence on $b \in B$.

Note that M is a manifold with a cylindrical end $N \times [0,\infty)$, where N is isometric to the disjoint union of six copies of the unit interval. Let (n,r) denote corresponding coordinates. Though $\mathrm{e}^{-B_t^2}$ is not of trace class we define

$$\mathrm{Tr}'_s \mathrm{e}^{-B_t^2} := \int_{M_c} \mathrm{tr}_s P_t(x,x)dx + \lim_{u\to\infty} \int_0^u \int_N \mathrm{tr}_s P_t((n,r),(n,r))dn\, dr \,.$$

We first argue that this limit exists. We claim that for some $C < \infty$, $c > 0$,

$$|\mathrm{tr}_s P_t((n,r),(n,r))| < C\mathrm{e}^{-cr^2} \,.$$

The constants C, c can be choosen uniformly for t varying in compact subsets of $(0,\infty)$. Consider the infinite cylinder $Z := N \times \mathbf{R}$. Let E^Z be the bundle on Z induced by E. Let $\{\tilde{\partial\!\!\!/}^Z(b)\}_{b\in B}$ denote the family of translation invariant operators on E^Z induced by $\{\tilde{\partial\!\!\!/}(b)\}_{b\in B}$. The domain $\mathcal{H}^Z$ of $\tilde{\partial\!\!\!/}(b)^Z$ is again independent of $b \in B$. We then obtain a translation invariant superconnection B_t^Z on the bundle $B \times L^2(Z, E^Z)$. Let $P_t^Z((n,r),(m,s))$ denote the corresponding heat kernel. The Clifford multiplication by $\imath\sigma_r$ is unitary, odd, and commutes with $(B_t^Z)^2$. Thus

$$\mathrm{tr}_s P_t^Z((n,r),(n,r)) = 0, \quad \forall (n,r) \in Z \,.$$

A standard finite propagation speed estimate [6] gives for $r > 0$

$$|P_t^Z((n,r),(n,r)) - P_t((n,r),(n,r))| < C\mathrm{e}^{-cr^2} \,,$$

and this proves the claim.

Let $\tilde{\mathcal{D}}^Z$ be the family of operators on $N \times V$ induced by $\tilde{\partial\!\!\!/}^Z$. Then $\tilde{\mathcal{D}}^Z$ can be identified with the direct sum of two copies of the direct sum of three copies of $\tilde{\mathcal{D}}$ with boundary conditions given by the families of pairs (L_0, L_1), (L_1, L_2), (L_2, L_0). By A_t^Z we denote the superconnection corresponding to $\tilde{\mathcal{D}}^Z$.

Set $\gamma := \sigma_r \in \mathrm{End}(E)$. It follows from Duhamel's formula that the integral kernel of $\mathrm{e}^{-B_t^2}$ depends smoothly on t. The comparison with the cylinder Z shows that $\mathrm{Tr}'_s \mathrm{e}^{-B_t^2}$ can be differentiated with respect to t, and that one can commute Tr'_s and d/dt.

Lemma 3.4.

$$\frac{d}{dt}\mathrm{Tr}'_s \mathrm{e}^{-B_t^2} = -\frac{1}{\sqrt{4\pi t}}\mathrm{Tr}_s \gamma \tilde{\mathcal{D}} \mathrm{e}^{-(A_t^Z)^2} - \frac{1}{2\sqrt{t}} d\mathrm{Tr}'_s \tilde{\partial\!\!\!/} \mathrm{e}^{-B_t^2}$$

Proof. First we claim that

$$\frac{d}{dt}\mathrm{Tr}'_s \mathrm{e}^{-B_t^2} = -\mathrm{Tr}'_s[B_t, \frac{dB_t}{dt}\mathrm{e}^{B_t^2}] .$$

Let ρ_u denote the characteristic function of $M_c \cup N \times [0,u]$. Using Duhamel's formula we get

$$\begin{aligned}
\frac{d}{dt}\mathrm{Tr}'_s \mathrm{e}^{-B_t^2} &= -\mathrm{Tr}'_s \int_0^1 \mathrm{e}^{-sB_t^2}\frac{dB_t^2}{dt}\mathrm{e}^{-(1-s)B_t^2}ds \\
&= -\lim_{u\to\infty}\int_0^1 \mathrm{Tr}_s \rho_u \mathrm{e}^{-sB_t^2}\frac{dB_t^2}{dt}\mathrm{e}^{-(1-s)B_t^2}ds \\
&= -\lim_{u\to\infty}\int_0^1 \mathrm{Tr}_s[\frac{dB_t}{dt}, B_t]\mathrm{e}^{-sB_t^2}\rho_u \mathrm{e}^{-(1-s)B_t^2}ds \\
&= -\lim_{u\to\infty}\lim_{v\to\infty}\int_0^1 \mathrm{Tr}_s\rho_v[\frac{dB_t}{dt}, B_t]\mathrm{e}^{-sB_t^2}\rho_u \mathrm{e}^{-(1-s)B_t^2}ds \\
&= -\lim_{v\to\infty}\lim_{u\to\infty}\int_0^1 \mathrm{Tr}_s\rho_v[\frac{dB_t}{dt}, B_t]\mathrm{e}^{-sB_t^2}\rho_u \mathrm{e}^{-(1-s)B_t^2}ds \\
&= -\lim_{v\to\infty}\int_0^1 \mathrm{Tr}_s\rho_v[\frac{dB_t}{dt}, B_t]\mathrm{e}^{-B_t^2}ds \\
&= -\mathrm{Tr}'_s[B_t, \frac{dB_t}{dt}\mathrm{e}^{-B_t^2}] .
\end{aligned}$$

In order to justify that $\lim_{v\to\infty}$ and $\lim_{u\to\infty}$ can be interchanged one can again use the comparison with the infinite cylinder Z. We use

$$\frac{dB_t}{dt} = \frac{1}{2\sqrt{t}}\tilde{\not\partial}$$

in order to write

$$\begin{aligned}
-\mathrm{Tr}'_s[B_t, \frac{dB_t}{dt}\mathrm{e}^{B_t^2}] &= -\frac{1}{2\sqrt{t}}\mathrm{Tr}'_s[\tilde{\nabla}, \tilde{\not\partial}\mathrm{e}^{-B_t^2}] - \frac{1}{2}\mathrm{Tr}'_s[\tilde{\not\partial}, \tilde{\not\partial}\mathrm{e}^{-B_t^2}] &(4)\\
&= -\frac{1}{2\sqrt{t}}d\mathrm{Tr}'_s\tilde{\not\partial}\mathrm{e}^{-B_t^2} - \frac{1}{2}\mathrm{Tr}'_s[\tilde{\not\partial}, \tilde{\not\partial}\mathrm{e}^{-B_t^2}] . &(5)
\end{aligned}$$

Before explaining the transition from (4) to (5) we consider the second term of (5). Let z denote the $\mathbf{Z}_2$ grading operator. By integration by parts we obtain

$$
\begin{aligned}
&-\frac{1}{2}\mathrm{Tr}_s'[\tilde{\partial}, \tilde{\partial\!\!\!/} \mathrm{e}^{-B_t^2}] \\
&= -\frac{1}{2}\mathrm{Tr}' z\tilde{\partial\!\!\!/}^2 \mathrm{e}^{-B_t^2} - \frac{1}{2}\mathrm{Tr}' z\tilde{\partial\!\!\!/} \mathrm{e}^{-B_t^2}\tilde{\partial\!\!\!/} \\
&= -\frac{1}{2}\lim_{u\to\infty}\int_0^u\int_N \mathrm{tr} z(\tilde{\partial\!\!\!/}^2 \mathrm{e}^{-B_t})((n,u),(n,u))dndu \\
&\quad -\frac{1}{2}\int_{M_c}\mathrm{tr} z(\tilde{\partial\!\!\!/}^2 \mathrm{e}^{-B_t^2})(m,m)dm \\
&\quad -\frac{1}{2}\lim_{u\to\infty}\int_0^u\int_N \mathrm{tr} z(\tilde{\partial\!\!\!/} \mathrm{e}^{-B_t}\tilde{\partial\!\!\!/})((n,u),(n,u))dndu \\
&\quad -\frac{1}{2}\int_{M_c}\mathrm{tr} z(\tilde{\partial\!\!\!/} \mathrm{e}^{-B_t^2}\tilde{\partial\!\!\!/})(m,m)dm \\
&= -\frac{1}{2}\lim_{u\to\infty}\int_N \mathrm{tr}(\gamma z\tilde{\partial\!\!\!/} \mathrm{e}^{-B_t})((n,u),(n,u)) \\
&= \frac{1}{2}\lim_{u\to\infty}\int_N \mathrm{tr}_s(\gamma\tilde{\partial\!\!\!/} \mathrm{e}^{-B_t})((n,u),(n,u)) .
\end{aligned}
$$

In order to evaluate this limit we can replace the kernel P_t by P_t^Z. We use the Volterra series [1], Prop.9.46, in order to compute P_t^Z. Note that on Z

$$
\mathrm{e}^{-t(\tilde{\partial\!\!\!/}^Z)^2}((n,r),(m,s)) = \mathrm{e}^{-t(\tilde{D}^Z)^2}(n,m)\frac{\mathrm{e}^{-(r-s)^2/4t}}{\sqrt{4\pi t}} , \tag{6}
$$

and

$$
P_t^Z = \mathrm{e}^{-t(\tilde{\partial\!\!\!/}^Z)^2} + \sum_{k=1}^{\infty}(-1)^k t^{k/2}\int_{\Delta^k}\mathrm{e}^{-t\sigma_0(\tilde{\partial\!\!\!/}^Z)^2}R^Z\dots R^Z\mathrm{e}^{-t\sigma_k(\tilde{\partial\!\!\!/}^Z)^2}d\sigma ,
$$

where Δ_k denotes the standard k-simplex. Inserting (6) we obtain

$$
\begin{aligned}
&P_t^Z((n,r),(m,s)) \\
&= \frac{\mathrm{e}^{-(r-s)^2/4t}}{\sqrt{4\pi t}}\Bigg(\mathrm{e}^{-t(\tilde{D}^Z)^2} + \sum_{k=1}^{\infty}(-1)^k t^{k/2} \\
&\qquad \int_{\Delta^k}\mathrm{e}^{-t\sigma_0(\tilde{D}^Z)^2}R^Z\dots R^Z\mathrm{e}^{-t\sigma_k(\tilde{D}^Z)^2}d\sigma\Bigg)(n,m) .
\end{aligned}
$$

We now apply

$$
\gamma\tilde{\partial\!\!\!/}^Z = -\frac{\partial}{\partial r} + \gamma\tilde{D}^Z
$$

and evaluate the result at $r = s$. We then obtain

$$\frac{1}{2}\lim_{u\to\infty}\int_N \mathrm{tr}_s\gamma(\tilde{\partial\!\!\!/}\mathrm{e}^{-B_t^2})(n,u)dn = \frac{1}{2}\lim_{u\to\infty}\int_N \mathrm{tr}_s\gamma(\tilde{\partial\!\!\!/}^Z\mathrm{e}^{-(B_t^Z)^2})(n,u)dn$$
$$= \frac{1}{4\sqrt{\pi t}}\mathrm{Tr}_s\gamma\tilde{\mathcal{D}}^Z\mathrm{e}^{-(A_t^Z)^2}.$$

Now we consider the first term of (5). By a similar computation as above one can show that on the cylinder Z

$$\mathrm{tr}_s[\tilde{\nabla}^Z,\tilde{\partial\!\!\!/}^Z\mathrm{e}^{-(B_t^Z)^2}]((n,r),(n,r)) \equiv 0\,.$$

Thus on M this quantity vanishes rapidly as $r\to\infty$. We can take Tr'_s and

$$-\frac{1}{2\sqrt{t}}\mathrm{Tr}'_s[\tilde{\nabla},\tilde{\partial\!\!\!/}\mathrm{e}^{-B_t^2}] = -\frac{1}{2\sqrt{t}}d\mathrm{Tr}'_s\tilde{\partial\!\!\!/}\mathrm{e}^{-B_t^2}$$

is an exact form. This finishes the proof of the lemma. □

Let ∇^0 denote the induced connection on the bundle $\ker(\tilde{\partial\!\!\!/}^+)$, and let $\mathbf{ch}(\nabla^0) := \mathrm{tr}_s\mathrm{e}^{-(\nabla^0)^2}$ be the corresponding Chern form.

Lemma 3.5. *Let $|.|$ be any continuous seminorm on the space of smooth forms on B. For $t\to\infty$ we have*

$$|\mathrm{Tr}'_s\mathrm{e}^{-B_t^2} - \mathbf{ch}(\nabla^0)| = O(t^{-1/2})$$
$$|\frac{1}{2\sqrt{t}}\mathrm{Tr}'_s\tilde{\partial\!\!\!/}\mathrm{e}^{-B_t^2}| = O(t^{-3/2})\,.$$

Proof. The proof is the same as that of [1], and Theorem 9.19, Corollary 9.22, Theorem 9.23. Since M is non-compact we better replace $\mathcal{K}(\mathcal{E})$ defined in [1], p.279, by smooth families of *finite-dimensional* operators. □

We conclude that

$$\int_s^\infty \frac{1}{4\sqrt{\pi t}}\mathrm{Tr}_s\gamma\tilde{\mathcal{D}}^Z\mathrm{e}^{-(A_t^Z)^2}dt =: \hat{\eta}(s)$$

exists and that

$$\mathbf{ch}(\nabla^0) = \mathrm{Tr}'_s\mathrm{e}^{-B_s^2} + \hat{\eta}(s) + d\alpha(s)\,,$$

where

$$\alpha(s) := -\int_s^\infty \frac{1}{2\sqrt{t}}\mathrm{Tr}'_s\tilde{\partial\!\!\!/}\mathrm{e}^{-B_t^2}dt\,. \tag{7}$$

Recall that we have included a parameter ϵ in our definition of B_s, and we will write $B_s(\epsilon)$ for a moment. Since $\frac{d}{d\epsilon}B_s(\epsilon)$ is finite-dimensional operator, [1], Theorem 9.25 applies, and we have

$$\frac{d}{d\epsilon}\mathrm{Tr}_s'\mathrm{e}^{-B_s(\epsilon)^2} = -d\mathrm{Tr}_s\frac{d}{d\epsilon}B_s(\epsilon)\mathrm{e}^{-B_s(\epsilon)^2} .$$

If we put

$$\beta(s) := -\int_0^1 \mathrm{Tr}_s\frac{d}{d\epsilon}B_s(\epsilon)\mathrm{e}^{-B_s(\epsilon)^2}d\epsilon ,$$

then

$$\mathrm{Tr}_s'\mathrm{e}^{-B_s(1)^2} - \mathrm{Tr}_s'\mathrm{e}^{-B_s(0)^2} = d\beta(s) .$$

We thus have

$$\mathbf{ch}(\nabla^0) = \mathrm{Tr}_s'\mathrm{e}^{-B_s(0)^2} + \hat{\eta}(s) + d\alpha(s) + d\beta(s) . \tag{8}$$

We now want to take the limit $s \to 0$. Since $\frac{d}{d\epsilon}B_s(\epsilon)$ is finite-dimensional, the limit $\beta(0) := \lim_{s\to 0}\beta(s)$ exists in $C^\infty(B, \Lambda^*T^*B)$.

Lemma 3.6. *Let $|.|$ be any continuous seminorm on the space of smooth forms on B. For $t \to 0$ we have*

$$|\mathrm{Tr}_s'\mathrm{e}^{-B_t(0)^2} - N| = o(1) \tag{9}$$

$$|\frac{1}{2\sqrt{t}}\mathrm{Tr}_s'\tilde{\partial\!\!\!/}\mathrm{e}^{-B_t(0)^2}| = O(t^{-1/2}) . \tag{10}$$

Proof. We first show (9). We employ finite propagation speed estimates and the comparison with the cylinder in order to show that on the end of M

$$|\mathrm{tr}_s P_t(n,r)| < C\mathrm{e}^{-r^2/t} .$$

The local index theorem [1], Ch.10, gives $|\mathrm{tr}_s P_t(x)| = o(1)$ for $x \in M$ since S is twisted with a bundle of the form $W \oplus W^{op}$. The contribution of the point $\{*\}$ is just $\mathrm{tr}_s P_t(*) = N$. This together yields (9).

Now we consider (10). On the cylinder Z we have $\mathrm{tr}_s\tilde{\partial\!\!\!/}^Z\mathrm{e}^{-(B_t^Z)^2}(n,r) \equiv 0$. We conclude that on $N \times [0,\infty) \subset M$ we have $|\mathrm{tr}_s\tilde{\partial\!\!\!/}\mathrm{e}^{-B_t^2}(n,r)| < C\mathrm{e}^{-r^2/t}$. Moreover, for x in a small neighbourhood of the boundary of M_c we have $|\mathrm{tr}_s\tilde{\partial\!\!\!/}\mathrm{e}^{-B_t^2}(x)| < C\mathrm{e}^{-c/t}$. If x is in the interior of M_c we can employ the method of [1], Ch. 10.5, in order to show that

$$|\frac{1}{2\sqrt{t}}\mathrm{tr}_s\tilde{\partial\!\!\!/}\mathrm{e}^{-B_t^2}(x)| = O(t^{-1/2}) .$$

The estimate (10) follows. □

Now we can take the limit $s \to 0$ in (7). From (8) we obtain

$$\mathbf{ch}(\nabla^0) - N = \hat{\eta}(0) + d\alpha(0) + d\beta(0) .$$

If $\eta(u, L_i, L_j)$ denotes the form (1) which is defined using the boundary condition given by family of pairs (L_i, L_j), then $\hat{\eta}(0) = \lim_{u\to 0}(\eta(u, L_0, L_1) + \eta(u, L_1, L_2) + \eta(u, L_2, L_0))$ (in particular this shows that the combination $\eta(u, L_0, L_1) + \eta(u, L_1, L_2) + \eta(u, L_2, L_0)$ is regular at $u = 0$). We conclude that

$$\tau(L_0, L_1, L_2) = [\mathbf{ch}(\nabla^0) - N] = \mathbf{ch}(\mathrm{index}(\tilde{\partial\!\!\!/}^+)) .$$

This proves Proposition 3.1. □

4. Computation of $\tau(L_0, L_1, L_2)$

Let $L_0, L_1, L_2 \subset V$ be pairwise transverse Lagrangian subspaces. Then $V = L_1 \oplus L_2$ and we can write $x_0 = x_1 + x_2$, $x_i \in L_i$, $i = 0, 1, 2$. We define a hermitean quadratic form Q on L_0 by

$$Q(x_0) := (Ix_1, x_2) ,$$

where $(I.,.)$ is the symplectic form on V associated to I and the Hilbert space structure of V. It is easy to see that Q is nondegenerate. Thus we can split $L_0 = L_0^+ \oplus L_0^-$ into the positive and negative eigenspace of Q. Returning now to the family case we obtain a decomposition $L_0 = L_0^+ \oplus L_0^-$ of the bundle of Lagrangian subspaces $L_0 \subset B \times V$ which is induced by the two other subbundles L_1, L_2.

Proposition 4.1. *We have*

$$\tau(L_0, L_1, L_2) = \mathbf{ch}(L_0^+) - \mathbf{ch}(L_0^-) \in H^*(B, \mathbf{R}) .$$

Proof. The proof of the proposition consists of two steps.

1. Using the K-theoretic relative index theorem [4] we reduce to an index problem for a family of Dirac operators on the disc. The parameter dependence of this family is again built in through the boundary conditions.
2. We then consider the "universal" family of such operators which is parametrized by a space which is homotopy equivalent to the space of all triples of pairwise transverse Lagrangian subspaces of V. It suffices to verify the assertion of the proposition in this special case.

First we want to compactify M by cutting off the cylindrical ends and gluing in half discs. The resulting manifold $\hat{M}$ is then topologically a disc. Let $\hat{\partial\!\!\!/}^+$ be the corresponding Dirac operator. We want to find a family of boundary conditions parametrized by B such that $\mathbf{ch}(\mathrm{index}(\partial\!\!\!/^+)) = \mathbf{ch}(\mathrm{index}(\hat{\partial\!\!\!/}^+))$.

Let $Y \subset \mathbf{R}^2$ denote the subset

$$Y := \{(s,t) \in \mathbf{R}^2 \mid s \geq 0, t \in [-1/2, 1/2] \text{ or } s \leq 0, t^2 + s^2 \leq 1/4\} .$$

Then Y is a Riemannian surface with C^1-boundary and one cylindrical end. Let S_Y be the spinor bundle of Y. Let $E_Y := S_Y \otimes (W \oplus W^{op})$ and $\partial\!\!\!/_Y$ be the Dirac operator on E_Y. We trivialize S_Y and E_Y using the flat Levi Civita connection and denote the typical fibre of E_Y by E. Then

$$\partial\!\!\!/^{\pm}_Y = \sigma_s^{\pm} \frac{\partial}{\partial s} + \sigma_t^{\pm} \frac{\partial}{\partial t} .$$

The space $V := E^+$ is a symplectic vector space with symplectic structure induced by $I = \sigma_s^- \sigma_t^+$.

Let now L_0, L_1 be transversal Lagrangian subspaces of V. We want to construct a family of Lagrangian subspaces $L(s,t) = L(s,t)(L_0, L_1)$ which is parametrized by $(s,t) \in \partial Y$, and which depends smoothly on L_0, L_1, such that $L(s,t) = L_0$ for $s \geq 0$, $t = 1/2$ and $L(s,t) = L_1$ for $s \geq 1$, $t = -1/2$. If $B \ni b \to (L_0(b), L_1(b))$ is a smooth family of pairs of transverse Lagrangian subspaces, then we require that the index of the Dirac operator $\partial\!\!\!/^+_Y$ subject to the family of boundary conditions $L(.,.)(L_0(b), L_1(b))$ is trivial.

We first set

$$\hat{L}(s,t)(L_0) := \begin{cases} L_0 & (s,t) \in \partial Y, t \geq 0 \\ IL_0 & (s,t) \in \partial Y, s \geq 0, t = -1/2 \\ \sigma_s^- n(s,t) L_0 & (s,t) \in \partial Y, s \leq 0, t \leq 0 \end{cases} ,$$

where $n(s,t) := 2(s\sigma_s^+ + t\sigma_t^+)$ is the Clifford multiplication by the normal vector. It is easy to see that $\hat{L}(s,t)(L_0)$ is C^1 with respect to (s,t). If $B \ni b \to L_0(b)$ is a smooth family of Lagrangian subspaces, then we consider the family $\{\hat{\partial\!\!\!/}^+_Y(b)\}_{b \in B}$ given by $\partial\!\!\!/^+_Y$ subject to the boundary conditions given by $\hat{L}(.,.)(L_0(b))$. Since L_0 and IL_0 are transverse, we see as in Section 3 that $\{\hat{\partial\!\!\!/}^+_Y(b)\}_{b \in B}$ gives rise to a family of Fredholm operators.

Lemma 4.2. *In* $K^0(B)$ *we have* $2\,\mathrm{index}(\hat{\partial\!\!\!/}^+_Y) = 0$.

Proof. We claim that $\hat{\partial\!\!\!/}^+_Y$ is equivalent with its adjoint $(\hat{\partial\!\!\!/}^+_Y)^*$. Thus $2\,\mathrm{index}(\hat{\partial\!\!\!/}^+_Y) = \mathrm{index}(\hat{\partial\!\!\!/}^+_Y) - \mathrm{index}(\hat{\partial\!\!\!/}^+_Y)^* = 0$.

We now show the claim. First we describe the adjoint $(\hat{\partial\!\!\!/}^+)^*$. Note that E^- is a hermitean symplectic vector space with symplectic structure induced by $\sigma_s^+ \sigma_t^-$. Then the adjoint of $\hat{\partial\!\!\!/}^+_Y$ is the operator $\hat{\partial\!\!\!/}^-_Y$ subject to the boundary conditions given by the family of Lagrangian subspaces of E^-

$$B \ni b \to \{(s,t) \in \partial Y \mapsto \tau(s,t) \hat{L}(s,t)(L_0(b))\} ,$$

where $\tau(s,t)$ is the Clifford multiplication by the tangent vector at $(s,t) \in \partial Y$. If $\psi^+(s,t)$ is a section of E_Y^+, then we set $(U^+\psi^+)(s,t) := \sigma_t^+ \psi^+(s,-t)$. Then U^+ is unitary, and $U^+\psi^+$ is a section of E_Y^-. It is easy to check that the equality of differential operators

$$U^+\hat{\partial\!\!\!/}_Y^- U^+ = \hat{\partial\!\!\!/}_Y^+$$

is compatible with the boundary conditions. This shows the claim. □

Let $\Omega(x,y) = (Ix, y)$ be the (hermitean) symplectic form on V. Let Λ denote the manifold of all Lagrangian subspaces of V. For $L \in \Lambda$ let $\mathcal{L}_L$ denote the subset of all Lagrangian subspaces $L' \in \Lambda$ which are transverse to L. The following discussion is parallel to that in [8] p. 117/118. Let $P_{L'}$ denote the projection from V to L' with kernel L. It is easy to check that $\Omega(P_{L'}x, y) + \Omega(x, P_{L'}y) = \Omega(x,y)$. We define the hermitean quadratic form

$$Q_{L'}(x,y) := \Omega(P_{L'}x, y) - \frac{1}{2}\Omega(x,y)\,. \tag{11}$$

Indeed

$$\begin{aligned} Q_{L'}(x,y) &= \Omega(P_{L'}x, y) - \frac{1}{2}\Omega(x,y) \\ &= -\Omega(x, P_{L'}y) + \frac{1}{2}\Omega(x,y) \\ &= \overline{\Omega(P_{L'}y, x) - \frac{1}{2}\Omega(y,x)} \\ &= \overline{Q_{L'}(y,x)}\,. \end{aligned}$$

We have

$$Q_{L'}(x,y) = -\frac{1}{2}\Omega(x,y), \quad \forall x \in L, y \in V\,. \tag{12}$$

Any hermitean quadratic form Q satisfying (12) determines a Lagrangian subspace L' such that $Q = Q_{L'}$. In fact let P' be determined by Q and (11), then L' is just the 1-eigenspace of P'. Thus we can identify $\mathcal{L}_L$ with the space of hermitean quadratic forms satisfying (12). In particular, $\mathcal{L}_L$ is an affine space where the affine structure only depends on L.

We now can construct the desired family $L(s,t)(L_0, L_1)$. Note that $IL_0, L_1 \in \mathcal{L}_{L_0}$, and there is a natural affine path $L(r) = L(r)(L_0, L_1)$ with $L(0) = IL_0$, $L(1) = L_1$. We choose a smooth cut-off function $\chi \in C^\infty([0,1])$ with $\chi(t) \in [0,1]$, $\chi(t) = 0$ near $t = 0$ and $\chi(t) = 1$ near $t = 1$. We set $L(s,t) = \hat{L}(s,t)$ for all $(s,t) \in \partial Y$ except for $t = -1/2$, where we set $L(s,t) := L(\chi(s))(L_0, L_1)$ for $s \in [0,1]$ and $L(s,t) := L_1$ for $s \geq 1$. Then $L(s,t)$ is C^1 with respect to (s,t) and depends smoothly on

L_0, L_1. Let $\{\not\partial_Y^+(b)\}_{b\in B}$ denote the family of Dirac operators given by $\not\partial_Y^+$ subject to the boundary conditions defined by $\{L(.,.)(L_0(b), L_1(b))\}_{b\in B}$. Then $\hat{\not\partial}_Y^+$ and $\not\partial_Y^+$ are homotopic families and thus $2\ \mathrm{index}(\not\partial_Y^+) = 0$ in $K^0(B)$. The upshot of the construction above is that we associated to a pair L_0, L_1 of transversal Lagrangian subspaces a canonical path $\gamma(L_0, L_1)$ from L_0 to L_1 which is parametrized by ∂Y. The path $\gamma(L_0, L_1)$ depends smoothly on the pair (L_0, L_1) and has (in a certain sense that will become clear below) the minimal winding number.

Now we can cut-off the six cylindrical ends of M and glue in the pieces $Y_i = \{(s,t) \in Y \mid s \geq 1\}$, $i = 0, \dots, 5$. The resulting manifold $\hat{M}$ is topologically a two-dimensional disc.

To be more precise let $Z_i = [0,\infty) \times [-1/2, 1/2]$, $i = 0, \dots, 5$, denote the cylindrical ends of M. Then we cut at $\{1\} \times [-1/2, 1/2]$. We identify $(s,t) \in Z_i$ with $(1-s, t) \in Y_i$, $s \in (0,1)$. Moreover, we use σ_s in order to glue the bundles. Then $\not\partial^+$ glues with $\not\partial_Y^-$. Assume that on the component $[0,\infty) \times \{-1/2\}$ of ∂Z_i we have the boundary condition given by L_i (resp. L_{i-3}) and on $[0,\infty) \times \{1/2\}$ we have the one given by L_{i+1} (resp. L_{i-2}), where $L_3 = L_0$. Then on the boundary part of $\hat{M}$ which comes from Y_i we choose the path $\sigma_s \tau \gamma(L_i, L_{i+1})$ (resp. $\sigma_s \tau \gamma(L_{i-3}, L_{i-2})$), where τ again denotes the Clifford multiplication with the unit vector tangent to the boundary. This path indeed connects L_i with L_{i+1}.

Thus we have constructed a closed path $\hat{\gamma}(L_0, L_1, L_2)$ of Lagrangian subspaces of V which is parametrized by $\partial\hat{M}$, and which depends smoothly on the triple (L_0, L_1, L_2). We use this path in order to define the boundary condition for the $W \oplus W^{op}$-twisted Dirac operator $\hat{\not\partial}^+$ on $\hat{M}$. Recall that we identify V with the fibres of the bundle $E^+_{\partial\hat{M}}$ using the parallel transport along $\partial\hat{M}$.

It follows from the K-theoretic relative index theorem [4] that $\mathrm{index}(\not\partial^+) = \mathrm{index}(\hat{\not\partial}^+) \in K^0(B)[1/2]$. Indeed, the relative index theorem states that

$$\mathrm{index}(\not\partial^+) + \sum_{i=0}^{5} \mathrm{index}(\not\partial_{Y_i}^+) = \mathrm{index}(\hat{\not\partial}^+) + \sum_{i=0}^{5} \mathrm{index}(\not\partial_{Z_i}^+)\,,$$

where $Z_i = [-1/2, 1/2] \times \mathbf{R}$ and $\not\partial_{Z_i}^+$ is the $W \oplus W^{op}$-twisted Dirac operator subject to the boundary conditions given by L_i at $\{-1/2\} \times \mathbf{R}$, L_{i+1} at $\{-1/2\} \times \mathbf{R}$ (resp. L_{i-3} at $\{-1/2\} \times \mathbf{R}$, L_{i-2} at $\{-1/2\} \times \mathbf{R}$). But $\mathrm{index}(\not\partial_{Z_i}^+) = 0$ in $K^0(B)[1/2]$ for symmetry reasons and $\mathrm{index}(\not\partial_{Y_i}^+) = 0$ by Lemma 4.2.

To be precise, the relative index theorem in [4] is stated for manifolds without boundary. But argument given there carries over to the present case without any essential modification.

Deforming the metric of $\hat{M}$ to the standard metric of the two disc we do not change the index. Below we will assume that $\hat{M}$ is isometric to the two disc. The parallel transport in E^+ along $\partial\hat{M}$ with respect to the globally flat metric gives an identification of V with the fibres of E^+ near $\partial\hat{M}$ which is topologically different from the one used above. This fact has to be taken into account below. We have now finished the first part of the proof of the proposition.

We start with the second part. Let Λ^3 be the space of triples (L_0, L_1, L_2) of pairwise transverse Lagrangian subspaces of V. Let $Sp(V)$ denote the group of symplectic automorphisms of V. Note that $i\Omega$ is a non-degenerate hermitean form of signature (l, l), where $l = \dim_{\mathbb{C}}(V)/2$. Thus $Sp(V) \cong U(l, l)$. The group $Sp(V)$ acts on Λ^3. We claim that Λ^3 is the disjoint union of orbits of $Sp(V)$.

First it is easy to see that $Sp(V)$ acts transitively on the space Λ. Let $L_0 \in \Lambda$. Then any $L_1 \in \mathcal{L}_{L_0}$ can be written as $\{Bx + x \mid x \in IL_0\}$ for some $B \in \mathrm{End}(IL_0, L_0)$. The condition that L_1 is Lagrangian translates to $\Omega(Bx, y) + \Omega(x, By) = 0$ for all $x, y \in IL_0$. This is equivalent to $(BI)^* = BI$, where $*$ is defined with respect to the hermitean metric of V. Thus we can parametrize $\mathcal{L}_{L_0}$ by the symmetric endomorphisms of L_0. Writing $V = L_0 \oplus IL_0$ it is easy to check that

$$A := \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \in Sp(V) ,$$

$AL_0 = L_0$ and $AIL_0 = L_1$. Thus $Sp(V)$ acts transitively on the set Λ^2 of pairs (L_0, L_1) of transverse Lagrangian subspaces.

Let G denote the stabilizer of the pair (L_0, IL_0). Let $j : Gl(L_0) \to Gl(IL_0)$ denote the unique isomorphism such that

$$Gl(L_0) \ni A \mapsto \begin{pmatrix} A & 0 \\ 0 & j(A) \end{pmatrix} \in G . \tag{13}$$

Then $j(A) = -I(A^{-1})^* I$. If $L_2 \in \Lambda$ is transverse to L_0 and IL_0, then we write

$$L_2 = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} IL_0$$

for some invertible B as above. If $g \in G$ is represented by $A \in Gl(L_0)$ according to (13), then

$$gL_2 = \begin{pmatrix} 1 & -ABIA^*I \\ 0 & 1 \end{pmatrix} IL_0 .$$

The action of G on $\mathcal{L}_{L_0}$ is hence given by action of $Gl(L_0)$ on the symmetric endomorphisms of L_0 by conjugation. Thus the signature of the symmetric

BI is the only invariant of the orbit of G generated by L_2 inside the space of Lagrangian subspaces which are transverse to L_0 and IL_0. We conclude that Λ^3 is the disjoint union of orbits of $Sp(V)$ of points (L_0, IL_0, L_2), which are distinguished by the signature of a matrix BI defined by L_2.

We now consider the orbit generated by a triple $(L_0, IL_0, L_2) \in \Lambda^3$. The stabilizer U of (L_0, IL_0, L_2) can be identified with the subgroup of $Gl(L_0)$ fixing the hermitean form on L_0 defined by BI. Let $K \subset U$ denote a maximal compact subgroup. We can choose K such that it fixes the metric $(.,.)$, hence K is a subgroup of the unitary group of V. But then it fixes I, too. Using the explicit formulas given above one checks that $\hat{\gamma}(L_0, IL_0, L_2) = k\hat{\gamma}(L_0, IL_0, L_2)$ for all $k \in K$.

We now globally trivialize S, E using the parallel transport given by the globally flat metric of $\hat{M}$. Along the boundary $\partial\hat{M}$ the old and the new trivialization of S are related by a twist of -2π in the structure group of S. Note that $S^{\pm}$ are the $\pm\imath$ eigenspaces of the Clifford multiplication by the volume form of $\hat{M}$. The image of the path $\hat{\gamma}(L_0, L_1, L_2)$ in the new trivialization can be obtained (up to homotopy) by $\gamma(L_0, L_1, L_2)(z) := z^{-\imath I}\hat{\gamma}(L_0, L_1, L_2)(z)$, $z \in S^1 = \partial\hat{M}$. We see that $\gamma(L_0, L_1, L_2)$ is K-invariant, too.

Let (x, y) be oriented, flat orthonormal coordinates on $\hat{M}$ and write $\hat{\not\partial} = \sigma_x\partial_x + \sigma_y\partial_y$. If we let K act on E^- by, say, $K \ni k \mapsto -\sigma_x^+ k \sigma_x^- \in \mathrm{End}(E^-)$, then $\hat{\not\partial}^+$ is K-equivariant. We now consider the family of Dirac operators parametrized by $Sp(V)$, given by $\hat{\not\partial}^+$ subject to the boundary conditions $Sp(V) \ni g \mapsto \gamma(gL_0, gIL_0, gL_2)$. This family is K-equivariant and we go over to the quotient family parametrized by $Sp(V)/K$ which we denote by $\check{\not\partial}^+$.

Let $X := \mathrm{index}(\hat{\not\partial}^+) \in R(K)$ be the K-equivariant index of $\hat{\not\partial}^+$ subject to the boundary conditions given by $\gamma(L_0, IL_0, L_2)$, where $R(K)$ denotes the representation ring of K. Then $\mathrm{index}(\check{\not\partial}^+) = [Sp(V) \times_K X] \in K^0(Sp(V)/K)$. The following Lemma implies the proposition for the family $\check{\not\partial}^+$.

Lemma 4.3. *Let (n, m) be the signature of the quadratic form defined by BI on L_0. Then $X = \mathbf{C}^n - \mathbf{C}^m$, where $\mathbf{C}^n, \mathbf{C}^m$ are the m- and n-dimensional standard representations of the corresponding factors of $K \cong U(n) \times U(m)$.*

Proof. Let $T \subset K$ be a maximal torus. If $Y \in R(K)$, then Y_T denotes the restriction to T. It is sufficient to show that $X_T = \mathbf{C}^n_T - \mathbf{C}^m_T$.

We first consider the case that $W \cong \mathbf{C}$. Then $\hat{\not\partial}^+$ can be expressed in terms of complex geometry. Indeed we have

$$\hat{\not\partial}^+ = \begin{pmatrix} -\partial & 0 \\ 0 & \bar{\partial} \end{pmatrix} .$$

Writing $\bar{\partial} = \partial_x + \imath\partial_y$, $\partial = \partial_x - \imath\partial_y$ we obtain

$$\sigma_x^+ = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_y^+ = \begin{pmatrix} \imath & 0 \\ 0 & \imath \end{pmatrix}$$

and hence

$$I = \begin{pmatrix} \imath & 0 \\ 0 & -\imath \end{pmatrix} .$$

A one-dimensional subspace $\mathbf{C}(a,b) \subset V$, $a,b \in \mathbf{C}$, is Lagrangian iff $|a| = |b|$. We parametrize the Lagrangian subspaces of V by S^1 associating to $\phi \in S^1$ the space $\mathbf{C}(\phi, 1)$.

The space of pairwise transverse triples Λ^3 consists of two components which are distinguished by the cyclic order of the parameters $\phi_i \in S^1$ of L_i. Let Λ^3_1 be the component with order $\phi_0 < \phi_1 < \phi_2$ and Λ^3_{-1} be the component with order $\phi_0 < \phi_2 < \phi_1$. The T-equivariant index of $\hat{\not\partial}^+$ only depends on the homotopy class of the path $\gamma(L_0, L_1, L_2)$.

We fix an identification $\pi_1(S^1) = \mathbf{Z}$, $\gamma \mapsto [\gamma]$, such that the path mapped to 1 has positive orientation. We leave to the reader to compute $[\gamma(L_0, L_1, L_2)] \in \mathbf{Z}$ for $(L_0, L_1, L_2) \in \Lambda^3_{\pm 1}$. The result is $[\gamma(L_0, L_1, L_2)] = 0$ on Λ^3_1 and $[\gamma(L_0, L_1, L_2)] = 2$ on Λ^3_{-1}.

Lemma 4.4. *Let $W \cong \mathbf{C}$, and let the boundary condition of $\hat{\not\partial}^+$ be given by a closed path γ of Lagrangian subspaces. Then* $\mathrm{index}(\hat{\not\partial}^+) = -[\gamma] + 1$.

Proof. Let $n \in \mathbf{Z}$ be represented by the path $\gamma_n(\phi) = \mathbf{C}(\phi^n, 1)$, $\phi \in S^1$. The kernel of $\hat{\not\partial}^+$ with boundary condition given by γ_n can be identified with the space of pairs (f,g) of functions on $\hat{M}$, where $\bar{\partial}g = 0$, $\partial f = 0$ and $z^n g(z) = f(z)$ at S^1. This space is non-trivial for $n \le 0$, and it is spanned by $(\bar{z}^{-n}, 1), (\bar{z}^{-n-1}, z), \dots, (1, z^{-n})$. One can check that $(\hat{\not\partial}^+)^*$ is given by

$$(\hat{\not\partial}^+)^* = \begin{pmatrix} \bar{\partial} & 0 \\ 0 & -\partial \end{pmatrix} ,$$

and that the boundary condition is given by the path γ_{n+2}. The kernel of $(\hat{\not\partial}^+)^*$ can be identified with the space of pairs (f,g) of functions on $\hat{M}$ with $\bar{\partial}f = 0$, $\partial g = 0$, and $z^{n-2}g(z) = f(z)$. The kernel of $(\hat{\not\partial}^+)^*$ is non-trivial for $n \ge 2$, and it is spanned by by $(z^{n-2}, 1), (z^{n-3}, \bar{z}), \dots, (1, \bar{z}^{n-2})$. It follows that $\mathrm{index}(\hat{\not\partial}^+) = -n + 1$. $\square$

We now finish the proof of Lemma 4.3 in the case $W = \mathbf{C}$. We must show that the signature of the quadratic form given by BI is $(1,0)$ on Λ^3_1 and $(0,1)$ on Λ^3_{-1}. Let $l_0 = (1,1)$, $l_1 = (-1,1)$, and $l_2 = (-i,1)$ generate the Lagrangian subspaces L_0, IL_0, L_2. Then $(L_0, IL_0, L_2) \in \Lambda^3_1$. We have $l_2 \sim l_0 + \frac{1+\imath}{1-\imath} l_1$ and $Il_0 = -il_1$. It follows that $BI = 1$. The other case is similar.

In order to complete the proof of Lemma 4.3 in the general case one reduces to the special case $W = \mathbf{C}$ by considering the direct sums. □

We now finish the proof of the proposition. Let $\{\Lambda^3_i\}$ denote the set of components of Λ^3 and choose $x_i \in \Lambda^3_i$ for all i. Let U_i be the stabilizer of x_i in $Sp(V)$. Then $\bigcup_i Sp(V)/U_i \cong \Lambda^3$ parametrizes the universal family of boundary conditions for $\hat{\not{\partial}}^+$ given by a family of path' $B \ni b \mapsto \gamma(L_0(b), L_1(b), L_2(b))$. Indeed, any such family can be pulled back from the universal one using the canonical map $B \to \bigcup_i Sp(V)/U_i$. Since the fibres of $\pi_i : Sp(V)/K \to Sp(V)/U_i$ are symmetric spaces of non-compact type, π_i is a homotopy equivalence. Since the proposition is proved for $\check{\not{\partial}}^+$ (for each component separately) it is also true for the universal family, and hence in general. □

Remark. We sketch another proof of Proposition 4.1 which avoids the use of the universal family and the equivariant index.

We first consider a model case where $V_{model} := \mathbf{C}^2$ and $I := \mathrm{diag}(\imath, -\imath)$. Let l_i, $i = 0,1,2$, be the Lagrangian subspaces of V_{model} parametrized by $\phi_0 := 0$, $\phi_1 := \pm\arg(\frac{1-i}{1+i})$, $\phi_2 := -\phi_1$, i.e., $l_i = \{(e^{\imath\phi_i}x, x) \in \mathbf{C}^2 \mid x \in \mathbf{C}\}$. Let $\not{\partial}^+_\pm$ be the Dirac operator on M with boundary conditions given by the triple l_0, l_1, l_2. We have $\mathrm{index}(\not{\partial}^+_\pm) = \pm 1$. This can be proved in the same way as Proposition 4.1, but the proof simplifies due to the facts that $\dim(V_{model}) = 2$, and that B is a point.

We now turn to the general case. Let V be any finite-dimensional Hilbert space with hermitean symplectic structure, and let $\{L_0, L_1, L_2\}_{b\in B}$ be a family of pairwise transverse Lagrangian subspaces of V. Then we can write $L_i = \{x + A_i Ix \mid x \in IL_0\}$, $i = 1,2$, where A_i are smooth symmetric bundle endomorphisms of the subbundle $L_0 \subset B \times V$ such that $A_1 - A_2$ is invertible.

We show that this family of triples is homotopic to a family in some standard form. Consider the family $A(t)_i := \frac{1}{2}((1-t)(A_i - A_{3-i}) + tA_i)$, $t \in [0,1]$. Then $A(t)_1 - A(t)_2 = A_1 - A_2$ is invertible for all t. Moreover, $A(1)_i = A_i$ and $A(0)_1 = A_1 - A_2 = -A(0)_2$. Thus up to homotopy we can assume that $A_1 = -A_2 = A$, where A is invertible. There is a further index bundle preserving homotopy of A to $A/|A|$. Thus we can assume that $A^2 = 1$.

Let $\partial\!\!\!/^+$ be the family of operators on M defined by the family $\{L_0, L_1, L_2\}_{b\in B}$ associated with A. Let $L_0^\pm$ be the ± 1-eigenspaces of A. We will define isomorphisms $\Phi : \ker(\partial\!\!\!/^+_+) \otimes L_0^+ \oplus \ker(\partial\!\!\!/^+_-) \otimes L_0^- \cong \ker(\partial\!\!\!/^+)$, $\Psi : \operatorname{coker}(\partial\!\!\!/^+_+) \otimes L_0^+ \oplus \operatorname{coker}(\partial\!\!\!/^+_-) \otimes L_0^- \cong \operatorname{coker}(\partial\!\!\!/^+)$.

First we fix a basis vector $v_{model} \in l_0$. Then any $v \in L_0^\pm$ defines an unique symplectic embedding $v_* : V_{model} \hookrightarrow V$ such that $v_*(v_{model}) = v$. If $v, w \in L_0^\pm$ and $\mu \in \mathbf{C}$, then we have $(\mu v + w)_* = \mu v_* + w_*$. One can check that $v_*(\ker(\partial\!\!\!/^+_\pm)) \subset \ker(\partial\!\!\!/^+)$, $v_*(\operatorname{coker}(\partial\!\!\!/^+_\pm)) \subset \operatorname{coker}(\partial\!\!\!/^+)$. We define $\Phi(f \otimes v \oplus f' \otimes v') := v_*(f) + v'_*(f')$, $\Psi(g \otimes w \oplus g' \otimes w') := w_*(g) + w'_*(g')$. In follows that

$$\operatorname{index}(\partial\!\!\!/^+) = \operatorname{index}(\partial\!\!\!/^+_+)[L_0^+] + \operatorname{index}(\partial\!\!\!/^+_-)[L_0^-] = [L_0^+] - [L_0^-] \in K^0(B)\,.$$

This finishes our sketch of an alternative proof of Proposition 4.1.

Example. For the purpose of illustration let us consider an example. Let $V := \mathbf{C}^4$ equipped with some complex structure I such that $L_0 = \mathbf{C}^2 \subset \mathbf{C}^4$ is Lagrangian. Let $B := P^2\mathbf{C}$. If $T \to B$ denotes the tautological bundle of B, then we have an orthogonal splitting of the trivial bundle $B \times L_0$ as $T \oplus T^\perp$. For $b \in B$ let Q_b be the quadratic form on L_0 given by the matrix $\operatorname{diag}(1, -1)$ with respect to the splitting $L_0 = T_b \oplus T_b^\perp$. The family of quadratic forms $\{Q_b\}_{b\in B}$ induces a family of Lagrangian subspaces $\{L_2(b)\}_{b\in B}$ such that $L_2(b)$ is transverse to L_0, IL_0 for all $b \in B$. Thus $\tau(L_0, IL_0, L_2)$ is defined, and we have

$$\tau(L_0, IL_0, L_2) = \mathbf{ch}(T) - \mathbf{ch}(T^*) = 2c_1(T)\,.$$

This class is non-trivial.

Remark. It would be desirable to have an explicit formula for the η-form generalizing the result of [9].

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