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Gerhard Burde · Heiner Zieschang

Knots

Second Revised and Extended Edition

Walter de Gruyter
Berlin · New York 2003

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Mathematics Subject Classification 2000: 57-02; 57M25, 20F34, 20F36

Keywords: knots; links; fibred knots; torus knots; factorization; braids; branched coverings; Montesinos links; knot groups; Seifert surfaces; Alexander polynomials; Seifert matrices; cyclic periods of knots; homfly polynomial

With 184 figures

♾ Printed on acid-free paper which falls within the guidelines of the ANSI to ensure permanence and durability.

Library of Congress Cataloging-in-Publication Data

Burde, Gerhard, 1931–
Knots / Gerhard Burde, Heiner Zieschang. – 2nd rev. and extended ed.
p. cm. – (De Gruyter studies in mathematics ; 5)
Includes bibliographical references and index.
ISBN 3 11 017005 1 (cloth : alk. paper)
I. Zieschang, Heiner. II. Title. III. Series.
QA612.2 .B87 2002
514′.224–dc21 2002034764

ISBN 3 11 017005 1

Bibliographic information published by Die Deutsche Bibliothek

Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie;
detailed bibliographic data is available in the Internet at <http://dnb.ddb.de>.

Cover design: Rudolf Hübler, Berlin.
Typeset using the authors' TEX files: I. Zimmermann, Freiburg.
Printing and binding: Hubert & Co. GmbH & Co. KG, Göttingen.

Preface to the First Edition

The phenomenon of a knot is a fundamental experience in our perception of three dimensional space. What is special about knots is that they represent a truly intrinsic and essential quality of 3-space accessible to intuitive understanding. No arbitrariness like the choice of a metric mars the nature of a knot – a trefoil knot will be universally recognizable wherever the basic geometric conditions of our world exist. (One is tempted to propose it as an emblem of our universe.)

There is no doubt that knots hold an important – if not crucial – position in the theory of 3-dimensional manifolds. As a subject for a mathematical textbook they serve a double purpose. They are excellent introductory material to geometric and algebraic topology, helping to understand problems and to recognize obstructions in this field. On the other hand they present themselves as ready and copious test material for the application of various concepts and theorems in topology.

The first nine chapters (excepting the sixth) treat standard material of classical knot theory. The remaining chapters are devoted to more or less special topics depending on the interest and taste of the authors and what they believed to be essential and alive. The subjects might, of course, have been selected quite differently from the abundant wealth of publications in knot theory during the last decades.

We have stuck throughout this book mainly to traditional topics of classical knot theory. Links have been included where they come in naturally. Higher-dimensional knot theory has been completely left out – even where it has a bearing on 3-dimensional knots such as slice knots. The theme of surgery has been rather neglected – excepting Chapter 15. Wild knots and Algebraic knots are merely mentioned.

This book may be read by students with a basic knowledge in algebraic topology – at least the first four chapters will present no serious difficulties to them. As the book proceeds certain fundamental results on 3-manifolds are used – such as the Papakyriakopoulos theorems. The theorems are stated in Appendix B and references are given where proofs may be found. There seemed to be no point in adding another presentation of these things. The reader who is not familiar with these theorems is, however, well advised to interrupt the reading to study them. At some places the theory of surfaces is needed – several results of Nielsen are applied. Proofs of these may be read in [ZVC 1980], but taking them for granted will not seriously impair the understanding of this book. Whenever possible we have given complete and self-contained proofs at the most elementary level possible. To do this we occasionally refrained from applying a general theorem but gave a simpler proof for the special case in hand.

There are, of course, many pertinent and interesting facts in knot theory – especially in its recent development – that were definitely beyond the scope of such a textbook. To be complete – even in a special field such as knots – is impossible today and was not aimed at. We tried to keep up with important contributions in our bibliography.

There are not many textbooks on knots. Reidemeister's "Knotentheorie" was conceived for a different purpose and level; Neuwirth's book "Knot Groups" and Hillman's monograph "Alexander Ideals of Links" have a more specialised and algebraic interest in mind. In writing this book we had, however, to take into consideration Rolfsen's remarkable book "Knots and Links". We tried to avoid overlappings in the contents and the manner of presentation. In particular, we thought it futile to produce another set of drawings of knots and links up to ten crossings – or even more. They can – in perfect beauty – be viewed in Rolfsen's book. Knots with less than ten crossings have been added in Appendix D as a minimum of ready illustrative material. The tables of knot invariants have also been devised in a way which offers at least something new. Figures are plentiful because we think them necessary and hope them to be helpful.

Finally we wish to express our gratitude to Colin Maclachlan who read the manuscript and expurgated it from the grosser lapsus linguae (this sentence was composed without his supervision). We are indebted to U. Lüdicke and G. Wenzel who wrote the computer programs and carried out the computations of a major part of the knot invariants listed in the tables. We are grateful to U. Dederek-Breuer who wrote the program for filing and sorting the bibliography. We also want to thank Mrs. A. Huck and Mrs. M. Schwarz for patiently typing, re-typing, correcting and re-correcting abominable manuscripts.

Frankfurt (Main)/Bochum, Summer 1985

Gerhard Burde
Heiner Zieschang

Preface to the Second Edition

The text has been revised, some mistakes have been eliminated and Chapter 15 has been brought up to date, especially taking into account the Gordon–Luecke Theorem on knot complements, although we have not included a proof. Chapter 16 was added, presenting an introduction to the HOMFLY polynomial, and including a self-contained account of the fundamental facts about Hecke algebras. A proof of Markov's theorem was added in Chapter 10 on braids. We also tried to bring the bibliography up to date. In view of the vast amount of recent and pertinent contributions even approximate completeness was out of the question.

We have decided not to deal with Vassiliev invariants, quantum group invariants and hyperbolic structures on knot complements, since a thorough treatment of these topics would go far beyond the space at our disposal. Adequate introductory surveys on these topics are available elsewhere.

Since the first edition of this book in 1985, a series of books on knots and related topics have appeared. We mention especially: [Kauffman 1987, 1991], [Kawauchi 1996], [Murasugi 1996], [Turaev 1994], [Vassiliev 1999].

Our heartfelt thanks go to Marlene Schwarz and Jörg Stümke for producing the LaTeX-file and to Richard Weidmann for proof-reading. We also thank the editors for their patience and pleasant cooperation, and Irene Zimmermann for her careful work on the final layout.

Frankfurt (Main)/Bochum, 2002

Gerhard Burde
Heiner Zieschang

Contents

Chapter 1

Knots and Isotopies

The chapter contains an elementary foundation of knot theory. Sections A and B define and discuss knots and their equivalence classes, and Section C deals with the regular projections of knots. Section D contains a short review of [Pannwitz 1933] and [Milnor 1950] intended to further an intuitive geometric understanding for the global quality of knotting in a simple closed curve in 3-space.

A Knots

A knot, in the language of mathematics, is an embedding of a circle S^1 into Euclidean 3-space, $\mathbb{R}^3$, or the 3-sphere, S^3. More generally embeddings of S^k into S^{n+k} have been studied in "higher dimensional knot theory", but this book will be strictly concerned with "classical" knots $S^1 \subset S^3$. (On occasion we digress to consider "links" or "knots of multiplicity $\mu > 1$" which are embeddings of a disjoint union of 1-spheres S_i^1, $1 \leq i \leq \mu$, into S^3 or $\mathbb{R}^3$.)

A single embedding $i: S^1 \to S^3$, is, of course, of little interest, and does not give rise to fruitful questions. The essential problem with a knot is whether it can be disentangled by certain moves that can be carried out in 3-space without damaging the knot. The topological object will therefore rather be a class of embeddings which are related by these moves (isotopic embeddings).

There will be a certain abuse of language in this book to avoid complicating the notation. A knot $\mathfrak{k}$ will be an embedding, a class of embeddings, the image $i(S^1) = \mathfrak{k}$ (a simple closed curve), or a class of such curves. There are different notions of isotopy, and we start by investigating which one of them is best suited to our purposes.

Let X and Y be Hausdorff spaces. A mapping $f: X \to Y$ is called an *embedding* if $f: X \to f(X)$ is a homeomorphism.

1.1 Definition (Isotopy). Two embeddings, $f_0, f_1 : X \to Y$ are *isotopic* if there is an embedding

$$F: X \times I \to Y \times I$$

such that $F(x,t) = (f(x,t),t)$, $x \in X$, $t \in I = [0,1]$, with $f(x,0) = f_0(x)$, $f(x,1) = f_1(x)$.

F is called a *level-preserving isotopy* connecting f_0 and f_1.

We frequently use the notation $f_t(x) = f(x,t)$ which automatically takes care of the boundary conditions. The general notion of isotopy as defined above is not good as far as knots are concerned. Any two embeddings $S^1 \to S^3$ can be shown to be isotopic although they evidently are different with regard to their knottedness. The idea of the proof is sufficiently illustrated by the sequence of pictures of Figure 1.1. Any area where knotting occurs can be contracted continuously to a point.

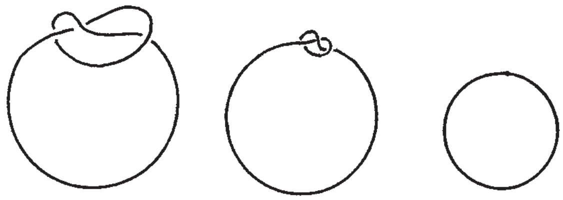


Figure 1.1

1.2 Definition (Ambient isotopy). Two embeddings $f_0, f_1 : X \to Y$ are *ambient isotopic* if there is a level preserving isotopy

$$H : Y \times I \to Y \times I, \quad H(y, t) = (h_t(y), t),$$

with $f_1 = h_1 f_0$ and $h_0 = \mathrm{id}_Y$. The mapping H is called an *ambient isotopy*.

An ambient isotopy defines an isotopy F connecting f_0 and f_1 by $F(x, t) = (h_t f_0(x), t)$. The difference between the two definitions is the following: An isotopy moves the set $f_0(X)$ continuously over to $f_1(X)$ in Y, but takes no heed of the neighbouring points of Y outside $f_1(X)$. An ambient isotopy requires Y to move continuously along with $f_t(X)$ such as a liquid filling Y will do if an object $(f_t(X))$ is transported through it.

The restriction

$$h_1| : (Y - f_0(X)) \to (Y - f_1(X))$$

of the homeomorphism $h_1 : Y \to Y$ is itself a homeomorphism of the complements of $f_0(X)$ resp. $f_1(X)$ in Y, if f_0 and f_1 are ambient isotopic. This is not necessarily true in the case of mere isotopy and marks the crucial difference. We shall see in Chapter 3 that the complement of the *trefoil* knot – see the first picture of Figure 1.1 – and the complement of the unknotted circle, the *trivial knot* or *unknot*, are not homeomorphic.

We are going to narrow further the scope of our interest. Topological embeddings $S^1 \to S^3$ may have a bizarre appearance as Figure 1.2 shows. There is an infinite sequence of similar meshes converging to a limit point L at which this knot is called *wild*. This example of a wild knot, invented by R.H. Fox, has indeed remarkable properties which show that at such a point of wildness something extraordinary may happen. In [Fox-Artin 1948] it is proved that the complement of the curve depicted in Figure 1.2 is different from that of a trivial knot. Nevertheless the knot can obviously be unravelled from the right – at least finitely many stitches can.

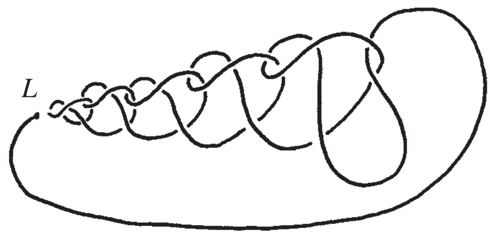


Figure 1.2

1.3 Definition (Tame knots). A knot $\mathfrak{k}$ is called *tame* if it is ambient isotopic to a simple closed polygon in $\mathbb{R}^3$ resp. S^3. A knot is *wild* if it is not tame.

If a knot is tame, any connected proper part α of it is ambient isotopic to a straight segment and therefore the complement $S^3 - \alpha$ is simply connected. Any proper subarc of the knot of Figure 1.2 which contains the limit point L can be shown [Fox-Artin 1948] to have a non-simply connected complement. From this it appears reasonable to call L a point at which the knot is wild. Wild knots are no exceptions – quite the contrary. Milnor proved: *"Most" knots are wild* [Milnor 1964]. One can even show that almost all knots are wild at every point [Brode 1981]. Henceforth we shall be concerned only with tame knots. *Consequently we shall work always in the p.l.-category* (*p.l. = piecewise linear*). All spaces will be compact polyhedra with a finite simplicial structure, unless otherwise stated. Maps will be piecewise linear. We repeat Definitions 1.1 and 1.2 in an adjusted version:

1.4 Definition (p.l. isotopy and p.l.-ambient isotopy). Let X, Y be polyhedra and $f_0, f_1\colon X \to Y$ p.l.-embeddings, f_0 and f_1 are p.l. *isotopic* if there is a level-preserving p.l.-embedding

$$F\colon X \times I \to Y \times I, \quad F(x,t) = (f_t(x), t), \qquad 0 \le t \le 1.$$

f_0 and f_1 are p.l.-*ambient isotopic* if there is a level-preserving p.l.-isotopy

$$H\colon Y \times I \to Y \times I, \quad H(y,t) = (h_t(y), t),$$

with $f_1 = h_1 f_0$ and $h_0 = \mathrm{id}_Y$.

In future we shall usually omit the prefix "p.l.".

We are now in a position to give the fundamental definition of a knot as a class of embeddings $S^1 \to S^3$ resp. $S^1 \to \mathbb{R}^3$:

1.5 Definition (Equivalence). Two (p.l.)-knots are *equivalent* if they are (p.l.)-ambient isotopic.

As mentioned before we use our terminology loosely in connection with this definition. A knot $\mathfrak{k}$ may be a representative of a class of equivalent knots or the class itself. If the knots $\mathfrak{k}$ and $\mathfrak{k}'$ are equivalent, we shall say they are the same, $\mathfrak{k} = \mathfrak{k}'$ and use the sign of equality. $\mathfrak{k}$ may mean a simple closed finite polygonal curve or a class of such curves.

The main topic of classical knot theory is the classification of knots with regard to equivalence.

Dropping "p.l." defines, of course, a broader field and a more general classification problem. The definition of tame knots (Definition 1.3) suggests applying the Definition 1.2 of "topological" ambient isotopy to define a topological equivalence for this class of knots. At first view one might think that the restriction to the p.l.-category will introduce equivalence classes of a different kind. We shall take up the subject in Chapter 3 to show that this is not true. In fact two tame knots are topologically equivalent if and only if the p.l.-representatives of their topological classes are p.l.-equivalent.

We have defined knots up to now without bestowing orientations either on S^1 or S^3. If S^1 is oriented (*oriented knot*) the notion of equivalence has to be adjusted: *Two oriented knots are equivalent,* if there is an ambient isotopy connecting them which respects the orientation of the knots. Occasionally we shall choose a fixed orientation in S^3 (for instance in order to define linking numbers). Ambient isotopies obviously respect the orientation of S^3.

B Equivalence of Knots

We defined equivalence of knots by ambient isotopy in the last paragraph. There are different notions of equivalence to be found in the literature which we propose to investigate and compare in this paragraph.

Reidemeister [Reidemeister 1926′] gave an elementary introduction into knot theory stressing the combinatorial aspect, which is also the underlying concept of his book "Knotentheorie" [Reidemeister 1932], the first monograph written on the subject. He introduced an isotopy by moves.

1.6 Definition (Δ-move). Let u be a straight segment of a polygonal knot $\mathfrak{k}$ in $\mathbb{R}^3$ (or S^3), and D a triangle in $\mathbb{R}^3$, $\partial D = u \cup v \cup w$; u, v, w 1-faces of D. If $D \cap \mathfrak{k} = u$, then $\mathfrak{k}' = (\mathfrak{k} - u) \cup v \cup w$ defines another polygonal knot. We say $\mathfrak{k}'$ *results from* $\mathfrak{k}$ *by a* Δ*-process or* Δ*-move*. If $\mathfrak{k}$ is oriented, $\mathfrak{k}'$ has to carry the orientation induced by that of $\mathfrak{k} - u$. The inverse process is denoted by Δ^{-1}. (See Figure 1.3.)

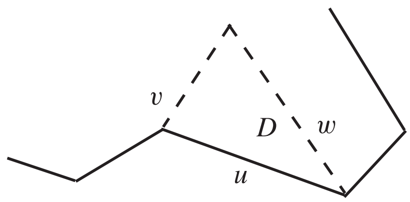


Figure 1.3

Remark. We allow Δ to degenerate as long as $\mathfrak{k}'$ remains simple. This means that Δ resp. Δ^{-1} may be a bisection resp. a reduction in dimension one.

1.7 Definition (Combinatorial equivalence). Two knots are *combinatorially equivalent* or *isotopic by moves*, if there is a finite sequence of Δ- and Δ^{-1}-moves which transforms one knot to the other.

There is a third way of defining equivalence of knots which takes advantage of special properties of the embedding space, $\mathbb{R}^3$ or S^3. Fisher [Fisher 1960] proved that an orientation preserving homeomorphism $h\colon S^3 \to S^3$ is isotopic to the identity. (A homeomorphism with this property is called a *deformation*.) We shall prove the special case of Fisher's theorem that comes into our province with the help of the following theorem which is well known, and will not be proved here.

1.8 Theorem of Alexander–Schoenflies. *Let $i\colon S^2 \to S^3$ be a (p.l.) embedding. Then*

$$S^3 = B_1 \cup B_2, \quad i(S^2) = B_1 \cap B_2 = \partial B_1 = \partial B_2,$$

where B_i, $i = 1, 2$, is a combinatorial 3-ball (B_i is p.l.-homeomorphic to a 3-simplex).

The theorem corresponds to the Jordan curve theorem in dimension two where it holds for topological embeddings. In dimension three it is not true in this generality [Alexander 1924], [Brown 1962].

We start by proving

1.9 Proposition (Alexander–Tietze). *Any (p.l.) homeomorphism f of a (combinatorial) n-ball B keeping the boundary fixed is isotopic to the identity by a (p.l.)-ambient isotopy keeping the boundary fixed.*

Proof. Define for $(x, t) \in \partial(B \times I)$

$$H(x,t) = \begin{cases} x & \text{for } t = 0 \\ x & \text{for } x \in \partial B \\ f(x) & \text{for } t = 1. \end{cases}$$

Every point $(x, t) \in B \times I$, $t > 0$, lies on a straight segment in $B \times I$ joining a fixed interior point P of $B \times 0$ and a variable point X on $\partial(B \times I)$. Extend $H|\partial(B \times I)$ linearly on these segments to obtain a p.l. level-preserving mapping $H\colon B \times I \to B \times I$, in fact, the desired ambient isotopy (*Alexander trick*, [Alexander 1923], Figure 1.4.) □

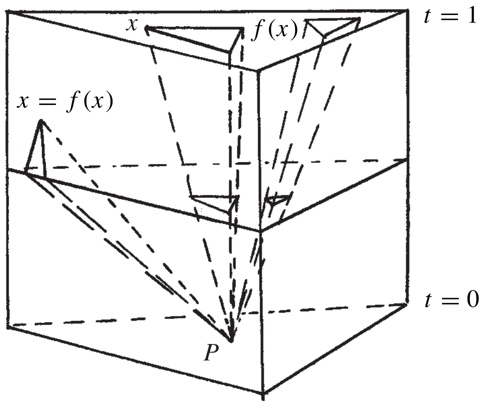


Figure 1.4

We are now ready to prove the main theorem of the paragraph:

1.10 Proposition (Equivalence of equivalences). *Let $\mathfrak{k}_0$ and $\mathfrak{k}_1$ be p.l.-knots in S^3. The following assertions are equivalent.*

(1) *There is an orientation preserving homeomorphism $f\colon S^3 \to S^3$ which carries $\mathfrak{k}_0$ onto $\mathfrak{k}_1$, $f(\mathfrak{k}_0) = \mathfrak{k}_1$.*

(2) *$\mathfrak{k}_0$ and $\mathfrak{k}_1$ are equivalent (ambient isotopic).*

(3) *$\mathfrak{k}_0$ and $\mathfrak{k}_1$ are combinatorially equivalent (isotopic by moves).*

Proof. (1) $\Longrightarrow$ (2): We begin by showing that there is an ambient isotopy $H(x, t) = (h_t(x), t)$ of S^3 such that $h_1 f$ leaves fixed a 3-simplex $[P_0, P_1, P_2, P_3]$. If $f\colon S^3 \to S^3$ has a fixed point, choose it as P_0; if not, let P_0 be any interior point of a 3-simplex $[s^3]$ of S^3. There is an ambient isotopy of S^3 which leaves $\overline{S^3 - [s^3]}$ fixed and carries P_0 over to any other interior point of $[s^3]$. If $[s^3]$ and $[s'^3]$ have a common 2-face, one can easily construct an ambient isotopy moving an interior point P_0 of $[s^3]$ to an interior point P_0' of $[s'^3]$ which is the identity outside $[s^3] \cup [s'^3]$ (Figure 1.5).

So there is an ambient isotopy H^0 with $h_1^0 f(P_0) = P_0$, since any two 3-simplices can be connected by a chain of adjoining ones. Next we choose a point $P_1 \neq P_0$ in the simplex star of P_0, and by similar arguments we construct an ambient isotopy H^1 with $h_1^1 h_1^0 f$ leaving fixed the 1-simplex $[P_0, P_1]$. A further step leads to $h_1^2 h_1^1 h_1^0 f$ with a fixed 2-simplex $[P_0, P_1, P_2]$. At this juncture the assumption comes in that

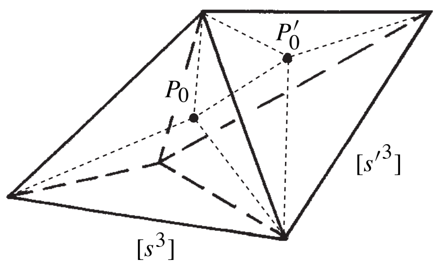


Figure 1.5

f is required to preserve the orientation. A point $P_3 \notin [P_0, P_1, P_2]$, but in the star of $[P_0, P_1, P_2]$, will be mapped by $h_1^2 h_1^1 h_1^0 f$ onto a point P_3' in the same half-space with regard to the plane spanned by P_0, P_1, P_2. This ensures the existence of the final ambient isotopy H^3 such that $h_1^3 h_1^2 h_1^1 h_1^0 f$ leaves fixed $[P_0, P_1, P_2, P_3]$. The assertion follows from the fact that $H = H^3 H^2 H^1 H^0$ is an ambient isotopy, $H(x,t) = (h_t(x), t)$.

By Theorem 1.8 the complement of $[P_0, P_1, P_2, P_3]$ is a combinatorial 3-ball and by Theorem 1.9 there is an ambient isotopy which connects $h_1 f$ with the identity of S^3.

(2) $\Longrightarrow$ (1) follows from the definition of an ambient isotopy.

Next we prove (1) $\Longrightarrow$ (3): Let $h\colon S^3 \to S^3$ be an orientation preserving homeomorphism and $\mathfrak{k}_1 = h(\mathfrak{k}_0)$. The preceding argument shows that there is another orientation preserving homeomorphism $g\colon S^3 \to S^3$, $g(\mathfrak{k}_0) = \mathfrak{k}_0$, such that hg leaves fixed some 3-simplex $[s^3]$ which will have to be chosen outside a regular neighbourhood of $\mathfrak{k}_0$ and $\mathfrak{k}_1$. For an interior point P of $[s^3]$ consider $S^3 - \{P\}$ as Euclidean 3-space $\mathbb{R}^3$. There is a translation τ of $\mathbb{R}^3$, which moves $\mathfrak{k}_0$ into $[s^3] - \{P\}$. It is easy to prove that $\mathfrak{k}_0$ and $\tau(\mathfrak{k}_0)$ are isotopic by moves (see Figure 1.6). We claim that $\mathfrak{k}_1 = hg(\mathfrak{k}_0)$ and $hg\tau(\mathfrak{k}_0) = \tau(\mathfrak{k}_0)$ are isotopic by moves also, which would complete the proof. Choose a subdivision of the triangulation of S^3 such that the triangles used in the isotopy by moves between $\mathfrak{k}_0$ and $\tau(\mathfrak{k}_0)$ form a subcomplex of S^3. There is an isotopy by moves $\mathfrak{k}_0 \to \tau(\mathfrak{k}_0)$ which is defined on the triangles of the subdivision. $hg\colon S^3 \to S^3$ maps the subcomplex onto another one carrying over the isotopy by moves, (see [Graeub 1949]).

(3) $\Longrightarrow$ (1). It is not difficult to construct a homeomorphism of S^3 onto itself which realizes a $\Delta^{\pm 1}$-move and leaves fixed the rest of the knot. Choose a regular neighbourhood U of the 2-simplex which defines the $\Delta^{\pm 1}$-move whose boundary meets the knot in two points. By linear extension one can obtain a homeomorphism producing the $\Delta^{\pm 1}$-move in U and leaving $S^3 - U$ fixed. $\square$

Isotopy by Δ-moves provides a means to formulate the knot problem on an elementary level. It is also useful as a method in proofs of invariance.

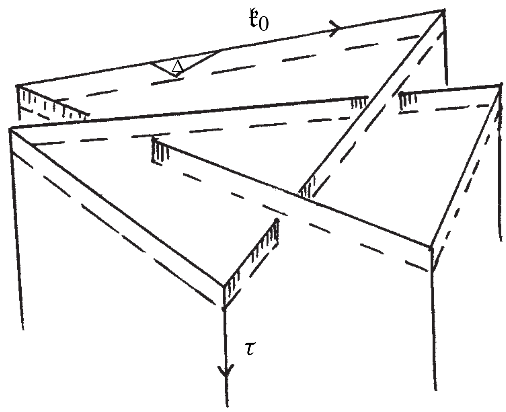


Figure 1.6

C Knot Projections

Geometric description in 3-spaces is complicated. The data that determine a knot are usually given by a *projection of* $\mathfrak{k}$ *onto a plane* E (projection plane) in $\mathbb{R}^3$. (In this paragraph we prefer $\mathbb{R}^3$ with its Euclidean metric to S^3; a knot $\mathfrak{k}$ will always be thought of as a simple closed polygon in $\mathbb{R}^3$.) A point $P \in p(\mathfrak{k}) \subset E$ whose preimage $p^{-1}(P)$ under the projection $p\colon \mathbb{R}^3 \to E$ contains more than one points of $\mathfrak{k}$ is called a *multiple* point.

1.11 Definition (Regular projection). A projection p of a knot $\mathfrak{k}$ is called *regular* if

(1) there are only finitely many multiple points $\{P_i \mid 1 \le i \le n\}$, and all multiple points are *double points*, that is, $p^{-1}(P_i)$ contains two points of $\mathfrak{k}$,

(2) no vertex of $\mathfrak{k}$ is mapped onto a double point.

The minimal number of double points or *crossings n* in a regular projection of a knot is called the *order* of the knot. A regular projection avoids occurrences as depicted in Figure 1.7.

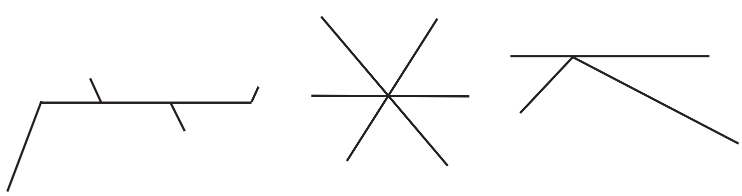


Figure 1.7

There are sufficiently many regular projections.

1.12 Proposition. *The set of regular projections is open and dense in the space of all projections.*

Proof. Think of directed projections as points on a unit sphere $S^2 \subset \mathbb{R}^3$ with the induced topology. A standard argument (general position) shows that singular (non-regular) projections are represented on S^2 by a finite number of curves. (The reader is referred to [Reidemeister 1926′] or [Burde 1978] for a more detailed treatment.) □

The projection of a knot does not determine the knot, but if at every double point in a regular projection the overcrossing line is marked, the knot can be reconstructed from the projection (Figure 1.8).

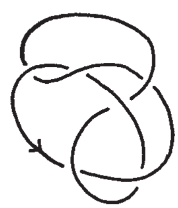


Figure 1.8

If the knot is oriented, the projection inherits the orientation. The projection of a knot with this additional information is called a *knot projection* or *knot diagram*. Two *knot diagrams* will be regarded as *equal* if they are isotopic in E as graphs, where the isotopy is required to respect overcrossing resp. undercrossing. Equivalent knots can be described by many different diagrams, but they are connected by simple operations.

1.13 Definition (Reidemeister moves). Two *knot diagrams* are called *equivalent*, if they are connected by a finite sequence of Reidemeister moves Ω_i, $i = 1, 2, 3$ or their inverses Ω_i^{-1}. The moves are described in Figure 1.9.

The operations $\Omega_i^{\pm 1}$ effect local changes in the diagram. Evidently all these operations can be realized by ambient isotopies of the knot; equivalent diagrams therefore define equivalent knots. The converse is also true:

1.14 Proposition. *Two knots are equivalent if and only if all their diagrams are equivalent.*

Proof. The first step in the proof will be to verify that any two regular projections p_1, p_2 of the same simple closed polygon $\mathfrak{k}$ are connected by $\Omega_i^{\pm 1}$-moves. Let p_1, p_2 again be represented by points on S^2, and choose on S^2 a polygonal path s from p_1 to p_2 in general position with respect to the lines of singular projections on S^2. When such a line is crossed the diagram will be changed by an operation $\Omega_i^{\pm 1}$, the actual

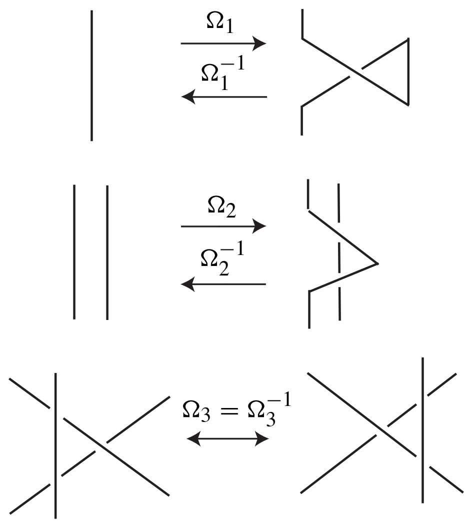


Figure 1.9

type depending on the type of singularity, see Figure 1.7, corresponding to the line that is crossed.

It remains to show that for a fixed projection equivalent knots possess equivalent diagrams. According to Proposition 1.10 it suffices to show that a $\Delta^{\pm 1}$-move induces $\Omega_i^{\pm 1}$-operations on the projection. This again is easily verified (Figure 1.10). □

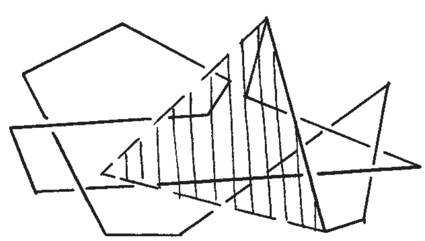


Figure 1.10

Proposition 1.14 allows an elementary approach to knot theory. It is possible to continue on this level and define invariants for diagrams with respect to equivalence [Burde 1978]. One might be tempted to look for a finite algorithm to decide equivalence of diagrams by establishing an a priori bound for the number of crossings. Such a bound is not known, and a simple counterexample shows that it can at least not be the maximum of the crossings that occur in the diagrams to be compared. The diagram of Figure 1.11 is that of a trivial knot, however, on the way to a simple closed projection via moves $\Omega_i^{\pm 1}$ the number of crossings will increase. This follows from the fact, that the diagram only allows operations Ω_1^{+1}, Ω_2^{+1} which increase the number of crossings. Figure 1.11 demonstrates one thing more: The operations Ω_i, $i = 1, 2, 3$ are "independent" – one cannot dispense with any of them (Exercise E 1.5), [Trace 1984].

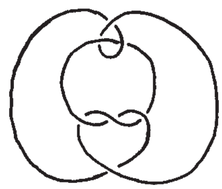


Figure 1.11

D Global Geometric Properties

In this section we will discuss two theorems (without giving proofs) which connect the property of "knottedness" and "linking" with other geometric properties of the curves in $\mathbb{R}^3$. The first is [Pannwitz 1933]:

1.15 Theorem (Pannwitz). *If $\mathfrak{k}$ is a non-trivial knot in $\mathbb{R}^3$, then there is a straight line which meets $\mathfrak{k}$ in four points.*

If a link of two components $\mathfrak{k}_i$, $i = 1, 2$ is not splittable, then there is a straight line which meets $\mathfrak{k}_1$ and $\mathfrak{k}_2$ in two points A_1, B_1 resp. A_2, B_2 each, with an ordering A_1, A_2, B_1, B_2 on the line. (*Such a line is called a four-fold chord of* $\mathfrak{k}$).

It is easy to see that the theorem does not hold for the trivial knot or a splittable linkage. (A link *is splittable or split* if it can be separated in $\mathbb{R}^3$ by a 2-sphere.)

What Pannwitz proved was actually something more general. For any knot $\mathfrak{k} \subset \mathbb{R}^3$ there is a singular disk $D \subset \mathbb{R}^3$ spanning $\mathfrak{k}$. For example one such disk can be constructed by erecting a cone over a regular projection of $\mathfrak{k}$ (Figure 1.12). If $D \subset \mathbb{R}^3$ is immersed in general position, there will be a finite number of singular points on $\mathfrak{k}$ (boundary singularities).

1.16 Definition (Knottedness). The minimal number of boundary singularities of a disk spanning a knot $\mathfrak{k}$ is called the *knottedness* k of $\mathfrak{k}$.

1.17 Theorem (Pannwitz). *The knottedness k of a non-trivial knot is an even number. A knot of knottedness k possesses $\frac{k^2}{2}$ four-fold chords.*

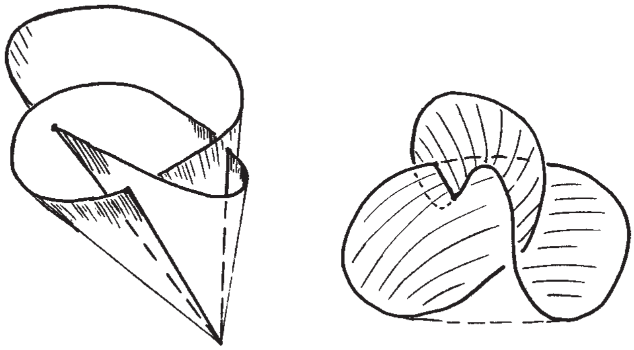


Figure 1.12

The *proof* of this theorem – which generalizes the first part of 1.15 – is achieved by cut-and-paste techniques as used in the proof of Dehn's Lemma. □

Figure 1.12 shows the trefoil spanned by a cone with 3 boundary singularities and by another disk with the minimal number of 2 boundary singularities. (The apex of the cone is not in general position, but a slight deformation will correct that.)

Another global theorem on of a knotted curve is due to J. Milnor [Milnor 1950]. If $\mathfrak{k}$ is smooth ($\mathfrak{k} \in C^{(2)}$), the integral

$$\kappa(\mathfrak{k}) = \int_{\mathfrak{k}} |\mathfrak{x}''(s)| ds$$

is called the *total curvature* of $\mathfrak{k}$. (Here $s \mapsto \mathfrak{x}(s)$ describes $\mathfrak{k}\colon S^1 \to \mathbb{R}^3$ with $s =$ arclength.) $\kappa(\mathfrak{k})$ is not an invariant of the knot type. Milnor generalized the notion of the total curvature so as to define it for arbitrary closed curves. In the case of a polygon this yields $\kappa(\mathfrak{k}) = \sum_{i=1}^{r} \alpha_i$, where the α_i are the angles of successive line segments (Figure 1.13).

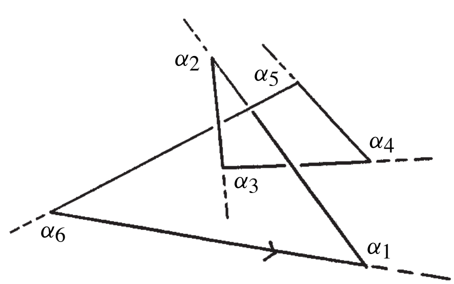


Figure 1.13

1.18 Theorem (Milnor). *The total curvature $\kappa(\mathfrak{k})$ of a non-trivial knot $\mathfrak{k} \subset \mathbb{R}^3$ exceeds* 4π. $\square$

We do not intend to copy Milnor's proof here. As an example, however, we give a realization of the trefoil in $\mathbb{R}^3$ with total curvature equal to $4\pi + \delta(\varepsilon)$, where $\delta(\varepsilon) > 0$ can be made arbitrarily small. This shows that the lower bound, 4π, is sharp.

In Figure 1.14 a diagram of the trefoil is given in the (x, y)-plane, the symbol at each vertex denotes the z-coordinate of the respective point on $\mathfrak{k}$. Six of eight angles α_i are equal to $\frac{\pi}{2}$, two of them, α and β, are larger, but tend to $\frac{\pi}{2}$ as $\varepsilon \to 0$.

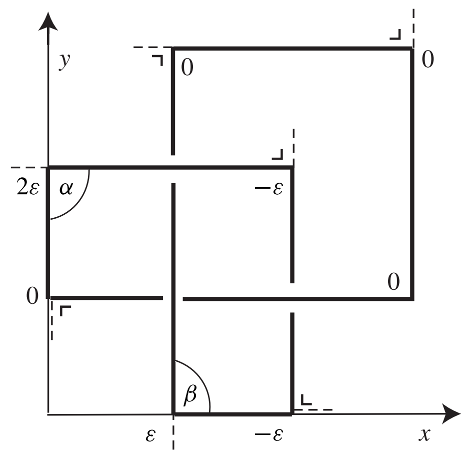


Figure 1.14

E History and Sources

A systematic and scientific theory of knots developed only in the last century when combinatorial topology came under way. The first contributions [Dehn 1910, 1914], [Alexander 1920, 1928] excited quite an interest, and a remarkable amount of work in this field was done which was reflected in the first monograph on knots, Reidemeister's Knotentheorie, [Reidemeister 1932]. The elementary approach to knots presented in this chapter stems from this source.

F Exercises

E 1.1. Let $\mathfrak{k}$ be a smooth oriented simple closed curve in $\mathbb{R}^2$, and let $-\mathfrak{k}$ denote the same curve with the opposite orientation. Show that $\mathfrak{k}$ and $-\mathfrak{k}$ are not isotopic in $\mathbb{R}^2$ whereas they are in $\mathbb{R}^3$.

E 1.2. Construct explicitly a p.l.-map of a complex K composed of two 3-simplices with a common 2-face onto itself which moves an interior point of one of the 3-simplices to an interior point of the other while keeping fixed the boundary ∂K of K (see Figure 1.5).

E 1.3. The suspension point P over a closed curve with n double points is called a *branch point of order* $n+1$. Show that there is an ambient isotopy in $\mathbb{R}^3$ which transforms the suspension into a singular disk with n branch points of order two [Papakyriakopoulos 1957′].

E 1.4. Let $p(t)$, $0 \le t \le 1$, be a continuous family of projections of a fixed knot $\mathfrak{k} \subset \mathbb{R}^3$ onto $\mathbb{R}^2$, which are singular at finitely many isolated points $0 < t_1 < t_2 < \cdots < 1$. Discuss by which of the operations Ω_i the two regular projections $p(t_k - \varepsilon)$ and $p(t_k + \varepsilon)$, $t_{k-1} < t_k - \varepsilon < t_k < t_k + \varepsilon < t_{k+1}$, are related according to the type of the singularity at t_k.

E 1.5. Prove that any projection obtained from a simple closed curve in $\mathbb{R}^3$ by using $\Omega_1^{\pm 1}$, $\Omega_2^{\pm 1}$ can also be obtained by using only Ω_1^{+1}, Ω_2^{+1}.

E 1.6. Let $p(\mathfrak{k})$ be a regular projection with n double points. By changing overcrossing arcs into undercrossing arcs at $k \le \frac{n-1}{2}$ double points, $p(\mathfrak{k})$ can be transformed into a projection of the trivial knot.

Chapter 2
Geometric Concepts

Some of the charm of knot theory arises from the fact that there is an intuitive geometric approach to it. We shall discuss in this chapter some standard constructions and presentations of knots and various geometric devices connected with them.

A Geometric Properties of Projections

Let $\mathfrak{k}$ be an oriented knot in oriented 3-space $\mathbb{R}^3$.

2.1 Definition (Symmetries). The knot obtained from $\mathfrak{k}$ by inverting its orientation is called the *inverted knot* and denoted by $-\mathfrak{k}$. The *mirror-image* of $\mathfrak{k}$ or *mirrored knot* is denoted by $\mathfrak{k}^*$; it is obtained by a reflection of $\mathfrak{k}$ in a plane.

A knot $\mathfrak{k}$ is called *invertible* if $\mathfrak{k} = -\mathfrak{k}$, and *amphicheiral* if $\mathfrak{k} = \mathfrak{k}^*$.

The existence of non-invertible knots was proved by H. Trotter [Trotter 1964]. The trefoil was shown to be non-amphicheiral in [Dehn 1914]; the trefoil is invertible. The *four-knot* 4_1 is both invertible and amphicheiral. The knot 8_{17} is amphicheiral but it is non-invertible [Kawauchi 1979], [Bonahon-Siebenmann 1979]. For more refined notions of symmetries in knot theory, see [Hartley 1983′].

2.2 Definition (Alternating knot). A *knot projection* is called *alternating*, if upper-crossings and undercrossings alternate while running along the knot. A *knot* is called *alternating*, if it possesses an alternating projection; otherwise it is *non-alternating*.

The existence of non-alternating knots was first proved by [Bankwitz 1930], see Proposition 13.30.

Alternating projections are frequently printed in knot tables without marking undercrossings. It is an easy exercise to prove that any such projection can be furnished in exactly two ways with undercrossings to become alternating; the two possibilities belong to mirrored knots. Without indicating undercrossings a closed plane curve does not hold much information about the knot whose projection it might be. Given such a curve there is always a trivial knot that projects onto it. To prove this assertion just choose a curve $\mathfrak{k}$ which ascends monotonically in $\mathbb{R}^3$ as one runs along the projection, and close it by a segment in the direction of the projection.

A finite set of closed plane curves defines a tessellation of the plane by simply connected *regions* bounded by arcs of the curves, and a single *infinite region*. (This can be avoided by substituting a 2-sphere for the plane.) The regions can be coloured by two colours like a chess-board such that regions of the same colour meet only at double points (Figure 2.1, E 2.2). The proof is easy. If the curve is simple, the fact is well known; if not, omit a simply closed partial curve s and colour the regions by an induction hypothesis. Replace s and exchange the colouring for all points inside s.

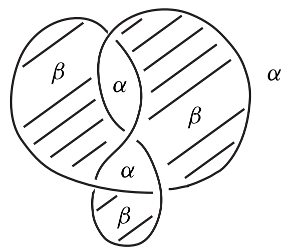


Figure 2.1

2.3 Definition (Graph of a knot). Let a regular knot diagram be chess-board coloured with colours α and β. Assign to every double point A of the projection an *index* $\theta(A) = \pm 1$ with respect to the colouring as defined by Figure 2.2. Denote by α_i, $1 \leqq i \leqq m$, the α-coloured regions of a knot diagram. Define a graph Γ whose vertices P_i correspond to the α_i, and whose edges a_{ij}^k correspond to the double points $A^k \subset \partial\alpha_i \cap \partial\alpha_j$, where a_{ij}^k joins P_i and P_j and carries the index $\theta(a_{ij}^k) = \theta(A^k)$.

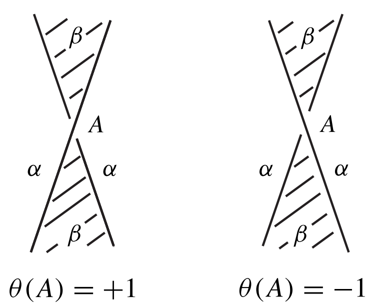


Figure 2.2

If β-regions are used instead of α-regions, a different graph is obtained from the regular projection. The Reidemeister moves Ω_i correspond to moves on graphs which can be used to define an equivalence of graphs. (Compare 1.13 and 1.14.) It is easy to prove (E 2.5) that the two graphs of a projection belonging to α- and β-regions are equivalent [Yajima-Kinoshita 1957]. Another Exercise (E 2.3) shows that a projection is alternating if and only if the index function $\theta(A)$ on the double points is a constant (Figure 2.3).

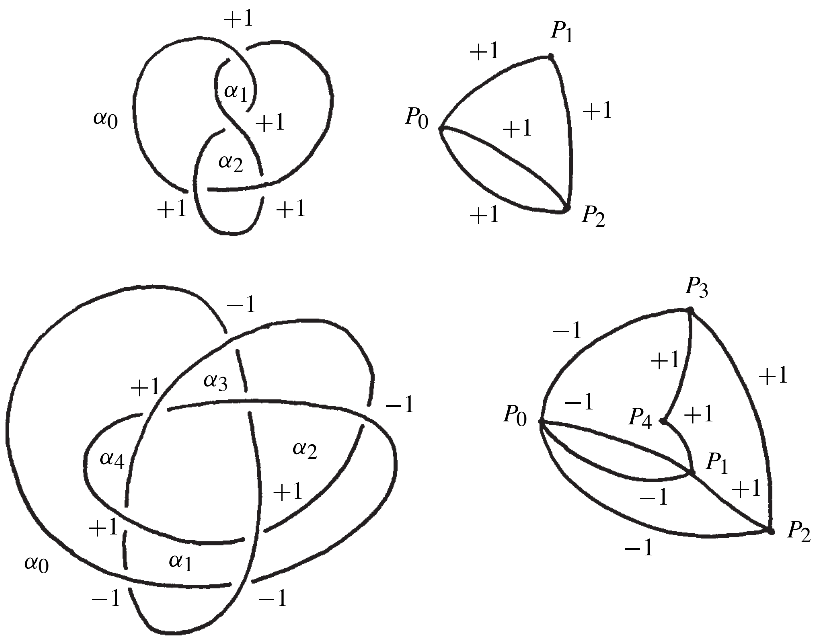


Figure 2.3

Graphs of knots have been repeatedly employed in knot theory [Aumann 1956], [Crowell 1959], [Kinoshita-Terasaka 1957]. We shall take up the subject again in Chapter 13 in connection with the quadratic form of a knot.

B Seifert Surfaces and Genus

A geometric fact of some consequence is the following:

2.4 Proposition (Seifert surface). *A simple closed curve $\mathfrak{k} \subset \mathbb{R}^3$ is the boundary of an orientable surface S, embedded in $\mathbb{R}^3$. It is called a Seifert surface.*

Proof. Let $p(\mathfrak{k})$ be a regular projection of $\mathfrak{k}$ equipped with an orientation. By altering $p(\mathfrak{k})$ in the neighbourhood of double points as shown in Figure 2.4, $p(\mathfrak{k})$ dissolves into a number of disjoint oriented simple closed curves which are called *Seifert cycles*. Choose an oriented 2-cell for each Seifert cycle, and embed the 2-cells in $\mathbb{R}^3$ as a disjoint union such that their boundaries are projected onto the Seifert cycles. The orientation of a Seifert cycle is to coincide with the orientation induced by the oriented 2-cell. We may place the 2-cells into planes z = const parallel to the projection plane ($z = 0$), and choose planes $z = a_1, z = a_2$ for corresponding Seifert cycles c_1, c_2 with $a_1 < a_2$ if c_1 contains c_2. Now we can undo the cut-and-paste-process described in Figure 2.4 by joining the 2-cells at each double point by twisted bands such as to obtain a connected surface S with $\partial S = \mathfrak{k}$ (see Figure 2.5).

Since the oriented 2-cells (including the bands) induce the orientation of $\mathfrak{k}$, they are coherently oriented, and hence, S is orientable. □

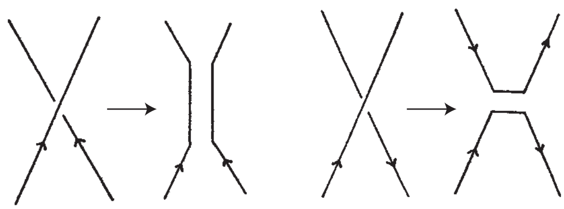


Figure 2.4

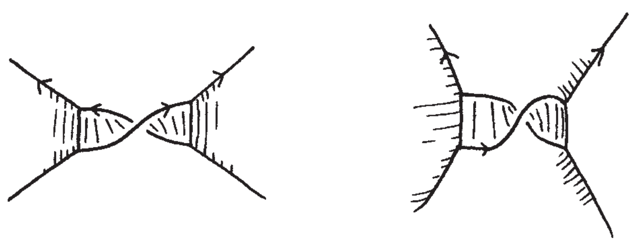


Figure 2.5

2.5 Definition (Genus). The minimal genus g of a Seifert surface spanning a knot $\mathfrak{k}$ is called the *genus of the knot* $\mathfrak{k}$.

Evidently the genus does not depend on the choice of a curve $\mathfrak{k}$ in its equivalence class: If $\mathfrak{k}$ and $\mathfrak{k}'$ are equivalent and S spans $\mathfrak{k}$, then there is a homeomorphism $h\colon S^3 \to S^3$, $h(\mathfrak{k}) = \mathfrak{k}'$ (Proposition 1.5), and $h(S) = S'$ spans $\mathfrak{k}'$. So the genus $g(\mathfrak{k})$ is a knot invariant, $g(\mathfrak{k}) = 0$ characterizes the trivial knot, because, if $\mathfrak{k}$ bounds a disk D which is embedded in $\mathbb{R}^3$ (or S^3), one can use Δ-moves over 2-simplices of D and reduce $\mathfrak{k}$ to the boundary of a single 2-simplex.

The notion of the genus was first introduced by H. Seifert in [Seifert 1934], it holds a central position in knot theory.

The method to construct a Seifert surface by Seifert cycles assigns a surface S' of genus g' to a given regular projection of a knot. We call g' the *canonical genus associated with the projection*. It is remarkable that in many cases the canonical genus coincides with the (minimal) genus g of the knot. It is always true for alternating projections (13.26(a)). In our table of knot projections up to nine crossings only the projections 8_{20}, 8_{21}, 9_{42}, 9_{44} and 9_{45} fail to yield $g' = g$: in these cases $g' = g + 1$.

This was already observed by H. Seifert; the fact that he lists 9_{46} instead of 9_{44} in [Seifert 1934] is due to the choice of different projections in Rolfsen's (and our table) and Reidemeister's.

There is a general algorithm to determine the genus of a knot [Schubert 1961], but its application is complicated. For other methods see E 4.10.

2.6 Definition and simple properties (Meridian and longitude). A tubular neighbourhood $V(\mathfrak{k})$ of a knot $\mathfrak{k} \subset S^3$ is homeomorphic to a solid torus. There is a simple closed curve m on $\partial V(\mathfrak{k})$ which is nullhomologous in $V(\mathfrak{k})$ but not on $\partial V(\mathfrak{k})$; we call m *meridian* of $\mathfrak{k}$. It is easy to see that any two meridians (if suitably oriented) in $\partial V(\mathfrak{k})$ are isotopic. A Seifert surface S will meet $\partial V(\mathfrak{k})$ in a simple closed curve ℓ, if $V(\mathfrak{k})$ is suitably chosen: ℓ is called a *longitude* of $\mathfrak{k}$. We shall see later on (Proposition 3.1) that ℓ, too, is unique up to isotopy on $\partial V(\mathfrak{k})$. If $\mathfrak{k}$ and S^3 are oriented, we may assign orientations to m and ℓ: The longitude ℓ is isotopic to $\mathfrak{k}$ in $V(\mathfrak{k})$ and will be oriented as $\mathfrak{k}$. The meridian will be oriented in such a way that its *linking number* $\mathrm{lk}(m, \mathfrak{k})$ with $\mathfrak{k}$ in S^3 is $+1$ or equivalently, its intersection number $\mathrm{int}(m, \ell)$ with ℓ is $+1$. From this it follows that ℓ is not nullhomologous on $\partial V(\mathfrak{k})$.

C Companion Knots and Product Knots

Another important idea was added by H. Schubert [1949]: the product of knots.

2.7 Definition (Product of knots). Let an oriented knot $\mathfrak{k} \subset \mathbb{R}^3$ meet a plane E in two points P and Q. The arc of $\mathfrak{k}$ from P to Q is closed by an arc in E to obtain a knot $\mathfrak{k}_1$; the other arc (from Q to P) is closed in the same way and so defines a knot $\mathfrak{k}_2$. The knot $\mathfrak{k}$ is called the *product* or *composition* of $\mathfrak{k}_1$ and $\mathfrak{k}_2$, and it is denoted by $\mathfrak{k} = \mathfrak{k}_1 \# \mathfrak{k}_2$; see Figure 2.6. $\mathfrak{k}$ is also called a *composite knot* when both knots $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are non-trivial. $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are called *factors* of $\mathfrak{k}$.

It is easy to see that for any given knots $\mathfrak{k}_1$, $\mathfrak{k}_2$ the product $\mathfrak{k} = \mathfrak{k}_1 \# \mathfrak{k}_2$ can be constructed; the product will not depend on the choice of representatives or on the plane E. A thorough treatment of the subject will be given in Chapter 7.

There are other procedures to construct more complicated knots from simpler ones.

2.8 Definition (Companion knot, satellite knot). Let $\tilde{\mathfrak{k}}$ be a knot in a 3-sphere $\tilde{S}^3$ and $\tilde{V}$ an unknotted solid torus in $\tilde{S}^3$ with $\tilde{\mathfrak{k}} \subset \tilde{V} \subset \tilde{S}^3$. Assume that $\tilde{\mathfrak{k}}$ is not contained

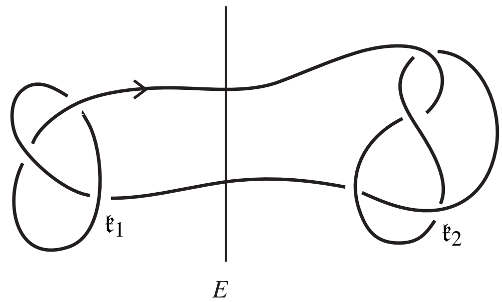


Figure 2.6

in a 3-ball of $\tilde{V}$. A homeomorphism $h\colon \tilde{V} \to \hat{V} \subset S^3$ onto a tubular neighbourhood $\hat{V}$ of a non-trivial knot $\hat{\mathfrak{k}} \subset S^3$ which maps a meridian of $\tilde{S}^3 - \tilde{V}$ onto a longitude of $\hat{\mathfrak{k}}$ maps $\tilde{\mathfrak{k}}$ onto a knot $\mathfrak{k} = h(\tilde{\mathfrak{k}}) \subset S^3$. The knot $\mathfrak{k}$ is called a *satellite* of $\hat{\mathfrak{k}}$, and $\hat{\mathfrak{k}}$ is its *companion* (Begleitknoten). The pair $(\tilde{V}, \tilde{\mathfrak{k}})$ is the *pattern* of $\mathfrak{k}$.

2.9 Remarks. The companion is the simpler knot, it forgets some of the tangles of its satellite. Each factor $\mathfrak{k}_i$ of a product $\mathfrak{k} = \mathfrak{k}_1 \# \mathfrak{k}_2$, for instance, is a companion of $\mathfrak{k}$. There are some special cases of companion knots: If $\tilde{\mathfrak{k}}$ is ambient isotopic in $\tilde{V}$ to a simple closed curve on $\partial\tilde{V}$, then $\mathfrak{k} = h(\tilde{\mathfrak{k}})$ is called a *cable knot* on $\hat{\mathfrak{k}}$. As another example consider $\tilde{\mathfrak{k}} \subset \tilde{V}$ as in Figure 2.7. Here the companion $\hat{\mathfrak{k}}$ is a trefoil, the satellite is called the *doubled knot* of $\hat{\mathfrak{k}}$. Doubled knots were introduced by J.H.C. Whitehead

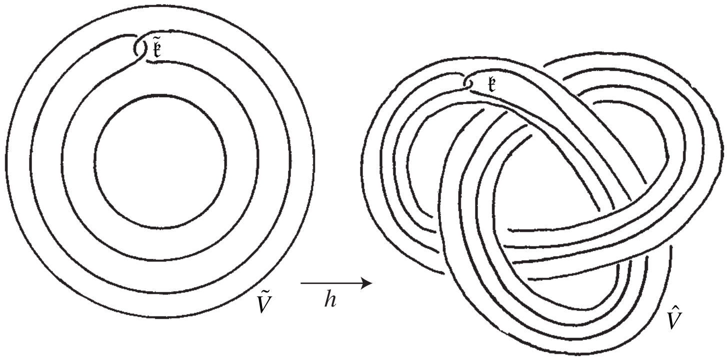


Figure 2.7

in [Whitehead 1937] and form an interesting class of knots with respect to certain algebraic invariants.

There is a relation between the genera of a knot and its companion.

2.10 Proposition (Schubert). *Let $\hat{\mathfrak{k}}$ be a companion of a satellite $\mathfrak{k}$ and $\tilde{\mathfrak{k}} = h^{-1}(\mathfrak{k})$ its preimage* (*as in* 2.8). *Denote by $\hat{g}$, g, $\tilde{g}$, the genera of $\hat{\mathfrak{k}}$, $\mathfrak{k}$, $\tilde{\mathfrak{k}}$, and by $n \geq 0$ the linking number of $\mathfrak{k}$ and a meridian $\hat{m}$ of a tubular neighbourhood $\hat{V}$ of $\hat{\mathfrak{k}}$ which contains $\mathfrak{k}$. Then*

$$g \geq n\tilde{g} + \hat{g}.$$

This result is due to H. Schubert [1953]. We start by proving the following lemma:

2.11 Lemma. *There is a Seifert surface S of minimal genus g spanning the satellite $\mathfrak{k}$ such that $S \cap \partial\hat{V}$ consists of n homologous (on $\partial\hat{V}$) longitudes of the companion $\hat{\mathfrak{k}}$. The intersection $S \cap (S^3 - \hat{V})$ consists of n components.*

Proof. Let S be a oriented Seifert surface of minimal genus spanning $\mathfrak{k}$. We assume that S is in general position with respect to $\partial\hat{V}$: that is, $S \cap \partial\hat{V}$ consists of a system of simple closed curves which are pairwise disjoint. If one of them, γ, is nullhomologous on $\partial\hat{V}$, it bounds a disk δ on $\partial\hat{V}$. We may assume that δ does not contain another simple closed curve with this property, $\delta \cap S = \gamma$. Cut S along γ and glue two disks δ_1, δ_2 (parallel to δ) to the curves obtained from γ. Since S was of minimal genus the new surface cannot be connected. Substituting the component containing $\mathfrak{k}$ for S reduces the number of curves. So we may assume that the curves $\{\gamma_1, \ldots, \gamma_r\} = S \cap \partial\hat{V}$ are not nullhomologous on the torus $\partial\hat{V}$; hence, they are parallel. The curves are supposed to follow each other on $\partial\hat{V}$ in the natural ordering $\gamma_1, \gamma_2, \ldots, \gamma_r$, and to carry the orientation induced by S. If for some index $\gamma_i \sim -\gamma_{i+1}$ on $\partial\hat{V}$ we may cut S along γ_i and γ_{i+1} and glue to the cuts two annuli parallel to one of the annuli on $\partial\hat{V}$ bounded by γ_i and γ_{i+1}. The resulting surface S' may not be connected but the Euler characteristic will remain invariant. Replace S by the component of S' that contains $\mathfrak{k}$. The genus g of S' can only be larger than that of S, if the other component is a sphere. In this case γ_i spans a disk in $\overline{S^3 - \hat{V}}$, and this means that the companion $\hat{\mathfrak{k}}$ is trivial which contradicts its definition. By the cut-and-paste process the pair $\gamma_i \sim -\gamma_{i+1}$ vanishes; so we may assume $\gamma_i \sim \gamma_{i+1}$ for all i. It is $\mathfrak{k} \sim r\gamma_1$ in $\hat{V}$, and $r\gamma_1 \sim 0$ in $\overline{S^3 - \hat{V}}$. We show that $S \cap (S^3 - \hat{V})$ consists of r components. If there is a component $\hat{S}$ of $S \cap S^3 - \hat{V}$ with $\hat{r} > 1$ boundary components then there are two curves $\gamma_i, \gamma_j \subset \partial\hat{S}$ such that $\gamma_k \cap \hat{S} = \emptyset$ for $i > k > j$. Connect γ_i and γ_j by a simple arc α in the annulus on $\partial\hat{V}$ bounded by γ_i and γ_j, and join its boundary points by a simple arc λ on $\hat{S}$. A curve u parallel to $\alpha \cup \lambda$ in $S^3 - \hat{V}$ will intersect $\hat{S}$ in one point (Figure 2.8), $\operatorname{int}(u, \hat{S}) = \pm 1$. Since u does not meet $\hat{V}$, we get $\pm 1 = \operatorname{int}(u, \hat{S}) = \operatorname{lk}(u, \partial\hat{S}) = k \cdot \hat{r}$, $k \in \mathbb{Z}$; hence $\hat{r} = 1$, a contradiction.

This implies that the γ_i are longitudes of $\hat{\mathfrak{k}}$; moreover

$$n = \operatorname{lk}(\hat{m}, \hat{\mathfrak{k}}) = \operatorname{lk}(\hat{m}, r\gamma_i) = r \cdot \operatorname{lk}(\hat{m}, \gamma_i) = r. \qquad \square$$

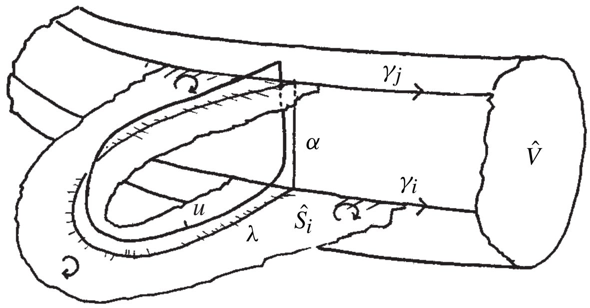


Figure 2.8

Proof of Proposition 2.10. Let S be a Seifert surface of $\mathfrak{k}$ according to Lemma 2.11. Each component $\hat{S}_i$ of $S \cap (S^3 - \hat{V})$ is a surface of genus $\hat{h}$ which spans a longitude γ_i of $\hat{\mathfrak{k}}$, hence $\hat{\mathfrak{k}}$ itself. The curves $\hat{l}_i = h^{-1}(\gamma_i)$ are longitudes of the unknotted solid torus $\tilde{V} \subset \tilde{S}^3$ bounding disjoint disks $\tilde{\delta}_i \subset \tilde{S}^3 - \tilde{V}$. Thus $h^{-1}(S \cap \hat{V}) \cup (\bigcup_i \tilde{\delta}_i)$ is a Seifert surface spanning $\tilde{\mathfrak{k}} = h^{-1}(\mathfrak{k})$. Its genus $\tilde{h}$ is the genus of $S \cap \hat{V}$. As $S = (S \cap \hat{V}) \cup \bigcup_{i=1}^n \hat{S}_i$ we get

$$g = n\hat{h} + \tilde{h} \geq n\hat{g} + \tilde{g}. \qquad \square$$

D Braids, Bridges, Plats

There is a second theme to our main theme of knots, which has developed some weight of its own: *the theory of braids*. E. Artin invented braids in [Artin 1925], and at the same time solved the problem of their classification. (The proof there is somewhat intuitive, Artin revised it to meet rigorous standards in a later paper [Artin 1947].) We shall occupy ourselves with braids in a special chapter but will introduce here the geometric idea of a braid, because it offers another possibility of representing knots (or links).

2.12. Place on opposite sides of a rectangle R in 3-space equidistant points P_i, Q_i, $1 \leq i \leq n$, (Figure 2.9). Let f_i, $1 \leq i \leq n$, be n simple disjoint polygonal arcs in $\mathbb{R}^3$, f_i starting in P_i and ending in $Q_{\pi(i)}$, where $i \mapsto \pi(i)$ is a permutation on $\{1, 2, \ldots, n\}$. The f_i are required to run "strictly downwards", that is, each f_i meets any plane perpendicular to the lateral edges of the rectangle at most once. The strings f_i constitute a *braid* $\mathfrak{z}$ (sometimes called an n-braid). The rectangle is called the *frame* of $\mathfrak{z}$, and $i \mapsto \pi(i)$ the *permutation of the braid*. In $\mathbb{R}^3$, *equivalent* or *isotopic braids* will be defined by "level preserving" isotopies of $\mathbb{R}^3$ relative to the endpoints $\{P_i\}$, $\{Q_i\}$, which will be kept fixed, but we defer a treatment of these questions to Chapter 10.

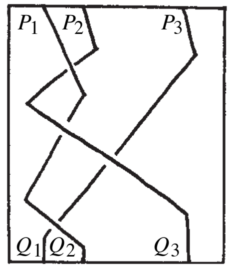


Figure 2.9

A *braid* can be *closed* with respect to an *axis h* (Figure 2.10). In this way every braid $\mathfrak{z}$ defines a *closed* braid $\hat{\mathfrak{z}}$ which represents a link of μ components, where μ

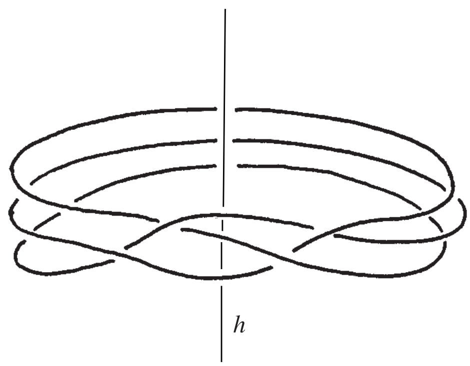


Figure 2.10

is the number of cycles of the permutation of $\mathfrak{z}$. We shall prove that every link can be presented as a closed braid. This mode of presentations is connected with another notion introduced by Schubert: the bridge-number of a knot (resp. link):

2.13 Definition (Bridge-number). Let $\mathfrak{k}$ be a knot (or link) in $\mathbb{R}^3$ which meets a plane $E \subset \mathbb{R}^3$ in $2m$ points such that the arcs of $\mathfrak{k}$ contained in each halfspace relative to E possess orthogonal projections onto E which are simple and disjoint. $(\mathfrak{k}, E)$ is called an *m-bridge presentation* of $\mathfrak{k}$; the minimal number m possible for a knot $\mathfrak{k}$ is called its *bridge-number*.

A regular projection $p(\mathfrak{k})$ of order n (see 1.11) admits an n-bridge presentation relative to the plane of projection (Figure 2.11 (a)). (If $p(\mathfrak{k})$ is not alternating, the number of bridges will even be smaller than n). The trivial knot is the only 1-bridge knot. The 2-bridge knots are an important class of knots which were classified by H. Schubert [1956]. Even 3-bridge knots defy classification up to this day.

2.14 Proposition (J.W. Alexander [1923′]). *A link $\mathfrak{k}$ can be represented by a closed braid.*

Proof. Choose $2m$ points P_i in a regular projection $p(\mathfrak{k})$, one on each arc between undercrossing and overcrossing (or vice versa). This defines an m-bridge presentation, $m \leq n$, with arcs s_i, $1 \leq i \leq m$, between P_{2i-1} and P_{2i} in the upper halfspace and arcs t_i, $1 \leq i \leq m$, joining P_{2i} and $P_{2i+1}(P_{2m+1} = P_1)$ in the lower halfspace of the projection plane (see Figure 2.11 (a)).

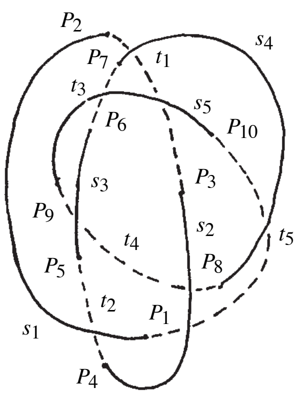


Figure 2.11 (a)

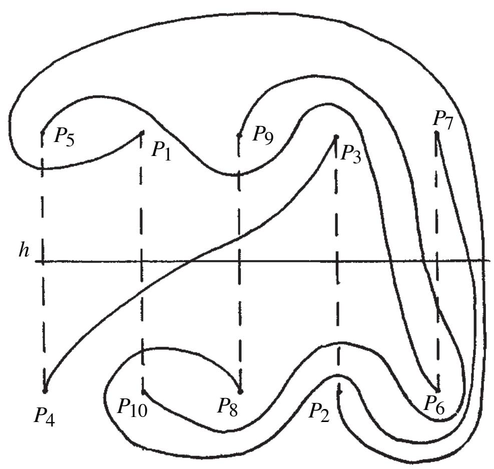


Figure 2.11 (b)

By an ambient isotopy of $\mathfrak{k}$ we arrange the $p(t_i)$ to form m parallel straight segments bisected by a common perpendicular line h such that all P_i with odd index are on one side of h (Figure 2.11 (b)). The arc $p(s_i)$ meets h in an odd number of points $P_{i1}, P_{i2}, \ldots$.

In the neighbourhood of a point P_{i2} we introduce a new bridge – we push the arc s_i in this neighbourhood from the upper halfspace into the lower one. Thus we obtain a bridge presentation where every arc $p(s_i)$, $p(t_i)$ meets h in exactly one point. Now choose s_i monotonically ascending over $p(s_i)$ from P_{2i-1} until h is reached, then descending to P_{2i}. Equivalently, the t_i begin by descending and ascend afterwards. The result is a closed braid with axis h. □

A $2m$-braid completed by $2m$ simple arcs to make a link as depicted in Figure 2.12 is called a *plat* or a $2m$-*plat*. A closed m-braid obviously is a special $2m$-plat, hence every link can be represented as a plat. The construction used in the proof of Proposition 2.14 can be modified to show that an m-bridge representation of a knot $\mathfrak{k}$ can be used to construct a $2m$-plat representing it. In Lemma 10.4 we prove the converse: *Every* $2m$*-plat allows an* m*-bridge presentation. The* 2*-bridge knots* (*and links*) *hence are the* 4*-plats* (*Viergeflechte*).

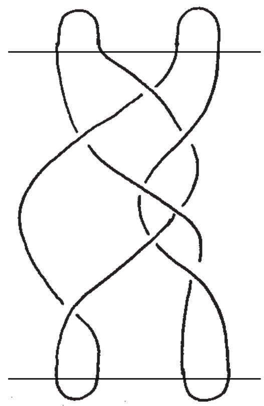


Figure 2.12

E Slice Knots and Algebraic Knots

R.H. Fox and J. Milnor introduced the notion of a slice knot. It arises from the study of embeddings $S^2 \subset S^4$ [Fox 1962].

2.15 Definition (Slice knot). A knot $\mathfrak{k} \subset \mathbb{R}^3$ is called a *slice knot* if it can be obtained as a cross section of a locally flat 2-sphere S^2 in $\mathbb{R}^4$ by a hyperplane $\mathbb{R}^3$. ($S^2 \subset \mathbb{R}^4$ is *embedded locally flat*, if it is locally a Cartesian factor.) The local flatness is essential: Any knot $\mathfrak{k} \subset \mathbb{R}^3 \subset \mathbb{R}^4$ is a cross section of a 2-sphere S^2 embedded in $\mathbb{R}^4$. Choose the double suspension of $\mathfrak{k}$ with suspension points P_+ and P_- respectively in the halfspace $\mathbb{R}^4_+$ and $\mathbb{R}^4_-$ defined by $\mathbb{R}^3$. The suspension S^2 is not locally flat at P_+ and P_-, (Figure 2.13).

There is a disk $D^2 = S^2 \cap \mathbb{R}^4_+$ spanning the knot $\mathfrak{k} = \partial D^2$ which will be locally flat if and only if S^2 can be chosen locally flat. This leads to an equivalent definition of slice knots:

2.16 Definition. A knot $\mathfrak{k}$ in the boundary of a 4-cell, $\mathfrak{k} \subset S^3 = \partial D^4$, is a *slice knot*, if there is a locally flat 2-disk $D^2 \subset D^4$, $\partial D^2 = \mathfrak{k}$, whose tubular neighbourhood intersects S^3 in a tubular neighbourhood of $\mathfrak{k}$.

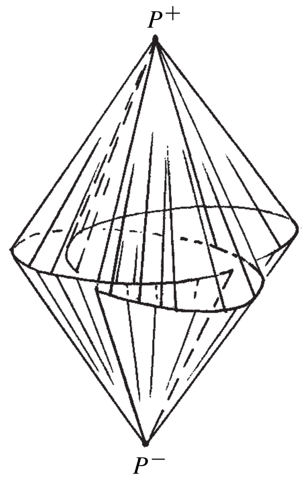


Figure 2.13

The last condition ensures that the intersection of $\mathbb{R}^3$ and D^2 resp. S^2 is transversal. We shall give some examples of knots that are *slice* and of some that are not.

Let $f\colon D^2 \to S^3$ be an immersion, and $\partial(f(D)) = \mathfrak{k}$ a knot. If the singularities of $f(D)$ are all double lines σ, $f^{-1}(\sigma) = \sigma_1 \cup \sigma_2$, such that at least one of the preimages σ_i, $1 \le i \le 2$, is contained in $\mathring{D}$, then $\mathfrak{k}$ is called a *ribbon knot*.

2.17 Proposition. *Ribbon knots are slice knots.*

Proof. Double lines with boundary singularities come in two types: The type required in a ribbon knot is shown in Figure 2.14 while the second type is depicted in Figure 1.12. In the case of a ribbon knot the hatched regions of $f(D)$ can be pushed into the fourth dimension without changing the knot $\mathfrak{k}$. □

It is not known whether all slice knots are ribbon knots. There are several criteria which allow to decide that a certain knot cannot be a slice knot [Fox-Milnor 1966], [Murasugi 1965]. The trefoil, for instance, is not a slice knot. In fact, of all knots of order ≤ 7 the knot 6_1 of Figure 2.14 is the only one which is a slice knot.

Knots turn up in connection with another higher-dimensional setting: a polynomial equation $f(z_1, z_2) = 0$ in two complex variables defines a complex curve C in $\mathbb{C}^2$. At a singular point $z_0 = (\mathring{z}_1, \mathring{z}_2)$, where $\left(\frac{\partial f}{\partial z_i}\right)_{z_0} = 0$, $i = 1, 2$, consider a small 3-sphere S^3_ε with centre z_0. Then $\mathfrak{k} = C \cap S^3_\varepsilon$ may be a knot or link. (If z_0 is a regular point of C, the knot $\mathfrak{k}$ is always trivial.)

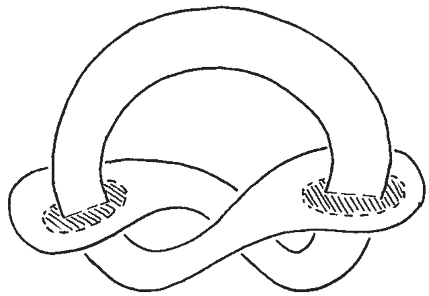


Figure 2.14

2.18 Proposition. *The algebraic surface* $f(z_1, z_2) = z_1^a + z_2^b = 0$ *with* $a, b \in \mathbb{Z}$, $a, b \geq 2$, *intersects the boundary* S_ε^3 *of a spherical neighbourhood of* $(0, 0)$ *in a torus knot* (*or link*) $\mathfrak{t}(a, b)$, *see* 3.26.

Proof. The equations

$$r_1^a e^{ia\varphi_1} = r_2^b e^{ib\varphi_2 + i\pi}, \quad r_1^2 + r_2^2 = \varepsilon^2, \quad z_j = r_j e^{i\varphi_j}$$

define the intersection $S_\varepsilon^3 \cap C$. Since $r_1^2 + r_1^{\frac{2a}{b}}$ is monotone, there are unique solutions $r_i = \varrho_i > 0$, $i = 1, 2$. Thus the points of the intersection lie on $\{(z_1, z_2) \mid |z_1| = \varrho_1, |z_2| = \varrho_2\}$, which is an unknotted torus in S_ε^3. Furthermore $a\varphi_1 \equiv b\varphi_2 + \pi \mod 2\pi$ so that $S_\varepsilon^3 \cap C = \{(\varrho_1 e^{ib\varphi}, \varrho_2 e^{ia\varphi + \frac{\pi i}{b}}) \mid 0 \leq \varphi \leq 2\pi\} = \mathfrak{t}(a, b)$. □

Knots that arise in this way at isolated singular points of algebraic curves are called *algebraic knots*. They are known to be *iterated torus knots*, that is, knots or links that are obtained by a repeated cabling process starting from the trivial knot. See [Milnor 1968], [Hacon 1976].

F History and Sources

To regard and treat a knot as an object of elementary geometry in 3-space was a natural attitude in the beginning, but proved to be very limited in its success. Nevertheless direct geometric approaches occasionally were quite fruitful and inspiring. H. Brunn [1897] prepared a link in a way which practically resulted in J.W. Alexander's theorem [Alexander 1923′] that every link can be deformed into a closed braid. The braids themselves were only invented by E. Artin in [Artin 1925] after closed braids were already in existence. H. Seifert then brought into knot theory the fundamental concept of the genus of a knot [Seifert 1934]. Another simple geometric idea led to the product of knots [Schubert 1949], and H. Schubert afterwards introduced and studied the theory of companions [Schubert 1953], and the notion of the bridge number of a knot [Schubert 1954]. Finally R.H. Fox and J. Milnor suggested looking at a knot from a 4-dimensional point of view which led to the slice knot [Fox 1962].

During the last decades geometric methods have gained importance in knot theory – but they are, as a rule, no longer elementary.

G Exercises

E 2.1. Show that the trefoil is symmetric, and that the four-knot is both symmetric and amphicheiral.

E 2.2. Let $p(\mathfrak{k}) \subset E^2$ be a regular projection of a link $\mathfrak{k}$. The plane E^2 can be coloured with two colours in such a way that regions with a common arc of $p(\mathfrak{k})$ in their boundary obtain different colours (chess-board colouring).

E 2.3. A knot projection is alternating if and only if $\theta(A)$ (see 2.3) is constant.

E 2.4. Describe the operations on graphs associated to knot projections which correspond to the Reidemeister operations Ω_i, $i = 1, 2, 3$.

E 2.5. Show that the two graphs associated to the regular projection of a knot by distinguishing either α-regions or β-regions are equivalent. (See Definition 2.3 and E 2.4.)

E 2.6. A regular projection $p(\mathfrak{k})$ (onto S^2) of a knot $\mathfrak{k}$ defines two surfaces $F_1, F_2 \subset S^3$ spanning $\mathfrak{k} = \partial F_1 = \partial F_2$ where $p(F_1)$ and $p(F_2)$ respectively cover the regions coloured by the same colour of a chess-board colouring of $p(\mathfrak{k})$ (see E 2.2). Prove that at least one of the surfaces F_1, F_2 is non-orientable.

E 2.7. Construct an orientable surface of genus one spanning the four-knot 4_1.

E 2.8. Give a presentation of the knot 6_3 as a 3-braid.

E 2.9. In Definition 2.7 the following condition was imposed on the knot $\tilde{\mathfrak{k}}$ embedded in the solid torus $\tilde{V}$:

(1) There is no ball $\tilde{B}$ such that $\tilde{\mathfrak{k}} \subset \tilde{B} \subset \tilde{V}$.

Show that (1) is equivalent to each of the following two conditions.

(2) $\tilde{\mathfrak{k}}$ intersects every disk $\delta \subset \tilde{V}$, $\partial\delta = \delta \cap \partial\tilde{V}$, $\partial\delta$ not contractible in $\partial\tilde{V}$.

(3) $\pi_1(\partial\tilde{V}) \to \pi_1(\tilde{V} - \tilde{\mathfrak{k}})$, induced by the inclusion, is injective.

Chapter 3
Knot Groups

The investigation of the complement of a knot in $\mathbb{R}^3$ or S^3 has been of special interest since the beginnings of knot theory. Tietze [Tietze 1908] was the first to prove the existence of non-trivial knots by computing the fundamental group of the complement of the trefoil. He conjectured that two knot types are equal if and only if their complements are homeomorphic. In 1988 Gordon and Luecke finally proved this conjecture – this proof is beyond the scope of this book. In the attempt to classify knot complements homological methods prove not very helpful. The fundamental group, however, is very effective and we will develop methods to present and study it. In particular, we will use it to show that there are non-trivial knots.

A Homology

$V = V(\mathfrak{k})$ denotes a tubular neighbourhood of the knot $\mathfrak{k}$ and $C = \overline{S^3 - V}$ is called the *complement* of the knot. H_j will denote the (singular) homology with coefficients in $\mathbb{Z}$.

3.1 Theorem (Homological properties).

(a) $H_0(C) \cong H_1(C) \cong \mathbb{Z}$, $H_n(C) = 0$ *for* $n \geq 2$.

(b) *There are two simple closed curves* m *and* ℓ *on* ∂V *with the following properties:*

(1) m *and* ℓ *intersect in one point,*

(2) $m \sim 0$, $\ell \sim \mathfrak{k}$ *in* $V(\mathfrak{k})$,

(3) $\ell \sim 0$ *in* $C = \overline{S^3 - V(\mathfrak{k})}$,

(4) $\mathrm{lk}(m, \mathfrak{k}) = 1$ *and* $\mathrm{lk}(\ell, \mathfrak{k}) = 0$ *in* S^3.

These properties determine m *and* ℓ *up to isotopy on* $\partial V(\mathfrak{k})$. *We call* m *a meridian and* ℓ *a longitude of the knot* $\mathfrak{k}$. *The knot* $\mathfrak{k}$ *and the longitude* ℓ *bound an annulus* $A \subset V$.

Proof. For (a) there are several proofs. Here we present one based on homological methods. We use the following well-known results:

$$H_n(S^3) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 3, \\ 0 & \text{otherwise,} \end{cases}$$

$$H_n(\partial V) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 2, \\ \mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$H_n(V) = H_n(S^1) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 1, \\ 0 & \text{otherwise;} \end{cases}$$

they can be found in standard books on algebraic topology, see [Spanier 1966], [Stöcker-Zieschang 1994].

Since C is connected, $H_0(X) = \mathbb{Z}$. For further calculations we use the Mayer–Vietoris sequence of the pair (V, C), where $V \cup C = S^3$, $V \cap C = \partial V$:

$$\begin{array}{ccccccc}
H_3(\partial V) & \longrightarrow & H_3(V) \oplus H_3(C) & \longrightarrow & H_3(S^3) & \longrightarrow & H_2(\partial V) \\
\| & & \| & & \|\wr & & \|\wr \\
0 & & 0 & & \mathbb{Z} & & \mathbb{Z} \\
 & & & & & & \\
 & \longrightarrow & H_2(V) \oplus H_2(C) & \longrightarrow & H_2(S^3) & \longrightarrow & H_1(\partial V) \\
 & & \| & & \| & & \|\wr \\
 & & 0 & & 0 & & \mathbb{Z} \oplus \mathbb{Z} \\
 & & & & & & \\
 & \longrightarrow & H_1(V) \oplus H_1(C) & \longrightarrow & H_1(S^3) & & \\
 & & \|\wr & & \| & & \\
 & & \mathbb{Z} & & 0 & &
\end{array}$$

It follows that $H_1(C) = \mathbb{Z}$. Since ∂V is the boundary of the orientable compact 3-manifold C, the group $H_2(\partial V)$ is mapped by the inclusion $\partial V \hookrightarrow C$ to $0 \in H_2(C)$. This implies that $H_2(C) = 0$ and that $H_3(S^3) \to H_2(\partial V)$ is surjective; hence, $H_3(C) = 0$.

Since C is a 3-manifold it follows that $H_n(C) = 0$ for $n > 3$; this is also a consequence of the Mayer–Vietoris sequence.

Consider the isomorphism

$$\mathbb{Z} \oplus \mathbb{Z} \cong H_1(\partial V) \to H_1(V) \oplus H_1(C)$$

in the Mayer–Vietoris sequence. The generators of $H_1(V) \cong \mathbb{Z}$ and $H_1(C) \cong \mathbb{Z}$ are determined up to their inverses. Choose the homology class of $\mathfrak{k}$ as a generator of $H_1(V)$ and represent it by a simple closed curve ℓ on ∂V which is homologous to 0 in $H_1(C)$. These conditions determine the homology class of ℓ in ∂V; hence, ℓ is unique up to isotopy on ∂V. A generator of $H_1(C)$ can be represented by a curve m on ∂V that is homologous to 0 in V. The curves ℓ and m determine a system of generators of $H_1(\partial V) \cong \mathbb{Z} \oplus \mathbb{Z}$. By a well-known result, we may assume that m is simple and intersects ℓ in one point, see e.g. [Stillwell 1980, 6.4.3], [ZVC 1980, E 3.22]. As m is homologous to 0 in V it is nullhomotopic in V, bounds a disk, and is a meridian of the solid torus V. The linking number of m and $\mathfrak{k}$ is 1 or -1. If necessary we reverse the direction of m to get (4). These properties determine m up to an isotopy of ∂V. A consequence is that ℓ and $\mathfrak{k}$ bound an annulus $A \subset V$.

Since $\ell \sim 0$ in C, ℓ bounds a surface, possibly with singularities, in C. (As we already know, see Proposition 2.4, ℓ even spans a surface without singularities: a Seifert surface.) □

Theorem 3.1 can be generalized to links (E 3.2). The negative aspect of the theorem is that complements of knots cannot be distinguished by their homological properties.

3.2 On the characterization of longitudes and meridians by the complement of a knot. With respect to the complement C of a knot the longitude ℓ and the meridian m have quite different properties: The longitude ℓ is determined up to isotopy and orientation by C; this follows from the fact that ℓ is a simple closed curve on ∂C which is not homologous to 0 on ∂C but homologous to 0 in C. The meridian m is a simple closed curve on ∂C that intersects ℓ in one point; hence, ℓ and m represent generators of $H_1(C) \cong \mathbb{Z}^2$. The meridian is not determined by C because simple closed curves on ∂C which are homologous to $m^{\pm 1}\ell^r$, $r \in \mathbb{Z}$, have the same properties (see E 3.3(a)).

B Wirtinger Presentation

The most important and effective invariant of a knot $\mathfrak{k}$ (or link) is its *group*: the fundamental group of its complement $\mathfrak{G} = \pi_1(S^3 - \mathfrak{k})$. Frequently $S^3 - \mathfrak{k}$ is replaced by $\mathbb{R}^3 - \mathfrak{k}$ or by $\overline{S^3 - V(\mathfrak{k})}$ or $\overline{\mathbb{R}^3 - V(\mathfrak{k})}$, respectively. The fundamental groups of these various spaces are obviously isomorphic, the isomorphisms being induced by inclusion. There is a simple procedure, due to Wirtinger, to obtain a presentation of a knot group.

3.3. Embed the knot $\mathfrak{k}$ into $\mathbb{R}^3$ such that its projection onto the plane $\mathbb{R}^2$ is regular. The projecting cylinder Z has self-intersections in n projecting rays a_i corresponding to the n double points of the regular projection. The a_i decompose Z into n 2-cells Z_i (see Figure 3.1), where Z_i is bounded by a_{i-1}, a_i and the overcrossing arc σ_i of $\mathfrak{k}$.

Choose the orientation of Z_i to induce on σ_i the direction of $\mathfrak{k}$. The complement of Z can be retracted parallel to the rays onto a halfspace above the knot; thus it is contractible.

To compute $\pi_1 C$ for some basepoint $P \in C$ observe that there is (up to a homotopy fixing P) exactly one polygonal closed path in general position relative to Z which intersects a given Z_i with intersection number ε_i and which does not intersect the other Z_j. *Paths of this type, taken for* $i = 1, 2, \ldots, n$ *and* $\varepsilon_i = 1$, *represent generators* $s_i \in \pi_1 C$. To see this, let a path in general position with respect to Z represent an arbitrary element of $\pi_1 C$. Move its intersection points with Z_i into the intersection of the curves s_i. Now the assertion follows from the contractibility of the complement of Z. Running through an arbitrary closed polygonal path ω yields the homotopy class as a word $w(s_i) = s_{i_1}^{\varepsilon_1} \ldots s_{i_r}^{\varepsilon_r}$ if in turn each intersection with Z_{i_j} and intersection number ε_j is put down by writing $s_{i_j}^{\varepsilon_j}$.

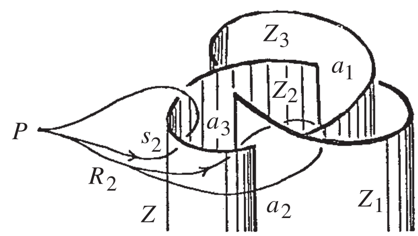


Figure 3.1

To obtain relators, consider a small path ϱ_j in C encircling a_j and join it with P by an arc λ_j. Then $\lambda_j \varrho_j \lambda_j^{-1}$ is contractible and the corresponding word $l_j r_j l_j^{-1}$ in the generators s_i is a relator. The word $r_j(s_j)$ can easily be read off from the knot projection. According to the characteristic $\eta \in \{1, -1\}$ of a double point, see Figure 3.2, we get the relator

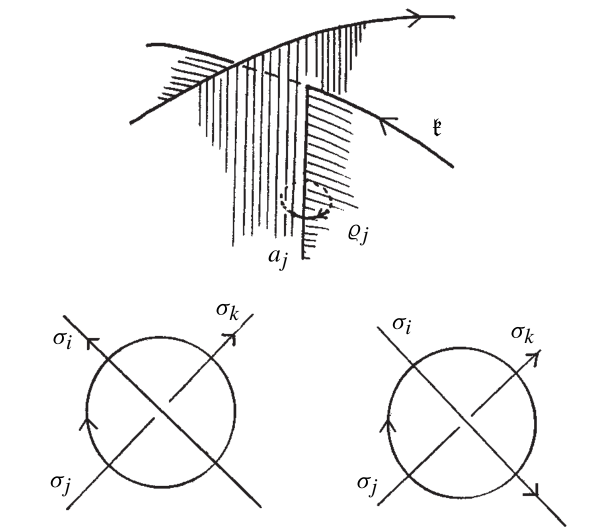
$$r_j = s_j s_i^{-\eta_j} s_k^{-1} s_i^{\eta_j}.$$

3.4 Theorem (Wirtinger presentation). *Let* σ_i, $i = 1, 2, \ldots, n$, *be the overcrossing arcs of a regular projection of a knot* (*or link*) $\mathfrak{k}$. *Then the knot group admits the following so-called Wirtinger presentation:*

$$\mathfrak{G} = \pi_1(\overline{S^3 - V(\mathfrak{k})}) = \langle s_1, \ldots, s_n \mid r_1, \ldots, r_n \rangle.$$

The arc σ_i *corresponds to the generator* s_i; *a crossing of characteristic* η_j *as in Figure* 3.2 *gives rise to the defining relator*

$$r_j = s_j s_i^{-\eta_j} s_k^{-1} s_i^{\eta_j}.$$



$\eta_j = +1,\ r_j = s_j s_i^{-1} s_k^{-1} s_i \qquad \eta_j = -1,\ r_j = s_j s_i s_k^{-1} s_i^{-1}$

Figure 3.2

Proof. It remains to check that $r_1, \ldots, r_n$ are defining relations. Consider $\mathbb{R}^3$ as a simplicial complex Σ containing Z as a subcomplex, and denote by Σ^* the dual complex. Let ω be a contractible curve in C, starting at a vertex P of Σ^*. By simplicial approximation ω can be replaced by a path in the 1-skeleton of Σ^* and the contractible homotopy by a series of homotopy moves which replace arcs on the boundary of 2-cells σ^2 of Σ^* by the inverse of the rest. If $\sigma^2 \cap Z = \emptyset$ the deformation over σ^2 has no effect on the words $\omega(s_i)$. If σ^2 meets Z in an arc then the deformation over σ^2 either cancels or inserts a word $s_i^{\varepsilon} s_i^{-\varepsilon}$, $\varepsilon \in \{1, -1\}$, in $\omega(s_i)$; hence, it does not effect the element of $\pi_1 C$ represented by ω. If σ^2 intersects a double line a_j then the deformation over σ^2 omits or inserts a relator: a conjugate of r_j or r_j^{-1} for some j. □

In the case of a link $\mathfrak{k}$ of μ components the relations ensure that generators s_i and s_j are conjugate if the corresponding arcs σ_i and σ_j belong to the same component. By abelianizing $\mathfrak{G} = \pi_1(S^3 - \mathfrak{k})$ we obtain from 3.4, see also E 3.2:

3.5 Proposition. *$H_1(S^3 - \mathfrak{k}) \cong \mathbb{Z}^{\mu}$ where μ is the number of components of $\mathfrak{k}$.* □

Using 3.5 and duality theorems for homology and cohomology one can calculate the other homology groups of $S^3 - \mathfrak{k}$, see E 3.2.

3.6 Corollary. *Let $\mathfrak{k}$ be a knot or link and $\langle s_1, \ldots, s_n \mid r_1, \ldots, r_n \rangle$ a Wirtinger presentation of $\mathfrak{G}$. Then each defining relation r_j is a consequence of the other defining relations r_i, $i \neq j$.*

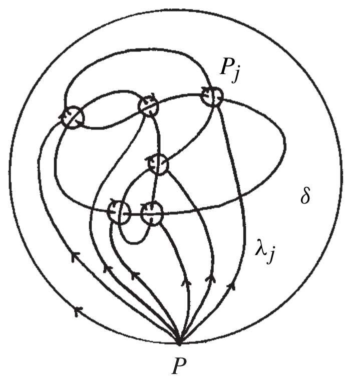


Figure 3.3

Proof. Choose the curves $\lambda_j \varrho_j \lambda_j^{-1}$ (see the paragraph before Theorem 3.4) in a plane E parallel to the projection plane and "far down" such that E intersects all a_i. Let δ be a disk in E such that $\mathfrak{k}$ is projected into δ, and let γ be the boundary of δ. We assume that P is on γ and that the λ_j have only the basepoint P in common. Then, see Figure 3.3,

$$\gamma \simeq \prod_{j=1}^{n} \lambda_j \varrho_j \lambda_j^{-1} \quad \text{in } E - \Big(\bigcup_j a_j \cap E\Big).$$

This implies the equation

$$1 \equiv \prod_{j=1}^{n} l_j r_j l_j^{-1}$$

in the free group generated by the s_i, where l_j is the word which corresponds to λ_j. Thus each relator r_j is a consequence of the other relators. □

3.7 Example (Trefoil knot = clover leaf knot). From Figure 3.4 we obtain Wirtinger generators s_1, s_2, s_3 and defining relators

$$\begin{array}{ll} s_1 s_2 s_3^{-1} s_2^{-1} & \text{at the vertex } A, \\ s_2 s_3 s_1^{-1} s_3^{-1} & \text{at the vertex } B, \\ s_3 s_1 s_2^{-1} s_1^{-1} & \text{at the vertex } C. \end{array}$$

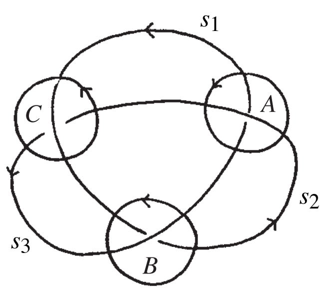


Figure 3.4

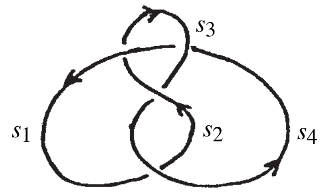


Figure 3.5

Since by 3.6 one relation is a consequence of the other two the knot group has the presentation

$$\begin{aligned}\langle\, s_1, s_2, s_3 \mid s_1s_2s_3^{-1}s_2^{-1}, s_3s_1s_2^{-1}s_1^{-1}\,\rangle &= \langle\, s_1, s_2 \mid s_1s_2s_1s_2^{-1}s_1^{-1}s_2^{-1}\,\rangle \\ &= \langle\, x, y \mid x^3y^2\,\rangle\end{aligned}$$

where $y = s_2^{-1}s_1^{-1}s_2^{-1}$ and $x = s_1s_2$. This group is not isomorphic to $\mathbb{Z}$, since the last presentation shows that it is a free product with amalgamated subgroup $\mathfrak{A}_1 *_{\mathfrak{B}} \mathfrak{A}_2$ where $\mathfrak{A}_i \cong \mathbb{Z}$ and $\mathfrak{B} = \langle\, x^3\,\rangle = \langle\, y^{-2}\,\rangle$ with $\mathfrak{B} \subsetneqq \mathfrak{A}_i$. Hence, it is not commutative. This can also be shown directly using the representation

$$\mathfrak{G} \to \mathrm{SL}_2(\mathbb{Z}), \quad x \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

since

$$\begin{aligned}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \\ &\neq \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.\end{aligned}$$

The reader should note that there for the first time in this book the existence of non-trivial knots has been proved, since the group of the trivial knot is cyclic.

We can approach the analysis of the group of the trefoil knot in a different manner by calculating its commutator subgroup using the Reidemeister–Schreier method. It turns out that $\mathfrak{G}'$ is a free group of rank 2, see E 4.2. We will use this method in the next example.

3.8 Example (Four-knot or figure eight knot, Figure 3.5).

$$\begin{aligned}\mathfrak{G} &= \langle\, s_1, s_2, s_3, s_4 \mid s_3 s_4^{-1} s_3^{-1} s_1,\ s_1 s_2^{-1} s_1^{-1} s_3,\ s_4 s_2^{-1} s_3^{-1} s_2 \,\rangle \\ &= \langle\, s_1, s_3 \mid s_3^{-1} s_1 s_3 s_1^{-1} s_3^{-1} s_1 s_3^{-1} s_1^{-1} s_3 s_1 \,\rangle \\ &= \langle\, s, u \mid u^{-1} sus^{-1} u^{-2} s^{-1} us \,\rangle,\end{aligned}$$

where $s = s_1$ and $u = s_1^{-1} s_3$.

The abelianizing homomorphism $\mathfrak{G} \to \mathfrak{Z}$ maps s onto a generator of $\mathfrak{Z}$ and u onto 0. Hence, $\{s^n \mid n \in \mathbb{Z}\}$ is a system of coset representatives and $\{x_n = s^n us^{-n} \mid n \in \mathbb{Z}\}$ the corresponding system of Schreier generators for the commutator subgroup $\mathfrak{G}'$ (see [ZVC 1980, 2.2]). The defining relations are

$$r_n = s^n (u^{-1} sus^{-1} u^{-2} s^{-1} us) s^{-n} = x_n^{-1} x_{n+1} x_n^{-2} x_{n-1},\ n \in \mathbb{Z}.$$

Using r_1, we obtain

$$x_2 = x_1 x_0^{-1} x_1^{+2};$$

hence, we may drop the generator x_2 and the relation r_1. Next we consider r_2 and obtain

$$x_3 = x_2 x_1^{-1} x_2^{+2}$$

and replace x_2 by the word in x_0, x_1 from above. Now we drop x_3 and r_2. By induction, we get rid of the relations $r_1, r_2, r_3, \dots$ and the generators $x_2, x_3, x_4, \dots$. Now, using the relation r_0 we obtain

$$x_{-1} = x_0^{+2} x_1^{-1} x_0;$$

thus we may drop the generator x_{-1} and the relation r_0. By induction we eliminate $x_{-1}, x_{-2}, x_{-3}, \dots$ and the relation $r_0, r_{-1}, r_{-2}, \dots$. Finally we are left with the generators x_0, x_1 and no relation, i.e. $\mathfrak{G}' = \langle\, x_0, x_1 | \,\rangle$ is a free group of rank 2. This proves that the figure eight knot is non-trivial. □

The fact that the commutator subgroup is finitely generated has a strong geometric consequence, namely that the complement can be fibered locally trivial over S^1 and the fibre is an orientable surface with one boundary component. In the case of a trefoil knot and the figure eight knot the fibre is a punctured torus. It turns out that these are the only knots that have a fibred complement with a torus as fibre, see Proposition 5.14. We will develop the theory of fibred knots in Chapter 5.

3.9 Example (2-bridge knot $\mathfrak{b}(7,3)$). From Figure 3.6 we determine generators for $\mathfrak{G}$ as before. It suffices to use the Wirtinger generators v, w which correspond to the bridges. One obtains the presentation

$$\begin{aligned}\mathfrak{G} &= \langle\, v, w \mid vwvw^{-1} v^{-1} wvw^{-1} v^{-1} w^{-1} vwv^{-1} w^{-1} \,\rangle \\ &= \langle\, s, u \mid susu^{-1} s^{-1} usu^{-1} s^{-1} u^{-1} sus^{-1} u^{-1} \,\rangle\end{aligned}$$

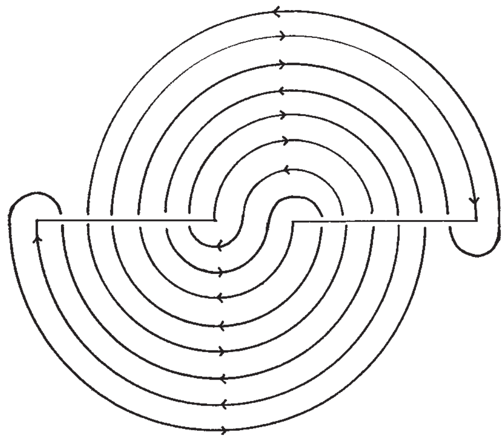


Figure 3.6

where $s = v$ and $u = wv^{-1}$. A system of coset representatives is $\{s^n \mid n \in \mathbb{Z}\}$ and they lead to the generators $x_n = s^n u s^{-n}$, $n \in \mathbb{Z}$, of $\mathfrak{G}'$ and the defining relations

$$x_{n+1}x_{n+2}^{-1}x_{n+1}x_{n+2}^{-1}x_n^{-1}x_{n+1}x_n^{-1}, \quad n \in \mathbb{Z}.$$

By abelianizing we obtain the relations $-2x_n + 3x_{n+1} - 2x_{n+2} = 0$, and now it is clear that this group is not finitely generated (E 3.4(a)).

From the above relations it follows that

$$\mathfrak{G}' = \cdots *_{\mathfrak{B}_{-2}} \mathfrak{A}_{-1} *_{\mathfrak{B}_{-1}} \mathfrak{A}_0 *_{\mathfrak{B}_0} \mathfrak{A}_1 *_{\mathfrak{B}_1 \ldots},$$

where $\mathfrak{A}_n = \langle x_n, x_{n+1}, x_{n+2} \rangle$ and $\mathfrak{B}_n = \langle x_{n+1}, x_{n+2} \rangle$ are free groups of rank 2 and $\mathfrak{A}_n \neq \mathfrak{B}_n \neq \mathfrak{A}_{n+1} = \langle x_{n+1}, x_{n+2} \rangle$. Proof as E 3.4(b).

A consequence is that the complement of this knot cannot be fibred over S^1 with a surface as fibre, see Theorem 5.1. This knot also has genus 1, i.e. it bounds a torus with one hole.

The background to the calculations in 3.8, 9 is discussed in Chapter 4.

3.10 Groups of satellites and companions. Recall the notation of 2.8: $\tilde{V}$ is an unknotted solid torus in a 3-sphere $\tilde{S}^3$ and $\tilde{\mathfrak{k}} \subset \tilde{V}$ a knot such that a meridian of $\tilde{V}$ is not contractible in $\tilde{V} - \tilde{\mathfrak{k}}$. As, by definition, a companion $\hat{\mathfrak{k}}$ is non-trivial the homomorphisms $i_\#\colon \pi_1\partial\hat{V} \to \pi_1(\hat{V} - \mathfrak{k})$, $j_\#\colon \pi_1\partial\hat{V} \to \pi_1(\overline{S^3 - \hat{V}})$ are injective, see 3.17. By the Seifert–van Kampen Theorem we get

3.11 Proposition. *With the above notation:*

$$\mathfrak{G} = \pi_1(S^3 - \mathfrak{k}) = \pi_1(\tilde{V} - \mathfrak{k}) *_{\pi_1\partial\hat{V}} \pi_1(S^3 - \hat{\mathfrak{k}}) = \mathfrak{H} *_{\langle \hat{t}, \hat{\lambda} \rangle} \hat{\mathfrak{G}},$$

where $\hat{t}$ and $\hat{\lambda}$ represent meridian and longitude of the companion knot, $\mathfrak{H} = \pi_1(\tilde{V} - \tilde{\mathfrak{k}})$ and $\hat{\mathfrak{G}}$ the knot group of $\hat{\mathfrak{k}}$. □

Remark. A satellite is never trivial.

3.12 Proposition (Longitude). *The longitude ℓ of a knot $\mathfrak{k}$ represents an element of the second commutator group of the knot group $\mathfrak{G}$:*

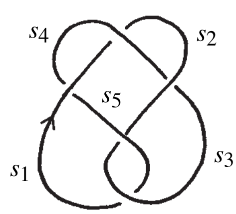
$$\ell \in \mathfrak{G}^{(2)}.$$

Proof. Consider a Seifert surface S spanning the knot $\mathfrak{k}$ such that for some regular neighbourhood V of $\mathfrak{k}$ the intersection $S \cap V$ is an annulus A with $\partial A = \mathfrak{k} \cup \ell$. Thus $\ell = \partial(S - A)$ implies that $\ell \sim 0$ in $C = \overline{S^3 - V}$. A 1-cycle z of C and S have intersection number r if $z \sim r \cdot m$ in C where m is a meridian of $\mathfrak{k}$. Hence, a curve ζ represents an element of the commutator subgroup $\mathfrak{G}'$ if and only if its intersection number with S vanishes. Since S is two-sided each curve on S can be pushed into $C - S$, and thus has intersection number 0 with S and consequently represents an element of $\mathfrak{G}'$. If $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ is a canonical system of curves on S then

$$\ell \simeq \prod_{n=1}^{g} [\alpha_n, \beta_n],$$

hence $\ell \in \mathfrak{G}^{(2)}$. □

3.13 Remark. The longitude ℓ of a knot $\mathfrak{k}$ can be read off a regular projection as a word in the Wirtinger generators as follows: run through the knot projection starting on the arc assigned to the generator s_k. Write down s_i (or s_i^{-1}) when undercrossing the arc from right to left (or from left to right) corresponding to s_i. Add s_k^{α} such that the sum of all exponents equals 0. See Figure 3.7, $\mathfrak{k} = 5_2$, $k = 1$, $\alpha = 5$.



$$\ell = s_4^{-1} s_5^{-1} s_2^{-1} s_1^{-1} s_3^{-1} \cdot s_1^5$$

Figure 3.7

C Peripheral System

In Definition 3.2 we assigned meridian and longitude to a given knot $\mathfrak{k}$. They define homotopy classes in the knot group. These elements are, however, not uniquely determined, but only up to a common conjugating factor. Meridian and longitude can be chosen as free abelian generators of $\pi_1 \partial V$. (In this section C always stands for the compact manifold $C = \overline{S^3 - V}$.)

3.14 Definition and Proposition (Peripheral system). *The peripheral system of a knot $\mathfrak{k}$ is a triple $(\mathfrak{G}, m, \ell)$ consisting of the knot group $\mathfrak{G}$ and the homotopy classes m, ℓ of a meridian and a longitude. These elements commute: $m \cdot \ell = \ell \cdot m$. The pair (m, ℓ) is uniquely determined up to a common conjugating element of $\mathfrak{G}$.*

The peripheral group system $(\mathfrak{G}, \mathfrak{P})$ consists of $\mathfrak{G}$ and the subgroup $\mathfrak{P}$ generated by m and ℓ, $\mathfrak{P} = \pi_1 \partial V$. As before, the inclusion $\partial V \subset C$ only defines a class of conjugate subgroups $\mathfrak{P}$ of $\mathfrak{G}$. □

The following theorem shows the strength of the peripheral system; unfortunately, its proof depends on a fundamental theorem of F. Waldhausen on 3-manifolds which we cannot prove here.

3.15 Theorem (Waldhausen). *Two knots $\mathfrak{k}_1$, $\mathfrak{k}_2$ in S^3 with the peripheral systems $(\mathfrak{G}_i, m_i, \ell_i)$, $i = 1, 2$, are equal if there is an isomorphism $\varphi\colon \mathfrak{G}_1 \to \mathfrak{G}_2$ with the property that $\varphi(m_1) = m_2$ and $\varphi(\ell_1) = \ell_2$.*

Proof. By the theorem of Waldhausen on sufficiently large irreducible 3-manifolds, see Appendix B.7, [Waldhausen 1968], [Hempel 1976, 13.6], the isomorphism φ is induced by a homeomorphism $h'\colon C_1 \to C_2$ mapping representative curves μ_1, λ_1 of m_1, ℓ_1 onto representatives μ_2, λ_2 of m_2, ℓ_2. The representatives can be taken on the boundaries ∂C_i. Waldhausen's theorem can be applied because $H_1(C_i) = \mathbb{Z}$ and $\pi_2(C_i) = 0$; the second condition is proved in Theorem 3.27. As h' maps the meridian of V_1 onto a meridian of V_2 it can be extended to a homeomorphism $h''\colon V_1 \to V_2$ mapping the 'core' $\mathfrak{k}_1$ onto $\mathfrak{k}_2$, see E 3.14. Together h' and h'' define the required homeomorphism $h\colon S^3 \to S^3$ which maps the (directed) knot $\mathfrak{k}_1$ onto the (directed) $\mathfrak{k}_2$. The orientation on S^3 defines orientations on V_1 and V_2, hence on the boundaries ∂V_1 and ∂V_2. Since $h(\mu_1) = \mu_2$ and $h(\lambda_1) = \lambda_2$ it follows that $h|\partial V_1\colon \partial V_1 \to \partial V_2$ is an orientation-preserving mapping. This implies that $h|V_1\colon V_1 \to V_2$ is also orientation-preserving; hence $h\colon S^3 \to S^3$ is orientation preserving. Thus $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are the 'same' knots. □

As direct consequence is the assertion 1.10, namely:

3.16 Corollary. *If two tame knots are topologically equivalent then they are p.l.-equivalent.* □

3.17 Proposition. *If $\mathfrak{k}$ is a non-trivial knot the inclusion $i\colon \partial V \to C = \overline{S^3 - V}$ induces an injective homomorphism $i_\#\colon \pi_1 \partial V \to \pi_1 C$.*

In particular, if $\pi_1 C \cong \mathbb{Z}$ then the knot $\mathfrak{k}$ is trivial.

Proof. Suppose $i_\#$ is not injective. Then the Loop Theorem of Papakyriakopoulos [1957], see Appendix B.5, [Hempel 1976, 4.2], guarantees the existence of a simple closed curve κ on ∂V and a disk δ in C such that

$$\kappa = \partial\delta \quad \text{(hence } \kappa \simeq 0 \text{ in } C\text{)}; \qquad \delta \cap V = \kappa \text{ and } \kappa \not\simeq 0 \text{ in } \partial V.$$

Since κ is simple and $\kappa \sim 0$ in C it is a longitude, see 3.2. So there is an annulus $A \subset V$ such that $A \cap \partial V = \kappa$, $\partial A = \kappa \cup \mathfrak{k}$, as has been shown in Theorem 3.1. This proves that $\mathfrak{k}$ bounds a disk in S^3 and, hence, is the trivial knot. □

The two trefoil knots can be distinguished by using the peripheral system. We will give a proof of this fact in a more general context in 3.29, but we suggest carrying out the calculations for the trefoil as an exercise.

The peripheral group system $(\mathfrak{G}, \langle m, \ell \rangle)$ has not – at first glance – the same strength as the peripheral system $(\mathfrak{G}, m, \ell)$ since it classifies only the complement of the knot [Waldhausen 1968]. The question whether different knots may have homomorphic complements was first posed by Tietze in 1908. In [Gordon-Luecke 1989] it is proved that a knot complement determines the knot. We cannot give the proof here which starts as follows: Let C be the complement of a knot, m the meridian and ℓ the longitude. Attach a solid torus W to C by identifying ∂W and ∂C in such a way that a meridian of W is identified with a simple closed curve $\kappa \sim m\ell^a$ on ∂C, $a \in \mathbb{Z}$. This yields a closed orientable 3-manifold M, its fundamental group is isomorphic to $\mathfrak{G}/\mathfrak{N}$, where $\mathfrak{N}$ is the normal closure of $m\ell^a$ in $\mathfrak{G}$. Gordon and Luecke show: $M = S^3$ implies $a = 0$.

A necessary condition is $\mathfrak{G}/\mathfrak{N} = 1$. However, as long as the Poincaré conjecture is not positively decided, this condition is not sufficient. The following definition avoids the Poincaré conjecture.

3.18 Definition (Property P). A knot $\mathfrak{k}$ with the peripheral system

$$(\langle s_1, \ldots, s_n \mid r_1, \ldots, r_n \rangle, m, \ell)$$

has *Property P* if $\langle s_1, \ldots s_n \mid r_1, \ldots, r_n, m\ell^a \rangle \neq 1$ for every integer $a \neq 0$.

Whether all knots have Property P is an open question. For this problem see E 3.12 and Chapter 15.

An immediate consequence of the proof of 3.15 is the following statement 3.19 (a); the assertion 3.19 (b) is obtained in the same way taking into account that $h|\partial V_1$ is orientation reversing.

3.19 Proposition (Invertible or amphicheiral knots). *Let* $(\mathfrak{G}, m, \ell)$ *be the peripheral system of the knot* $\mathfrak{k}$.

(a) $\mathfrak{k}$ *is invertible if and only if there is an automorphism* $\varphi\colon \mathfrak{G} \to \mathfrak{G}$ *such that* $\varphi(m) = m^{-1}$ *and* $\varphi(\ell) = \ell^{-1}$.

(b) $\mathfrak{k}$ *is amphicheiral if and only if there is an automorphism* $\psi\colon \mathfrak{G} \to \mathfrak{G}$ *such that* $\psi(m) = m^{-1}$ *and* $\psi(\ell) = \ell$. □

The only knot with the minimal number 4 of crossings, the four-knot is invertible and amphicheiral. The latter property is shown in Figure 3.8.

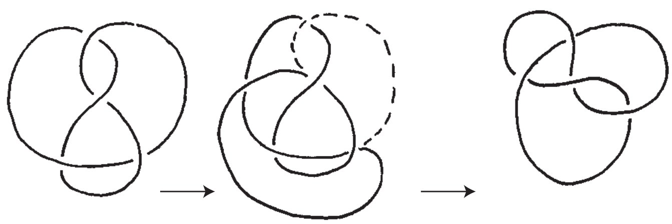


Figure 3.8

D Knots on Handlebodies

The Wirtinger presentation of a knot group is easily obtained and is most frequently applied in the study of examples. It depends, however, strongly on the knot projection and, in general, it does not reflect geometric symmetries of the knot nor does it afford much insight into the structure of the knot group as we have seen in the preceding Sections B and C. In this section, we describe another method. In the simplest case, for solid tori, a detailed treatment will be given in Section E.

3.20 Definition (Handlebody, Heegaard splitting).

(a) A *handlebody* V of *genus* g is obtained from a 3-ball $\mathbb{B}^3$ by attaching g handles $\mathbb{D}^2 \times I$ such that the boundary ∂V is an orientable closed surface of genus g, see Figure 3.9:

$$\begin{aligned} V &= \mathbb{B}^3 \cup H_1 \cup \cdots \cup H_g, \quad && H_i \cap H_j = \emptyset \quad && (i \neq j), \\ H_i \cap \mathbb{B}^3 &= D_{i1} \cup D_{i2}, && D_{i1} \cap D_{i2} = \emptyset, && D_{ij} \cong \mathbb{D}^2, \end{aligned}$$

and $(\overline{\partial \mathbb{B}^3 - \bigcup_{i,j} D_{ij}}) \cup \bigcup_i (\overline{\partial H_i - (D_{i1} \cup D_{i2})})$ is a closed orientable surface of genus g.

Another often-used picture of a handlebody is shown in Figure 3.10.

(b) The decomposition of a closed orientable 3-manifold M^3 into two handlebodies V, W: $M^3 = V \cup W$, $V \cap W = \partial V = \partial W$, is called a *Heegaard splitting* or *decomposition* of M^3 of *genus* g.

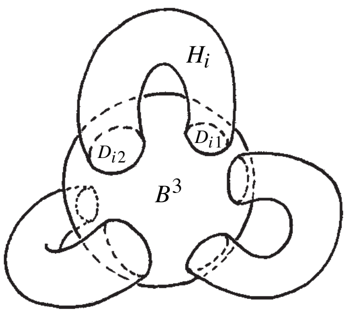


Figure 3.9

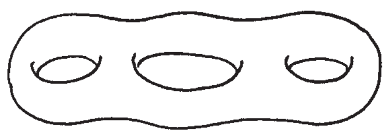


Figure 3.10

A convenient characterization of handlebodies is

3.21 Proposition. *Let W be an orientable 3-manifolds. If W contains a system $D_1, \dots, D_g$ of mutually disjoint disks such that $\partial W \cap D_i = \partial D_i$ and $\overline{W - \cup_i U(D_i)}$ is a closed 3-ball then W is a handlebody of genus g. (By $U(D_i)$ we denote closed regular neighbourhoods of D_i with $U(D_i) \cap U(D_j) = \emptyset$ for $i \neq j$.)*

Proof as Exercise E 3.9. □

Each orientable closed 3-manifold M^3 admits Heegaard splittings; one of them can be constructed as follows: Consider the 1-skeleton of a triangulation of M^3, define V as a regular neighbourhood of it and put $W = \overline{M^3 - V}$. Then V and W are handlebodies and form a Heegaard decomposition of M. (Proof as Exercise E 3.10; that V is a handlebody is obvious, that W is also can be proved using Proposition 3.21.) The classification problem of 3-manifolds can be reformulated as a problem on Heegaard decompositions, see [Reidemeister 1933], [Singer 1933]. F. Waldhausen has shown in [Waldhausen 1968′] that Heegaard splittings of S^3 are unique. We quote his theorem without proof.

3.22 Theorem (Heegaard splittings of S^3). *Any two Heegaard decompositions of S^3 of genus g are homeomorphic; more precisely: If (V, W) and (V', W') are Heegaard splittings of this kind then there exists an orientation preserving homeomorphism $h\colon S^3 \to S^3$ such that $h(V) = V'$ and $h(W) = W'$.* □

Next a direct application to knot theory.

3.23 Proposition. *Every knot in S^3 can be embedded in the boundary of the handlebodies of a Heegaard splitting of S^3.*

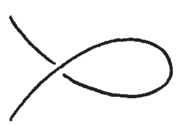


Figure 3.11

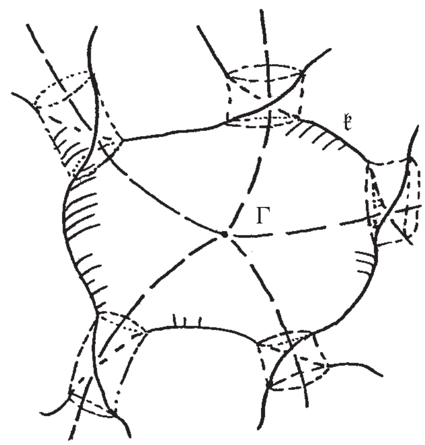


Figure 3.12

Proof. A (tame) knot $\mathfrak{k}$ can be represented by a regular projection onto S^2 which does not contain loops (see Figure 3.11). Let Γ be a graph of $\mathfrak{k}$ with vertices in the α-coloured regions of the projection (comp. 2.3), and let W be a regular neighbourhood of Γ. Obviously the knot $\mathfrak{k}$ can be realized by a curve on ∂W, see Figure 3.12. $\mathfrak{k}$ can serve as a canonical curve on ∂W – if necessary add a handle to ensure $\mathfrak{k} \not\sim 0$ on ∂W

W is a handlebody. To see this choose a tree T in Γ that contains all the vertices of Γ. It follows by induction on the number of edges of T that a regular neighbourhood of T is a 3-ball B. A regular neighbourhood W of Γ is obtained from B by attaching handles; for each of the segments of $\Gamma - T$ attach one handle.

$\overline{S^3 - W}$ also is a handlebody: The finite β-regions represent disks D_i such that $D_i \cap W = \partial D_i$. If one dissects $\overline{S^3 - W}$ along the disks D_i one obtains a ball, see Figure 3.13. □

We can now obtain a new presentation of the group of the knot $\mathfrak{k}$:

3.24 Proposition. *Let W, W' be a Heegaard splitting of S^3 of genus g. Assume that the knot $\mathfrak{k}$ is represented by a curve on the surface $F = \partial W = \partial W'$. Choose free generators s_i, s'_i, $1 \leqq i \leqq g$, $\pi_1 W = \langle s_1, \dots, s_g \mid - \rangle$, $\pi_1 W' = \langle s'_i, \dots, s'_g \mid - \rangle$, and a canonical system of curves κ_l, $1 \leqq l \leqq 2g$, on $F = W \cap W'$ with a common base point P, such that $\kappa_2 = \mathfrak{k}$. If κ_i is represented by a word $w_i(s_j) \in \pi_1 W$ and by $w'_i(s'_j) \in \pi_1 W'$, then*

(a) $\mathfrak{G} = \pi_1(S^3 - V(\mathfrak{k})) = \langle s_1, \dots, s_g, s'_1, \dots, s'_g \mid w_i(w'_i)^{-1},\ 2 \leqq i \leqq 2g \rangle$,

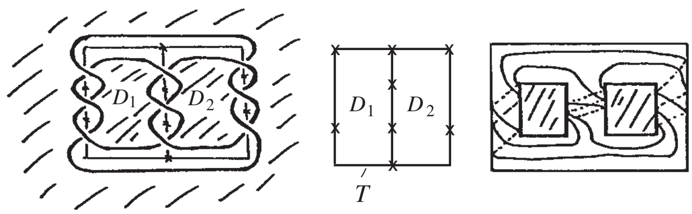


Figure 3.13

(b) $w_1(s_j)(w_1'(s_j'))^{-1}$ *can be represented by a meridian, and, for some (well-defined) integer r*, $w_2(s_j)(w_1(w_1')^{-1})^r$ *can be represented by a longitude, if the base point is suitably chosen.*

Proof. Assertion (a) is an immediate consequence of van Kampen's theorem (see Figure 3.14). For the proof of (b) let D be a disk in the tubular neighbourhood $V(\mathfrak{k})$, spanning a meridian m of $\mathfrak{k}$, and let D meet κ_1 in a subarc κ_1' which contains the base point P.

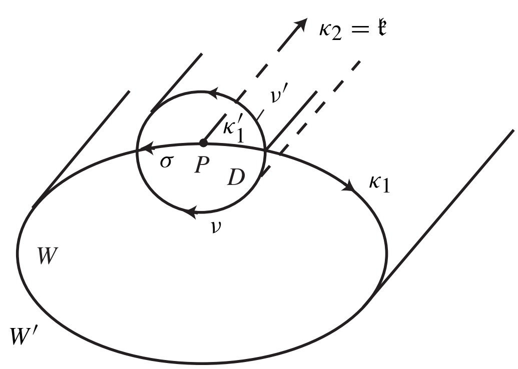


Figure 3.14

The boundary ∂D is composed of two arcs $\nu = \partial D \cap W$, $\nu' = \partial D \cap W'$, $\partial D = \nu^{-1}\nu'$, such that $\sigma\nu^{-1}\nu'\sigma^{-1}$ is a Wirtinger generator. For $\kappa_1'' = \kappa_1 - \kappa_1'$, the paths $\sigma\nu^{-1}\kappa_1''\sigma^{-1}$ resp. $\sigma\nu'^{-1}\kappa_1''\sigma^{-1}$ represent $w_1(s_j)$ resp. $w_1'(s_j)$; hence $w_1(w_1')^{-1} = \sigma\nu^{-1}\nu'\sigma^{-1}$. A longitude ℓ is represented by a simple closed curve λ on ∂V, $\lambda \sim \kappa_2$ in V, which is nullhomologous in $C = \overline{S^3 - V}$. Hence $\sigma\lambda\sigma^{-1}$ represents $w_2(s_j) \cdot (w_1(w_1')^{-1})^r$ for some (uniquely determined) integer r (see Remark 3.13). If the endpoint of σ is chosen as a base point assertion (b) is valid. $\square$

3.25 Corollary. *Assume* $S^3 = W \cup W'$, $W \cap W' = F \supset \mathfrak{k}$ *as in* 3.24. *If the inclusions* $i\colon \overline{F - V} \to W$, $i'\colon \overline{F - V} \to W'$ *induce injective homomorphisms of the corresponding fundamental groups, then*

$$\mathfrak{G} = \pi_1(\overline{S^3 - V(\mathfrak{k})}) = \pi_1 W *_{\pi_1(F-V)} \pi_1 W' = \mathfrak{F}_g *_{\mathfrak{F}_{2g-1}} \mathfrak{F}_g.$$

There is a finite algorithm by which one can decide whether the assumption of the corollary is valid. In this case the knot group $\mathfrak{G}$ *has a non-trivial centre if and only if* $g = 1$.

Proof. Since $F - V$ is connected, it is an orientable surface of genus $g - 1$ with two boundary components. $\pi_1(\overline{F - V})$ is a free group of rank $2(g - 1) + 1$.

There is an algorithm due to Nielsen [1921], see [ZVC 1980, 1.7], by which the rank of the finitely generated subgroup $i_\# \pi_1(\overline{F - V})$ in the free group $\pi_1 W = \mathfrak{F}_g$ can be determined. The remark about the centre follows from the fact that the centre of a proper product with amalgamation is contained in the amalgamating subgroup. $\square$

We propose to study the case $g = 1$, the torus knots, in the following paragraph. They form the simplest class of knots and can be classified. For an intrinsic characterization of torus knots see Theorem 6.1.

E Torus Knots

Let $S^3 = \mathbb{R}^3 \cup \{\infty\} = W \cup W'$ be a 'standard' Heegaard splitting of genus 1 of the oriented 3-sphere S^3. We may assume W to be an unknotted solid torus in $\mathbb{R}^3$ and $F = W \cap W'$ a torus carrying the orientation induced by that of W. There are meridians μ and ν of W and W' on F which intersect in the basepoint P with intersection number 1 on F, see Figure 3.15.

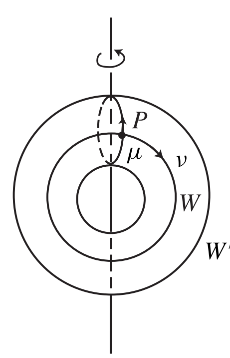


Figure 3.15

Any closed curve κ on F is homotopic to a curve $\mu^a \cdot \nu^b$, $a, b \in \mathbb{Z}$. Its homotopy class on F contains a (non-trivial) simple closed curve iff a and b are relatively prime. Such a simple curve intersects μ resp. ν exactly b resp. a times with intersection number $+1$ or -1 according to the signs of a and b. Two simple closed curves $\kappa = \mu^a \nu^b$, $\lambda = \mu^c \nu^d$ on F intersect, eventually after an isotopy, in a single point if and only if $\left|\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right| = \pm 1$, where the exact value of the determinant is the intersection number of κ with λ.

3.26 Definition (Torus knots). Let (W, W') be the Heegaard splitting of genus 1 of S^3 described above. If $\mathfrak{k}$ is a simple closed curve on F with the intersection numbers a, b with ν and μ, respectively, and if $|a|, |b| \geq 2$ then $\mathfrak{k}$ is called a torus knot, more precisely, *the torus knot* $\mathfrak{t}(a, b)$.

3.27 Proposition.

(a) $\mathfrak{t}(-a, -b) = -\mathfrak{t}(a, b)$, $\mathfrak{t}(a, -b) = \mathfrak{t}^*(a, b)$ *(see Definition* 2.1*)*.

(b) $\mathfrak{t}(a, b) = \mathfrak{t}(-a, -b) = \mathfrak{t}(b, a)$*: torus knots are invertible.*

Proof. The first assertion of (a) is obvious. A reflection in a plane and a rotation through π illustrate the other equations, see Figure 3.15 □

3.28 Proposition. (a) *The group $\mathfrak{G}$ of the torus knot $\mathfrak{t}(a, b)$ can be presented as follows:*

$$\mathfrak{G} = \langle\, u, v \mid u^a v^{-b} \,\rangle = \langle\, u | \,\rangle *_{\langle u^a \rangle = \langle v^b \rangle} \langle\, v | \,\rangle, \ \ \mu, \nu \text{ representing } u, v.$$

The amalgamating subgroup $\langle\, u^a \,\rangle$ is an infinite cyclic group; it represents the centre $\mathfrak{Z} = \langle\, u^a \,\rangle \cong \mathbb{Z}$ of $\mathfrak{G}$ and $\mathfrak{G}/\mathfrak{Z} \cong \mathbb{Z}_{|a|} * \mathbb{Z}_{|b|}$.

(b) *The elements $m = u^c v^d$, $\ell = u^a m^{-ab}$, where $ad + bc = 1$, describe meridian and longitude of $\mathfrak{t}(a, b)$ for a suitable chosen basepoint.*

(c) *$\mathfrak{t}(a, b)$ and $\mathfrak{t}(a', b')$ have isomorphic groups if and only if $|a| = |a'|$ and $|b| = |b'|$ or $|a| = |b'|$ and $|b| = |a'|$.*

Proof. The curve $\mathfrak{t}(a, b)$ belongs to the homotopy class u^a of W and to v^b of W'. This implies the first assertion of (a) by 3.24 (a). From 3.24 (b) it follows that the meridian of $\mathfrak{t}(a, b)$ belongs to the homotopy class $u^c v^d$ with $\left|\begin{smallmatrix} a & -b \\ c & d \end{smallmatrix}\right| = 1$. Since the classes u^a and $u^c v^d$ can be represented by two simple closed curves on ∂V intersecting in one point the class $(u^a)^1 (u^c v^d)^{-ab}$ can be represented by a simple closed curve on ∂V. Since it becomes trivial by abelianizing it is the class of a longitude. This implies (b).

It is clear that u^a belongs to the centre of the knot group $\mathfrak{G}$. If we introduce the relation $u^a = 1$ we obtain the free product

$$\langle\, u, v \mid u^a, v^b \,\rangle = \langle\, u \mid u^a \,\rangle * \langle\, v \mid v^b \,\rangle.$$

Since this group has a trivial centre, see [ZVC 1980, 2.3.9], it follows that u^a generates the centre. Moreover, $\mathfrak{G} = \langle u | \rangle *_{\langle u^a \rangle} \langle v | \rangle$ implies that each of the factor subgroups is free.

(c) is a consequence of the fact that u and v generate non-conjugate maximal finite cyclic subgroups in the free product $\mathbb{Z}_{|a|} * \mathbb{Z}_{|b|}$, comp. [ZVC 1980, 2.3.10]. □

3.29 Theorem (Classification of torus knots). (a) $\mathfrak{t}(a, b) = \mathfrak{t}(a', b')$ *if and only if* (a', b') *is equal to one of the following pairs:* (a, b), (b, a), $(-a, -b)$, $(-b, -a)$.

(b) *Torus knots are invertible, but not amphicheiral.*

Proof. Sufficiency follows from 3.27. Suppose $\mathfrak{t}(a, b) = \mathfrak{t}(a', b')$. Since the centre $\mathfrak{Z}$ is a characteristic subgroup, $\mathfrak{G}/\mathfrak{Z}$ is a knot invariant. The integers $|a|$ and $|b|$ are in turn invariants of $\mathbb{Z}_{|a|} * \mathbb{Z}_{|b|}$; they are characterized by the property that they are the orders of maximal finite subgroups of $\mathbb{Z}_{|a|} * \mathbb{Z}_{|b|}$ which are not conjugate. Hence, $\mathfrak{t}(a, b) = \mathfrak{t}(a', b')$ implies that $|a| = |a'|$, $|b| = |b'|$ or $|a| = |b'|$, $|b| = |a'|$.

By 3.27 (b) it remains to prove that torus knots are not amphicheiral. Let us assume $a, b > 0$ and $\mathfrak{t}(a, b) = \mathfrak{t}(a, -b)$. By 3.14 there is an isomorphism

$$\varphi\colon \mathfrak{G} = \langle u, v \mid u^a v^{-b} \rangle \to \langle u', v' \mid u'^a v'^b \rangle = \mathfrak{G}^*$$

mapping the peripheral system $(\mathfrak{G}, m, \ell)$ onto $(\mathfrak{G}^*, m', \ell')$:

$$m' = \varphi(u^c v^d) = u'^{c'} v'^{d'}, \;\; \ell' = \varphi(u^a (u^c v^d)^{-ab}) = u'^a (u'^{c'} v'^{d'})^{+ab}$$

with $ad + bc = ad' - bc' = 1$.

It follows that

$$d' = d + jb \quad \text{and } c' = -c + ja \quad \text{for some } j \in \mathbb{Z}.$$

The isomorphism φ maps the centre $\mathfrak{Z}$ of $\mathfrak{G}$ onto the centre $\mathfrak{Z}^*$ of $\mathfrak{G}^*$. This implies that $\varphi(u^a) = (u'^a)^\varepsilon$ for $\varepsilon \in \{1, -1\}$. Now,

$$\begin{aligned} u'^a (u'^{c'} v'^{d'})^{ab} &= \varphi(u^a (u^c v^d)^{-ab}) = \varphi(u^a) \varphi(u^c v^d))^{-ab} \\ &= (u'^a)^\varepsilon (u'^{c'} v'^{d'})^{-ab}; \end{aligned}$$

hence, $(u'^a)^{1-\varepsilon} = (u'^{c'} t'^{d'})^{-2ab}$. This equation is impossible: the homomorphism $\mathfrak{G}^* \to \mathfrak{G}^*/\mathfrak{Z}^* \cong \mathbb{Z}_a * \mathbb{Z}_b$ maps the term on the left onto unity, whereas the term on the right represents a non-trivial element of $\mathbb{Z}_a * \mathbb{Z}_b$ because $a \nmid c'$ and $b \nmid d'$. This follows from the solution of the word problem in free products, see [ZVC 1980, 2.3.3]. □

F Asphericity of the Knot Complement

In this section we use some notions and deeper results from algebraic topology, in particular, the notion of a $K(\pi, 1)$-space, π a group: X is called a $K(\pi, 1)$-*space* if $\pi_1 X = \pi$ and $\pi_n X = 0$ for $n \neq 1$. X is also called *aspherical*.

3.30 Theorem. *Let $\mathfrak{k} \subset S^3$ be a knot, C the complement of an open regular neighbourhood V of $\mathfrak{k}$. Then*

(a) $\pi_n C = 0$ *for* $n \neq 1$*; in other words, C is a $K(\pi_1 C, 1)$-space.*

(b) $\pi_1 C$ *is torsionfree.*

Proof. $\pi_0 C = 0$ since C is connected. Assume that $\pi_2 C \neq 0$. By the Sphere Theorem [Papakyriakopoulos 1957′], see Appendix B.6, [Hempel 1976, 4.3], there is an embedded p.l.-2-sphere $S \subset C$ which is not nullhomotopic. By the theorem of Schoenflies, see [Moise 1977, p. 117], S divides S^3 into two 3-balls B_1 and B_2. Since $\mathfrak{k}$ is connected it follows that one of the balls, say B_2, contains V and $B_1 \subset C$. Therefore S is nullhomotopic, contradicting the assumption. This proves $\pi_2 C = 0$.

To calculate $\pi_3 C$ we consider the universal covering $\tilde{C}$ of C. As $\pi_1 C$ is infinite $\tilde{C}$ is not compact, and this implies $H_3(\tilde{C}) = 0$. As $\pi_1 \tilde{C} = 0$ and $\pi_2 \tilde{C} = \pi_2 C = 0$ it follows from the Hurewicz Theorem, see [Spanier 1966, 7.5.2], [Stöcker-Zieschang 1994, 16.8.4], that $\pi_3 \tilde{C} = \pi_3 C = 0$. By the same argument $\pi_n C = \pi_n \tilde{C} = H_n(\tilde{C}) = 0$ for $n \geqq 4$.

This proves (a). To prove (b) assume that $\pi_1 C$ contains a non-trivial element x of finite order $m > 1$. The cyclic group generated by x defines a covering $p\colon \bar{C} \to C$ with $\pi_1 \bar{C} = \mathbb{Z}_m$. As $\pi_n \bar{C} = 0$ for $n > 1$ it follows that $\bar{C}$ is a $K(\mathbb{Z}_m, 1)$-space hence, $H_n(\bar{C}) = \mathbb{Z}_m$ for n odd, see [Maclane 1963, IV Theorem 7.1]. This contradicts the fact that $\bar{C}$ is a 3-manifold. □

G History and Sources

The knot groups became very early an important tool in knot theory. The method presenting groups by generators and defining relations has been developed by W. Dyck [1882], pursuing a suggestion of A. Cayley [1878]. The best known knot group presentations were introduced by W. Wirtinger; however, in the literature only the title "Über die Verzweigung bei Funktionen von zwei Veränderlichen" of this talk at the Jahresversammlung der Deutschen Mathematiker Vereinigung in Meran 1905 in Jahresber. DMV 14, 517 (1905) is mentioned. His student K. Brauner later used the Wirtinger presentations again in the study of singularities of algebraic surfaces in $\mathbb{R}^4$ and mentioned that these presentations were introduced by Wirtinger, see [Brauner 1928]. M. Dehn [1910] introduced the notion of a knot group and implicitly used the peripheral system to show that the two trefoils are inequivalent in [Dehn 1914]. (He used a different presentation for the knot group, see E 3.15.) O. Schreier [1924] classified the groups $\langle\, A, B \mid A^a B^b = 1 \,\rangle$ and determined their automorphism groups; this permitted to classify the torus knots. R.H. Fox [1952] introduced the peripheral system and showed its importance by distinguishing the square and the granny knot. These knots have isomorphic groups: there is, however, no isomorphism preserving the peripheral system.

Dehn's Lemma, the Loop and the Sphere Theorem, proved in [Papakyriakopoulos 1957, 1957′], opened new ways to knot theory, in particular, C.D. Papakyriakopoulos showed that knot complements are aspherical. F. Waldhausen [1968] found the full strength of the peripheral system, showing that it determines the knot complement and, hence, the knot type (see 15.5 and [Gordon-Luecke 1989]). New tools for the study and use of knot groups have been made available by R. Riley and W. Thurston discovering a hyperbolic structure in many knot complements.

H Exercises

E 3.1. Compute the relative homology $H_i(S^3, \mathfrak{k})$ for a knot $\mathfrak{k}$ and give a geometric interpretation of the generator of $H_2(S^3, \mathfrak{k}) \cong \mathbb{Z}$.

E 3.2. Calculate the homology $H_i(S^3 - \mathfrak{k})$ of the complement of a link $\mathfrak{k}$ with μ components.

E 3.3. Let $\mathfrak{k}$ be a knot with meridian m and longitude ℓ. Show:

(a) Attaching a solid torus with meridian m' to the complement of $\mathfrak{k}$ defines a homology sphere if and only if m' is mapped to $m^{\pm 1}\ell^r$.

(b) If $\mathfrak{k}$ is a torus knot then the fundamental group of the space obtained above is non-trivial if $r \neq 0$. (Hint: Use Proposition 3.28.)

E 3.4. Let $\mathfrak{G}' = \langle \{x_n, n \in \mathbb{Z}\} \mid \{x_{n+1}x_{n+2}^{-1}x_{n+1}x_{n+2}^{-1}x_n^{-1}x_{n+1}x_n^{-1}, n \in \mathbb{Z}\} \rangle$. Prove:

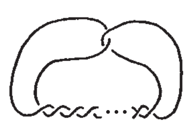
(a) $\mathfrak{G}'$ is not finitely generated.

(b) The subgroups $\mathfrak{A}_n = \langle x_n, x_{n+1}, x_{n+2} \rangle$, $\mathfrak{B}_n = \langle x_{n+1}, x_{n+2} \rangle$ of $\mathfrak{G}'$ are free groups of rank 2, and

$$\mathfrak{G}' = \cdots *_{\mathfrak{B}_{-2}} \mathfrak{A}_{-1} *_{\mathfrak{B}_{-1}} \mathfrak{A}_0 *_{\mathfrak{B}_0} \mathfrak{A}_1 *_{\mathfrak{B}_1} \cdots .$$

(For this exercise compare 3.9 and 4.6.)

E 3.5. Calculate the groups and peripheral systems of the knots in Figure 3.16.



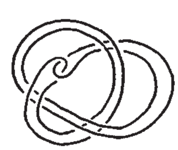


Figure 3.16

E 3.6. Express the peripheral system of a product knot in terms of those of the factor knots.

E 3.7. Let $\mathfrak{G}$ be a knot group and $\varphi\colon \mathfrak{G} \to \mathbb{Z}$ a non-trivial homomorphism. Then ker $\varphi = \mathfrak{G}'$.

E 3.8. Show that the two trefoil knots can be distinguished by their peripheral systems.

E 3.9. Prove Proposition 3.21.

E 3.10. Show that a regular neighbourhood V of the 1-skeleton of a triangulation of S^3 (or any closed orientable 3-manifold M) and $\overline{S^3 - V}$ ($\overline{M - V}$, respectively) form a Heegaard splitting of S^3 (or M).

E 3.11. Prove that $\mathfrak{F}_g *_{\mathfrak{F}_{2g-1}} \mathfrak{F}_g$ has a trivial centre for $g > 1$. (Here $\mathfrak{F}_g$ is the free group of rank g.)

E 3.12. Show Property P for torus knots.

E 3.13. Let $h\colon S^3 \to S^3$ be an orientation preserving homeomorphism with $h(\mathfrak{k}) = \mathfrak{k}$ for a knot $\mathfrak{k} \subset S^3$. Show that h induces an automorphism $h_*\colon \mathfrak{G}'/\mathfrak{G}'' \to \mathfrak{G}'/\mathfrak{G}''$ which commutes with $\alpha\colon \mathfrak{G}'/\mathfrak{G}'' \to \mathfrak{G}'/\mathfrak{G}''$, $x \mapsto t^{-1}xt$, where t represents a meridian of $\mathfrak{k}$.

E 3.14. Let V_1 and V_2 be solid tori with meridians m_1 and m_2. A homeomorphism $h\colon \partial V_1 \to \partial V_2$ can be extended to a homeomorphism $H\colon V_1 \to V_2$ if and only if $h(m_1) \sim m_2$ on ∂V_2.

E 3.15. (*Dehn presentation*) Derive from a regular knot projection a presentation of the knot group of the following kind: Assign a generator to each of the finite regions of the projection, and a defining relator to each double point.

Chapter 4
Commutator Subgroup of a Knot Group

There is no practicable procedure to decide whether two knot groups, given, say by Wirtinger presentations, are isomorphic. It has proved successful to investigate instead certain homomorphic images of a knot group $\mathfrak{G}$ or distinguished subgroups. The abelianized group $\mathfrak{G}/\mathfrak{G}' \cong H_1(C)$, though, is not helpful, since it is infinite cyclic for all knots, see 3.1. However, the commutator subgroup $\mathfrak{G}'$ together with the action of $\mathfrak{Z} = \mathfrak{G}/\mathfrak{G}'$ is a strong invariant which nicely corresponds to geometric properties of the knot complement; this is studied in Chapter 4. Another fruitful invariant is the metabelian group $\mathfrak{G}/\mathfrak{G}''$ which is investigated in the Chapters 8–9. All these groups are closely related to cyclic coverings of the complement.

A Construction of Cyclic Coverings

For the group $\mathfrak{G}$ of a knot $\mathfrak{k}$ the property $\mathfrak{G}/\mathfrak{G}' \cong \mathfrak{Z}$ implies that there are epimorphisms $\mathfrak{G} \to \mathfrak{Z}$ and $\mathfrak{G} \to \mathfrak{Z}_n$, $n \geq 2$, such that their kernels $\mathfrak{G}'$ and $\mathfrak{G}_n$ are characteristic subgroups of $\mathfrak{G}$, hence, invariants of $\mathfrak{k}$. Moreover, $\mathfrak{G}$ and $\mathfrak{G}_n$ are semidirect products of $\mathfrak{Z}$ and $\mathfrak{G}'$:

$$\mathfrak{G} = \mathfrak{Z} \ltimes \mathfrak{G}' \quad \text{and} \quad \mathfrak{G}_n = n\mathfrak{Z} \ltimes \mathfrak{G}',$$

where $n\mathfrak{Z}$ denotes the subgroup of index n in $\mathfrak{Z}$ and the operation of $n\mathfrak{Z}$ on $\mathfrak{G}'$ is the induced one.

The following Proposition 4.1 is a consequence of the general theory of coverings. However, in 4.4 we give an explicit construction and reprove most of 4.1.

4.1 Proposition and Definition (Cyclic coverings). *Let C denote the complement of a knot $\mathfrak{k}$ in S^3. Then there are regular coverings*

$$p_n \colon C_n \to C, \quad 2 \leq n \leq \infty,$$

such that $p_{n\#}(\pi_1 C_n) = \mathfrak{G}_n$ and $p_{\infty\#}(\pi_1 C_\infty) = \mathfrak{G}'$. The n-fold covering is uniquely determined.

The group of covering transformations is $\mathfrak{Z}$ for $p_\infty \colon C_\infty \to C$ and $\mathfrak{Z}_n$ for $p_n \colon C_n \to C$, $2 \leq n < \infty$.

The covering $p_\infty \colon C_\infty \to C$ is called the infinite cyclic covering, the coverings $p_n \colon C_n \to C$, $2 \leq n < \infty$, are called the finite cyclic coverings of the knot complement (or, inexactly, of the knot $\mathfrak{k}$). □

The main tool for the announced construction is the *cutting of the complement along a surface*; this process is inverse to pasting parts together.

4.2 Cutting along a surface. Let M be a 3-manifold and S a two-sided surface in M with $\partial S = S \cap \partial M$. Let U be a regular neighbourhood of S; then $U - S = U_1 \cup U_2$ with $U_1 \cap U_2 = \emptyset$ and $U_i \cong S \times (0, 1]$. Let M_0', U_1', U_2' be homeomorphic copies of $\overline{M - U}, \overline{U_1}, \overline{U_2}$, respectively, and let $f_0 \colon \overline{M - U} \to M_0'$, $f_i \colon \overline{U_i} \to U_i'$ be homeomorphisms. Let M' be obtained from the disjoint union $U_1' \cup M_0' \cup U_2'$ by identifying $f_0(x)$ and $f_i(x)$ when $x \in \overline{M - U} \cap \overline{U_i} = \partial(\overline{M - U}) \cap \partial U_i$, $i \in \{1, 2\}$. *The result M' is a 3-manifold* and we say that M' is *obtained by cutting M along S*. There is a natural mapping $j \colon M' \to M$.

Cutting along a one-sided surface can be described in a slightly more complicated way (Exercise E 4.1). The same construction can be done in other dimensions; in fact, the classification of surfaces is usually based on cuts of surfaces along curves, see Figure 4.1. A direct consequence of the definition is the following proposition.

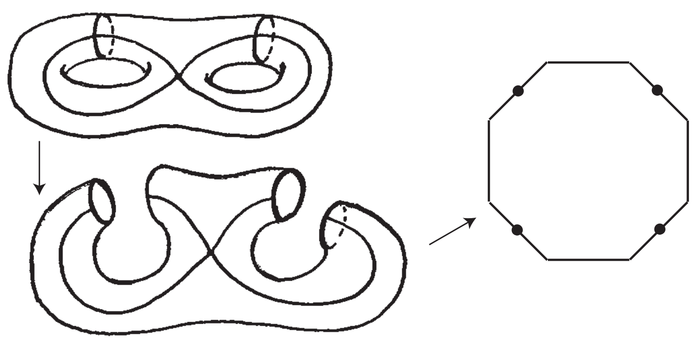


Figure 4.1

4.3 Proposition.

(a) *M' is a 3-manifold homeomorphic to $\overline{M - U} = M_0'$.*

(b) *There is an identification map $j \colon M' \to M$ which induces a homeomorphism $M' - j^{-1}(S) \to M - S$.*

(c) *The restriction $j \colon j^{-1}(S) \to S$ is a two-fold covering. When S is two-sided $j^{-1}(S)$ consists of two copies of S; when S is one-sided $j^{-1}(S)$ is connected.*

(d) *When S is two-sided an orientation of M' induces orientations on both components of $j^{-1}(S)$. They are projected by j onto opposite orientations of S, if M' is connected.* □

4.4 Construction of the cyclic coverings. The notion of cutting now permits a convenient description of the cyclic coverings $p_n \colon C_n \to C$: Let V be a regular neighbourhood of the knot $\mathfrak{k}$ and S' a Seifert surface. Assume that $V \cap S'$ is an annulus and that $\lambda = \partial V \cap S'$ is a simple closed curve, that is a longitude of $\mathfrak{k}$. Define $C = \overline{S^3 - V}$ and $S = S' \cap C$. Cutting C along S defines a 3-manifold C^*. The boundary of C^* is a connected surface and consists of two disjoint parts S^+ and S^-, both homeomorphic to S, and an annulus R which is obtained from the torus $\partial V = \partial C$ by cutting along λ:

$$\partial C^* = S^+ \cup R \cup S^-, \quad S^+ \cap R = \lambda^+, \quad S^- \cap R = \lambda^-, \quad \partial R = \lambda^+ \cup \lambda^-,$$

see Figure 4.2. (C^* is homeomorphic to the complement of a regular neighbourhood of the Seifert surface S.) Let $r\colon S^+ \to S^-$ be the homeomorphism mapping a point from S^+ to the point of S^- which corresponds to the same point of S. Let $i^+\colon S^+ \to C^*$ and $i^-\colon S^- \to C^*$ denote the inclusions.

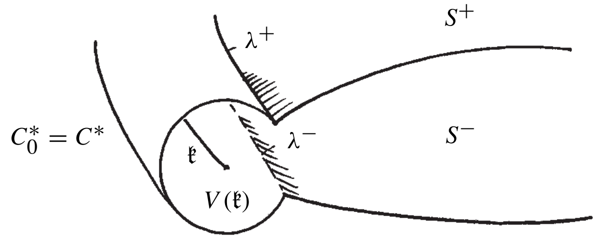


Figure 4.2

Take homeomorphic copies C_j^* of C^* ($j \in \mathbb{Z}$) with homeomorphisms $h_j\colon C^* \to C_j^*$. The topological space C_∞ is obtained from the disjoint union $\bigcup_{j=-\infty}^{\infty} C_j^*$ by identifying $h_j(x)$ and $h_{j+1}(r(x))$ when $x \in S^+$, $j \in \mathbb{Z}$; see Figure 4.3. The space C_n is defined by starting with $\bigcup_{j=0}^{n-1} C_j^*$ and identifying $h_j(x)$ with $h_{j+1}(r(x))$ and $h_n(x)$ with $h_1(r(x))$ when $x \in S^+$, $1 \leq j \leq n-1$. For $2 \leq n \leq \infty$ define $p_n(x) = \iota(h_i^{-1}(x))$ if $x \in C_\iota^*$; here ι denotes the identification mapping $C^* \to C$, see 4.3 (b). It easily follows that $p_n\colon C_n \to C$ is an n-fold covering.

By $t|C_j^* = h_{j+1}h_j^{-1}$, $j \in \mathbb{Z}$, a covering transformation $t\colon C_\infty \to C_\infty$ of the covering $p_\infty\colon C_\infty \to C$ is defined. For any two points $x_1, x_2 \in C_\infty$ with the same p_∞-image in C there is an exponent m such that $t^m(x_1) = x_2$. Thus the covering $p_\infty\colon C_\infty \to C$ is regular, the group of covering transformations is infinite cyclic and t generates it. Hence, $p_\infty\colon C_\infty \to C$ is *the* infinite cyclic covering of 4.1. In the same way it follows that $p_n\colon C_n \to C$ ($2 \leq n < \infty$) is *the* n-fold cyclic covering. The generating covering transformation t_n is defined by

$$t_n|C_j = h_{j+1}h_j^{-1} \quad \text{for } 1 \leq j \leq n-1,$$
$$t_n|C_n = h_1h_n^{-1}.$$

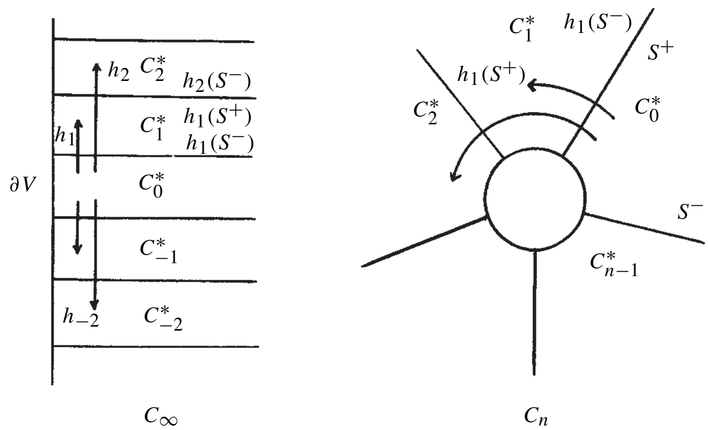


Figure 4.3

B Structure of the Commutator Subgroup

Using the Seifert–van Kampen Theorem the groups $\mathfrak{G}' = \pi_1 C_\infty$ and $\mathfrak{G}_n = \pi_1 C_n$ can be calculated from $\pi_1(C^*)$ and the homomorphisms $i_\#^\pm : \pi_1 S^\pm \to \pi_1 C^*$.

4.5 Lemma (Neuwirth). *When S is a Seifert surface of minimal genus spanning the knot $\mathfrak{k}$ the inclusions $i^\pm : S^\pm \to C^*$ induce monomorphisms $i_\#^\pm : \pi_1 S^\pm \to \pi_1 C^*$.*

Proof. If, e.g., $i_\#^+$ is not injective, then, by the Loop Theorem (see Appendix B.5) there is a simple closed curve ω on S^+, $\omega \not\simeq 0$ in S^+, and a disk $\delta \subset C$ such that $\partial\delta = \omega = \delta \cap \partial C = \delta \cap S^+$. Replace S^+ by $S_1^+ = (S^+ - U(\delta)) \cup \delta_1 \cup \delta_{-1}$, where

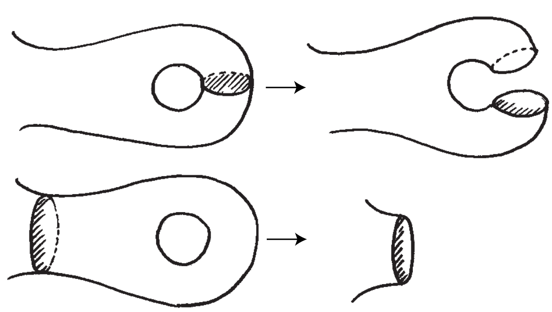


Figure 4.4

$U(\delta) = [-1, +1] \times \delta$ is a regular neighbourhood of δ in C with $\delta_i = i \times \delta, 0 \times \delta = \delta$. Then $g(S_1^+) + 1 = g(S^+)$, g the genus, contradicting the minimality of $g(S)$, if S_1^+

is connected. If not, the component of S_1^+ containing ∂S^+ has smaller genus than S^+, since $\omega \not\simeq 0$ in S^+; again this leads to a contradiction to the assumption on S. Compare Figure 4.4. □

Next we prove the main theorem of this chapter:

4.6 Theorem (Structure of the commutator subgroups). (a) *If the commutator subgroup $\mathfrak{G}'$ of a knot group $\mathfrak{G}$ is finitely generated, then $\mathfrak{G}'$ is a free group of rank $2g$ where g is the genus of the knot. In fact, $\mathfrak{G}' = \pi_1 S$, S a Seifert surface of genus g.*

(b) *If $\mathfrak{G}'$ cannot be finitely generated, then*

$$\mathfrak{G}' = \cdots \mathfrak{A}_{-1} *_{\mathfrak{B}_{-1}} \mathfrak{A}_0 *_{\mathfrak{B}_0} \mathfrak{A}_1 *_{B_1} \mathfrak{A}_2 \cdots$$

and the generator t of the group of covering transformations of $p_\infty\colon C_\infty \to C$ induces an automorphism τ of $\mathfrak{G}'$ such that $\tau(\mathfrak{A}_j) = \mathfrak{A}_{j+1}$, $\tau(\mathfrak{B}_j) = \mathfrak{B}_{j+1}$. Here $\mathfrak{A}_j \cong \pi_1 C^*$, $\mathfrak{B}_j \cong \pi_1 S \cong \mathfrak{F}_{2g}$ and $\mathfrak{B}_j$ is a proper subgroup of $\mathfrak{A}_j$ and $\mathfrak{A}_{j+1}$. (The subgroups $\mathfrak{B}_j$ and $\mathfrak{B}_{j+1}$ do not coincide.)

Proof. We apply the construction of 4.4, for a Seifert surface of minimal genus. By 4.5, the inclusions $i^\pm\colon S^\pm \to C_\infty$ induce monomorphisms $i_\#^\pm\colon \pi_1 S^\pm \to \pi_1 C_\infty$. By the Seifert–van Kampen Theorem (Appendix B.3), $\mathfrak{G}' = \pi_1 C_\infty$ is the direct $\lim_{n\to\infty} \mathfrak{P}_n$ of the following free products with amalgamation:

$$\mathfrak{P}_n = \mathfrak{A}_{-n} *_{\mathfrak{B}_{-n}} \mathfrak{A}_{-n+1} *_{\mathfrak{B}_{-n+1}} \cdots *_{\mathfrak{B}_0} \mathfrak{A}_1 *_{\mathfrak{B}_1} \mathfrak{A}_2 \cdots *_{\mathfrak{B}_{n-1}} \mathfrak{A}_n;$$

here $\mathfrak{A}_j$ corresponds to the sheet C_j^* and $\mathfrak{B}_j$ to $h_j(S^+)$ if considered as a subgroup of $\mathfrak{A}_j$ and to $h_{j+1}(S^-)$ as a subgroup of $\mathfrak{A}_{j+1}$. Thus for different j the pairs $(\mathfrak{A}_j, \mathfrak{B}_j)$ are isomorphic and the same is true for the pairs $(\mathfrak{A}_{j+1}, \mathfrak{B}_j)$.

When $\mathfrak{G}'$ is finitely generated there is an n such that the generators of $\mathfrak{G}'$ are in $\mathfrak{P}_n$. This implies that $\mathfrak{B}_n = \mathfrak{A}_{n+1}$ and $\mathfrak{B}_{-n-1} = \mathfrak{A}_{-n-1}$; hence, $\pi_1 S^+ \cong \pi_1 C^* \cong \pi_1 S^- \cong \mathfrak{F}_{2g}$ where g is the genus of S (and $\mathfrak{k}$). Now it follows that $\pi_1 C_\infty \cong \pi_1 C^* \cong \pi_1 S \cong \mathfrak{F}_{2g}$.

There remain the cases where $i_\#^+(\pi_1 S^+) \neq \pi_1 C^*$ or $i_\#^-(\pi_1 S^-) \neq \pi_1 C^*$. Then $\mathfrak{G}'$ cannot be generated by a finite system of generators. Lemma 4.7, due to [Brown-Crowell 1965], shows that these two inequalities are equivalent; hence, $i_\#^+(\pi_1 S^+) \neq \pi_1 C^* \neq i_\#^-(\pi_1 S^-)$, and now the situation is as described in (b). (That $\mathfrak{B}_j$ and $\mathfrak{B}_{j+1}$ do not coincide can be deduced using facts from the proof of Theorem 5.1). □

Section C is devoted to the proof of the Lemma 4.7 of [Brown-Crowell 1965] and can be neglected at first reading.

C A Lemma of Brown and Crowell

The following lemma is a special case of a result in [Brown-Crowell 1965]:

4.7 Lemma (Brown–Crowell). *Let M be an orientable compact 3-manifold where ∂M consists of two surfaces S^+ and S^- of genus g with common boundary*

$$\partial S^+ = \partial S^- = S^+ \cap S^- = \bigcup_{i=1}^{r} \kappa_i \neq \emptyset, \ \kappa_i \cap \kappa_j = \emptyset \quad \text{for } i \neq j.$$

If the inclusion $i^+ \colon S^+ \to M$ induces an isomorphism $i^+_\# \colon \pi_1 S^+ \to \pi_1 M$ so does $i^- \colon S^- \to M$.

Proof by induction on the Euler characteristic of the surface S^+. As $\partial S^+ \neq \emptyset$ the Euler characteristic $\chi(S^+)$ is maximal for $r = 1$ and $g = 0$; in this case $\chi(S^+) = 1$ and S^+ and S^- are disks, $\pi_1 S^-$ and $\pi_1 S^+$ are trivial; hence, $\pi_1 M$ is trivial too, and nothing has to be proved.

If $\chi(S^+) = \chi(S^-) < 1$ there is a simple arc α on S^- with $\partial\alpha = \{A, B\} = \alpha \cap \partial S^-$ which does not separate S^-, see Figure 4.5. We want to prove that there is an arc β on S^+ with the same properties such that $\alpha^{-1}\beta$ bounds a disk δ in M.

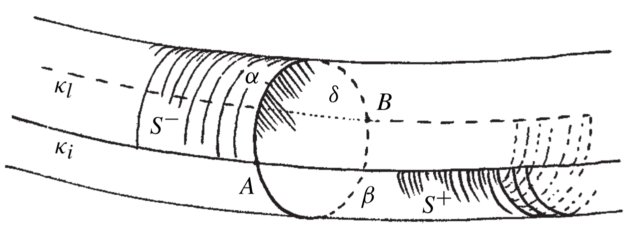


Figure 4.5

$i^+_\# \colon \pi_1 S^+ \to \pi_1 M$ is an isomorphism by assumption, thus there is an arc β' in S^+ connecting A and B such that $(\alpha, A, B) \simeq (\beta', A, B)$ in M. In general, the arc β' is not simple. The existence of a simple arc is proved using the following *doubling trick*: Let M_1 be a homeomorphic copy of M with $\partial M_1 = S_1^+ \cup S_1^-$. Let M' be obtained from the disjoint union $M \cup M_1$ by identifying S^+ and S_1^+ and let $\alpha_1 \subset M_1$ be the arc corresponding to α. In M', $\alpha\alpha_1^{-1} \simeq \beta'\beta'^{-1} \simeq 1$. By Dehn's Lemma (Appendix B.4), there is a disk δ' in M' with boundary $\alpha\alpha_1^{-1}$. We may assume that δ' is in general position with respect to $S^+ = S_1^+$ and that $\delta' \cap \partial M' = \partial\delta' = \alpha\alpha_1^{-1}$. The disk δ' intersects S^+ in a simple arc β connecting A and B and, perhaps, in a number of closed curves. The simple closed curve $\alpha\beta^{-1}$ is nullhomotopic in δ', hence in M'. By the Seifert–van Kampen Theorem,

$$\pi_1 M' = \pi_1 M *_{\pi_1 S^+} \pi_1 M_1 \cong \pi_1 M;$$

thus the inclusion $M \hookrightarrow M'$ induces an isomorphism $\pi_1 M \to \pi_1 M'$. Since $\alpha\beta^{-1}$ is contained in M it follows that $\alpha\beta^{-1} \simeq 0$ in M. By Dehn's Lemma, there is a disk $\delta \subset M$ with $\delta \cap \partial M = \partial\delta = \alpha \cup \beta$, see Figure 4.6.

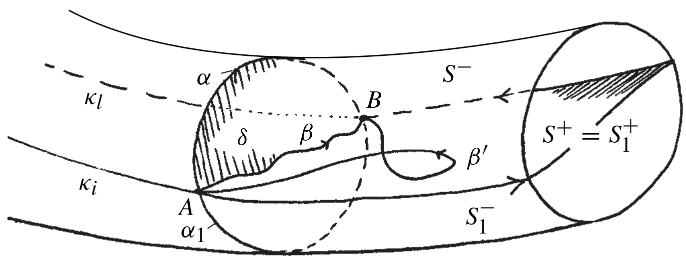


Figure 4.6

The arc β does not separate S^+. To prove this let C and D be points of S^+ close to β on different sides. There is an arc λ in M connecting C and D without intersecting δ; this is a consequence of the assumption that α does not separate S^-. Now deform λ into S^+ by a homotopy that leaves fixed C and D. The resulting path $\lambda' \subset S^+$ again connects C and D and has intersection number 0 with δ, the intersection number calculated in M; hence, also 0 with β when the calculation is done in S^+. This proves that β does not separate S^+.

Cut M along δ, see Figure 4.7. The result is a 3-manifold M_*. We prove that the boundary of M_* fulfils the assumptions of the lemma and that $\chi(\partial M_*) > \chi(\partial M)$. Then induction can be applied.

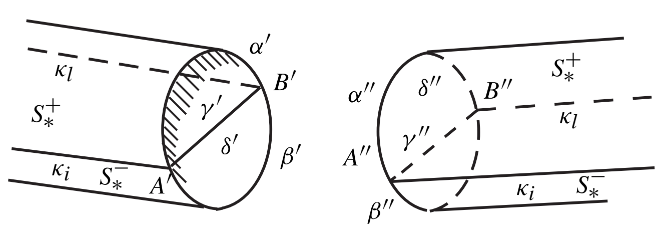


Figure 4.7

Assume that $A \in \kappa_i$, $B \in \kappa_\ell$. Let γ be a simple arc in δ such that $\gamma \cap \partial\delta = \partial\gamma = \{A, B\}$. By cutting M along δ, γ is cut into two arcs γ', γ'' which join the points A', B' and A'', B'' corresponding to A and B. The curves κ_i and κ_ℓ of ∂S^+ are replaced by one new curve κ_i' if $i \neq \ell$ or by two new curves $\kappa_{i,1}, \kappa_{i,2}$ if $i = \ell$. These new curves together with those κ_m that do not intersect δ decompose ∂M_* into two homeomorphic surfaces. They contain homeomorphic subsets S_*^+, S_*^- which result from removing the two copies of δ in ∂M_*. The surfaces S_*^+, S_*^- are obtained from

S^+ and S^- by cutting along $\partial\delta$. It follows that

$$\chi(S_*^+) = \chi(S^+) + 1,$$

since for $i = \ell$ the number r of boundary components increases by 1 and the genus decreases by 1: $r_* = r+1$, $g_* = g-1$, and for $i \neq \ell$ one has $r_* = r-1$ and $g_* = g$.

The inclusions and identification mappings form the following commutative diagrams:

$$\begin{tikzcd} S_*^+ \arrow[r, "j^+"] \arrow[d, hook, "i_*^+"'] & S^+ \arrow[d, hook, "i^+"] \\ M_* \arrow[r, "j"] & M \end{tikzcd} \qquad \begin{tikzcd} S_*^- \arrow[r, "j^-"] \arrow[d, hook, "i_*^-"'] & S^- \arrow[d, hook, "i^-"] \\ M_* \arrow[r, "j"] & M \end{tikzcd}$$

From the second version of the Seifert–van Kampen Theorem, see Appendix B.3 (b), [ZVC 1980, 2.8.3], [Stöcker-Zieschang 1994, 5.3.11], it follows that

$$\begin{aligned} &\pi_1 M = j_\#(\pi_1 M_*) * \mathfrak{Z}, \\ &\pi_1 S^+ = j_\#^+(\pi_1 S_*^+) * \mathfrak{Z}, \quad \pi_1 S^- = j_\#^-(\pi_1 S_*^-) * \mathfrak{Z}, \end{aligned}$$

where $\mathfrak{Z}$ is the infinite cyclic group generated by κ_i. By assumption the inclusion $i^+ \colon S^+ \to M$ induces an isomorphism $i_\#^+$ which maps $j_\#^+(\pi_1 S_*^+)$ to $j_\# i_{*\#}^+(\pi_1 S^+) \subset j_\#(\pi_1 M_*)$ and $\mathfrak{Z}$ onto $\mathfrak{Z}$. From the solution of the word problem in free products, see [ZVC 1980, 2.3], it follows that i_*^+ bijectively maps $j_\#^+(\pi_1 S_*^+)$ onto $j_\#(\pi_1 M_*)$; hence, $i_{*\#}^+$ is an isomorphism.

As induction hypothesis we may assume that $i_{*\#}^-$ is an isomorphism. By arguments similar to those above, it follows that $i_\#^-$ can be described by the following commutative diagram:

$$\begin{tikzcd} j_\#^-(\pi_1 S_*^-) * \mathfrak{Z} \arrow[r, phantom, "="] \arrow[d, "i_{*\#}^- * (i_\#^- | \mathfrak{Z})"', "\cong"] & \pi_1 S^- \arrow[d, "i_\#^-"] \\ j_\#(\pi_1 M_*) * \mathfrak{Z} \arrow[r, phantom, "="] & \pi_1 M. \end{tikzcd}$$

Since the mapping on the left side is bijective, $i_\#^-$ is an isomorphism. □

D Examples and Applications

Theorem 4.6 now throws some light on the results in 3.7–3.9: the trefoil (E 4.2) and the figure eight knot (Figure 3.8) have finitely generated commutator subgroups. The 2-bridge knot $\mathfrak{b}(7,3)$ has a commutator subgroup of infinite rank; in 3.9 we have already calculated $\mathfrak{G}'$ in the form of 4.6 (b) using the Reidemeister–Schreier method.

We will prove that all torus knots have finitely generated commutator subgroups. Let us begin with some consequences of Theorem 4.6.

4.8 Corollary. *Let the knot $\mathfrak{k}$ have a finitely generated commutator subgroup and let S be an orientable surface spanning $\mathfrak{k}$. If S is incompressible in the knot complement* (*this means that the inclusion $i\colon S \hookrightarrow C$ induces a monomorphism $i_\#\colon \pi_1 S \to \pi_1 C = \mathfrak{G}$*) *then S and $\mathfrak{k}$ have the same genus.* □

In the following $\mathfrak{G}$ always denotes a knot group.

4.9 Corollary. *The centre of $\mathfrak{G}'$ is trivial.*

Proof. If $\mathfrak{G}'$ cannot be finitely generated, by 4.6 there are groups $\mathfrak{A}$ and $\mathfrak{B}$ with $\mathfrak{G}' = \mathfrak{A} *_{\mathfrak{F}_{2g}} \mathfrak{B}$ where g is the genus of $\mathfrak{k}$ and $\mathfrak{A} \neq \mathfrak{F}_{2g} \neq \mathfrak{B}$. From the solution of the word problem it follows that the centre is contained in the amalgamated subgroup and is central in both factors, see [ZVC 1980, 2.3.9]. But $\mathfrak{F}_{2g}$ has trivial centre ([ZVC 1980, E1.5]). The last argument also applies to finitely generated $\mathfrak{G}'$ because they are free groups. □

4.10 Proposition. (a) *If the centre $\mathfrak{C}$ of $\mathfrak{G}$ is non-trivial then $\mathfrak{G}'$ is finitely generated.*

(b) *The centre $\mathfrak{C}$ of $\mathfrak{G}$ is trivial or infinite cyclic. When $\mathfrak{C} \neq 1$, $\mathfrak{C}$ is generated by an element $t^n \cdot u$, $n > 1$, $u \in \mathfrak{G}'$.* (*The coset $t\mathfrak{G}'$ generates the first homology group $\mathfrak{G}/\mathfrak{G}' \cong \mathfrak{Z}$.*)

Proof. (a) Assume that $\mathfrak{G}'$ cannot be generated by finitely many elements. Then, by Theorem 4.6, $\mathfrak{G}' = \cdots * \mathfrak{A}_{-1} *_{\mathfrak{B}_{-1}} \mathfrak{A}_0 *_{\mathfrak{B}_0} \mathfrak{A}_1 * \cdots$ where $\mathfrak{A}_j \not\supseteq \mathfrak{B}_j \not\subseteq \mathfrak{A}_{j+1}$. Denote by $\mathfrak{H}_r$ the subgroup of $\mathfrak{G}'$ which is generated by $\{\mathfrak{A}_j \mid j \leqq r\}$. Then $\mathfrak{H}_{r+1} = \mathfrak{H}_r *_{\mathfrak{B}_r} \mathfrak{A}_{r+1}$ and $\mathfrak{H}_r \neq \mathfrak{B}_r \neq \mathfrak{A}_{r+1}$; hence $\mathfrak{H}_r \neq \mathfrak{H}_{r+1}$ and

$$\mathfrak{H}_r \neq \mathfrak{H}_s \quad \text{if } r < s. \tag{1}$$

Let $t \in \mathfrak{G}$ be an element which is mapped onto a generator of $\mathfrak{G}/\mathfrak{G}' = \mathfrak{Z}$. Assume that $t^{-1}\mathfrak{A}_r t = \mathfrak{A}_{r+1}$; hence, $t^{-1}\mathfrak{H}_r t = \mathfrak{H}_{r+1}$.

Consider $z \in \mathfrak{C}$, $1 \neq z$. Then $z = ut^m$ where $u \in \mathfrak{G}'$. By 4.9, $m \neq 0$; without loss of generality: $m > 0$. Choose s such that $u \in \mathfrak{H}_s$. Then

$$\mathfrak{H}_s = z^{-1}\mathfrak{H}_s z$$

since $z \in \mathfrak{C}$, and

$$z^{-1}\mathfrak{H}_s z = t^{-m}u^{-1}\mathfrak{H}_s u t^m = t^{-m}\mathfrak{H}_s t^m = \mathfrak{H}_{s+m}.$$

This implies $\mathfrak{H}_s = \mathfrak{H}_{s+m}$, contradicting (1).

(b) By (a), a non-trivial centre $\mathfrak{C}$ contains an element $t^n \cdot u$, $n > 0$, $u \in \mathfrak{G}'$ and n minimal. By 4.9,

$$\mathfrak{C}\mathfrak{G}' \cong n\mathfrak{Z} \times \mathfrak{G}',$$
$$\mathfrak{C}\mathfrak{G}'/\mathfrak{G}' \cong \mathfrak{C}/\mathfrak{C} \cap \mathfrak{G}' \cong \mathfrak{C} \cong n\mathfrak{Z}.$$

If $n = 1$ then $\mathfrak{G} = \mathfrak{C} \times \mathfrak{G}'$ which contradicts the fact that $\mathfrak{G}$ collapses if the relator $t = 1$ is introduced. □

Since the group of a torus knot has non-trivial centre we have proved the first statement of the following theorem:

4.11 Corollary (Genus of torus knots). (a) *The group* $\mathfrak{G}_{a,b} = \langle\, x, y \mid x^a y^{-b}\,\rangle$ *of the torus knot* $\mathfrak{t}(a,b)$, $a, b \in \mathbb{N}$, $(a,b) = 1$ *has a finitely generated commutator subgroup. It is, following* 4.6 (a), *a free group of rank* $2g$ *where* g *is the genus of* $\mathfrak{t}(a,b)$.

(b) $g = \frac{(a-1)\cdot(b-1)}{2}$.

Proof. It remains to prove (b). Consider the commutative diagram

$$\begin{tikzcd} \mathfrak{G}_{a,b} \arrow[r, "\varphi"] \arrow[d, "\lambda"'] & \langle t \mid \rangle \arrow[d, "\kappa"] \\ \mathfrak{Z}_a * \mathfrak{Z}_b \arrow[r, "\psi"] & \langle t \mid t^{ab} \rangle \end{tikzcd} \qquad t = \varphi(x^r y^s),\ as + br = 1$$

where φ, ψ are the abelianizing homomorphisms, λ and κ the natural projections. The centre $\mathfrak{C}$ of $\mathfrak{G}_{a,b}$ is generated by $x^a = y^b$, and it is $\mathfrak{C} = \ker\lambda$. Now

$$\begin{aligned} &\ker(\psi\lambda) = \lambda^{-1}(\ker\psi) \cong \mathfrak{C} \times \ker\psi \\ &\quad \| \\ &\ker(\kappa\varphi) = \varphi^{-1}(\langle\, t^{ab}\,\rangle) \cong \mathfrak{C} \times \ker\varphi; \end{aligned}$$

the last isomorphism is a consequence of

$$t^{ab} = \varphi((x^r y^s)^{ab}) = \varphi(x^{rab} \cdot x^{a^2 s}) = \varphi(x^a).$$

Hence, $\ker\varphi \cong \ker\psi$.

We prove next that $(\mathfrak{Z}_a * \mathfrak{Z}_b)' = \ker\varphi \cong \mathfrak{F}_{(a-1)(b-1)}$. Consider the 2-complex C^2 consisting of one vertex, two edges ξ, η and two disks δ_1, δ_2 with the boundaries ξ^a and η^b, respectively. Then $\pi_1 C^2 \cong \mathfrak{Z}_a * \mathfrak{Z}_b$. Let $\tilde{C}^2$ be the covering space of C^2 with fundamental group the commutator subgroup of $\mathfrak{Z}_a * \mathfrak{Z}_b$. Each edge of $\tilde{C}^2$ over η (or ξ) belongs to the boundaries of exactly b (resp. a) disks of $\tilde{C}^2$ which have the same boundary. It suffices to choose one to get a system of defining relations of $\pi_1\tilde{C}^2 \cong (\mathfrak{Z}_a * \mathfrak{Z}_b)'$. Then there are $\frac{ab}{b}$ disks over δ_2 and $\frac{ab}{a}$ disks over δ_1. The new complex $\hat{C}^2$ contains

$$ab \text{ vertices}, \quad 2ab \text{ edges}, \quad a+b \text{ disks},$$

and each edge is in the boundary of exactly one disk of $\hat{C}^2$. Thus $\pi_1\hat{C}^2$ is a free group of rank

$$2ab - (ab-1) - (a+b) = (a-1)(b-1).$$

Theorem 4.6 implies that the genus of $\mathfrak{t}(a,b)$ is $\frac{1}{2}(a-1)(b-1)$. □

The isomorphism $(\mathfrak{Z}_a * \mathfrak{Z}_b)' \cong \mathfrak{F}_{(a-1)(b-1)}$ can also be proved using the (modified) Reidemeister–Schreier method, see [ZVC 1980, 2.2.8]; in the proof above the geometric background of the algebraic method has directly been used.

E Commutator Subgroups of Satellites

According to 3.11, the groups of a satellite $\mathfrak{k}$, its companion $\hat{\mathfrak{k}}$ and the pattern $\tilde{\mathfrak{k}} \subset \tilde{V}$ are related by $\mathfrak{G} = \hat{\mathfrak{G}} *_{\mathfrak{A}} \pi_1(\hat{V} - \mathfrak{k}) \cong \hat{\mathfrak{G}} *_{\mathfrak{A}} \pi_1(\tilde{V} - \tilde{\mathfrak{k}}) = \hat{\mathfrak{G}} *_{\mathfrak{A}} \mathfrak{H}$, where $\mathfrak{A} = \pi_1(\partial \hat{V}) \cong \mathbb{Z}^2$ and $\mathfrak{H} = \pi_1(\tilde{V} - \tilde{\mathfrak{k}})$. For the calculation of $\mathfrak{G}'$ we need a refined presentation, which we will also use in Chapter 9 for the calculation of Alexander polynomials of satellites.

4.12 Presentation of the commutator subgroup of a satellite. Let $\tilde{m}$ and $\tilde{\ell}$ be meridian and longitude of $\tilde{V}$ where $\tilde{\ell}$ is a meridian of $\overline{S^3 - \tilde{V}}$. Starting with a Wirtinger presentation for the link $\tilde{\mathfrak{k}} \cup \tilde{m}$ and after replacing all meridional generators of $\tilde{\mathfrak{k}}$ except t by elements of $\mathfrak{H}'$ one obtains a presentation

$$\begin{aligned} \mathfrak{H} = \pi_1(\tilde{V} - \tilde{\mathfrak{k}}) &= \langle\, t, \tilde{u}_i, \hat{\lambda} \mid \tilde{R}_j(\tilde{u}_i^{t^\nu}, \hat{\lambda})\,\rangle \\ &= \langle\, t, \hat{t}, \tilde{u}_i, \hat{\lambda} \mid \tilde{R}_j(\tilde{u}_i^{t^\nu}, \hat{\lambda}),\ \hat{t}^{-1} \cdot t^n \tilde{v}(\tilde{u}_i^{t^\nu},\ \hat{\lambda}),\ [\hat{t}, \hat{\lambda}]\,\rangle \end{aligned} \tag{1}$$

where $\tilde{u}_i \in \mathfrak{H}'$, $\tilde{u}_i^{t^\nu} = t^\nu \tilde{u}_i t^{-\nu}$, $\nu \in \mathbb{Z}$, $i \in I$, $j \in J$; I, J finite sets. The $\hat{t}$ represents a meridian of $\hat{V}$ on $\partial \hat{V}$, $\hat{t} = t^n \cdot \tilde{v}(\tilde{u}_i^{t^\nu}, \hat{\lambda})$ with $\tilde{v}(\tilde{u}_i^{t^\nu}, \hat{\lambda}) \in \mathfrak{G}'$ and $n = \mathrm{lk}(\hat{t}, \hat{\mathfrak{k}})$. The generator $\hat{\lambda}$ represents the longitude $\hat{\ell}$, hence $\hat{\lambda} \in \hat{\mathfrak{G}}''$. The relation $[\hat{t}, \hat{\lambda}]$ is a consequence of the remaining relations. The group of the knot $\tilde{\mathfrak{k}}$ is:

$$\begin{aligned} \tilde{\mathfrak{G}} = \pi_1(\tilde{S}^3 - \tilde{\mathfrak{k}}) &= \langle\, t, \tilde{u}_i \mid \tilde{R}_j(\tilde{u}_i^{t^\nu}, 1)\,\rangle \\ &= \langle\, t, \tilde{u}_i, \tilde{\lambda} \mid R_j(\tilde{u}_i^{t^\nu},\ \hat{\lambda}),\ \hat{\lambda}\,\rangle. \end{aligned} \tag{2}$$

The group of the companion has a presentation

$$\hat{\mathfrak{G}} = \pi_1(S^3 - \hat{\mathfrak{k}}) = \langle\, \hat{t}, \hat{u}_k, \hat{\lambda} \mid \hat{R}_\iota(\hat{u}_k^{\hat{t}^\nu}),\ \hat{\lambda}^{-1} \cdot \hat{w}(\hat{u}_k^{\hat{t}^\nu}),\ [\hat{t}, \hat{\lambda}]\,\rangle \tag{3}$$

for $\hat{u}_k \in \hat{\mathfrak{G}}'$ and some $\hat{w}$. By assumption $\hat{t}, \hat{\lambda}$ generate a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ in $\mathfrak{H}$ as well as in $\hat{\mathfrak{G}}$ since $\hat{\mathfrak{k}}$ is not trivial. By the Seifert–van Kampen theorem

$$\begin{aligned} \mathfrak{G} = \pi_1(S^3 - \mathfrak{k}) &= \hat{\mathfrak{G}} *_{\pi_1(\partial \hat{V})} \pi_1(\hat{V} - \hat{\mathfrak{k}}) = \hat{\mathfrak{G}} *_{\langle \hat{t}, \hat{\lambda} \rangle} \mathfrak{H} \\ &\cong \langle\, t, \tilde{u}_i, \hat{t}, \hat{u}_k, \hat{\lambda} \mid \tilde{R}_j(\tilde{u}_i^{t^\nu}, \hat{\lambda}),\ \hat{t}^{-1} \cdot t^n \tilde{v}(\tilde{u}_i^{t^\nu}, \hat{\lambda}),\ \hat{R}_\iota(\hat{u}_k^{\hat{t}^\nu}), \hat{\lambda}^{-1} \cdot \hat{w}(\hat{u}_k^{\hat{t}^\nu}), [\hat{t}, \hat{\lambda}]\,\rangle, \end{aligned} \tag{4}$$

a result already obtained in 3.11.

To determine $\mathfrak{G}'$ we drop the generator $\hat{t}$ using the relation $\hat{t} = t^n \tilde{v}(\tilde{u}_i^{t^\nu}, \hat{\lambda})$; however, we will still write $\hat{t}$ for the expression on the right side. Now

$$\mathfrak{G}' = \langle\, \tilde{u}_i^{t^\varrho}, \hat{u}_k^{t^\varrho}, \hat{\lambda}^{t^\varrho} \mid \tilde{R}_j^{t^\varrho}(\tilde{u}_i^{t^\nu} \cdot \hat{\lambda}),\ \hat{R}_\iota^{t^\varrho}(\hat{u}_k^{t^\nu}),\ (\hat{\lambda}^{t^\varrho})^{-1} \cdot \hat{w}^{t^\varrho}(\hat{u}_k^{t^\nu}),\ [\hat{t}^{t^\varrho}, \hat{\lambda}^{t^\varrho}]\,\rangle$$

where ϱ ranges over $\mathbb{Z}$, $\tilde{u}_i^{t^\varrho} = t^\varrho \tilde{u}_i t^{-\varrho}$, $\tilde{R}_j^{t^\varrho}(\tilde{u}_i^{t^\nu}, \hat{\lambda}) = t^\varrho \tilde{R}_j(\tilde{u}_i^{t^\nu}, \hat{\lambda}) t^{-\varrho}$ etc. For $n > 0$ write

$$\varrho = \mu + \sigma \cdot n, \quad 0 \leqq \mu < n,$$

and

$$t^\varrho = t^{\sigma n} t^\mu = \tilde{v}_\sigma(\tilde{u}_i^{t^\nu}, \hat{\lambda}) \hat{t}^\sigma t^\mu.$$

Define $\hat{u}_{\mu,k} = t^\mu \hat{u}_k t^{-\mu}$, $\hat{\lambda}_\mu = t^\mu \hat{\lambda} t^{-\mu}$. Now

$$\begin{aligned} \mathfrak{G}' = \langle\, \tilde{u}_i^{t^\varrho}, \hat{u}_{\mu,k}^{\hat{t}^\sigma}, \hat{\lambda}_\mu^{\hat{t}^\sigma} \mid \tilde{R}_j^{t^\varrho}(\tilde{u}_i^{t^\nu}, \hat{\lambda}),\ \hat{R}_\iota^{\hat{t}^\sigma}(\hat{u}_{\mu,k}^{\hat{t}^\nu}), \\ (\hat{\lambda}_\mu^{\hat{t}^\sigma})^{-1} \hat{w}^{\hat{t}^\sigma}(\hat{u}_{\mu,k}^{\hat{t}^\nu}),\ \hat{t}^\sigma[\hat{t}^{t^\mu}, \hat{\lambda}_\mu]\hat{t}^{-\sigma}\,\rangle; \end{aligned} \tag{5}$$

here $\sigma \in \mathbb{Z}$ and $0 \leqq \mu < n$.

On the other hand,

$$\begin{aligned} \hat{\mathfrak{G}}' &= \langle\, \hat{u}_k^{\hat{t}^\sigma}, \hat{\lambda}^{\hat{t}^\sigma} \mid \hat{R}_\iota^{\hat{t}^\sigma}(\hat{u}_k^{\hat{t}^\nu}),\ (\hat{\lambda}^{\hat{t}^\sigma})^{-1} \cdot \hat{w}^{\hat{t}^\sigma}(\hat{u}_k^{\hat{t}^\nu}),\ [\hat{t}, \hat{\lambda}^{\hat{t}^\sigma}]\,\rangle \\ &= \langle\, \hat{u}_k^{\hat{t}^\sigma}, \hat{\lambda} \mid \hat{R}_\iota^{\hat{t}^\sigma}(\hat{u}_k^{\hat{t}^\nu}),\ (\hat{\lambda}^{\hat{t}^\sigma})^{-1} \cdot \hat{w}^{\hat{t}^\sigma}(\hat{u}_k^{\hat{t}^\nu})\,\rangle \end{aligned} \tag{6}$$

since the relation $[\hat{t}, \hat{\lambda}^{\hat{t}^\sigma}]$ implies that $\hat{\lambda}^{\hat{t}^\sigma} = \hat{\lambda}^{\hat{t}^{\sigma+1}}$. By conjugation with t^μ we obtain

$$\hat{\mathfrak{G}}'^{t^\mu} = t^\mu \hat{\mathfrak{G}}' t^{-\mu} = \langle\, \hat{u}_{\mu,k}^{\hat{t}^\sigma}, \hat{\lambda}_\mu \mid \hat{R}_\iota^{\hat{t}^\sigma}(\hat{u}_{\mu,k}^{\hat{t}^\nu}),\ (\hat{\lambda}_\mu^{\hat{t}^\sigma})^{-1} \cdot \hat{w}^{\hat{t}^\sigma}(\hat{u}_{\mu,k}^{\hat{t}^\nu})\,\rangle. \tag{6μ}$$

Define

$$\mathfrak{K} = \langle\, \tilde{u}_i^{t^\varrho}, \hat{\lambda}^{t^\mu} \mid \tilde{R}_j^{t^\varrho}(\tilde{u}_i^{t^\nu}, \hat{\lambda})\,\rangle = \langle\, \tilde{u}_i^{t^\varrho}, \hat{\lambda}_\mu \mid \tilde{R}_j^{t^\varrho}(\tilde{u}_i^{t^\nu}, \hat{\lambda}_\mu)\,\rangle. \tag{7}$$

Since the presentations of $\hat{\mathfrak{G}}', \hat{\mathfrak{G}}'^t, \dots, \hat{\mathfrak{G}}'^{t^{n-1}}$ have disjoint sets of generators, it follows from (6) that

$$\langle\, \hat{u}_k^{t^\sigma}, \hat{\lambda}_\mu \mid \hat{R}_\iota^{\hat{t}^\sigma}(\hat{u}_{\mu,k}^{\hat{t}^\nu}),\ (\hat{\lambda}_\mu^{\hat{t}^\sigma})^{-1} \cdot \hat{w}^{\hat{t}^\sigma}(\hat{u}_{\mu,k}^{\hat{t}^\nu})\,\rangle = \hat{\mathfrak{G}}' * \hat{\mathfrak{G}}'^t * \dots * \hat{\mathfrak{G}}'^{t^{n-1}} \tag{8}$$

and that $\hat{\lambda}_0, \dots, \hat{\lambda}_{n-1}$ generate a free group of rank n. Moreover,

$$\langle\, \hat{\lambda}^{t^\varrho} \mid \varrho \in \mathbb{Z}\,\rangle = \langle\, \hat{\lambda}_0, \dots, \hat{\lambda}_{n-1}\,\rangle, \tag{9}$$

as follows from the commutator relations $[\hat{t}, \hat{\lambda}]^{t^\varrho}$. If $n = 0$ then $\langle\, \hat{\lambda}^{t^\varrho}\,\rangle$ is of infinite rank. Now (5), (7) and (8) imply that

$$\mathfrak{G}' = \mathfrak{K} *_{\langle \hat{\lambda}^{t^\varrho} \rangle} (\hat{\mathfrak{G}}' * \hat{\mathfrak{G}}'^t * \dots * \hat{\mathfrak{G}}'^{t^{n-1}}). \tag{10}$$

4.13 Lemma. *For $n \neq 0$, $\mathfrak{G}'$ is finitely generated if and only if $\mathfrak{K}$ and $\hat{\mathfrak{G}}'$ are finitely generated.*

Proof. This is a consequence of

$$\operatorname{rank}\left(\hat{\mathfrak{G}}' * \cdots * \hat{\mathfrak{G}}'^{t^{n-1}}\right) = n \cdot \operatorname{rank} \hat{\mathfrak{G}}'$$

and the following Lemma 4.14. □

4.14 Lemma. *Let $\mathfrak{G} = \mathfrak{G}_1 *_{\mathfrak{S}} \mathfrak{G}_2$ where $\mathfrak{S}$ is finitely generated. Then $\mathfrak{G}$ is finitely generated if and only if $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are finitely generated.*

Proof (R. Bieri). When $\mathfrak{G}$ is finitely generated there are finite subsets $X_i \subset \mathfrak{G}_i$ $(i = 1, 2)$ with $\langle X_1, X_2 \rangle = \mathfrak{G}$. Since $\mathfrak{S}$ is finitely generated we may assume that both X_1 and X_2 contain generators for $\mathfrak{S}$. Let $\mathfrak{H}_i = \langle X_i \rangle \subset \mathfrak{G}_i$. Then $\mathfrak{S} = \mathfrak{G}_1 \cap \mathfrak{G}_2 \supset \mathfrak{H}_1 \cap \mathfrak{H}_2$, but on the other hand $\mathfrak{H}_1 \cap \mathfrak{H}_2 \supset \mathfrak{S}$, so that $\mathfrak{S} = \mathfrak{G}_1 \cap \mathfrak{G}_2 = \mathfrak{H}_1 \cap \mathfrak{H}_2$. It follows that the map $\mathfrak{H}_1 *_{\mathfrak{S}} \mathfrak{H}_2 \to \mathfrak{G}_1 *_{\mathfrak{S}} \mathfrak{G}_2$ induced by the embeddings $\mathfrak{H}_i \to \mathfrak{G}_i$ is an isomorphism. Now the solution of the word problem implies that $\mathfrak{H}_i = \mathfrak{G}_i$. □

4.15 Corollary. *If $\mathfrak{G}'$ is finitely generated, then $n \neq 0$ and $\tilde{\mathfrak{G}}'$ and $\hat{\mathfrak{G}}'$ are finitely generated. If $n \neq 0$ and $\mathfrak{K}$, see* (7), *and $\hat{\mathfrak{G}}'$ are finitely generated, so is $\mathfrak{G}'$.*

Proof. If $n = 0$, then $\mathfrak{G}'$ contains the subgroup $\langle \hat{t}, \hat{\lambda} \rangle \cong \mathbb{Z} \oplus \mathbb{Z}$, so $\mathfrak{G}'$ cannot be finitely generated, because this would, by 4.6 (a), imply that $\mathfrak{G}'$ is free, a contradiction. For the remaining assertions see Lemma 4.13 and look at the presentation (1) of $\mathfrak{H}$. The relators $\tilde{R}_j(\tilde{u}_i^{t^\nu})$ can be split into a set $\tilde{Q}_j(\tilde{u}_i^{t^\nu})$ not containing $\hat{\lambda}$, and a relator of the form $[\hat{\lambda}, \tilde{v}_s(\tilde{u}_i^{t^\nu})]$:

$$\mathfrak{H} = \langle t, \tilde{u}_i, \tilde{\lambda} \mid \tilde{Q}_r(\tilde{u}_i^{t^\nu}),\ [\tilde{\lambda}, \tilde{v}_s(\tilde{u}_i^{t^\nu}]\,\rangle.$$

The augmentation $\varphi\colon \mathfrak{G} \to \mathbb{Z}$ induces a homomorphism

$$\varphi\colon \mathfrak{H} \to \mathbb{Z} \quad \text{with } t \mapsto 1,\ \tilde{u}_i, \tilde{\lambda} \mapsto 0$$

and

$$\ker \varphi = \langle \tilde{u}_i^{t^\varrho}, \tilde{\lambda}^{t^\varrho} \mid \tilde{Q}_r^{t^\varrho}(\tilde{u}_i^{t^\nu}),\ [\tilde{\lambda}, \tilde{v}_s(\tilde{u}_i^{t^\nu})]^{t^\varrho} \rangle = \mathfrak{K}.$$

Moreover,

$$\tilde{\mathfrak{G}} = \langle t, \tilde{u}_i, \tilde{\lambda} \mid \tilde{Q}_r(\tilde{u}_i^{t^\nu}),\ \tilde{\lambda} \rangle$$

and

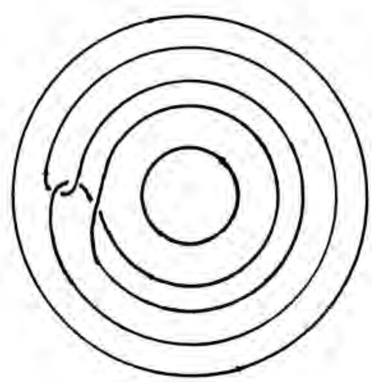
$$\tilde{\mathfrak{G}}' = \langle \tilde{u}_i^{t^\varrho}, \tilde{\lambda}^{t^\varrho} \mid \tilde{Q}_r^{t^\varrho}(\tilde{u}_i^{t^\nu}),\ \tilde{\lambda}^{t^\varrho} \rangle.$$

Consider the canonical homomorphism $\psi\colon \mathfrak{H} \to \tilde{\mathfrak{G}}$ with $\ker\psi$ the normal closure of $\hat{\lambda}$ in $\mathfrak{H}$. One has $\ker\psi \subset \mathfrak{K}$, and

$$1 \to \ker\psi \to \mathfrak{K} \to \tilde{\mathfrak{G}}' \to 1$$

is exact. For $n \neq 0$, $\ker \psi$ is the normal closure of $\langle \hat{\lambda}_0, \ldots, \hat{\lambda}_{n-1} \rangle$, see (9). Hence $\mathfrak{G}'$ is finitely generated, if $\tilde{\mathfrak{G}}'$ and $\ker \psi$ are. $\square$

Remark. In the first edition of this book it was wrongly assumed that $\ker \psi$ was always finitely generated. D. Silver pointed out the mistake and he supplied the following counterexample: No satellite with pattern $\tilde{\mathfrak{k}}$ (Figure 4.8) has a finitely generated commutator subgroup $\mathfrak{G}'$ since $\mathfrak{K}$ is not finitely generated even if $\tilde{\mathfrak{G}}'$ and $\hat{\mathfrak{G}}'$ are.



$\tilde{\mathfrak{k}}$

Figure 4.8

F History and Sources

The study of the commutator subgroup $\mathfrak{G}'$ concentrated on $\mathfrak{G}'/\mathfrak{G}''$ in the early years of knot theory. This will be the object of Chapters 8, 9. In [Reidemeister 1932, § 6] there is a group presentation of $\mathfrak{G}'$. But the structure of $\mathfrak{G}'$ eluded the purely algebraic approach.

Neuwirth made the first important step by investigating the infinite cyclic covering space C_∞, $\pi_1 C_\infty = \mathfrak{G}'$, using the then (relatively) new tools *Dehn's Lemma* and *Loop Theorem* [Neuwirth 1960]: Lemma 4.6. The analysis of $\mathfrak{G}'$ resulted in splitting off a special class of knots, whose commutator subgroups are finitely generated. In this case $\mathfrak{G}'$ proves to be a free group of rank $2g$, g the genus of the knot. These knots will be treated separately in the next chapter. There remained two different possible types of infinitely generated commutator groups in Neuwirth's analysis, and it took some years till one of them could be excluded [Brown-Crowell 1965]: Lemma 4.7. The remaining one, an infinite free product with amalgamations does occur. This group is rather complicated and its structure surely could do with some further investigation.

G Exercises

E 4.1. Describe the process of cutting along a one-sided surface.

E 4.2. Prove that the commutator subgroup of the group of the trefoil is free of rank 2.

E 4.3. Prove that the commutator subgroup of the group of the knot 6_1 cannot be finitely generated.

If the bands of a Seifert surface spanning $\mathfrak{k}$ form a plat (Figure 4.9), we call $\mathfrak{k}$ a *braid-like* knot (compare 8.2).

E 4.4. Show that for a braid-like knot the group $\mathfrak{A} = \pi_1 C^*$ is always free. (For the notation see 4.4–4.6).

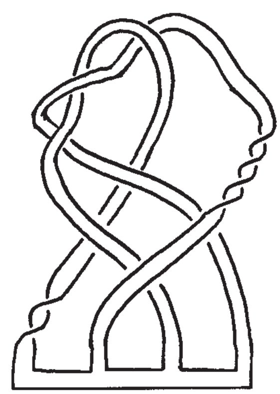


Figure 4.9

E 4.5. Doubled knots are not braid-like. (See 2.9.)

E 4.6. If $\mathfrak{k}$ is braid-like with respect to a Seifert surface of minimal genus, then there is an algorithm by which one can decide whether $\mathfrak{G}'$ is finitely generated or not. Apply this to E 4.2.

E 4.7. Let $\mathfrak{Z}_a$ and $\mathfrak{Z}_b$ be cyclic groups of order a resp. b. Use the Reidemeister–Schreier method to prove that the commutator subgroup $(\mathfrak{Z}_a * \mathfrak{Z}_b)'$ of the free product is a free group of rank $(a-1)(b-1)$.

E 4.8. Let C^* be the space obtained by cutting a knot complement along a Seifert surface of minimal genus. Prove that in the case of a trefoil or 4-knot C^* is a handlebody of genus two.

E 4.9. If $\mathfrak{G}$ is the group of a link of multiplicity μ, $\mathfrak{G} \xrightarrow{\kappa} \mathfrak{G}/\mathfrak{G}' \cong \mathfrak{Z}^\mu \xrightarrow{\Delta} \mathfrak{Z}$.

Generalize the construction of C_∞ to links by replacing $\mathfrak{G}'$ by $\ker(\Delta \circ \kappa)$. ($\Delta$ is the diagonal map.)

E 4.10. Let $\mathfrak{p}(2p+1, 2q+1, 2r+1) = \mathfrak{k}$ be a pretzel-knot, $p, q, r \in \mathbb{Z}$, Figure 8.9. Compute $i_\#^\pm \colon \pi_1 S \to \pi_1 C^*$ and $i_*^\pm \colon H_1(S) \to H_1(C^*)$ for a Seifert surface S of minimal genus spanning $\mathfrak{k}$ and decide which of these knots have a finitely generated commutator subgroup.

E 4.11. Consider the (generalized) "pretzel-knot $\mathfrak{p}(3, 1, 3, -1, -3)$", and show that it spans a Seifert surface F which is not of minimal genus such that the inclusions $i^\pm \colon F \to C^*$ induce injections $i_\#^\pm$. (The homomorphisms $i_*^\pm$ are necessarily not injective, compare E 8.1.)

Chapter 5
Fibred Knots

By the theorem of Brown, Crowell and Neuwirth, knots fall into two different classes according to the structure of their commutator subgroups. The first of them comprises the knots whose commutator subgroups are finitely generated, and hence free, the second one those whose commutator subgroups cannot be finitely generated. We have seen that all torus-knots belong to the first category and we have given an example – the 2-bridge knot $\mathfrak{b}(7,3)$ – of the second variety. The aim of this chapter is to demonstrate that the algebraic distinction of the two classes reflects an essential difference in the geometric structure of the knot complements.

A Fibration Theorem

5.1 Theorem (Stallings). *The complement $C = \overline{S^3 - V(\mathfrak{k})}$ of a knot $\mathfrak{k}$ fibres locally trivially over S^1 with Seifert surfaces of genus g as fibres if the commutator subgroup $\mathfrak{G}'$ of the knot group is finitely generated, $\mathfrak{G}' \cong \mathfrak{F}_{2g}$. Incidentally g is the genus of the knot.*

Theorem 5.1 is a special version of the more general Theorem 5.6 of [Stallings 1961]. The following proof of 5.1 is based on Stalling's original argument but takes advantage of the special situation, thus reducing its length and difficulty.

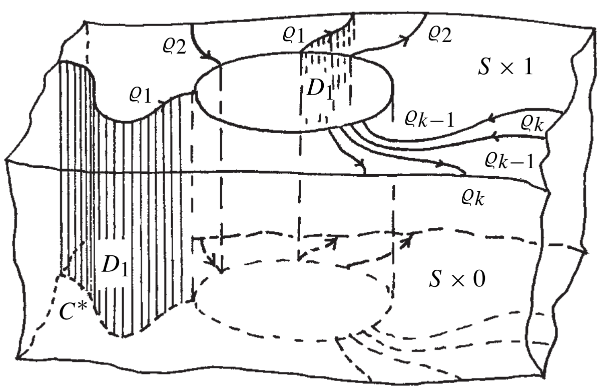


Figure 5.1

5.2. To prepare the setting imagine C fibred as described in 5.1. Cut along a Seifert surface S of $\mathfrak{k}$. The resulting space C^* is a fibre space with base-space the interval I, hence $C^* \cong S \times I$. The space C is reobtained from C^* by an identification of $S \times 0$ and $S \times 1$: $(x, 0) = (h(x), 1)$, $x \in S$, where $h\colon S \to S$ is an orientation preserving homeomorphism. We write in short:

$$C = S \times I/h.$$

Choose a base point P on ∂S and let $\sigma = P \times I$ denote the path leading from $(P, 1)$ to $(P, 0)$. For $w^0 = (w, 0)$, $w^1 = (w, 1)$ and $w \in \pi_1(S, P)$ there is an equation

$$w^1 = \sigma w^0 \sigma^{-1} \quad \text{in } \pi_1(C^*, (P, 1)).$$

Let $\kappa_1, \dots, \kappa_{2g}$ be simple closed curves representing canonical generators of S. Then obviously

$$\sigma \kappa_i^0 \sigma^{-1} (\kappa_i^1)^{-1} = \varrho_i' \simeq 0 \quad \text{in } C^*.$$

The curves $\{\varrho_i' \mid 1 \leqq i \leqq 2g\}$ coincide in σ; they can be replaced by a system of simple closed curves $\{\varrho_i\}$ on ∂C^* which are pairwise disjoint, where each ϱ_i is obtained from ϱ_i' by an isotopic deformation near σ, see Figure 5.1. There are disks D_i embedded in C^*, such that $\partial D_i = \varrho_i$. Cut C^* along the disks D_i to obtain a 3-ball C^{**} (Figure 5.2).

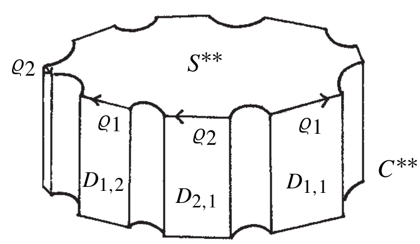


Figure 5.2

5.3. *Proof of* 5.1. We cut C along a Seifert surface S of minimal genus and get C^* with $S^\pm = S \times 1, 0$ in its boundary as in Chapter 4. Our aim is to produce a 3-ball C^{**} by cutting C^* along disks. The inclusions $i^+\colon S^+ \to C^*$ and $i^-\colon S^- \to C^*$ induce isomorphisms $i_\#^\pm$ of the fundamental groups. Let $m \subset \partial C$ be a meridian through the base point P on ∂S. Then, by the cutting process $C \to C^*$ m will become a path σ leading from $P^+ = (P, 1)$ to $P^- = (P, 0)$. Assign to $\sigma w^- \sigma^{-1}$ for $w^- \in \pi_1(S^-, P^-)$ the element $w^+ \in \pi_1(S^+, P^+)$, $w^+ = \sigma w^- \sigma^{-1}$ in $\pi_1(C^*, P^+)$. We know the map $f_\#(w^-) = w^+$ to be an isomorphism $f_\#\colon \pi_1(S^-, P^-) \to \pi_1(S^+, P^+)$. So by Nielsen's theorem [Nielsen 1927], [ZVC 1980, 5.7] there is a homeomorphism $f\colon S^- \to S^+$ inducing $f_\#$. There are canonical curves κ_i^+, κ_i^- on S^+ and S^- with $f(\kappa_i^-) = \kappa_i^+$ and $\sigma\kappa_i^-\sigma^{-1} \simeq \kappa_i^+$ in C^*. Again the system $\{\sigma\kappa_i^-\sigma^{-1}(\kappa_i^+)^{-1} \mid 1 \leqq$

$i \leqq 2g\}$ is replaced by an homotopic system $\{\varrho_i \mid 1 \leqq i \leqq 2g\}$ of disjoint simple curves, which by Dehn's Lemma [Papakyriakopoulos 1957] (see Appendix B.4) span non-singular disks D_i, $\partial D_i = \varrho_i = D_i \cap \partial C^*$ which can be chosen disjoint.

Cut C^* along the D_i. The resulting space C^{**} is a 3-ball (Figure 5.2) by Alexander's Theorem (see [Graeub 1950]), because its boundary is a 2-sphere in S^3 composed of an annulus $\partial S \times I$ and two 2-cells $(S^+)^*$ and $(S^-)^*$ where S^*, $(S^+)^*$ and $(S^-)^*$ are, respectively, obtained from S, S^+, S^- by the cutting of C^*. So C^{**} can be fibred over I, $C^{**} = S^* \times I$, $(S^*)^+ = S^* \times 1$, $(S^*)^- = S^* \times 0$. It remains to show that the identification $C^{**} \to C^*$ inverse to the cutting-process can be changed by an isotopy such as to be compatible with the fibration. Let g_i' be the identifying homeomorphisms, $g_i'(D_{i1}) = D_{i2} = D_i$, $i = 1, 2, \ldots, 2g$. The fibration of C^{**} induces a fibration on D_{ij}, the fibres being parallel to $D_{ij} \cap (S^*)^{\pm}$. There are fibre preserving homeomorphisms $g_i\colon D_{i1} \to D_{i2}$ which coincide with g_i' on the top $(S^*)^+$ and the bottom $(S^*)^-$. Since the D_{i1}, D_{i2} are 2-cells, the g_i are isotopic to the g_i'; hence, $C^* \cong S \times I$ and $C = S \times I/h$ (compare Lemma 5.7). □

5.4 Corollary. *The complement C of a fibred knot of genus g is obtained from $S \times I$, S a compact surface of genus g with a connected nonempty boundary, by the identification*

$$(x, 0) = (h(x), 1), \quad x \in S,$$

where $h\colon S \to S$ is an orientation preserving homeomorphism:

$$C = S \times I/h.$$

Now $\mathfrak{G} = \pi_1 C$ is a semidirect product $\mathfrak{G} = \mathfrak{Z} \ltimes_\alpha \mathfrak{G}'$, where $\mathfrak{G}' = \pi_1 S \simeq \mathfrak{F}_{2g}$. The automorphism $\alpha = \alpha(t)\colon \mathfrak{G}' \to \mathfrak{G}'$, $a \mapsto t^{-1}at$, and $h_\#^{-1}$ belong to the same class of automorphisms, in other words, $\alpha(t) \cdot h_\#^{-1}$ or $\alpha(t) \cdot h_\#$ is an inner automorphism of $\mathfrak{G}'$.

The *proof* follows from the construction used in proving 5.1. □

Observe that σ after identification by h becomes a generator of $\mathfrak{Z}$. If t is replaced by another coset representative t^* mod $\mathfrak{G}'$, $\alpha(t^*)$ and $\alpha(t)$ will be in the same class of automorphisms. Furthermore $\alpha(t^{-1}) = \alpha^{-1}(t)$. The ambiguity $h_\#^{\pm 1}$ can be avoided if σ as well as t are chosen to represent a meridian of $\mathfrak{k}$. ($h_\#$ is called the monodromy map of C.)

There is an addendum to Theorem 5.1.

5.5 Proposition. *If the complement C of a knot $\mathfrak{k}$ of genus g fibres locally trivially over S^1 then the fibre is a compact orientable surface S of genus g with one boundary component, and $\mathfrak{G}' = \pi_1 S \cong \mathfrak{F}_{2g}$.*

Proof. Since the fibration $C \to S^1$ is locally trivial the fibre is a compact 2-manifold S. There is an induced fibration $\partial C \to S^1$ with fibre ∂S. Consider the exact fibre sequences

$$\begin{array}{ccccccccc} 1 \longrightarrow & \pi_1(\partial S) & \longrightarrow & \pi_1(\partial C) & \longrightarrow & \pi_1 S^1 & \longrightarrow & \pi_0(\partial S) & \longrightarrow 1 \\ & \downarrow & & \downarrow & & \| & & \downarrow & \\ 1 \longrightarrow & \pi_1 S & \longrightarrow & \pi_1 C & \longrightarrow & \pi_1 S^1 & \longrightarrow & \pi_0 S & \longrightarrow 1. \end{array}$$

The diagram commutes, and $\pi_1(\partial C) \to \pi_1 S^1$ is surjective. Hence $\pi_1 C \to \pi_1 S^1$ is surjective and $\pi_0(\partial S) = \pi_0 S = 1$, that is, S and ∂S are connected. (See E 5.1.)

Now the second sequence pins down $\pi_1 S$ as $(\pi_1 C)'$. □

We conclude this paragraph by stating the general theorem of Stallings without proof:

5.6 Theorem (Stallings). *Let M be a compact irreducible* 3*-manifold* (*this means that in M every* 2*-sphere bounds a* 3*-ball*). *Assume that $\varphi\colon \pi_1 M \to \mathbb{Z}$ is an epimorphism with a finitely generated kernel. Then:*

(a) $\ker\varphi$ *is isomorphic to the fundamental group of a compact surface S.*

(b) *M can be fibred locally trivially over S^1 with fibre S if* $\ker\varphi \not\cong \mathbb{Z}_2$. □

B Fibred Knots

The knots of the first class whose commutator subgroups are finitely generated – in fact are free groups of rank $2g$ – are called *fibred knots* by virtue of Theorem 5.1. The fibration of their complements affords additional mathematical tools for the treatment of these knots. They are in a way the simpler knots and in their case the original 3-dimensional problem can to some extent be played down to two dimensions. This is a phenomenon also known in the theory of braids (see Chapter 10) or Seifert fibre spaces.

We shall study the question: How much information on the fibred knot $\mathfrak{k}$ do we get by looking at $h\colon S \to S$ in the formula $S \times I/h = \overline{S^3 - V(\mathfrak{k})}$?

5.7 Lemma (Neuwirth). *If h_0, $h_1\colon S \to S$ are isotopic homeomorphisms then there is a fibre preserving homeomorphism*

$$H\colon S \times I/h_0 \to S \times I/h_1.$$

Proof. Let h_t be the isotopy connecting h_0 and h_1. Put $g_t = h_t h_0^{-1}$ and define a homeomorphism

$$H'\colon S \times I \to S \times I$$

by $H'(x,t) = (g_t(x), t)$, $x \in S$, $t \in I$. Since $H'(x,0) = (x,0)$ and

$$H'(h_0(x), 1) = (g_1 h_0(x), 1) = (h_1(x), 1),$$

H' induces a homeomorphism H as desired. □

5.8 Lemma. *Let $f\colon S \to S$ be a homeomorphism. Then there is a fibre preserving homeomorphism $F\colon S \times I/h \to S \times I/fhf^{-1}$. If f is orientation preserving then there is a homeomorphism F which also preserves the orientation.*

Proof. Take $F(x,t) = (f(x), t)$. □

5.9 Definition (Similarity). Homeomorphisms $h_1\colon S_1 \to S_1, h_2\colon S_2 \to S_2$ of homeomorphic oriented compact surfaces S_1 and S_2 are called *similar*, if there is a homeomorphism $f\colon S_1 \to S_2$ respecting orientations, such that fh_1f^{-1} and h_2 are isotopic.

The notion of similarity enables us to characterize homeomorphic complements C_1 and C_2 of fibred knots $\mathfrak{k}_1$ and $\mathfrak{k}_2$ of equal genus g by properties of the gluing homeomorphisms.

5.10 Proposition. *Let $\mathfrak{k}_1$, $\mathfrak{k}_2$ be two (oriented) fibred knots of genus g with (oriented) complements C_1 and C_2. There is an orientation preserving homeomorphism $H\colon C_1 = S_1 \times I/h_1 \to C_2 = S_2 \times I/h_2$, $\lambda_1 = \partial S_1 \simeq \mathfrak{k}_1$, $H(\partial S_1) = \partial S_2 = \lambda_2 \simeq \mathfrak{k}_2$, if and only if there is a homeomorphism $h\colon S_1 \to S_2$, respecting orientations, $h(\lambda_1) = \lambda_2$, such that hh_1h^{-1} and h_2 are isotopic, that is, h_1 and h_2 are similar.*

Proof. If h exists and hh_1h^{-1} and h_2 are isotopic then by Lemma 5.7 there is a homeomorphism which preserves orientation and fibration:

$$F\colon S_2 \times I/hh_1h^{-1} \to S_2 \times I/h_2.$$

Now $F'\colon\ S_1 \times I/h_1 \to S_2 \times I/hh_1h^{-1}$, $(x,t) \mapsto (h(x), t)$, gives $H = FF'$ as desired.

To show the converse let $H\colon C_1 = S_1 \times I/h_1 \to S_2 \times I/h_2 = C_2$ be an orientation preserving homeomorphism, $H(\lambda_1) = \lambda_2$. There is an isomorphism

$$H_\#\colon \pi_1 C_1 = \mathfrak{G}_1 \to \mathfrak{G}_2 = \pi_1 C_2$$

which induces an isomorphism

$$h_\#\colon \pi_1 S_1 = \mathfrak{G}_1' \to \mathfrak{G}_2' = \pi_1 S_2.$$

By Nielsen ([ZVC 1970, Satz V.9], [ZVC 1980, 5.7.2]), there is a homeomorphism $h\colon S_1 \to S_2$ respecting the orientations induced on ∂S_1 and ∂S_2. We can choose representatives m_1 and m_2 of meridians of $\mathfrak{k}_1, \mathfrak{k}_2$, such that

$$h_{i\#}\colon \pi_1 S_i \to \pi_1 S_i, \quad x \mapsto m_i^{-1} x m_i,\ i = 1, 2.$$

Since H preserves the orientation, $H_\#(m_2) = m_2$. Now

$$\begin{aligned} h_\# h_{1\#}(x) &= h_\#(m_1^{-1} x m_1) = H_\#(m_1^{-1}) h_\#(x)(H_\#(m_1)) \\ &= m_2^{-1} h_\#(x) m_2 = (h_{2\#} h_\#(x)). \end{aligned}$$

By Baer's Theorem ([ZVC 1970, Satz V.15], [ZVC 1980, 5.13.1]), hh_1 and h_2h are isotopic; hence h_1 and h_2 are similar. □

Proposition 5.10 shows that the classification of fibred knot complements can be formulated in terms of the fibring surfaces and maps of such surfaces. The proof also shows that if fibred complements are homeomorphic then there is a fibre preserving homeomorphism. This means: different fibrations of a complement C admit a fibre preserving autohomeomorphism. Indeed, by [Waldhausen 1968], there is even an isotopy connecting both fibrations.

In the case of fibred knots invertibility and amphicheirality can be excluded by properties of surface mappings.

5.11 Proposition. *Let $C = S \times I/h$ be the complement of a fibred knot $\mathfrak{k}$.*

(a) *$\mathfrak{k}$ is amphicheiral only if h and h^{-1} are similar.*

(b) *$\mathfrak{k}$ is invertible only if there is a homeomorphism $f: S \to S$, reversing orientation, such that h and $fh^{-1}f^{-1}$ are similar.*

Proof [Burde-Zieschang 1967].

(a) The map $(x, t) \mapsto (x, 1-t)$, $x \in S$, $t \in I$ induces a mapping

$$C = S \times I/h \to S \times I/h^{-1} = C'$$

onto the mirror image C' of C satisfying the conditions of Proposition 5.10.

(b) If $f: S \to S$ is any homeomorphism inverting the orientation of S, then $(x, t) \mapsto (f(x), 1-t)$ induces a homeomorphism

$$S \times I/h \to S \times I/fh^{-1}f^{-1}$$

which maps ∂S onto its inverse. Again apply Proposition 5.10. □

C Applications and Examples

The fibration of a non-trivial knot complement is not easily visualized, even in the simplest cases. (If $\mathfrak{k}$ is trivial, C is a solid torus, hence trivially fibred by disks D^2, $C = S^1 \times D^2$.)

5.12 Fibring the complement of the trefoil. Let C be the complement of a trefoil $\mathfrak{k}$ sitting symmetrically on the boundary of an unknotted solid torus $T_1 \subset S^3$ (Figure 5.3). $T_2 = \overline{S^3 - T_1}$ is another unknotted solid torus in S^3. A Seifert surface S (hatched regions in Figure 5.3) is composed of two disks D_1 and D_2 in T_2 and three twisted 2-cells in T_1. (Figure 5.4 shows T_1 and the twisted 2-cells in a straightened position.) A rotation about the core of T_1 through φ and, at the same time, a rotation about the core of T_2 through $2\varphi/3$ combine to a mapping $f_\varphi \colon S^3 \to S^3$. Now C is fibred by $\{f_\varphi(S) \mid 0 \leqq \varphi \leqq \pi\}$ (see [Rolfson 1976, p. 329]).

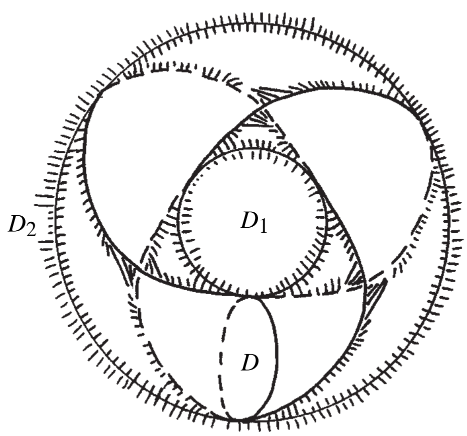


Figure 5.3

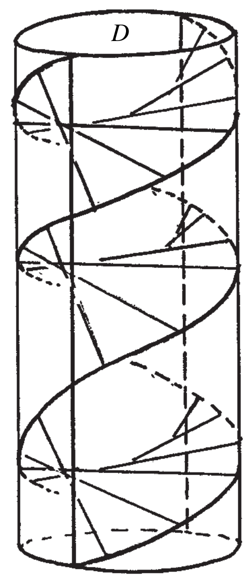


Figure 5.4

5.13 Fibring the complement of the four-knot The above construction of a fibration takes advantage of the symmetries of the trefoil as a torus knot. It is not so easy to convince oneself of the existence of a fibration of the complement of the figure-eight knot $\mathfrak{k}$ by geometric arguments. The following sequence of figures (5.5(a)–(g)) tries to do it: (a) depicts a Seifert surface S spanning the four-knot in a tolerably symmetric fashion. (b) shows S thickened up to a handlebody V of genus 2. The knot $\mathfrak{k}$ is a curve on its boundary. (c) presents $V' = \overline{S^3 - V}$. In order to find $\mathfrak{k}$ on $\partial V'$ express $\mathfrak{k}$ on ∂V by canonical generators $\alpha, \beta, \gamma, \delta$ of $\pi_1(\partial V)$, $\mathfrak{k} = \beta\alpha^{-1}\gamma^{-1}\delta^{-1}\alpha\beta^{-1}\gamma\delta$. Replace every generator by its inverse to get $\mathfrak{k} = \beta^{-1}\alpha\gamma\delta\alpha^{-1}\beta\gamma^{-1}\delta^{-1}$ on $\partial V'$. The knot $\mathfrak{k}$ divides $\partial V'$ into two surfaces S^+ and S^- of genus one, Figure (d). (e) just simplifies (d); the knot is pushed on the outline of the figure as far as possible. By way of (f) we

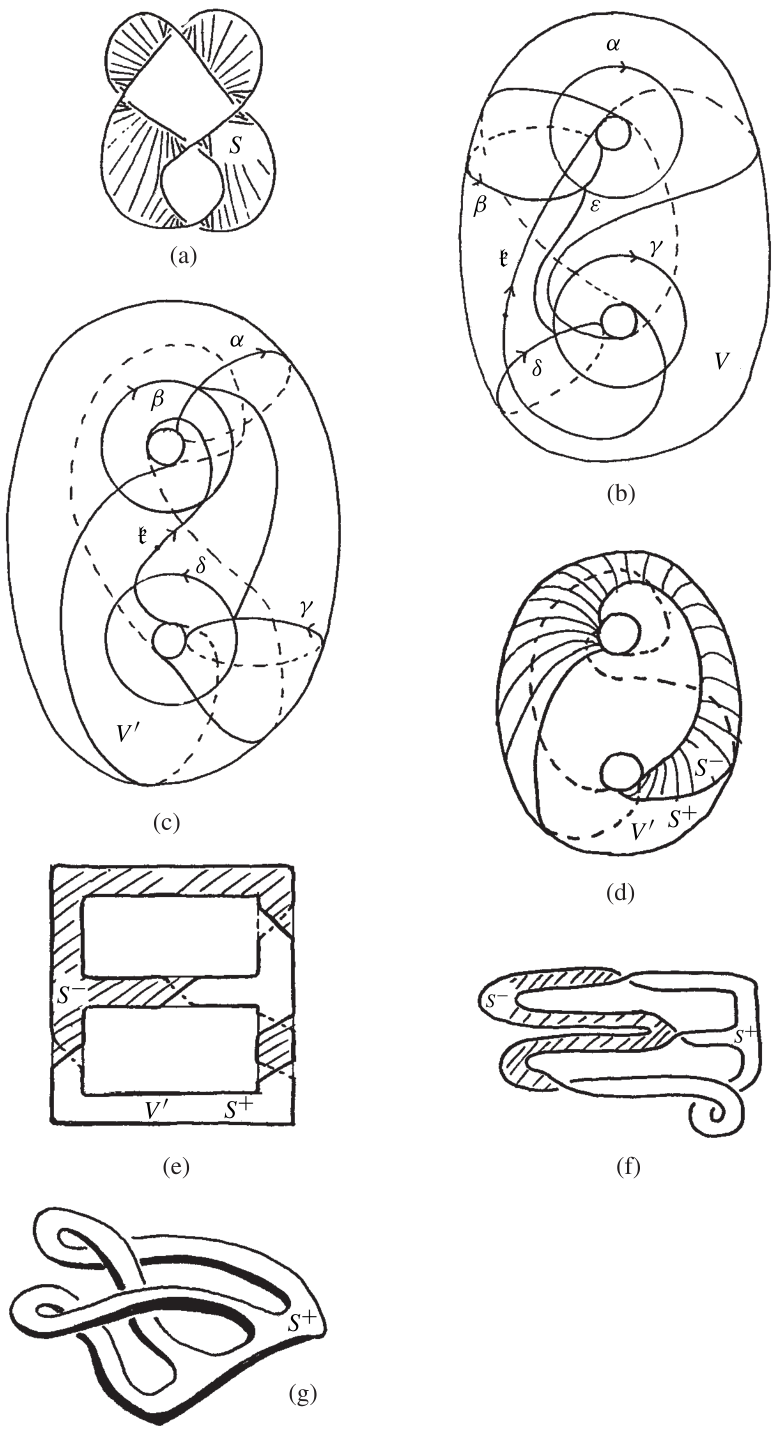


Figure 5.5

finally reach (g), where the fibres of $V' - \mathfrak{k}$ are Seifert surfaces parallel to S^+ and S^-. The fibration extends to $V - \mathfrak{k}$ by the definition of V.

The following proposition shows that the trefoil and the four-knot are not only the two knots with the fewest crossings, but constitute a class that can be algebraically characterized.

5.14 Proposition. *The trefoil knot and the four-knot are the only fibred knots of genus one.*

At this stage we only prove a weaker result: *A fibred knot of genus one has the same complement as the trefoil or the four-knot.*

Proof (see [Burde-Zieschang 1967]). Let $C = S \times I/h$ be the complement of a knot $\mathfrak{k}$ and assume that S is a torus with one boundary component. Then h induces automorphisms $h_\#\colon \pi_1 S \to \pi_1 S$ and $h_*\colon H_1(S) \to H_1(S) \cong \mathbb{Z}^2$. Let A denote the 2×2-matrix corresponding to h_* (after the choice of a basis).

$$\det A = 1, \tag{1}$$

since h preserves the orientation. The automorphism $h_\#$ describes the effect of the conjugation with a meridian of $\mathfrak{k}$ and it follows that $\pi_1 S$ becomes trivial by introducing the relations $h_\#(x) = x \in \pi_1 S$. This implies:

$$\det\left(A - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \pm 1. \tag{2}$$

From (1) and (2) it follows that

$$\operatorname{trace} A \in \{1, 3\}. \tag{3}$$

A matrix of trace +1 is conjugate in $\mathrm{SL}(2, \mathbb{Z})$ to $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ and a matrix with trace 3 is conjugate to $\begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}$, [Zieschang 1981, 21.15]. Two automorphisms of $\mathfrak{F}_2$ which induce the same automorphism on $\mathbb{Z} \oplus \mathbb{Z}$ differ by an inner automorphism ([Nielsen 1918], [Lyndon-Schupp 1977, I.4.5]). The Baer Theorem now implies that the gluing mappings are determined up to isotopy; hence, by Lemma 5.7, the complement of the knot is determined up to homeomorphism by the matrix above. The matrices $\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ are obtained when the complements of the trefoil knots are fibred, see 5.13. The matrix $\begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}$ results in the case of the figure-eight knot as follows from the fact that in 3.8 the conjugation by s induces on $\mathfrak{G}'$ the mapping $x_0 \mapsto x_1, x_1 \mapsto x_1 x_0^{-1} x_1^2$.

Thus we have proved that the complement of a fibred knot $\mathfrak{k}$ of genus 1 is homeomorphic to the complement of a trefoil knot or the figure-eight knot. Later we shall show that $\mathfrak{k}$ is indeed a trefoil knot (Theorem 6.1) or a four-knot (Theorem 15.8). □

5.15. We conclude this section with an application of Proposition 5.11 and reprove the fact (see 3.29 (b)) that the trefoil knot is not amphicheiral. This was first proved by M. Dehn [1914].

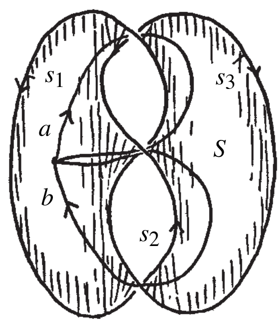


Figure 5.6

Figure 5.6 shows a trefoil bounding a Seifert surface S of genus one. The Wirtinger presentation of the knot group $\mathfrak{G}$ is

$$\mathfrak{G} = \langle\, s_1, s_2, s_3 \mid s_3 s_1 s_3^{-1} s_2^{-1},\ s_1 s_2 s_1^{-1} s_3^{-1},\ s_2 s_3 s_2^{-1} s_1^{-1} \,\rangle.$$

The curves a and b in Figure 5.6 are free generators of $\pi_1 S = \mathfrak{F}_2 = \langle\, a, b \,\rangle$. They can be expressed by the Wirtinger generators s_i (see 3.7):

$$a = s_1^{-1} s_2, \quad b = s_2^{-1} s_3.$$

Using the relations we get (with $t = s_1$):

$$t^{-1} a t = s_1^{-1} s_1^{-1} s_2 s_1 = s_1^{-1} s_2 s_3^{-1} s_1 = s_1^{-1} s_2 s_3^{-1} s_2 s_2^{-1} s_1 = a b^{-1} a^{-1},$$
$$t^{-1} b t = s_1^{-1} s_2^{-1} s_3 s_1 = s_1^{-1} s_2^{-1} s_2 s_3 = s_1^{-1} s_2 \cdot s_2^{-1} s_3 = ab.$$

Let $C = S \times I/h$ be the complement of the trefoil. Relative to the basis $\{a, b\}$ of $H_2(S) = \mathbb{Z} \otimes \mathbb{Z}$ the homomorphism $h_*\colon H_1(S) \to H_1(S)$ is given by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$. (See Corollary 5.4). If the trefoil were amphicheiral then by Proposition 5.11 there would be a unimodular matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad 1 = \alpha\delta - \beta\gamma,$$

such that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

$$\begin{pmatrix} \alpha - \beta & \alpha \\ \gamma - \delta & \gamma \end{pmatrix} = \begin{pmatrix} -\gamma & -\delta \\ \alpha + \gamma & \beta + \delta \end{pmatrix}.$$

This means: $\delta = -\alpha$, $\gamma = \beta - \alpha$. However, $1 = \alpha\delta - \beta\gamma = -\alpha^2 - \beta(\beta - \alpha) = -(\alpha^2 - \alpha\beta + \beta^2)$ has no integral solution.

D History and Sources

The material of this chapter is for the larger part based on J. Stallings' theorem on fibring 3-manifold [Stallings 1962]. The fibred complement $C = S \times I/f$ of a "fibred knot" was further investigated in [Neuwirth 1961′] and [Burde-Zieschang 1967]. In the first paper the complement was shown to be determined by the peripheral system of the knot group while in the second one C was characterized by properties of the identifying surface map f.

Neuwirth's result is a special case of the general theorems of Waldhausen [1968]. In this fundamental paper manifolds with a Stallings fibration play an important role.

E Exercises

E 5.1. Construct a fibration of a compact orientable 3-manifold M over S^1 such that $\pi_1 M \to \pi_1 S^1$ is not surjective. Observe that the fibre is not connected in this case.

E 5.2. Find a 2×2-matrix A representing $f_*\colon H_1(S) \to H_1(S)$ in the case of the complement $C = S \times I/f$ of the four-knot. Show that A and A^{-1} are conjugate.

E 5.3. Compute the powers of the automorphism $f_\#\colon \pi_1 S \to \pi_1 S$

$$f_\#(a) = t^{-1}at = b^{-1}$$
$$f_\#(b) = t^{-1}at = ba$$

induced by the identifying map of the trefoil (see 5.15). Describe the manifolds $S \times I/f^i$, $i \in \mathbb{Z}$.

E 5.4. Show that the knot 5_2 can be spanned by a Seifert surface S of minimal genus such that the knot complement C cut along S is a handlebody C^*. Apply the method used in 5.13 to show that nevertheless 5_2 is not fibred!

E 5.5. Show that the knot 8_{20} is fibred.

Chapter 6
A Characterization of Torus Knots

Torus knots have been repeatedly considered as examples in the preceding chapters. If knots are placed on the boundaries of handlebodies as in Chapter 3, the least possible genus of a handlebody carrying a knot defines a hierarchy for knots where the torus knots form the simplest class excepting the trivial knot. Torus knots admit a simple algebraic characterization; see Theorem 6.1.

A Results and Sources

6.1 Theorem (Burde–Zieschang). *A non-trivial knot whose group $\mathfrak{G}$ has a non-trivial centre is a torus knot.*

The theorem was first proved in [Burde-Zieschang 1966], and had been proved for alternating knots in [Murasugi 1961] and [Neuwirth 1961]. Since torus knots have Property P (Chapter 15), Theorem 6.1 together with Theorem 3.29 shows: any knot group with a non-trivial centre determines its complement, and the complement in turn admits just one torus knot $\mathfrak{t}(a, b)$ and its mirror image $\mathfrak{t}(a, -b)$.

F. Waldhausen later proved a more general theorem which includes Theorem 6.1 by the way of Seifert's theory of fibred 3-manifolds, see [Waldhausen 1967]:

6.2 Theorem (Waldhausen). *Let M be an orientable compact irreducible 3-manifold. If either $H_1(M)$ is infinite or $\pi_1 M$ a non-trivial free product with amalgamation, and if $\pi_1 M$ has a non-trivial centre, then M is homeomorphic to a Seifert fibred manifold with orientable orbit-manifold (Zerlegungsfläche).* □

Because of Theorem 3.30, Theorem 6.2 obviously applies to knot complements $C = M$. A closer inspection of the Seifert fibration of C shows that it can be extended to S^3 in such a way that the knot becomes a normal fibre. Theorem 6.1 now follows from a result of [Seifert 1933] which contains a complete description of all fibrations of S^3.

6.3 Theorem (Seifert). *A fibre of a Seifert fibration of S^3 is a torus knot or the trivial knot. Exceptional fibres are always unknotted.* □

We propose to give now a proof of Theorem 6.1 which makes use of a theorem by Nielsen [1942] on mappings of surfaces. (This theorem is also used in Waldhausen's proof.) We do not presuppose Waldhausen's theory or Seifert's work on fibred manifolds, though Seifert's ideas are applied to the special case in hand. The proof is also different from that given in the original paper [Burde-Zieschang 1966].

Proof of Theorem 6.1. Let $\mathfrak{k}$ be a non-trivial knot whose group $\mathfrak{G}$ has a centre $\mathfrak{C} \neq 1$. Then by 4.10(a) its commutator subgroup $\mathfrak{G}'$ is finitely generated, and hence by 5.1 the complement C is a fibre space over S^1 with a Seifert surface S of minimal genus g as a fibre. Thus $C = S \times I/h$ as defined in 5.2. Let t and $r = \partial(S \times 0)$ represent a meridian and a longitude on ∂C, and choose their point of intersection P as base point for $\pi_1(C) = \mathfrak{G}$. The homeomorphism $h\colon S \to S$ induces the automorphism:

$$h_\#\colon \mathfrak{G}' \to \mathfrak{G}' = \pi_1(S \times 0), \quad x \mapsto t^{-1}xt,$$

since by 5.7 we may assume that $h(P) = P$. Again by 4.10, $\mathfrak{C} \cong \mathbb{Z}$. In the following we use the notation of Proposition 4.10.

6.4 Proposition. *Let $z = t^n u$, $n > 1$, be a generator of the centre $\mathfrak{C}$ of $\mathfrak{G}$. Then u is a power of the longitude r, $u = r^{-m}$, $m \neq 0$, and $h_\#^n$ is the inner automorphism $h_\#^n(x) = r^m x r^{-m}$. The exponent n is the smallest one with this property. The powers of r are the only fixed elements of $h_\#^i$, $i \neq 0$.*

Proof. By assumption $t^{-n}xt^n = uxu^{-1}$ for all $x \in \mathfrak{G}'$. From $h_\#(r) = t^{-1}rt = r$ it follows that u commutes with r. The longitude r is a product of commutators of free generators of $\mathfrak{G}' \cong \mathfrak{F}_{2g}$ and it is easily verified that r is not a proper power of any other element of $\mathfrak{G}'$; hence, $u = r^{-m}$, $m \in \mathbb{Z}$ (see [ZVC 1980, E 1.5]). We shall see $\gcd(n, m) = 1$ in 6.8 (2). Fixed elements of $h_\#^i$, $i \neq 0$, are also fixed elements of $h_\#^{in}$, hence they commute with r and are therefore powers of r.

Now assume that $t^{-k}xt^k = vxv^{-1}$ for all $x \in \mathfrak{G}'$ and some $k \neq 0$ and $v \in \mathfrak{G}'$. Then $t^k v \in \mathfrak{C}$, thus $t^k v = (t^n u)^l = t^{nl}u^l$. This proves that n is the smallest positive exponent such that $h_\#^n$ is an inner automorphism of $\mathfrak{G}'$. □

We now state without proof two theorems on periodic mappings of surfaces due to [Nielsen 1942, 1937]. A proof of a generalization of the first one can be found in [Zieschang 1981]; it is a deep result which requires a considerable amount of technicalities in its proof. A different approach was used by [Fenchel 1948, 1950], a combinatorial proof of his theorem was given by [Zimmermann 1977], for a more general result see also [Kerckhoff 1980, 1983].

6.5 Theorem (Nielsen). *Let S be a compact surface different from the sphere with less than three boundary components. If $h\colon S \to S$ is a homeomorphism such that h^n is isotopic to the identity, then there is a periodic homeomorphism f of order n isotopic to h.* □

We need another theorem which provides additional geometric information on periodic surface mappings:

6.6 Theorem (Nielsen). *Let $f: S \to S$ be an orientation preserving periodic homeomorphism of order n, $f^n = \mathrm{id}$, of a compact orientable surface S. Let $q \in S$ be some point with $f^k(q) = q$ for some k with $0 < k < n$, and let k be minimal with this property. Then there is a neighbourhood $U(q)$ of q in S, homeomorphic to an open 2-cell, such that $f^l(U(q)) \cap U(q) = \emptyset$ for $0 < l < k$. Furthermore $f^k|U(q)$ is a topological rotation of order $\frac{n}{k}$ with fixed point q.* □

For a proof of Theorem 6.6 see [Nielsen 1937] or [Nielsen 1984]. Points q of S for which such a k exists are called *exceptional* points.

6.7 Corollary (Nielsen). *A periodic mapping $f: S \to S$ as in Theorem 6.6 has at most finitely many exceptional points, none of them on $r = \partial S$.* □

At this point the reader may take the short cut via Seifert manifolds to Theorem 6.1: By Lemma 5.7,

$$C = S \times I/h \cong S \times I/f.$$

The trivial fibration of $S \times I$ with fibre I defines a Seifert fibration of C. Exceptional points in S correspond to exceptional fibres by Theorem 6.6. Since a fibre on ∂C is not isotopic to a meridian, the Seifert fibration of C extended to give a Seifert fibration of S^3, where $\mathfrak{k}$ is a fibre, normal or exceptional. By Theorem 6.3 normal fibres of Seifert fibrations of S^3 are torus knots or trivial knots, while exceptional fibres are always unknotted. So $\mathfrak{k}$ has to be a normal fibre, i.e. a torus knot.

B Proof of the Main Theorem

We shall now give a proof of 6.1 by making use only of the theory of regular coverings.

6.8. The orbit of an exceptional point of S relative to the cyclic group $\mathfrak{Z}_n$ generated by f consists of k_j points, $1 \leqq k_j \leqq n$, $k_j|n$. We denote exceptional points accordingly by $Q_{j\nu}$, $1 \leqq j \leqq s$, $0 \leqq \nu \leqq k_j - 1$, where $Q_{j,\nu+1} = f(Q_{j\nu})$, $\nu + 1 \bmod k_j$. By deleting the neighbourhoods $U(Q_{j\nu})$ of 6.6 we obtain $S_0 = S - \bigcup U(Q_{j\nu})$, which is a compact surface of genus g with $1 + \sum_{j=1}^{s} k_j$ boundary components, on which $\mathfrak{Z}_n = \langle f \rangle$ operates freely. So there is a regular cyclic n-fold covering $p_0: S_0 \to S_0^*$ with $\langle f \rangle$ as its group of covering transformations. We define a covering

$$p: C_0 = S_0 \times I/f \to S_0^* \times I/\mathrm{id} \cong S_0^* \times S^1 = C_0^*$$

by

$$p(u, v) = (p_0(u), v), \quad u \in S_0, v \in I.$$

This covering is also cyclic of order n, and $f \times \mathrm{id}$ generates its group of covering transformations. Let $r_{j\nu}$ represent the boundary of $U(Q_{j\nu})$ in $\pi_1(S_0)$ in such a way that

$$\partial S = r = \prod_{i=1}^{g}[a_i, b_i] \cdot \prod_{j=1}^{s} \prod_{\nu=0}^{k_j-1} r_{j\nu}.$$

The induced homomorphism $p_\# \colon \pi_1(C_0) \to \pi_1(C_0^*)$ then gives

$$p_\#(r) = r^{*n}, \quad p_\#(r_{j\nu}) = (r_j^*)^{m_j}, \quad m_j k_j = n, \tag{1}$$

where r^* and r_j^* represent the boundaries of S_0^* in $\pi_1(S_0^*)$ such that

$$r^* = \prod_{i=1}^{g^*}[a_i^*, b_i^*] \cdot \prod_{j=1}^{s} r_j^*.$$

Let z^* be a simple closed curve on $r^* \times S^1$ representing a generator of $\pi_1(S^1)$, such that $p_\#^{-1}(z^{*n}) = (t^n v, v) \in \pi_1(S_0)$. Then $t^n v$ is a simple closed curve on the torus $r \times I/f$ and it is central in $\pi_1(C_0)$, since z^* is central in $\pi_1(C_0^*)$. Therefore $t^n v$ is central in $\pi_1(C) \cong \mathfrak{G}$, too; hence, $p_\#^{-1}(z^{*n}) = z = t^n \cdot r^{-m}$, see 6.4. Since $t^n v = t^n r^{-m}$ represents a simple closed curve on the torus $r \times I/f$ it follows that

$$\gcd(m, n) = 1. \tag{2}$$

Furthermore, $z^{*n} = p_\#(z) = (p_\#(t))^n \cdot r^{*-mn}$. Putting $p_\#(t) = t^*$, we obtain

$$z^* = t^* r^{*-m}. \tag{3}$$

For $\alpha, \beta \in \mathbb{Z}$, satisfying

$$\alpha m + \beta n = 1, \tag{4}$$

$$q = t^\alpha r^\beta \quad \text{and} \quad p_\#(q) = q^* = t^{*\alpha} r^{*n\beta} \tag{5}$$

are simple closed curves on ∂C and $r^* \times S^1$, respectively. From these formulas we derive:

$$t^* = z^{*n\beta} \cdot q^{*m}, \tag{6}$$

$$r^* = z^{*-\alpha} \cdot q^*. \tag{7}$$

Since $f|r$ is a rotation of order n (see 6.6), the powers $\{r^{*\mu} \mid 0 \leqq \mu \leqq n-1\}$ are coset representatives in $\pi_1(C_0^*)$ mod $p_\#\pi_1(C_0)$. From (3) it follows that $\{z^{*\mu}\}$ also represent these cosets. By (7),

$$z^{*\alpha} r^* = q^* \in p_\#\pi_1(C_0). \tag{8}$$

We shall show that there are similar formulas for the boundaries r_j^*.

6.9 Lemma. *There are* $\alpha_j \in \mathbb{Z}$, $\gcd(\alpha_j, m_j) = 1$ *such that*

$$z^{*\alpha_j k_j} r_j^* = q_j^* \in p_\# \pi_1(C_0).$$

The α_j *are determined* mod m_j.

Proof. For some $\nu \in \mathbb{Z}$: $z^{*\nu} r_j^* \in p_\# \pi_1(C_0)$. Now $q_j^* = z^{*\nu} r_j^*$ and z^* generate $\pi_1(r_j^* \times S^1)$, $q_j = p_\#^{-1}(q_j^*)$ and $z = p_\#^{-1}(z^{*n})$ are generators of $\pi_1(p^{-1}(r_j^* \times S^1))$. Hence,

$$r_{j0} = z^{-a_j} q_j^{\beta_j}, \quad \gcd(\alpha_j, \beta_j) = 1$$

and

$$z^{*-n\alpha_j} (q_j^*)^{\beta_j} = p_\#(r_{j0}) = (r_j^*)^{m_j} = z^{*-\nu m_j} (q_j^*)^{m_j},$$

thus

$$k_j \alpha_j = \nu, \quad m_j = \beta_j.$$

(Remember that $m_j k_j = n$, see 6.8 (1).) □

6.10. Now let $\hat{C}_0^* = \hat{S}_0^* \times \hat{S}^1$ be a homeomorphic copy of $C_0^* = S_0^* \times S^1$ with $\hat{z}^*$ generating $\pi_1(\hat{S}^1)$ and $\hat{a}_i^*, \hat{b}_i^*, \hat{r}^*, \hat{r}_j^*$ representing canonical generators of $\pi_1(\hat{S}_0^*)$, and $\hat{r}^* = \prod_{i=1}^{g^*} [\hat{a}_i^*, \hat{b}_i^*] \cdot \prod_{j=1}^{s} \hat{r}_j^*$. Define an isomorphism

$$\kappa_\#^* \colon \pi_1(C_0^*) \to \pi_1(\hat{C}_0^*)$$

by

$$\kappa_\#^*(z^*) = \hat{z}^*, \ \kappa_\#^*(r_j^*) = \hat{z}^{*-\alpha_j k_j} \cdot \hat{r}_j^*,$$
$$\kappa_\#^*(a_i^*) = \hat{z}^{*-\varrho_i} \cdot \hat{a}_i^*, \ \kappa_\#^*(b_i^*) = \hat{z}^{*-\sigma_i} \cdot \hat{b}_i^*,$$

where ϱ_i, σ_i are chosen in such a way that $z^{*\varrho_i} a_i^*, z^{*\sigma_i} b_i^* \in \pi_1(S_0^*)$. (The ϱ_i, σ_i will play no role in the following.)

6.11 Lemma. $\kappa_\#^* p_\# \pi_1(C_0) = \pi_1(\hat{S}_0^*) \times \langle z^{*n} \rangle$.

Proof. By construction we have $\kappa_\#^{*-1}(\hat{a}_i^*) = z^{*\varrho_i} a_i^* \in p_\# \pi_1(C_0)$, and likewise $\kappa_\#^{*-1}(b_i^*)$, $\kappa_\#^{*-1}(\hat{r}_j^*) \in p_\# \pi_1(C_0)$. Since $\kappa_\#^*$ is an isomorphism, $\kappa_\#^* p_\# \pi_1(C_0)$ is a normal subgroup of index n in $\pi_1(\hat{S}_0^*) \times \langle \hat{z}^* \rangle$, which contains $\pi_1(\hat{S}_0^*)$, because it contains its generators. This proves Lemma 6.11. □

We shall now see that $\kappa_\#^*$ can be realized by a homeomorphism $\kappa^* \colon S_0^* \times S^1 \to \hat{S}_0^* \times \hat{S}^1$, and that there is a homeomorphism $\kappa \colon C_0 \to \hat{C}_0 = \hat{S}_0 \times \hat{\hat{S}}^1$ covering κ^*

such that the following diagram is commutative

$$\begin{array}{ccc} C_0 = S_0 \times I/f & \xrightarrow{\kappa} & \hat{S}_0 \times \hat{\hat{S}}^1 = \hat{C}_0 \\ \Big\downarrow p & & \Big\downarrow \hat{p} \\ S_0^* \times S^1 & \xrightarrow{\kappa^*} & \hat{S}_0^* \times \hat{S}^1 . \end{array} \tag{9}$$

Here $\hat{p}$ is the n-fold cyclic covering defined by $\hat{p}(x, \zeta) = (x, \zeta^n)$, if the 1-spheres $\hat{S}^1$, $\hat{\hat{S}}^1$ are described by complex numbers ζ of absolute value one.

6.12 Lemma. *There exists a homeomorphism* $\kappa^*\colon S_0^* \times S^1 \to \hat{S}_0^* \times \hat{S}^1$ *inducing the isomorphim* $\kappa_\#^* : \pi_1(S_0^* \times S^1) \to \pi_1(\hat{S}_0^* \times \hat{S}^1)$*, and a homeomorphism* $\kappa\colon C_0 \to \hat{C}_0$ *covering* κ^*.

Proof. First observe that S_0^* is not a disk because in this case the Seifert surface S would be a covering space of S_0^* and therefore a disk. The $2g^*+s$ simple closed curves $\{a_i^*, b_i^*, r_j^* \mid 1 \le i \le g^*, 1 \le j \le s\}$ joined at the base point $P^* = p(P)$ represent a deformation retract R^* of S_0 as well as the respective generators $\{\hat{a}_i^*, \hat{b}_i^*, \hat{r}_j^*\} = \hat{R}^*$ in $\hat{S}_0^*$. It is now easy to see that there is a homeomorphism

$$\kappa^*|\colon R^* \times S^1 \to \hat{R}^* \times \hat{S}^1$$

inducing $\kappa_\#^*$ (Figure 6.1), because the homeomorphism obviously exists on each of

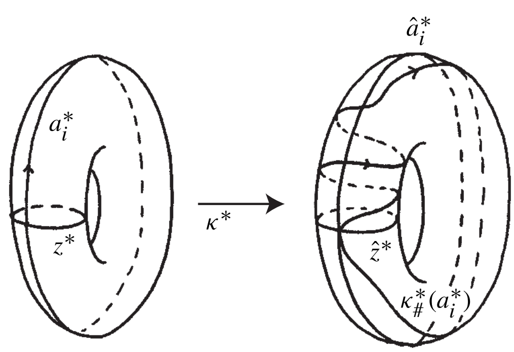


Figure 6.1

the tori $a_i^* \times S^1$, $b_i^* \times S^1$ and $r_j^* \times S^1$. The extension of $\kappa^*|R^*$ to

$$\kappa^*\colon S_0^* \times S^1 \to \hat{S}_0^* \times \hat{S}^1$$

presents no difficulty. Lemma 6.11 ensures the existence of a covering homeomorphism κ. □

We obtain by $\kappa_\#\colon \pi_1(C_0) \to \pi_1(\hat{C}_0)$ a new presentation of $\pi_1(C_0) \cong \pi_1(\hat{C}_0) = \langle \{\hat{r}_j, \hat{a}_i, \hat{b}_i \mid 1 \le i \le g^*, 1 \le j \le s\} \mid \rangle \times \langle \hat{z} \rangle$ such that

$$\hat{p}_\#(\hat{r}_j) = \hat{r}_j^*, \quad \hat{p}_\#(\hat{a}_i) = \hat{a}_i^*, \quad \hat{p}_\#(\hat{b}_i) = \hat{b}_i^*, \quad \hat{p}_\#(\hat{z}) = \hat{z}^{*n}. \tag{10}$$

From this presentation we can derive a presentation of $\mathfrak{G} \cong \pi_1(C)$ by introducing the defining relators $\kappa_\#(r_{j\nu}) = 1$. It suffices to choose $\nu = 0$ for all $j = 1, \dots, s$.

We get from 6.8 (1), 6.10, (9):

$$\kappa_\#(r_{j0}) = \hat{p}_\#^{-1}\kappa_\#^*(r_j^{*m_j}) = \hat{z}^{-a_j}\hat{r}_j^{m_j}. \tag{11}$$

Furthermore (see 6.8 (1), 6.10):

$$\kappa_\#^*(r^*) = \hat{z}^{*-\sum_{j=1}^s k_j\alpha_j} \cdot \hat{r}^*.$$

(8) and 6.10 imply

$$\kappa_\#^*(r^*) = \hat{z}^{*-\alpha}\kappa_\#^*(q^*).$$

By (5) and (9), $\kappa_\#^*(q^*) \in \hat{p}_\#\pi_1(\hat{C}_0)$, and by (1) and (10), $\hat{r}^* \in \hat{p}_\#\pi_1(\hat{C}_0)$. Now the definition of $\hat{p}_\#$ (see (9)) yields

$$\alpha \equiv \sum_{j=1}^{s} k_j\alpha_j \mod n.$$

By 6.9 we may replace α_1 by an element of the same coset mod m_1, such that the equation

$$\alpha = \sum_{j=1}^{s} k_j\alpha_j \tag{12}$$

is satisfied. By (8), $\kappa_\#^*(q^*) = \hat{r}^*$, and, since $p_\#(t) = t^*$, it follows from (6), (9) that

$$\kappa_\#(t) = \hat{z}^\beta \cdot \hat{r}^m. \tag{13}$$

6.13 Lemma. *S_0^* is a sphere with two boundary components: $g^* = 0$, $s = 2$. Moreover $m_1 \cdot m_1 = n$, $\gcd(m_1, m_2) = 1$. It is possible to choose $m = 1$, $\alpha = 1$, $\beta = 0$.*

There is a presentation

$$\mathfrak{G} = \langle \hat{z}, \hat{r}_1, \hat{r}_2 \mid \hat{z}^{-\alpha_1}\hat{r}_1^{m_1},\ \hat{z}^{-\alpha_2}\hat{r}_2^{m_2},\ [\hat{z}, \hat{r}_1],\ [\hat{z}, \hat{r}_2] \rangle$$

of the knot group $\mathfrak{G}$.

Proof. We have to introduce the relators $\hat{z}^{-\alpha_j}\hat{r}_j^{m_j}$ (see (11)) in

$$\pi_1(\hat{C}_0) = \langle \{\hat{r}_j, \hat{a}_i, \hat{b}_i \mid 1 \le i \le g^*, 1 \le j \le s\} \mid \rangle \times \langle \hat{z} \rangle.$$

The additional relator $\kappa_\#(t) = \hat{z}^\beta \cdot \hat{r}^m = 1$ must trivialize the group. This remains true, if we put $\hat{z} = 1$.

Now $g^* = 0$ follows. For $s \geq 3$ the resulting groups

$$\left\langle \{\hat{r}, \hat{r}_j \mid 1 \leq j \leq s\} \mid \hat{r}^{-m}, \hat{r}_j^{m_j}, \hat{r}^{-1} \prod_{j=1}^{s} \hat{r}_j \right\rangle \tag{14}$$

are known to be non-trivial [ZVC 1980, 4.16.4] since by definition $m_j > 1$. For $s = 2$ by the same argument (14) describes the trivial group only if $m = \pm 1$. The cases $s < 2$ cannot occur as $\mathfrak{k}$ was assumed to be non-trivial. By a suitable choice of the orientation of $r = \partial S$ we get $m = 1$. Thus by $\alpha = 1$, $\beta = 0$ equation (4) is satisfied. Now (12) takes the form

$$\alpha_1 k_1 + \alpha_2 k_2 = 1. \tag{15}$$

It follows that

$$\langle \hat{z}, \hat{r}_1, \hat{r}_2 \mid \hat{z}^{-\alpha_1} \hat{r}_1^{m_1},\ \hat{z}^{-\alpha_2} \hat{r}_2^{m_2},\ \hat{r}_1 \hat{r}_2,\ [\hat{z}, \hat{r}_1],\ [\hat{z}, \hat{r}_2] \rangle = 1$$

is a presentation of the trivialized knot group. By abelianizing this presentation yields

$$\alpha_1 m_2 + \alpha_2 m_1 = \pm 1. \tag{16}$$

The equations (15) and (16) are proportional since $m_2 k_2 - m_1 k_1 = n - n = 0$, by (1). As $m_j, k_j > 0$, they are indeed identical, $m_2 = k_1, m_1 = k_2$. □

It is a consequence of Lemma 6.13 that C_0 is obtained from a 3-sphere S^3 by removing three disjoint solid tori. Equation (13) together with $m = 1, \beta = 0$ shows $\kappa_\#(t) = \hat{r}$. We use this equation to extend $\kappa\colon C_0 \to \hat{C}_0$ to a homeomorphism $\hat{\kappa}$ defined on $C_0 \cup V(\mathfrak{k})$, obtained from C_0 by regluing the tubular neighbourhood $V(\mathfrak{k})$ of $\mathfrak{k}$. We get

$$\hat{\kappa}\colon C_0 \cup V(\mathfrak{k}) \to B \times \hat{\hat{S}}^1$$

where B is a ribbon with boundary $\partial B = \hat{r}_1 \cup \hat{r}_2$. The fundamental group $\pi_1(B \times \hat{\hat{S}}^1)$ is a free abelian group generated by $\hat{z}$ and $\hat{r}_1 = \hat{r}_2^{-1}$. Define $\hat{q}_1$ and $\hat{q}_2$ by

$$\begin{aligned} &\hat{\kappa}_\#(r_{10}) = \hat{z}^{-\alpha_1} \hat{r}_1^{m_1} = \hat{q}_1^{-1}, \\ &\hat{\kappa}_\#(r_{20}) = \hat{z}^{-\alpha_2} \hat{r}_1^{m_2} = \hat{q}_2^{-1}, \quad \alpha_1 m_2 + \alpha_2 m_1 = 1. \end{aligned} \tag{17}$$

(For the notation compare 6.8.) Now we glue two solid tori to $B \times \hat{\hat{S}}^1$ such that their meridians are identified with $\hat{q}_1, \hat{q}_2$, respectively, and obtain a closed manifold $\hat{\hat{S}}^3$. Thus $\hat{\kappa}$ can be extended to a homeomorphism $\hat{\hat{\kappa}}\colon S^3 \to \hat{\hat{S}}^3$. From (17) we see that $\hat{q}_1$ and $\hat{q}_2$ are a pair of generators of $\pi_1(\hat{r}_1 \times \hat{\hat{S}}^1)$. Therefore the torus $\hat{r}_1 \times \hat{\hat{S}}^1$ defines a Heegaard-splitting of $\hat{\hat{S}}^3$ which is the same as the standard Heegaard-splitting of genus one of the 3-sphere. The knot $\mathfrak{k}$ is isotopic (in S^3) to $z \subset \partial C_0$. Its image $\hat{\mathfrak{k}} = \hat{\kappa}(\mathfrak{k})$ can be represented by any curve $(Q \times \hat{\hat{S}}^1) \subset \hat{S}_0^* \times \hat{\hat{S}}^1$, where Q is a point of $\hat{S}_0^*$. Take $Q \in \hat{r}_1$ then $\hat{\mathfrak{k}}$ is represented by a simple closed curve on the unknotted torus $\hat{r}_1 \times \hat{\hat{S}}^1$ in $\hat{\hat{S}}^3$. This finishes the proof of Theorem 6.1. □

C Remarks on the Proof

In Lemma 6.13 we have obtained a presentation of the group of the torus knot which differs from the usual one (see Proposition 3.28). The following substitution connects both presentations:

$$u = \hat{r}_1^{m_2} \cdot \hat{z}^{\alpha_2}$$
$$v = \hat{r}_2^{m_1} \cdot \hat{z}^{\alpha_1}.$$

First observe that $\hat{r}_1$ and $\hat{r}_2$ generate $\mathfrak{G}$:

$$\hat{r}_1^n \cdot \hat{r}_2^n = \hat{r}_1^{m_1 k_1} \cdot \hat{r}_2^{m_2 k_2} = \hat{z}^{\alpha_1 k_1 + \alpha_2 k_2} = \hat{z},$$

as follows from (16), the presentation before (16), and (15). It follows that u and v are also generators:

$$u^{\alpha_1} = \hat{r}_1^{\alpha_1 m_2} \cdot \hat{z}^{\alpha_1 \alpha_2} = \hat{r}_1 \cdot \hat{r}_1^{-\alpha_2 m_1} \cdot \hat{z}^{\alpha_1 \alpha_2} = \hat{r}_1,$$

and similarly, $v^{\alpha_2} = \hat{r}_2$. The relation $u^{m_1} = v^{m_2}$ is easily verified:

$$u^{m_1} = \hat{r}_1^{m_1 m_2} \hat{z}^{\alpha_2 m_1} = \hat{z}^{\alpha_1 m_2 + \alpha_2 m_1} = \hat{z} = v^{m_2}.$$

Starting with the presentation

$$\mathfrak{G} = \langle u, v \mid u^a = v^b \rangle, \quad a = m_1,\ b = m_2,$$

one can re-obtain the presentation of 6.13 by introducing

$$\hat{z} = u^a = v^b \quad \text{and} \quad \hat{r}_1 = u^{a_1},\ \hat{r}_2 = v^{a_2}.$$

The argument also identifies the $\mathfrak{k}$ of 6.1 as the torus knot $\mathfrak{k}(m_1, m_2)$: for the definition of m_1, m_2 see 6.8 (1).

6.14. The construction used in the proof gives some additional information. The Hurwitz-formula [ZVC 1980, 4.14.23] of the covering $p_0\colon S_0 \to S_0^*$ gives

$$2g + \sum_{j=1}^{s} k_j = n(2g^* + s - 1) + 1.$$

Since $g^* = 0$, $s = 2$, $k_1 = b$, $k_2 = a$, $ab = n$ it follows that $2g + a + b = ab + 1$, hence

$$g = \frac{(a-1)(b-1)}{2}$$

and, by 4.6, this reproves the genus formula from 4.11.

6.15 On cyclic coverings of torus knots. The q-fold cyclic coverings $C^q_{a,b}$ of the complement $C_{a,b}$ of the knot $\mathfrak{t}(a,b)$ obviously have a period $n = ab$:

$$C^q_{a,b} \cong C^{q+kn}_{a,b}.$$

This is a consequence of the realization of $C_{a,b} \cong S \times I/f$ by a mapping f of period n. The covering transformation of $C^q_{a,b} \to C_{a,b}$ can be interpreted geometrically as a shift along the fibre $z = t^{ab} \cdot r^{-1} \simeq \mathfrak{t}(a,b)$ such that a move from one sheet of the covering to the adjoining one shifts $\mathfrak{t}(a,b)$ through $\frac{1}{ab}$ of its "length". There is an $(ab+1)$-fold cyclic covering of $C_{a,b}$ onto itself:

$$C_{a,b} \cong C^{ab+1}_{a,b} \to C_{a,b}.$$

All its covering transformations $\neq$ id map $\mathfrak{t}(a,b)$ onto itself but no point of $\mathfrak{t}(a,b)$ is left fixed. There is no extension of the covering transformation to the $(ab+1)$-fold cyclic covering $\bar{p}\colon S^3 \to S^3$ branched along $\mathfrak{t}(a,b)$, in accordance with Smith's Theorem [Smith 1934], see also Appendix B.8, [Zieschang 1981, 36.4]. The covering transformations can indeed only by extended to a manifold $\hat{C}_{a,b}$ which results from gluing to $C_{a,b}$ a solid torus whose meridian is tr^{-1} instead of t. The manifold $\hat{C}_{a,b}$ is always different from S^3 as long as $\mathfrak{t}(a,b)$ is a non-trivial torus knot. In fact, one can easily compute

$$\pi_1(\hat{C}_{a,b}) = \langle\, \hat{z}, \hat{r}_1, \hat{r}_2, \hat{r} \mid \hat{r}\hat{r}_1\hat{r}_2, \hat{r}^{ab+1}, \hat{r}_1^a, \hat{r}_2^b, [\hat{z}, \hat{r}_1], [\hat{z}, \hat{r}_2] \,\rangle$$

by using again the generators $\hat{r}_1, \hat{r}_2$ and $\hat{z}$. The group $\pi_1(\hat{C}_{a,b})$ is infinite since $|a| > 1$, $|b| > 1$, $|ab+1| > 6$, see [ZVC 1980, 6.4.7].

In the case of the trefoil $\mathfrak{t}(3,2)$ the curves, surfaces and mappings constructed in the proof can be made visible with the help of Figure 5.3. The mapping f of order $6 = 3 \cdot 2$, $a = m_1 = 3$, $b = m_2 = 2$ is the one given by f_φ (at the end of Chapter 5) for $\varphi = \pi$. Its exceptional points Q_{10}, Q_{11} are the centres of the disks D_1 and D_2 (Figure 5.3 and 6.2) while Q_{20}, Q_{21}, Q_{22} are the points in which the core of T_1 meets the Seifert surface S.

Figure 6.2 shows a fundamental domain of S relative to $\mathfrak{Z}_6 = \langle\, f \,\rangle$. If its edges are identified as indicated in Figure 6.2, one obtains as orbit manifold (Zerlegungsfläche) a 2-sphere or a twice punctured 2-sphere S_0^*, if exceptional points are removed.

Figure 6.3 finally represents the ribbon B embedded in S^3. One of its boundaries is placed on ∂T_1. The ribbon B represents the orbit manifold minus two disks. The orbit manifold itself can, of course, not be embedded in S^3, since there is no 2-sphere in S^3 which intersects a fibre z in just one point. The impossibility of such embeddings is also evident because B is twisted by 2π.

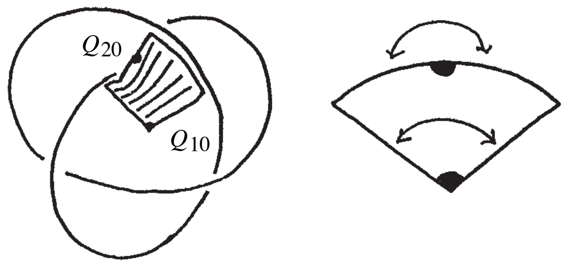


Figure 6.2

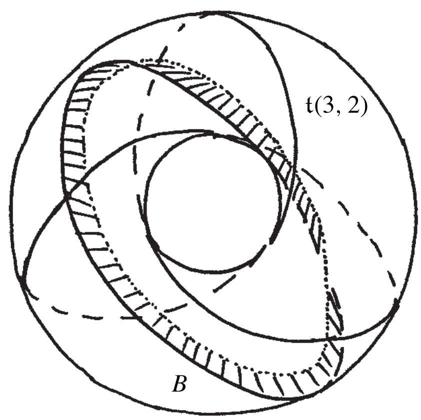


Figure 6.3

D History and Sources

Torus knots and their groups have been studied in [Dehn 1914] and [Schreier 1924]. The question of whether torus knots are determined by their groups was treated in [Murasugi 1961] and [Neuwirth 1961], and answered in the affirmative for alternating torus knots. This was proved in the general case in [Burde-Zieschang 1967], where torus knots were shown to be the only knots the groups of which have a non-trivial centre. A generalization of this theorem to 3-manifolds with non-trivial centre is due to Waldhausen [1967], and, as an application of it, the case of link groups with a centre $\neq 1$ was investigated in [Burde-Murasugi 1970].

E Exercises

E 6.1. Let a lens space $L(p,q)$ be given by a Heegaard splitting of genus one, $L(p,q) = V_1 \cup V_2$. Define a torus knot in $L(p,q)$ by a simple closed curve on $\partial V_1 = \partial V_2$. Determine the links in the universal covering S^3 of $L(p,q)$ which cover a torus knot in $L(p,q)$. (Remark: The links that occur in this way classify the genus one Heegaard-splittings of lens spaces.)

E 6.2. Show that the q-fold cyclic covering $C^q_{a,b}$ of a torus knot $\mathfrak{t}(a,b)$ is a Seifert fibre space, and that the fibration can be extended to the branched covering $\hat{C}^q_{a,b}$ without adding another exceptional fibre. Compute Seifert's invariants of fibre spaces for $\hat{C}^q_{a,b}$. (Remark: The 3-fold cyclic branched covering of a trefoil is a Seifert fibre space with three exceptional fibres of order two.)

Chapter 7

Factorization of Knots

In Chapter 2 we have defined a composition of knots. The main result of this chapter states that each tame knot is composed of finitely many indecomposable (prime) knots and that these factors are uniquely determined.

A Composition of Knots

In the following we often consider parts of knots, arcs, embedded in balls, and it is convenient to have the concept of knotted arcs:

7.1 Definition. Let $B \subset S^3$ be a closed ball carrying the orientation induced by the standard orientation of S^3. A simple path $\alpha\colon I \to B$ with $\alpha(\partial I) \subset \partial B$ and $\alpha(\mathring{I}) \subset \mathring{B}$ is called a *knotted arc*. Two knotted arcs $\alpha \subset B_1$, $\beta \subset B_2$ are called *equivalent* if there exists an orientation preserving homeomorphism $f\colon B_1 \to B_2$ such that $\beta = f\alpha$. An *arc* equivalent to a line segment is called *trivial*.

If α is a knotted arc in B and γ some simple curve on ∂B which connects the endpoints of α then $\alpha\gamma$ – with the orientation induced by α – represents the *knot corresponding to* α. This knot does not depend on the choice of γ and it follows easily that equivalent knotted arcs correspond to equivalent knots.

By a slight alteration in the definition of the composition of knots we get the following two alternative versions of its description. Figures 7.1 and 7.2 show that the different definitions are equivalent.

Figure 7.1

7.2 (a) Figure 7.1 describes the composition $\mathfrak{k} \# \mathfrak{l}$ of the knots $\mathfrak{k}$ and $\mathfrak{l}$ by joining representing arcs.

(b) Let $V(\mathfrak{k})$ be the tubular neighbourhood of the knot $\mathfrak{k}$, and $B \subset V(\mathfrak{k})$ some ball such that $\kappa' = \mathfrak{k} \cap B$ is a trivial arc in B, $\kappa = \mathfrak{k} - \kappa'$. If κ' is replaced by a knotted arc λ defining the knot $\mathfrak{l}$, then $\kappa \cup \lambda$ represents the product $\mathfrak{k} \# \mathfrak{l} = \kappa \cup \lambda$.

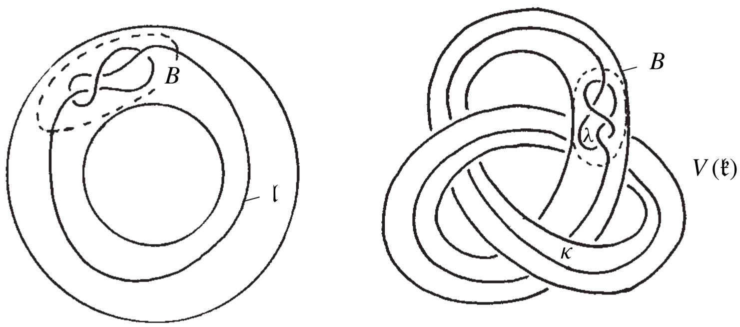


Figure 7.2

The following lemma is a direct consequence of the construction in 7.2 and is proved by Figures 7.3 and 7.4.

7.3 Lemma. (a) $\mathfrak{l} \# \mathfrak{k} = \mathfrak{k} \# \mathfrak{l}$.

(b) $\mathfrak{k}_1 \# (\mathfrak{k}_2 \# \mathfrak{k}_3) = (\mathfrak{k}_1 \# \mathfrak{k}_2) \# \mathfrak{k}_3$.

(c) *If* $\mathfrak{i}$ *denotes the trivial knot then* $\mathfrak{k} \# \mathfrak{i} = \mathfrak{k}$.

Proof. (a) Figure 7.3. (b) Figure 7.4. □

Figure 7.3

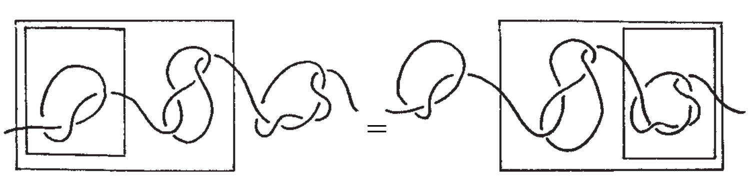


Figure 7.4

Associativity now permits us to define $\mathfrak{k}_1 \# \cdots \# \mathfrak{k}_n$ for an arbitrary $n \in \mathbb{N}$ without using brackets.

7.4 Proposition (Genus of knot compositions). *Let $\mathfrak{k}$, $\mathfrak{l}$ be knots and let $g(\mathfrak{x})$ denote the genus of the knot $\mathfrak{x}$. Then*

$$g(\mathfrak{k} \# \mathfrak{l}) = g(\mathfrak{k}) + g(\mathfrak{l}).$$

Proof. Let $B \subset S^3$ be a (p.l.-)ball. Since any two (p.l.-)balls in S^3 are ambient isotopic, see [Moise 1977, Chap. 17], we can describe $\mathfrak{k} \# \mathfrak{l}$ in the following way. Let $S_\mathfrak{k}$ and $S_\mathfrak{l}$ be Seifert surfaces of minimal genus of $\mathfrak{k}$ resp. $\mathfrak{l}$ such that $S_\mathfrak{k}$ is contained in some ball $B \subset S^3$, and $S_\mathfrak{l}$ in $S^3 - B$. Furthermore we assume $S_\mathfrak{k} \cap \partial B = S_\mathfrak{l} \cap \partial B = \alpha$ to be a simple arc. (See Figure 7.5.) Obviously $S_\mathfrak{k} \cup S_\mathfrak{l}$ is a Seifert surface spanning $\mathfrak{k} \# \mathfrak{l}$, hence:

$$g(\mathfrak{k} \# \mathfrak{l}) \leqq g(\mathfrak{k}) + g(\mathfrak{l}). \tag{1}$$

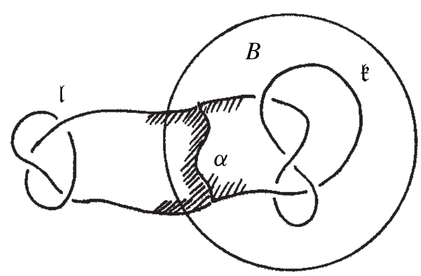


Figure 7.5

Let S be a Seifert surface of minimal genus spanning $\mathfrak{k} \# \mathfrak{l}$. The 2-sphere $S^2 = \partial B$ is supposed to be in general position with respect to S. Since $\mathfrak{k} \# \mathfrak{l}$ meets ∂B in two points, $\partial B \cap S$ consists of a simple arc α joining these points, and, possibly, a set of pairwise disjoint simple closed curves. An 'innermost' curve σ on ∂B bounds a disk $\delta \subset \partial B$ such that $\delta \cap S = \sigma$. Let us assume that σ does not bound a disk on S. In the case where σ separates S replace the component not containing $\mathfrak{k} \# \mathfrak{l}$ by δ. If σ does not separate S, cut S along σ, and attach two copies of δ along their boundaries to the cuts. (See proof of Lemma 4.5.) In both cases we obtain a Seifert surface for $\mathfrak{k} \# \mathfrak{l}$ of a genus smaller than that of S, contradicting the assumption of minimality.

Thus σ bounds a disk on S as well as on ∂B, and there is an isotopy of S which removes σ. So we may assume $S \cap \partial B = \alpha$, which means

$$g(\mathfrak{k}) + g(\mathfrak{l}) \leqq g(\mathfrak{k} \# \mathfrak{l}). \qquad \square$$

7.5 Corollary. (a) $\mathfrak{k} \# \mathfrak{l} = \mathfrak{k}$ *implies that* $\mathfrak{l}$ *is the trivial knot.*

(b) *If* $\mathfrak{k} \# \mathfrak{l}$ *is the trivial knot then* $\mathfrak{k}$ *and* $\mathfrak{l}$ *are trivial* $\square$

Corollary 7.5 motivates the following definition.

7.6 Definition (Prime knot). A knot $\mathfrak{k}$ which is the composition of two non-trivial knots is called *composite*; a non-trivial knot which is not composite is called a *prime knot*.

7.7 Corollary. *Genus* 1 *knots are prime.* □

7.8 Proposition. *Every* 2*-bridge knot* $\mathfrak{b}$ *is prime.*

Proof. Let δ_1 and δ_2 be disks spanning the arcs of $\mathfrak{b}$ in the upper half-space, and suppose that the other two arcs λ_i', $i \in \{1, 2\}$, of $\mathfrak{b}$ are contained in the boundary E of the half-space. The four "endpoints" of $E \cap \mathfrak{b}$ are joined pairwise by the simple arcs λ_i' and $\lambda_i = E \cap \delta_i$. We suppose the separating sphere S to be in general position with respect to E and δ_i. The intersections of S with $\mathfrak{b}$ may be pushed into two endpoints. Simple closed curves of $\delta_i \cap S$ and those of $E \cap S$ which do not separate endpoints can be removed by an isotopy of S. The remaining curves in $E \cap S$ must now be parallel, separating the arcs λ_1' and λ_2'. If there are more than one of these curves, there is a pair of neighbouring curves bounding annuli on E and S which together form a torus T. The torus T intersects δ_i in simple closed curves, not null-homotopic on T, bounding disks δ in δ_i with $\delta \cap T = \partial\delta$. So T bounds a solid torus which does not intersect $\mathfrak{b}$. There is an isotopy which removes the pair of neighbouring curves. We may therefore assume that $E \cap S$ consists of one simple closed curve separating λ_1' and λ_2'. The ball B bounded by S in $\mathbb{R}^3$ now intersects $\mathfrak{b}$ in, say, λ_1', and λ_1' is isotopic in $E \cap B$ to an arc of $S \cap E$. Hence this factor is trivial. □

A stronger result was proved in [Schubert 1954, Satz 7]:

7.9 Theorem (Schubert). *The minimal bridge number $b(\mathfrak{k})$ minus* 1 *is additive with respect to the product of knots:*

$$b(\mathfrak{k}_1 \# \mathfrak{k}_2) = b(\mathfrak{k}_1) + b(\mathfrak{k}_2) - 1. \qquad \square$$

7.10 Proposition (Group of composite knots). *Let $\mathfrak{k} = \mathfrak{k}_1 \# \mathfrak{k}_2$ and denote by $\mathfrak{G}$, $\mathfrak{G}_1$, $\mathfrak{G}_2$ the corresponding knot groups. Then $\mathfrak{G} = \mathfrak{G}_1 *_{\mathfrak{Z}} \mathfrak{G}_2$, where $\mathfrak{Z}$ is an infinite cyclic group generated by a meridian of $\mathfrak{k}$, and $\mathfrak{G}' = \mathfrak{G}_1' * \mathfrak{G}_2'$. Here $\mathfrak{G}_i$ und $\mathfrak{G}_i'$ are – in the natural way – considered as subgroups of $\mathfrak{G} = \mathfrak{G}_1 *_{\mathfrak{Z}} \mathfrak{G}_2$, $i = 1, 2$.*

Proof. Let S be a 2-sphere that defines the product $\mathfrak{k} = \mathfrak{k}_1 \# \mathfrak{k}_2$. Assume that there is a regular neighbourhood V of $\mathfrak{k}$ such that $S \cap V$ consists of two disks. Then $S \cap C$ is an annulus. The complement $C = \overline{S^3 - V}$ is divided by $S \cap C$ into C_1 and C_2 with $C = C_1 \cup C_2$ and $S \cap C = C_1 \cap C_2$. Since $\pi_1(S \cap C) \cong \mathbb{Z}$ is generated by a meridian it is embedded into $\pi_1(C_i)$ and the Seifert–van Kampen Theorem implies that

$$\pi_1(C) = \pi_1(C_1) *_{\pi_1(C_1 \cap C_2)} \pi_1(C_2) = \mathfrak{G}_1 *_{\mathfrak{Z}} \mathfrak{G}_2.$$

Applying Schreier's normal form $hg_1'g_2'\dots,\ h \in \mathfrak{Z},\ g_1' \in \mathfrak{G}_1',\ g_2' = \mathfrak{G}_2'$, the equation $\mathfrak{G}' = \mathfrak{G}_1' * \mathfrak{G}_2'$ follows from the fact that both groups are characterized by $h = 1$. □

7.11 Corollary. *Torus knots are prime.*

Proof. For this fact we give a geometric and a short algebraic proof.

1. *Geometric proof.* Let the torus knot $\mathfrak{t}(a,b)$ lie on an unknotted torus $T \subset S^3$ and let the 2-sphere S define a decomposition of $\mathfrak{t}(a,b)$. (By definition, $|a|, |b| \geq 2$.) We assume that S and T are in general position, that is, $S \cap T$ consists of finitely many disjoint simple closed curves. Such a curve either meets $\mathfrak{t}(a,b)$, is parallel to it or it bounds a disk D on T with $D \cap \mathfrak{t}(a,b) = \emptyset$. Choose γ as an innermost curve of the last kind, i.e., $D \cap S = \partial D = \gamma$. Then γ divides S into two disks D', D'' such that $D \cup D'$ and $D \cup D''$ are spheres, $(D \cup D') \cap (D \cup D'') = D$; hence, D' or D'' can be deformed into D by an isotopy of S^3 which leaves $\mathfrak{t}(a,b)$ fixed. By a further small deformation we get rid of one intersection of S with T.

Consider the curves of $T \cap S$ which intersect $\mathfrak{t}(a,b)$. There are one or two curves of this kind since $\mathfrak{t}(a,b)$ intersects S in two points only. If there is one curve it has intersection numbers $+1$ and -1 with $\mathfrak{t}(a,b)$ and this implies that it is either isotopic to $\mathfrak{t}(a,b)$ or nullhomotopic on T. In the first case $\mathfrak{t}(a,b)$ would be the trivial knot. In the second case it bounds a disk D_0 on T and $D_0 \cap \mathfrak{t}(a,b)$, plus an arc on S, represents one of the factor knots of $\mathfrak{t}(a,b)$; this factor would be trivial, contradicting the hypothesis.

The case remains where $S \cap T$ consists of two simple closed curves intersecting $\mathfrak{t}(a,b)$ exactly once. These curves are parallel and bound disks in one of the solid tori bounded by T. But this contradicts $|a|, |b| \geq 2$.

2. *Algebraic proof.* Let the torus knot $\mathfrak{t}(a,b)$ be the product of two knots. By 7.10,

$$\mathfrak{G} = \langle u, v \mid u^a v^{-b} \rangle = \mathfrak{G}_1 *_{\mathfrak{Z}} \mathfrak{G}_2,$$

where $\mathfrak{Z}$ is generated by a meridian t. The centre of the free product of groups with amalgamated subgroup is the intersection of the centres of the factors, see [ZVC 1980, 2.3.9]; hence, it is generated by a power of t. Since u^a is the generator of the center of $\mathfrak{G}$ it follows from 3.28 (b) that

$$u^a = (u^c v^d)^m \qquad \text{where} \quad \begin{vmatrix} a & -b \\ c & d \end{vmatrix} = 1,\ m \in \mathbb{Z}.$$

From the solution of the word problem it follows that this equation is impossible. □

Now we formulate the main theorem of this chapter which was first proved in [Schubert 1949].

7.12 Theorem (Unique prime decomposition of knots). *Each non-trivial knot $\mathfrak{k}$ is a finite product of prime knots and these factors are uniquely determined. More precisely:*

(a) $\mathfrak{k} = \mathfrak{k}_1 \# \cdots \# \mathfrak{k}_n$ *where each* $\mathfrak{k}_i$ *is a prime knot.*

(b) *If* $\mathfrak{k} = \mathfrak{k}_1 \# \cdots \# \mathfrak{k}_n = \mathfrak{k}'_1 \# \cdots \# \mathfrak{k}'_m$ *are two decompositions into prime factors* $\mathfrak{k}_i$ *or* $\mathfrak{k}'_j$*, respectively, then* $n = m$ *and* $\mathfrak{k}'_i = \mathfrak{k}_{j(i)}$ *for some permutation* $\begin{pmatrix} 1 & \dots & n \\ j(1) & \dots & j(n) \end{pmatrix}$.

Assertion (a) is a consequence of 7.4; part (b) will be proved in Section B. The results can be summarized as follows:

7.13 Corollary (Semigroup of knots). *The knots in* S^3 *with the operation # form a commutative semigroup with a unit element such that the law of unique prime decomposition is valid.* □

B Uniqueness of the Decomposition into Prime Knots: Proof

We will first describe a general concept for the construction of prime decompositions of a given knot $\mathfrak{k}$. Then we show that any two decompositions can be connected by a chain of 'elementary processes'.

7.14 Definition (Decomposing spheres). Let S_j, $1 \leqq j \leqq m$, be a system of disjoint 2-spheres embedded in S^3, bounding $2m$ balls B_i, $1 \leqq i \leqq 2m$, in S^3, and denote by B_j, $B_{c(j)}$ the two balls bounded by S_j. If B_i contains the s balls $B_{\ell(1)}, \dots, B_{\ell(s)}$ as proper subsets, $R_i = (B_i - \bigcup_{q=1}^{s} \mathring{B}_{\ell(q)})$ is called the *domain* R_i. The *spheres* S_j are said to be *decomposing* with respect to a knot $\mathfrak{k} \subset S^3$ if the following conditions are fulfilled:

(1) Each sphere S_j meets $\mathfrak{k}$ in two points.

(2) The arc $\kappa_i = \mathfrak{k} \cap R_i$, oriented as $\mathfrak{k}$, and completed by simple arcs on the boundary of R_i to represent a knot $\mathfrak{k}_i \subset R_i \subset B_i$, is prime. $\mathfrak{k}_i$ is called the *factor of* $\mathfrak{k}$ *determined by* B_i. By $\mathfrak{S} = \{(S_j, \mathfrak{k}) \mid 1 \leqq j \leqq m\}$ we denote a decomposing sphere system with respect to $\mathfrak{k}$; if $\mathfrak{k}$ itself is prime we put $\mathfrak{S} = \emptyset$.

It is immediately clear that $\mathfrak{k}_i$ does not depend on the choice of the arcs on ∂R_i. The following lemma connects this definition with our definition of the composition of a knot.

7.15 Lemma. *If* $\mathfrak{S} = \{(S_j, \mathfrak{k}) \mid 1 \leqq j \leqq m\}$ *is a decomposing system of spheres, then there are* $m + 1$ *balls* B_i *determining prime knots* $\mathfrak{k}_i$, $1 \leqq i \leqq m + 1$*, such that*

$$\mathfrak{k} = \mathfrak{k}_{j(1)} \# \cdots \# \mathfrak{k}_{j(m+1)}, \quad i \mapsto j(i) \text{ a permutation.}$$

Proof by induction on m. For $m = 0$ the assertion is obviously true and for $m = 1$ Definition 7.14 reverts to the original definition of the product of knots. For $m > 1$ let B_l be a ball not containing any other ball B_i and determining the prime knot $\mathfrak{k}_l$. Replacing the knotted arc $\kappa_l = B_l \cap \mathfrak{k}$ in $\mathfrak{k}$ by a simple arc on ∂B_l defines a (non-trivial) knot $\mathfrak{k}' \subset S^3$. The induction hypothesis applied to $\{(S_j, \mathfrak{k}') \mid 1 \leqq j \leqq m, j \neq l\}$ gives $\mathfrak{k}' = \mathfrak{k}_{j(1)} \# \cdots \# \mathfrak{k}_{j(m)}$. Now $\mathfrak{k} = \mathfrak{k}' \# \mathfrak{k}_l = \mathfrak{k}_{j(1)} \# \cdots \# \mathfrak{k}_{j(m+1)}$, $j(m+1) = l$. □

Figure 7.6 illustrates Definition 7.14 and Lemma 7.15.

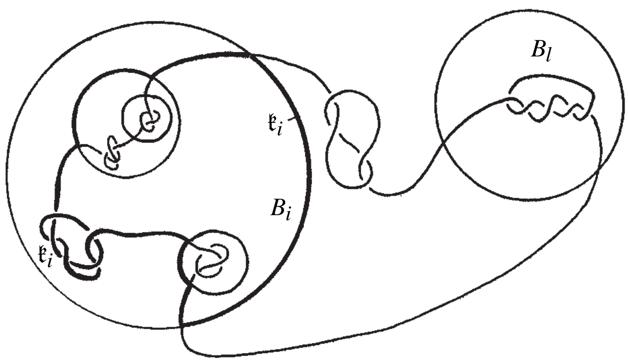


Figure 7.6

7.16 Definition. Two *decomposing systems of spheres* $\mathfrak{S} = \{S_j, \mathfrak{k}\}$, $\mathfrak{S}' = \{S_j', \mathfrak{k}\}$, $1 \leqq j \leqq m$, are called *equivalent* if they define the same (unordered) $(m+1)$ factor knots $\mathfrak{k}_{\ell(j)}$.

The following lemma is the crucial tool used in the proof of the Uniqueness Theorem. It describes a process by which one can pass over from a decomposing system to an equivalent one.

7.17 Lemma. *Let $\mathfrak{S} = \{(S_j, \mathfrak{k}) \mid 1 \leqq j \leqq m\}$ be a decomposing system of spheres, and let S' be another 2-sphere embedded in S^3, disjoint from $\{S_j \mid 1 \leqq j \leqq m\}$, bounding the balls B' and B'' in S^3. If B_i, $\partial B_i = S_i$, is a maximal ball contained in B', that is $B_i \subset B'$ but there is no B_j such that $B_i \subset B_j \subset B'$ for any $j \neq i$, and if B' determines the knot $\mathfrak{k}_i$ relative to the spheres $\{S_j \mid 1 \leqq j \leqq m, j \neq i\} \cup \{S'\}$, then these spheres define a decomposing system of spheres with respect to $\mathfrak{k}$ equivalent to $\mathfrak{S} = \{(S_j, \mathfrak{k}) \mid 1 \leqq j \leqq m\}$.*

Proof. Denote by $\mathfrak{k}_j$ the knot determined by B_j relative to $\mathfrak{S}$, and assume $B_j \subset B'$. For $i \neq j$, B_j determines the same knot $\mathfrak{k}_j$ relative to $\mathfrak{S}'$ since no inclusion $B_i \subset B_j \subset B'$, $i \neq j$, exists. If there is a ball B_ℓ, $B_j \subset B_\ell \subset B_i$, then $B_{c(j)}$ determines $\mathfrak{k}_\ell$ relative to $\mathfrak{S}$ and $\mathfrak{S}'$. If there is no such B_ℓ we have $\mathfrak{k}_{c(i)} = \mathfrak{k}_{c(j)}$ (see Figure 7.7). Now $B_{c(j)}$

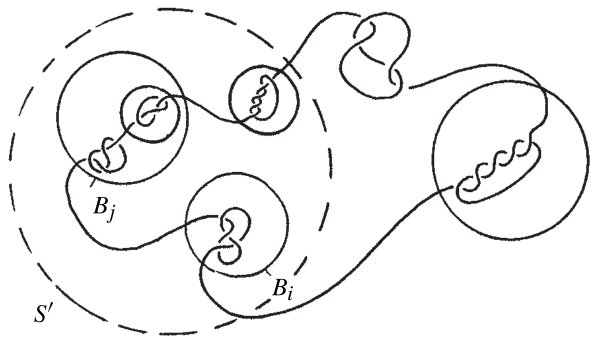


Figure 7.7

determines $\mathfrak{k}_i$ and B'' the knot $\mathfrak{k}_{c(j)}$. So instead of $\mathfrak{k}_i, \mathfrak{k}_{c(i)}, \mathfrak{k}_j, \mathfrak{k}_{c(j)} = \mathfrak{k}_{c(i)}$ determined by $B_i, B_{c(i)}, B_j, B_{c(j)}$ in $\mathfrak{S}$, we get $\mathfrak{k}_i, \mathfrak{k}_{c(j)}, \mathfrak{k}_j, \mathfrak{k}_i$ determined by $B', B'', B_j, B_{c(j)}$ in $\mathfrak{S}'$. The case $B_j \subset B''$ is dealt with in a similar way. □

7.18. *Proof* of the Uniqueness Theorem 7.12 (b). The proof consists in verifying the assertion that any two decomposing systems $\mathfrak{S} = \{(S_j, \mathfrak{k}) \mid 1 \leqq j \leqq m\}$, $\mathfrak{S}' = \{(S'_j, \mathfrak{k}) \mid 1 \leqq j \leqq m'\}$ with respect to the same knot $\mathfrak{k}$ are equivalent. We prove this by induction on $m + m'$. For $m + m' = 0$ nothing has to be proved. The spheres S_j and S'_j can be assumed to be in general position relative to each other.

To begin with, suppose there is a ball $B_i \cap \mathfrak{S}' = \emptyset$ not containing any other B_j or B'_j. Then by 7.17 some S'_j can be replaced by S_i and induction can be applied to $\mathfrak{k} \cap B_{c(i)}$.

If there is no such B_i (or B'_i), choose an innermost curve λ' of $S'_j \cap \mathfrak{S}$ bounding a disk $\delta' \subset S'_j = \partial B'_j$ such that B'_j contains no other ball B_k or B'_l. The knot $\mathfrak{k}$ meets δ' in at most two points. The disk δ' divides B_i into two balls B_i^1 and B_i^2, and in the first two cases of Figure 7.8 one of them determines a trivial knot or does not meet $\mathfrak{k}$ at all, and the other one determines the prime knot $\mathfrak{k}_i$ with respect to $\mathfrak{S}$, because otherwise δ' would effect a decomposition of $\mathfrak{k}_i$.

If B_i^1 determines $\mathfrak{k}_i$, replace S_i by ∂B_i^1 or rather by a sphere S' obtained from ∂B_i^1 by a small isotopy such that λ' disappears and general position is restored. The new decomposing system is equivalent to the old one by 7.17. If $\mathfrak{k}$ meets δ' in two points – the third case of Figure 7.8 – one may choose $\delta'' = S_j - \delta'$ instead of δ' if λ' is the only intersection curve on S'_j. If not, there will be another innermost curve $\lambda'' = S'_j \cap S_k$ on S'_j bounding a disk $\delta'' \subset S'_j$. In both events the knot $\mathfrak{k}$ will not meet δ'' and we are back to case one of Figure 7.8. Thus we obtain finally an innermost ball without intersections. This proves the theorem. □

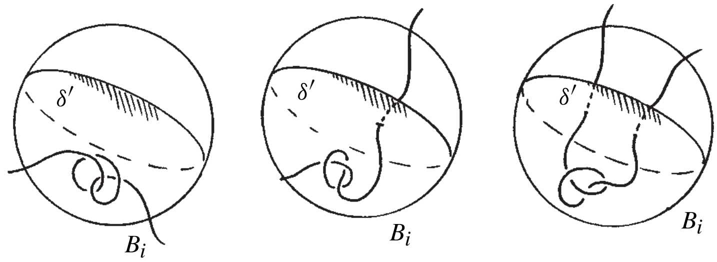


Figure 7.8

The theorem on the existence and uniqueness of decomposition carries over to the case of links without major difficulties [Hashizume 1958].

C Fibred Knots and Decompositions

It is easily seen that the product of two fibred knots is also fibred. It is also true that factor knots of a fibred knot are fibred. We present two proofs of this assertion, an algebraic one which is quite short, and a more complicated geometric one which affords a piece of additional insight.

7.19 Proposition (Decomposition of fibred knots). *A composite knot* $\mathfrak{k} = \mathfrak{k}_1 \# \mathfrak{k}_2$ *is a fibred knot if and only if* $\mathfrak{k}_1$ *and* $\mathfrak{k}_2$ *are fibred knots.*

Proof. Let $\mathfrak{G}, \mathfrak{G}_1, \mathfrak{G}_2$ etc. denote the groups of $\mathfrak{k}$, $\mathfrak{k}_1$ and $\mathfrak{k}_2$, respectively. By Proposition 7.10, $\mathfrak{G}' = \mathfrak{G}_1' * \mathfrak{G}_2'$. From the Grushko Theorem, see [ZVC 1980, 2.9.2], it follows that $\mathfrak{G}'$ is finitely generated if and only if $\mathfrak{G}_1'$ and $\mathfrak{G}_2'$ are finitely generated. Now the assertion 7.19 is a consequence of Theorem 5.1 □

7.20 Theorem (Decomposition of fibred knots). *Let* $\mathfrak{k}$ *be a fibred knot,* V *a regular neighbourhood of* $\mathfrak{k}$*,* $C = \overline{S^3 - V}$ *its complement, and* $p\colon C \to S^1$ *a fibration of* C*. Let a* 2*-sphere* $S \subset S^3$ *decompose* $\mathfrak{k}$ *into two non-trivial factors. Then there is an isotopy of* S^3 *deforming* S *into a sphere* S' *with the property that* $S' \cap V$ *consists of two disks and* $S' \cap C$ *intersects each fibre* $p^{-1}(t)$*,* $t \in S^1$*, in a simple arc. Moreover, the isotopy leaves the points of* $\mathfrak{k}$ *fixed.*

Proof. It follows by standard arguments that there is an isotopy of S^3 that leaves the knot pointwise fixed and maps S into a sphere that intersects V in two disks. Moreover, we may assume that p maps the boundary of each of these disks bijectively onto S^1. Suppose that S already has these properties. Consider the annulus $A = C \cap S$ and the fibre $F = p^{-1}(*)$ where $* \in S^1$. We may assume that S and F are in general position and that $A \cap \partial F$ consists of two points; otherwise S can be deformed by an ambient isotopy to fulfil these conditions.

Now $A \cap F$ is composed of an arc joining the points of $A \cap \partial F$ (which are on different components of ∂A) and, perhaps, further simple closed curves. Each of them bounds a disk on A, hence also a disk on F, since $\pi_1(F) \to \pi_1(C)$ is injective. Starting with an innermost disk δ on F we find a 2-sphere $\delta \cup \delta'$ consisting of disks $\delta \subset F$ and $\delta' \subset A$ such that $\delta \cap \delta'$ is the curve $\partial\delta = \partial\delta'$ and $\delta \cap A = \partial\delta$. Now δ' can be deformed to a disk not intersecting A and the number of components of $A \cap F$ becomes smaller. Thus we may assume that $A \cap F$ consists of an arc α joining the boundary components of A, see Figure 7.9.

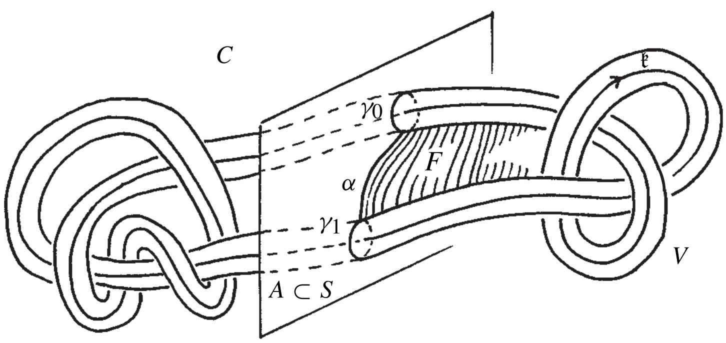


Figure 7.9

We cut C along F and obtain a space homeomorphic to $F \times I$. The cut transforms the annulus A into a disk D, $\partial D = \alpha_0\gamma_0\alpha_1^{-1}\gamma_1^{-1}$, where the $\alpha_i \subset F \times \{i\}$, $i = 0, 1$, are obtained from α and the γ_i from the meridians $\partial V \cap S$.

Let $q\colon F \times I \to F$ be the projection. The restriction $q|D$ defines a homotopy $q \circ \alpha_0 \simeq q \circ \alpha_1$. Since $q \circ \alpha_0$ and $q \circ \alpha_1$ are simple arcs with endpoints on ∂F it follows that these arcs are ambient isotopic and the isotopy leaves the endpoints fixed. (This can be proved in the same way as the refined Baer Theorem (see [ZVC 1980, 5.12.1]) which respects the basepoint; it can, in fact, be derived from that theorem by considering ∂F as the boundary of a 'small' disk around the basepoint of a closed surface F' containing F.) Thus there is a homeomorphism

$$H\colon (I \times I, (\partial I) \times I) \to (F \times I, (\partial F) \times I)$$

with

$$H(t, 0) = \alpha_0(t), \quad H(t, 1) = \alpha_1(t)$$

which is level preserving:

$$H(x, t) = (q(H(x, t)), t) \quad \text{for } (x, t) \in I \times I.$$

Therefore $D' = H(I \times I)$ is a disk and intersects each fibre $F \times \{t\}$ in a simple arc. It is transformed by re-identifying $F \times \{0\}$ and $F \times \{1\}$ into an annulus A' which intersects each fiber $p^{-1}(t)$, $t \in S^1$, in a simple closed curve. In addition $\partial A' = \partial A$.

It remains to prove that A' is ambient isotopic to A. An ambient isotopy takes D into general position with respect to D' while leaving its boundary ∂D fixed. Then $\mathring{D} \cap \mathring{D}'$ consists of simple closed curves. Take an innermost (relative to D') curve β. It bounds disks $\delta \subset D$ and $\delta' \subset D'$. The sphere $\delta \cup \delta' \subset F \times I \subset S^3$ bounds a 3-ball by the Theorem of Alexander. Thus there is an ambient isotopy of $F \times I$ which moves δ to δ' and a bit further to diminish the number of components in $D \cap D'$; during the deformation the boundary $\partial(F \times I)$ remains fixed. After a finite number of such deformations we may assume that $D \cap D' = \partial D = \partial D'$. Now $D \cup D'$ bounds a ball in $F \times I$ and D can be moved into D' by an isotopy which is the identity on $\partial(F \times I)$. Therefore the isotopy induces an isotopy of C that moves A to A'. (See Figure 7.10.) □

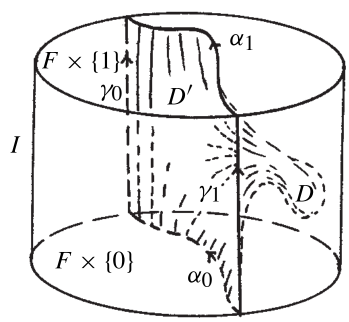


Figure 7.10

D History and Sources

The concept and the main theorem concerning products of knots are due to H. Schubert, and they are contained in his thesis [Schubert 1949]. His theorem was shown to be valid for links in [Hashizume 1958] where a new proof was given which in some parts simplified the original one. A further simplification can be derived from Milnor's uniqueness theorem for the factorization of 3-manifolds [Milnor 1962]. The proof given in this chapter takes advantage of it.

Compositions of knots of a more complicated nature have been investigated in [Kinoshita-Terasaka 1957] and [Hashizume-Hosokawa 1958], see E 14.3 (b).

Schubert used Haken's theory of incompressible surfaces to give an algorithm which effects the decomposition into prime factors for a given link [Schubert 1961].

In the case of a fibred knot primeness can be characterized algebraically: The subgroup of fixed elements under the automorphism $\alpha(t)\colon \mathfrak{G}' \to \mathfrak{G}'$, $\alpha(x) = t^{-1}xt$, $x \in G'$, t a meridian, consists of an infinite cyclic group generated by a longitude if and only if the knot is prime [Whitten 1972‴].

For higher dimensional knots the factorization is not unique: [Kearton 1979′], [Bayer 1980′], see also [Bayer-Hillman-Kearton 1981].

E Exercises

E 7.1. Show that in general the product of two links $\mathfrak{l}_1 \# \mathfrak{l}_2$ (use an analogous definition) will depend on the choice of the components which are joined.

E 7.2. An m-tangle $\mathfrak{t}_m$ consists of m disjoint simple arcs α_i, $1 \leqq i \leqq m$, in a (closed) 3-ball B, $\partial B \cap \bigcup_{i=1}^m \alpha_i = \underline{\bigcup_{i=1}^m \partial\alpha_i}$. An m-tangle t_m is called m-rational, if there are disjoint disks $\delta_i \subset B$, $\alpha_i = \overset{\circ}{B} \cap \partial\delta_i$. Show that $\mathfrak{t}_m$ is m-rational if and only if there is an m-tangle $\mathfrak{t}_m^C$ in the complement $C = \overline{S^3 - B}$ such that $\mathfrak{t}_m \cup \mathfrak{t}_m^c$ is the trivial knot. (Observe that the complementary tangle $\mathfrak{t}_m^C$ is rational.) 2-rational tangles are called just rational.

E 7.3. Let S^3 be composed of two balls B_1, B_2, $S^3 = B_1 \cup B_2$, $B_1 \cap B_2 = S^2 \subset S^3$. If a knot (or link) $\mathfrak{k}$ intersects the B_i in m-rational tangles $\mathfrak{t}_i = \mathfrak{k} \cap B_i$, $i = 1, 2$, then $\mathfrak{k}$ has a bridge number $\leqq m$.

E 7.4. Prove 7.5 (b) using 7.10 and 3.17.

E 7.5. Show that the groups of the product knots $\mathfrak{k}_1 \# \mathfrak{k}_2$ and $\mathfrak{k}_1 \# \mathfrak{k}_2^*$ are isomorphic, where $\mathfrak{k}_2^*$ is the mirror image of $\mathfrak{k}_2$. The knots are non-equivalent if $\mathfrak{k}_2$ is not amphicheiral.

Chapter 8
Cyclic Coverings and Alexander Invariants

One of the most important invariants of a knot (or link) is known as the Alexander polynomial. Sections A and B introduce the Alexander module, which is closely related to the homomorphic image $\mathfrak{G}/\mathfrak{G}''$ of the knot group modulo its second commutator subgroup $\mathfrak{G}''$. The geometric background is the infinite cyclic covering C_∞ of the knot complement and its homology (Section C). Section D is devoted to the Alexander polynomials themselves. Finite cyclic coverings are investigated in 8 E – they provide further invariants of knots.

Let $\mathfrak{k}$ be a knot, U a regular neighbourhood of $\mathfrak{k}$, $C = \overline{S^3 - U}$ the complement of the knot.

A The Alexander Module

We saw in Chapter 3 that the knot group $\mathfrak{G}$ is a powerful invariant of the knot, and the peripheral group system was even shown (compare 3.15) to characterize a knot. Torus knots could be classified by their groups (see 3.28). In general, however, knot groups are difficult to treat algebraically, and one tries to simplify matters by looking at homomorphic images of knot groups.

The knot group $\mathfrak{G}$ is a semidirect product $\mathfrak{G} = \mathfrak{Z} \ltimes \mathfrak{G}'$, where $\mathfrak{Z} \cong \mathfrak{G}/\mathfrak{G}'$ is a free cyclic group, and we may choose $t \in \mathfrak{G}$ (representing a meridian of $\mathfrak{k}$) as a representative of a generating coset of $\mathfrak{Z}$. The knot group $\mathfrak{G}$ can be described by $\mathfrak{G}'$ and the operation of $\mathfrak{Z}$ on $\mathfrak{G}'$: $a \mapsto a^t = t^{-1}at$, $a \in \mathfrak{G}'$. In Chapter 4 we studied the group $\mathfrak{G}'$; it is a free group, if finitely generated, but if not, its structure is rather complicated. We propose to study in this chapter the abelianized commutator subgroup $\mathfrak{G}'/\mathfrak{G}''$ together with the operation of $\mathfrak{Z}$ on it. We write $\mathfrak{G}'/\mathfrak{G}''$ additively and the induced operation as a multiplication:

$$a \mapsto ta, \quad a \in \mathfrak{G}'/\mathfrak{G}''.$$

(Note that the induced operation does not depend on the choice of the representative t in the coset $t\mathfrak{G}'$.) The operation $a \mapsto ta$ turns $\mathfrak{G}'/\mathfrak{G}''$ into a module over the group ring $\mathbb{Z}\mathfrak{Z} = \mathbb{Z}(t)$ of $\mathfrak{Z} \cong \langle\, t \,\rangle$ by

$$\Big(\sum_{i=-\infty}^{+\infty} n_i t^i\Big)a = \sum_{i=-\infty}^{+\infty} n_i (t^i a), \quad a \in \mathfrak{G}'/\mathfrak{G}'',\ n_i \in \mathbb{Z}.$$

8.1 Definition (Alexander module). The $\mathfrak{Z}$-module $\mathfrak{G}'/\mathfrak{G}''$ is called the *Alexander module* $M(t)$ of the knot group where t denotes either a generator of $\mathfrak{Z} = \mathfrak{G}/\mathfrak{G}'$ or a representative of its coset in $\mathfrak{G}$.

$M(t)$ is uniquely determined by $\mathfrak{G}$ except for the change from t to t^{-1}. We shall see, however, that the operations t and t^{-1} are related by a duality in $M(t)$, and that the invariants of $M(t)$ (see Appendix A.6) prove to be symmetric with respect to the substitution $t \mapsto t^{-1}$.

B Infinite Cyclic Coverings and Alexander Modules

The commutator subgroup $\mathfrak{G}' \triangleleft \mathfrak{G}$ defines an infinite cyclic covering $p_\infty \colon C_\infty \to C$ of the knot complement, $\mathfrak{G}' \cong \pi_1 C_\infty$. The Alexander module $M(t)$ is the first homology group $H_1(C_\infty) \cong \mathfrak{G}'/\mathfrak{G}''$, and the group of covering transformations which is isomorphic to $\mathfrak{Z} = \mathfrak{G}/\mathfrak{G}'$ induces on $H_1(C_\infty)$ the module operation. Following [Seifert 1934] we investigate $M(t) \cong H_1(C_\infty)$ in a similar way as we did in the case of the fundamental group $\pi_1 C_\infty \cong \mathfrak{G}'$, see 4.4.

Choose a Seifert surface $S \subset S^3$, $\partial S = \mathfrak{k}$ of genus h (not necessarily minimal), and cut C along S to obtain a bounded manifold C^*. Let $\{a_i \mid 1 \leqq i \leqq 2h\}$ be a canonical system of curves on S which intersect in a basepoint P. We may assume that $a_i \cap \mathfrak{k} = \emptyset$, and that $\prod_{i=1}^{h}[a_{2i-1}, a_{2i}] \simeq \mathfrak{k}$ on S, see 3.12. Retract S onto a regular neighbourhood B of $\{a_i \mid 1 \leqq i \leqq 2h\}$ consisting of $2h$ bands that start and end in a neighbourhood of P. Figure 8.1 shows two examples.

Choosing a suitable orientation we obtain $\partial B \simeq \prod_{i=1}^{h}[a_{2i-1}, a_{2i}]$ in B, and ∂B represents $\mathfrak{k}$ in S^3. The second assertion is proved as follows: by cutting S along $a_1, \dots, a_{2h}$ we obtain an annulus with boundaries $\mathfrak{k}$ and $\prod_{i=1}^{h}[a_{2i-1}, a_{2i}]$. This proves the first two parts of the following proposition:

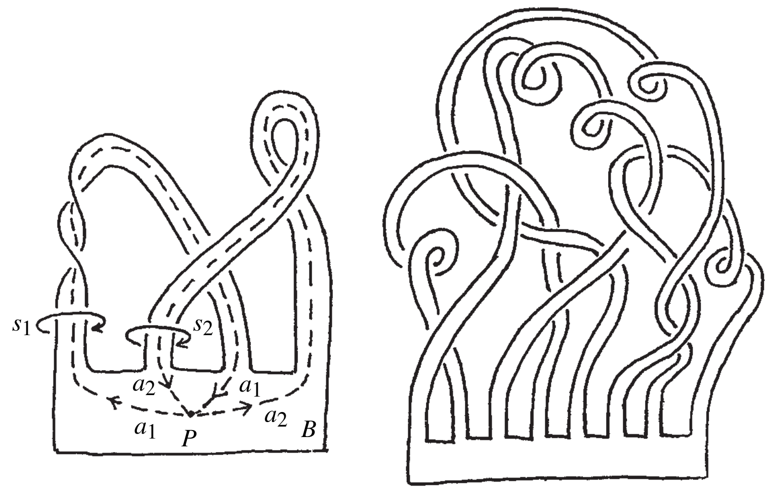
8.2 Proposition (Band projection of a knot). *Every knot can be represented as the boundary of an orientable surface S embedded in 3-space with the following properties:*

(a) $S = D^2 \cup B_1 \cup \dots \cup B_{2h}$ *where* D^2 *and each* B_j *is a disk.*

(b) $B_i \cap B_j = \emptyset$ *for* $i \neq j$, $\partial B_i = \alpha_i \gamma_i \beta_i \gamma_i'^{-1}$, $D^2 \cap B_i = \alpha_i \cup \beta_i$, $\partial D^2 = \alpha_1 \delta_1 \beta_2^{-1} \delta_2 \beta_1^{-1} \delta_3 \alpha_2 \delta_4 \dots \alpha_{2h-1} \delta_{4h-3} \beta_{2h}^{-1} \delta_{4h-2} \beta_{2h-1}^{-1} \delta_{4h-1} \alpha_{2h} \delta_{4h}$.

(c) *There is a projection which is locally homeomorphic on S (there are no twists in the bands B_i.)*

A projection of this kind is called *band projection of S* or *of* $\mathfrak{k}$ (see Figure 8.1 (b)).



(a) (b)

Figure 8.1

Proof. It remains to verify assertion (c). Since S is orientable every band is twisted through multiples of 2π (full twists). A full twist can be changed into a loop of the band (see Figure 8.2). □

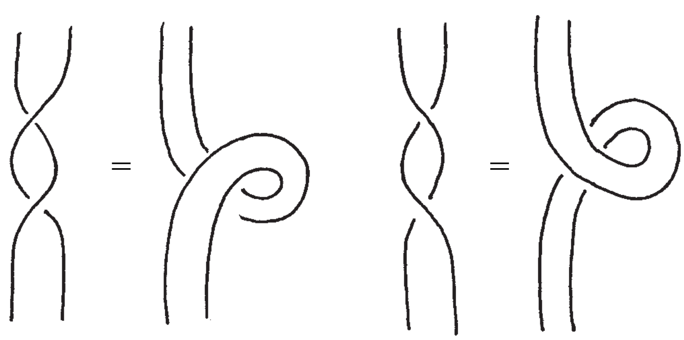


Figure 8.2

8.3 There is, obviously, a handlebody W of genus $2h$ contained in a regular neighbourhood of S with the following properties:

(a) $S \subset W$,

(b) $\partial W = S^+ \cup S^-$, $S^+ \cap S^- = \partial S^+ = \partial S^- = S \cap \partial W = \mathfrak{k}$, $S^+ \cong S^- \cong S$,

(c) S is a deformation retract of W.

We call S^+ the upside and S^- the downside of W. The curves $a_1, \dots, a_{2h}$ of S are projected onto curves $a_1^+, \dots, a_{2h}^+$ on S^+, and $a_1^-, \dots, a_{2h}^-$ on S^-, respectively. After connecting the basepoints of S^+ and S^- with an arc, they define together a canonical system of curves on the closed orientable surface ∂W of genus $2h$; in particular, they define a basis of $H_1(\partial W) \cong \mathbb{Z}^{4h}$. Clearly

$$a_i^+ \sim a_i^- \quad \text{in } W.$$

Choose a curve s_i on the boundary of the neighbourhood of the band B_i such that s_i bounds a disk in W. The orientations of the disk and of s_i are chosen such that the intersection number is $+1$, $\operatorname{int}(a_i, s_i) = 1$ (right-hand-rule), see Figure 8.3.

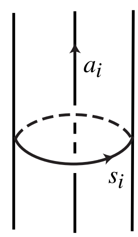


Figure 8.3

8.4 Lemma. (a) *The sets* $\{a_1^+, \dots, a_{2h}^+, a_1^-, \dots, a_{2h}^-\}$ *and* $\{s_1, \dots, s_{2h}, a_1^\varepsilon, \dots, a_{2h}^\varepsilon\}$ *with* $\varepsilon = +$ *or* $\varepsilon = -$ *are bases of* $H_1(\partial W) \cong \mathbb{Z}^{4h}$.

(b) $\{a_1^\varepsilon, \dots, a_{2h}^\varepsilon\}$ *(*$\varepsilon \in \{+, -\}$*) is a basis of* $H_1(W)$*, and* $\{s_1, \dots, s_{2h}\}$ *is a basis of* $H_1(\overline{S^3 - W}) = \mathbb{Z}^{2h}$.

Proof. The first statements in (a) and (b) follow immediately from the definition of W. The second one of (a) is a consequence of the fact that either system of curves $\{s_1, \dots, s_{2h}, a_1^\varepsilon, \dots, a_{2h}^\varepsilon\}$, $\varepsilon = +$ or $-$, is canonical on ∂W, that is, cutting ∂W along these curves transforms ∂W into a disk. Finally $\{s_1, \dots, s_{2h}\}$ is a basis of $H_1(\overline{S^3 - W})$, since W can be retracted to a $2h$-bouquet in S^3. The fundamental group and, hence, the first homology group of its complement can be computed in the same way as for the complement of a knot, see Appendix B.3. One may also apply the Mayer–Vietoris sequence:

$$0 = H_2(S^3) \to H_1(\partial W) \xrightarrow{\varphi} H_1(W) \oplus H_1(\overline{S^3 - W}) \to H_1(S^3) = 0.$$

Here $\varphi(s_i) = (0, s_i)$. From $H_1(\partial W) \cong \mathbb{Z}^{4h}$ and $H_1(W) \cong \mathbb{Z}^{2h}$ we get $H_1(\overline{S^3 - W}) = \mathbb{Z}^{2h}$. Now it follows from (a) that $\{s_1, \dots, s_{2h}\}$ is a basis of $H_1(\overline{S^3 - W})$. $\square$

8.5 Definition (Seifert matrix). (a) Let $v_{jk} = \operatorname{lk}(a_j^-, a_k)$ be the linking number of a_j^- and a_k. The $(2h \times 2h)$-matrix $V = (v_{jk})$ is called a *Seifert matrix* of $\mathfrak{k}$.

(b) Define $f_{jk} = \operatorname{lk}(a_j^- - a_j^+, a_k)$ and $F = (f_{jk})$.

A Seifert matrix (v_{jk}) can be read off a band projection in the following way: Consider the j-th band B_j endowed with the direction of its core a_j. Denote by l_{jk} (resp. r_{jk}) the number of times when B_j overcrosses B_k from left to right (resp. from right to left), then put $v_{jk} = l_{jk} - r_{jk}$.

8.6 Lemma. (a) *Let $i^\varepsilon \colon S^\varepsilon \to \overline{S^3 - W}$ denote the inclusion. Then*

$$i_*^+(a_j^+) = \sum_{k=1}^{2h} v_{kj} s_k \quad \textit{and} \quad i_*^-(a_j^-) = \sum_{k=1}^{2h} v_{jk}\, s_k.$$

$$\text{(b)} \qquad F = \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{pmatrix}.$$

Proof. (a) Let Z_j^- be a projecting cylinder of the curve a_j^-, and close Z_j^- by a point at infinity. $Z_j^- \cap (S^3 - W)$ represents a 2-chain realizing $a_j^- \sim \sum_{k=1}^{2h} v_{jk} s_k$, Figure 8.4. The same construction applied to a_j^+, using a projecting cylinder Z_k^+ directed upward,

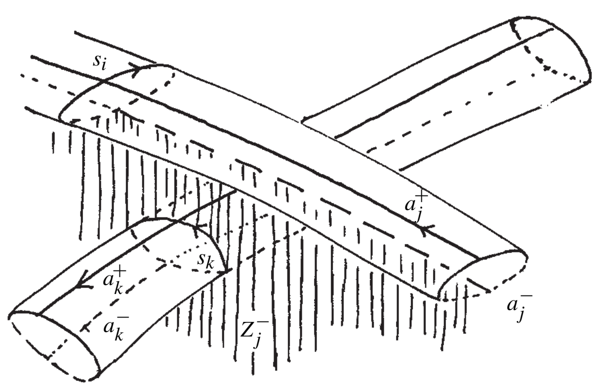


Figure 8.4

yields

$$a_j^+ \sim \sum_k v_{kj}\ s_k.$$

(b) There is an annulus bounded by $a_i^- - a_i^+$. It follows from the definition of the canonical system $\{a_j\}$ that

$$f_{2n-1,2n} = \mathrm{lk}(a_{2n-1}^- - a_{2n-1}^+, a_{2n}) = \mathrm{int}(a_{2n-1}, a_{2n}) = +1,$$
$$f_{2n,2n-1} = \mathrm{lk}(a_{2n}^- - a_{2n}^+, a_{2n-1}) = \mathrm{int}(a_{2n}, a_{2n-1}) = -1,$$

$f_{ik} = 0$ otherwise (Figure 8.5). (A compatible convention concerning the sign of the intersection number is supposed to have been agreed on.) The matrix $F = (f_{jk})$ is the intersection matrix of the canonical curves $\{a_j\}$ (Figure 8.5). □

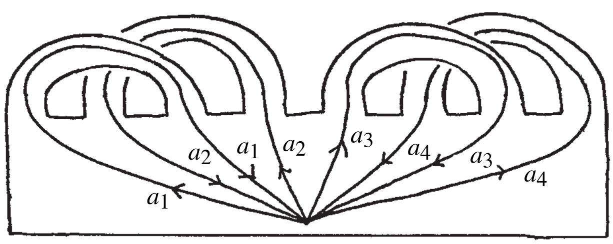


Figure 8.5

We write these equations frequently in matrix form, $a^- = Vs$, $a^+ = V^T s$, where a^+, a^-, s denote the $2h$-columns of the elements a_j^+, a_j^-, s_j.

8.5 and 8.6 imply that Seifert matrices have certain properties. The following proposition uses these properties to characterize Seifert matrices:

8.7 Proposition (Characterization of Seifert matrices). *A Seifert matrix V of a knot $\mathfrak{k}$ satisfies the equation $V - V^T = F$. (V^T is the transposed matrix of V and F is the intersection matrix defined in* 8.6 (b)).

Every square matrix V of even order satisfying $V - V^T = F$ is a Seifert matrix of a knot.

Proof. Figure 8.5 shows a realization of the matrix

$$V_0 = \begin{pmatrix} 0 & 1 & & & & & \\ 0 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & 0 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & 0 & 0 \end{pmatrix}.$$

Any $2h \times 2h$ matrix V satisfying $V - V^T = F$ is of the form $V = V_0 + Q$, $Q = Q^T$. A realization of V is easily obtained by an inductive argument on h as shown in Figure 8.6. (Here a $(2h-2) \times (2h-2)$ matrix V_1 and a 2×2 matrix V_2 are assumed to be already realized; the bands are represented just by lines.) The last two bands can be given arbitrary linking numbers with the first $2h-2$ bands. □

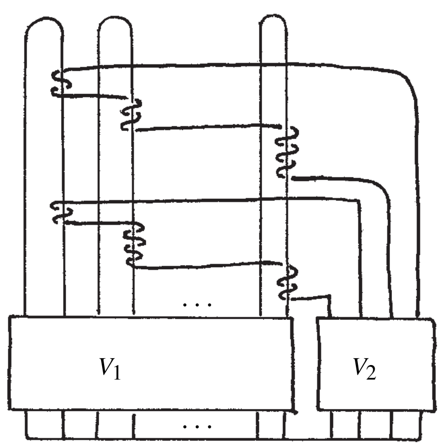


Figure 8.6

C Homological Properties of C_∞

Let S be a Seifert surface, W its closed regular neighbourhood, $C^* = \overline{S^3 - W}$, and let C_i^* $(i \in \mathbb{Z})$ be copies of C^*. For the expressions a^+, a^-, s see 8.3.

8.8 Theorem. *Let V be a Seifert matrix of a knot. Then $A(t) = V^T - tV$ is a presentation matrix of the Alexander module $H_1(C_\infty) = M(t)$. More explicitly: $H_1(C_\infty)$ is generated by the elements*

$$t^i s_j,\ i \in \mathbb{Z}, 1 \le j \le 2h, \quad \textit{and} \quad t^i a_j^+ = \sum_{k=1}^{2h} t^i v_{kj} s_k = \sum_{k=1}^{2h} t^{i+1} v_{jk} s_k = t^{i+1} a_j^-$$

are defining relations.

Proof. We use the notation of 4.4. By 8.4 (b) the elements $\{t^i s_j \mid 1 \leqq j \leqq 2h\}$ represent a basis of $H_1(C_i^*)$. The defining relations $t^i a_j^+ = t^{i+1} a_j^-$ are obtained from the identification $S_i^+ = S_{i+1}^-$ by abelianizing the Seifert–van Kampen theorem. □

We call a presentation matrix of the Alexander module an *Alexander matrix*.

For further use we are interested in the other homology groups of C_∞. (This paragraph may be skipped at first reading.)

8.9 Proposition.

$$\begin{aligned} H_m(C_\infty) &= 0 \quad \textit{for } m > 1,\\ H_1(C_\infty, \partial C_\infty) &\cong H_1(C_\infty),\\ H_2(C_\infty, \partial C_\infty) &\cong \mathbb{Z},\\ H_m(C_\infty, \partial C_\infty) &= 0 \quad \textit{for } m > 2. \end{aligned}$$

Proof. C_∞ is a 3-dimensional non-compact manifold, and ∂C_∞ is an open 2-manifold. Thus: $H_m(C_\infty) = H_m(C_\infty, \partial C_\infty) = 0$ for $m \geqq 3$, and $H_2(\partial C_\infty) = 0$. In 3.1 (a) we showed $H_i(C) = 0$ for $i \geqq 2$. The exact homology sequence of the pair (C, C^*), for C^* see 4.4, then gives

$$0 = H_3(C) \to H_3(C, C^*) \to H_2(C^*) \to H_2(C) = 0,$$

or, $H_3(C, C^*) \cong H_2(C^*)$. Now $(C, C^*) \to (W, \partial W)$ is an excision, and $(W, \partial W) \to (S, \partial S)$ a homotopy equivalence. It follows that

$$0 = H_3(S, \partial S) \cong H_2(C^*).$$

We apply the Mayer–Vietoris sequence to the decomposition

$$E_0 \cup E_1 = C_\infty, \quad E_0 = \bigcup_{i \in \mathbb{Z}} C_{2i}^*, \quad E_1 = \bigcup_{i \in \mathbb{Z}} C_{2i+1}^*:$$

$$\begin{aligned} 0 = H_2(E_0) \oplus H_2(E_1) \to H_2(C_\infty) \to H_1(E_0 \cap E_1)\\ \xrightarrow{j_*} H_1(E_0) \oplus H_1(E_1) \to H_1(C_\infty) \to H_0(E_0 \cap E_1). \end{aligned}$$

(Observe that $E_0 \cap E_1 = \bigcup_{i\in\mathbb{Z}} S_i$.)

Since E_0 and E_1 consist of disjoint copies of C^*, we have $H_2(E_0) = H_2(E_1) = 0$. The homomorphism $H_0(\bigcup_i S_i) \to H_0(E_0) \oplus H_0(E_1)$ is injective, since for $i \neq j$ the surfaces S_i and S_j belong to different components of E_0 or E_1. This implies that

$$\begin{array}{ccccccccc} 0 \longrightarrow H_2(C_\infty) \longrightarrow & H_1(\bigcup_i S_i) & \xrightarrow{j_*} & H_1(E_0) \oplus H_1(E_1) & \longrightarrow H_1(C_\infty) \longrightarrow 0 \\ & \| & & \| & \\ & \bigoplus_{i\in\mathbb{Z}} H_1(S_i) & & \bigoplus_{i\in\mathbb{Z}} H_1(C_i) & \end{array}$$

is exact. We prove that j_* is an isomorphism. The inclusion $i\colon S^+ \cup S^- \to C^*$ induces a homomorphism $i_*\colon H_1(S^+ \cup S^-) \to H_1(C^*)$ which can be computed by the equations

$$i_*^+(a^+) = V^T s, \quad i_*^-(a^-) = Vs$$

of 8.6:

$$i_*(a^-, a^+) = i_*^-(a^-) - i_*^+(a^+) = (V - V^T)s = Fs.$$

It follows that i_*, and hence, j_* is an isomorphism, since det $F = 1$. (The sign in $-i_*^+(a^+)$ is due to the convention that the orientation induced by the orientation of C^* on S^- resp. S^+ coincides with that of S^- but is opposite to that of S^+.)

We conclude: $H_2(C_\infty) = 0$. The homology sequence then yields

$$0 = H_2(C_\infty) \to H_2(C_\infty, \partial C_\infty) \to H_1(\partial C_\infty) \underset{e_*}{\to} H_1(C_\infty) \to H_1(C_\infty, \partial C_\infty) \to 0.$$

∂C_∞ is an annulus: $\partial S_1 \times \mathbb{R}$; this implies that e_* is the null-homomorphism, thus

$$H_2(C_\infty, \partial C_\infty) \cong H_1(\partial C_\infty) \cong \mathbb{Z},$$

$$H_1(C_\infty, \partial C_\infty) \cong H_1(C_\infty). \qquad \square$$

D Alexander Polynomials

The Alexander module $M(t)$ of a knot is a finitely presented $\mathfrak{Z}$-module. In the preceding section we have described a method of obtaining a presentation matrix $A(t)$ (an Alexander matrix) of $M(t)$. An algebraic classification of Alexander modules is not known, since the group ring $\mathbb{Z}(t)$ is not a principal ideal domain. But the theory of finitely generated modules over principal ideal domains can nevertheless be applied to obtain algebraic invariants of $M(t)$.

We call two *Alexander matrices* $A(t)$, $A'(t)$ *equivalent*, $A(t) \sim A'(t)$, if they present isomorphic modules.

Let R be a commutative ring with a unity element 1, and A an $m \times n$-matrix over R. We define *elementary ideals* $E_k(A) \subset R$ for $k \in \mathbb{Z}$ by

$$E_k(A) = \begin{cases} 0, & \text{if } n-k > m \text{ or } k < 0, \\ R, & \text{if } n-k \leqq 0, \\ \text{ideal, generated by the } (n-k)\times(n-k) \text{ minors of } A \text{ if } 0 < n-k \leqq m. \end{cases}$$

It follows from the Laplace expansion theorem that the elementary ideals form an ascending chain

$$0 = E_{-1}(A) \subset E_0(A) \subset E_1(A) \subset \cdots \subset E_n(A) = E_{n+1}(A) = \cdots = R.$$

Given a knot $\mathfrak{k}$, its Alexander module $M(t)$ and an Alexander matrix $A(t)$ we call $E_k(t) = E_{k-1}(A(t))$ the k-th elementary ideal of $\mathfrak{k}$. The proper ideals $E_k(t)$ are invariants of $M(t)$, and hence, of $\mathfrak{k}$. Compare Appendix A.6, [Crowell-Fox 1963, Chapter VII].

8.10 Definition (Alexander polynomials). The greatest common divisor $\Delta_k(t)$ of the elements of $E_k(t)$ is called the *k-th Alexander polynomial of* $M(t)$, resp. of the knot. Usually the first Alexander polynomial $\Delta_1(t)$ is simply called *the Alexander polynomial* and is denoted by $\Delta(t)$ (without an index). If there are no proper elementary ideals, we say that the Alexander polynomials are trivial, $\Delta_k(t) = 1$.

Remark. $\mathbb{Z}(t)$ is a unique factorization ring. So $\Delta_k(t)$ exists, and it is determined up to a factor $\pm t^\nu$, a unit of $\mathbb{Z}(t)$. It will be convenient to introduce the following notation:

$$f(t) \doteq g(t) \quad \text{for } f(t), g(t) \in \mathbb{Z}(t),\ f(t) = \pm t^\nu g(t),\ \nu \in \mathbb{Z}.$$

8.11 Proposition. *The (first) Alexander polynomial $\Delta(t)$ is obtained from the Seifert matrix V of a knot by*

$$|V^T - tV| = \det(V^T - tV) = \Delta(t).$$

The first elementary ideal $E_1(t)$ is a principal ideal.

Proof. $V^T - tV = A(t)$ is a $2h \times 2h$-matrix. $|A(t)|$ generates the elementary ideal $E_0(A(t)) = E_1(t)$. Since $\det(A(1)) = 1$, the ideal does not vanish, $E_1(t) \neq 0$. □

8.12 Proposition. *The Alexander matrix $A(t)$ of a knot $\mathfrak{k}$ satisfies*

(a) $A(t) \sim A^T(t^{-1})$ (*Duality*).

The Alexander polynomials $\Delta_k(t)$ are polynomials of even degree with integral coefficients subject to the following conditions:

(b) $\Delta_k(t) | \Delta_{k-1}(t)$,

(c) $\Delta_k(t) \doteq \Delta_k(t^{-1})$ (*Symmetry*),

(d) $\Delta_k(1) = \pm 1$.

Remark. The symmetry (c) implies, together with deg $\Delta_k(t) \equiv 0 \bmod 2$, that $\Delta_k(t)$ is a symmetric polynomial:

$$\Delta_k(t) = \sum_{i=0}^{2r} a_i t^i, \quad a_{2r-i} = a_i,\ a_0 = a_{2r} \neq 0$$

Proof. Duality follows from the fact that $A(t) = V^T - tV$ is an Alexander matrix by 8.8, $(V^T - t^{-1}V)^T = -t^{-1}(V^T - tV)$. This implies $E_k(t) = E_k(t^{-1})$ and (c). For $t = 1$ we get: $A(1) = F^T$, and since det $F = 1$, we have $E_k(1) = \mathbb{Z}(1) = \mathbb{Z}$, which proves (d). The fact that $\Delta_k(t)$ is of even degree is a consequence of (c) and (d). Property (b) follows from the definition. □

The symmetry of $\Delta(t)$ suggests a transformation of variables in order to describe the function $\Delta(t)$ by an arbitrary polynomial in $\mathbb{Z}(t)$ of half the degree of $\Delta(t)$. Write

$$\Delta(t) \doteq a_r + a_{r+1}(t + t^{-1}) + \cdots + a_{2r}(t^r + t^{-r}),$$

and note that $t^k + t^{-k}$ is a polynomial in $(t + t^{-1})$ with coefficients in $\mathbb{Z}$. The proof is by induction on k, using the Bernoulli formula. For the sake of normalizing we introduce $u = t + t^{-1} - 2$ as a new variable, and obtain $\Delta(t) \doteq \sum_{i=0}^{r} c_i u^i$, $c_0 = 1, c_i \in \mathbb{Z}$. Starting from $\Delta(t) = |V^T - tV|$ we may express the Alexander polynomial as a characteristic polynomial: By $V^T = V - F$, we get $\Delta(t) \doteq |F^T V - \lambda E|, \lambda^{-1} = 1 - t$. Now $(\lambda(\lambda - 1))^{-1} = u$, hence,

$$|F^T V - \lambda E| \doteq \sum_{i=0}^{r} c_{r-i}(\lambda(\lambda - 1))^i.$$

Clearly, every polynomial $\sum_{i=0}^{r} c_i u^i$ yields a "symmetric polynomial" putting $u = t + t^{-1} - 2$.

8.13 Theorem. *The Alexander polynomial $\Delta(t) = \sum_{i=0}^{2r} a_i t^i$, $a_{2r-i} = a_i$ of a knot can be written in the form*

$$\Delta(t) \doteq \sum_{i=0}^{r} c_i u^i = u^{2r} \sum_{i=0}^{r} c_{r-i}(\lambda(\lambda - 1))^i = \pm|F^T V - \lambda E| = \chi(\lambda),$$

with $u = t + t^{-2} - 2$, $\lambda^{-1} = 1 - t$, $c_0 = 1$ and $c_i \in \mathbb{Z}$. Given arbitrary integers $c_i \in \mathbb{Z}$, $1 \leqq i \leqq r$, there is a knot $\mathfrak{k}$ with Alexander polynomial

$$\Delta(t) \doteq \sum_{i=0}^{r} c_i u^i, \quad c_0 = 1.$$

Consequently, every symmetric polynomial $\Delta(t) = \sum_{i=0}^{2r} a_i t^i$ with $\Delta(1) = \pm 1$ is the Alexander polynomial of some knot $\mathfrak{k} \subset S^3$.

Proof. The following $(2r \times 2r)$-matrix

$$V = \begin{pmatrix} c_1 & c_1 & 0 & 1 & & & & & & \\ c_1 - 1 & c_1 & 0 & 1 & & & & & & \\ 0 & 0 & c_2 & c_2 & 0 & 1 & & & & \\ 1 & 1 & c_2 - 1 & c_2 - 1 & 0 & 1 & & & & \\ & & 0 & 0 & & & & & & \\ & & 1 & 1 & \ddots & & & & & \\ & & & & & & & & 0 & 1 \\ & & & & & & & \ddots & 0 & 1 \\ & & & & & & 0 & 0 & c_r & c_r \\ & & & & & & 1 & 1 & c_r - 1 & c_r - 1 \end{pmatrix}$$

is a Seifert matrix (compare Theorem 8.7). We propose to show:

$$\chi(\lambda) = |F^T V - \lambda E| = \sum_{i=0}^{r-1} c_{r-i} (-1)^{r-i-1} \cdot (\lambda(\lambda - 1))^i + (\lambda(\lambda - 1))^r$$

by induction on r.

Denote the determinant consisting of the first $2i$ rows and columns of $(F^T V - \lambda E)$ by D_{2i}, and by D'_{2i} resp. D''_{2i} the determinants that result from D_{2i} when the last column – resp. the last but one – of D_{2i} is replaced by $(0, \dots, -1, 1)^T$. Then, by expanding D_{2r} by the 2×2-minors of the last two rows, we obtain:

$$D_{2r} = D_{2(r-1)} \cdot \lambda(\lambda - 1) - c_r (D'_{2(r-1)} + D''_{2(r-1)}).$$

Again by expanding $D'_{2(r-1)}$ and $D''_{2(r-1)}$ in the same way:

$$D'_{2(r-1)} + D''_{2(r-1)} = -(D'_{2(r-2)} + D''_{2(r-2)}).$$

By induction:

$$D'_{2(r-1)} + D''_{2(r-1)} = (-1)^{r-2} (D'_2 + D''_2) = (-1)^{r-2}.$$

Hence,

$$D_{2r} = D_{2(r-1)} \lambda(\lambda - 1) + (-1)^{r-1} \cdot c_r.$$

By induction again:

$$\begin{aligned} D_{2r} &= (\lambda(\lambda - 1))^r + \lambda(\lambda - 1) \cdot \sum_{i=0}^{r-2} c_{r-1-i} (-1)^{r-2-i} \cdot (\lambda(\lambda - 1))^i + (-1)^{r-1} \cdot c_r \\ &= (\lambda(\lambda - 1))^r + \sum_{i=0}^{r-1} c_{r-i} (-1)^{r-i-1} (\lambda(\lambda - 1))^i. \end{aligned}$$

□

Remark. It is possible to construct a knot with given arbitrary polynomials $\Delta_k(t)$ subject to the conditions (b)–(d) of 8.12 [Levine 1965].

The presentation of the Alexander polynomial in the concise form

$$\Delta(t) \doteq \sum_{i=0}^{n} c_i u^i$$

was first given in [Crowell-Fox 1963, Chapter IX, Exercise 4] and employed later in [Burde 1966] where the coefficients c_i represented twists in a special knot projection. This connection between the algebraic invariant $\Delta(t)$ and the geometry of the knot projection has come to light very clearly through Conway's discovery [Conway 1970]. The Conway polynomial is closely connected to the form $\Sigma c_i u^i$ of the Alexander polynomial. It is, however, necessary to include links in order to get a consistent theory. This will be done in Chapter 13

8.14 Proposition. *Let $V_{\mathfrak{k}}$ and $V_{\mathfrak{l}}$ be Seifert matrices for the knots $\mathfrak{k}$ and $\mathfrak{l}$, and let $\Delta^{(\mathfrak{k})}(t)$ and $\Delta^{(\mathfrak{l})}(t)$ denote their Alexander polynomials. Then*

$$\begin{pmatrix} V_{\mathfrak{k}} & 0 \\ 0 & V_{\mathfrak{l}} \end{pmatrix} = V$$

is a Seifert matrix of the product knot $\mathfrak{k} \# \mathfrak{l}$, and

$$\Delta^{(\mathfrak{k}\#\mathfrak{l})}(t) = \Delta^{(\mathfrak{k})}(t) \cdot \Delta^{(\mathfrak{l})}(t).$$

Proof. The first assertion is an immediate consequence of the construction of a Seifert surface of $\mathfrak{k} \# \mathfrak{l}$ in 7.4. The second one follows from

$$|V^T - tV| = |V^{(\mathfrak{k})^T} - tV^{(\mathfrak{k})}| \cdot |V^{(\mathfrak{l})^T} - tV^{(\mathfrak{l})}|. \qquad \square$$

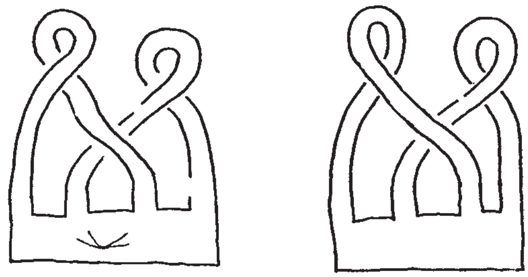
8.15 Examples (a) The Alexander polynomials of a trivial knot are trivial: $\Delta_k(t) = 1$. (In this case $\mathfrak{G} = \mathfrak{G}/\mathfrak{G}' \cong \mathfrak{Z}$, $\mathfrak{G}' = 1$, $M(t) = (0)$.)

(b) Figure 8.7 (a) and (b) show band projections of the trefoil 3_1 and the four-knot 4_1. The Seifert matrices are:

$$V_{3_1} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}, \qquad V_{4_1} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix},$$

$$|V_{3_1}^T - tV_{3_1}| \doteq t^2 - t + 1, \quad |V_{4_1}^T - tV_{4_1}| \doteq t^2 - 3t + 1.$$

(For further examples see E 8.6.)



(a) (b)

Figure 8.7

8.16 Proposition (Alexander polynomials of fibred knots). *The Alexander polynomial* $\Delta(t) = \sum_{i=0}^{2g} a_i t^i$ *of a fibred knot* $\mathfrak{k}$ (*see Chapter* 5 B) *satisfies the conditions*

(a) $\Delta(0) = a_0 = a_{2g} = \pm 1$,

(b) $\deg \Delta(t) = 2g$, *g the genus of* $\mathfrak{k}$.

Proof. If S is a Seifert surface of minimal genus g spanning $\mathfrak{k}$, the inclusion $i^{\pm}\colon S^{\pm} \to C^*$ induces isomorphisms $i^{\pm}_*\colon \pi_1 S^{\pm} \to \pi_1 C^*$ (by 4.6). Hence, $i^{\pm}_*\colon H_1(S^{\pm}) \to H_1(C^*)$ are also isomorphisms. This means (by 8.6) that the corresponding Seifert matrix V is invertible. By 8.11: $\Delta(t) \doteq |V^T V^{-1} - tE|$, $\Delta(t)$ is the characteristic polynomial of a $2g \times 2g$ regular matrix $V^T V^{-1}$. □

Conditions (a) and (b) of 8.16 characterize Alexander polynomials of fibred knots: *There is a fibred knot with Alexander polynomial* $\Delta(t)$, *if* $\Delta(t)$ *is any polynomial satisfying* (a) *and* (b), [Burde 1966], [Quach 1981]. Moreover, it was proved in [Burde-Zieschang 1967], [Bing-Martin 1971] that the trefoil and the four-knot are the only fibred knots of genus one. The conjecture that fibred knots are classified by their Alexander polynomials has proved to be false in the case of genus $g > 1$ [Morton 1978]. There are infinitely many different fibred knots to each Alexander polynomial of degree > 2 satisfying 8.16 (a) [Morton 1983′]. The methods used in Morton's paper are beyond the scope of this book; results of [Johannson 1979], [Jaco-Shalen 1979] and Thurston are employed.

It has been checked that the knots up to ten crossings are fibred if (and only if) $\Delta(0) = \pm 1$ [Kanenobu 1979].

E Finite Cyclic Coverings

Beyond the infinite cyclic covering C_∞ of the knot complement $C = \overline{S^3 - V(\mathfrak{k})}$ the finitely cyclic coverings of C are of considerable interest in knot theory. The topological invariants of these covering spaces yield new and powerful knot invariants.

Let m be a meridian of a tubular neighbourhood $V(\mathfrak{k})$ of $\mathfrak{k}$ representing the element t of the knot group $\mathfrak{G} = \mathfrak{Z} \ltimes \mathfrak{G}'$, $\mathfrak{Z} = \langle t \rangle$. For $n \geqq 0$ there are surjective homomorphisms:

$$\psi_n \colon \mathfrak{G} \to \mathfrak{Z}_n, \quad (\mathfrak{Z}_0 = \mathfrak{Z}).$$

8.17 Proposition. $\ker \psi_n = n\mathfrak{Z} \ltimes \mathfrak{G}' = \mathfrak{G}_n$, $n\mathfrak{Z} = \langle t^n \rangle$.
If $\varphi_n \colon \mathfrak{G} \to \mathfrak{Z}_n \cong \mathfrak{Z}/n\mathfrak{Z}$ *is a surjective homomorphism, then* $\ker \varphi_n = \ker \psi_n$.

Proof. Since $\mathfrak{Z}_n$ is abelian, every homomorphism $\varphi_n \colon \mathfrak{G} \to \mathfrak{Z}_n$ can be factorized, $\varphi_n = j_n \kappa$, $\ker \kappa = \mathfrak{G}'$:

$$\begin{tikzcd}
\mathfrak{G} \arrow[rr, "\varphi_n"] \arrow[dr, "\kappa"'] & & \mathfrak{Z}_n \\
& \mathfrak{G}/\mathfrak{G}' = \mathfrak{Z}. \arrow[ur, "j_n"'] &
\end{tikzcd}$$

One has $\langle \kappa(t) \rangle = \mathfrak{G}/\mathfrak{G}'$, $\ker j_n = \langle n \cdot \kappa(t) \rangle$, and

$$\ker \psi_n = \ker \varphi_n = n\mathfrak{Z} \ltimes \mathfrak{G}' = \mathfrak{G}_n. \qquad \square$$

It follows that for each $n \geqq 0$ there is a (uniquely defined) regular covering space C_n, ($C_0 = C_\infty$), with $\pi_1 C_n = \mathfrak{G}_n$, and a group of covering transformations isomorphic to $\mathfrak{Z}_n$.

8.18 Branched coverings $\hat{C}_n$. In C_n the n-th ($n > 0$) power m^n of the meridian is a simple closed curve on the torus ∂C_n. By attaching a solid torus T_n to C_n, $h \colon \partial T_n \to \partial C_n$, such that the meridian of T_n is mapped onto m^n, we obtain a closed manifold $\hat{C}_n = C_n \cup_h T_n$ which is called the *n-fold branched covering of* $\mathfrak{k}$. Obviously $p_n \colon C_n \to C$ can be extended to a continuous surjective map $\hat{p}_n \colon \hat{C} \to S^3$ that fails to be locally homeomorphic (that is, to be a covering map) only in the points of the core $\hat{p}^{-1}(\mathfrak{k}) = \hat{\mathfrak{k}}$ of T_n. The restriction $p| \colon \hat{\mathfrak{k}} \to \mathfrak{k}$ is a homeomorphism. $\mathfrak{k}$ resp. $\hat{\mathfrak{k}}$ is called the branching set of S^3 resp. $\hat{C}_n$, and $\hat{\mathfrak{k}}$ is said to have *branch index n*. As $\hat{C}_n$ is also uniquely determined by $\mathfrak{k}$, the spaces $\hat{C}_n$ as well as C_n are knot invariants; we shall be concerned especially with their homology groups $H_1(\hat{C}_n)$.

8.19 Proposition.

(a) $\mathfrak{G}_n \cong \pi_1 C_n \cong (n\mathfrak{Z}) \ltimes \mathfrak{G}'$ *with* $n\mathfrak{Z} = \langle t^n \rangle$.

(b) $H_1(C_n) \cong (n\mathfrak{Z}) \oplus (\mathfrak{G}'/\mathfrak{G}'_n)$.

(c) $H_1(\hat{C}_n) \cong \mathfrak{G}'/\mathfrak{G}'_n$.

(d) $H_1(C_n) \cong (n\mathfrak{Z}) \oplus H_1(\hat{C}_n)$.

Proof. (a) by definition, (b) follows since $\mathfrak{G}'_n \lhd \mathfrak{G}'$. Assertion (c) is a consequence of the Seifert–van Kampen theorem applied to $\hat{C}_n = C_n \cup_h T_n$. □

8.20 Proposition (Homology of branched cyclic coverings $\hat{C}_n$). *Let V be a $2h \times 2h$ Seifert matrix of a knot $\mathfrak{k}$, $V - V^T = F$, $G = F^T V$, and $\mathfrak{Z}_n = \langle t \mid t^n \rangle$.*

(a) *$R_n = (G - E)^n - G^n$ is a presentation matrix of $H_1(\hat{C}_n)$ as an abelian group. In the special case $n = 2$ one has $R_2 \sim V + V^t = A(-1)$.*

(b) *As a $\mathfrak{Z}_n$-module $H_1(\hat{C}_n)$ is annihilated by $\varrho_n(t) = 1 + t + \cdots + t^{n-1}$.*

(c) $(R_n F)^T = (-1)^n (R_n F)$.

(d) *$(V^T - tV)$ is a presentation matrix of $H_1(\hat{C}_n)$ as a $\mathfrak{Z}_n$-module.*

Proof. Denote by τ the covering transformation of the covering $p_n \colon C_n \to C$ corresponding to $\psi_n(t) \in \mathfrak{Z}_n$, see 8.17. Select a sheet C_0^* of the covering, then $\{C_i^* = \tau^i C_0^* \mid 0 \leqq i \leqq n-1\}$ are then n sheets of C_n (see Figures 4.2 and 8.8). Let

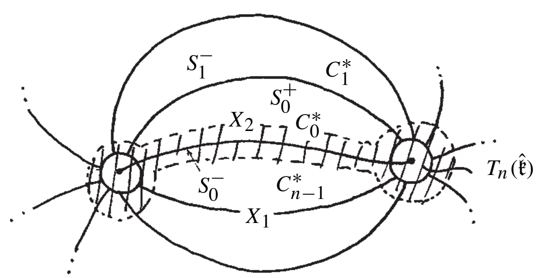


Figure 8.8

$s_j, a_i^{\pm}$ be defined as in 8.3. Apply the Seifert–van Kampen theorem to

$$X_1 = \mathring{C}_0^* \cup C_1^* \cup \cdots \cup C_{n-2}^* \cup \mathring{C}_{n-1}^* \quad \text{and} \quad X_2 = U(S_0^- \cup T_n)$$

a tubular neighbourhood of $S_0^- \cup T_n$. As in the proof of 4.6 one gets

$$\pi_1 X_1 \cong \pi_1 C_0^* *_{\pi_1 S_0^+} \pi_1 C_1^* *_{\pi_1 S_1^+} \cdots *_{\pi_1 S_{n-2}^+} \pi_1 C_{n-1}^*,$$
$$\pi_1 X_2 \cong \pi_1 S_0^-, \ \pi_1(X_1 \cap X_2) \cong \pi_1 S_{n-1}^+ *_{\langle \hat{\ell} \rangle} \pi_1 S_0^-,$$

where $\hat{\ell} = \partial S_0^-$ is a longitude of $\hat{\mathfrak{k}}$ in T_n. It follows by abelianizing $\pi_1(\hat{C}_n) = \pi_1(X_1 \cup X_2)$ that $H_1(\hat{C}_n) \cong \pi_1(\hat{C}_n)/\pi_1'(\hat{C}_n)$ is generated by $\{t^i s_j \mid 1 \leqq j \leqq 2h, 0 \leqq i \leqq n-1\}$, and its defining relations are

$$(t^i V^T - t^{i+1} V)s = 0,\ 0 \leqq i \leqq n-1, \quad t^n = 1, \quad s^T = (s_1, s_2, \ldots, s_{2h}),$$

see 8.8. (Observe that in $H_1(\hat{C}_n)$ the longitude $\hat{\ell}$ is 0-homologous.) This proves (d).

Multiply the relations by F^T and introduce the abbreviation $F^T V = G$ (see 8.20). One gets:

$$Gt^i s - t^i s - Gt^{i+1} s = 0, \quad 0 \leqq i \leqq n-1. \qquad (K_i)$$

Adding these equations gives

$$(1 + t + \cdots + t^{n-1})s = 0,$$

and proves (b).

Add (K_1) to (K_0) to obtain

$$(G - E)s - ts - Gt^2 s = 0. \qquad (E_1)$$

Multiply (E_1) by $G - E$ and add to (K_1): The result is

$$(G - E)^2 s - G^2 t^2 s = 0. \qquad (R_2)$$

The relations (K_0), (K_1) can be replaced by the relations (E_1) and (R_2), and (E_1) can be used to eliminate ts. This procedure can be continued. Assume that after $(i-1)$ steps the generators $ts, t^2 s, \ldots, t^{i-1} s$ are eliminated, and the equations (K_j), $i \leqq j \leqq n-1$ together with

$$(G - E)^i s - G^i t^i s = 0 \qquad (R_i)$$

form a set of defining relations. Now multiply (K_i) by $\sum_{j=0}^{i-1} G^j$ and add to (R_i). One obtains

$$(G - E)^i s - t^i s - G \sum_{j=0}^{i-1} G^j t^{i+1} s = 0. \qquad (E_i)$$

Multiply (E_i) by $(G - E)$ and add to (K_i). The result is

$$(G - E)^{i+1} s - G^{i+1} t^{i+1} s = 0. \qquad (R_{i+1})$$

The relations (R_i), (K_i) have thus been replaced by (E_i), (R_{i+1}). Eliminate $t^i s$ by (E_i) and omit (E_i).

The procedure stops when only the generators $s = (s_j)$ are left, and the defining relations

$$G^n s - (G - E)^n s = 0$$

remain. This proves (a).

(c) is easily verified using the definition of R_n and F. □

Remark. It follows from 8.20 (b) for $n = 2$ that $1 + t$ is the 0-endomorphism of $H_1(\hat{C}_2)$. This means

$$a \mapsto ta = -a \quad \text{for } a \in H_1(\hat{C}_2).$$

8.21 Theorem. *$H_1(\hat{C}_n)$ is finite if and only if no root of the Alexander polynomial $\Delta(t)$ of $\mathfrak{k}$ is an n-th root of unity ζ_i, $1 \leqq i \leqq n$. In this case*

$$|H_1(\hat{C}_n)| = \Big|\prod_{i=1}^{n} \Delta(\zeta_i)\Big|.$$

In general, the Betti number of $H_1(\hat{C}_n)$ is even and equals the number of roots of the Alexander polynomial which are also roots of unity; each such root is counted ν-times, if it occurs in ν different elementary divisors $\varepsilon_k(t) = \Delta_k(t)\Delta_{k+1}^{-1}(t)$, $k = 1, 2, \ldots$.

Proof. Since the matrices $G - E$ and G commute,

$$R_n = (G - E)^n - G^n = \prod_{i=1}^{n}[(G - E) - \zeta_i G].$$

By 8.8,

$$(G - E) - tG = F^T(V^T - tV) = F^T A(t)$$

is a presentation matrix of the Alexander module $M(t)$; thus, by 8.11,

$$\det((G - E) - tG) \doteq \Delta(t).$$

This implies that $\det R_n = \prod_{i=1}^{n} \Delta(\zeta_i)$. The order of $H_1(\hat{C}_n)$ is $\det R_n$, if $\det R_n \neq 0$.

In the general case the Betti number of $H_1(\hat{C}_n)$ is equal to $2h - \text{rank } R_n$. To determine the rank of R_n we study the Jordan canonical form $G_0 = L^{-1}GL$ of G, where L is a non-singular matrix with coefficients in $\mathbb{C}$. Then $L^{-1}R_nL = (G_0 - E)^n - G_0^n$. The diagonal elements of G_0 are the roots $\lambda_i = (1 - t_i)^{-1}$ of the characteristic polynomial $\chi(\lambda) = \det(G - \lambda E)$, where the t_i are the roots of the Alexander polynomial, see 8.13. The nullity of $L^{-1}R_nL$ equals the number of λ_i which have the property $(\lambda_i - 1)^n - \lambda_i^n = 0 \iff t_i^n = 1, t_i \neq 1$, once counted in each Jordan block of G_0.

From $\Delta(1) = 1$ and the symmetry of the Alexander polynomial it follows that only non-real roots of unity may be roots of $\chi(\lambda)$ and those occur in pairs. □

The following property of $H_1(\hat{C}_n)$ is a consequence of 8.20 (c).

8.22 Proposition ([Plans 1953]). *$H_1(\hat{C}_n) \cong A \oplus A$ if $n \equiv 1 \bmod 2$.*

Proof. $Q = R_n F$ is equivalent to R_n, and hence a presentation matrix of $H_1(\hat{C}_n)$. For odd n the matrix Q is skew symmetric, $Q = -Q^T$. Proposition 8.22 follows from the fact that Q has a canonical form

$$L^T Q L = \begin{pmatrix} 0 & a_1 & & & & & & & & & \\ -a_1 & 0 & & & & & & & & & \\ & & 0 & a_2 & & & & & & & \\ & & -a_2 & 0 & & & & & & & \\ & & & & \ddots & & & & & & \\ & & & & & 0 & a_s & & & & \\ & & & & & -a_s & 0 & & & & \\ & & & & & & & 0 & & & \\ & & & & & & & & 0 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 0 \end{pmatrix},$$

where L is unimodular (invertible over $\mathbb{Z}$). A proof is given in Appendix A.1. □

8.23 Proposition (Alexander modules of satellites). *Let* $\mathfrak{k}$ *be a satellite,* $\hat{\mathfrak{k}}$ *its companion, and* $\tilde{\mathfrak{k}}$ *the preimage of* $\mathfrak{k}$ *under the embedding* $h\colon \tilde{V} \to \hat{V}$ *as defined in* 2.8. *Denote by* $M(t)$, $\hat{M}(t)$, $\tilde{M}(t)$ *resp.* $\Delta(t)$, $\hat{\Delta}(t)$, $\tilde{\Delta}(t)$ *the Alexander modules resp. Alexander polynomials of* $\mathfrak{k}$, $\hat{\mathfrak{k}}$ *and* $\tilde{\mathfrak{k}}$.

(a) $M(t) = \tilde{M}(t) \oplus [\mathbb{Z}(t) \otimes_{\mathbb{Z}(t^n)} \hat{M}(t^n)]$ *with* $n = \operatorname{lk}(\hat{\mathfrak{m}}, \mathfrak{k})$, $\hat{\mathfrak{m}}$ *a meridian of* $\hat{\mathfrak{k}}$.

(b) $\Delta(t) = \tilde{\Delta}(t) \cdot \hat{\Delta}(t^n)$.

Proof. The proposition is a consequence of 4.12, but a direct proof of 8.23 using the trivialization of $\mathfrak{G}''$ in $M(t)$ shows that 8.23 is much simpler than 4.12. We use the notation of 4.12. Let $\mathfrak{G}$, $\hat{\mathfrak{G}}$, $\tilde{\mathfrak{G}}$ denote the knot groups of $\mathfrak{k}$, $\hat{\mathfrak{k}}$ and $\tilde{\mathfrak{k}}$. There are presentations:

$$\begin{aligned} &\hat{\mathfrak{G}} = \langle\, \hat{t}, \hat{u}_j \mid \hat{R}_k(\hat{u}, \hat{t})\,\rangle,\ \ \hat{u}_j \in \hat{\mathfrak{G}}', \\ &\mathfrak{H} = h_\# \pi_1(\tilde{V} - \tilde{\mathfrak{k}}) = \langle\, t, \hat{\lambda}, \tilde{u}_j \mid \tilde{R}_l(\tilde{u}_j, \hat{\lambda}, t)\,\rangle, \end{aligned}$$

with $\mathfrak{H}/\langle\, \hat{\lambda}\,\rangle \cong \tilde{\mathfrak{G}}$, $\tilde{u}_j \in \tilde{\mathfrak{G}}'$. Here t resp. $\hat{t}$ represent meridians of $\mathfrak{k}$ and $\hat{\mathfrak{k}}$, and $\hat{\lambda}$ a longitude of $\hat{\mathfrak{k}}$. It follows that $\hat{t} \in t^n \tilde{\mathfrak{G}}'$. The Seifert–van Kampen Theorem gives that

$$\mathfrak{G} = \hat{\mathfrak{G}} *_{\langle \hat{t}, \hat{\lambda}\rangle} \mathfrak{H}, \quad \text{where } \langle\, \hat{t}, \hat{\lambda}\,\rangle = \pi_1(\partial \hat{V})$$

is a free abelian group of rank 2. (The Definition 2.8 of a companion knot ensures that $\langle\, \hat{t}, \hat{\lambda}\,\rangle$ is embedded in both factors.) We apply the Reidemeister–Schreier method to

$\mathfrak{G}$, $\hat{\mathfrak{G}}$, $\mathfrak{H}$ with respect to the commutator subgroups $\mathfrak{G}'$, $\hat{\mathfrak{G}}'$, $\mathfrak{H}'$ and representatives t^ν, $\hat{t}^\mu$. One obtains generators $\tilde{u}_i^{t^\nu}$, $\hat{u}_j^{\hat{t}^\mu}$, $\nu, \mu \in \mathbb{Z}$, and presentations

$$\hat{\mathfrak{G}}'/\hat{\mathfrak{G}}'' = \langle\, \hat{u}_j^{\hat{t}^\mu} \mid \hat{R}_k(\hat{u}_j^{\hat{t}^\mu})\,\rangle \quad \text{and} \quad \tilde{\mathfrak{G}}'/\tilde{\mathfrak{G}}'' \cong \langle\, \tilde{u}_i^{t^\nu} \mid \tilde{R}_l(\tilde{u}_i^{t^\nu}, 1)\,\rangle.$$

Since $\mathfrak{G}' \supset \hat{\mathfrak{G}}'$, $\mathfrak{G}'' \supset \hat{\mathfrak{G}}'' \ni \hat{\lambda}$, $\mathfrak{G}' \supset \tilde{\mathfrak{G}}'$,

$$\mathfrak{G}'/\mathfrak{G}'' = \langle\, \hat{u}_j^{\hat{t}^\mu}, \tilde{u}_i^{t^\nu} \mid \hat{R}_k(\hat{u}_j^{\hat{t}^\mu}),\, \tilde{R}_l(\tilde{u}_i^{t^\nu}, 1)\,\rangle \cong \hat{\mathfrak{G}}'/\hat{\mathfrak{G}}'' \oplus \tilde{\mathfrak{G}}'/\tilde{\mathfrak{G}}'',$$

where the amalgamation is reduced to the fact that the operations $\hat{t}$ resp. t on the first resp. the second summand are connected by $\hat{t} = t^n$. □

F History and Sources

J.W. Alexander [1928] first introduced Alexander polynomials. H. Seifert [1934] investigated the matter from the geometric point of view and was able to prove the characterizing properties of the Alexander polynomial (Proposition 8.12, 8.13). The presentation of the homology of the finite cyclic coverings in Proposition 8.20 is also due to him [Seifert 1934].

G Exercises

E 8.1. Prove: $\deg \Delta(t) \leqq 2g$, where g is the genus of a knot, and $\Delta(t)$ its Alexander polynomial. (For knots up to ten crossings equality holds.)

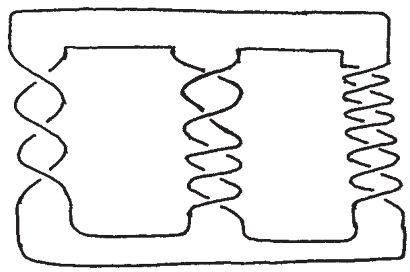
E 8.2. Write $\Delta(t) = t^4 - 2t^3 + t^2 - 2t - 1$ in the reduced form $\sum_{i=0}^{2} c_i u^i$ (Proposition 8.13). Construct a knot with $\Delta(t)$ as its Alexander polynomial. Construct a fibred knot with $\Delta(t)$ as its Alexander polynomial. (Hint: use braid-like knots as defined in E 4.4.)

E 8.3. Show that $H_1(C_\infty) = 0$ if and only if $\Delta(t) = 1$. Prove that $\pi_1 C_\infty$ is of finite rank, if it is free.

E 8.4. Prove: $H_1(\hat{C}_n) = 0$ for $n \geqq 2$ if and only if $H_1(C_\infty) = 0$.

E 8.5. Show $|H_1(\hat{C}_2)| \equiv 1 \bmod 2$; further, for a knot of genus one with $|H_1(\hat{C}_2)| = 4a \pm 1$, show that $H_1(\hat{C}_3) \cong \mathbb{Z}_{3a\pm1} \oplus \mathbb{Z}_{3a\pm1}$, $a \in \mathbb{N}$.

E 8.6. By $\mathfrak{p}(p, q, r)$, p, q, r odd integers, we denote a pretzel knot (Figure 8.9). (The sign of the integers defines the direction of the twist.) Construct a band projection of $\mathfrak{p}(p, q, r)$, and compute its Seifert matrix V and its Alexander polynomial. (Figure 8.10 shows how a band projection may be obtained.)



$\mathfrak{p}(3, -5, -7)$

Figure 8.9

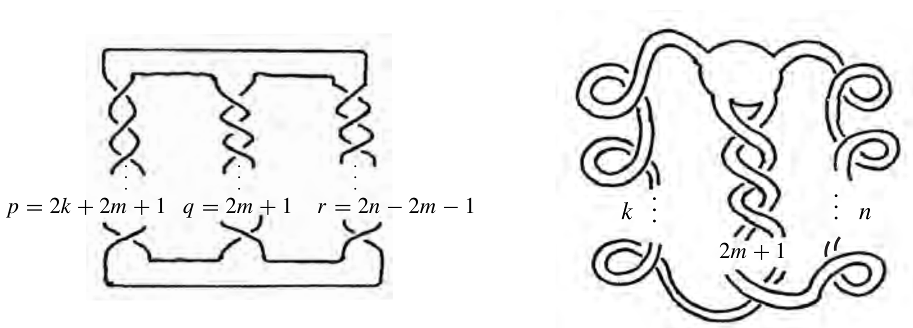


Figure 8.10

E 8.7. Let $\mathfrak{k}$ be a link of $\mu > 1$, components. Show that there is a homomorphism φ of its group $\mathfrak{G} = \pi_1(S^3 - \mathfrak{k})$ onto a free cyclic group $\mathfrak{Z} = \langle\, t \,\rangle$ which maps every Wirtinger generator of $\mathfrak{G}$ onto t. Construct an infinite cyclic covering C_∞ of the link complement using a Seifert surface S of $\mathfrak{k}$, compute its Seifert matrix and define its Alexander polynomial following the lines developed in this chapter in the case of a knot. (See also E 9.5.)

E 8.8. Let $\hat{C}_3$ be the 3-fold cyclic branched covering of a knot. If $H_1(\hat{C}_3) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ for some prime p, then there are generators a, b of $\mathbb{Z}_p \oplus \mathbb{Z}_p$ such that $t\colon H_1(\hat{C}_3) \to H_1(\hat{C}_3)$ is given by $ta = b$, $tb = -a - b$. For all knots one has $p \neq 3$.

E 8.9. Construct a knot of genus one with the Alexander polynomials of the trefoil but not fibred – and hence different from the trefoil.

E 8.10. Show that $\Delta_1(t)\Delta_2^{-1}(t)$ annihilates the Alexander module $M(t)$ of a knot [Crowell 1964].

E 8.11. Let $\mathfrak{k}$ be a fibred knot of genus g, and let $F \times I/h$ denote its complement. Describe $h_*\colon H_1(F, \mathbb{Q}) \to H_1(F, \mathbb{Q})$ by a matrix $A = \oplus A_i$ where A_i is a companion matrix determined by the Alexander polynomials of $\mathfrak{k}$. (For the notion of a companion matrix see, e.g. [van der Waerden 1955, §117].)

E 8.12. Prove that a satellite is never trivial. Show, that doubled knots (see 2.9) have trivial Alexander modules, and therefore trivial Alexander polynomials.

Chapter 9
Free Differential Calculus and Alexander Matrices

In Chapter 8 we studied the homology of the cyclic coverings of the knot complement. Alexander polynomials were defined, and a general method of computing these invariants via a band projection of the knot was developed. Everyone who actually wants to carry out this task will soon find out that the calculations involved increase rapidly with the genus of the knot. There are, however, knots of arbitrary genus with groups of a relatively simple structure (for instance: torus knots). We shall present in this chapter another method of computing Alexander's knot invariants which will prove to be considerably simpler in this case – and in many other cases. The method is based on the theory of Fox derivations in the group ring of a free group. There is a geometric background to the Fox calculus with which we intend to start. It is the *theory of homotopy chains* [Reidemeister 1935′], or, to use the modern terminology, *equivariant homology*.

A Regular Coverings and Homotopy Chains

The one-to-one correspondence between finitely presented groups and fundamental groups of 2-complexes, and between (normal) subgroups and (regular) coverings of such complexes has been exploited in combinatorial group theory to prove group theoretical theorems (as for instance the Reidemeister–Schreier method or the Kurosh subgroup theorem [ZVC 1980, 2.6]) by topological methods. In the case of homology these relationships are less transparent, but some can be retained for the first homology groups.

9.1 On the homology of a covering space. Let $p\colon \tilde{X} \to X$ be a regular covering of a connected 2-complex. We assume X to be a finite CW-complex with one 0-cell P. Then a presentation

$$\mathfrak{G} = \pi_1(X, P) = \langle\, s_1, \dots s_n \mid R_1, \dots, R_m \,\rangle$$

of the fundamental group of X is obtained by assigning a generator s_i to each (oriented) 1-cell (also denoted by s_i), and a defining relation to (the boundary of) each 2-cell e_j of X. Choose a base point $\tilde{P} \subset \tilde{X}$ over P, $p_\#(\pi_1(\tilde{X}, \tilde{P})) = \mathfrak{N} \triangleleft \mathfrak{G}$, and let $\mathfrak{D} \cong \mathfrak{G}/\mathfrak{N}$ denote the group of covering transformations.

Let φ: $\mathfrak{G} \to \mathfrak{D}$, $w \mapsto w^\varphi$ be the canonical homomorphism. The linear extension to the group ring is also denoted by $\varphi\colon \mathbb{Z}\mathfrak{G} \to \mathbb{Z}\mathfrak{D}$. Observe: $(w_1 w_2)^\varphi = w_1^\varphi w_2^\varphi$.

Our aim is to present $H_1(\tilde{X}, \tilde{X}^0)$ as a $\mathbb{Z}\mathfrak{D}$-module. (We follow a common convention by writing merely $\mathfrak{D}$-module instead of $\mathbb{Z}\mathfrak{D}$-module. $\tilde{X}^0$ denotes the 0-skeleton of $\tilde{X}$.)

The (oriented) edges s_i lift to edges $\tilde{s}_i$ with initial point $\tilde{P}$. By w we denote a closed path in the 1-skeleton X^1 of X, and, at the same time, the element it represents in the free group $\mathfrak{F} = \pi_1(X^1, P) = \langle s_1, \dots, s_n \mid - \rangle$. There is a unique lift $\tilde{w}$ of w starting at $\tilde{P}$. Clearly $\tilde{w}$ is a special element of the relative cycles $Z_1(\tilde{X}, \tilde{X}^0)$ which are called *homotopy* 1-*chains*. Every 1-chain can be written in the form $\sum_{j=1}^n \xi_j \tilde{s}_j$, $\xi_j \in \mathbb{Z}\mathfrak{D}$. (The expression $g\tilde{s}_j$ denotes the image of the edges $\tilde{s}_j$ under the covering transformation g.) There is a rule

$$\widetilde{w_1 w_2} = \tilde{w}_1 + w_1^{\varphi} \cdot \tilde{w}_2. \tag{1}$$

To understand it, first lift w_1 to $\tilde{w}_1$. Its endpoint is $w_1^{\varphi} \cdot \tilde{P}$. The covering transformation w_1^{φ} maps $\tilde{w}_2$ onto a chain $w_1^{\varphi}\tilde{w}_2$ over w_2 which starts at $w_1^{\varphi}\tilde{P}$. If $\tilde{w}_k = \sum_{j=1}^n \xi_{kj}\tilde{s}_j$ with $\xi_{kj} \in \mathbb{Z}\mathfrak{D}$, $k = 1, 2$, then $\widetilde{w_1 w_2} = \sum_{j=1}^n \xi_j \tilde{s}_j$ with

$$\xi_j = \xi_{1j} + w_1^{\varphi} \cdot \xi_{2j}, \quad 1 \leqq j \leqq n. \tag{2}$$

(The coefficient ξ_{kj} is the algebraic intersection number of the path $\tilde{w}_k$ with the covers of s_j.) This defines mappings

$$\left(\frac{\partial}{\partial s_j}\right)^{\varphi} : \mathfrak{G} = \pi_1(X, P) \to \mathbb{Z}\mathfrak{D}, \quad w \mapsto \xi_j, \quad \text{with } \tilde{w} = \sum_{j=1}^n \xi_j \tilde{s}_j, \tag{3}$$

satisfying the rule

$$\left(\frac{\partial}{\partial s_j}(w_1 w_2)\right)^{\varphi} = \left(\frac{\partial}{\partial s_j} w_1\right)^{\varphi} + w_1^{\varphi} \cdot \left(\frac{\partial}{\partial s_j} w_2\right)^{\varphi}. \tag{4}$$

There is a linear extension to the group ring $\mathbb{Z}\mathfrak{G}$:

$$\left(\frac{\partial}{\partial s_j}(\eta + \xi)\right)^{\varphi} = \left(\frac{\partial}{\partial s_j}\eta\right)^{\varphi} + \left(\frac{\partial}{\partial s_j}\xi\right)^{\varphi} \quad \text{for } \eta, \xi \in \mathbb{Z}\mathfrak{G}. \tag{5}$$

From the definition it follows immediately that

$$\left(\frac{\partial}{\partial s_j} s_k\right)^{\varphi} = \delta_{jk}, \quad \tilde{w} = \sum \left(\frac{\partial w}{\partial s_j}\right)^{\varphi} \tilde{s}_j, \quad \delta_{jk} = \begin{cases} 1 & j = k, \\ 0 & j \neq k. \end{cases} \tag{6}$$

We may now use this terminology to present $H_1(\tilde{X}, \tilde{X}^0)$ as a $\mathfrak{D}$-module: The 1-chains $\tilde{s}_i$, $1 \leqq i \leqq n$, are generators, and the lifts $\tilde{R}_j$ of the boundaries $R_j = \partial e_j$ of the 2-cells are defining relations. (The boundary of an arbitrary 2-cell of $\tilde{X}$ is of the form $\delta(\tilde{R}_j)$, $\delta \in \mathfrak{D}$. Hence, in a presentation of $H_1(\tilde{X}, \tilde{X}^0)$ as a $\mathfrak{D}$-module is suffices to include the $\tilde{R}_j$, $1 \leqq j \leqq m$, as defining relations.)

9.2 Proposition.

$$H_1(\tilde{X}, \tilde{X}^0) = \langle \tilde{s}_1, \dots, \tilde{s}_n \mid \tilde{R}_1, \dots, \tilde{R}_m \rangle, \quad 0 = \tilde{R}_j = \sum_{i=1}^{n} \left(\frac{\partial R_j}{\partial s_i}\right)^{\varphi} \tilde{s}_i, \quad 1 \leqq j \leqq m$$

is a presentation of $H_1(\tilde{X}, \tilde{X}^0)$ *as a* $\mathfrak{D}$*-module.* □

B Fox Differential Calculus

In this section we describe a purely algebraic approach to the mapping $\left(\frac{\partial}{\partial s_j}\right)^{\varphi}$ [Fox 1953, 1954, 1956]. Let $\mathfrak{G}$ be a group and $\mathbb{Z}\mathfrak{G}$ its group ring (with integral coefficients); $\mathbb{Z}$ is identified with the multiples of the unit element 1 of $\mathfrak{G}$.

9.3 Definition (Derivation). (a) There is a homomorphism

$$\varepsilon\colon \mathbb{Z}\mathfrak{G} \to \mathbb{Z}, \quad \tau = \Sigma n_i g_i \mapsto \Sigma n_i = \tau^{\varepsilon}.$$

We call ε the *augmentation homomorphism*, and its kernel $I\mathfrak{G} = \varepsilon^{-1}(0)$ the *augmentation ideal*.

(b) A mapping $\Delta\colon \mathbb{Z}\mathfrak{G} \to \mathbb{Z}\mathfrak{G}$ is called a *derivation* (of $\mathbb{Z}\mathfrak{G}$) if

$$\Delta(\xi + \eta) = \Delta(\xi) + \Delta(\eta) \qquad \text{(linearity)},$$

and

$$\Delta(\xi \cdot \eta) = \Delta(\xi) \cdot \eta^{\varepsilon} + \xi \cdot \Delta(\eta) \qquad \text{(product rule)},$$

for $\xi, \eta \in \mathbb{Z}\mathfrak{G}$.

From the definition it follows by simple calculations:

9.4 Lemma. (a) *The derivations of* $\mathbb{Z}\mathfrak{G}$ *form a* (*right*) $\mathfrak{G}$*-module under the operations defined by*

$$(\Delta_1 + \Delta_2)(\tau) = \Delta_1(\tau) + \Delta_2(\tau), \quad (\Delta\gamma)(\tau) = \Delta(\tau) \cdot \gamma.$$

(b) *Let* Δ *be a derivation. Then:*

$$\begin{aligned}
&\Delta(m) = 0 \quad \textit{for } m \in \mathbb{Z},\\
&\Delta(g^{-1}) = -g^{-1} \cdot \Delta(g),\\
&\Delta(g^n) = (1 + g + \cdots + g^{n-1}) \cdot \Delta(g),\\
&\Delta(g^{-n}) = -(g^{-1} + g^{-2} + \cdots + g^{-n}) \cdot \Delta(g) \quad \textit{for } n \geqq 1.
\end{aligned}$$ □

9.5 Examples (a) $\Delta_\varepsilon\colon \mathbb{Z}\mathfrak{G} \to \mathbb{Z}\mathfrak{G}$, $\tau \mapsto \tau - \tau^\varepsilon$, is a derivation.

(b) If $a, b \in \mathfrak{G}$ commute, $ab = ba$, then $(a-1)\Delta b = (b-1)\Delta a$. (We write Δa instead of $\Delta(a)$ when no confusion can arise.) It follows that a derivation $\Delta\colon \mathbb{Z}\mathfrak{Z}^n \to \mathbb{Z}\mathfrak{Z}^n$ of the group ring of a free abelian group $\mathfrak{Z}^n = \langle S_1 \rangle \times \cdots \times \langle S_n \rangle$, $n \geqq 2$, with $\Delta S_i \neq 0$, $1 \leqq i \leqq n$, is a multiple of Δ_ε in the module of derivations.

Contrary to the situation in group rings of abelian groups the group ring of a free group admits a great many derivations.

9.6 Proposition. *Let $\mathfrak{F} = \langle \{S_i \mid i \in J\}\rangle$ be a free group. There exists a uniquely determined derivation $\Delta\colon \mathbb{Z}\mathfrak{F} \to \mathbb{Z}\mathfrak{F}$ with $\Delta S_i = w_i$ for arbitrary elements $w_i \in \mathbb{Z}\mathfrak{F}$.*

Proof. $\Delta(S_i^{-1}) = -S_i^{-1}w_i$ follows from $\Delta(1) = 0$ and the product rule. Linearity and product rule imply uniqueness. Define $\Delta(S_{i_1}^{\eta_1} \dots S_{i_k}^{\eta_k})$ using the product rule:

$$\Delta(S_{i_1}^{\eta_1} \dots S_{i_k}^{\eta_k}) = \Delta S_{i_1}^{\eta_1} + S_{i_1}^{\eta_1}\Delta S_{i_2}^{\eta_2} + \cdots + S_{i_1}^{\eta_1} \dots S_{i_{k-1}}^{\eta_{k-1}}\Delta S_{i_k}^{\eta_k}.$$

The product rule then follows for combined words $w = uv$, $\Delta w = \Delta u + u\Delta v$. The equation

$$\begin{aligned}\Delta(uS_i^\eta S_i^{-\eta}v) &= \Delta u + u\Delta S_i^\eta + uS_i^\eta \Delta S_i^{-\eta} + u\Delta v = \Delta u + u\Delta v\\ &= \Delta(uv), \quad \eta = \pm 1,\end{aligned}$$

shows that Δ is well defined on $\mathfrak{F}$. □

9.7 Definition (Partial derivations). The derivations

$$\frac{\partial}{\partial S_i}\colon\ \mathbb{Z}\mathfrak{F} \to \mathbb{Z}\mathfrak{F}, \quad S_j \mapsto \begin{cases} 1 & \text{for } i = j\\ 0 & \text{for } i \neq j,\end{cases}$$

of the group ring of a free group $\mathfrak{F} = \langle \{S_i \mid i \in J\} \mid - \rangle$ are called *partial derivations*.

The partial derivations form a basis of the module of derivations:

9.8 Proposition. (a) $\Delta = \sum_{i \in J} \frac{\partial}{\partial S_i} \cdot \Delta(S_i)$ *for every derivation* $\Delta\colon \mathbb{Z}\mathfrak{F} \to \mathbb{Z}\mathfrak{F}$. (*The sum may be infinite, however, for any $\tau \in \mathbb{Z}\mathfrak{F}$ there are only finitely many* $\frac{\partial \tau}{\partial S_i} \neq 0$.)

(b) $\sum_{i \in J} \frac{\partial}{\partial S_i} \cdot \tau_i = 0 \iff \tau_i = 0,\ i \in J.$

(c) $\Delta_\varepsilon(\tau) = \tau - \tau^\varepsilon = \sum_{i \in J} \frac{\partial \tau}{\partial S_i}(S_i - 1)$ (*Fundamental formula*).

(d) $\tau - \tau^\varepsilon = \sum_{i \in J} v_i(S_i - 1) \iff v_i = \frac{\partial \tau}{\partial S_i}$, $i \in J$.

Proof. $\left(\sum_i \frac{\partial}{\partial S_i} \Delta S_i\right) S_j = \sum_i \frac{\partial S_j}{\partial S_i} \Delta S_i = \Delta S_j$ proves (a) by 9.6. For $\Delta =$ 0-map, and $\Delta = \Delta_\varepsilon$ one gets (b) and (c). To prove (d) apply $\frac{\partial}{\partial S_j}$ to the equation. □

The theory of derivations in $\mathbb{Z}\mathfrak{F}$ (free derivations) has been successfully used to study $\mathbb{Z}\mathfrak{F}$ and $\mathfrak{F}$ itself [Zieschang 1962]. There are remarkable parallels to the usual derivations used in analysis. For instance, the fundamental formula resembles a Taylor expansion. If $(S_1, \ldots, S_n)$, $(S_1', \ldots, S_n')$, $(S_1'', \ldots, S_n'')$ are bases of a free group $\mathfrak{F}_n$, there is a chain rule for the Jacobian matrices:

$$\frac{\partial S_k''}{\partial S_i} = \sum_{j=1}^{n} \frac{\partial S_k''}{\partial S_j'} \cdot \frac{\partial S_j'}{\partial S_i}.$$

(Apply 9.8 (a) in the form $\Delta = \sum_{j=1}^{n} \frac{\partial}{\partial S_j'} \Delta S_j'$ for $\Delta = \frac{\partial}{\partial S_i}$ to S_k''.)

J. Birman [1973′] proves that $(S_1', \ldots, S_m')$ is a basis of $\mathfrak{F} = \langle\, S_1, \ldots, S_n \mid - \,\rangle$ if and only if the Jacobian $\left(\frac{\partial S_j'}{\partial S_i}\right)$ is invertible over $\mathbb{Z}\mathfrak{F}$.

For further properties of derivations see E 9.7,8.

C Calculation of Alexander Polynomials

We return to the regular covering $p\colon \tilde{X} \to X$ of 9.1. Let

$$\psi : \mathfrak{F} = \langle\, S_1, \ldots, S_n \mid - \,\rangle \to \langle\, S_1, \ldots, S_n \mid R_1, \ldots, R_m \,\rangle = \mathfrak{G}$$

denote the canonical homomorphism of the groups and, at the same time, its extension to the group rings:

$$\psi : \mathbb{Z}\mathfrak{F} \to \mathbb{Z}\mathfrak{G}, \quad \left(\sum n_i f_i\right)^{\psi} = \sum n_i f_i^{\psi} \quad \text{for } f_i \in \mathfrak{F},\ n_i \in \mathbb{Z}.$$

Combining ψ with the map $\varphi\colon \mathbb{Z}\mathfrak{G} \to \mathbb{Z}\mathfrak{D}$ of 9.1 (we use the notation $(\xi)^{\varphi\psi} = (\xi^{\psi})^{\varphi}$, $\xi \in \mathbb{Z}\mathfrak{F}$), we may state Proposition 9.2 in terms of the differential calculus.

9.9 Proposition. $\left(\left(\frac{\partial R_k}{\partial S_j}\right)^{\varphi\psi}\right)$, $1 \leqq k \leqq m$, $1 \leqq j \leqq n$, *is a presentation matrix of* $H_1(\tilde{X}, \tilde{X}^0)$ *as a* $\mathfrak{D}$*-module.* ($k =$ *row index,* $j =$ *column index.*)

Proof. Comparing the linearity and the product rule of the Fox derivations 9.3 with (4) and (5) of 9.1, we deduce from 9.6 that the mappings $\left(\frac{\partial}{\partial S_i}\right)^{\varphi}$ in (3) coincide with those defined by $\left(\frac{\partial}{\partial S_i}\right)^{\varphi\psi}$ in 9.7 □

Remark. The fact that the partial derivation of (5) and 9.7 are the same lends a geometric interpretation also to the fundamental formula: For $w \in \mathfrak{G}$ and $\tilde{w}$ its lift,

$$\partial \tilde{w} = (w^{\varphi\psi} - 1)\tilde{P} = \sum_i \left(\frac{\partial w}{\partial S_i}\right)^{\varphi\psi} (S_i^{\varphi\psi} - 1)\tilde{P} = \sum_i \left(\frac{\partial w}{\partial S_i}\right)^{\varphi\psi} \partial \tilde{s}_i .$$

To obtain information about $H_1(\tilde{X})$ we consider the exact homology sequence

$$\begin{array}{ccccccccc} H_1(\tilde{X}^0) & \longrightarrow & H_1(\tilde{X}) & \longrightarrow & H_1(\tilde{X}, \tilde{X}^0) & \xrightarrow{\partial} & H_0(\tilde{X}^0) & \xrightarrow{i_*} & H_0(\tilde{X}) \longrightarrow 0. \\ \| & & & & & & \|\wr & & \|\wr \\ 0 & & & & & & \mathbb{Z}\mathfrak{D} & & \mathbb{Z} \end{array} \tag{7}$$

$H_0(\tilde{X}^0)$ is generated by $\{w^{\varphi\psi} \cdot \tilde{P} \mid w \in \mathfrak{F}\}$ as an abelian group. The kernel of i_* is the image $(I\mathfrak{F})^{\varphi\psi}$ of the augmentation ideal $I\mathfrak{F} \subset \mathbb{Z}\mathfrak{F}$ (see 9.3 (a)). The fundamental formula shows that $\ker i_*$ is generated by $\{(S_j^{\varphi\psi} - 1)\tilde{P} \mid 1 \leqq j \leqq n\}$ as a $\mathfrak{D}$-module.

Thus we obtain from (7) a short exact sequence:

$$0 \longrightarrow H_1(\tilde{X}) \longrightarrow H_1(\tilde{X}, \tilde{X}^0) \xrightarrow{\partial} \ker i_* \longrightarrow 0. \tag{8}$$

In the case of a knot group $\mathfrak{G}$, and its infinite cyclic covering C_∞ ($\mathfrak{N} = \mathfrak{G}'$) the group of covering transformations is cyclic, $\mathfrak{D} = \mathfrak{Z} = \langle t \rangle$, and $\ker i_*$ is a free $\mathfrak{Z}$-module generated by $(t-1)\tilde{P}$. The sequence (8) splits, and

$$H_1(\tilde{X}, \tilde{X}^0) \cong H_1(\tilde{X}) \oplus \sigma(\mathbb{Z}\mathfrak{Z} \cdot (t-1)\tilde{P}), \tag{9}$$

where σ is a homomorphism $\sigma\colon \ker i_* \to H_1(\tilde{X}, \tilde{X}^0)$, $\partial\sigma = \mathrm{id}$. This yields the following

9.10 Theorem. *For $\mathfrak{G} = \langle S_1, \dots, S_n \mid R_1, \dots, R_n \rangle$, its Jacobian $\left(\left(\frac{\partial R_j}{\partial S_i}\right)^{\varphi\psi}\right)$ and $\varphi\colon \mathfrak{G} \to \mathfrak{G}/\mathfrak{G}' = \mathfrak{D} = \langle t \rangle$, a presentation matrix (Alexander matrix) of $H_1(\tilde{X}) \cong H_1(C_\infty)$ as a $\mathfrak{D}$-module is obtained from the Jacobian by omitting its i-th column, if $S_i^{\varphi\psi} = t^{\pm 1}$. (In the case of a Jacobian derived from a Wirtinger presentation any column may be omitted.)*

Proof. It remains to show that the homomorphism $\sigma\colon \ker i_* \to H_1(\tilde{X}, \tilde{X}^0)$ can be chosen in such a way that $\sigma(\ker i_*) = \mathbb{Z}\mathfrak{D}\tilde{s}_i$. Put $\sigma(t-1)\tilde{P} = \pm t^\mu \tilde{s}_i$, $S_i^{\varphi\psi} = t^\nu$, $\partial\sigma = \mathrm{id}$. Then

$$(t-1)\tilde{P} = \partial\sigma(t-1)\tilde{P} = \partial(\pm t^\mu \tilde{s}_i) = \pm t^\mu (S_i^{\varphi\psi} - 1)\tilde{P} = \pm t^\mu (t^\nu - 1)\tilde{P},$$

that is, $(t-1) = \pm t^\mu (t^\nu - 1)$. It follows $\nu = \pm 1$, and in these cases σ can be chosen as desired. □

If $\mathfrak{D}$ is not free cyclic, the sequence (8) does not necessarily split, and $H_1(\tilde{X})$ cannot be identified as a direct summand of $H_1(\tilde{X}, \tilde{X}^0)$. We shall treat the cases $\mathfrak{D} \cong \mathbb{Z}_n$ and $\mathfrak{D} \cong \mathbb{Z}^\mu$ in Section D.

There is a useful corollary to Theorem 9.10:

9.11 Corollary. *Every $(n-1) \times (n-1)$ minor Δ_{ij} of the $n \times n$ Jacobian of a Wirtinger presentation $\langle S_i \mid R_j \rangle$ of a knot group $\mathfrak{G}$ is a presentation matrix of $H_1(C_\infty)$. Furthermore,* det $\Delta_{ij} \doteq \Delta(t)$. *The elementary ideals of the Jacobian are the elementary ideals of the knot.*

Proof. Every Wirtinger relator R_k is a consequence of the remaining ones (Corollary 3.6). Thus, by 9.10 a presentation matrix of $H_1(C_\infty) = M(t)$ is obtained from the Jacobian by leaving out an arbitrary row and arbitrary column. □

Corollary 9.11 shows that a Jacobian of a Wirtinger presentation has nullity one. The following lemma explicitly describes the linear dependence of the rows and columns of the Jacobian of a Wirtinger presentation:

9.12 Lemma. (a) $\sum_{i=1}^n \left(\frac{\partial R_j}{\partial S_i}\right)^{\varphi\psi} = 0.$

(b) $\sum_{j=1}^n \eta_j \left(\frac{\partial R_j}{\partial S_i}\right)^{\varphi\psi} = 0$, $\eta_j = t^{\nu_j}$ *for suitable $\nu_j \in \mathbb{Z}$ for a Wirtinger presentation $\langle S_1, \dots, S_n \mid R_1, \dots, R_n \rangle$ of a knot group.*

Proof. Equation (a) follows from the fundamental formula 9.8 (c) applied to R_j:

$$0 = (R_j - 1)^{\varphi\psi} = \Big[\sum_{i=1}^n \left(\frac{\partial R_j}{\partial S_i}\right)(S_i - 1)\Big]^{\varphi\psi} = \sum_{i=1}^n \left(\frac{\partial R_j}{\partial S_i}\right)^{\varphi\psi} (t-1).$$

Since $\mathbb{Z}\mathfrak{Z}$ has no divisors of zero (E 9.1) equation (a) is proved. To prove (b) we use the identity of Corollary 3.6 which expresses the dependence of Wirtinger relators by the equation $\prod_{j=1}^n L_j R_j L_j^{-1} = 1$ in the free group $\langle S_1, \dots, S_n \mid - \rangle$. Now

$$\begin{aligned}(\tfrac{\partial}{\partial S_i} L_j R_j L_j^{-1})^{\varphi\psi} &= \left(\frac{\partial L_j}{\partial S_i}\right)^{\varphi\psi} + L_j^{\varphi\psi}\left(\frac{\partial R_j}{\partial S_i}\right)^{\varphi\psi} - (L_j R_j L_j^{-1})^{\varphi\psi}\left(\frac{\partial L_j}{\partial S_i}\right)^{\varphi\psi} \\ &= L_j^{\varphi\psi}\left(\frac{\partial R_j}{\partial S_i}\right)^{\varphi\psi},\end{aligned}$$

as $(L_j R_j L_j^{-1})^{\varphi\psi} = 1$. By the product rule

$$\begin{aligned}0 &= \frac{\partial}{\partial S_i}\Big(\prod_{j=1}^n L_j R_j L_j^{-1}\Big)^{\varphi\psi} = \sum_{j=1}^n \Big(\prod_{k=1}^{j-1} \big(L_k R_k L_k^{-1}\big)\Big)^{\varphi\psi} L_j^{\varphi\psi}\left(\frac{\partial R_j}{\partial S_i}\right)^{\varphi\psi} \\ &= \sum_{j=1}^n L_j^{\varphi\psi}\left(\frac{\partial R_j}{\partial S_i}\right)^{\varphi\psi},\end{aligned}$$

which proves (b) with $L_j^{\varphi\psi} = t^{\nu_j} = \eta_j$. □

9.13 Example. A Wirtinger presentation of the group of the trefoil is

$$\langle\, S_1, S_2, S_3 \mid S_1S_2S_3^{-1}S_2^{-1},\ S_2S_3S_1^{-1}S_3^{-1},\ S_3S_1S_2^{-1}S_1^{-1}\,\rangle,$$

see 3.7. If $R = S_1S_2S_3^{-1}S_2^{-1}$ then

$$\frac{\partial R}{\partial S_1} = 1, \quad \frac{\partial R}{\partial S_2} = S_1 - S_1S_2S_3^{-1}S_2^{-1}, \quad \frac{\partial R}{\partial S_3} = -S_1S_2S_3^{-1}$$

and

$$\left(\frac{\partial R}{\partial S_1}\right)^{\varphi\psi} = 1, \quad \left(\frac{\partial R}{\partial S_2}\right)^{\varphi\psi} = t-1, \quad \left(\frac{\partial R}{\partial S_3}\right)^{\varphi\psi} = -t.$$

By similar calculations we obtain the matrix of derivatives and apply $\varphi\psi$ to get an Alexander matrix

$$\begin{pmatrix} 1 & t-1 & -t \\ -t & 1 & t-1 \\ t-1 & -t & 1 \end{pmatrix}.$$

It is easy to verify 9.12 (a) and (b). The 2×2 minor $\Delta_{11} = \begin{pmatrix} 1 & t-1 \\ -t & 1 \end{pmatrix}$, for instance, is a presentation matrix; $|\Delta_{11}| = 1 - t + t^2 = \Delta(t)$, $E_1(t) = (1 - t + t^2)$. For $k > 1$: $E_k(t) = (1) = \mathbb{Z}(t)$, $\Delta_k(t) = 1$.

9.14 Proposition. *Let*

$$\langle\, S_1, \dots, S_n \mid R_1, \dots, R_m\,\rangle = \mathfrak{G} = \langle\, S_1', \dots, S_{n'}' \mid R_1', \dots, R_{m'}'\,\rangle$$

be two finite presentations of a knot group. The elementary ideals of the respective Jacobians $\left(\left(\frac{\partial R_j}{\partial S_i}\right)^{\varphi\psi}\right)$ *and* $\left(\left(\frac{\partial R_j'}{\partial S_i'}\right)^{\varphi\psi}\right)$ *coincide, and are those of the knot.*

Proof. This follows from 9.11, and from the fact (Appendix A.6) that the elementary ideals are invariant under Tietze processes. □

9.15 Example (Torus knots). $\mathfrak{G} = \langle\, x, y \mid x^a y^{-b}\,\rangle$, $a > 0$, $b > 0$, $\gcd(a, b) = 1$, is a presentation of the group of the knot $\mathfrak{t}(a, b)$ (see 3.28). The projection homomorphism $\varphi\colon \mathfrak{G} \to \mathfrak{G}/\mathfrak{G}' = \mathfrak{Z} = \langle\, t\,\rangle$ is defined by: $x^\varphi = t^b$, $y^\varphi = t^a$ (Exercise E 9.3). The Jacobian of the presentation is:

$$\left(\frac{\partial(x^a y^{-b})}{\partial x}, \frac{\partial(x^a y^{-b})}{\partial y}\right)^{\varphi\psi} = \left(\frac{t^{ab}-1}{t^b-1}, -\frac{t^{ab}-1}{t^a-1}\right).$$

The greatest common divisor

$$\gcd\left(\frac{t^{ab}-1}{t^b-1}, \frac{t^{ab}-1}{t^a-1}\right) = \frac{(t^{ab}-1)(t-1)}{(t^a-1)(t^b-1)} = \Delta_{a,b}(t)$$

is the Alexander polynomial of $\mathfrak{t}(a,b)$, $\deg \Delta_{a,b}(t) = (a-1)(b-1)$. One may even prove something more: The Alexander module $M_{a,b}(t)$ of a torus knot $\mathfrak{t}(a,b)$ is cyclic: $M_{a,b}(t) \cong \mathbb{Z}(t)/(\Delta_{a,b}(t))$.

Proof. There are elements $\alpha(t), \beta(t) \in \mathbb{Z}(t)$ such that

$$\alpha(t)(t^{a-1}+t^{a-2}+\cdots+t+1)+\beta(t)(t^{b-1}+t^{b-2}+\cdots+t+1)=1. \tag{10}$$

This is easily verified by applying the Euclidean algorithm. It follows that

$$\alpha(t)\frac{t^{ab}-1}{t^b-1}+\beta(t)\frac{t^{ab}-1}{t^a-1}=\Delta_{a,b}(t).$$

Hence, the Jacobian can be replaced by an equivalent one:

$$\left(\frac{t^{ab}-1}{t^b-1}, -\frac{t^{ab}-1}{t^b-1}\right)\begin{pmatrix}\alpha(t) & t^{b-1}+\cdots+1\\ -\beta(t) & t^{a-1}+\cdots+1\end{pmatrix}=(\Delta_{a,b}(t),0).$$

We may interpret by 9.9 the Jacobian as a presentation matrix of $H_1(\tilde{X},\tilde{X}^0)$:

$$\left(\frac{t^{ab}-1}{t^b-1}, -\frac{t^{ab}-1}{t^a-1}\right)\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}=0,$$

where $\tilde{x}, \tilde{y}$ are the 1-chains that correspond to the generators x, y (see 9.1).

The transformation of the Jacobian implies a contragredient (dual) transformation of the generating 1-chains:

$$\tilde{u}=(t^{a-1}+\cdots+1)\tilde{x}-(t^{b-1}+\cdots+1)\tilde{y},$$
$$\tilde{v}=\beta(t)\tilde{x}+\alpha(t)\tilde{y}.$$

These 1-chains form a new basis with:

$$(\Delta_{a,b}(t),0)\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix}=0.$$

Since $\partial\tilde{x}=(t^b-1)\tilde{P}$, $\partial\tilde{y}=(t^a-1)\tilde{P}$, one has $\partial\tilde{u}=0$ and

$$\partial\tilde{v}=(\beta(t)(t^b-1)+\alpha(t)(t^a-1))\tilde{P}=(t-1)\tilde{P}$$

by (10). Thus $\tilde{v}$ generates a free summand $\sigma(\ker i_*)$ (see (9)), and $\tilde{u}$ generates $M(t)$, subject to the relation $\Delta_{a,b}(t)\tilde{u}=0$. □

Torus knots are fibred knots, by 4.10 and 5.1. We proved in 4.11 that the commutator subgroup $\mathfrak{G}'$ of a torus knot $\mathfrak{t}(a,b)$ is free of rank $(a-1)(b-1)$. By Theorem 4.6 the genus of $\mathfrak{t}(a,b)$ is $g=\frac{(a-1)(b-1)}{2}$, a fact which is reproved by 8.16, and $\deg \Delta_{a,b}(t)=(a-1)(b-1)$.

D Alexander Polynomials of Links

Let $\mathfrak{k}$ be an oriented link of $\mu > 1$ components, and $\mathfrak{G} = \pi_1(\overline{S^3 - V(\mathfrak{k})})$ its group. $\varphi\colon \mathfrak{G} \to \mathfrak{G}/\mathfrak{G}' = \mathfrak{Z}^\mu = \langle t_1 \rangle \times \cdots \times \langle t_\mu \rangle$ maps $\mathfrak{G}$ onto a free abelian group of rank μ. For each component we choose a meridian t_i with $\operatorname{lk}(\mathfrak{k}_i, t_i) = +1$. We assume, as in the case of a knot, that t_i, $1 \leqq i \leqq \mu$, denotes at the same time a free generator of $\mathfrak{Z}^\mu$ or a representative in $\mathfrak{G}$ mod $\mathfrak{G}'$, representing a meridian of the i-th component $\mathfrak{k}_i$ of $\mathfrak{k}$ with $\varphi(t_i) = t_i$. We may consider $\mathfrak{G}'/\mathfrak{G}''$ as module over the group ring $\mathbb{Z}\mathfrak{Z}^\mu$ using the operation $a \mapsto t_i^{-1} a t_i$, $a \in \mathfrak{G}'$, to define the operation of $\mathbb{Z}\mathfrak{Z}^\mu$ on $\mathfrak{G}'/\mathfrak{G}''$. Proposition 9.2 applies to the situation with $\mathfrak{N} = \mathfrak{G}'$, $\mathfrak{D} \cong \mathfrak{Z}^\mu$. Denote by ψ the canonical homomorphism

$$\psi : \mathfrak{F} = \langle S_1, \dots, S_n \mid - \rangle \to \langle S_1, \dots, S_n \mid R_1, \dots, R_n \rangle = \mathfrak{G}$$

onto the link group $\mathfrak{G}$, described by a Wirtinger presentation. The Jacobian $\left(\left(\frac{\partial R_j}{\partial S_i}\right)^{\varphi\psi}\right)$, then is a presentation matrix of $H_1(\tilde{X}, \tilde{X}^0)$. The exact sequence (8) does not split, so that a submodule isomorphic to $H_1(\tilde{X}) \cong H_1(C_\infty)$ cannot easily be identified. Following [Fox 1954] we call $H_1(\tilde{X}, \tilde{X}^0)$ the *Alexander module* of $\mathfrak{k}$ and denote it by $M(t_1, \dots, t_\mu)$.

9.16 Proposition. *The first elementary ideal $E_1(t_1, \dots, t_\mu)$ of the Alexander module $M(t_1, \dots, t_\mu)$ of a μ-component link $\mathfrak{k}$ is of the form:*

$$E_1(t_1, \dots, t_\mu) = J_0 \cdot (\Delta(t_1, \dots, t_\mu))$$

where J_0 is the augmentation ideal of $\mathbb{Z}\mathfrak{Z}^\mu$ (see 9.3), and the second factor is a principal ideal generated by the greatest common divisor of $E_1(t_1, \dots, t_\mu)$; it is called the Alexander polynomial $\Delta(t_1, \dots, t_\mu)$ of $\mathfrak{k}$, and it is an invariant of $\mathfrak{k}$ – up to multiplication by a unit of $\mathbb{Z}\mathfrak{Z}^\mu$.

Proof. Corollary 3.6 is valid in the case of a link. The $(n-1) \times n$-matrix $\mathfrak{R}$ resulting from the Jacobian $\left(\left(\frac{\partial R_j}{\partial S_i}\right)^{\varphi\psi}\right)$ by omitting its last row is, therefore, a presentation matrix of $H_1(\tilde{X}, \tilde{X}^0)$, and defines its elementary ideals. Let $\Delta_i' = \det(\mathfrak{a}_1, \dots, \mathfrak{a}_{i-1}, \mathfrak{a}_{i+1}, \dots, \mathfrak{a}_n)$ be the determinant formed by the column-vectors $\mathfrak{a}_j$, $i \neq j$, of $\mathfrak{R}$. The fundamental formula $R_k - 1 = \sum_{j=1}^n \frac{\partial R_k}{\partial S_j}(S_j - 1)$ yields $\sum_{j=1}^n \mathfrak{a}_j(S_j^{\varphi\psi} - 1) = 0$. Hence,

$$\begin{aligned}
\Delta_j'(S_i^{\varphi\psi} - 1) &= \det\left(\mathfrak{a}_1, \dots, \mathfrak{a}_i(S_i^{\varphi\psi} - 1), \dots, \mathfrak{a}_{j-1}, \mathfrak{a}_{j+1}, \dots\right) \\
&= \det\left(\mathfrak{a}_1, \dots, -\sum_{k \neq i} \mathfrak{a}_k(S_k^{\varphi\psi} - 1), \dots, \mathfrak{a}_{j-1}, \mathfrak{a}_{j+1}, \dots\right) \\
&= \det\left(\mathfrak{a}_1, \dots, -\mathfrak{a}_j(S_j^{\varphi\psi} - 1), \dots, \mathfrak{a}_{j-1}, \mathfrak{a}_{j+1}, \dots\right) \\
&= \pm \Delta_i'(S_j^{\varphi\psi} - 1);
\end{aligned}$$

thus

$$\Delta'_j(S_i^{\varphi\psi} - 1) = \pm\Delta'_i(S_j^{\varphi\psi} - 1). \tag{11}$$

The $S_i^{\varphi\psi}$, $1 \leqq i \leqq n$ take the value of all t_k, $1 \leqq k \leqq \mu$. Now it follows that $(S_i^{\varphi\psi} - 1)|\Delta'_i$. Define Δ_i by $(S_i^{\varphi\psi} - 1)\Delta_i = \Delta'_i$. Since $\mathbb{Z}\mathfrak{Z}^\mu$ is a unique factorization ring, (11) implies that $\Delta_i = \pm\Delta$ for $1 \leqq i \leqq n$. The first elementary ideal, therefore, is a product $J_0 \cdot (\Delta)$, where J_0 is generated by the elements $(t_k - 1)$, $1 \leqq k \leqq \mu$. It is easy to prove (E 9.1) that J_0 is the augmentation ideal $I\mathfrak{Z}^\mu$ of $\mathbb{Z}\mathfrak{Z}^\mu$.

The elementary ideal E_1 is an invariant of $\mathfrak{G}$ (Appendix A.6); hence, its greatest common divisor is an invariant of $\mathfrak{G}$ – up to multiplication by a unit $\pm t_1^{r_1} \dots t_\mu^{r_\mu}$ of $\mathbb{Z}\mathfrak{Z}^\mu$. The polynomial $\Delta(t_1, \dots, t_\mu) = \Delta$, though, depends on the choice of a basis of $\mathfrak{Z}^\mu$. But it is possible to distinguish a basis of $\mathfrak{Z}^\mu \cong H_1(\overline{S^3 - V(\mathfrak{k})})$ geometrically by choosing a meridian for each component $\mathfrak{k}_i$ to represent t_i. □

For more information on Alexander modules of links see [Crowell-Strauss 1969], [Hillman 1981′], [Levine 1975].

A link $\mathfrak{k}$ is called *splittable*, if it can be separated by a 2-sphere embedded in S^3.

9.17 Corollary. *The Alexander polynomial of splittable link of multiplicity* $\mu \geq 2$ *vanishes, i.e.* $\Delta(t_1, \dots, t_\mu) = 0$.

Proof. A splittable link $\mathfrak{k}$ allows a Wirtinger presentation of the following form: There are two disjoint finite sets of Wirtinger generators, $\{S_i \mid i \in I\}$, $\{T_j \mid j \in J\}$, and correspondingly, two sets of relators $\{R_k(S_i)\}$, $\{N_l(T_j)\}$. For $i \in I$, $j \in J$ consider

$$\Delta'_i(T_j^{\varphi\psi} - 1) = \pm\Delta'_j(S_i^{\varphi\psi} - 1).$$

The column $\mathfrak{a}_i(S_i^{\varphi\psi} - 1)$ in $\pm\Delta'_j(S_i^{\varphi\psi} - 1)$ is by $\sum_{k\in I} \mathfrak{a}_k(S_i^{\varphi\psi} - 1) = 0$ a linear combination of other columns. It follows that $\Delta'_i(T_j^{\varphi\psi} - 1) = 0$, i.e. $\Delta'_i = 0$. □

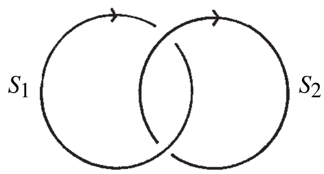
Alexander polynomials of links retain some properties of knot polynomials. In [Torres-Fox 1954] they are shown to be symmetric. The conditions (Torres-conditions) do not characterize Alexander polynomials of links ($\mu \geqq 2$), as J.A. Hillman [1981, VII, Theorem 5] showed.

9.18 There is a simplified version of the Alexander polynomial of a link. Consider the homomorphism $\chi\colon \mathfrak{Z}^\mu \to \mathfrak{Z} = \langle t \rangle$, $t_i \mapsto t$. Put $\mathfrak{N} = \ker \chi\varphi$. The sequence (9) now splits, and, as in the case of a knot, any $(n-1) \times (n-1)$ minor of the Jacobian $\left(\left(\frac{\partial R_j}{\partial S_i}\right)^{\chi\varphi\psi}\right)$ is a presentation matrix of $H_1(\tilde{X}) \cong H_1(C_\infty)$, where C_∞ is the infinite cyclic covering of the complement of the link which corresponds to the normal subgroup $\mathfrak{N} = \ker \chi\varphi \lhd \mathfrak{G}$. The first elementary ideal is generated by

$(t-1)\cdot\Delta(t,\dots,t)$ (see 9.16) where $\Delta(t_1,\dots,t_\mu)$ is the Alexander polynomial of the link. The polynomial $\Delta(t,\dots,t)$ (*the so-called reduced Alexander polynomial*) is of the form $\Delta(t,\dots,t)=(t-1)^{\mu-2}.\ \nabla(t)$, and $\nabla(t)$ is called the *Hosokawa polynomial of the link* (E 9.5). In [Hosokawa 1958] it was shown that $\nabla(t)$ *is of even degree and symmetric*. Furthermore, any such polynomial $f(t)\in\mathbb{Z}\mathfrak{Z}$ is the Hosokawa polynomial of a link for any $\mu>1$.

9.19 Examples

(a) For the link of Figure 9.1:

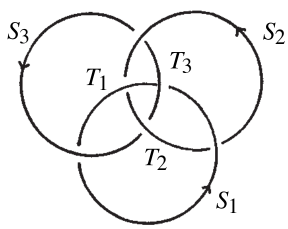


$$\mathfrak{G}=\langle S_1,S_2\mid S_1S_2S_1^{-1}S_2^{-1}\rangle$$

Figure 9.1

$$\mathfrak{R}=((1-S_1S_2S_1^{-1})^{\varphi\psi},\quad(S_1-S_1S_2S_1^{-1}S_2^{-1})^{\varphi\psi})=(1-t_2,t_1-1)\quad\text{and}\quad\Delta=1.$$

(b) Borromean link (Figure 9.2).



Generators: $S_1,\ S_2,\ S_3,\ T_1,\ T_2,\ T_3$

Relators: $T_1^{-1}S_2^{-1}T_1T_2,\ T_2^{-1}S_3T_2T_3^{-1},$
$S_2^{-1}S_3S_2T_3^{-1},\ S_1^{-1}S_3T_1S_3^{-1}$
$S_1^{-1}S_2S_1T_2^{-1}$

Figure 9.2

Eliminate $T_1=S_3^{-1}S_1S_3$, $T_2=S_1^{-1}S_2S_1$ and $T_3=S_2^{-1}S_3S_2$, and obtain the presentation

$$\mathfrak{G}=\langle S_1,S_2,S_3\mid S_3^{-1}S_1^{-1}S_3S_2^{-1}S_3^{-1}S_1S_3S_1^{-1}S_2S_1,\ S_1^{-1}S_2^{-1}S_1S_3S_1^{-1}S_2S_1S_2^{-1}S_3^{-1}S_2\rangle.$$

From this we get

$$\mathfrak{R}=\begin{pmatrix} -t_1^{-1}t_3^{-1}+t_1^{-1}t_2^{-1}t_3^{-1}-t_1^{-1}t_2^{-1}+t_1^{-1}\ ,0\ ,\ -t_3^{-1}+t_1^{-1}t_3^{-1}-t_1^{-1}t_2^{-1}t_3^{-1}+t_2^{-1}t_3^{-1} \\ -t_1^{-1}+t_1^{-1}t_2^{-1}-t_1^{-1}t_2^{-1}t_3+t_1^{-1}t_3\ ,-t_1^{-1}t_2^{-1}+t_1^{-1}t_2^{-1}t_3-t_2^{-1}t_3+t_2^{-1}\ ,\ 0 \end{pmatrix}$$

$$=\begin{pmatrix} -t_1^{-1}t_2^{-1}t_3^{-1}(t_2-1)(t_3-1)\ ,\ 0\ ,\ -t_1^{-1}t_2^{-1}t_3^{-1}(t_2-1)(t_1-1) \\ t_1^{-1}t_2^{-1}(t_2-1)(t_3-1)\ ,\ -t_1^{-1}t_2^{-1}(t_1-1)(t_3-1)\ ,\ 0 \end{pmatrix}.$$

Therefore

$$\Delta_1' = -t_1^{-2}t_2^{-2}t_3^{-1}(t_1-1)(t_2-1)(t_3-1)(t_1-1) \doteq -\Delta \cdot (t_1-1)$$
$$\Delta_2' = t_1^{-2}t_2^{-2}t_3^{-1}(t_1-1)(t_2-1)(t_3-1)(t_2-1) \doteq \Delta \cdot (t_2-1)$$
$$\Delta_3' = t_1^{-2}t_2^{-2}t_3^{-1}(t_1-1)(t_2-1)(t_3-1)(t_3-1) \doteq \Delta \cdot (t_3-1),$$

where $\Delta = \Delta(t_1, t_2, t_3) = (t_1-1)(t_2-1)(t_3-1)$.

E Finite Cyclic Coverings Again

The theory of Fox derivations may also be utilized to compute the homology of finite branched cyclic coverings of knots. (For notations and results compare 8.17–22, 9.1.)

Let C_N, $0 < N \in \mathbb{Z}$, be the N-fold cyclic (unbranched) covering of the complement C. We know (see 8.20 (d)) that $(V^T - tV)s = 0$ are defining relations of $H_1(\hat{C}_N)$ as a $\mathfrak{Z}_N$-module, $\mathfrak{Z}_N = \langle t \mid t^N \rangle$.

9.20 Proposition. (a) *Any Alexander matrix $A(t)$ (that is a presentation matrix of $H_1(C_\infty)$ as a $\mathfrak{Z}$-module, $\mathfrak{Z} = \langle t \rangle$) is a presentation matrix of $H_1(\hat{C}_N)$ as a $\mathfrak{Z}_N$-module. $\mathfrak{Z}_N = \langle t \mid t^N \rangle$.*

(b) *The matrix*

$$\begin{pmatrix} & A(t) & & \\ \varrho_N & 0 & \dots & 0 \\ 0 & \varrho_N & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \varrho_N \end{pmatrix} = B_N(t), \quad \varrho_N = 1 + t + \cdots + t^{N-1},$$

is a presentation matrix of $H_1(\hat{C}_N)$ as a $\mathfrak{Z}$-module.

Proof. The first assertion follows from the fact that, if two presentation matrices $A(t)$ and $A'(t)$ are equivalent over $\mathbb{Z}\mathfrak{Z}$, they are equivalent over $\mathbb{Z}\mathfrak{Z}_N$. The second version is a consequence of 8.20 (b). Observe that $(t^N - 1) = \varrho_N(t)(t-1)$. □

9.21 Corollary. *The homology groups $H_1(\hat{C}_N)$ of the N-fold cyclic branched coverings of a torus knot $\mathfrak{t}(a, b)$ are periodic with the period ab:*

$$H_1(\hat{C}_{N+kab}) \cong H_1(\hat{C}_N), \quad k \in \mathbb{N}.$$

Moreover

$$H_1(\hat{C}_N) \cong H_1(\hat{C}_{N'}) \quad \textit{if } N' \equiv -N \mod ab.$$

Proof. By 9.20 (b), $B_N(t) = \begin{pmatrix} \Delta(t) \\ \varrho_N(t) \end{pmatrix}$ is a presentation matrix for the $\mathbb{Z}(t)$-module $H_1(\hat{C}_N)$. Since $\Delta(t)|\varrho_{ab}(t)$ and $\varrho_{N+kab} = \varrho_N + t^N \cdot \varrho_k(t^{ab}) \cdot \varrho_{ab}$, the presentation matrices $B_N(t)$ and $B_{N+kab}(t)$ are equivalent. The second assertion is a consequence of

$$\varrho_{ab} - \varrho_N = t^N \cdot \varrho_{ab-N} \quad \text{for } 0 < N < ab. \qquad \square$$

9.22 Example. For the trefoil $\mathfrak{t}(3,2)$ the homology groups of the cyclic branched coverings are:

$$H_1(\hat{C}_N) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } N \equiv 0 \mod 6 \\ 0 & \text{for } N \equiv \pm 1 \mod 6 \\ \mathbb{Z}_3 & \text{for } N \equiv \pm 2 \mod 6 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } N \equiv 3 \mod 6. \end{cases}$$

Proof.

$$N \equiv 0 \mod 6\colon \begin{pmatrix} 1-t+t^2 \\ 0 \end{pmatrix} \sim (1-t+t^2).$$

$$N \equiv 1 \mod 6\colon \begin{pmatrix} 1-t+t^2 \\ 1 \end{pmatrix} \sim (1).$$

$$N \equiv 2 \mod 6\colon \begin{pmatrix} 1-t+t^2 \\ 1+t \end{pmatrix} \sim \begin{pmatrix} 3 \\ 1+t \end{pmatrix}$$

$$N \equiv 3 \mod 6\colon \begin{pmatrix} 1-t+t^2 \\ 1+t+t^2 \end{pmatrix} \sim \begin{pmatrix} 2 \\ 1+t+t^2 \end{pmatrix}.$$

$N \equiv 0\colon\ H_1(\hat{C}_N) \cong \langle\, s \,\rangle \oplus \langle\, ts \,\rangle$ where s is the generator.

$N \equiv 1\colon\ H_1(\hat{C}_N) = 0.$

$N \equiv 2\colon\ H_1(\hat{C}_N) \cong \langle\, s \mid 3s \,\rangle.$

$N \equiv 3\colon\ H_1(\hat{C}_N) \cong \langle\, s \mid 2s \,\rangle \oplus \langle\, ts \mid 2ts \,\rangle.$ $\square$

9.23 Remark. In the case of a two-fold covering $\hat{C}_2$ we get a result obtained already in 8.20 (a):

$$B_2(t) = \begin{pmatrix} & & A(t) & \\ 1+t & & & \\ & 1+t & & \\ & & \ddots & \\ & & & 1+t \end{pmatrix} \sim A(-1).$$

Proposition 8.20 gives a presentation matrix for $H_1(\hat{C}_N)$ as an abelian group (8.20 (a)) derived from the presentation matrix $A(t) = (V^T - tV)$ for $H_1(\hat{C}_N)$ as a $\mathfrak{Z}_N$-module. This can also be achieved by the following trick: Blow up $A(t)$ by replacing every matrix element $r_{ik}(t) = \sum_j c_{ik}^{(j)} t^j$ by an $N \times N$-matrix $R_{ik} = \sum_j c_{ik}^{(j)} \mathfrak{T}_N^j$,

$$\mathfrak{T}_N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ \vdots & \vdots & \vdots & & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

This means introducing N generators $s_i, ts_i, \dots, t^{N-1}s_i$ for each generator s_i, observing $t(t^\nu s_i) = t^{\nu+1}s_i$, $t^N = 1$. The blown up matrix is a presentation matrix of $H_1(\hat{C}_N)$ as an abelian group. For practical calculations of $H_1(\hat{C}_N)$ this procedure is not very useful, because of the high order of the matrices. It may be used, though, to give an alternative proof of 8.21, see [Neuwirth 1965, 5.3.1].

F History and Sources

Homotopy chains were first introduced by Reidemeister [1934], and they were used to classify lens spaces [Reidemeister 1935], [Franz 1935]. R.H. Fox gave an algebraic foundation and generalization of the theory in his free differential calculus [Fox 1953, 1954, 1956], and introduced it to knot theory. Most of the material of this chapter is connected with the work of R.H. Fox. In connection with the Alexander polynomials of links the contribution of [Crowell-Strauss 1969] and [Hillman 1981′] should be mentioned.

G Exercises

E 9.1. Show

(a) that the augmentation ideal $I\mathfrak{Z}^\mu$ of $\mathbb{Z}\mathfrak{Z}^\mu$ is generated by the elements $(t_i - 1)$, $1 \leqq i \leqq \mu$,

(b) $\mathbb{Z}\mathfrak{Z}^\mu$ is a unique factorization ring with no divisors of zero,

(c) the units of $\mathbb{Z}\mathfrak{Z}$ are $\pm g$, $g \in \mathfrak{Z}^\mu$.

E 9.2. The Alexander module of a 2-bridge knot $\mathfrak{b}(a, b)$ is cyclic. Deduce from this that $\Delta_k(t) = 1$ for $k > 1$.

E 9.3. Let $\varphi\colon \mathfrak{G} \to \mathfrak{G}/\mathfrak{G}' = \langle t \rangle$ be the abelianizing homomorphism of the group $\mathfrak{G} = \langle x, y | x^a y^{-b} \rangle$ of a torus knot $\mathfrak{t}(a, b)$. Show that $x^\varphi = t^b$, $y^\varphi = t^a$.

E 9.4. Compute the Alexander polynomial $\Delta(t_1, t_2)$ of the two component link $\mathfrak{k}_1 \cup \mathfrak{k}_2$, where $\mathfrak{k}_1$ is a torus knot, $\mathfrak{k}_1 = \mathfrak{t}(a, b)$, and $\mathfrak{k}_2$ the core of the solid torus T on whose boundary $\mathfrak{t}(a, b)$ lies. Hint: Prove that $\langle x, y, z \mid [x, z], x^a y^{-b} z^b \rangle$ is a presentation of the group of $\mathfrak{k}_1 \cup \mathfrak{k}_2$.

Result: $\Delta(t_1, t_2) = \frac{(t_1^a t_2)^b - 1}{t_1^a t_2 - 1}$.

E 9.5. Let C_∞ be the infinite cyclic covering of a link $\mathfrak{k}$ of μ components (see 9.18). Show that $H_1(C_\infty)$ has a presentation matrix of the form $(V^t - tV)$ with

$$V - V^T = F' = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}.$$

F is a $2g \times 2g$ matrix (g the genus of $\mathfrak{k}$), and the order of F' is $2g + \mu - 1$. Deduce from this that the reduced Alexander polynomial of $\mathfrak{k}$ is divisible by $(t-1)^{\mu-2}$ (compare 9.18), and from this: $H_1(\hat{C}_2; \mathbb{Z}_2) = \bigoplus_{i=1}^{\mu-1} \mathbb{Z}_2$.

Prove that $|\nabla(1)|$ equals the absolute value of a $(\mu - 1) \times (\mu - 1)$ principal minor of the linking matrix $(\mathrm{lk}(\mathfrak{k}_i, \mathfrak{k}_j))$, $1 \leqq i, j \leqq \mu$. Show that $\nabla(t)$ is symmetric.

E 9.6. Compute the Alexander polynomial of the doubled knot with m half-twists (Figure 9.3). (Result: $\Delta(t) = kt^2 - (2k+1)t + k$ for $m = 2k$, $\Delta(t) = kt^2 - (2k-1)t + k$ for $m = 2k - 1$, $k = 1, 2, \dots$)

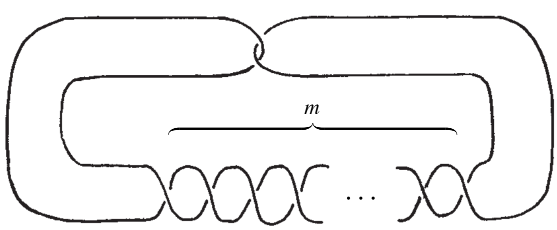


Figure 9.3

E 9.7. For $\mathfrak{F} = \langle \{s_i \mid i \in I\} \mid - \rangle$ let l denote the usual length of words with respect to the free generators $\{s_i \mid i \in I\}$. Extend it to $\mathbb{Z}\mathfrak{F}$ by $l(n_1 x_1 + \cdots + n_k x_k) = \max\{l(x_j) \mid$

$1 \leqq j \leqq k, n_j \neq 0\}$; here $n_j \in \mathbb{Z}$ and $x_j \in \mathfrak{F}$ with $x_j \neq x_i$ for $i \neq j$. Introduce the following derivations:

$$\frac{\partial}{\partial s_i^{-1}} : \mathbb{Z}\mathfrak{F} \to \mathbb{Z}\mathfrak{F}, \quad s_j \mapsto \begin{cases} -s_i & \text{for } j = i, \\ 0 & \text{for } j \neq i. \end{cases}$$

Prove:

(a) $\frac{\partial}{\partial s_i^{-1}}(s_i^{-1}) = 1$, $\frac{\partial}{\partial s_i^{-1}} = -\frac{\partial}{\partial s_i} \cdot s_i$.

(b) $l\left(\frac{\partial \tau}{\partial s_i}\right) \leqq l(\tau)$, $l\left(\frac{\partial \tau}{\partial s_i^{-1}}\right) \leqq l(\tau)$ for all $i \in I$, $\tau \in \mathbb{Z}\mathfrak{F}$.

(c) $l\left(\frac{\partial}{\partial s_i^{-1}} \frac{\partial}{\partial s_i}(\tau)\right) < l(\tau)$, $l\left(\frac{\partial}{\partial s_i} \frac{\partial}{\partial s_i^{-1}}(\tau)\right) < l(\tau)$.

(d) $\frac{\partial}{\partial s_i} = \left(\frac{\partial}{\partial s_i^{-1}} \frac{\partial}{\partial s_i}\right) \cdot s_i^{-1} - \frac{\partial}{\partial s_i} \frac{\partial}{\partial s_i^{-1}}$.

E 9.8. (a) With the notation of E 9.7 prove: Let $\tau, \gamma \in \mathbb{Z}\mathfrak{F}$, $\gamma \neq 0$ and $l(\tau\gamma) \leqq l(\tau)$. Then either $\gamma \in \mathbb{Z}$ or there is a s_i^δ, $i \in I$, $\delta \in \{1, -1\}$ such that $l(\tau s_i^\delta) \leqq l(\tau)$. All elements $f \in \mathfrak{F}$ with $l(f) = l(\tau)$ that have a non-trivial coefficient in τ end with $s_i^{-\delta}$.

(b) If $l(\tau\gamma) < l(\tau)$ and $\gamma \neq 0$ then there is a s_i^δ, $i \in I$, $\delta \in \{1, -1\}$ such that $l(\tau s_i^\delta) < l(\tau)$.

(c) If $\tau\varrho \in \mathbb{Z}$ then either τ or ϱ is 0 or τ and ϱ have the form af with $f \in \mathfrak{F}$, $a \in \mathbb{Z}$.

Chapter 10
Braids

In this chapter we will present the basic theorems of the theory of braids including their classification or, equivalently, the solution of the word problem for braid groups, but excluding a proof of the conjugation problem (see Makanin [1968], Garside [1969], Birman [1974]). In Section C we shall consider the Fadell–Neuwirth configuration spaces which present a different aspect of the matter. Geometric reasoning will prevail, as seems appropriate in a subject of such simple beauty.

A The Classification of Braids

Braids were already defined in Chapter 2, Section D. We start by defining an isotopy relation for braids, using combinatorial equivalence. We apply Δ- and Δ^{-1}-moves to the strings f_i , $1 \leqq i \leqq n$, of the braid (see Definition 1.6) assuming that each process preserves the braid properties and keeps fixed the points P_i, Q_i, $1 \leqq i \leqq n$. (See Figure 10.1.)

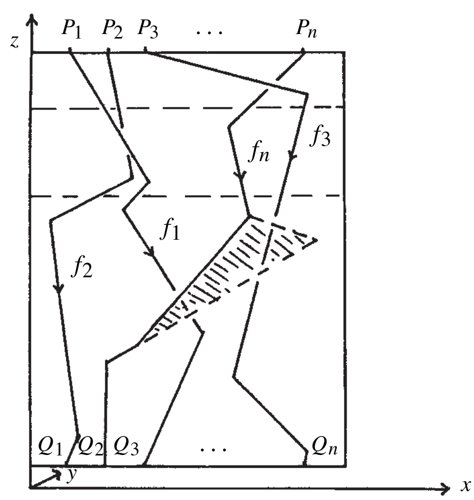


Figure 10.1

10.1 Definition (Isotopy of braids). Two braids $\mathfrak{z}$ and $\mathfrak{z}'$ are called *isotopic* or *equivalent*, if they can be transformed into each other by a finite sequence of $\Delta^{\pm 1}$-processes.

It is obvious that a theorem similar to Proposition 1.10 can be proved. Various notions of isotopy have been introduced [Artin 1947] and shown to be equivalent. As in the case of knots we shall use the term braid and the notation $\mathfrak{z}$ also for a class of equivalent braids. All braids in this section are supposed to be n-braids for some fixed $n > 1$. There is an obvious composition of two braids $\mathfrak{z}$ and $\mathfrak{z}'$ by identifying the endpoints Q_i of $\mathfrak{z}$ with the initial points P_i' of $\mathfrak{z}'$ (Figure 10.2). The composition of

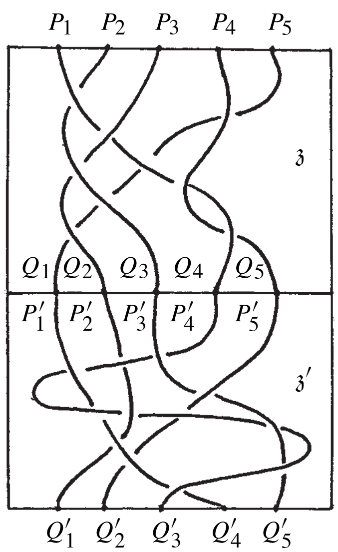


Figure 10.2

representatives defines a composition of equivalence classes. Since there is also a unit with respect to this composition and an inverse $\mathfrak{z}^{-1}$ obtained from $\mathfrak{z}$ by a reflection in a plane perpendicular to the braid, we obtain a group:

10.2 Proposition and Definition (Braid group $\mathfrak{B}_n$). *The isotopy classes $\mathfrak{z}$ of n-braids form a group called the braid group $\mathfrak{B}_n$.* □

We now undertake to find a presentation of $\mathfrak{B}_n$. It is easy to see that $\mathfrak{B}_n$ is generated by $n-1$ generators σ_i (Figure 10.3).

For easier reference let us introduce cartesian coordinates (x, y, z) with respect to the frames of the braids. The frames will be parallel to the plane $y = 0$ and those of their sides which carry the points P_i and Q_i will be parallel to the x-axis. Now every class of braids contains a representative such that its y-projection (onto the plane $y = 0$) has finitely many double points, all of them with different z-coordinates. Choose planes z = const which bound slices of $\mathbb{R}^3$ containing parts of $\mathfrak{z}$ with just one double point in their y-projection. If the intersection points of $\mathfrak{z}$ with each of those planes z = const are moved into equidistant positions on the line in which the frame meets $z = c$ (without introducing new double points in the y-projection) the braid $\mathfrak{z}$ appears as a product of the *elementary braids* σ_i, σ_i^{-1}, compare Figure 10.3.

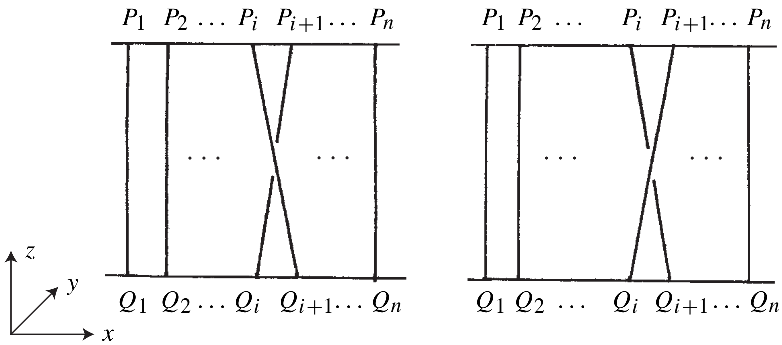
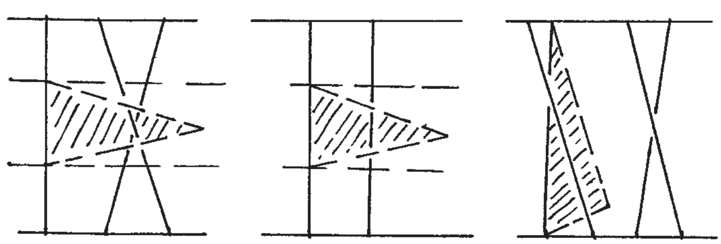


Figure 10.3

To obtain defining relations for $\mathfrak{B}_n$ we proceed as we did in Chapter 3, Section B, in the case of a knot group. Let $\mathfrak{z} = \sigma_{i_1}^{\varepsilon_1} \dots \sigma_{i_r}^{\varepsilon_r}$, $\varepsilon_i = \pm 1$, be a braid and consider its y-projection. We investigate how a Δ-process will effect the word $\sigma_{i_1}^{\varepsilon_1} \dots \sigma_{i_r}^{\varepsilon_r}$ representing $\mathfrak{z}$. We may assume that the y-projection of the generating triangle of the Δ-process contains one double point or no double points in its interior; in the second case one can assume that the projection of at most one string intersects the interior. Figure 10.4 demonstrates the possible configurations; in the first two positions it is possible to choose the triangle in a slice which contains one (Figure 10.4 (a)), or no double point (Figure 10.4 (b)) in the y-projection.



(a) (b) (c)

Figure 10.4

In Figure 10.4 (a), σ_{i+1} is replaced by $\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}$; (b) describes an elementary expansion, and in (c) a double point is moved along a string which may lead to a commutator relation $\sigma_i\sigma_k = \sigma_k\sigma_i$ for $|i-k| \geqq 2$. It is easy to verify that any process of type (a) with differently chosen over- and undercrossings leads to the same relation $\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} = 1$; (a) describes, in fact, an Ω_3-process (see 1.13), and one can always think of the uppermost string as the one being moved.

10.3 Proposition (Presentation of the braid group). *The braid group $\mathfrak{B}_n$ can be presented as follows:*

$$\mathfrak{B}_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_j\sigma_{j+1}\sigma_j\sigma_{j+1}^{-1}\sigma_j^{-1}\sigma_{j+1}^{-1} \text{ for } 1 \leqq j \leqq n-2, \qquad (10.1)$$
$$[\sigma_j, \sigma_k] \text{ for } 1 \leq j < k-1 \leqq n-2) \rangle. \qquad \square$$

In the light of this theorem the classification problem of braids can be understood as the word problem for $\mathfrak{B}_n$. We shall, however, solve the classification problem by a direct geometric approach and thereby reach a solution of the word problem, rather than vice versa.

As before, let (x, y, z) be the cartesian coordinates of a point in Euclidean 3-space. We modify the geometric setting by placing the frame of the braid askew in a cuboid K. The edges of K are supposed to be parallel to the coordinate axes; the upper side of the frame which carries the points P_i coincides with an upper edge of K parallel to the x-axis, the opposite side which contains the Q_i is assumed to bisect the base-face of K (see Figure 10.5).

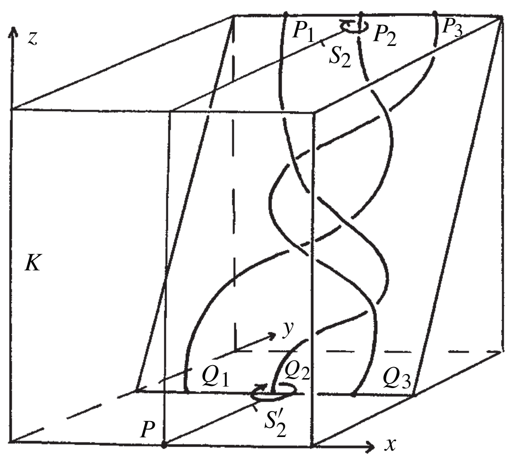


Figure 10.5

10.4 Lemma. *Every class of braids contains a representative the z-projection of which is simple* (*without double points*).

Proof. The representative $\mathfrak{z}$ of a class of braids can be chosen in such a way that its y-projection and z-projection yield the same word $\sigma_{i_1}^{\varepsilon_1}\sigma_{i_2}^{\varepsilon_2}\dots\sigma_{i_r}^{\varepsilon_r}$. This can be achieved by placing the strings in a neighbourhood of the frame, compare 2.12.

Consider the double point in $z = 0$ corresponding to $\sigma_{i_r}^{\varepsilon_r}$, push the overcrossing along the undercrossing string over its endpoint Q_j (Figure 10.6) while preserving the z-level. Obviously this process is an isotopy of $\mathfrak{z}$ which can be carried out without

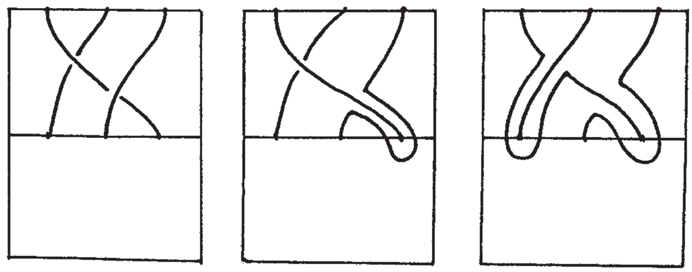


Figure 10.6

disturbing the upper part of the braid which is projected onto $\sigma_{i_1}^{\varepsilon_1}\dots\sigma_{i_{r-1}}^{\varepsilon_{r-1}}$. Proceed by removing the double point in $z=0$ corresponding to $\sigma_{i_{r-1}}^{\varepsilon_{r-1}}$. The procedure eventually leads to a braid with a simple z-projection as claimed in the lemma. □

Remark. Every $2m$-plat has an m-bridge presentation.

Let us denote the base-face of K by D, and the z-projections of f_i, P_i by f_i', P_i'. The simple projection of a braid then consists of a set of simple and pairwise disjoint arcs f_i' leading from P_i' to $Q_{\pi(i)}'$, $1\leqq i\leqq n$, where π is the permutation associated with the braid $\mathfrak{z}$ (see 2.12). We call $\{f_i' \mid 1\leqq i\leqq n\}$ a *normal dissection* of the punctured rectangle $D-\bigcup_{j=1}^{n} Q_j = D_n$. By Lemma 10.4 every braid can be represented by a set of strings which projects onto a normal dissection of D_n, and obviously every normal dissection of D_n is a z-projection of some braid in K. Two normal dissections are called isotopic if they can be transformed into each other by a sequence of $\Delta^{\pm1}$-processes in D_n. The defining triangle of such a Δ-process intersects $\{f_i'\}$ in one or two of its sides, line segments of some f_k'. Any two braids projecting onto isotopic normal dissections evidently are isotopic. The groups $\pi_1(K-\mathfrak{z})$ as well as $\pi_1 D_n$ are free of rank n. This is clear from the fact that the projecting cylinders of a braid with a simple z-projection dissect $K-\mathfrak{z}$ into a 3-cell. Every braid $\mathfrak{z}$ in K defines two sets of free generators $\{S_i\}$, $\{S_i'\}$, $1\leqq i\leqq n$, of $\pi_1(K-\mathfrak{z})$: Choose a basepoint P an the x-axis and let S_i be represented by a loop on ∂K consisting of a small circle around P_i and a (shortest) arc connecting it to P. Similarly define S_i' by encircling Q_i instead of P_i (Figure 10.5).

Since every isotopy $\mathfrak{z}\mapsto\mathfrak{z}'$ can be extended to an ambient isotopy in K leaving ∂K pointwise fixed (Proposition 1.10), a class of braids defines an *associated braid automorphism* of $\mathfrak{F}_n\cong\pi_1(K-\mathfrak{z})$, $\zeta:\mathfrak{F}_n\to\mathfrak{F}_n$, $S_i\mapsto S_i'$. All information on ζ can be obtained by looking at the normal dissection of D_n associated to $\mathfrak{z}$. Every normal dissection defines a set of free generators of $\pi_1 D_n$. A loop in D_n which intersects $\{f_i'\}$ once positively in f_k' represents a free generator $S_k\in\pi_1 D_n$ which is mapped onto $S_k\in\pi_1(K-\mathfrak{z})$ by the isomorphism induced by the inclusion. Hence, $S_i'(S_j)$ as a word in the S_j is easily read off the normal dissection:

$$S_i' = L_i S_{\pi^{-1}(i)} L_i^{-1}. \tag{1}$$

To determine the word $L_i(S_j)$, run through a straight line from P to Q_i, noting down S_k or S_k^{-1} if the line is crossed by f_k' from left to right or otherwise.

The braid automorphism (1) can also be interpreted as an automorphism of $\pi_1 D_n$ with $\{S_i\}$ associated to the normal dissection $\{f_i'\}$, and $\{S_i'\}$ associated to the standard normal dissection consisting of the straight segments from P_i' to Q_i.

The solution of the classification problem of n-braids is contained in the following

10.5 Proposition (E. Artin). *Two n-braids are isotopic if and only if they define the same braid automorphism.*

Proof. Assigning a braid automorphism ζ to a braid $\mathfrak{z}$ defines a homomorphism

$$\mathfrak{B}_n \to \operatorname{Aut} \mathfrak{F}_n, \quad \mathfrak{z} \mapsto \zeta.$$

To prove Proposition 10.5 we must show that this homomorphism is injective. This can be done with the help of

10.6 Lemma. *Two normal dissections define the same braid automorphism if and only if they are isotopic.*

Proof. A Δ-process does not change the $S_i' = L_i S_{\pi^{-1}(i)} L_i^{-1}$ as elements in the free group. This follows also from the fact that isotopic normal dissections are $\mathfrak{z}$-projections of isotopic braids, and the braid automorphism is assigned to the braid class. Now let $\{f_i'\}$ be some normal dissection of D_n and $S_i' = L_i S_{\pi^{-1}(i)} L_i^{-1}$ read off it as described before. If $L(S_i)$ contains a part of the form $S_j^{\varepsilon} S_j^{-\varepsilon}$, the two points on f_i' corresponding to S_j^{ε} and $S_j^{-\varepsilon}$ are connected by two simple arcs on f_i' and the loop in D_n representing S_i'. These arcs bound a 2-cell in D which contains no point Q_k, because otherwise $f_{\pi^{-1}(k)}'$ would have to meet one of the arcs which is impossible. Hence the two arcs bound a 2-cell in D_n, and there is an isotopy moving f_j' across it causing the elementary contraction in L_i which deletes $S_j^{\varepsilon} S_j^{-\varepsilon}$. Thus we can replace the normal dissection by an isotopic one such that the corresponding words $L_i(S_j)$ are reduced. Similarly we can assume $L_i S_{\pi^{-1}(i)} L_i^{-1}$ to be reduced. If the last symbol of $L_i(S_j)$ is $S_{\pi^{-1}(i)}^{\varepsilon}$, there is an isotopy of f_i' which deletes $S_{\pi^{-1}(i)}^{\varepsilon}$ in $L_i(S_j)$ (Figure 10.7).

Suppose now that two normal dissections $\{f_i'\}$, $\{f_i''\}$ define the same braid automorphism $S_i \mapsto S_j' = L_i S_{\pi^{-1}(i)} L_i^{-1}$. Assume the $L_i S_{\pi^{-1}(i)} L_i^{-1}$ to be reduced, and let the points of intersection of $\{f_i'\}$ and $\{f_i''\}$ with the loops representing the S_j' coincide. It follows that two successive intersection points on some f_k' are also successive on f_k'', and, hence, the two connecting arcs on f_k' resp. f_k'' can be deformed into each

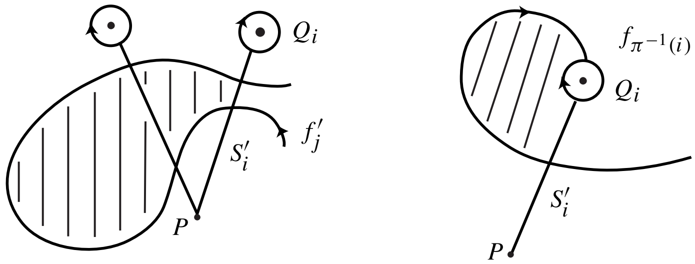


Figure 10.7

other by an isotopy of $\{f_i'\}$. This is clear if $\{f_i'\}$ is the standard normal dissection and this suffices to prove Lemma 10.6. □

We return to the proof of Proposition 10.5. Let $\mathfrak{z}$ and $\mathfrak{z}'$ be n-braids inducing the same braid automorphism. By Lemma 10.4 we may assume that their z-projections are simple. Lemma 10.6 ensures that the z-projections are isotopic; hence $\mathfrak{z}$ and $\mathfrak{z}'$ are isotopic. □

Proposition 10.5 solves, of course, the word problem of the braid group $\mathfrak{B}_n$: *Two braids $\mathfrak{z}$, $\mathfrak{z}'$ are isotopic if and only if their automorphisms coincide* – a matter which can be checked easily, since $\mathfrak{F}_n$ is free.

Propositions 10.5 and 10.6 moreover imply that there is a one-to-one correspondence between braids, braid automorphisms and isotopy classes of normal dissections. These classes represent elements of *the mapping class group of D_n*; its elements are homeomorphisms of D_n which keep ∂D_n pointwise fixed, modulo deformations of D_n.

The injective image of $\mathfrak{B}_n$ in the group Aut $\mathfrak{F}_n$ of automorphisms of the free group of rank n is called the *group of braid automorphisms*. We shall also denote it by $\mathfrak{B}_n$. The injection $\mathfrak{B}_n \to$ Aut $\mathfrak{F}_n$ depends on a set of distinguished free generators S_i of $\mathfrak{F}_n$. It is common use to stick to these distinguished generators or rather their class modulo braid automorphisms, and braid automorphisms will always be understood in this way. We propose to study these braid automorphisms more closely.

Figure 10.8 illustrates the computations of the braid automorphisms corresponding to the elementary braids $\sigma_i^{\pm 1}$ – we denote the automorphisms by the same symbols:

$$\sigma_i(S_j) = S_j' = \begin{cases} S_i S_{i+1} S_i^{-1}, & j = i \\ S_i, & j = i+1 \\ S_j, & j \neq i, i+1 \end{cases} \tag{2}$$

$$\sigma_i^{-1}(S_j) = S_j' = \begin{cases} S_{i+1}, & j = i \\ S_{i+1}^{-1} S_i S_{i+1}, & j = i+1 \\ S_j, & j \neq i, i+1 \end{cases} \tag{2'}$$

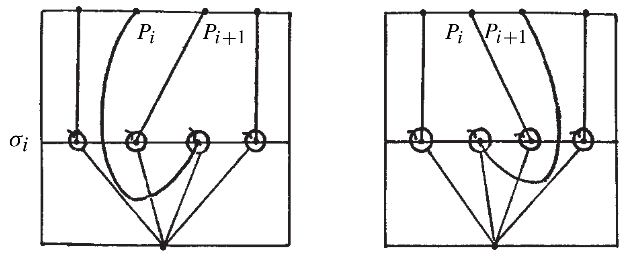


Figure 10.8

From these formulas the identity in $\mathfrak{F}_n$

$$\prod_{i=1}^{n} S_i' = \prod_{i=1}^{n} S_i \tag{3}$$

follows for any braid automorphism $\zeta : S_i \to S_i'$, as well as

$$S_i' = L_i S_{\pi^{-1}(i)} L_i^{-1}. \tag{1}$$

This is also geometrically evident, since $\prod S_i$ as well as $\prod S_i'$ is represented by a loop which girds the whole braid.

At this point it seems necessary to say a few words about the correct interpretation of the symbols σ_i. If $\mathfrak{z} = \sigma_{i_1}^{\varepsilon_1} \dots \sigma_{i_r}^{\varepsilon_r}$ is understood as a braid, the composition is defined from left to right. Denote by $\mathfrak{z}_k = \sigma_{i_1}^{\varepsilon_1} \dots \sigma_{i_k}^{\varepsilon_k}$, $0 \leqq k \leqq r$, the k-th *initial section* of $\mathfrak{z}$ and by ζ_k the braid automorphism associated to $\mathfrak{z}_k$ (operating on the original generators S_i). The injective homomorphism $\mathfrak{B}_n \to \mathrm{Aut}\mathfrak{F}_n$ then maps a factor $\sigma_{i_j}^{\varepsilon_j}$ of $\mathfrak{z}$ onto an automorphism of $\mathfrak{F}_n$ defined by (2) where $\zeta_{j-1}(S_i)$ takes the place of S_i.

There is an identity in the free group generated by the $\{\sigma_i\}$:

$$\mathfrak{z} = \prod_{k=1}^{r} \sigma_{i_k}^{\varepsilon_k} = \prod_{k=1}^{r} \mathfrak{z}_{r-k} \sigma_{i_{r-k+1}}^{\varepsilon_{r-k+1}} \mathfrak{z}_{r-k}^{-1}, \quad \mathfrak{z}_0 = 1.$$

The automorphism $\zeta_{r-k} \sigma_{i_{r-k+1}}^{\varepsilon_{r-k+1}} \zeta_{r-k}^{-1}$ (carried out from right to left!) is the automorphism $\sigma_{i_{r-k+1}}^{\varepsilon_{r-k+1}}$ defined by (2) on the original generators S_i (from the top of the braid). We may therefore understand $\mathfrak{z} = \sigma_{i_1}^{\varepsilon_1} \dots \sigma_{i_r}^{\varepsilon_r}$ either as a product (from right to left) of automorphisms $\sigma_{i_j}^{\varepsilon_j}$ in the usual sense, or, performed from left to right, as a successive application of a rule for a substitution according to (2) with varying arguments. The last one was originally employed by Artin, and it makes the mapping $\mathfrak{B}_n \to \mathrm{Aut}\mathfrak{F}_n$ a homomorphism rather than an anti-homomorphism. The two interpretations are dual descriptions of the same automorphism.

Braid automorphisms of $\mathfrak{F}_n(S_i)$ can be characterized by (1) and (3). Artin [1925] even proved a slightly stronger theorem where he does not presuppose that the given substitution is an automorphism:

10.7 Proposition. *Let $\mathfrak{F}_n(S_j)$ be a free group on a given set $\{S_j \mid 1 \leqq j \leqq n\}$ of free generators, and let π be a permutation on $\{1, 2, \dots, n\}$. Any set of words $S'_i(S_j)$, $1 \leqq i \leqq n$, subject to the following conditions:*

(1) $S'_i = L_i S_{\pi(i)} L_i^{-1}$,

(3) $\prod_{i=1}^n S'_i = \prod_{i=1}^n S_i$,

generates $\mathfrak{F}_n$; the homomorphism defined by $S_i \mapsto S'_i$ is a braid automorphism.

Proof. Assume S'_i to be reduced and call $\lambda(\zeta) = \sum_{i=1}^n l(L_i)$ the length of the substitution $\zeta : S_i \to S'_i$, where $l(L_i)$ denotes the length of L_i. If $\lambda = 0$, it follows from (3) that ζ is the identity. We proceed by induction an λ. For $\lambda > 0$ there will be reductions in

$$\prod_{i=1}^n S_i = L_1 S_{\pi(1)} L_1^{-1} \dots L_n S_{\pi(n)} L_n^{-1}$$

such that some $S_{\pi(i)}$ is cancelled by $S_{\pi(i)}^{-1}$ contained in L_{i-1}^{-1} or L_{i+1}. (If all L_i cancel out, they have to be all equal, and hence empty, since L_1 and L_n have to be empty). Suppose L_{i+1} cancels $S_{\pi(i)}$, then

$$l(L_i S_{\pi(i)} L_i^{-1} L_{i+1}) < l(L_{i+1}).$$

Apply σ_i to S'_j, $\sigma_i(S'_j) = S''_j$, to obtain $\lambda(\zeta\sigma_i) < \lambda(\zeta)$ while $\zeta\sigma_i$ still fulfils conditions (1) and (3). Thus, by induction, $\zeta\sigma_i$ is a braid automorphism and so is ζ. (If $S_{\pi(i)}$ is cancelled by L_{i-1}^{-1}, one has to use σ_{i-1}^{-1} instead of σ_i.) □

B Normal Form and Group Structure

We have derived a presentation of the braid group $\mathfrak{B}_n$, and solved the word problem by embedding $\mathfrak{B}_n$ into the group of automorphisms of the free group of rank n. For some additional information on the group structure of $\mathfrak{B}_n$ first consider the surjective homomorphism of the braid group onto the symmetric group:

$$\mathfrak{B}_n \to \mathfrak{S}_n, \quad \mathfrak{z} \mapsto \pi,$$

which assigns to each braid $\mathfrak{z}$ its permutation π. We propose to study the kernel $\mathfrak{I}_n \lhd \mathfrak{B}_n$ of this homomorphism.

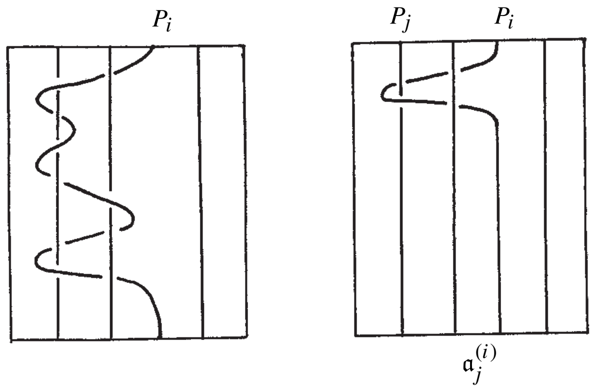


Figure 10.9

10.8 Definition (Pure braids). A braid of $\mathfrak{Z}_n$ is called a *pure i-braid* if there is a representative with the strings f_j, $j \neq i$, constant (straight lines), and if its y-projection only contains double points concerning f_i and f_j, $j < i$, see Figure 10.9.

10.9 Proposition. *The pure i-braids of $\mathfrak{Z}_n$ form a free subgroup $\mathfrak{F}^{(i)}$ of $\mathfrak{Z}_n$ of rank $i-1$.*

Proof. It is evident that $\mathfrak{F}^{(i)}$ is a subgroup of $\mathfrak{Z}_n$. Furthermore $\mathfrak{F}^{(i)}$ is obviously generated by the braids $\mathfrak{a}_j^{(i)}$, $1 \leqq j < i$, as defined in Figure 10.9. Let $\mathfrak{z}^{(i)} \in \mathfrak{F}^{(i)}$ be an arbitrary pure i-braid. Note down $(\mathfrak{a}_{j_k}^{(i)})^{\varepsilon_k}$ as you traverse f_i at every double point in the y-projection where f_i overcrosses f_{j_k}, while choosing $\varepsilon_k = +1$ resp. $\varepsilon_k = -1$ according to the characteristic of the crossing. Then $\mathfrak{z}^{(i)} = \mathfrak{a}_{j_1}^{(i)\varepsilon_1}\mathfrak{a}_{j_2}^{(i)\varepsilon_2}\dots\mathfrak{a}_{j_r}^{(i)\varepsilon_r}$.

It is easy to see that the $\mathfrak{a}_j^{(i)}$ are free generators. It follows from the fact that the loops formed by the strings f_i of $\mathfrak{a}_j^{(i)}$ combined with an arc on ∂Q can be considered as free generators of $\pi_1(Q - \bigcup_{j=1}^{i-1} f_j) \cong \mathfrak{F}^{(i-1)}$. □

10.10 Proposition. *The subgroup $\mathfrak{B}_{i-1} \subset \mathfrak{B}_n$ generated by $\{\sigma_r \mid 1 \leqq r \leqq i-2\}$ operates on $\mathfrak{F}^{(i)}$ by conjugation.*

$$\sigma_r^{-1}\mathfrak{a}_j^{(i)}\sigma_r = \begin{cases} \mathfrak{a}_j^{(i)}, & j \neq r, r+1, \\ \mathfrak{a}_r^{(i)}\mathfrak{a}_{r+1}^{(i)}\mathfrak{a}_r^{(i)-1}, & j = r, \\ \mathfrak{a}_r^{(i)}, & j = r+1. \end{cases}$$

The *proof* is given in Figure 10.10. □

It is remarkable that σ_r induces on $\mathfrak{F}^{(i)}$ the braid automorphism σ_r with respect to the free generators $\mathfrak{a}_j^{(i)}$.

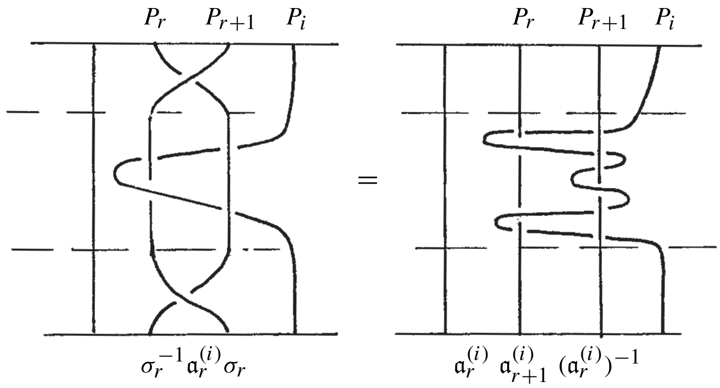


Figure 10.10

The following theorem describes the group structure of $\mathfrak{J}_n$ to a certain extent.

10.11 Proposition. *The braids $\mathfrak{z}$ of $\mathfrak{J}_n$ admit a unique decomposition:*

$$\mathfrak{z} = \mathfrak{z}_2 \dots \mathfrak{z}_n, \quad \mathfrak{z}_i \in \mathfrak{F}^{(i)},\ \mathfrak{F}^{(1)} = 1.$$

This decomposition is called the normal form of $\mathfrak{z}$. There is a product rule for normal forms:

$$\Big(\prod_{i=2}^{n} \mathfrak{x}_i\Big)\Big(\prod_{i=2}^{n} \mathfrak{y}_i\Big) = (\mathfrak{x}_2\mathfrak{y}_2)(\mathfrak{x}_3^{\eta_2}\mathfrak{y}_3)\dots(\mathfrak{x}_n^{\eta_{n-1}\dots\eta_3\eta_2}\mathfrak{y}_n),$$

where η_i denotes the braid automorphism associated to the braid $\mathfrak{y}_i \in \mathfrak{F}^{(i)}$.

Proof. The existence of a normal form for $\mathfrak{z} \in \mathfrak{J}_n$ is an immediate consequence of Lemma 10.4. One has to realize $\mathfrak{z} \in \mathfrak{J}_n$ from a simple z-projection by letting first f_n ascend over its z-projection while representing the f_j, $j < n$, by straight lines over the endpoints Q_j. This defines the factor $\mathfrak{z}_n$. The remaining part of f_n is projected onto P_n' and therefore has no effect on the rest of the braid. Thus the existence of the normal form follows by induction on n, Figure 10.11.

The product rule is a consequence of Proposition 10.10. Uniqueness follows from the fact that, if $\mathfrak{x}_2 \dots \mathfrak{x}_n \cdot \mathfrak{y}_n^{-1} \dots \mathfrak{y}_2^{-1} = 1$, then $(\mathfrak{x}_n \cdot \mathfrak{y}_n^{-1})^{\eta_2^{-1}\dots\eta_{n-1}^{-1}}$ is its component in $\mathfrak{F}^{(n)}$. Its string f_n is homotopic to some arc on ∂Q in $Q - \bigcup_{j=1}^{n-1} f_j$; hence $(\mathfrak{x}_n \cdot \mathfrak{y}_n^{-1})^{\eta_2^{-1}\dots\eta_{n-1}^{-1}} = 1$, $\mathfrak{x}_n = \mathfrak{y}_n$. The rest follows by induction. □

The normal form affords some insight into the structure of $\mathfrak{J}_n$. By definition $\mathfrak{F}^{(1)} = 1$; the group $\mathfrak{J}_n$ is a repeated semidirect product of free groups with braid automorphisms operating according to Proposition 10.10:

$$\mathfrak{J}_n = \mathfrak{F}^{(1)} \ltimes (\mathfrak{F}^{(2)} \ltimes (\cdots(\mathfrak{F}^{(n-1)} \ltimes \mathfrak{F}^{(n)})\cdots)).$$

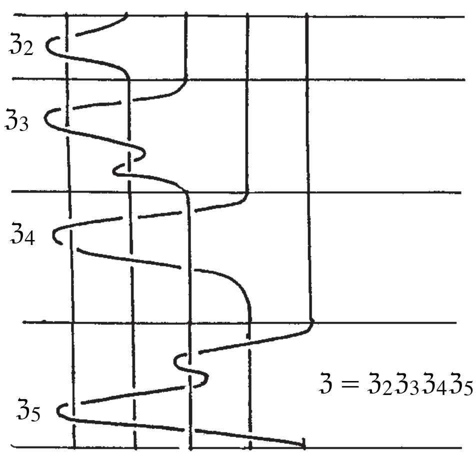


Figure 10.11

There is some more information contained in the normal form:

10.12 Proposition. *$\mathfrak{I}_n$ contains no elements $\neq 1$ of finite order. The centre of $\mathfrak{I}_n$ and of $\mathfrak{B}_n$ is an infinite cyclic group generated by $(\sigma_1\sigma_2\dots\sigma_{n-1})^n$ for $n > 2$.*

Proof. Suppose for the normal forms of $\mathfrak{z}$ and $\mathfrak{z}^m$

$$(\mathfrak{x}_2\dots\mathfrak{x}_n)^m = \mathfrak{y}_2\dots\mathfrak{y}_n = 1, m > 1.$$

By 10.11, $\mathfrak{y}_2 = (\mathfrak{x}_2)^m = 1$. Now $\mathfrak{x}_2 = 1$ follows from Proposition 10.9. In the same way we get $\mathfrak{x}_i = 1$ successively for $i = 3, 4, \dots, n$. This proves the first assertion.

The braid $\mathfrak{z}^0 = (\sigma_1\sigma_2\dots\sigma_{n-1})^n$ of Figure 10.12 obviously is an element of the centre $Z(\mathfrak{B}_n)$. lt is obtained from the trivial braid by a full twist of the lower side of the frame while keeping the upper one fixed. The normal form of $\mathfrak{z}^0$ is given on the right of Figure 10.12:

$$\mathfrak{z}^0 = \mathfrak{z}_2^0\dots\mathfrak{z}_n^0, \quad \mathfrak{z}_i^0 = \mathfrak{a}_1^{(i)}\dots\mathfrak{a}_{i-1}^{(i)}.$$

(For the definition of $\mathfrak{a}_j^{(i)}$ see Figure 10.9.) It is easily verified that $\mathfrak{z}^0$ determines the braid automorphism

$$\zeta^0\colon S_i \mapsto \Big(\prod_{j=1}^n S_j\Big) S_i \Big(\prod_{j=1}^n S_j\Big)^{-1},$$

and that by (3) $\mathfrak{B}_n \cap \mathfrak{T}_n = \langle\,\zeta^0\,\rangle \cong \mathfrak{Z}$, $\mathfrak{T}_n$ the inner automorphisms of $\mathfrak{F}_n$.

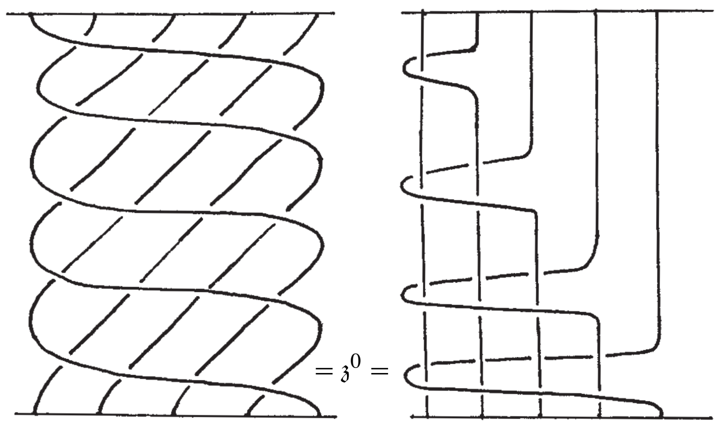


Figure 10.12

Note that Proposition 10.10 yields, for $1 \leqq i \leqq r < n$

$$\begin{aligned}(\sigma_1 \dots \sigma_{r-1})^{-r}\mathfrak{a}_i^{(r+1)}(\sigma_1 \dots \sigma_{r-1})^r &= (\mathfrak{a}_i^{(r+1)})^{(\sigma_1 \dots \sigma_{r-1})^r} \\ &= (\mathfrak{a}_1^{(r+1)} \dots \mathfrak{a}_r^{(r+1)})\mathfrak{a}_i^{(r+1)}(\mathfrak{a}_1^{(r+1)} \dots \mathfrak{a}_r^{(r+1)})^{-1}.\end{aligned} \tag{4}$$

For $n > 2$ the symmetric group $\mathfrak{S}_n$ has a trivial centre; hence, $Z(\mathfrak{B}_n) < Z(\mathfrak{J}_n)$ for the centres $Z(\mathfrak{B}_n)$ and $Z(\mathfrak{J}_n)$ of $\mathfrak{B}_n$ and $\mathfrak{J}_n$. We may therefore write an arbitrary central element $\mathfrak{z}$ of $\mathfrak{J}_n$ or $\mathfrak{B}_n$ in normal form $\mathfrak{z} = \mathfrak{z}_2 \dots \mathfrak{z}_n$, $\mathfrak{z}_i \in \mathfrak{F}^{(i)}$. (We denote by ζ_i, ξ_i, η_i the braid automorphisms associated to the braids $\mathfrak{z}_i, \mathfrak{x}_i, \mathfrak{y}_i$.)

For every $\mathfrak{x}_3 \in \mathfrak{F}^{(3)}$:

$$\mathfrak{z}_2\mathfrak{x}_3^{\zeta_2}\mathfrak{z}_3 \dots \mathfrak{z}_n = \mathfrak{x}_3\mathfrak{z}_2 \dots \mathfrak{z}_n = \mathfrak{z}_2 \dots \mathfrak{z}_n\mathfrak{x}_3 = \mathfrak{z}_2\mathfrak{z}_3\mathfrak{x}_3\mathfrak{z}_4^{\xi_3} \dots \mathfrak{z}_n^{\xi_3}.$$

It follows that $\mathfrak{x}_3^{\zeta_2}\mathfrak{z}_3 = \mathfrak{z}_3\mathfrak{x}_3$, or $\mathfrak{x}_3^{\zeta_2} = \mathfrak{z}_3\mathfrak{x}_3\mathfrak{z}_3^{-1}$. Now $\mathfrak{z}_2 = (\mathfrak{a}_1^{(2)})^k = (\mathfrak{z}_2^0)^k$ for some $k \in \mathbb{Z}$. Apply (4) for $r = 2$, $\zeta_2 = \sigma_1^{2k}$: $(\mathfrak{a}_i^{(3)})^{\sigma_1^{2k}} = (\mathfrak{a}_1^{(3)}\mathfrak{a}_2^{(3)})^k\mathfrak{a}_i^{(3)}(\mathfrak{a}_1^{(3)}\mathfrak{a}_2^{(3)})^{-k}$. Hence, for $\mathfrak{x}_3 \in \mathfrak{F}^{(3)}$:

$$\mathfrak{z}_3\mathfrak{x}_3\mathfrak{z}_3^{-1} = \mathfrak{x}_3^{\sigma_1^{2k}} = (\mathfrak{a}_1^{(3)}\mathfrak{a}_2^{(3)})^k\mathfrak{x}_3(\mathfrak{a}_1^{(3)}\mathfrak{a}_2^{(3)})^{-k}.$$

Since $\mathfrak{F}^{(3)}$ is free, we get $\mathfrak{z}_3 = (\mathfrak{a}_1^{(3)}\mathfrak{a}_2^{(3)})^k = (\mathfrak{z}_3^0)^k$.

The next step determines $\mathfrak{z}_4$ by the following property. For $\mathfrak{x}_4 \in \mathfrak{F}^{(4)}$:

$$\mathfrak{x}_4\mathfrak{z}_2 \dots \mathfrak{z}_n = \mathfrak{z}_2\mathfrak{z}_3\mathfrak{x}_4^{\zeta_3\zeta_2}\mathfrak{z}_4 \dots \mathfrak{z}_n = \mathfrak{z}_2 \dots \mathfrak{z}_n\mathfrak{x}_4 = \mathfrak{z}_2\mathfrak{z}_3\mathfrak{z}_4\mathfrak{x}_4\mathfrak{z}_5^{\xi_4} \dots \mathfrak{z}_n^{\xi_4}.$$

The uniqueness of the normal form gives

$$\mathfrak{x}_4^{\zeta_3\zeta_2} = \mathfrak{z}_4\mathfrak{x}_4\mathfrak{z}_4^{-1}.$$

The braids $\mathfrak{z}_2$ and $\mathfrak{z}_3$ commute – draw a figure – and so do the corresponding automorphisms: $\zeta_2\zeta_3 = \zeta_3\zeta_2$.

We already know $\mathfrak{z}_2\mathfrak{z}_3 = (\mathfrak{a}_1^{(2)})^k(\mathfrak{a}_1^{(3)}\mathfrak{a}_2^{(3)})^k$, $\zeta_2\zeta_3 = (\sigma_1\sigma_2)^{3k}$. By (4) we get

$$\begin{aligned}\mathfrak{x}_4^{\xi_1\xi_2} &= (\sigma_1\sigma_2)^{-3k}\ \mathfrak{x}_4(\sigma_1\sigma_2)^{3k}\\ &= (\mathfrak{a}_1^{(4)}\mathfrak{a}_2^{(4)}\mathfrak{a}_3^{(4)})^k\ \mathfrak{x}_4(\mathfrak{a}_1^{(4)}\mathfrak{a}_2^{(4)}\mathfrak{a}_3^{(4)})^{-k}\end{aligned}$$

and hence, $\mathfrak{z}_4 = (\mathfrak{a}_1^{(4)}\mathfrak{a}_2^{(4)}\mathfrak{a}_3^{(4)})^k = (\mathfrak{z}_4^0)^k$. The procedure yields $\mathfrak{z}_i = (\mathfrak{z}_i^0)^k$, $\mathfrak{z} = (\mathfrak{z}^0)^k$. □

The braid group $\mathfrak{B}_n$ itself is also torsion free. This was first proved in [Fadell-Neuwirth 1962]. A different proof is contained in [Murasugi 1982]. We discuss these proofs in Section C.

C Configuration Spaces and Braid Groups

In [Fadell-Neuwirth 1962] and [Fox-Neuwirth 1962] a different approach to braids was developed. We shall prove some results of it here. For details the reader is referred to the papers mentioned above.

A braid $\mathfrak{z}$ meets a plane $z = c$ in n points $(p_1, p_2, \dots, p_n)$ if $0 \leqq c \leqq 1$, and $z = 1$ $(z = 0)$ contains the initial points P_i (endpoints Q_i) of the strings f_i. One may therefore think of $\mathfrak{z}$ as a simultaneous motion of n points in a plane E^2, $\{(p_1(t), \dots, p_n(t)) \mid 1 \leqq t \leqq 1\}$. We shall construct a $2n$-dimensional manifold where $(p_1, \dots, p_n)$ represents a point and $(p_1(t), \dots, p_n(t))$ a loop such that the braid group $\mathfrak{B}_n$ becomes the fundamental group of the manifold.

Every n-tuple $(p_1, \dots, p_n)$ represents a point $P = (x_1, y_1, x_2, y_2, \dots, x_n, y_n)$ in Euclidean $2n$-space E^{2n}, where (x_i, y_i) are the coordinates of $p_i \in E^2$. Let $i \prec j$ stand for the inequality $x_i < x_j$, $i \hat{=} j$ for $x_i = x_j$, $y_i < y_j$, and $i = j$ for $x_i = x_j$, $y_i = y_j$. Any distribution of these symbols in a sequence, e.g. $\pi(1) = \pi(2) \hat{=} \pi(3) \prec \pi(4) \dots \pi(n)$, $\pi \in \mathfrak{S}_n$, then describes a set of linear inequalities and, hence, a convex subset of E^{2n}. Obviously these cells form a cell division of E^{2n}. There are $n!$ cells of dimension $2n$, defined by $(\pi(1) \prec \pi(2) \prec \dots \prec \pi(n))$.

The dimension of a cell defined by some sequence is easily calculated from the number of times the different signs $\prec$, $\hat{=}$, $=$ are employed in the sequence. The permutations $\pi \in \mathfrak{S}_n$ under $\pi(p_1, \dots, p_n) = (p_{\pi(1)}, \dots p_{\pi(n)})$ form a group of cellular operations on E^{2n}. The quotient space $\hat{E}^{2n} = E^{2n}/\mathfrak{S}_n$ inherits the cell decomposition. The following example shows how we denote the projected cells:

$$(\pi(1) \prec \pi(2) \hat{=} \pi(3) \cdots = \pi(n)) \mapsto (\prec \hat{=} \cdots =).$$

(Just omit the numbers $\pi(i)$.) $\mathfrak{S}_n$ operates freely on $E^{2n} - \Lambda$, where Λ is the $(2n-2)$-dimensional subcomplex consisting of cells defined by sequences in which the sign

= occurs at least once. The projection $q\colon E^{2n} \to \hat{E}^{2n}$ then maps Λ onto a $(2n-2)$-subcomplex $\hat{\Lambda}$ of $\hat{E}^{2n}$ and $q\colon E^{2n} - \Lambda \to \hat{E}^{2n} - \hat{\Lambda}$ describes a regular covering of an open $2n$-dimensional manifold with $\mathfrak{S}_n$ as its group of covering transformations. $\hat{E}^{2n}$ is called a *configuration space*.

10.13 Proposition. $\pi_1(\hat{E}^{2n} - \hat{\Lambda}) \cong \mathfrak{B}_n$, $\pi_1(E^{2n} - \Lambda) \cong \mathfrak{J}_n$.

Proof. Choose a base point $\hat{P}$ in the (one) $2n$-cell of $\hat{E}^{2n} - \hat{\Lambda}$ and some $P, q(P) = \hat{P}$. A braid $\mathfrak{z} \in \mathfrak{B}_n$ then defines a loop in $\hat{E}^{2n} - \hat{\Lambda}$, with base point $\hat{P} = q(P_1, \ldots, P_n) = q(Q_1, \ldots, Q_n)$. Two such loops $\mathfrak{z}_t = q(p_1(t), \ldots, p_n(t)), \mathfrak{z}'_t = q(p'_1(t), \ldots, p'_n(t))$, $0 \leqq t \leqq 1$, are homotopic relative to $\hat{P}$, if these is a continuous family $\mathfrak{z}_t(s)$, $0 \leqq s \leqq 1$, with $\mathfrak{z}_t(0) = \mathfrak{z}_t$, $\mathfrak{z}_t(1) = \mathfrak{z}'_t$, $\mathfrak{z}_0(s) = \mathfrak{z}_1(s) = \hat{P}$. This homotopy relation $\mathfrak{z}_t \sim \mathfrak{z}'_t$ coincides with Artin's definition of s-isotopy for braids $\mathfrak{z}_t, \mathfrak{z}'_t$ [Artin 1947].

It can be shown by using simplicial approximation arguments that s-isotopy is equivalent to the notion of isotopy as defined in Definition 10.1, which would prove 10.13. We shall omit the proof, instead we show that $\pi_1(\hat{E}^{2n} - \hat{\Lambda})$ can be computed directly from its cell decomposition (see [Fox-Neuwirth 1962]).

We already chose the base point $\hat{P}$ in the interior of the only $2n$-cell $\hat{\lambda} = (\prec \cdots \prec)$. There are $n-1$ cells $\hat{\lambda}_i$ of dimension $2n-1$ corresponding to sequences where the sign $\hat{=}$ occurs once $(\prec \ldots \hat{=} \cdots \prec)$ at the i-th position.

Think of $\hat{P}$ as a 0-cell dual to $\hat{\lambda}$, and denote by σ_i, $1 \leqq i \leqq n-1$, the 1-cells dual to $\hat{\lambda}_i$. By a suitable choice of the orientation σ_i will represent the elementary braid. Figure 10.13 describes a loop σ_i which intersects $\hat{\lambda}_i$ at $t = \frac{1}{2}$.

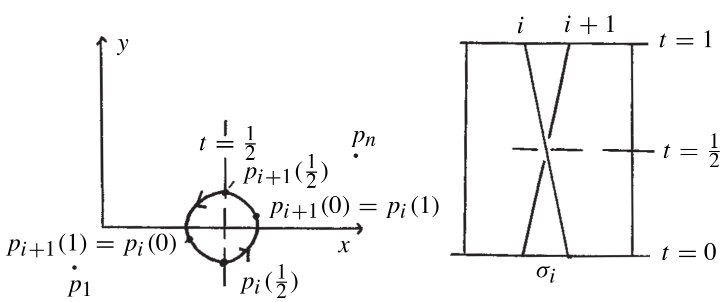


Figure 10.13

It follows that the σ_i are generators of $\pi_1(\hat{E}^{2n} - \hat{\Lambda})$. Defining relators are obtained by looking at the 2-cells $\hat{r}_{ik}$ dual to the $(2n-2)$-cells $\hat{\lambda}_{ik}$ of $\hat{E}^{2n} - \hat{\Lambda}$ which are characterized by sequences in which two signs $\hat{=}$ occur: $\hat{\lambda}_{ik} = (\prec \cdots \prec \hat{=} \prec \cdots \prec \hat{=} \prec \ldots)$ at position i and k, $1 \leqq i < k \leqq n-1$. The geometric situation will be quite different in the two cases $k = i+1$ and $k > i+1$.

Consider a plane γ transversal to $\hat{\lambda}_{i,i+1}$ in $\hat{E}^{2n} - \hat{\Lambda}$. One may describe it as the plane defined by the equations $x_i + x_{i+1} + x_{i+2} = 0$, $x_j = 0$, $j \neq i, i+1, i+2$. Figure 10.14 shows γ as an (x_i, x_{i+1})-plane with lines defined by $x_i = x_{i+1}$, $x_i = x_{i+2}$, $x_{i+1} = x_{i+2}$.

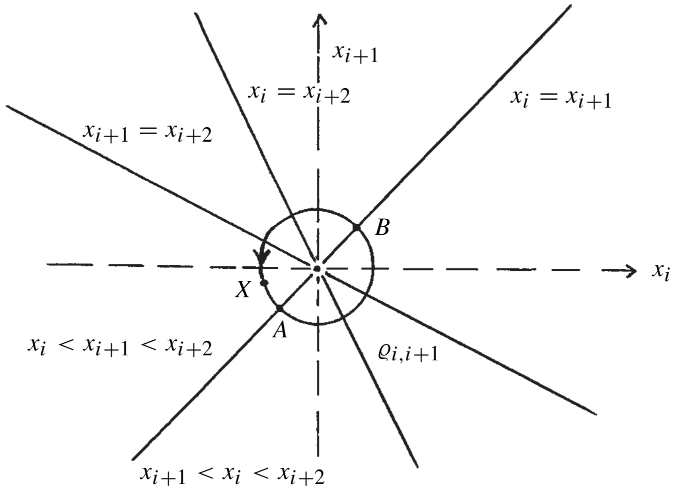


Figure 10.14

The origin of the (x_i, x_{i+1})-plane is $\gamma \cap \hat{\lambda}_{i,i+1}$ and the half rays of the lines are $\gamma \cap \hat{\lambda}_j$, $i \leqq j \leqq i+2$. We represent the points of $\gamma \cap \hat{\lambda}$ by ordered triples. We choose some point X in $x_i < x_{i+1} < x_{i+2}$ to begin with, and let it run along a simple closed curve $\varrho_{i,i+1}$ around the origin (Figure 10.14). Traversing $x_i = x_{i+1}$ corresponds to a generator $\sigma_i = (\ldots \hat{=} \prec \ldots)$, the point on $\varrho_{i,i+1}$ enters the $2n$-cell $x_{i+1} < x_i < x_{i+2}$ after that. Figure 10.15 describes the whole circuit $\varrho_{i,i+1}$.

Thus we get: $\varrho_{i,i+1} = \sigma_i \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1} \sigma_{i+1}^{-1}$. Whether to use σ_i or σ_i^{-1} can be decided in the following way. In the cross-section γ coordinates x_j, y_j different from x_i, x_{i+1}, x_{i+2} are kept fixed. Thus we have always $y_i < y_{i+1} < y_{i+2}$. Now Figure 10.16 shows the movement of the points $p_i, p_{i+1}, p_{i+2} \in E^2$ at the points A and B of Figure 10.14.

The same procedure applies to the case $k > i+1$. Here the cross-section to $\hat{\lambda}_{i,k}$ can be described by the solutions of the equations $x_i + x_{i+1} = x_k + x_{k+1} = 0$. We use an (x_i, x_k)-plane and again $\gamma \cap \hat{\lambda}_{i,k}$ is the origin and the coordinate half-rays represent $\gamma \cap \hat{\lambda}_i$, $\gamma \cap \hat{\lambda}_k$ (Figure 10.17).

It is left to the reader to verify for $i+1 < k$ that

$$\varrho_{ik} = \sigma_i \sigma_k \sigma_i^{-1} \sigma_k^{-1}.$$

The boundaries $\partial \hat{r}_{ik}$ are homotopic to ϱ_{ik}, thus we have again obtained the standard presentation $\mathfrak{B}_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \varrho_{ik}\ (1 \leqq i < k \leqq n-1) \rangle$ of the braid group (see 10.3). By definition $\pi_1(E^{2n} - \Lambda) \cong \mathfrak{Z}_n$. □

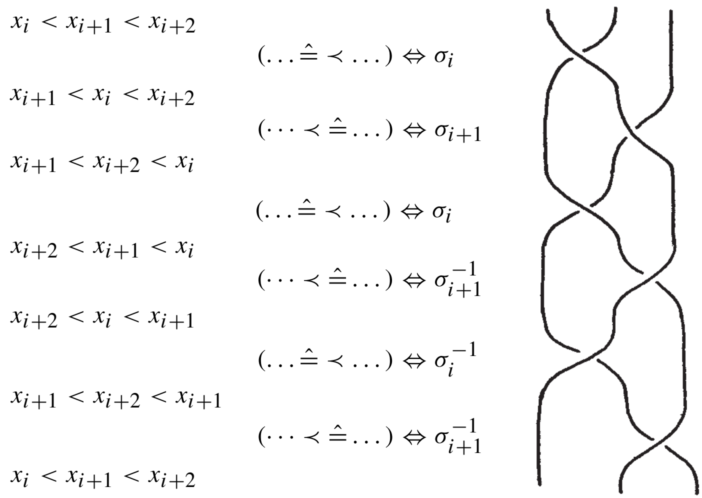


Figure 10.15

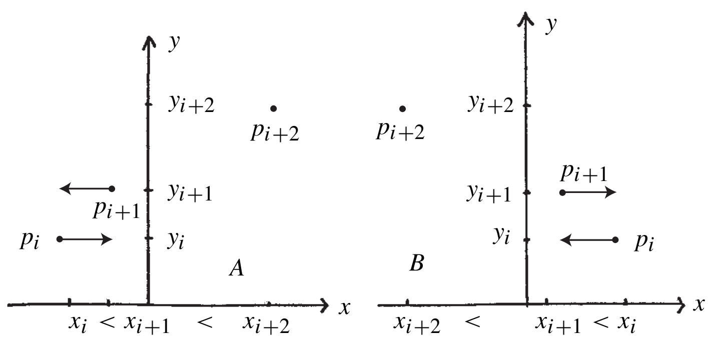


Figure 10.16

A presentation of $\mathfrak{Z}_n$ might be obtained in the same way by studying the cell complex $E^{2n} - \Lambda$, but it is more easily derived from the normal form (Proposition 10.11).

Fadell and Neuwirth [1962] have shown that $\hat{E}^{2n} - \hat{\Lambda}$ is aspherical; in fact, $\hat{E}^{2n} - \hat{\Lambda}$ is a $2n$-dimensional open manifold and a $K(\mathfrak{B}_n, 1)$ space. From this it follows by the argument used in 3.30 that $\mathfrak{B}_n$ has no elements $\neq 1$ of finite order.

10.14 Proposition. *The braid group $\mathfrak{B}_n$ is torsion free.*

We give a proof of this theorem using a result of Waldhausen [1967].

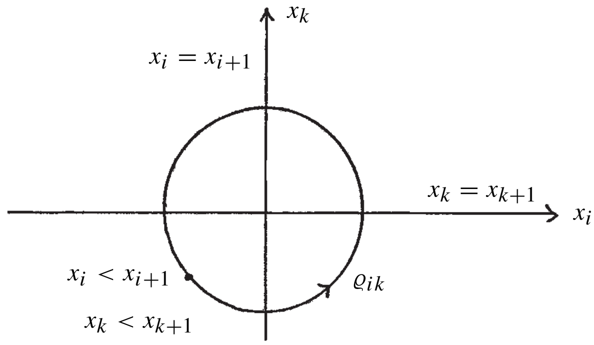


Figure 10.17

Proof. Let V be a solid torus with meridian m and longitude ℓ and $\hat{\mathfrak{z}} \subset V$ a closed braid derived from an n-braid $\mathfrak{z}$ of finite order k, $\mathfrak{z}^k = 1$. The embedding $\hat{\mathfrak{z}} \subset V$ is chosen in such a way that $\hat{\mathfrak{z}}$ meets each meridional disk D in exactly n points. For some open tubular neighbourhood $U(\hat{\mathfrak{z}})$,

$$\pi_1(V - U(\hat{\mathfrak{z}})) \cong \mathfrak{Z} \ltimes \pi_1 D_n \quad \text{with } D_n = D \cap (V - U(\hat{\mathfrak{z}})),$$

where $\mathfrak{Z}(= \langle t \rangle)$ resp. $\pi_1 D_n (= \mathfrak{N})$ are free groups of rank 1 resp. n. The generator t can be represented by the longitude ℓ (compare Corollary 5.4). There is a k-fold cyclic covering

$$p\colon (\hat{V} - \hat{U}(\hat{\mathfrak{z}}^k)) \to (V - U(\hat{\mathfrak{z}}))$$

corresponding to the normal subgroup $\langle t^k \rangle \ltimes \mathfrak{N} \lhd \langle t \rangle \ltimes \mathfrak{N}$. Now $\langle t^k \rangle \ltimes \mathfrak{N} = \langle t^k \rangle \times \mathfrak{N}$ since $\mathfrak{z}^k$ is the trivial braid. From this it follows that $\pi_1(V - U(\hat{\mathfrak{z}}))$ has a non-trivial centre containing the infinite cyclic subgroup $\langle t^k \rangle$ generated by t^k which is not contained in $\mathfrak{N} \cong \pi_1 D_n$. ($D_n$ is an incompressible surface in $V - U(\hat{\mathfrak{z}})$.)

By [Waldhausen 1967, Satz 4.1] $V - U(\hat{\mathfrak{z}})$ is a Seifert fibre space, $\langle t^k \rangle$ is the centre of $\pi_1(V - U(\hat{\mathfrak{z}}))$ and t^k represents a fibre $\simeq \ell^k$. The fibration of $V - U(\hat{\mathfrak{z}})$ can be extended to a fibration of V [Burde-Murasugi 1970]. This means that $\hat{\mathfrak{z}}$ is a torus link $\mathfrak{t}(a,b) = \hat{\mathfrak{z}}$. It follows that $\hat{\mathfrak{z}}^k = \mathfrak{t}(a, kb)$. Since $\mathfrak{z}^k$ is trivial, we get $kb = 0$, $b = 0$, and $\mathfrak{z} = 1$. □

The proof given above is a special version of an argument used in the proof of a more general theorem in [Murasugi 1982].

D Braids and Links

In Chapter 2, Section D we have described the procedure of *closing* a braid $\mathfrak{z}$ (see Figure 2.10). The closed braid obtained from $\mathfrak{z}$ is denoted by $\hat{\mathfrak{z}}$ and its axis by h.

10.15 Definition. Two *closed braids* $\hat{\mathfrak{z}}, \hat{\mathfrak{z}}'$ in $\mathbb{R}^3$ are called *equivalent*, if they possess a common axis h, and if there is an orientation preserving homeomorphism $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(\hat{\mathfrak{z}}) = \hat{\mathfrak{z}}'$, which keeps the axis h pointwise fixed.

Of course, $\mathbb{R}^3$ may again be replaced by S^3 and the axis by a trivial knot. Artin [1925] already noticed the following:

10.16 Proposition. *Two closed braids $\hat{\mathfrak{z}}, \hat{\mathfrak{z}}'$ are equivalent if and only if $\mathfrak{z}$ and $\mathfrak{z}'$ are conjugate in $\mathfrak{B}_n$.*

Proof. If $\mathfrak{z}$ and $\mathfrak{z}'$ are conjugate, the equivalence of $\hat{\mathfrak{z}}$ and $\hat{\mathfrak{z}}'$ is evident. Observe that a closed braid $\hat{\mathfrak{z}}$ can be obtained from several braids which differ by a cyclic permutation of their words in the generators σ_i, and hence are conjugate.

If $\hat{\mathfrak{z}}$ and $\hat{\mathfrak{z}}'$ are equivalent, we may assume that the homeomorphism $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f(\hat{\mathfrak{z}}) = \hat{\mathfrak{z}}'$, is constant outside a sufficiently large cube containing $\hat{\mathfrak{z}}$ and $\hat{\mathfrak{z}}'$. Since h is also kept fixed, we may choose an unknotted solid torus V containing $\hat{\mathfrak{z}}, \hat{\mathfrak{z}}'$ and restrict f to $f: V \to V$ with $f(x) = x$ for $x \in \partial V$. (We already used this construction at the end of the preceding section.) Let t again be a longitude of ∂V, and $\mathfrak{F}_n \cong \pi_1 D_n$ the free group of rank n where D_n is a disk with n holes. There is a homeomorphism $z: D_n \to D_n$, $z|\partial D_n = \text{id}$, inducing the braid automorphism ζ of $\mathfrak{z}$, and $V - U(\hat{\mathfrak{z}}) = (D_n \times I)/z$, $\pi_1(V - U(\hat{\mathfrak{z}})) \cong \langle t \rangle \ltimes \mathfrak{F}_n$, compare 5.2, 10.5, 10.6. For the presentation

$$\pi_1(V - U(\hat{\mathfrak{z}})) = \langle t, u_i \mid t u_i t^{-1} = \zeta(u_i) \rangle, \quad 1 \leqq i \leqq n,$$

choose a base point on $\partial V \cap D_n$ and define the generators $\{u_i\}$ of $\pi_1 D_n$ by a normal dissection of D_n (see 10.4).

The automorphism ζ is then defined with respect to these geometrically distinguished generators up to conjugation in the group of braid-automorphisms. The class of braid automorphisms conjugate to ζ is then invariant under the mapping

$$f: (V - U(\hat{\mathfrak{z}})) \to (V - U(\hat{\mathfrak{z}}')).$$

and, hence, the defining braids $\mathfrak{z}, \mathfrak{z}'$ must be conjugate. $\square$

The conjugacy problem in $\mathfrak{B}_n$ is thus equivalent to the problem of classifying closed braids. There have been, therefore, many attempts since Artin's paper in 1925 to solve it, and some partial solutions had been attained [Fröhlich 1936], until in [Makanin 1968], [Garside 1969] the problem was solved completely. Garside invented an ingenious though rather complicated algorithm by which he can decide whether two braids are conjugate or not. This solution implies a new solution of the word problem by way of a new normal form. We do not intend to copy his proof which does not seem to allow any essential simplification (see also [Birman 1974]).

Alexander's theorem (Proposition 2.9) can be combined with Artin's characterization of braid automorphisms (Proposition 10.7) to give a characterization of link groups in terms of special presentations.

10.17 Proposition. *A group $\mathfrak{G}$ is the fundamental group $\pi_1(S^3 - \mathfrak{l})$ for some link $\mathfrak{l}$ (a link group) if and only if there is a presentation of the form*

$$\mathfrak{G} = \langle\, S_1, \dots, S_n \mid S_i^{-1} L_i S_{\pi(i)} L_i^{-1},\ 1 \leqq i \leqq n \,\rangle,$$

with π a permutation and $\prod_{i=1}^{n} S_i = \prod_{i=1}^{n} L_i S_{\pi(i)} L_i^{-1}$ in the free group generated by $\{S_i \mid 1 \leqq i \leqq n\}$. □

A group theoretical characterization of knot groups $\pi_1(S^n - S^{n-2})$ has been given by Kervaire [1965] for $n \geqq 5$ only. Kervaire's characterization includes $H_1(S^n - S^{n-2}) \cong \mathbb{Z}$, $H_2(\pi_1(S^n - S^{n-2})) = 0$, and that $\pi_1(S^n - S^{n-2})$ is finitely generated and is the normal closure of one element. All these conditions are fulfilled in dimensions 3 and 4 too. For $n = 4$ the characterization is correct modulo a Poincaré conjecture, but for $n = 3$ it is definitely not sufficient. There is an example $G = \langle\, x, y \mid x^2yx^{-1}y^{-1} \,\rangle$ given in [Rolfson 1976] which satisfies all conditions, but its Jacobian (see Proposition 9.10)

$$\left(\left(\frac{\partial(x^2yx^{-1}y^{-1})}{\partial x}\right)^{\varphi\psi}, \left(\frac{\partial(x^2yx^{-1}y^{-1})}{\partial y}\right)^{\varphi\psi}\right) = (2 - t, 0),$$

$x^{\varphi\psi} = 1$. $y^{\varphi\psi} = t$, lacks symmetry. It seems to be a natural requirement to include a symmetry condition in a characterization of classical knot groups $\pi_1(S^3 - S^1)$. An infinite series of Wirtinger presentations satisfying Kervaire's conditions is constructed in [Rosebrock 1994]. These presentations do not belong to knot groups although they have symmetric Alexander polynomials.

We conclude this chapter by considering the relation between closed braids and the links defined by them.

Let $\mathfrak{B}_n$ be the group of braids resp. braid automorphisms ζ operating on the free group $\mathfrak{F}_n$ of rank n with free generators $\{S_i\}$, $\{S_i'\}$, $S_i' = \zeta(S_i)$, such that (1) and (3) in 10.7 are valid. There is a ring homomorphism

$$\varphi\colon \mathbb{Z}\mathfrak{F}_n \to \mathbb{Z}\mathfrak{Z},\ \mathfrak{Z} = \langle\, t \,\rangle,$$

defined by: $\varphi(S_i) = t$, mapping the group ring $\mathbb{Z}\mathfrak{F}_n$ onto the group ring $\mathbb{Z}\mathfrak{Z}$ of an infinite cyclic group $\mathfrak{Z}$ generated by t.

10.18 Proposition ([Burau 1936]). *The mapping $\beta\colon \mathfrak{B}_n \to \mathrm{GL}(n, \mathbb{Z}\mathfrak{Z})$ defined by $\zeta \mapsto \left(\left(\frac{\partial\zeta(S_j)}{\partial S_i}\right)^{\varphi}\right)$ is a homomorphism of the braid group $\mathfrak{B}_n$ into the group of*

$(n \times n)$-matrices over $\mathbb{Z}\mathfrak{Z}$. Then

$$\beta(\sigma_i) = \begin{array}{c} \begin{array}{cc} i & i+1 \end{array} \\ \begin{pmatrix} E & & & \\ & 1-t & t & \\ & 1 & 0 & \\ & & & E \end{pmatrix} \end{array} \begin{array}{l} \\ i \\ i+1 \\ \end{array} \quad , 1 \le i \le n.$$

β is called the Burau representation.

The proof of 10.18 is a simple consequence of the chain rule for Jacobians:

$$\zeta(S_i) = S_i', \quad \zeta'(S_k') = S_k'', \quad \frac{\partial S_k''}{\partial S_i} = \sum_{j=1}^{n} \frac{\partial S_k''}{\partial S_j'} \frac{\partial S_j'}{\partial S_i}.$$

The calculation of $\beta(\sigma_i)$ (and $\beta(\sigma_i^{-1})$) is left to the reader. □

10.19 Proposition. $\sum_{j=1}^{n} \left(\frac{\partial \zeta(S_i)}{\partial S_j}\right)^{\varphi} = 1, \quad \sum_{i=1}^{n} t^{i-1} \cdot \left(\frac{\partial \zeta(S_i)}{\partial S_j}\right)^{\varphi} = t^{j-1}.$

Again the *proof* becomes trivial by using the Fox calculus. The fundamental formula yields

$$(\zeta(S_i) - 1)^{\varphi} = t - 1 = \sum_{j=1}^{n} \left(\frac{\partial \zeta(S_i)}{\partial S_j}\right)^{\varphi} (t-1).$$

For the second equation we exploit $\prod_{i=1}^{n} \zeta(S_i) = \prod_{i=1}^{n} S_i$ in $\mathfrak{F}_n$:

$$\sum_{i=1}^{n} t^{i-1} \left(\frac{\partial \zeta(S_i)}{\partial S_j}\right)^{\varphi} = \left(\frac{\partial}{\partial S_j} \prod_{i=1}^{n} \zeta(S_i)\right)^{\varphi} = \left(\frac{\partial}{\partial S_j} \prod_{i=1}^{n} S_i\right)^{\varphi} = t^{j-1}.$$

□

The equations of 10.19 express a linear dependence between the rows and columns of the representing matrices. This makes it possible to reduce the degree n of the representation by one. If $C(t)$ is a representing matrix, we get:

$$S^{-1} C(t) S = \begin{pmatrix} & & 0 \\ & B(t) & \vdots \\ & & 0 \\ * & \cdots * & 1 \end{pmatrix} \tag{5}$$

$$S = \begin{pmatrix} 1 & 1 & & 1 \\ & 1 & & 1 \\ & & & \\ 0 & & & 1 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & 1 & -1 \\ 0 & \cdots & & 0 & 1 \end{pmatrix}.$$

This is easily verified and it follows that by setting $\hat{\beta}(\zeta) = B(t)$ we obtain a representation of $\mathfrak{B}_n$ in $\mathrm{GL}(n-1, \mathbb{Z}\mathfrak{Z})$ which is called the *reduced Burau representation*. Note that

$$\hat{\beta}(\sigma_1) = \begin{pmatrix} -t & 0 & \\ 1 & 1 & \\ & & E \end{pmatrix}$$

$$\hat{\beta}(\sigma_i) = \begin{array}{c} \begin{array}{ccccc} & & i & & \end{array} \\ \begin{pmatrix} E & & & & \\ & 1 & t & 0 & \\ & 0 & -t & 0 & \\ & 0 & 1 & 1 & \\ & & & & E \end{pmatrix} \end{array}, \quad 1 < i < n-1$$

$$\hat{\beta}(\sigma_{n-1}) = \begin{pmatrix} E & & \\ & 1 & t \\ & 0 & -t \end{pmatrix}$$

($\hat{\beta}(\sigma_1) = (t)$ for $n = 2$).

In addition to the advantage of reducing the degree from n to $n-1$, the reduced representation $\hat{\beta}$ has the property of mapping the centre of $\mathfrak{B}_n$ into the centre of $\mathrm{GL}(n-1, \mathbb{Z}\mathfrak{Z})$; thus

$$\hat{\beta}(\sigma_1 \ldots \sigma_{n-1})^n = \begin{pmatrix} t^n & & & 0 \\ & t^n & & \\ & & \ddots & \\ 0 & & & t^n \end{pmatrix}$$

The original $\hat{\beta}$ maps the centre on non-diagonal matrices.

The algebraic level of these representations is clearly that of the Alexander module (Chapter 8 A). There should be a connection.

10.20 Proposition. *For* $\mathfrak{z} \in \mathfrak{B}_n$, $\beta(\mathfrak{z}) = C(t)$, *the matrix* $(C(t) - E)$ *is a Jacobian* (*see* 9.10) *of the link* $\hat{\mathfrak{z}}$. *Furthermore:*

$$\det(B(t) - E) \doteq \nabla(t)(1 + t + \cdots + t^{n-1})(1 - t)^{\mu - 1},$$

where $\nabla(t)$ *is the Hosokawa polynomial of* $\hat{\mathfrak{z}}$ (*see* 9.18), *and* μ *the multiplicity of* $\hat{\mathfrak{z}}$.

Proof. The first assertion is an immediate consequence of 10.17. The second part – first proved in [Burau 1936] – is a bit harder:

The matrix $(C(t) - E)\,S$, see (5), is a matrix with the n-th column consisting of zeroes – this is a consequence of the first identity in 10.19. If the vector $\mathfrak{a}_i$ denotes the i-th row of $(C(t) - E)$, then the second identity can be expressed by $\sum_{i=1}^{n} t^{i-1}\mathfrak{a}_i = 0$. Hence

$$\sum_{i=1}^{n} t^{i-1}\mathfrak{d}_i = 0, \tag{6}$$

where $\mathfrak{d}_i$ denotes the vector composed of the first $n - 1$ components of $\mathfrak{a}_i S$.

By multiplying the $\mathfrak{a}_i S$ by S^{-1} we obtain

$$\det(B(t) - E) = \det(\mathfrak{d}_1 - \mathfrak{d}_2, \mathfrak{d}_2 - \mathfrak{d}_3, \ldots, \mathfrak{d}_{n-1} - \mathfrak{d}_n)$$

(compare (5)). From this we get that

$$\begin{aligned}
\pm\det(B(t) - E) &= \det(\mathfrak{d}_2 - \mathfrak{d}_1, \mathfrak{d}_3 - \mathfrak{d}_1, \ldots, \mathfrak{d}_n - \mathfrak{d}_1)\\
&= \det(\mathfrak{d}_2, \mathfrak{d}_3, \ldots \mathfrak{d}_n) + \sum_{i=1}^{n-1} \det(\mathfrak{d}_2, \ldots, \mathfrak{d}_i, (-\mathfrak{d}_1), \mathfrak{d}_{i+2}, \ldots, \mathfrak{d}_n)\\
&= \det(\mathfrak{d}_2, \ldots, \mathfrak{d}_n) + \sum_{i=1}^{n-1} \det(\mathfrak{d}_2, \ldots, t^i\mathfrak{d}_{i+1}, \ldots, \mathfrak{d}_n)\\
&= (1 + t + \cdots + t^{n-1}) \cdot \nabla(t) \cdot (t - 1)^{\mu - 1}.
\end{aligned}$$

The last equation follows from 9.18 since $\det(\mathfrak{d}_2, \ldots, \mathfrak{d}_n)$ by (6) generates the first elementary ideal of $\hat{\mathfrak{z}}$. □

The question of the faithfulness of the Burau-presentation has received some attention: In [Magnus-Peluso 1969] faithfulness was proved for $n \leq 3$; only recently this was shown to be wrong for $n \geq 5$ in [Moody 1991], [Long-Paton 1993], [Bigelow 2001].

It is evident that two non-equivalent closed braids may represent equivalent knots or links. For instance, $\hat{\sigma}_1$ and $\hat{\sigma}_1^{-1}$ both represent the unknot, but σ_1 and σ_1^{-1} are not conjugate in $\mathfrak{B}_2$. Figure 10.18 shows two closed n-braids which are isotopic to $\hat{\mathfrak{z}}$ as links for any $\mathfrak{z} \in \mathfrak{B}_{n-1}$.

A.A. Markov proved in 1936 a theorem [Markoff 1936] which in the case of oriented links controls the relationship between closed braids and links represented by them. The orientation in a closed braid is always defined by assuming that the strings of the braid run downward.

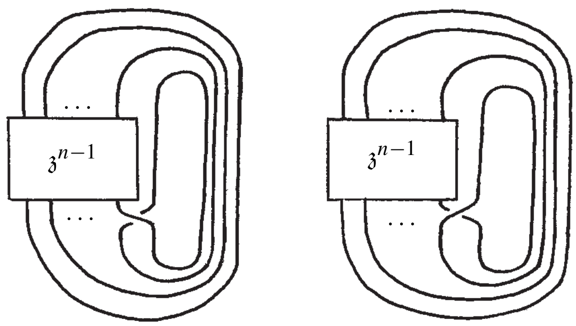


Figure 10.18

10.21 Definition (Markov move). The process which replaces $\mathfrak{z} \in \mathfrak{B}_{n-1}$ by $\mathfrak{z}\sigma_{n-1}^{\pm 1}$ (Figure 10.18) or vice versa is called a *Markov move*. Two braids $\mathfrak{z}$ and $\mathfrak{z}'$ are *Markov-equivalent*, if they are connected by a finite chain of braids:

$$\mathfrak{z} = \mathfrak{z}_0 \to \mathfrak{z}_1 \to \mathfrak{z}_2 \to \cdots \to \mathfrak{z}_r = \mathfrak{z}',$$

where either two consecutive braids $\mathfrak{z}_i$ are conjugate or related by a Markov move.

10.22 Theorem (Markov). *Two oriented links represented by the closed braids $\hat{\mathfrak{z}}$ and $\hat{\mathfrak{z}}'$ are isotopic, if and only if the braids $\mathfrak{z}$ and $\mathfrak{z}'$ are Markov-equivalent.*

Proof. We revert to Alexander's theorem and its proof in 2.14. Starting with an oriented link $\mathfrak{k}$ the procedure automatically gives a closed braid with all strings oriented in the same direction, assuming that the orientation of $\mathfrak{k}$ goes along with increasing indices of the intersection points P_i, $1 \le i \le 2m$. We denote the oriented projections of the overcrossing arcs from P_{2i-1} to P_{2i} by s_i, and the undercrossing ones from P_{2i} to P_{2i+1} by t_i; we also give an orientation to the axis h from left to right, Figure 2.11 (b). Let S denote the set $\{P_{2i-1}\}$ of the starting point of the arcs s_i, and $F = \{P_{2i}\}$ their finishing points.

We now consider different axes for a given fixed projection $p(\mathfrak{k})$. Choose any simply closed oriented curve h' in the projection plane $\mathbb{R}^2 = p(\mathbb{R}^3)$ which separates the sets S and F and meets the projection $p(\mathfrak{k})$ transversely such that S is on the left and F on the right side of h'. We arrange that overcrossing arcs always cross h' from left to right and undercrossing ones from right to left, Figure 10.19. This can be achieved by

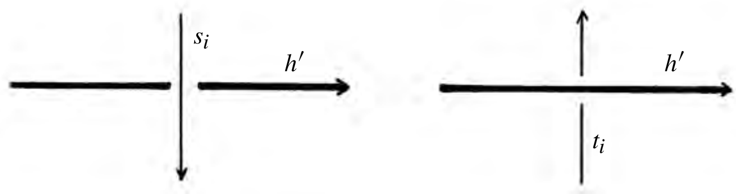


Figure 10.19

changing $\mathfrak{k}$ while keeping $p(\mathfrak{k})$ fixed by introducing new pairs of intersection points. But the new sets $S' \supset S$, $F' \supset F$ are still separated by h' in the same way. The original axis h of Figure 2.11 (b) can be replaced by an axis in the form h' by using an arc far off the projection, and, on the other hand, any axis h' defines a closed braid $\hat{\mathfrak{z}} \cong \mathfrak{k}$ in the same way as h.

We now study the effect of changing h' while keeping $p(\mathfrak{k})$ fixed.

We first look at isotopies of h' in $\mathbb{R}^2 - S \cup F$ by Δ-moves, see Definition 1.7. The following cases may occur: In Figure 10.20 the fat line always is h' while the others belong to $p(\mathfrak{k})$. Intersection points are not marked.

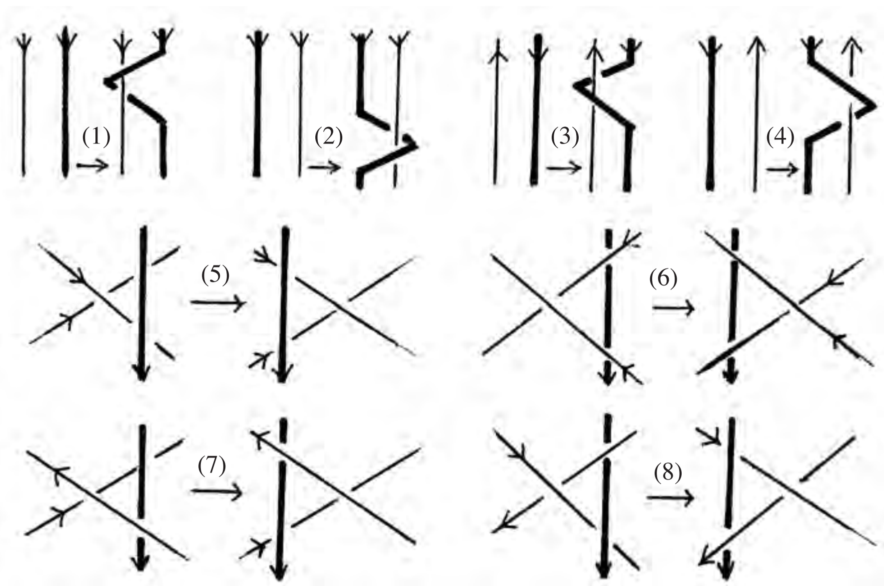


Figure 10.20

The moves (5), (6), (7) are isotopies of the closed braid. Move (8) is a sequence of the remaining moves, Figure 10.21:

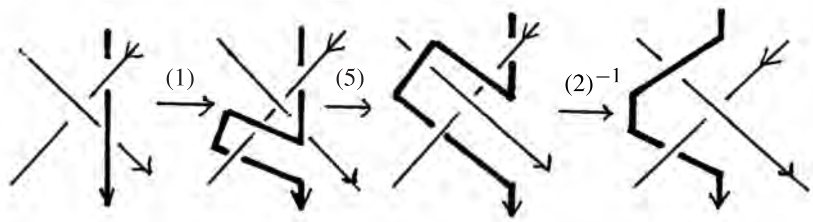


Figure 10.21

Assume h' to be in the position of the x-axis of $\mathbb{R}^2$ and $S = \{P_{2i-1}\}$ in the upper half-plane H^+. We investigate the effect of the moves (1)–(4). The arcs $s'_i = H^+ \cap p(s_i)$ form a normal dissection of $H^+ - S$ and, hence, define a braid $\mathfrak{z}^+$ (the four braids corresponding to $H^\pm \cap S$, $H^\pm \cap F$ form the closed braid $\hat{\mathfrak{z}}$ determined by h').

In Figure 10.22 the move (4) is applied in a special position: we assume that on the right side of Q_m there are no intersections $h' \cap p(\mathfrak{k})$, and that the Δ-move is executed in the neighbourhood of Q_m. This special position can always be produced by an isotopy of $\hat{\mathfrak{z}}$.

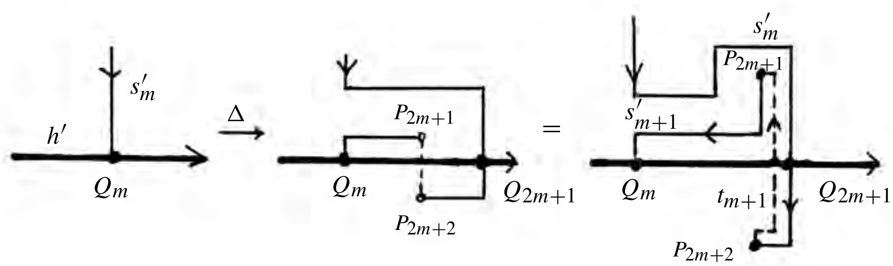


Figure 10.22

A comparison of Figure 10.22 with Figure 10.8 shows that $\mathfrak{z}^+$ is replaced by $\mathfrak{z}^+\sigma_m$ while the other three constituents of $\hat{\mathfrak{z}}$ just obtain an additional trivial string. By similar arguments we see that the moves of Figure 10.20 result in Markov moves.

In general the isotopy which moves an axis h' into another axis h in $\mathbb{R}^2$ will not be an isotopy of $\mathbb{R}^2 - (S \cup F)$. Figure 10.23 shows the general case:

Suppose that F is contained in the interior of the closed curve h', and let the dissection lines $\{\ell_{2i+1}\}$ start in the $\{P_{2i+1}\}$ and run upwards to infinity. Consider the local process which pushes two segments of h', oppositely oriented, simultaneously

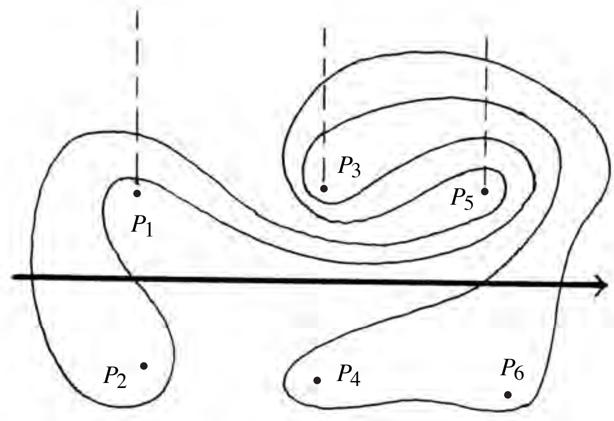


Figure 10.23

over an intersection point P_{2i+1}, Figure 10.24:

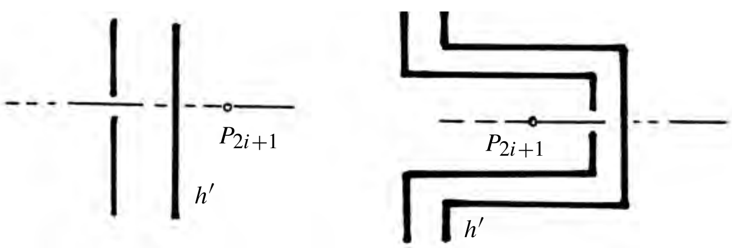


Figure 10.24

This process again is an isotopy of $\hat{\mathfrak{z}}$. Applying it, we can deform h' into h'' with $h'' \cap (\bigcup \ell_{2i+1}) = \emptyset$. Then h'' is a simple closed curve containing F in its interior, S in its exterior and isotopic to h in $\mathbb{R}^2 - (S \cup F)$. So we have proved the following:

10.23 Claim. *Given a diagram $p(\mathfrak{k})$ of an oriented link which intersection points (S, F), and two axes h and h' separating S and F, then the closed braids defined by S, F, h and h' are Markov-equivalent.*

10.24 Remarks. Starting with $p(\mathfrak{k})$ and (S, F), a separating axis h will in general enforce additional intersection points $S^* \supset S$, $F^* \supset F$, but the separating property will be preserved.

If two closed braids for $p(\mathfrak{k})$ and axes h and h' with different intersection points (S, F) and (S', F') are given, they are still Markov-equivalent, because by the claim those given by h, (S'', F'') and h', (S'', F'') are, for a common refinement (S'', F'') of (S, F) and (S', F').

To complete the proof of Markov's theorem, we have to check the effect of the Reidemeister moves Ω_i, $1 \leq i \leq 3$, see 1.13, on $p(\mathfrak{k})$. We take advantage of Claim 10.23 in choosing a suitable axis: for Ω_1 and Ω_2 the axis can be chosen away from the local region where the move is applied. In a situation where Ω_3 can be executed, Figure 10.25, the line s_i will cross the axis h'. According to the orientations, two cases arise which are shown in Figure 10.25.

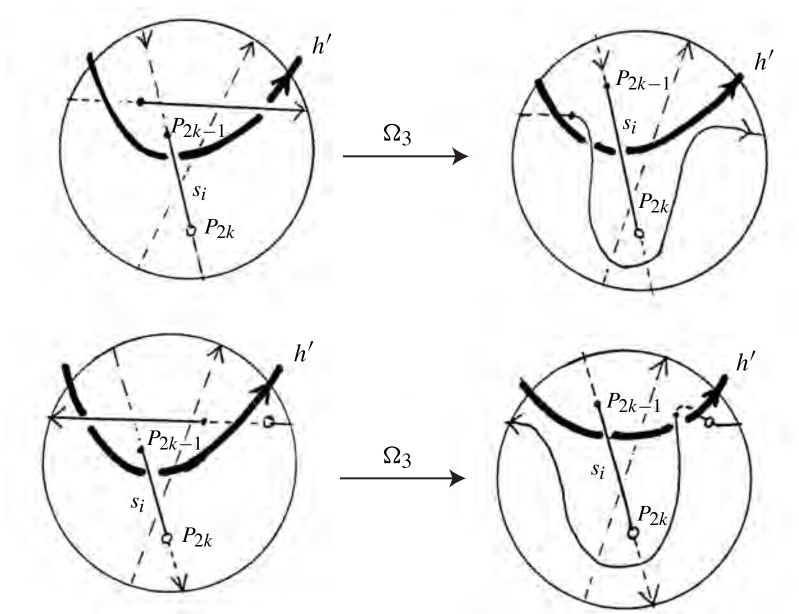


Figure 10.25

It suffices to ascertain that we can in each case place the intersection points in the region of the Ω_3-move in such a way that S and F are separated by h'. The intersection points outside the region are not changed. This completes the proof which is due to H. Morton, [Morton 1986″]. □

A Markov-theorem for unoriented links was proved by Anja Simon [Simon 1998]; in addition to conjugation and Markov moves a further move (Markov*) is necessary which operates not on the braid group but on the monoid of pseudobraids.

E History and Sources

There are few theories in mathematics the origin and author of which can be named so definitely as in the case of braids: Emil Artin invented them in his famous paper

"Theorie der Zöpfe" in 1925. (O. Schreier, who was helpful with some proofs, should, nevertheless be mentioned.) This paper already contains the fundamental isomorphism between braids and braid automorphisms by which braids are classified. The proof, though, is not satisfying. Artin published a new paper on braids in 1947 with rigorous definitions and proofs including the normal form of a braid. The remaining problem was the conjugacy problem. The importance of the braid group in other fields became evident in Magnus' paper on the mapping class groups of surfaces [Magnus 1934]. Further contributions in that direction were made by J. Birman and H. Hilden. There have been continual contributions to braid theory by several authors. For a bibliography see [Birman 1974]. The outstanding work was doubtless Makanin's and Garside's solution of the conjugacy problem [Makanin 1968], [Garside 1969]. Braid theory from the point of view of configuration spaces [Fadell-Neuwirth 1962] assigns braid groups to manifolds – the original braid group then is the braid group of the plane $\mathbb{R}^2$. This approach has been successfully applied [Arnol'd 1969] to determine the homology and cohomology groups of braid groups.

F Exercises

E 10.1. (Artin) Prove: $\mathfrak{B}_n = \langle \sigma, \tau \mid \sigma^{-n}(\sigma\tau)^{n-1}, [\sigma^i\tau\sigma^{-i}, \tau],\ 2 \leqq i \leqq \frac{n}{2} \rangle$, $\sigma = \sigma_1\sigma_2 \ldots \sigma_{n-1}$, $\tau = \sigma_1$. Derive from this presentation a presentation of the symmetric groups $\mathfrak{S}_n$.

E 10.2. $\mathfrak{B}_n/\mathfrak{B}_n' \cong \mathfrak{Z}$, $\mathfrak{J}_n/\mathfrak{J}_n' \cong \mathfrak{Z}^{\binom{n}{2}}$.

E 10.3. Let $Z(\mathfrak{B}_n)$ be the centre of $\mathfrak{B}_n$. Prove that $\mathfrak{z}^m \in Z(\mathfrak{B}_n)$ and $\mathfrak{z} \in \mathfrak{J}_n$ imply $\mathfrak{z} \in Z(\mathfrak{B}_n)$.

E 10.4. Interpret $\mathfrak{J}_n$ as a group of automorphisms of $\mathfrak{F}_n$ and denote by $\mathfrak{T}_n$ the inner automorphisms of $\mathfrak{F}_n$. Show that

$$\mathfrak{T}_n\mathfrak{J}_n/\mathfrak{T}_n = \mathfrak{T}_{n-1}\mathfrak{J}_{n-1}, \quad \mathfrak{T}_n \cap \mathfrak{J}_n = Z(\mathfrak{J}_n) = \text{centre of } \mathfrak{J}_n.$$

Derive from this that $\mathfrak{T}_n\mathfrak{J}_n$ has no elements of finite order $\neq 1$.

E 10.5. (Garside) Show that every braid $\mathfrak{z}$ can be written in the form

$$\sigma = \sigma_{i_1}^{a_1} \ldots \sigma_{i_r}^{a_r} \Delta^k, a_k \geqq 1, \quad \text{with } \Delta = (\sigma_1 \ldots \sigma_{n-1})(\sigma_1 \ldots \sigma_{n-2}) \ldots (\sigma_2\sigma_2)\sigma_1$$

the *fundamental braid*, k an integer.

E 10.6. Show that the Burau representation β and its reduced version $\hat{\beta}$ are equivalent under $\beta(\mathfrak{z}) \mapsto \hat{\beta}(\mathfrak{z})$. The representations are faithful for $n \leqq 3$.

E 10.7. Show that the notion of isotopy of braids as defined in 10.1 is equivalent to s-isotopy of braids as used in the proof of 10.13.

Chapter 11
Manifolds as Branched Coverings

The first section contains a treatment of Alexander's theorem [Alexander 1920] (Theorem 11.1). It makes use of the theory of braids and plats. The second part of this chapter is devoted to the Hilden–Montesinos theorem (Theorem 11.11) which improves Alexander's result in the case of 3-manifolds. We give a proof following H. Hilden [1976], but prefer to think of the links as plats. This affords a more transparent description of the geometric relations between the branch sets and the Heegaard splittings of the covering manifolds. The Dehn–Lickorish theorem (Theorem 11.7) is used but not proved here.

A Alexander's Theorem

11.1 Theorem (Alexander [1920]). *Every orientable closed* 3*-manifold is a branched covering of* S^3*, branched along a link with branching indices* $\leqq 2$. (*Compare* 8.18.)

Proof. Let M^3 be an arbitrary closed oriented manifold with a finite simplicial structure. Define a map p on its vertices $\hat{P}_i$, $1 \leqq i \leqq N$, $p(\hat{P}_i) = P_i \in S^3$, such that the P_i are in general position in S^3. After choosing an orientation for S^3 we extend p to a map $p\colon M^3 \to S^3$ by the following rule. For any positively oriented 3-simplex $[\hat{P}_{i_1}, \hat{P}_{i_2}, \hat{P}_{i_3}, \hat{P}_{i_4}]$ of M^3 we define p as the affine mapping

$$p\colon [\hat{P}_{i_1}, \hat{P}_{i_2}, \hat{P}_{i_3}, \hat{P}_{i_4}] \to [P_{i_1}, P_{i_2}, P_{i_3}, P_{i_4}].$$

if $[P_{i_1}, P_{i_2}, P_{i_3}, P_{i_4}]$ is positively oriented in S^3; if not, we choose the complement $C[P_{i_1}, P_{i_2}, P_{i_3}, P_{i_4}]$ as image,

$$p\colon [\hat{P}_{i_1}, \hat{P}_{i_2}, \hat{P}_{i_3}, \hat{P}_{i_4}] \to C[P_{i_1}, P_{i_2}, P_{i_3}, P_{i_4}].$$

We will show that p is a branched covering, the 1-skeleton M^1 of M^3 being mapped by p onto the branching set $p(M^1) = T \subset S^3$. For every point $P \in S^3 - p(M^2)$, M^2 the 2-skeleton of M^3, there is a neighbourhood $U \subset S^3 - p(M^2)$ containing P such that $p^{-1}(U(P))$ consists of n disjoint neighbourhoods $\hat{U}_j$ of the points $p^{-1}(P)$. Suppose $\hat{P}$ is contained in the interior of the 2-simplex $[\hat{P}_1, \hat{P}_2, \hat{P}_3]$, in the boundary of $[\hat{P}_0, \hat{P}_1, \hat{P}_2, \hat{P}_3]$ and $[\hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4]$. Let $[P_0, P_1, P_2, P_3]$ be positively oriented in S^3. If P_0 and P_4 are separated by the plane defined by $[P_1, P_2, P_3]$, we get that

$$p[\hat{P}_0, \hat{P}_1, \hat{P}_2, \hat{P}_3] = [P_0, P_1, P_2, P_3], \quad p[\hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4] = [P_1, P_2, P_3, P_4];$$

if not,

$$p[\hat{P}_0, \hat{P}_1, \hat{P}_2, \hat{P}_3] = [P_0, P_1, P_2, P_3], \quad p[\hat{P}_1, \hat{P}_2, \hat{P}_3, \hat{P}_4] = C[P_1, P_2, P_3, P_4].$$

In both cases there is a neighbourhood $\hat{U}$ of $\hat{P}$ which is mapped onto a neighbourhood U of $P = p(\hat{P})$, see Figure 11.1.

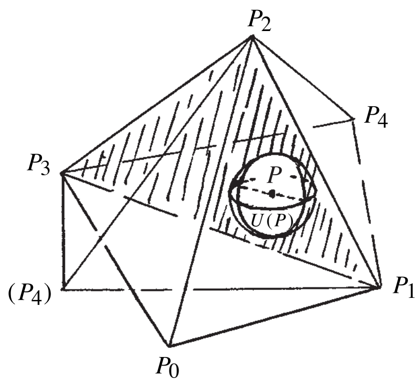


Figure 11.1

As a consequence $p \colon M^3 \to S^3$ is surjective, otherwise the compact polyhedron $p(M^3) \subset S^3$ would have boundary points on $p(M^2) - p(M^1)$. It follows from the construction that the restriction $p|M^2 \colon M^2 \to S^3$ is injective. The preimage $p^{-1}(P_i)$ of a vertex P_i consists of $\hat{P}_i$ and may-be several other points with branching index one. The same holds for the images $[P_i, P_j] = p[\hat{P}_i, \hat{P}_j]$ of edges. It remains to show that p can be modified in such a way that the branching set $T = p(M^1)$ is transformed into a link (without changing M^3).

By $U(T)$ we denote a tubular neighbourhood of $T \subset S^3$, consisting of (closed) balls B_i with centres P_i and cylindrical segments Z_{ij} with axes on $[P_i, P_j]$. The intersection $Z_{ij} \cap B_k$ is a disk δ_k for $k = i, j$ and empty otherwise. With $I = [0, 1]$, $Z_{ij} = I \times \delta$, and for $Y \in I$ the disk $Y \times \delta$ is covered by a collection of disjoint disks in M^3, of which at most one may contain a branching point $\hat{Y} \in M^1$ of index $r > 1$. The branched covering $p| \colon \hat{Y} \times \hat{\delta} \to Y \times \delta$ (for short: $p \colon \hat{\delta} \to \delta$) is cyclic (Figure 11.2).

A cycle of length r may be written as a product of r–1 transpositions, $(1, 2, \dots, r) = (1, 2)(2, 3) \dots (r-1, r)$. Correspondingly there is a branched covering $p' \colon \hat{\delta}' \to \delta$ with $r - 1$ branchpoints $\hat{Y}_i$, $1 \leqq i \leqq r - 1$, of index two. $\hat{\delta}'$ is a disk, and $p' \mid \partial\hat{\delta}' = p|\partial\hat{\delta}$. We substitute p' for p on all cylindrical segments $\hat{Z}_{ij} \subset p^{-1}Z_{ij}$ and obtain a new branched covering

$$p' \colon \overline{M^3 - \bigcup p^{-1}(B_i)} \to \overline{S^3 - \bigcup B_i}.$$

We denote by $\hat{B}_i$ the component of $p^{-1}(B_i)$ which contains $\hat{P}_i$. The branching set consists of lines in the cylindrical segments parallel to the axis of the cylinder.

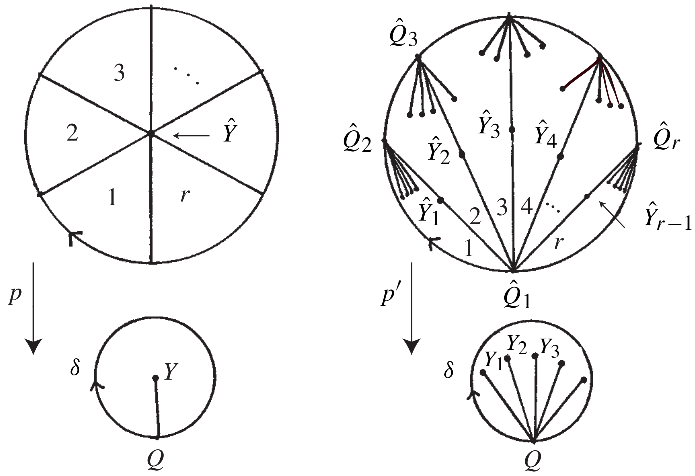


Figure 11.2

$p'|\partial\hat{B}_i = \hat{S}^2 \to S^2 = \partial B_i$ is a branched covering with branching points Q_j, $1 \leqq j \leqq q$, of index two where the sphere S^2 meets the branching lines contained in the adjoining cylinders. To describe this covering we use a normal dissection of $S^2 - \bigcup_{j=1}^{q} Q_j = \Sigma_q$ joining the Q_j by simple arcs s_j to some $Q \in \Sigma_q$. (The arcs are required to be disjoint save for their common endpoint Q, Figure 11.3.)

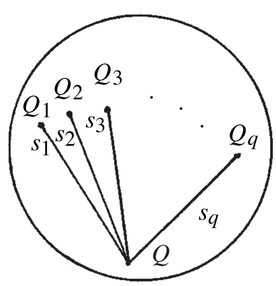


Figure 11.3

We assign to each s_j a transposition $\tau_j \in \mathfrak{S}_n$, where n is the number of sheets of the covering $p'\colon \hat{S}^2 \to S^2$, and $\mathfrak{S}_n$ is the symmetric group of order $n!$. Crossing an arc of $p^{-1}(s_j)$ in $\hat{S}^2$ means changing from the k-th sheet to the $\tau_j(k)$-th sheet of the covering. Since Q is not a branch point, $\prod_{j=1}^{q} \tau_j = \mathrm{id}$, $q = 2m$. Computing $\chi(\hat{S}^2) = 2$ gives $(n-1) = m$. On the other hand, any set of transpositions $\{\tau_j \mid 1 \leqq j \leqq 2m\}$ which generate a transitive subgroup of $\mathfrak{S}_n$, $n = m + 1$, defines a covering $p'\colon \hat{S}^2 \to S^2$, if $\prod_{j=1}^{2m} \tau_j = \mathrm{id}$. We may assign generators $S_j \in \pi_1(\Sigma_{2m})$ to the arcs s_j (see the text preceding 10.5), $\prod_{j=1}^{2m} S_j = 1$, and there is a homomorphism $\varphi\colon \pi_1\Sigma_{2m} \to \mathfrak{S}_n$, $\varphi(S_j) = \tau_j$. Given two normal dissections $\{s_i\}$ and $\{s_j'\}$ of Σ_{2m} with respect to Q_i, Q_j there is a homeomorphism $h\colon \Sigma_{2m} \to \Sigma_{2m}$, $h(s_i) = s_j'$ which induces a braid automorphism $\zeta: S_j \mapsto \zeta(S_j) = S_j' = L_j S_i L_j^{-1}$, $\pi(i) = j$, where π is the permutation of the braid. The generator S_j' corresponds to the arc s_j'. The commutative diagram

$$\begin{tikzcd}
\pi_1\Sigma_{2m} \arrow[r, "\zeta"] \arrow[d, "\varphi"'] & \pi_1\Sigma_{2m} \arrow[d, "\varphi'"] \\
\mathfrak{S}_n \arrow[r, "\zeta^*"] & \mathfrak{S}_n
\end{tikzcd}$$

defines a mapping ζ^* called the *induced braid substitution in* $\mathfrak{S}_n$. This can be used to compute the transpositions $\tau_j' = \zeta^*(\tau_j)$ which have to be assigned to the arcs s_j' in order to define the covering $p'\colon \hat{S}^2 \to S^2$. It follows that the homeomorphism $h\colon \Sigma_{2m} \to \Sigma_{2m}$ can be extended and lifted to a homeomorphism $\hat{h}$:

$$\begin{tikzcd}
\hat{S}^2 \arrow[r, "\hat{h}"] \arrow[d, "p'"'] & \hat{S}^2 \arrow[d, "p'"] \\
S^2 \arrow[r, "h"] & S^2
\end{tikzcd}$$

We interrupt our proof to show that there are homeomorphisms h, $\hat{h}$ such that the τ_j are replaced by τ_j' with a special property.

11.2 Lemma. *If* $2m$ *transpositions* $\tau_i \in \mathfrak{S}_n$, $1 \leqq i \leqq 2m$, *satisfy* $\prod_{i=1}^{2m} \tau_i = \mathrm{id}$, *then there is a braid substitution* $\zeta^*: \tau_i \mapsto \tau_i'$, *such that*

$$\tau_{2j-1}' = \tau_{2j}', \ 1 \leqq j \leqq m.$$

Proof. Denote by $\sigma_k^{*\pm 1}$ the braid substitutions in $\mathfrak{S}_n$ induced by the elementary braids $\sigma_k^{\pm 1}$ (Chapter 10 A, (2) resp. (2)$'$). If $\tau_k = (a, b)$, $\tau_{k+1} = (c, d)$, a, b, c, d all different, the effect of $\sigma_k^{*\pm 1}$ is to interchange the transpositions:

$$\tau_k' = \sigma_k^{*\pm 1}(\tau_k) = \tau_{k+1},\ \tau_{k+1}' = \sigma_k^{*\pm 1}(\tau_{k+1}) = \tau_k.$$

If $\tau_k = (a, b)$, $\tau_{k+1} = (b, c)$ then

$$\sigma_k^*(\tau_k) = (a, c),\ \sigma_k^*(\tau_{k+1}) = (a, b) \text{ and } \sigma_k^{*-1}(\tau_k) = (b, c),\ \sigma_k^{*-1}(\tau_{k+1}) = (a, c).$$

Assume $\tau_1 = (1, 2)$. Let $\tau_j = (1, a)$ be the transposition containing the figure 1, with minimal $j > 1$. (There is such a τ_j because $\prod \tau_i = \mathrm{id}$.) If $j > 2$, $\tau_{j-1} = (b, c)$, $b, c \neq 1$, the braid substitution $\sigma_{j-1}^{*\pm 1}$ will interchange τ_{j-1} and τ_j, if a, b, c are different. A pair $(a, b) = \tau_{j-1}$, $(1, a) = \tau_j$ is replaced by $(1, b)$, (a, b) if σ_{j-1}^{*} is applied, and by $(1, a)$, $(1, b)$, if σ_{j-1}^{*-1} is used.

Thus the sequence $\tau_1, \tau_2, \dots, \tau_{2m}$ can be transformed by a braid substitution into (1,2), $(1, i_2) \dots (1, i_\nu)$, $\tau''_{\nu+1}, \dots, \tau''_{2m}$, where the τ''_j, $j > \nu$, do not contain the figure 1. There is an $i_j = 2$, $2 \leqq j \leqq \nu$. If $j = 2$, the lemma is proved by induction. Otherwise we may replace $(1, i_{j-1})$, $(1, 2)$ by $(1, 2)$, $(2, i_{j-1})$ using σ_{j-1}^{*-1}. □

We are now in a position to extend $p' \colon (M^3 - \bigcup p^{-1}(\mathring{B}_i)) \to (S^3 - \bigcup \mathring{B}_i)$ to a covering $\tilde{p} \colon M^3 \to S^3$ and complete the proof of Theorem 11.1.

We choose a homeomorphism

$$h \colon \Sigma_{2m} \to \Sigma_{2m}$$

which induces a braid automorphism $\zeta \colon \pi_1 \Sigma_{2m} \to \pi_1 \Sigma_{2m}$ satisfying Lemma 11.2: $\zeta^*(\tau_k) = \tau'_k$, $\tau'_{2j-1} = \tau'_{2j}$, $1 \leqq j \leqq m$. The homeomorphism $h \colon S^2 \to S^2$ is orientation preserving and hence there is an isotopy

$$H \colon S^2 \times I \to S^2,\ H(x, 0) = x,\ H(x, 1) = h(x).$$

Lift H to an isotopy

$$\hat{H} \colon \hat{S}^2 \times I \to \hat{S}^2,\ \hat{H}(x, 0) = x,\ \hat{H}(x, 1) = \hat{h}(x).$$

Now identify $S^2 \times 0$ and $\hat{S}^2 \times 0$ with ∂B_i and $\partial \hat{B}_i$ and extend p' to $\hat{S}^2 \times I$ by setting $p'(x, t) = (p'(x), t)$.

It is now easy to extend p' to a pair of balls $\hat{B}'_i, B'_i$ with $\partial \hat{B}'_i = \hat{S}^2 \times 1$, $\partial B'_i = S^2 \times 1$. We replace the normal dissection $\{s'_j\}$ of $(S^2 \times 1) - U\{Q'_j\}$, $h(Q_j) = Q'_{\pi^{-1}(j)}$, by disjoint arcs t_j, $1 \leqq j \leqq m$, which connect Q'_{2j-1} and Q'_{2j} (Figure 11.4).

There is a branched covering $p'' \colon \hat{B}'_i \to B'_i$ with a branching set consisting of m simple disjoint unknotted arcs t'_j, $t'_j \cap \partial B'_j = Q'_{2j-1} \cup Q'_{2j}$, and there are m disjoint disks $\delta_j \subset B'_j$ with $\partial \delta_j = t_j \cup t'_j$ (Figure 11.4).

Passing through a disk of $(p'')^{-1}(\delta_j)$ in $\hat{B}'_i$ means changing from sheet number k to sheet number $\tau'_j(k)$. Since $p'' | \partial \hat{B}'_i = p'$ we may thus extend p' to a covering $\tilde{p} \colon M^3 \to S^3$. (There is no problem in extending p' to the balls of $p^{-1}(B_i)$ different from $\hat{B}_i$, since the covering is not branched in these.) □

The branching set of $\tilde{p}$ in $B'_i \cup (S^2 \times I)$ is described in Figure 11.5: The orbits $\{(Q_i, t) \mid 0 \leqq t \leqq 1\} \subset S^2 \times I$ form a braid to which in B'_i the arcs $\partial \delta_i - t_i$ are added as in the case of a plat.

□

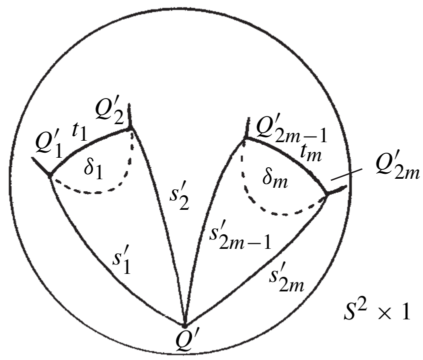


Figure 11.4

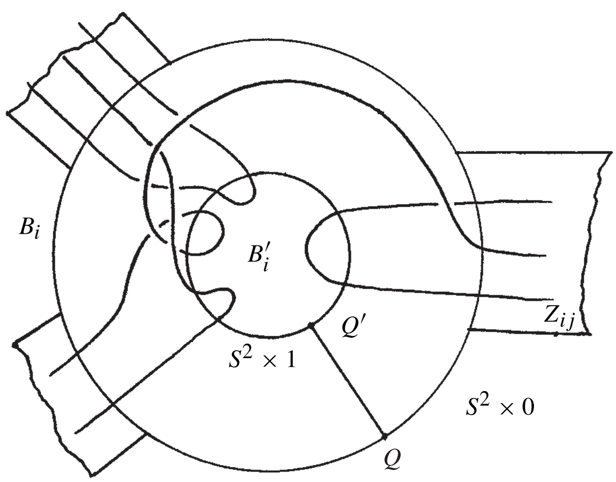


Figure 11.5

The braids that occur depend on the braid automorphisms required in Lemma 11.2. They can be chosen in a rather special way. It is easy to verify from the operations used in Lemma 11.2 that braids $\mathfrak{z}_{2m}$ of the type depicted in Figure 11.6 suffice. One can see that the tangle in B_i then consists of m unknotted and unlinked arcs.

B Branched Coverings and Heegaard Diagrams

By Alexander's theorem every closed oriented 3-manifold is an n-fold branched covering $p\colon M^3 \to S^3$ of the sphere. Suppose the branching set $\mathfrak{k}$ is a link of multiplicity μ,

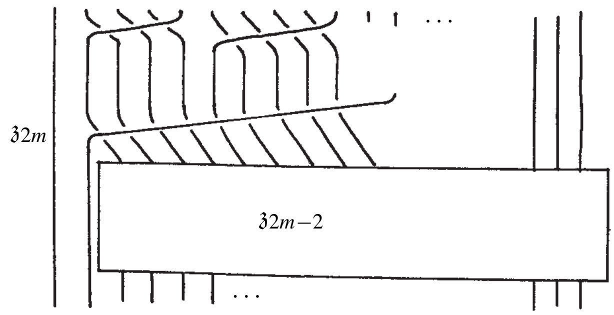


Figure 11.6

$\mathfrak{k} = \bigcup_{i=1}^{\mu} \mathfrak{k}_i$, and it is presented as a $2m$-plat (see Chapter 2 D), where m is the bridge number of $\mathfrak{k}$. A component $\mathfrak{k}_i$ is then presented as a $2\lambda_i$-plat, $\sum_{i=1}^{\mu} \lambda_i = m$. Think of S^3 as the union of two disjoint closed balls B_0, B_1, and $I \times S^2$, $\{j\} \times S^2 = \partial B_j = S_j^2$, $j = 0, 1$. Let the plat $\mathfrak{k}$ intersect $\mathring{B}_0$ and $\mathring{B}_1$ in m unknotted arcs spanning disjoint disks δ_i^j, $1 \leqq i \leqq m$, in B_j, $j = 0, 1$, and denote by $\mathfrak{z} = \mathfrak{k} \cap (I \times S^2)$ the braid part of $\mathfrak{k}$ (Figure 11.7). Every point of $\mathfrak{k}_i \cap (S_0^2 \cup S_1^2)$ is covered by the same number $\mu_i \leqq n$ of points in M^3.

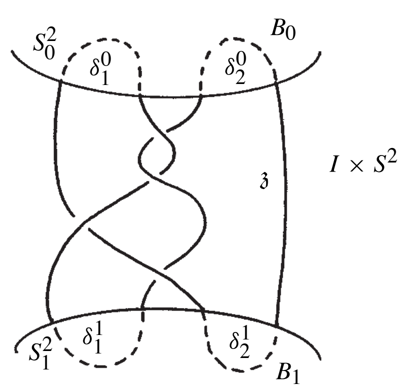


Figure 11.7

11.3 Proposition. *A manifold M^3 which is an n-fold branched covering of S^3 branched along the plat $\mathfrak{k}$ possesses a Heegaard splitting of genus*

$$g = m \cdot n - n + 1 - \sum_{i=1}^{\mu} \lambda_i \mu_i .$$

Proof. The 2-spheres S_j^2 are covered by orientable closed surfaces $\hat{F}_j = p^{-1}(S_j^2)$, $j = 0, 1$. The group $\pi_1(S^3 - \mathfrak{k})$ can be generated by m Wirtinger generators s_i, $1 \leqq i \leqq m$, encircling the arcs $\mathfrak{k} \cap B_0$. Similarly one may choose generators s_i' assigned to $\mathfrak{k} \cap B_1$; the s_i can be represented by curves in S_0^2, the s_i' by curves in S_1^2. It follows that $\hat{F}_0$ and $\hat{F}_1$ are connected. $p| \colon \hat{F}_j \to S_j^2$, $j = 0, 1$, are branched coverings with $2m$ branchpoints $\mathfrak{k} \cap S_j^2$ each. The genus g of $\hat{F}_0$ and $\hat{F}_1$ can easily be calculated via the Euler characteristic as follows: $p^{-1}(S_j^2 \cap \mathfrak{k}_i)$ consists of $2\lambda_i \mu_i$ points. Hence,

$$\chi(\hat{F}_0) = \chi(\hat{F}_1) = n + 2 \cdot \sum_{j=1}^{\mu} \lambda_i \mu_i - 2m \cdot n + n = 2 - 2g.$$

The balls B_j are covered by handlebodies $p^{-1}(B_j) = \hat{B}_j$ of genus g. This is easily seen by cutting the B_j along the disk δ_i^j and piecing copies of the resulting space together to obtain $\hat{B}_j$. The manifold M^3 is homeomorphic to the Heegaard splitting $\hat{B}_0 \cup_{\hat{h}} \hat{B}_1$. □

The homeomorphism $\hat{h} \colon \hat{F}_0 \to \hat{F}_1$ can be described in the following way. The braid $\mathfrak{z}$ determines a braid automorphism ζ which is induced by a homeomorphism $h \colon [S_0^2 - (\mathfrak{k} \cap S_0^2)] \to [S_1^2 - (\mathfrak{k} \cap S_1^2)]$. One may extend h to a homeomorphism $h \colon S_0^2 \to S_1^2$ and lift it to obtain $\hat{h}$:

$$\begin{tikzcd}
\hat{F}_0 \arrow[r, "\hat{h}"] \arrow[d, "p"'] & \hat{F}_1 \arrow[d, "p"'] \\
S_0^2 \arrow[r, "h"] & S_1^2
\end{tikzcd}$$

Proposition 11.3 gives an upper bound for the *Heegaard genus* (g minimal) of a manifold M^3 obtained as a branched covering.

11.4 Proposition. *The Heegaard genus g^* of an n-fold branched covering of S^3 along the $2m$-plat $\mathfrak{k}$ satisfies the inequality*

$$g^* \leqq m \cdot n - n + 1 - \sum_{i=1}^{\mu} \lambda_i \mu_i \leqq (m-1)(n-1).$$

Proof. The second part of the inequality is obtained by putting $\mu_i = 1$. □

The 2-fold covering of knots or links with two bridges ($n = m = 2$) have Heegaard genus one – a well-known fact. (See Chapter 12, [Schubert 1956]). Of special interest are coverings with $g = 0$. In this case the covering space M^3 is a 3-sphere. There are many solutions of the equation $0 = mn - n + 1 - \sum_{i=1}^{\mu} \lambda_i \mu_i$; for instance, the 3-sheeted irregular coverings of 2-bridge knots, $m = 2$, $n = 3$, $\mu_i = 2$, [Fox 1962′], [Burde 1971]. The braid $\mathfrak{z}$ of the plat then lifts to the braid $\hat{\mathfrak{z}}$ of the plat $\hat{\mathfrak{k}}$. Since $\hat{\mathfrak{z}}$ can be determined via the lifted braid automorphism $\hat{\zeta}$, $p\hat{\zeta} = \zeta p$, one can actually find $\hat{\mathfrak{k}}$. This was done for the trefoil [Kinoshita 1967] and the four-knot [Burde 1971].

A simple calculation shows that our construction never yields genus zero for regular coverings – except in the trivial cases $n = 1$ or $m = 1$.

For fixed m and n the Heegaard genus of the covering space M^3 is minimized by choosing $\mu_i = n - 1$, $g = m + 1 - n$. These coverings are of the type used in our version of Alexander's Theorem 11.1. From this we get

11.5 Proposition. *An orientable closed 3-manifold M^3 of Heegaard genus g^* is an n-fold branched covering with branching set a link $\mathfrak{k}$ with at least $g^* + n - 1$ bridges.* □

We propose to investigate the relation between the Heegaard splitting and the branched-covering description of a manifold M^3 in the special case of a 2-fold covering, $n = 2$. Genus and bridge-number are then related by $m = g + 1$.

The covering $p| : \hat{F}_0 \to S_0^2$ is described in Figure 11.8.

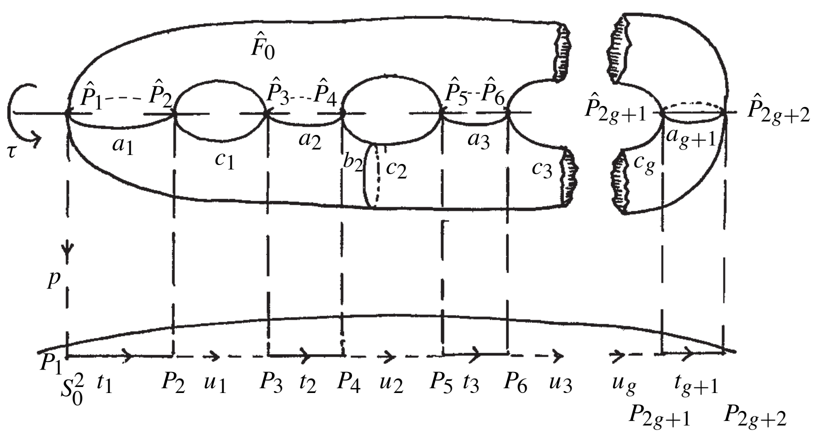


Figure 11.8

Connect P_{2j} and P_{2j+1}, $1 \le j \le g$, by simple arcs u_j, such that $t_1 u_1 t_2 u_2 \dots u_g t_{g+1}$ is a simple arc, $t_i = S_0^2 \cap \delta_i^0$. A rotation through π about an axis which pierces $\hat{F}_0$ in the branch points $\hat{P}_j = p^{-1}(P_j)$, $1 \leqq j \leqq 2g + 2$ is easily seen to be the covering transformation. The preimages $a_i = p^{-1}(t_i)$, $c_j = p^{-1}(u_j)$, $1 \leqq i \leqq g+1$, $1 \leqq j \leqq g$ are simple closed curves on $\hat{F}_0$. We consider homeomorphisms of the punctured sphere $S_0^2 - \bigcup_{j=1}^{2g+2} P_j$ which induce braid automorphisms, especially the homeomorphisms that induce the elementary braid automorphisms σ_k, $1 \leqq k \leqq 2g+1$. We extend them to S_0^2 and still denote them by σ_k. We are going to show that $\sigma_k\colon S_0^2 \to S_0^2$ lifts to a homeomorphism of $\hat{F}_0$, a so-called Dehn-twist.

11.6 Definition (Dehn twist). Let a be a simple closed (unoriented) curve on a closed oriented surface F, and $U(a)$ a closed tubular neighbourhood of a in F. A right-handed 2π-twist of $U(a)$ (Figure 11.9), extended by the identity map to F is called a *Dehn twist* α about a.

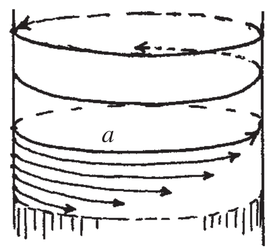


Figure 11.9

Up to isotopy a Dehn twist is well defined by the simple closed curve a and a given orientation of F. Dehn twists are important because a certain finite set of Dehn twists generates the mapping class group of F – the group of autohomeomorphisms of F modulo the deformations (the homeomorphisms homotopic to the identity) [Dehn 1938].

11.7 Theorem (Dehn, Lickorish). *The mapping class group of a closed orientable surface F of genus g is generated by the Dehn twists $\alpha_i, \beta_k, \gamma_j$, $1 \leqq i \leqq g+1$, $2 \leqq k \leqq g-1$, $1 \leqq j \leqq g$, about the curves a_i, b_k, c_j as depicted in Figure* 11.8.

For a *proof* see [Lickorish 1962, 1964, 1966]. We remark that a left-handed twist about a is the inverse α^{-1} of the right-handed Dehn twist α about the same simple closed curve a. □

11.8 Lemma. *The homeomorphisms σ_{2i-1}, $1 \leqq i \leqq g+1$ lift to Dehn twists α_i about $a_i = p^{-1}(t_i)$ and the homeomorphisms σ_{2j}, $1 \leqq j \leqq g$, lift to Dehn twists γ_j about $c_j = p^{-1}(u_j)$.*

Proof. We may realize σ_{2i-1} by a half twist of a disk δ_i containing t_i (Figure 11.10), keeping the boundary $\partial\delta_i$ fixed.

The preimage $p^{-1}(\delta_i)$ consists of two annuli A_i and $\tau(A_i)$, $A_i \cap \tau(A_i) = p^{-1}(t_i) = a_i$. The half twist of δ_i lifts to a half twist of A_i, and to a half twist of $\tau(A_i)$ in the opposite direction. Since $A_i \cap \tau(A_i) = a_i$, these two half twists add up to a full Dehn

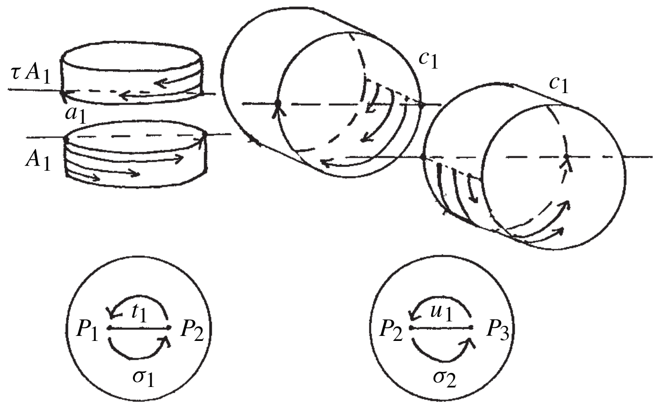


Figure 11.10

twist α_i along a_i. A similar construction shows that σ_{2j} is covered by a Dehn twist γ_j along $c_j = p^{-1}(u_j)$. □

There is an immediate corollary to 11.5, 11.7 and 11.8.

11.9 Corollary. *A closed oriented* 3*-manifold M^3 of Heegaard genus $g \leqq 2$ is a two-fold branched covering of S^3 with branching set a link $\mathfrak{k} \subset S^3$ with $g+1$ bridges.* □

There are, of course, closed oriented 3-manifolds which are not 2-fold coverings, if their Heegaard genus is at least three. $S^1 \times S^1 \times S^1$ is a well-known example [Fox 1972].

11.10 Proposition (R.H. Fox). *The manifold $S^1 \times S^1 \times S^1$ is not a two-fold branched covering of S^3; its Heegaard genus is three.*

Proof. We have seen earlier that for any n-fold branched cyclic covering $\hat{C}_n$ of a knot the endomorphism $1 + t + \cdots + t^{n-1}$ annihilates $H_1(\hat{C}_n)$ (Proposition 8.20 (b)). This holds equally for the second homology group, even if the branching set is merely a 1-complex. (It is even true for higher dimensions, see [Fox 1972].) Let M^3 be a closed oriented manifold which is an n-fold cyclic branched covering of S^3. Let $\hat{c}_q = \sum_{i=0}^{n-1} \sum_k n_{ik} t^{\nu_{ik}} \hat{c}_k^q$, $\partial \hat{c}_q = 0$, be a q-cycle of $H_q(M^3)$, $q \in \{1, 2\}$, with $\hat{c}_k^q$ a simplex over c_k^q, $p\hat{c}_k^q = c_k^q$, $\langle t \rangle$ the covering transformations. For a $(q+1)$-chain

c^{q+1} of S^3

$$\begin{aligned}\Big(\sum_{j=0}^{n-1} t^j\Big)\hat{c}_q &= \sum_{i,j,k} n_{ik} t^{\nu_{ik}+j}\hat{c}_k^q = \sum_{i,k} n_{ik}\hat{c}_k^q \sum_j t^{\nu_{ik}+j}\\ &= \Big(\sum_j t^j\Big)\Big(\sum_{i,k} n_{ik}\hat{c}_k^q\Big) = p^{-1}\Big(\sum_{i,k} n_{ik} c_k^q\Big) = p^{-1}\partial c^{q+1}\\ &= \partial p^{-1} c^{q+1} \sim 0.\end{aligned}$$

Suppose $M = S_1^1 \times S_2^1 \times S_3^1$ is a 2-fold covering of S^3. One has

$$\pi_1(S_1^1 \times S_2^1 \times S_3^1) \cong H_1(S_1^1 \times S_2^1 \times S_3^1) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z},$$

and t can be described by the (3×3)-matrix $-E$ with respect to the basis represented by the three factors. As $S_1^1 \times S_2^1 \times S_3^1$ is aspherical the covering transformation τ which induces t in the homology is homotopic to a map which inverts each of the 1-spheres S_i^1 [Spanier 1966, Chapter 8, Theorem 11]. Poincaré duality assigns to each S_i^1 a torus $S_j^1 \times S_k^1$, i, j, k all different, which represents a free generator of $H_2(S_1^1 \times S_2^1 \times S_3^1)$. Thus t operates on $H_2(S_1^1 \times S_2^1 \times S_3^1)$ as the identity which contradicts $1 + t = 0$.

It is easy to see that $S^1 \times S^1 \times S^1$ can be presented by a Heegaard splitting of genus three – identify opposite faces of a cube K (Figure 11.11). After two pairs are identified one gets a thickened torus. Identifying its two boundary tori obviously gives $S^1 \times S^1 \times S^1$. On the other hand K_1 and $K_2 = \overline{K - K_1}$ become handlebodies of genus three under the identifying map. □

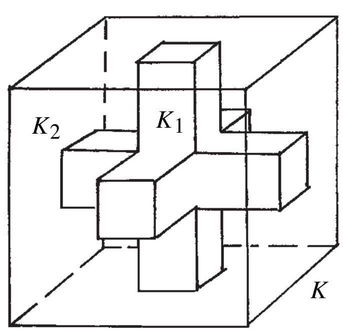


Figure 11.11

The method developed in this section can be used to study knots with two bridges by looking at their 2-fold branched covering spaces – a tool already used by H. Seifert [Schubert 1956]. It was further developed by Montesinos who was able to classify a set of knots comprising knots with two bridges and bretzel knots by similar means. We shall take up the matter in Chapter 12.

We conclude this section by proving the following

11.11 Theorem (Hilden–Montesinos). *Every closed orientable* 3*-manifold* M *is an irregular* 3*-fold branched covering of* S^3*. The branching set* $\mathfrak{k}$ *can be chosen in different ways, for instance as a knot or a link with unknotted components. If* g *is the Heegaard genus of* M*, it suffices to use a* $(g+2)$*-bridged branching set* $\mathfrak{k}$.

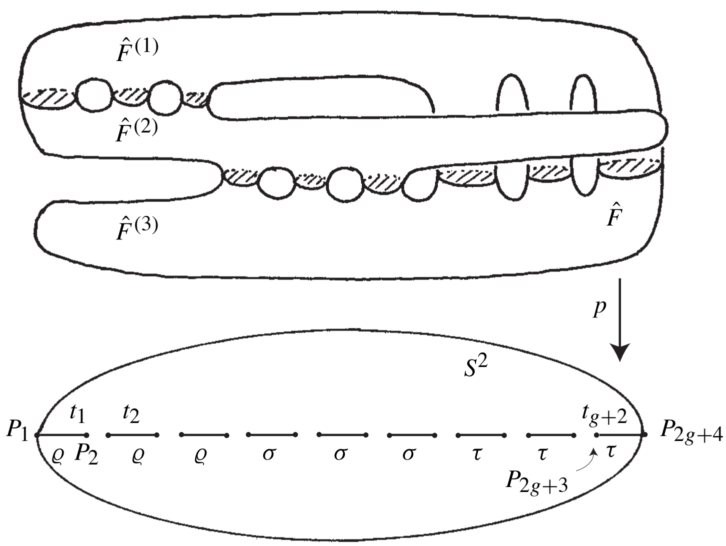


Figure 11.12

Before starting on the actual proof in 11.14 we study irregular 3-fold branched coverings $p\colon \hat{F} \to S^2$ of S^2 with branch indices $\leqq 2$. If $\hat{F}$ is an orientable closed surface of genus g, a calculation of $\chi(\hat{F})$ shows that the branching set in S^2 consists of $2(g+2)$ points P_i, $1 \leqq i \leqq 2(g+2)$. Let us denote by ϱ, σ, τ the transpositions $(1,2)$, $(2,3)$, $(1,3)$. Then by choosing $g+2$ disjoint simple arcs t_i, $1 \leqq i \leqq g+2$ in S^2, t_i connecting P_{2i-1} and P_{2i} (Figure 11.12), and assigning to each t_i one of the transpositions ϱ, σ, τ, we may construct a 3-fold branched covering $p\colon \hat{F} \to S^2$ (see Figure 11.12). The sheets $\hat{F}^{(j)}$, $1 \leqq j \leqq 3$, of the covering are homeomorphic to a 2-sphere with $g+2$ boundary components obtained from S^2 by cutting along the t_i, $1 \leqq i \leqq g+2$. Traversing an arc of $p^{-1}(t_i)$ in $\hat{F}$ means changing from $\hat{F}^{(j)}$ to $\hat{F}^{(\sigma(j))}$, if σ is assigned to t_i. (For $\hat{F}$ to be connected it is necessary and sufficient that at least two of the three transpositions are used in the construction.)

It will be convenient to use a very special version of such a covering. We assign ϱ to t_i, $1 \leqq i \leqq g+1$, and σ to t_{g+2} (Figure 11.13).

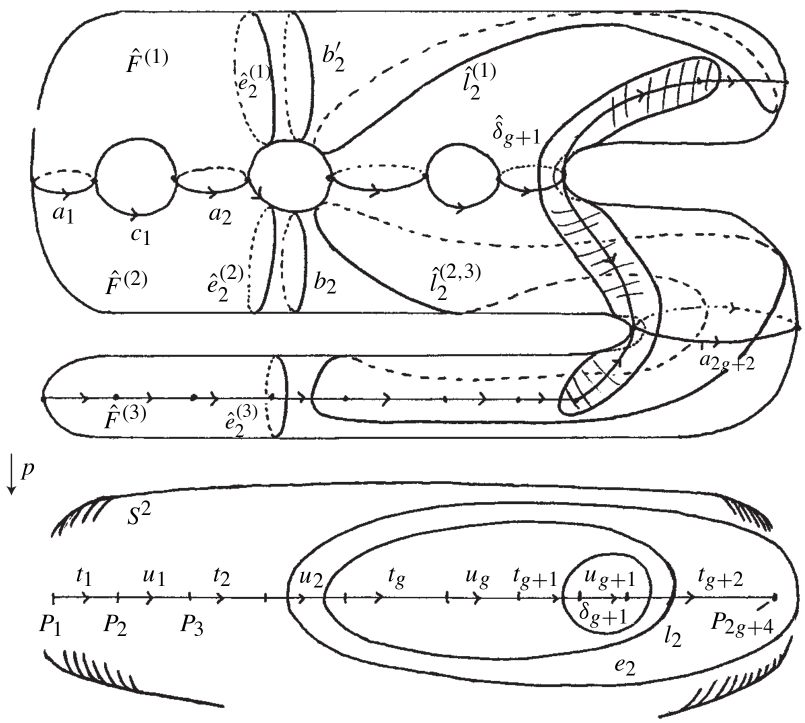


Figure 11.13

As in Figure 11.8 we introduce arcs u_j, $1 \leqq j \leqq g+1$, connecting P_{2j} and P_{2j+1}. We direct the t_i,u_j coherently (Figure 11.13) and lift these orientations. $p^{-1}(t_i)$, $1 \leqq i \leqq g+2$, consists of a closed curve a_i which will be regarded as unoriented, since its two parts carry opposite orientations, and an arc in $\hat{F}_3$ for $1 \leqq i \leqq g+1$, resp. in $\hat{F}_1$ for $i = g+2$. By the Dehn–Lickorish Theorem 11.7 the mapping class group of $\hat{F}$ is generated by the Dehn twists $\alpha_i, \beta_k, \gamma_j$, $1 \leqq i \leqq g+1$, $2 \leqq k \leqq g-1$, $1 \leqq j \leqq g$ about the curves a_i, b_k, c_j. Lemma 11.8 can be applied to the situation in hand: σ_{2i-1} in S^2 lifts to α_i, $1 \leqq i \leqq g+1$ and σ_{2j} lifts to γ_j, $1 \leqq j \leqq 2g$, because the effect of the lifting in $\hat{F}_3$ is isotopic to the identity. (Observe that σ_{2g+3} lifts to a deformation.) The only difficulty to overcome is to find homeomorphisms of $S^2 - \bigcup_{i=1}^{2g+4} P_i$ that lift to homeomorphisms of $\hat{F}$ isotopic to the Dehn twists β_k, $2 \leqq k \leqq g-1$. These are provided by the following

11.12 Lemma. *Let $p\colon \hat{F} \to S^2$ be the 3-fold branched covering described in Figure* 11.13.

(a) σ_{2i-1} *lifts to* α_i, $1 \leqq i \leqq g+1$; σ_{2j} *lifts to* γ_j, $1 \leqq j \leqq g$.

(b) $\omega_k = (\sigma_{2g+2}\sigma_{2g+1}\dots\sigma_{2k+2}\sigma_{2k+1}^2\sigma_{2k+2}\dots\sigma_{2g+2})^2$ *lifts to* β_k *for* $2 \leqq k \leqq g-1$.

(c) *The lifts of* ω_1 *resp.* ω_g *are isotopic to* α_1 *resp.* α_{g+1}.

(d) σ_{2g+2}^3 *and* σ_{2g+3} *lift to mappings isotopic to the identity.*

Proof. (a) was proved in 11.8 (b): consider simple closed curves e_i, l_i, $1 \leqq i \leqq g+1$, in S^2 (Figure 11.13). The curve e_i lifts to three simple closed curves $\hat{e}_i^{(j)} \in \hat{F}^{(j)}$, $1 \leqq j \leqq 3$, while l_i^2 is covered by two curves $(\hat{l}_i^{(1)})^2$, $\hat{l}_i^{(2,3)}$ (Figure 11.13). This is easily checked by looking at the intersections of e_i and l_i with t_i and u_i, resp. at those of $\hat{e}_i^{(j)}$ and $\hat{l}_i^{(1)}, \hat{l}_i^{(2,3)}$ with a_i and c_i. Since $\hat{e}_k^{(1)} \simeq b'_k \simeq \hat{l}_k^{(1)}$, $\hat{e}_k^{(2)} \simeq b_k$ for $2 \leqq k \leqq g-1$, and $\hat{e}_i^{(3)} \simeq 1$, $1 \leqq i \leqq g+1$, a Dehn twist ε_k in S^2 along e_k lifts to the composition of the Dehn twists β_k and β'_k, while the square of the Dehn twist λ_k along l_k lifts to the composition of $(\beta'_k)^2$ and β_k. Thus $\varepsilon_k^2\lambda_k^{-2}$ lifts to $\beta_k^2{\beta'_k}^2(\beta'_k)^{-2}\beta_k^{-1} = \beta_k$. ε_k induces a braid automorphism resp. a $(2g+4)$-braid with strings $\{f_i \mid 1 \leqq i \leqq 2g+4\}$ represented by a full twist of the strings $f_{2k+1}, f_{2k+2}, \dots, f_{2g+4}$ (Figure 11.13). ε_k^2 is then a double twist and λ_k^{-2} a double twist in the opposite direction leaving out the last string f_{2g+4}. It follows that $\varepsilon_k^2\lambda_k^{-2}$ defines a braid $(\sigma_{2g+3}\sigma_{2g+2}\dots\sigma_{2k+1}^2\sigma_{2k+2}\dots\sigma_{2g+3})^2$ in which only the last string f_{2g+4} is not constant, encircling its neighbours $f_{2k+1}, \dots, f_{2g+3}$ to the left, twice. Since obviously σ_{2g+3} lifts to a deformation, (b) is proved.

Assertion (c) follows in the same way as (b). To prove (d) consider a disk δ_{g+1} which is a regular neighbourhood of u_{g+1}. The third power $(\partial\delta_{g+1})^3$ of its boundary lifts to a simple closed curve in $\hat{F}$ bounding a disk $\hat{\delta}_{g+1} = p^{-1}(\delta_{g+1})$. The deformation σ_{2g+2}^3 in S^3 lifts to a “half-twist” of $\hat{\delta}_{g+1}$, a deformation of $\hat{F}$ which leaves the boundary $\partial\hat{\delta}_{g+1}$ pointwise fixed, and thus is isotopic to the identity. □

An easy consequence of Lemma 11.12 is the following

11.13 Corollary. *For a given permutation* $\pi \in \mathfrak{S}_{2g+4}$ *there is a braid automorphism* $\sigma \in \mathfrak{B}_{2g+4}$ *with permutation* π *induced by a homeomorphism of* $S^2 - \bigcup_{i=1}^{2g+4} P_i$ *which lifts to a deformation of* $\hat{F}$.

Proof. Together with σ_{2g+2}^3 the conjugates

$$\sigma_i\sigma_{i+1}\dots\sigma_{2g+1}\sigma_{2g+2}^3\sigma_{2g+1}^{-1}\dots\sigma_i^{-1}, \quad 1 \leqq i \leqq 2g+1,$$

lift to deformations. Hence, the transpositions $(i, 2g+3) \in \mathfrak{S}_{2g+4}$ can be realized by deformations. Since σ_{2g+3} also lifts to a deformation, the lemma is proved. □

11.14. *Proof of Theorem* 11.11. Let $M = \hat{B}_0 \cup_{\hat{h}} \hat{B}_1$ be a Heegaard splitting of genus g, and $p_j\colon \hat{F}_j \to S_j^2$, $j \in \{0, 1\}$, be 3-fold branched coverings of the type described in Figure 11.13, $\partial \hat{B}_j = \hat{F}_j$. Extend p_j to a covering $p_j\colon \hat{B}_j \to B_j$, $\partial B_j = S_j^2$, B_j a ball, in the same way as in the proof of Theorem 11.1. (Compare Figure 11.4). The branching set of p_j consists in B_j of $g+2$ disjoint unknotted arcs, each joining a pair P_{2i-1}, P_{2i} of branch points.

By the Lemmas 11.8 and 11.12, there is a braid $\mathfrak{z}$ with given permutation π defining a homeomorphism $h\colon S_0^2 \to S_1^2$ which lifts to a homeomorphism isotopic to $\hat{h}\colon \hat{F}_0 \to \hat{F}_1$. The plat $\mathfrak{k}$ defined by $\mathfrak{z}$ is the branching set of a 3-fold irregular covering $p\colon M \to S^3$, and if π is suitably chosen, $\mathfrak{k}$ is a knot. In the case $\pi = \mathrm{id}$ the branching set $\mathfrak{k}$ consists of $g+2$ trivial components. □

There are, of course, many plats $\mathfrak{k}$ defined by braids $\mathfrak{z} \in \mathfrak{B}_{2g+4}$ which by this construction lead to equivalent Heegaard diagrams and, hence, to homeomorphic manifolds. Replace $\mathfrak{k}$ by $\mathfrak{k}'$ with a defining braid $\mathfrak{z}' = \mathfrak{z}_1\mathfrak{z}\mathfrak{z}_0$ such that $\mathfrak{z}_i \subset B_i$, and $\mathfrak{k}' \cap B_i$ is a trivial half-plat (E 11.3). Then $\mathfrak{z}'$ lifts to a map $\hat{h}' = \hat{h}_1\hat{h}\hat{h}_0\colon \hat{F}_0 \to \hat{F}_1$, and there are homeomorphisms $\hat{H}_i\colon \hat{B}_i \to \hat{B}_i$ extending the homeomorphisms $\hat{h}_i\colon \hat{F}_i \to \hat{F}_i = \partial \hat{B}_i$, $i \in \{0, 1\}$. Obviously $\hat{B}_0 \cup_{\hat{h}'} \hat{B}_1$ and $\hat{B}_0 \cup_{\hat{h}} \hat{B}_1$ are homeomorphic. The braids $\mathfrak{z}_i$ of this type form a finitely generated subgroup in $\mathfrak{B}_{2g+4}$ (Exercise E 11.3).

Lemma 11.8 and 11.12 can be exploited to give some information on the mapping class group $M(g)$ of an orientable closed surface of genus g. The group $M(1)$ is well known [Goeritz 1932], and will play an important role in Chapter 12. By Lemma 11.8 and Corollary 11.9, $M(2)$ is a homomorphic image of the braid group $\mathfrak{B}_6$. A presentation is know [Birman 1974]. Since one string of the braids of $\mathfrak{B}_6$ can be kept constant, $M(2)$ is even a homomorphic image of $\mathfrak{B}_5$. For $g > 2$ the group $M(g)$ is a homomorphic image of the subgroup $\mathfrak{I}^*_{2g+3}$ of $\mathfrak{I}_{2g+3}$ generated by $\mathfrak{I}_{2g+2} \subset \mathfrak{I}_{2g+3}$ and the pure $(2g+3)$-braids ω_k, $2 \leqq k \leqq g-1$, of Lemma 11.12 (b). There is, however, a kernel $\neq 1$, which was determined in [Birman-Wajnryb 1985]. This leads to a presentation of $M(g)$, see also [McCool 1975], [Hatcher-Thurston 1980], [Wajnryb 1983].

C History and Sources

J.W. Alexander [1920] proved that every closed oriented n-manifold M is a branched covering of the n-sphere. The branching set is a $(n-2)$-subcomplex. Alexander claims in his paper (without giving a proof) that for $n = 3$ the branching set can be assumed to be a closed submanifold – a link in S^3. J.S. Birman and H.M. Hilden [1975] gave a proof, and, at the same time, obtained some information on the relations between the Heegaard genus of M, the number of sheets of the covering and the bridge number of the link. Finally Hilden [1976] and Montesinos [1976′] independently showed that every orientable closed 3-manifold is a 3-fold irregular covering of S^3

over a link $\mathfrak{k}$. It suffices to confine oneself to rather special types of branching sets $\mathfrak{k}$ [Hilden-Montesinos-Thickstun 1976].

D Exercises

E 11.1. Show that a Dehn-twist α of an orientable surface F along a simple closed (unoriented) curve a in F is well defined (up to a deformation) by a and an orientation of F. Dehn-twists α and α' represent the same element of the mapping class group ($\alpha' = \delta\alpha$, δ a deformation) if the corresponding curves are isotopic.

E 11.2. Apply the method of Lemma 11.2 to the following situation: Let $p\colon S^3 \to S^3$ be the cyclic 3-fold covering branched along the triangle A, B, C (Figure 11.14). Replace the branch set outside the balls around the vertices of the triangle as was done in the proof of Theorem 11.1. It follows that the 3-fold irregular covering along a trefoil is also a 3-sphere.

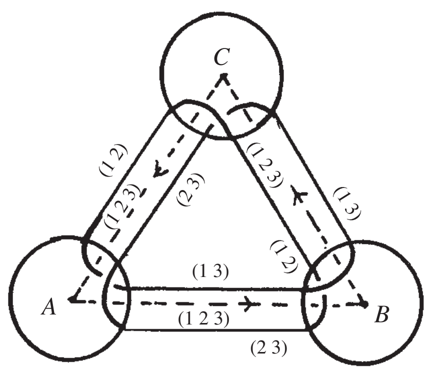


Figure 11.14

E 11.3. Let $\mathfrak{k}$ be a $2m$-plat in 3-space $\mathbb{R}^3$ and (x, y, z) cartesian coordinates of $\mathbb{R}^3$. Suppose $z = 0$ meets $\mathfrak{k}$ transversally in the $2m$ points $P_i = (i, 0, 0)$, $1 \leqq i \leqq 2m$. We call the intersection of $\mathfrak{k}$ with the upper half-space $\mathbb{R}^3_0 = \{(x, y, z | z \geqq 0\}$ a *half-plat* $\mathfrak{k}_0$, and denote its defining braid by $\mathfrak{z}_0 \in \mathfrak{B}_{2m}$. *The half-plat* $\mathfrak{k}_0$ *is trivial* if it is isotopic in $\mathbb{R}^3_0$ to m straight lines α_i in $x = 0$, $\partial a_i = \{P_{2i-1}, P_{2i}\}$.

Show that the braids $\mathfrak{z}_0 \in \mathfrak{B}_{2m}$ defining trivial half-plats form a subgroup of $\mathfrak{B}_{2m}$ generated by the braids σ_{2i-1}, $1 \leqq i \leqq m$, $\varrho_k = \sigma_{2k}\sigma_{2k-1}\sigma_{2k+1}\sigma_{2k}$, $\tau_k = \sigma_{2k}\sigma_{2k-1}\sigma_{2k+1}^{-1}\sigma_{2k}^{-1}$, $1 \leqq k \leqq m-1$.

E 11.4. Construct $S^1 \times S^1 \times S^1$ as a 3-fold irregular covering of S^3 along a 5-bridged knot.

Chapter 12

Montesinos Links

This chapter contains a study of a special class of knots. Section A deals with the 2-bridge knots which are classified by their twofold branched coverings – a method due to H. Seifert.

Section B looks at 2-bridge knots as 4-plats (Viergeflechte). This yields interesting geometric properties and new normal forms [Siebenmann 1975]. They are used in Section C to derive some properties concerning the genus and the possibility of fibring the complement, [Funcke 1978], [Hartley 1979′].

Section D is devoted to the classification of the Montesinos links which generalize knots and links with two bridges with respect to the property that their twofold branched coverings are Seifert fibre spaces. These knots have been introduced by Montesinos [1973, 1979], and the classification, conjectured by him, was given in [Bonahon 1979]. Here we present the proof given in [Zieschang 1984]. The last part deals with results of Bonahon–Siebenmann and Boileau from 1979 on the symmetries of Montesinos links. We prove these results following the lines of [Boileau-Zimmermann 1987] where a complete classification of all nonelliptic Montesinos links is given.

Montesinos knots include also the so-called pretzel knots which furnished the first examples of non-invertible knots [Trotter 1964].

A Schubert's Normal Form of Knots and Links with Two Bridges

H. Schubert [1956] classified knots and links with two bridges. His proof is a thorough and quite involved geometric analysis of the problem, his result a complete classification of these oriented knots and links. Each knot is presented in a normal form – a distinguished projection.

If one considers these knots as unoriented, their classification can be shown to rest on the classification of 3-dimensional lens spaces. This was already noticed by Seifert [Schubert 1956].

12.1. We start with some geometric properties of a 2-bridge knot, using Schubert's terminology. The knot $\mathfrak{k}$ meets a projection plane $\mathbb{R}^2 \subset \mathbb{R}^3$ in four points: A, B, C, D. The plane $\mathbb{R}^2$ defines an upper and a lower halfspace, and each of them intersects $\mathfrak{k}$ in two arcs. Each pair of arcs can be projected onto $\mathbb{R}^2$ without double points (see 2.13). We may assume that one pair of arcs is projected onto straight segments $w_1 = AB$, $w_2 = CD$ (Figure 12.1); the other pair is projected onto two disjoint simple curves v_1 (from B to C) and v_2 (from D to A). The diagram can be reduced in the following

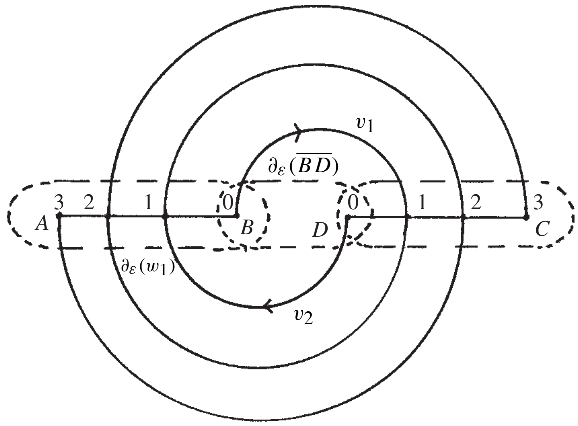


Figure 12.1

way: v_1 first meets w_2. A first double point on w_1 can be removed by an isotopy. In the same way one can arrange for each arc v_i to meet the w_j alternately, and for each w_j to meet the v_i alternately. The number of double points, hence, is even in a reduced diagram with $\alpha - 1$ ($\alpha \in \mathbb{N}$) double points on w_1 and on w_2. We attach numbers to these double points, counting against the orientation of w_1 and w_2 (Figure 12.1). Observe that for a knot α is odd; α even and $\partial v_1 = \{A, B\}$, $\partial v_2 = \{C, D\}$ yields a link.

12.2. We now add a point ∞ at infinity, $S^3 = \mathbb{R}^3 \cup \{\infty\}$, $S^2 = \mathbb{R}^2 \cup \{\infty\}$, and consider the two-fold branched covering T of S^2 with the branch set $\{A, B, C, D\}$, $\hat{p}\colon T \to S^2$, see Figure 12.2. The covering transformation $\tau\colon T \to T$ is a rotation

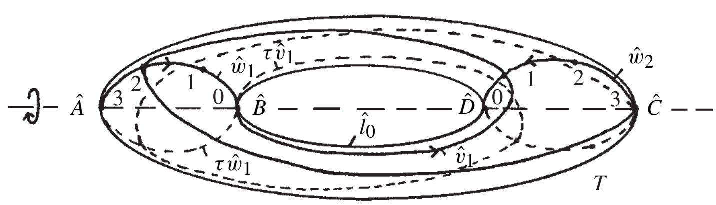


Figure 12.2

through π about an axis which pierces T in the points $\hat{A} = \hat{p}^{-1}(A)$, $\hat{B} = \hat{p}^{-1}(B)$, $\hat{C} = \hat{p}^{-1}(C)$, $\hat{D} = \hat{p}^{-1}(D)$.

w_1 and w_2 lift to $\{\hat{w}_1, \tau\hat{w}_1\}$, $\{\hat{w}_2, \tau\hat{w}_2\}$ and in the notation of homotopy chains, see 9.1, $(1-\tau)\hat{w}_1$ and $(1-\tau)\hat{w}_2$ are isotopic simple closed curves on T. Likewise, $(1-\tau)\hat{v}_1$, $(1-\tau)\hat{v}_2$ are two isotopic simple closed curves on T, each mapped onto its inverse by τ. They intersect with the $(1-\tau)\hat{w}_i$ alternately:

$$\operatorname{int}((1-\tau)\hat{v}_i, (1-\tau)\hat{w}_j) = \alpha.$$

Denote by $\partial_\varepsilon(c)$ the boundary of a small tubular neighbourhood of an arc c in $\mathbb{R}^2$. We choose an orientation on $\mathbb{R}^2$, and let $\partial_\varepsilon(c)$ have the induced orientation. The curve $\partial_\varepsilon(w_i)$ lifts to two curves isotopic to $\pm(1-\tau)\hat{w}_i$, $1 \leqq i \leqq 2$. The preimage $p^{-1}(\partial_\varepsilon(\overline{BD}))$ consists of two curves; one of them, $\hat{\ell}_0$ together with $\hat{m}_0 = (1-\tau)\hat{w}_1$ can be chosen as canonical generators of $H_1(T)$ – we call $\hat{m}_0$ a meridian, and $\hat{\ell}_0$ a longitude. Equally $p^{-1}(\partial_\varepsilon(v_i))$ consists of two curves isotopic to $\pm(1-\tau)\hat{v}_i$.

We assume for the moment $\alpha > 1$. (This excludes the trivial knot and a splittable link with two trivial components.) Then $(1-\tau)\hat{v}_i = \beta\hat{m}_0 + \alpha\hat{\ell}_0$ where $\beta \in \mathbb{Z}$ is positive, if at the first double point of v_1 the arc w_2 crosses from left to right in the double point $|\beta|$, and negative otherwise. From the construction it follows that $|\beta| < \alpha$ and that $\gcd(\alpha, \beta) = 1$.

12.3 Proposition. *For any pair α, β of integers subject to the conditions*

$$\alpha > 0, \quad -\alpha < \beta < +\alpha, \quad \gcd(\alpha, \beta) = 1, \ \beta \text{ odd}, \tag{1}$$

there is a knot or link with two bridges $\mathfrak{k} = \mathfrak{b}(\alpha, \beta)$ with a reduced diagram with numbers α, β. We call α the torsion, and β the crossing number of $\mathfrak{b}(\alpha, \beta)$. The number of components of $\mathfrak{b}(\alpha, \beta)$ is $\mu \equiv \alpha \bmod 2$, $1 \leqq \mu \leqq 2$. The 2-fold covering of S^3 branched along $\mathfrak{b}(\alpha, \beta)$ is the lens space $L(\alpha, \beta)$.

Proof. We first prove the last assertion. Suppose $\mathfrak{k} = \mathfrak{b}(\alpha, \beta)$ is a knot with two bridges whose reduced diagram determines the numbers α and β. We try to extend the covering $p\colon T \to S^2$ to a covering of S^3 branched along $\mathfrak{b}(\alpha, \beta)$. Denote by B_0, B_1 the two balls bounded by S^2 in S^3 with $\mathfrak{k} \cap B_0 = w_1 \cup w_2$. The 2-fold covering $\hat{B}_i$ of B_i branched along $B_i \cap \mathfrak{k}$ can be constructed by cutting B_i along two disjoint disks δ_1^i, δ_2^i spanning the arcs $B_i \cap \mathfrak{k}$, $i = 0, 1$.

This defines a sheet of the covering, and $\hat{B}_i$ itself is obtained by identifying corresponding cuts of two such sheets. $\hat{B}_i, 0 \leqq i \leqq 1$, is a solid torus, and $(1-\tau)\hat{w}_1 = \hat{m}_0$ represents a meridian of $\hat{B}_0$ while $\hat{m}_1 = (1-\tau)\hat{v}_1$ represents a meridian of $\hat{B}_1$. This follows from the definition of the curves $\partial_\varepsilon(v_i)$, $\partial_\varepsilon(w_i)$. Since

$$\hat{m}_1 = (1-\tau)\hat{v}_1 \simeq \beta\hat{m}_0 + \alpha\hat{\ell}_0, \tag{2}$$

the covering $\hat{B}_0 \cup_T \hat{B}_1$ is the Heegaard splitting of the lens space $L(\alpha, \beta)$.

Further information is obtained by looking at the universal covering $\tilde{T} \cong \mathbb{R}^2$ of T. The curve $\hat{v}_1$ is covered by $\tilde{v}_1$ which may be drawn as a straight line through a lattice point over $\hat{B}$ and another over $\hat{C}$ (resp. $\hat{A}$) for α odd (resp. α even). If cartesian coordinates are introduced with $\tilde{B}_{00}$ as the origin and $\tilde{D}_{00} = (0, \alpha)$, $\tilde{A}_{00} = (\alpha, 0)$, see Figure 12.3, $\tilde{v}_1$ is a straight line through $(0,0)$ and (β, α), and $\tilde{v}_2$ is a parallel

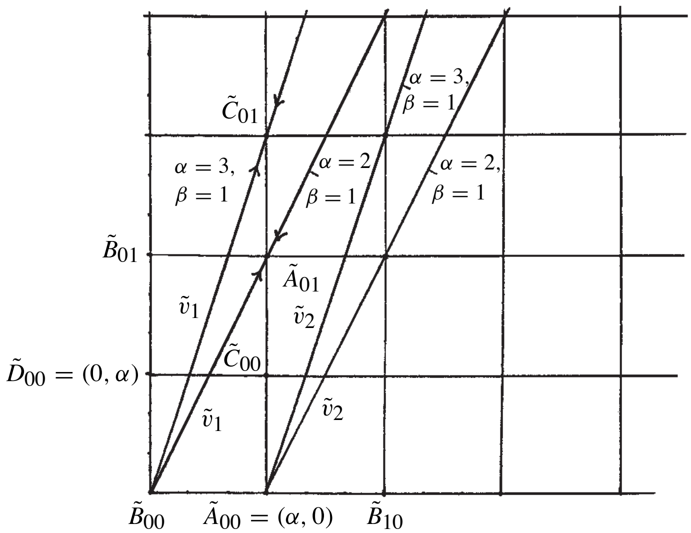


Figure 12.3

through $(\alpha, 0)$ and $(\alpha + \beta, \alpha)$. The $2\alpha \times 2\alpha$ square is a fundamental domain of the covering $\tilde{p}\colon \tilde{T} \to T$. Any pair of coprime integers (α, β) defines such curves which are projected onto simple closed curves of the form $(1 - \tau)\hat{v}_i$ on T, and, by $\hat{p}\colon T \to S^2$, onto a reduced diagram. $\square$

One may choose $\alpha > 0$. If $\tilde{v}_1$ starts in $\tilde{B}_{00}$, it ends in $(\beta\alpha, \alpha^2)$. Thus $\beta \equiv 1 \bmod 2$, since v_1 ends in C or A.

We attached numbers γ to the double points of the reduced projection of $\mathfrak{b}(\alpha, \beta)$ (Figure 12.1). To take into account also the characteristic of the double point we assign a residue class modulo 2α to it, represented by γ (resp. $-\gamma$) if w_i crosses v_j from left to right (resp. from right to left). Running along v_i one obtains the sequence:

$$0,\ \beta,\ 2\beta,\ \ldots,\ (\alpha - 1)\beta \ \text{ modulo } \ 2\alpha. \tag{3}$$

This follows immediately by looking at the universal covering $\tilde{T}$ (Figure 12.3). Note that $\tilde{v}_i$ is crossed from right to left in the strips where the attached numbers run from right to left, and that $-(\alpha - \delta) \equiv \alpha + \delta$ modulo 2α.

12.4 Remark. It is common use to normalize the invariants α, β of a lens space in a different way. In this usual normalization, $L(\alpha, \beta)$ is given by $L(\alpha, \beta^*)$ where $0 < \beta^* < \alpha$, $\beta^* \equiv \beta \bmod \alpha$.

12.5 Proposition. *Knots and links with two bridges are invertible.*

Proof. A rotation through π about the core of the solid torus $\hat{B}_0$ (or $\hat{B}_1$) commutes with the covering transformation τ. It induces therefore a homeomorphism of $S^2 = p(T)$ – a rotation through π about the centres of w_1 and w_2 (resp. v_1 and v_2) if the reduced diagram is placed symmetrically on S^2. This rotation can be extended to an isotopy of S^3 which carries $\mathfrak{k}$ onto $-\mathfrak{k}$. □

12.6 Theorem (H. Schubert). (a) $\mathfrak{b}(\alpha, \beta)$ *and* $\mathfrak{b}(\alpha', \beta')$ *are equivalent as oriented knots* (*or links*), *if and only if*

$$\alpha = \alpha', \quad \beta^{\pm 1} \equiv \beta' \bmod 2\alpha.$$

(b) $\mathfrak{b}(\alpha, \beta)$ *and* $\mathfrak{b}(\alpha', \beta')$ *are equivalent as unoriented knots* (*or links*), *if and only if*

$$\alpha = \alpha', \quad \beta^{\pm 1} \equiv \beta' \bmod \alpha.$$

Here β^{-1} denotes the integer with the properties $0 < \beta^{-1} < 2\alpha$ and $\beta\beta^{-1} \equiv 1 \bmod 2\alpha$. For the *proof* of (a) we refer to [Schubert 1956]. The weaker statement (b) follows from the classification of lens spaces [Reidemeister 1935], [Brody 1960]. □

12.7 Remark. In the case of knots (α odd) 12.6 (a) and (b) are equivalent – this follows also from 12.5. For links Schubert gave examples which show that *one can obtain non-equivalent links* (*with linking number zero*) *by reversing the orientation of one component*. (A link $\mathfrak{b}(\alpha, \beta)$ is transformed into $\mathfrak{b}(\alpha, \beta')$, $\beta' \equiv \alpha + \beta \bmod 2\alpha$, if one component is reoriented). The link $\mathfrak{b}(32, 7)$ is an example. The sequence (3) can be used to compute the linking number $\operatorname{lk}(\mathfrak{b}(\alpha, \beta))$ of the link:

$$\operatorname{lk}(\mathfrak{b}(\alpha, \beta)) = \sum_{\nu=1}^{\frac{\alpha}{2}} \varepsilon_\nu, \quad \varepsilon_\nu = (-1)^{\left[\frac{(2\nu-1)\beta}{\alpha}\right]}.$$

($[a]$ denotes the integral part of a.) One obtains for $\alpha = 32$, $\beta = 7$:

$$\sum_{v=1}^{16} \varepsilon_v = 1 + 1 - 1 - 1 - 1 + 1 + 1 - 1 - 1 + 1 + 1 - 1 - 1 - 1 + 1 + 1 = 0.$$

12.8. Lastly, our construction has been unsymmetric with respect to B_0 and B_1. If the balls are exchanged, $(\hat{m}_0, \hat{\ell}_0)$ and $(\hat{m}_1, \hat{\ell}_1)$ have to change places, where $\hat{m}_1$ is defined by (2) and forms a canonical basis together with $\hat{\ell}_1$:

$$\begin{aligned} \hat{m}_1 &= \beta \hat{m}_0 + \alpha \hat{\ell}_0, \\ \hat{\ell}_1 &= \alpha' \hat{m}_0 + \beta' \hat{\ell}_0, \end{aligned} \qquad \begin{vmatrix} \beta & \alpha \\ \alpha' & \beta' \end{vmatrix} = 1.$$

It follows $\hat{m}_0 = \beta'\hat{m}_1 - \alpha\hat{\ell}_1$. Since B_0 and B_1 induce on their common boundary opposite orientations, we may choose $(\hat{m}_1, -\hat{\ell}_1)$ as canonical curves on T. Thus $\mathfrak{b}(\alpha, \beta) = \mathfrak{b}(\alpha, \beta')$, $\beta\beta' - \alpha\alpha' = 1$, i.e. $\beta\beta' \equiv 1 \bmod \alpha$.

A reflection in a plane perpendicular to the projection plane and containing the straight segments w_i transforms a normal form $\mathfrak{b}(\alpha, \beta)$ into $\mathfrak{b}(\alpha, -\beta)$. Therefore $\mathfrak{b}^*(\alpha, \beta) = \mathfrak{b}(\alpha, -\beta)$.

B Viergeflechte (4-Plats)

Knots with two bridges were first studied in the form of 4-plats (see Chapter 2 D), [Bankwitz-Schumann 1934], and certain advantages of this point of view will become apparent in the following. We return to the situation described in 11 B (Figure 11.7).

12.9. S^3 now is composed of two balls B_0, B_1 and $I \times S^2$ in between, containing a 4-braid $\mathfrak{z}$ which defines a 2-bridge knot $\mathfrak{b}(\alpha, \beta)$. The 2-fold branched covering M^3 is by 12.3 a lens space $L(\alpha, \beta)$. (In this section we always choose $0 < \beta < \alpha$, β odd or even.) Lemma 11.8 shows that the braid operations σ_1, σ_2 lift to Dehn twists δ_1, δ_2 such that

$$\begin{aligned} \delta_1(\hat{m}_0) &= \hat{m}_0, & \delta_2(\hat{m}_0) &= \hat{m}_0 + \hat{\ell}_0 \\ \delta_1(\hat{\ell}_0) &= -\hat{m}_0 + \hat{\ell}_0, & \delta_2(\hat{\ell}_0) &= \hat{\ell}_0. \end{aligned}$$

Thus we may assign to σ_1, σ_2 matrices

$$\sigma_1 \mapsto A_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \sigma_2 \mapsto A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

which describe the linear mappings induced on $H_1(\hat{F}_0)$ by δ_1, δ_2 with respect to the basis $\hat{m}_0, \hat{\ell}_0$. A braid $\zeta = \sigma_2^{a_1}\sigma_1^{-a_2}\sigma_2^{a_3}\ldots\sigma_2^{a_m}$ induces the transformation

$$A = \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix}\cdots\begin{pmatrix} 1 & a_{m-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ a_m & 1 \end{pmatrix}. \tag{1}$$

Suppose the 2-fold covering M^3 of a 4-plat as in Figure 11.7 is given by a Heegaard splitting $M^3 = T_0 \cup_{\hat{h}} T_1$, $\partial T_j = \hat{F}_j$. Relative to bases $(\hat{m}_0, \hat{\ell}_0)$, $(\hat{m}_1, \hat{\ell}_1)$ of $H_1(\hat{F}_0)$, $H_1(\hat{F}_1)$, the isomorphism $\hat{h}_*\colon H_1(\hat{F}_0) \to H_1(\hat{F}_1)$ is represented by a unimodular matrix:

$$A = \begin{pmatrix} \beta & \alpha' \\ \alpha & \beta' \end{pmatrix}; \qquad \alpha, \alpha', \beta, \beta' \in \mathbb{Z};\ \beta\beta' - \alpha\alpha' = 1.$$

The integers α' and β' are determined up to a change $\alpha' \mapsto \alpha' + c\beta$, $\beta' \mapsto \beta' + c\alpha$ which can be achieved by

$$\begin{pmatrix} \beta & \alpha' \\ \alpha & \beta' \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \beta & \alpha' + c\beta \\ \alpha & \beta' + c\alpha \end{pmatrix}.$$

This corresponds to a substitution $\zeta \mapsto \zeta\sigma_1^c$ which does not alter the plat. The product (1) defines a sequence of equations ($r_0 = \alpha$, $r_1 = \beta$):

$$\begin{aligned} r_0 &= a_1 r_1 + r_2 \\ r_1 &= a_2 r_2 + r_3 \\ &\vdots \\ r_{m-1} &= a_m r_m + 0, \quad |r_m| = 1, \end{aligned} \tag{2}$$

following from

$$\begin{pmatrix} 1 & 0 \\ -a_i & 1 \end{pmatrix} \begin{pmatrix} r_i & * \\ r_{i-1} & * \end{pmatrix} = \begin{pmatrix} r_i & * \\ r_{i-1} - a_i r_i & * \end{pmatrix} = \begin{pmatrix} r_i & * \\ r_{i+1} & * \end{pmatrix},$$

$$\begin{pmatrix} 1 & -a_{i+1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r_i & * \\ r_{i+1} & * \end{pmatrix} = \begin{pmatrix} r_i - a_{i+1} r_{i+1} & * \\ r_{i+1} & * \end{pmatrix} = \begin{pmatrix} r_{i+2} & * \\ r_{i+1} & * \end{pmatrix}.$$

If we postulate $0 \leqq r_i < r_{i-1}$, the equations (2) describe an euclidean algorithm which is uniquely defined by $\alpha = r_0$ and $\beta = r_1$.

12.10 Definition. We call a system of equations (2) with $r_i, a_j \in \mathbb{Z}$, a *generalized euclidean algorithm* of length m if $0 < |r_i| < |r_{i-1}|$, $1 \leqq i \leqq m$, and $r_0 \geqq 0$.

Such an algorithm can also be expressed by a continued fraction:

$$\frac{\beta}{\alpha} = \frac{r_1}{r_0} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots \quad \cdots + a_{m-1} + \frac{1}{a_m}}}} = [a_1, a_2, \ldots, a_m].$$

The integers a_i are called the *quotients of the continued fraction*. From $0 < |r_m| < |r_{m-1}|$ it follows that $|a_m| \geqq 2$. We allow the augmentation

$$[a_1, a_2, \ldots, (a_m \pm 1), \mp 1] = [a_1, a_2, \ldots, a_m], \tag{3}$$

since
$$(a_m \pm 1) + \frac{1}{\mp 1} = a_m.$$
Thus, by allowing $|r_{m-1}| = |r_m| = 1$, we may assume m to be odd.

12.11. To return to the 2-bridge knot $\mathfrak{b}(\alpha, \beta)$ we assume $\alpha > 0$ and $0 \leqq \beta < \alpha$, $\gcd(\alpha, \beta) = 1$. For any integral solution of (2) with $r_0 = \alpha$, $r_1 = \beta$, one obtains a matrix equation:

$$\begin{pmatrix} \beta & \alpha' \\ \alpha & \beta' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ a_m & 1 \end{pmatrix} \begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix}, \quad m \text{ odd}, \tag{4}$$

$$\begin{pmatrix} \beta & \alpha' \\ \alpha & \beta' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & a_m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & * \end{pmatrix}, \quad m \text{ even}. \tag{5}$$

The first equation (m odd) shows that a 4-plat defined by the braid

$$\mathfrak{z} = \sigma_2^{a_1} \sigma_1^{-a_2} \sigma_2^{a_3} \ldots \sigma_2^{a_m}$$

is the knot $\mathfrak{b}(\alpha, \beta)$, since its 2-fold branched covering is the (oriented) lens space $L(\alpha, \beta)$. The last factor on the right represents a power of σ_1 which does not change the knot, and which induces a homeomorphism of $\hat{B}_1$. In the case when m is even observe that

$$\begin{pmatrix} 0 & -1 \\ 1 & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

From this it follows (Figure 12.4) that $\mathfrak{b}(\alpha, \beta)$ is defined by $\mathfrak{z} = \sigma_2^{a_1} \sigma_1^{-a_2} \ldots \sigma_2^{-a_m}$ but that the plat has to be closed at the lower end in a different way, switching meridian $\hat{m}_1$ and longitude $\hat{\ell}_1$ corresponding to the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

12.12 Remark. The case $\alpha = 1$, $\beta = 0$, is described by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \beta & \alpha' \\ \alpha & \beta' \end{pmatrix}.$$

The corresponding plat (Figure 12.5) is a trivial knot. The matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \beta & \alpha' \\ \alpha & \beta' \end{pmatrix}$$

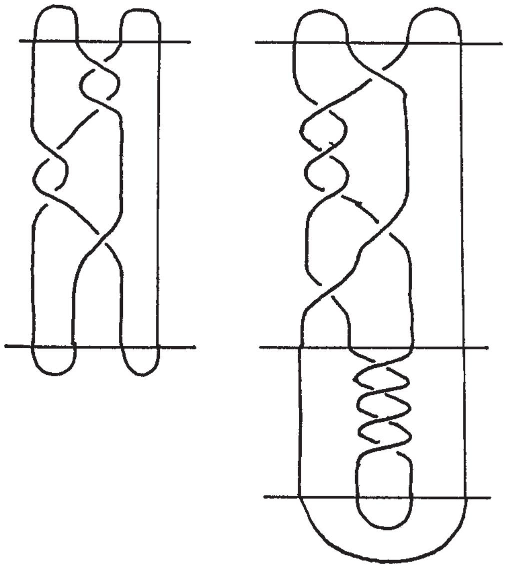


Figure 12.4

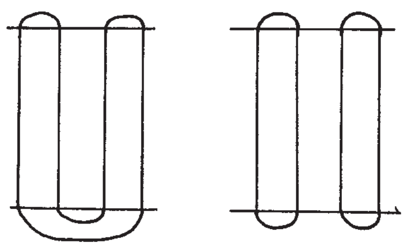


Figure 12.5

is characterized by the pair $(0, 1) = (\alpha, \beta)$ (Figure 12.5). It is therefore reasonable to denote by $\mathfrak{b}(1, 0)$ resp. $\mathfrak{b}(0, 1)$ the unknot resp. two split unknotted components, and to put: $L(1, 0) = S^3$, $L(0, 1) = S^1 \times S^2$. The connection between the numbers a_i and the quotient $\beta\alpha^{-1}$ allows to invent many different normal forms of (unoriented) knots with two bridges as 4-plats. All it requires is to make the algorithm (2) unique and to take into account that the balls B_0 and B_1 are exchangeable.

12.13 Proposition. *The (unoriented) knot (or link) $\mathfrak{b}(\alpha, \beta)$, $0 < \beta < \alpha$, has a presentation as a 4-plat with a defining braid $\mathfrak{z} = \sigma_2^{a_1}\sigma_1^{-a_2} \dots \sigma_2^{a_m}$, $a_i > 0$, m odd, where the a_i are the quotients of the continued fraction $[a_1, \dots, a_m] = \beta\alpha^{-1}$. Sequences $(a_1, \dots, a_m)$ and $(a'_1, \dots, a'_{m'})$ define the same knot or link if and only if $m = m'$, and $a_i = a'_i$ or $a_i = a'_{m+1-i}$, $1 \leqq i \leqq m$.*

Proof. The algorithm (2) is unique, since $a_i > 0$ implies that $r_i > 0$ for $m \geqq i \geqq 1$. The expansion of $\beta\alpha^{-1}$ as a continued fraction of odd length m is unique [Perron 1954]. A rotation through π about an axis in the projection plane containing $\overline{AB}$ and $\overline{CD}$ finally exchanges B_0 and B_1; its lift exchanges $\hat{B}_0$ and $\hat{B}_1$. □

12.14 Remark. It is an easy exercise in continued fractions (E 12.3) to prove $\beta'\alpha^{-1} = [a_m \dots, a_1]$ if $\beta\alpha^{-1} = [a_1, \dots, a_m]$, and $\beta\beta' \equiv 1 \mod \alpha$.

Note that the normal form of 4-plats described in 12.13 represents alternating plats, hence:

12.15 Proposition (Bankwitz–Schumann). *Knots and links with two bridges are alternating.* □

12.16 Examples. Consider $\mathfrak{b}(9, 5) = 6_1$ as an example: $5/9 = [1, 1, 4]$. The corresponding plat is defined by $\sigma_2\sigma_1^{-1}\sigma_2^4$ (Figure 12.6). (Verify: $2/9 = [4, 1, 1]$, $2 \cdot 5 \equiv 1 \mod 9$.) Figure 12.6 also shows the normal forms of the two trefoils: $\mathfrak{z} = \sigma_2^3$ resp. $\mathfrak{z}' = \sigma_2\sigma_1^{-1}\sigma_2$, according to $1/3 = [3]$, $2/3 = [1, 1, 1]$. A generalized euclidean

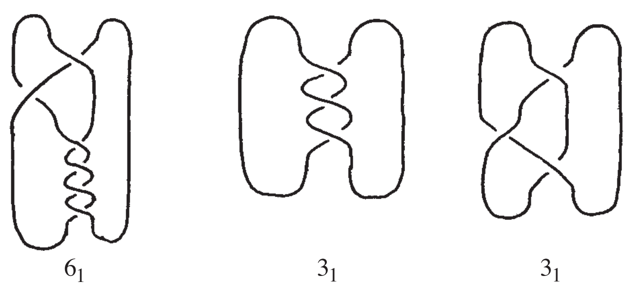


Figure 12.6

algorithm is, of course, not unique. One may impose various conditions on it to make it so, for instance, the quotients a_i, $1 \leqq i < m$ can obviously be chosen either even or odd. Combining such conditions for the quotients with $r_j > 0$ for some j gives multifarious possibilities for normal forms of 4-plats.

We choose from each pair of mirror images the one with $\beta > 0$, β odd.

12.17 Proposition. *There is a unique generalized euclidean algorithm*

$$r_{i-1} = c_i r_i + r_{i+1}$$

of length m with

$$r_0 = \alpha > 0, \quad r_1 = \beta > 0, \quad \gcd(\alpha, \beta) = 1, \ \beta \textit{ odd},$$

$$r_{2j} > 0, \quad c_{2j} = 2b_j \quad \textit{for a suitable } b_j,\ 1 \leqq 2j \leqq m,$$
$$|r_{i-1}| > |r_i| \quad \textit{for } 0 \leqq i < m, \quad |r_{m-1}| \geqq |r_m|.$$
$$\textit{If } |r_{m-1}| = |r_m|, \ \textit{then } c_{m-1}c_m > 0.$$

The length m of the algorithm is odd ($r_{m+1} = 0$), $r_{2j-1} \equiv 1 \bmod 2$, *and* $a_j b_j > 0$ *for* $a_j = c_{2j-1}$, $1 \leqq j \leqq \frac{m+1}{2}$.

Proof. The algorithm is easily seen to be unique, and $r_{2j-1} \equiv m \equiv 1 \bmod 2$ is an immediate consequence. From

$$r_{2j-2} = a_j r_{2j-1} + r_{2j},$$
$$r_{2j-1} = 2b_j r_{2j} + r_{2j+1}$$

one derives

$$(r_{2j-1} - r_{2j+1})a_j = 2a_j b_j r_{2j}$$

and that the sign of the left hand expression is the same as the sign of

$$r_{2j-1}a_j = r_{2j-2} - r_{2j} > 0,$$

since $|r_{2j-1}| > |r_{2j+1}|$. □

12.18 Remark. The quotients a_i obtained from the generalized algorithm of 12.17 may change if $r_1 = \beta$ is replaced by β' with $\beta\beta' \equiv \pm 1 \bmod \alpha$. We are, however, only interested in the fact that there is always a presentation according to 12.17 of any knot $\mathfrak{b}(\alpha, \beta)$ or $\mathfrak{b}^*(\alpha, \beta) = \mathfrak{b}(\alpha, -\beta)$, and we shall exploit this to get information about the Alexander polynomial and the genus of $\mathfrak{b}(\alpha, \beta)$.

C Alexander Polynomial and Genus of a Knot with Two Bridges

We have shown in 8.13 that the Alexander polynomial $\Delta(t)$ of a knot may be written as a polynomial with integral coefficients in $u = t + t^{-1} - 2$, $\Delta(t) = f(u)$. Hence, $\Delta(t^2)$ is a polynomial in $z = t - t^{-1}$. (It is even a polynomial in z^2.) J.H. Conway [1970] defined a polynomial $\nabla_{\mathfrak{k}}(z)$ with integral coefficients for (oriented) links which can be inductively computed from a regular projection of a link $\mathfrak{k}$ in the following way:

12.19 (Conway potential function).

(1) $\nabla_{\mathfrak{k}}(z) = 1$, if $\mathfrak{k}$ is the trivial knot.

(2) $\nabla_{\mathfrak{k}}(z) = 0$, if $\mathfrak{k}$ is a split link.

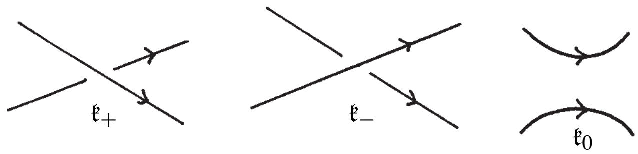


Figure 12.7

(3) $\nabla_{\mathfrak{k}_+} - \nabla_{\mathfrak{k}_-} = z \cdot \nabla_{\mathfrak{k}_0}$, if $\mathfrak{k}_+$, $\mathfrak{k}_-$, and $\mathfrak{k}_0$ differ by a local operation of the kind depicted in Figure 12.7.

Changing overcrossings into undercrossings eventually transforms any regular projection into that of a trivial knot or splittable link, compare 2.2. Equation (3) may therefore be used as an algorithm (*Conway algorithm*) to compute $\nabla_{\mathfrak{k}}(z)$ with initial conditions (1) and (2). Thus, if there is a function $\nabla_{\mathfrak{k}}(z)$ satisfying conditions (1), (2), (3) which is an invariant of the link, it must be unique.

12.20 Proposition. (a) *There is a unique integral polynomial* $\nabla_{\mathfrak{k}}(z)$ *satisfying* (1), (2), (3)*; it is called the Conway potential function and is an invariant of the link.*

(b) $\nabla_{\mathfrak{k}}(t - t^{-1}) \doteq \Delta(t^2)$ *for* $\mu = 1$, $\nabla_{\mathfrak{k}}(t - t^{-1}) \doteq (t^2 - 1)^{\mu - 1}\nabla(t^2)$ *for* $\mu > 1$.

(Here μ is the number of components of $\mathfrak{k}$, $\Delta(t)$ denotes the Alexander polynomial, and $\nabla(t)$ the Hosokawa polynomial of $\mathfrak{k}$, see 9.18.)

We shall prove 12.20 in 13.33 by defining an invariant function $\nabla_{\mathfrak{k}}(t)$. Observe that the equations that relate $\nabla_{\mathfrak{k}}(t - t^{-1})$ with the Alexander polynomial and the Hosokawa polynomial suffice to show the invariance of $\nabla_{\mathfrak{k}}(z)$ in the case of knots, whereas for $\mu > 1$ there remains the ambiguity of the sign.

12.21 Definition. The polynomials $f_n(z)$, $n \in \mathbb{Z}$, defined by

$$f_{n+1}(z) = zf_n(z) + f_{n-1}(z), \quad f_0(z) = 0, \quad f_1(z) = 1,$$
$$f_{-n}(z) = (-1)^{n+1} f_n(z) \quad \text{for } n \geqq 0$$

are called *Fibonacci polynomials*.

12.22 Lemma. *The Fibonacci polynomials are of the form:*

$$f_{2n-1} = 1 + a_1 z^2 + a_2 z^4 + \cdots + a_{n-1} z^{2(n-1)}$$
$$f_{2n} = z \cdot (b_0 + b_1 z^2 + b_2 z^4 + \cdots + b_{n-1} z^{2(n-1)}),$$

$a_i, b_i \in \mathbb{Z}$, $n \geqq 0$, $a_{n-1} = b_{n-1} = 1$.

Consequence: $\deg f_n = \deg f_{-n} = n - 1$.

Proof. An easy Exercise E 12.6. □

Let $\mathfrak{b}(\alpha, \beta)$, $\alpha > \beta > 0$, $\alpha \equiv \beta \equiv 1 \bmod 2$, be represented by the 4-plat defined by the braid

$$\mathfrak{z} = \sigma_2^{a_1}\sigma_1^{-2b_1}\sigma_2^{a_2}\sigma_1^{-2b_2}\ldots\sigma_1^{-2b_{k-1}}\sigma_2^{a_k}, \quad k = \frac{m+1}{2},$$

with $\beta/\alpha = [a_1, 2b_1, a_2, 2b_2, \ldots, a_k]$ according to the algorithm of 12.17. By 12.17, $a_j b_j > 0$, but $b_j a_{j+1}$ may be positive or negative. Assign a sequence $(i_1, i_2, \ldots, i_r)$ to the sequence of quotients noting down i_j, if $b_{i_j} a_{i_{j+1}} < 0$. The normalizations in 12.17 imply that $b_1 > 0$.

12.23 Proposition. *Let $\mathfrak{b}(\alpha, \beta)$ be defined as a 4-plat by the braid*

$$\mathfrak{z} = \sigma_2^{a_1}\sigma_1^{-2b_1}\ldots\sigma_1^{-2b_{k-1}}\sigma_2^{a_k}, \quad m = 2k - 1,$$

and let $i_1, i_2, \ldots, i_r$ denote the sequence of indices where a change of sign occurs in the sequence of quotients.

(a) $\deg \nabla_{\mathfrak{b}}(z) = \left(\sum_{j=1}^{k} |a_j|\right) - 1$ *where $\nabla_{\mathfrak{b}}(z)$ is the Conway polynomial of $\mathfrak{b}(\alpha, \beta) = \mathfrak{b}$.*

(b) *The absolute value of the leading coefficient $C(\nabla_{\mathfrak{b}})$ of $\nabla_{\mathfrak{b}}(z)$ is*

$$\prod_{j=1}^{k-1}(|b_j| + 1 - \eta_j) = |C(\nabla_{\mathfrak{b}})|, \quad \eta_j = \begin{cases} 1, & j \in \{i_1, \ldots, i_r\}, \\ 0 & \textit{otherwise}. \end{cases}$$

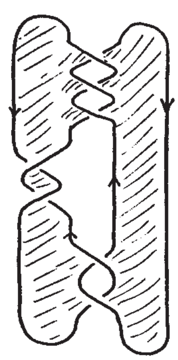


Figure 12.8

Proof. Orient the 4-plat defined by $\mathfrak{z}$ as in Figure 12.8 – the fourth string downward. By applying the Conway algorithm, it is easy to compute the Conway polynomial

$\nabla_a(z)$ of the 4-plat defined by $\mathfrak{z} = \sigma_2^a$ with $a > 0$: $\nabla_a = (-1)^{a+1} f_a$, f_a the a-th Fibonacci polynomial. Equally $\nabla_{-a} = (-1)^{a+1}\nabla_a$. Now assume $a > 0$, $b > 0$, $\mathfrak{z} = \sigma_2^a\sigma_1^{-2b}\sigma_2^c$. (The conditions of 12.17 exclude $c = -1$.) The Conway polynomial of the 4-plat defined by $\mathfrak{z}$ is denoted by ∇_{abc}. Apply again the Conway algorithm to the double points of σ_2^a, working downward from the top of the braid:

$$\begin{aligned}\nabla_{abc} &= (-1)^a f_{a-1}\nabla_c + (-1)^{a+1} f_a \nabla_{c+1} - b(-1)^{a+1} z \cdot f_a \nabla_c \\ &= \big((-1)^a f_{a-1} + (-1)^a b \cdot z f_a\big)(-1)^{c+1} f_c + (-1)^{a+1+c} f_a f_{c+1} \\ &= \nabla_{a-1}\nabla_c + \nabla_a \nabla_{c+1} - bz\nabla_a\nabla_c.\end{aligned}$$

Using 12.22 one obtains

$$\begin{aligned} c > 0: &\quad \deg \nabla_{abc} = a + c - 1,\ |C(\nabla_{abc})| = |b+1|; \\ c < 0: &\quad \deg \nabla_{abc} = 1 + a - 1 - c - 1 = a - c - 1, \quad |C(\nabla_{abc})| = |b|.\end{aligned}$$

In the same way the case $a < 0$, $b < 0$, $c \neq 1$ can be treated:

$$\nabla_{abc} = \nabla_{a+1}\nabla_c + \nabla_a\nabla_{c-1} - bz\nabla_a\nabla_c.$$

Again

$$\begin{aligned} &\deg \nabla_{abc} = |a| + |c| - 1, \\ &C(\nabla_{abc}) = |b| + 1 - \eta, \quad \eta = \begin{cases} 1, & c > 0 \\ 0, & c < 0 \end{cases}\end{aligned}$$

Now suppose $\mathfrak{z} = \sigma_2^{a_1}\sigma_1^{-2b_1} \cdot \mathfrak{z}'$, $\mathfrak{z}' = \sigma_2^{a_2} \cdot \sigma_1^{-2b_2} \dots$, $a_1 > 0$, $a_2 > 0$. One has

$$\nabla_{\mathfrak{z}} = \nabla_{a_1}\nabla_{\sigma_2\mathfrak{z}'} + \nabla_{a_1-1}\nabla_{\mathfrak{z}'} - b_1 z \nabla_{a_1}\nabla_{\mathfrak{z}'},\ \ \deg\nabla_{\mathfrak{z}} = \deg \nabla_{a_1}\nabla_{\sigma_2\mathfrak{z}'}.$$

($\nabla_{\mathfrak{z}}$ is the polynomial of the 4-plat defined by $\mathfrak{z}$.)

It follows by induction that

$$\deg \nabla_{\mathfrak{z}} = |a_1| - 1 + \sum_{j>1} |a_j| = \Big(\sum_{j=1}^{k} |a_j|\Big) - 1,$$

and

$$|C(\nabla_{\mathfrak{z}})| = \prod_{j=1}^{k-1} (|b_j| + 1 - \eta_j).$$

Similarly, for $a_1 > 0$, $a_2 < 0$

$$\deg \nabla_{\mathfrak{z}} = \deg \nabla_{a_1}\nabla_{\mathfrak{z}'} + 1 = (|a_1| - 1) + \Big[\Big(\sum_{j>1}^{k} |a_j|\Big) - 1\Big] + 1 = \Big(\sum_{j=1}^{k} |a_j|\Big) - 1.$$

$\square$

Since 2-bridge knots are alternating, $\deg \nabla_{\mathfrak{b}}(z) = 2g + \mu - 1$ where g is the genus of $\mathfrak{b}(\alpha, \beta)$ [Crowell 1959]. Moreover, $|C(\nabla_{\mathfrak{b}}(z))| = |C(\Delta(t))| = 1$ characterizes fibred knots [Murasugi 1960, 1963]. A proof of both results is given in 13.26. From this it follows

12.24 Proposition. *The genus of a* 2*-bridge knot* $\mathfrak{b}(\alpha, \beta)$ *of multiplicity* μ *is*

$$g(\alpha, \beta) = \frac{1}{2}\Big[\Big(\sum_{j=1}^{k} |a_j|\Big) - \mu\Big].$$

The knot $\mathfrak{b}(\alpha, \beta)$ *is fibred if and only if its defining braid is of the form*

$$\mathfrak{z} = \sigma_2^{a_1}\sigma_1^{-2}\sigma_2^{-a_2}\sigma_1^{2}\sigma_2^{a_3}\sigma_1^{-2}\dots\sigma_2^{\pm a_k}, \quad a_j > 0,\ k > 0.$$

(*The quotients* a_j, b_j *of* $\beta\alpha^{-1}$ *are determined by the algorithm of* 12.17.)

Proof. It remains to prove the second assertion. It follows from 12.23 that $|b_j| = 1$, $\eta_j = 1$ for $1 \leqq j < k$. Since $b_1 = 1$, one has $b_j = (-1)^{j-1}$. □

Using 12.13 we obtain

12.25 Corollary. *There are infinitely many knots* $\mathfrak{b}(\alpha, \beta)$ *of genus* $g > 0$*, and infinitely many fibred knots with two bridges. However, for any given genus there are only finitely many knots with two bridges which are fibred.* □

12.26 Proposition. *A knot with two bridges of genus one or its mirror image is of the form* $\mathfrak{b}(\alpha, \beta)$ *with*

$$\beta = 2n, \quad \alpha = 2m\beta \pm 1, \quad m, n \in \mathbb{N}.$$

The trefoil and the four-knot are the only fibred 2*-bridge knots of genus one.*

Proof. This is a special case of 12.24 and the proof involves only straight forward computations. By 12.24, $k < 4$.

For $k = 1$ one obtains the sequence [3] which defines the trefoil (see Figure 12.6).

For $k = 2$ there are two types of sequences, see 12.24 and 12.17:

$$[2, 2b, 1], \quad [1, 2b, \pm 2], \quad b \in \mathbb{N}.$$

The sequence $[1, 2, -2]$ defines a fibred knot – the four-knot.

For $k = 3$ the sequences are of the form:

$$[1, 2b, 1, 2c, 1] \quad \text{or} \quad [1, 2b, -1, -2c, -1], \quad b, c \in \mathbb{N}.$$

Using 12.17 again, this leads to

$$\begin{aligned} \alpha &= 4(b+1)(c+1)-1, \quad & \beta &= 2(c+1)(2b+1)-1, \quad \text{resp.} \\ \alpha &= 4b(c+1)+1, & \beta &= 2(c+1)(2b-1)+1. \end{aligned}$$

The simpler formulae of Proposition 12.26 is obtained by replacing β by $\alpha - \beta$ what corresponds to the replacement of the knot by its mirror image, see 12.8. □

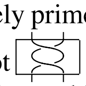
12.27 Remark. If $\mathfrak{b}(\alpha, \beta)$ is given in a normal form according to 12.17 the band marked as a hatched region in Figure 12.8 is an orientable surface of minimal genus spanning $\mathfrak{b}(\alpha, \beta)$.

Proposition 12.24 is a version of a theorem proved first in [Funcke 1978] and [Hartley 1979]. R. Hartley also proves in this paper a monotony property of the coefficients of the Alexander polynomial of $\mathfrak{b}(\alpha, \beta)$. See also [Burde 1984, 1985].

D Classification of Montesinos Links

The classification of knots and links with two bridges was achieved by classifying their twofold branched coverings – the lens spaces. It is natural to use this tool in the case of a larger class of manifolds which can be classified. Montesinos [1973, 1979] defined a set of links whose twofold branched covering spaces are Seifert fibre spaces. Their classification is a straight forward generalization of Seifert's idea in the case of 2-bridge knots.

We start with a definition of Montesinos links, and formulate the classification theorem of [Bonahon 1979]. Then we show that the twofold branched covering is a Seifert fibre space. Those Seifert fibre spaces are classified by their fundamental groups. By repeating the arguments for the classification of those groups we classify the Seifert fibre space together with the covering transformation. This then gives the classification of Montesinos links.

12.28 Definition (Montesinos link). A *Montesinos link* (or *knot*) has a projection as shown in Figure 12.9. The numbers e, a_i', a_i'' denote numbers of half-twists. A box $\boxed{\alpha, \beta}$ stands for a so-called *rational tangle* as illustrated in Figure 12.9 (b), and α, β are defined by the continued fraction $\frac{\beta}{\alpha} = [a_1, -a_2, a_3, \dots, \pm a_m]$, $a_j = a_j' + a_j''$ together with the conditions that α and β are relatively prime and $\alpha > 0$. A further assumption is that $\frac{\beta}{\alpha}$ is not an integer, that is $\boxed{\alpha, \beta}$ is not ; in this case the knot has a simpler projection. The above Montesinos link is denoted by $\mathfrak{m}(e; \alpha_1/\beta_1, \dots, \alpha_r/\beta_r)$.

In Figure 12.9 (a): $e = 3$; in Figure 12.9 (b): $n = 5$, $a_1' = 2$, $a_1'' = 0 \Longrightarrow a_1 = 2$; $a_2' = -1$, $a_2'' = -2 \Longrightarrow a_2 = -3$; $a_3 = -1$, $a_4 = 3$, $a_5 = 5$ and $\beta/\alpha = -43/105$.

As before in the case of 2-bridge knots we think of $\mathfrak{m}$ as unoriented. It follows from Section B that *the continued fractions* $\frac{\beta}{\alpha} = [a_1, \dots, \pm a_m]$ (*including* $1/0 = \infty$)

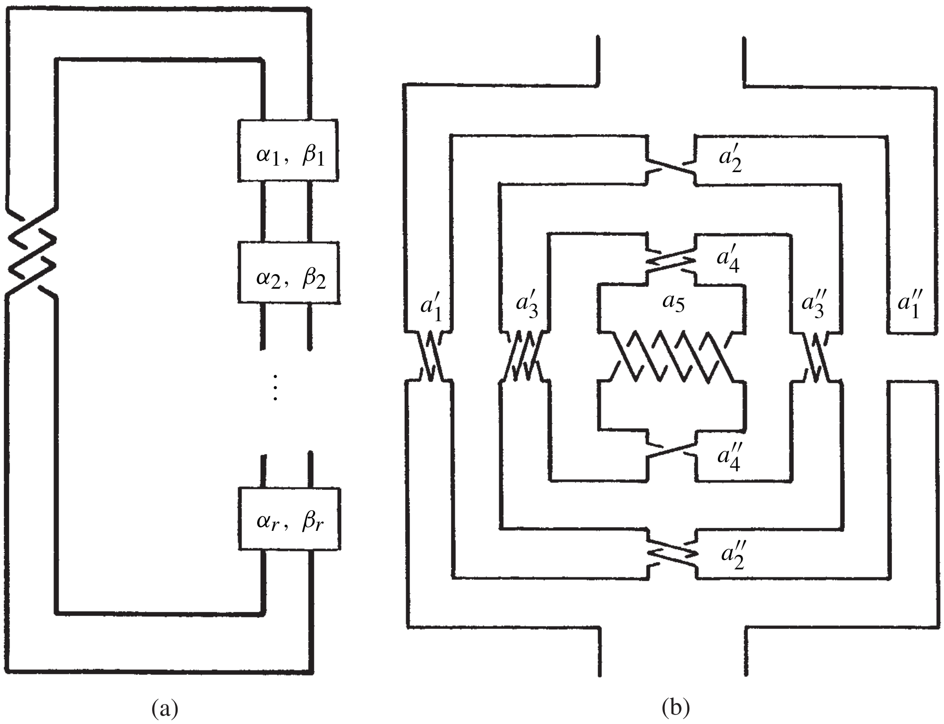


Figure 12.9

classify the rational tangles up to isotopies which leave the boundary of the box pointwise fixed.

It is easily seen that a rational tangle (α, β) is the intersection of the box with a 4-plat: there is an isotopy which reduces all twists a_j'' to 0-twists. A tangle in this position may gradually be deformed into a 4-plat working from the outside towards the inside. A rational tangle closed by two trivial bridges is a knot or link $\mathfrak{b}(\alpha, \beta)$, see the definition in 12.1 and Proposition 12.13. (Note that we excluded the trivial cases $\mathfrak{b}(0, 1)$ and $\mathfrak{b}(1, 0)$.)

12.29 Theorem (Classification of Montesinos links). *Montesinos links with r rational tangles, $r \geqq 3$ and $\sum_{j=1}^{r} \frac{1}{\alpha_j} \leqq r-2$, are classified by the ordered set of fractions $\left(\frac{\beta_1}{\alpha_1} \bmod 1, \ldots, \frac{\beta_r}{\alpha_r} \bmod 1\right)$, up to cyclic permutations and reversal of order, together with the rational number $e_0 = e + \sum_{j=1}^{r} \frac{\beta_j}{\alpha_j}$.*

This result was obtained by Bonahon [1979]. Another proof was given by Boileau and Siebenmann [1980]. The proof here follows the arguments of the latter, based on the method Seifert used to classify 2-bridge knots. We give a self-contained proof [Zieschang 1984] which does not use the classification of Seifert fibre spaces. We prove a special case of the Isomorphiesatz 3.7 in [Zieschang-Zimmermann 1982].

The *proof* of Theorem 12.29 will be finished in 12.38.

12.30 Another construction of Montesinos links. For the following construction we use Proposition 12.3. From S^3 we remove $r+1$ disjoint balls $B_0, B_1, \dots, B_r$ and consider two disjoint disks δ_1 and δ_2 in $\overline{S^3 - \bigcup_{i=1}^r B_i} = W$ where the boundary $\partial\delta_j$ intersects B_i in an arc $\varrho_{ji} = \partial B_i \cap \delta_j = B_i \cap \delta_j$. Assume that $\partial\delta_j = \varrho_{j0}\lambda_{j0}\varrho_{j1}\lambda_{j1}\dots\varrho_{jr}\lambda_{jr}$. In B_i let κ_{1i} and κ_{2i} define a tangle of type (α_i, β_i). We assume that in B_0 there is only an e-twist, that is $\alpha_0 = 1$, $\beta_0 = e$. *Then* $\bigcup(\lambda_{ji} \cup \kappa_{ji})$ $(j = 1, 2;\ i = 0, \dots, r)$ *is the Montesinos link* $\mathfrak{m}(e; (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r))$, see Figure 12.10 and Proposition 12.13.

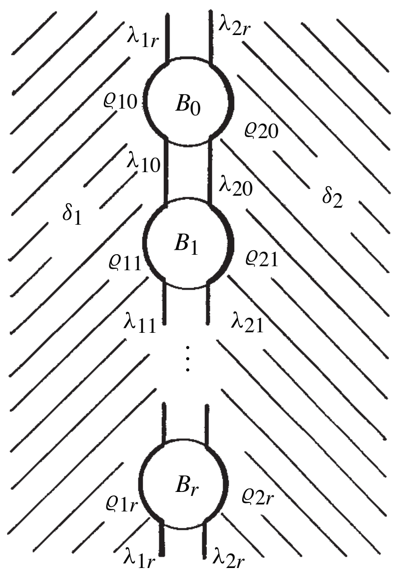


Figure 12.10

12.31 Proposition. (a) *The twofold branched covering* $\hat{C}_2$ *of* S^3 *branched over the Montesinos link* $\mathfrak{m}(e; \alpha_1/\beta_1, \dots, \alpha_r/\beta_r)$ *is a Seifert fibre space with the fundamental group*

12.32 $$\pi_1\hat{C}_2 = \langle h, s_1, \dots, s_r \mid s_i^{\alpha_i}h^{\beta_i}, [s_i, h]\ (1 \leqq i \leqq r), s_1 \dots s_r h^{-e} \rangle.$$

(b) *The covering transformation* Φ *of the twofold covering induces the automorphism*

12.33 $$\varphi\colon \pi_1\hat{C}_2 \to \pi_1\hat{C}_2, h \mapsto h^{-1}, s_i \mapsto s_1 \dots s_{i-1}s_i^{-1}s_{i-1}^{-1} \dots s_1^{-1} \quad (1 \leqq i \leqq r).$$

(c) *The covering transformations of the universal cover of $\hat{C}_2$ together with the lift c of Φ form a group $\mathfrak{H}$ with the following presentations:*

12.34
$$\begin{aligned}\mathfrak{H} &= \langle h, s_1, \dots, s_r, c \mid chc^{-1}h,\ cs_ic^{-1}\cdot(s_1\dots s_{i-1}s_is_{i-1}^{-1}\dots s_1^{-1}),\\ &\qquad [s_i,h], s_i^{\alpha_i}h^{\beta_i}\ (1 \leqq i \leqq r),\ s_1\dots s_rh^{-e},\ c^2\rangle\\ &= \langle h, c_0, \dots, c_r \mid c_i^2,\ c_ihc_i^{-1}h\ (0 \leqq i \leqq r),\\ &\qquad (c_{i-1}c_i)^{\alpha_i}h^{\beta_i}\ (1 \leqq i \leqq r),\ c_0^{-1}c_rh^{-e}\rangle.\end{aligned}$$

Proof. We use the notation of 12.30 and repeat the arguments of the proof 12.3. Cutting along δ_1, δ_2 turns W into the cartesian product $(\overline{D^2 - \bigcup_{i=1}^r D_i)}) \times I$ where D^2 is a 2-disk and the D_i are disjoint disks in D^2. The twofold covering T_r of W branched over the λ_{ji} is a solid torus with r parallel solid tori removed: $T_r = (\overline{D^2 - \bigcup_{i=1}^r D_i}) \times S^1$. The product defines an S^1-fibration of T_r. The covering transformation Φ is the rotation through 180^0 about the axis containing the arcs λ_{ji}, compare Figure 12.11.

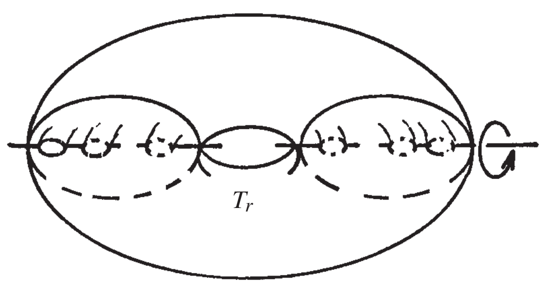


Figure 12.11

To calculate the fundamental group we choose the base point on the axis and on ∂B_0. Generators of $\pi_1 T_r$ are obtained from the curves shown in Figure 12.12, and

$$\pi_1 T_r = \langle h, s_0, s_1, \dots, s_r \mid [h, s_i]\ (0 \leqq i \leqq r),\ s_0s_1\dots s_r\rangle.$$

The covering transformation Φ maps the generators as described in Figure 12.12; hence $\Phi_*\colon \pi_1T_r \to \pi_1T_r$, $h \mapsto h^{-1}$, $s_0 \mapsto s_0^{-1}$, $s_1 \mapsto s_1^{-1}$, $s_2 \mapsto s_1s_2^{-1}s_1^{-1}, \dots, s_r \mapsto s_1\dots s_{r-1}s_r^{-1}s_{r-1}^{-1}\dots s_1^{-1}$. The twofold covering of B_i $(0 \leqq i \leqq r)$, branched over the arcs κ_{ji}, is a solid torus $\hat{V}_i$, see 12.3. Thus the twofold covering $\hat{C}_2$ of S^3 branched over $\mathfrak{m} = \mathfrak{m}(e; \alpha_1/\beta_1, \dots, \alpha_r/\beta_r)$ is $T_r \cup \bigcup_{i=0}^r \hat{V}_i = \hat{C}_2$ with corresponding boundaries identified. The fibration of T_r can be extended to the solid tori $\hat{V}_i$ as we have excluded the case $(\alpha_i, \beta_i) = (0, 1)$, and $\hat{C}_2$ obtains a Seifert fibration. Adding the solid tori $\hat{V}_i$ introduces the relations $s_i^{\alpha_i}h^{\beta_i}$ for $1 \leqq i \leqq r$ and s_0h^e. This finishes the proof of (a).

The proof of (b) follows from the effect of Φ on π_1T_r. The first presentation of 12.34 follows from 12.32 and 12.33 by interpreting $\pi_1\hat{C}_2$ as the group of covering

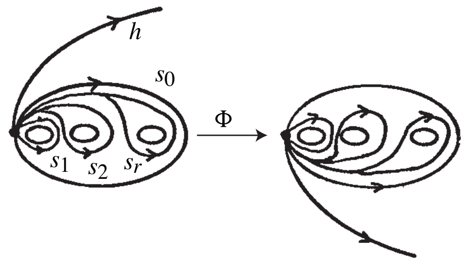


Figure 12.12

transformations of the universal covering of $\hat{C}_2$. It remains to show $c^2 = 1$. This follows from the fact that Φ has order 2 and admits the base point as a fixed point.

Define $c_i = cs_1 \dots s_i$ $(1 \leqq i \leqq r)$ and $c_0 = c$. Then $s_i = c_{i-1}^{-1} c_i$ $(1 \leqq i \leqq r)$ and

$$\begin{aligned}\mathfrak{H} = \langle\, h, c_0, \dots, c_r \mid\, & c_0 h c_0^{-1} h,\ c_0(c_{i-1}^{-1} c_i) c_0^{-1} \cdot c_0^{-1}(c_i c_{i-1}^{-1}) c_0, \\ & [c_{i-1}^{-1} c_i, h],\ (c_{i-1}^{-1} c_i)^{\alpha_i} h^{\beta_i}\ (1 \leqq i \leqq r), c_0^{-1} c_r h^{-e}, c_0^2 \,\rangle \\ = \langle\, h, c_0, \dots, c_r \mid\, & c_i h c_i^{-1} h, c_i^2\ (0 \leqq i \leqq r), \\ & (c_{i-1} c_i)^{\alpha_i} h^{\beta_i}\ (1 \leqq i \leqq r),\ c_0^{-1} c_r h^{-e} \,\rangle. \qquad \square\end{aligned}$$

12.35 Remark. For later use we note a geometric property of the twofold branched covering: The branch set $\hat{\mathfrak{m}}$ in $\hat{C}_2$ is the preimage of the Montesinos link $\mathfrak{m}$. From the construction of $\hat{C}_2$ it follows that $\hat{\mathfrak{m}}$ intersects each exceptional fibre exactly twice, in the "centres" of the pair of disks in Figure 12.11 belonging to one $\hat{V}_i$.

12.36 Lemma. *For* $\sum_{i=1}^{r} \frac{1}{\alpha_i} \leqq r-2$ *the element* h *in the presentation* 12.32 *of* $\pi_1 \hat{C}_2$ *generates an infinite cyclic group* $\langle\, h \,\rangle$*, the centre of* $\pi_1 \hat{C}_2$.

Proof. $\pi_1 \hat{C}_2 / \langle\, h \,\rangle$ is a discontinuous group with compact fundamental domain of motions of the euclidean plane, if equality holds in the hypothesis, otherwise of the non-euclidean plane, and all transformations preserve orientation; see [ZVC 1980, 4.5.6, 4.8.2]. In both cases the group is generated by rotations and there are r rotation centres which are pairwise non-equivalent under the action of $\pi_1 \hat{C}_2$. A consequence is that the centre of $\pi_1 \hat{C}_2 / \langle\, h \,\rangle$ is trivial, see [ZVC 1980, 4.8.1 c)]; hence, $\langle\, h \,\rangle$ is the centre of $\pi_1 \hat{C}_2$.

The proof that h has infinite order is more complicated. It is simple for $r > 3$. Then

$$\begin{aligned}\pi_1 \hat{C}_2 = {} & \langle\, h, s_1, s_2 \mid s_i^{\alpha_i} h^{\beta_i},\ [h, s_i]\ (1 \leqq i \leqq 2) \,\rangle \\ & *_{\mathbb{Z}^2} \langle\, h, s_3, \dots, s_r \mid s_i^{\alpha_i} h^{\beta_i},\ [h, s_i]\ (3 \leqq i \leqq r) \,\rangle\end{aligned}$$

where $\mathbb{Z}^2 \cong \langle h, s_1s_2 \rangle \cong \langle h, (s_3 \dots s_r)^{-1} \rangle$. (It easily follows by arguments on free products that the above subgroups are isomorphic to $\mathbb{Z}^2$.) In particular, $\langle h \rangle \cong \mathbb{Z}$.

To show the lemma for $r = 3$ we prove the following Theorem 12.37 by repeating the arguments of the proof of Theorem 3.30.

12.37 Theorem. *Let M be an orientable 3-manifold with no sphere in its boundary. If $\pi_1 M$ is infinite, non-cyclic, and not a free product then M is aspherical and $\pi_1 M$ is torsion-free.*

Proof. If $\pi_2 M \neq 0$ there is, by the Sphere Theorem [Papakyriakopoulos 1957′], Appendix B.6, [Hempel 1976, 4.3], an S^2, embedded in M, which is not nullhomotopic in M. If S^2 does not separate M then there is a simple closed curve λ that properly intersects S^2 in exactly one point. The regular neighbourhood U of $S^2 \cup \lambda$ is bounded by a separating 2-sphere. One has

$$\pi_1 M = \pi_1 U * \pi_1(\overline{N - U}) \cong \mathbb{Z} * \pi_1(\overline{N - U})$$

contradicting the assumptions that $\pi_1 M$ is neither cyclic nor a free product. Thus S^2 separates M into two manifolds M', M''. Since $\pi_1 M$ is not a free product we may assume that $\pi_1 M' = 1$. It follows that $\partial M' = S^2$, since by assumption every other boundary component is a surface of genus $\geqq 1$ and, therefore, $H_1(M') \neq 0$, see [Seifert–Threfall 1934, p. 223 Satz IV], contradicting $\pi_1 M' = 1$. This proves that S^2 is null-homologous in M'. Since $\pi_1 M' = 1$, it follows by the Hurewicz theorem, see [Spanier 1966, 7.5.2], that S^2 is nullhomotopic – a contradiction. This proves $\pi_2 M = 0$.

Now consider the universal cover $\tilde{M}$ of M. Since $|\pi_1 M| = \infty$, $\tilde{M}$ is not compact and this implies that $H_3(\tilde{M}) = 0$. Moreover

$$1 = \pi_1 \tilde{M}, \quad H_2(\tilde{M}) = \pi_2 \tilde{M} = \pi_2 M = 0.$$

By the Hurewicz theorem, $\pi_3 \tilde{M} \cong H_3(\tilde{M}) = 0$, and by induction $\pi_j \tilde{M} \cong H_j(\tilde{M}) = 0$ for $j \geqq 3$. Since $\pi_j M \cong \pi_j \tilde{M}$, the manifold M is aspherical and a $K(\pi_1 M, 1)$-space.

Assume that $\pi_1 M$ contains an element of finite order r. Then there is a cover M^+ of M with $\pi_1 M^+ \cong \mathbb{Z}_r$. Since $[\pi_1 M : \pi_1 M^+] = \infty$ we can apply the same argument as above to prove that M^+ is a $K(\mathbb{Z}_r, 1)$-space. This implies that $H_j(\mathbb{Z}_r) = H_j(M^+)$ for all $j \in \mathbb{N}$. Since the sequence of homology groups of a cyclic group has period 2, there are non-trivial homology groups in arbitrary high dimensions. (These results can be found in [Spanier 1966, 9.5].) This contradicts the fact that $H_j(M^+) = 0$ for $j \geqq 3$. □

To complete the proof of Lemma 12.36 it remains to show that $\pi_1 \hat{C}_2$ is not a proper free product. Otherwise it cannot have a non-trivial centre, that is, in that case $h = 1, \pi_1 \hat{C}_2 = \langle s_1, s_2 \mid s_1^{\alpha_1}, s_2^{\alpha_2}, (s_1s_2)^{\alpha_3} \rangle$. By the Grushko Theorem [ZVC 1980, 2.9.2, E 4.10] both factors of the free product have rank $\leqq 1$, and $\pi_1 \hat{C}_2$ is one of the groups $\mathbb{Z}_n * \mathbb{Z}_m$, $\mathbb{Z}_n * \mathbb{Z}$ or $\mathbb{Z} * \mathbb{Z}$. But in the group $\langle s_1, s_2 \mid s_1^{\alpha_1}, s_2^{\alpha_2}, (s_1 s_2)^{\alpha_3} \rangle$ there are three non-conjugate maximal finite subgroups, namely those generated by s_1, s_2 and $s_1 s_2$, respectively, (for a proof see [ZVC 1980, 4.8.1]), while there are at most 2 in the above free products of cyclic groups. This proves also that h is non-trivial; hence, by Theorem 12.37, h has infinite order. □

12.38. *Proof of the Classification Theorem* 12.29. Let $\mathfrak{H}'$ and $\mathfrak{H}$ be groups presented in the form of 12.34, and let $\psi\colon \mathfrak{H}' \to \mathfrak{H}$ be an isomorphism. By Lemma 12.36, $\psi(h') = h^{\varepsilon}$, $\varepsilon \in \{1, -1\}$, and ψ induces an isomorphism

$$\bar{\psi}\colon \mathfrak{C}' = \mathfrak{H}'/\langle h' \rangle \to \mathfrak{H}/\langle h \rangle = \mathfrak{C}.$$

The groups $\mathfrak{C}'$ and $\mathfrak{C}$ are crystallographic groups of the euclidean or non-euclidean plane E with compact fundamental region. Hence, $\bar{\psi}$ is induced by a homeomorphism $\chi\colon E/\mathfrak{C}' \to E/\mathfrak{C}$, see [ZVC 1980, 6.6.11]. Both surfaces $E/\mathfrak{C}'$ and $E/\mathfrak{C}$ are compact and have one boundary component, on which the images of the centres of the rotations $\bar{c}_1'\bar{c}_2', \bar{c}_2'\bar{c}_3', \dots, \bar{c}_r', \bar{c}_1'$ and $\bar{c}_1\bar{c}_2, \bar{c}_2\bar{c}_3, \dots, \bar{c}_r\bar{c}_1$, respectively, follow in this order, see [ZVC 1980, 4.6.3, 4]. (The induced mappings on the surfaces are denoted by *a* bar.) Now χ preserves or reverses this order up to a cyclic permutation, and it follows that $(\alpha_1', \dots, \alpha_{r'}')$ differs from $(\alpha_1, \dots, \alpha_r)$ or $(\alpha_r, \dots, \alpha_1)$ only by a cyclic permutation. In the first case χ preserves the direction of the rotations, in the second case it reverses it, and we obtain the following equations:

$$\bar{\psi}(\bar{s}_i') = \bar{x}_i \bar{s}_{j(i)}^{\eta} \bar{x}_i^{-1}, \quad \eta \in \{1, -1\},$$

$$\begin{pmatrix} 1 & \dots & r \\ j(1) & \dots & j(r) \end{pmatrix} \quad \text{a permutation with } \alpha_i' = \alpha_{j(i)}.$$

Moreover,

$$\bar{x}_1 \bar{s}_{j(1)}^{\eta} \bar{x}_1^{-1} \dots \bar{x}_r \bar{s}_{j(r)}^{\eta} \bar{x}_r^{-1} = \bar{x}(\bar{s}_1 \dots \bar{s}_r)^{\eta} \bar{x}^{-1}$$

in the free group generated by the $\bar{s}_i$, see [ZVC 1980, 5.8.2]. Hence, ψ is of the following form:

$$\psi(h') = h^{\varepsilon}, \quad \psi(s_i') = x_i s_{j(i)}^{\eta} x_i^{-1} h^{\lambda_i}, \quad \lambda_i \in \mathbb{Z},$$

where the x_i are the same words in the s_i as the $\bar{x}_i$ in the $\bar{s}_i$.

The orientation of S^3 determines orientations on the twofold branched covering spaces $\hat{C}_2'$ and $\hat{C}_2$. When the links $\mathfrak{m}'$ and $\mathfrak{m}$ are isotopic then there is an orientation preserving homeomorphism from $\hat{C}_2'$ to $\hat{C}_2$. This implies that $\varepsilon\eta = 1$, since the orientations of $\hat{C}_2'$ and $\hat{C}_2$ are defined by the orientations of the fibres and the bases and $\varepsilon = -1$ corresponds to a change of the orientation in the fibres while $\eta = -1$ corresponds to a change of the bases. Therefore,

$$\begin{aligned} h^{\varepsilon\beta_i'} &= \psi(h'^{\beta_i'}) = \psi(s_i'^{-\alpha_i'}) = x_i (s_{j(i)}^{\varepsilon} h^{\lambda_i})^{-\alpha_i'} x_i^{-1} \\ &= x_i s_{j(i)}^{-\varepsilon a_{j(i)}} x_i^{-1} h^{-\alpha_{j(i)}\lambda_i} = h^{\varepsilon\beta_{j(i)} - \alpha_{j(i)}\lambda_i}, \end{aligned}$$

that is,

$$\beta_i' = \beta_{j(i)} - \varepsilon\alpha_{j(i)}\lambda_i \quad \text{for } 1 \leqq i \leqq r.$$

This proves the invariance of the β_i/α_i and their ordering.

From the last relation we obtain:

$$\begin{aligned} h^{\varepsilon e'} &= \psi(h'^{e'}) = \psi(s_1' \dots s_r') = x_1 s_{j(1)}^{\varepsilon} x_1^{-1} h^{\lambda_1} \dots x_r s_{j(r)}^{\varepsilon} x_r^{-1} h^{\lambda_r} \\ &= h^{\lambda_1+\dots+\lambda_r} x(s_1 \dots s_r)^{\varepsilon} x^{-1} = h^{\lambda_1+\dots+\lambda_r+\varepsilon e}; \end{aligned}$$

hence,

$$e' = e + \varepsilon(\lambda_1 + \dots + \lambda_r).$$

Now,

$$e' + \sum_{i=1}^{r} \frac{\beta_i'}{\alpha_i'} = e + \varepsilon(\lambda_1 + \dots + \lambda_r) + \sum_{i=1}^{r} \frac{\beta_{j(i)} - \varepsilon\alpha_{j(i)}\lambda_i}{\alpha_{j(i)}} = e + \sum_{j=1}^{r} \frac{\beta_j}{\alpha_j}. \qquad \square$$

12.39 Remark. The "orbifold" $E/\mathfrak{C}$ of fibres is a disk with r marked vertices on the boundary. A consequence of 12.35 is that the image of $\hat{\mathfrak{m}}$ consists of the edges of the boundary of $E/\mathfrak{C}$. In other words, the fundamental domain of $\mathfrak{C}$ is an r-gon, the edges of which are the images of $\hat{\mathfrak{m}}$. Each component $\hat{\mathfrak{k}}$ of $\hat{\mathfrak{m}}$ determines an element of $\mathfrak{C}$ which is fixed when conjugated with a suitable reflection of $\mathfrak{C}$. The reflections of $\mathfrak{C}$ are conjugate to the reflections in the (euclidean or non-euclidean) lines containing the edges of the fundamental domain. From geometry we know that the reflection $\bar{c}$ with axis l fixes under conjugation the following orientation preserving mappings of E:

i) the rotations of order 2 with centres on l,

ii) the hyperbolic transformations with axis l.

Since the image of $\hat{\mathfrak{k}}$ contains the centres of different non-conjugate rotations of $\mathfrak{C}$ it follows that $\hat{\mathfrak{k}}$ determines, up to conjugacy, a hyperbolic transformation in $\mathfrak{C}$.

Improving slightly the proof of the Classification Theorem one obtains

12.40 Corollary. *If $\sum_{i=1}^{r} \frac{1}{\alpha_i} < r-2$, that is, $\mathfrak{C} = \mathfrak{H}/\langle h \rangle$ is a non-euclidean crystallographic group, each automorphism of $\mathfrak{H}$ is induced by a homeomorphism of $E \times \mathbb{R}$.*

Proofs can be found in [Conner-Raymond 1970, 1977], [Kamishima-Lee-Raymond 1983], [Lee-Raymond 1984], [Zieschang-Zimmermann 1982, 2.10]. $\square$

Moreover, the outer automorphism group of $\mathfrak{H}$ can be realized by a group of homeomorphisms. This can be seen directly by looking at the corresponding extensions of $\mathfrak{H}$ and realizing them by groups of mappings of $E \times \mathbb{R}$, see the papers mentioned above.

E Symmetries of Montesinos Links

Using the Classification Theorem 12.29 and 12.40 we can easily decide about amphicheirality and invertibility of Montesinos links.

12.41 Proposition (Amphicheiral Montesinos links). (a) *The Montesinos link*

$$\mathfrak{m}(e_0; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r), \quad r \geqq 3,$$

is amphicheiral if and only if

1. $e_0 = 0$ *and*
2. *there is a permutation* π *– an* r*-cycle or a reversal of the ordering – such that*

$$\beta_{\pi(i)}/\alpha_{\pi(i)} \equiv -\beta_i/\alpha_i \mod 1 \qquad \text{for } 1 \leqq i \leqq r.$$

(b) *For* $r \geqq 3$*,* r *odd, Montesinos knots are never amphicheiral.*

Proof. The reflection in the plane maps $\mathfrak{m}$ to the Montesinos link

$$\mathfrak{m}(-e_0; -\beta_1/\alpha_1, \ldots, -\beta_r/\alpha_r);$$

hence, (a) is a consequence of the Classification Theorem 12.29. Proof of (b) as Exercise E 12.7 □

A *link* $\mathfrak{l}$ is called *invertible*, see [Whitten 1969, 1969′], if there exists a homeomorphism f of S^3 which maps each component of $\mathfrak{l}$ into itself reversing the orientation. Let us use this term also for the case where f maps each component of $\mathfrak{l}$ into itself and reverses the orientation of at least one of them. In the following proof we will see that both concepts coincide for Montesinos links.

12.42 Theorem (Invertible Montesinos links). *The Montesinos link*

$$\mathfrak{m} = \mathfrak{m}(e_0; \beta_1/\alpha_1, \ldots, \beta_r/\alpha_r), \quad r \geqq 3,$$

is invertible if and only if, with an appropriate enumeration,

(a) *at least one of the* α_i*,* $1 \leqq i \leqq r$*, is even, or*

(b) *all* α_i *are odd and there are three possibilities:*

$\mathfrak{m} = \mathfrak{m}(e_0; \beta_1/\alpha_1, \ldots, \beta_p/\alpha_p, \beta_p/\alpha_p, \ldots, \beta_1/\alpha_1)$ *when* $r = 2p$*, or*
$\mathfrak{m} = \mathfrak{m}(e_0; \beta_1/\alpha_1, \ldots, \beta_p/\alpha_p, \beta_{p+1}/\alpha_{p+1}, \beta_p/\alpha_p, \ldots, \beta_1/\alpha_1)$ *when* $r = 2p + 1$;
$\mathfrak{m} = \mathfrak{m}(e_0; \beta_1/\alpha_1, \ldots, \beta_p/\alpha_p, \beta_{p+1}/\alpha_{p+1}, \beta_p/\alpha_p, \ldots, \beta_2/\alpha_2)$ *when* $r = 2p$.

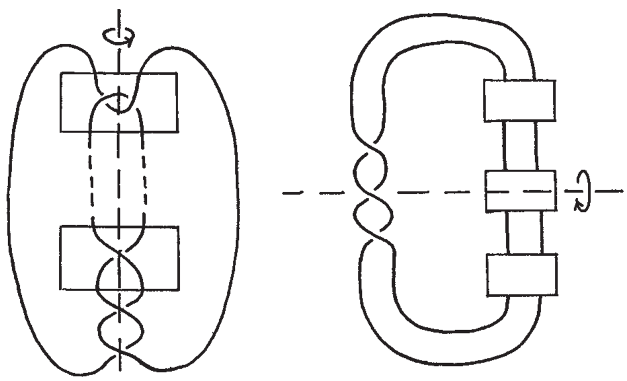


Figure 12.13

Proof. That the conditions (a) or (b) are sufficient follows easily from 12.40 (and the corresponding result for the euclidean cases) or from Figure 12.13.

For case (a), the rotation through 180° about the dotted line maps the Montesinos link onto an equivalent one. If α_i is even, a component of $\mathfrak{m}$ enters the i-th box from above and leaves it in the same direction. The rotation inverts the components. In case (b) the rotation through 180° shown in Figure 12.13 (b) gives the required symmetry.

For the proof that the conditions are necessary we may restrict ourselves to the case where $\mathfrak{C}$ operates on the hyperbolic plane $\mathbb{H}$, since in the euclidean cases either an exponent 2 occurs or all α_i are equal to 3 and the links are invertible. Let $f\colon S^3 \to S^3$ be an orientation preserving homeomorphism that maps $\mathfrak{m}$ onto $\mathfrak{m}$ and maps one component $\mathfrak{k}$ of $\mathfrak{m}$ onto itself, but reverses the orientation on $\mathfrak{k}$. Then, after a suitable choice of the base point, f induces an automorphism φ of $\mathfrak{C}$ that maps the element $k \in \mathfrak{C}$ defined by $\mathfrak{k}$ into its inverse. By 12.39, k is a hyperbolic transformation.

If φ is the inner automorphism $x \mapsto g^{-1}xg$ of $\mathfrak{C}$ then g has a fixed point on the axis A of k. Hence, g is either a rotation of order 2 with centre on A or g is the reflection with an axis perpendicular to A. In both cases $\mathfrak{C}$ contains an element of even order, i.e. one of the α_i is even.

If φ is not an inner automorphism then φ corresponds to a rotation or a reflection of the disk $\mathbb{H}/\mathfrak{C}$ that preserves the fractions β_i/α_i. It must reverse the orientation since the direction of one of the edges of $\mathbb{H}/\mathfrak{C}$ is reversed. Therefore φ corresponds to a reflection of the disk and this implies (b). □

Next we study the isotopy classes of symmetries of a Montesinos link $\mathfrak{m}$ with $r \geqq 3$ tangles, in other words, we study the group $\mathfrak{M}(S^3, \mathfrak{m})$ of mapping classes of the pair $(S^3, \mathfrak{m})$. This group can be described as follows: using the compact-open topology on the set of homeomorphisms or diffeomorphisms of $(S^3, \mathfrak{m})$ we obtain topological spaces $\mathrm{Homeo}(S^3, \mathfrak{m})$ and $\mathrm{Diff}(S^3, \mathfrak{m})$, respectively. Now $\mathfrak{M}(S^3, \mathfrak{m})$ equals the set of path-components of the above spaces:

12.43 $\mathfrak{M}(S^3, \mathfrak{m}) \cong \pi_0 \text{ Homeo}(S^3, \mathfrak{m}) \cong \pi_0 \text{ Diff}(S^3, \mathfrak{m})$.

Each symmetry induces an automorphism of the knot group $\mathfrak{G}$ which maps the kernel of the homomorphism $\mathfrak{G} \to \mathbb{Z}_2$ onto itself, and maps meridians to meridians; hence, symmetries and isotopies can be lifted to the twofold branched covering $\hat{C}_2$ such that the liftings commute with the covering transformation of $\hat{C}_2 \to S^3$. Lifting a symmetry to the universal cover $\mathbb{H} \times \mathbb{R}$ of $\hat{C}_2$ yields a homeomorphism

12.44 $\gamma : \mathfrak{M}(S^3, \mathfrak{m}) \to \text{Out } \mathfrak{H} = \text{Aut } \mathfrak{H}/\text{Inn } \mathfrak{H}$,

where $\mathfrak{H}$ has a presentation of the form 12.34. The fundamental assertion is:

12.45 Proposition. $\gamma : \mathfrak{M}(S^3, \mathfrak{m}) \to \text{Out } \mathfrak{H}$ *is an isomorphism.*

Unfortunately, we cannot give a self-contained proof here, but have to use results of Thurston and others which are not common knowledge. But this proof shows the influence of these theorems on knot theory. An explicit and simple description of Out $\mathfrak{H}$ is given afterwards in 12.47.

Proof ([Boileau-Zimmermann 1987]). Consider first the case $\sum_{i=1}^{r} \frac{1}{\alpha_i} < r-2$. From 12.40 it follows that γ is surjective and it remains to show that γ is injective. By Bonahon–Siebenmann, $\mathfrak{m}$ is a simple knot, that means, $\mathfrak{m}$ does not have a companion. By [Thurston 1997], $S^3 - \mathfrak{m}$ has a complete hyperbolic structure with finite volume. Mostow's rigidity theorem [Mostow 1968] implies that $\mathfrak{M}(S^3, \mathfrak{m})$ is finite and that every element of $\mathfrak{M}(S^3, \mathfrak{m})$ can be represented by an isometry of the same order as its homotopy class. Now we represent a non-trivial element of the kernel of γ by a homeomorphism f with the above properties. Let $\bar{f}$ be the lift of f to $\mathbb{H} \times \mathbb{R}$; then $\bar{f}^m \in \mathfrak{H}$ for a suitable $m > 0$. Since the class of f is in the kernel of γ we may assume that the conjugation by $\bar{f}$ yields the identity in $\mathfrak{H}$. As the centre of $\mathfrak{H}$ is trivial it follows that $\bar{f}^m = \text{id}_{\mathbb{H} \times \mathbb{R}}$ and, thus, that $\bar{f}$ is a periodic diffeomorphism commuting with the operation of $\pi_1 \hat{C}_2$. Therefore $\bar{f}$ is a rotation of the hyperbolic 3-space and its fixed point set is a line. The elements of $\pi_1 \hat{C}_2$ commute with $\bar{f}$; hence, they map the axis of $\bar{f}$ onto itself and it follows from the discontinuity that $\pi_1 \hat{C}_2$ is infinite cyclic or dihedral. This is a contradiction. Therefore γ is injective.

The euclidean cases $\left(\sum_{i=1}^{r} \frac{1}{\alpha_i} = r-2\right)$ are left. There are four cases: (3,3,3), (2,3,6), (2,4,4) and (2,2,2,2). They can be handled using the results of Bonahon–Siebenmann and [Zimmermann 1982]. The last paper depends strongly on Thurston's approach [1997] which we used above, and, furthermore, on [Jaco-Shalen 1979], [Johannson 1979]. □

Next we determine Out $\mathfrak{H}$ for the knot $\mathfrak{m}(e; (\alpha_1, \beta_1), \ldots, (\alpha_r, \beta_r))$. We assume $0 < \beta_j < \alpha_j$, $1 \leqq j \leqq r$, for the sake of simplicity.

12.46 Definition. Let $\mathbb{D}_r$, denote the dihedral group of order $2r$, realized as a group of rotations and reflections of a regular polygon with vertices (1,2,. . . ,r). Define

$$\tilde{\mathbb{D}}_r = \{\varrho \in \mathbb{D}_r \mid \alpha_{\varrho(i)} = \alpha_i \text{ for } 1 \leqq i \leqq r\}.$$

Let $\breve{\mathbb{D}}_r \subset \tilde{\mathbb{D}}_r$ consist of

(i) the rotations ϱ with $\alpha_{\varrho(i)} = \alpha_i$ and $\beta_{\varrho(i)} = \beta_i$ and the reflections ϱ with $\alpha_{\varrho(i)} = \alpha_i$ and $\beta_{\varrho(i)} = \alpha_i - \beta_i$ if $e_0 \neq 0$,

(ii) the rotations ϱ with $(\alpha_{\varrho(i)}, \beta_{\varrho(i)}) = (\alpha_i, \alpha_i - \beta_i)$ and the reflections ϱ with $(\alpha_{\varrho(i)}, \beta_{\varrho(i)}) = (\alpha_i, \beta_i)$ if $e_0 = 0$.

12.47 Proposition. Out $\mathfrak{H}$ *is an extension of* $\mathbb{Z}_2$ *or* $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *by the finite dihedral or cyclic group* $\breve{\mathbb{D}}_r$.

Proposition 12.47 is a direct consequence of the following Lemmas 12.48 and 12.50.

Since $\langle h \rangle$ is the centre of $\mathfrak{H}$, the projection $\mathfrak{H} \to \mathfrak{H}/\langle h \rangle = \mathfrak{C}$ is compatible with every automorphism of $\mathfrak{H}$ and we obtain a homomorphism $\chi :$ Out $\mathfrak{H} \to$ Out $\mathfrak{C}$. It is easy to determine the image of χ; thus the main problem is to calculate the kernel.

Consider an automorphism ψ of $\mathfrak{H}$ which induces the identity on $\mathfrak{C}$. Then $\psi(c_j) = c_j h^{m_j}$, $\psi(h) = h^\varepsilon$ where $\varepsilon \in \{1, -1\}$, and

$$\begin{aligned} h^{\varepsilon\beta_1} &= \psi(h^{\beta_1}) = \psi((c_0c_1)^{-\alpha_1}) = (c_0h^{m_0}c_1h^{m_1})^{-\alpha_1} \\ &= (c_0c_1)^{-\alpha_1}h^{-\alpha_1(m_1-m_0)} = h^{\beta_1-\alpha_1(m_1-m_0)}. \end{aligned} \tag{1}$$

1. *Case* $\varepsilon = 1$. Since h has infinite order it follows that $m_1 = m_0$ and, by copying this argument, $m_0 = m_1 = \cdots = m_r = 2l + \eta$ with $\eta \in \{0, 1\}$. Now multiply ψ by the inner automorphism $x \mapsto h^l x h^{-l}$:

$$c_j \mapsto h^l c_j h^{-l} \mapsto h^l c_j h^{m_0} h^{-l} = c_j h^\eta;$$

hence, these automorphisms define a subgroup of ker χ isomorphic to $\mathbb{Z}_2$.

2. *Case* $\varepsilon = -1$. Now $2\beta_1 = -\alpha_1(m_0 - m_1)$ by (1). Since α_1 and β_1 are relatively prime and $0 < \beta_1 < \alpha_1$ it follows that $\alpha_1 = 2$, $\beta_1 = 1$ and $m_0 = m_1 - 1$. By induction: $\alpha_1 = \cdots = \alpha_r = 2$, $\beta_1 = \cdots = \beta_r = 1$, $m_j = m_0 + j$ for $1 \leqq j \leqq r$. Now

$$h^{-e} = \psi(h^e) = \psi(c_0^{-1}c_r) = h^{-m_0}c_0^{-1}c_r h^{m_r} = h^{e+m_r-m_0} = h^{e+r}.$$

It follows that $e = -\frac{r}{2}$ and that the Euler number e_0 vanishes:

$$e_0 = e + \sum_{j=1}^{r} \frac{\beta_j}{\alpha_j} = 0.$$

Thus we have proved

12.48 Lemma. $\ker \chi \cong \mathbb{Z}_2$ *is generated by* $\psi_0 \colon \mathfrak{H} \to \mathfrak{H}$, $c_j \mapsto c_j h$, $h \mapsto h$, *except in the case where* $(\alpha_j, \beta_j) = (2, 1)$ *for* $1 \leqq j \leqq r$ *and* $e_0 = 0$*; then* $\ker \chi \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, *generated by* ψ_0 *and* $\psi_1 \colon \mathfrak{H} \to \mathfrak{H}$, $h \mapsto h^{-1}$, $c_j \mapsto c_j h^{r-j}$. □

Using the generalized Nielsen theorem, see [ZVC 1980, 5.8.3, 6.6.9], Out $\mathfrak{C}$ is easily calculated:

12.49 Lemma. (a) *An automorphism* $\varphi \colon \mathfrak{C} \to \mathfrak{C}$ *mapping each conjugacy class of elliptic subgroups* $\langle (c_j c_{j+1}) \rangle$ *onto itself is an inner automorphism of* $\mathfrak{C}$.

(b) *The canonical mapping* $\tilde{\mathbb{D}}_r \to \operatorname{Out} \mathfrak{C}$ *is an isomorphism.*

Proof. By the generalized Nielsen theorem, see [ZVC 1980, 6.6.11], φ is induced by a homeomorphism f of $\mathbb{H}/\mathfrak{C} \cong D^2$ onto itself which fixes the rotation centres lying on ∂D^2. Now the Alexander trick [Alexander 1923] can be used to isotope f into the identity. This implies that φ is an inner automorphism. □

12.50 Lemma. *The image of* $\operatorname{Out} \mathfrak{H}$ *in* $\operatorname{Out} \mathfrak{C}$ *is the subgroup* $\breve{\mathbb{D}}_r$ *of* $\tilde{\mathbb{D}}_r$.

Proof. Let φ be an automorphism of $\mathfrak{H}$. By 12.49 (b), φ induces a 'dihedral' permutation π of the cyclic set $\bar{c}_1, \dots, \bar{c}_{r-1}, \bar{c}_r = \bar{c}_0$. We discuss the cases with $\varphi(h) = h$.

1. π is a rotation. Then

$$\begin{aligned} h^{\beta_1} &= \varphi(h^{\beta_1}) = \varphi((c_0 c_1)^{-\alpha_1}) = (c_{i-1} h^{m_0} c_i h^{m_1})^{-\alpha_1} \\ &= h^{\beta_i - \alpha_1(m_1 - m_0)} \end{aligned}$$

and $\alpha_i = \alpha_1$. Since, by assumption, $0 < \beta_i < \alpha_i$, it follows that $m_1 = m_0$ and $\beta_1 = \beta_i$. Therefore φ preserves the pairs (α_j, β_j) and maps c_j to $c_j h^m$ for a fixed m. By multiplication with an inner automorphism and, if necessary, with ψ_1 from 12.48 we obtain $m = 0$. The image of φ in $\operatorname{Out} \mathfrak{C}$ is in $\breve{\mathbb{D}}_r$, and each rotation $\pi \in \breve{\mathbb{D}}_r$ is obtained from a $\varphi \in \operatorname{Out} \mathfrak{H}$.

2. π is a reflection. Then

$$\begin{aligned} h^{\beta_1} &= \varphi(h^{\beta_1}) = \varphi((c_0 c_1)^{-\alpha_1}) = (c_i h^{m_0} c_{i-1} h^{m_1})^{-\alpha_1} \\ &= h^{-\beta_i - \alpha_1(m_1 - m_0)} \end{aligned}$$

and $\alpha_1 = \alpha_i$. Therefore $m_1 - m_0 = -1$, $\beta_i + \beta_1 = \alpha_1$, and φ assigns to a pair (α_k, β_k) a pair $(\alpha_j, \beta_j) = (\alpha_k, \alpha_k - \beta_k)$. The generators c_i are mapped as follows:

$$\begin{array}{ccccccccc} c_0 & c_1 & \dots & c_i & & c_{i+1} & & \dots & c_r \\ \downarrow & \downarrow & & \downarrow & & \downarrow & & & \downarrow \\ c_i h^m & c_{i-1} h^{m-1} & \dots & c_0 h^{m-1} & & c_{r-1} h^{-e+m-i-1} & & \dots & c_i h^{-e+m-r} \\ & & & \| & & & & & \\ & & & c_r h^{-e+m+i} & & & & & \end{array}$$

and
$$c_i h^{-e+m-r} = \varphi(c_r) = \varphi(c_0 h^e) = c_i h^{m+e}.$$
This implies $e = -\frac{r}{2}$ and
$$e_0 = e + \sum_{j=1}^{r} \frac{\beta_j}{\alpha_j} = e + \frac{1}{2} \sum_{j=1}^{r} \left(\frac{\beta_j}{\alpha_j} + \frac{a_{i-j} - \beta_{i-j}}{\alpha_j} \right) = 0;$$
here $i - j$ is considered $\bmod r$. By normalizing as before we obtain $m = 0$.

The cases for $\varphi(h) = h^{-1}$ can be handled the same way; proof as E 12.8. □

Lemmas 12.48 and 12.50 imply Proposition 12.47. As a corollary of Proposition 12.45 and 12.47 we obtain the following results of Bonahon and Siebenmann (for $r \geqq 4$) and Boileau (for $r = 3$).

12.51 Corollary. *The symmetry group $\mathfrak{M}(S^3, \mathfrak{m})$ is an extension of $\mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, by the finite dihedral or cyclic group $\mathbb{D}_r$.* □

F History and Sources

4-plats (Viergeflechte) were first investigated in [Bankwitz-Schumann 1934] where they were shown to be alternating and invertible. They were classified by H. Schubert [1956] as knots and links with two bridges. A different proof using linking numbers of covering spaces was given in [Burde 1975]. Special properties of 2-bridge knots (genus, Alexander polynomial, fibring, group structure) were studied in [Funcke 1975, 1978], [Hartley 1979′], [Mayland 1976].

J. Montesinos then introduced a more general class of knots and links which could nevertheless be classified by essentially the same trick that H. Seifert had used to classify (unoriented) knots with two bridges: Montesinos links are links with 2-fold branched covering spaces which are Seifert fibre spaces, see [Montesinos 1973, 1979], [Boileau-Siebenmann 1980], [Zieschang 1984]. In other papers on Montesinos links their group of symmetries was determined in most cases (Bonahon and Siebenmann, [Boileau-Zimmermann 1987]).

G Exercises

E 12.1. Show that a reduced diagram of $\mathfrak{b}(\alpha, \beta)$ leads to the following Wirtinger presentations:
$$\mathfrak{G}(\alpha, \beta) = \langle\, S_1, S_2 \mid S_2^{-1} L_1^{-1} S_1 L_1 \,\rangle,$$

with

$$L_1 = S_2^{\varepsilon_1} S_1^{\varepsilon_2} \dots S_2^{\varepsilon_{\alpha-2}} S_1^{\varepsilon_{\alpha-1}}, \quad \alpha \equiv 1 \mod 2,$$

$$\mathfrak{G}(\alpha, \beta) = \langle\, S_1, S_2 \mid S_1^{-1} L_1^{-1} S_1 L_1 \,\rangle,$$

with

$$L_1 = S_2^{\varepsilon_1} S_1^{\varepsilon_2} \dots S_1^{\varepsilon_{\alpha-2}} S_2^{\varepsilon_{\alpha-1}}, \quad \alpha \equiv 1 \mod 2,$$

here $\varepsilon_i = (-1)^{\left[\frac{i\beta}{\alpha}\right]}$, $[\alpha] =$ integral part of α.

E 12.2. The matrices

$$A_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

generate the mapping class group of the torus (Section B). Show that $\langle\, A_1, A_2 \mid A_1A_2A_1 = A_2A_1A_2, (A_1A_2)^6 \,\rangle$ is a presentation of the group $\mathrm{SL}(2, \mathbb{Z})$ and connect it with the classical presentation

$$\mathrm{SL}(2, \mathbb{Z}) = \langle\, S, T, Z \mid S^2 = T^3 = Z,\ Z^2 = 1 \,\rangle.$$

E 12.3. Let α, β, β' be positive integers, $\gcd(\alpha, \beta) = \gcd(\alpha, \beta') = 1$ and $\beta\beta' \equiv 1$ mod α. If $\beta \cdot \alpha^{-1} = [a_1, \dots, a_m]$, are the quotients of the continued fraction $\beta \cdot \alpha^{-1}$ of odd length m, then $\beta' \cdot \alpha^{-1} = [a_m, \dots, a_1]$. (Find an algebraic proof.)

E 12.4. Let α, β, β' (α odd) be integers as in E 12.3, and let $\beta \cdot \alpha^{-1} = [a_1, \dots, a_k]$ be the quotients obtained from the generalized algorithm 12.17. Prove: If $\mathfrak{b}(\alpha, \beta)$ is a fibred knot, then for $\varepsilon = (-1)^{k+1}$:

$$\beta'\alpha^{-1} = [\varepsilon a_k - 1, \varepsilon a_{k-1}, \dots, \varepsilon a_2, \varepsilon a_1 + 1].$$

E 12.5. Compute a Seifert matrix $V(\alpha, \beta)$ for $\mathfrak{b}(\alpha, \beta)$ using a Seifert surface as described in 12.27. Prove

(a) $|\det V(\alpha, \beta)| = \prod_{i=1}^{k-1} \left[b_i + \frac{1}{2} \left(\frac{a_i}{|a_i|} + \frac{a_{i+1}}{|a_{i+1}|} \right) \right]$,

(b) $\sigma[V(\alpha, \beta) + V^T(\alpha, \beta)] = \left(\sum_{i=1}^{k} a_i \right) - \frac{a_k}{|a_k|}$.

(σ denotes the signature of a matrix, see Appendix A.2.) Deduce 12.24 from (a).

E 12.6. Prove 12.22.

E 12.7. Prove 12.41 (b).

E 12.8. Finish the proof of 12.50 for the case $\varphi(h) = h^{-1}$.

E 12.9. Prove that Montesinos knots are prime. (Use the Smith conjecture for involutions.)

Chapter 13
Quadratic Forms of a Knot

In this chapter we propose to reinvestigate the infinite cyclic covering C_∞ of a knot and to extract another knot invariant from it: the quadratic form of the knot. The first section gives a cohomological definition of the quadratic form $q(x)$ of a knot. The main properties of $q(x)$ and its signature are derived. The second part is devoted to the description of a method of computation of $q(x)$ from a special knot projection. Part C then compares the different quadratic forms of Goeritz [1933], Trotter [1962], Murasugi [1965], and Milnor–Erle [1969]. Some examples are discussed.

A The Quadratic Form of a Knot

In Proposition 8.9 we have determined the integral homology groups $H_i(C_\infty)$, $H_i(C_\infty, \partial C_\infty)$ of the infinite cyclic covering C_∞ of a knot $\mathfrak{k}$. It will become necessary to consider these homology groups with more general coefficients. Let A be an integral domain with identity. Then:

$$H_i(X, Y; A) = H_i(X, Y) \otimes_{\mathbb{Z}} A$$

for a pair $Y \subset X$. So we have:

13.1 Proposition. *Let A be an integral domain with identity and C_∞ the infinite cyclic covering of a knot $\mathfrak{k}$, Then*

$$\begin{aligned} &H_1(C_\infty; A) \cong H_1(C_\infty, \partial C_\infty; A), \\ &H_2(C_\infty, \partial C_\infty; A) \cong A. \end{aligned} \qquad \square$$

As we use throughout this chapter homology (and cohomology) with coefficients in A, this will be omitted in our notation.

Again, as in Chapter 8, we start by cutting the knot complement C along a Seifert surface S. Let $\{a_i \mid 1 \leqq i \leqq 2g\}$ (see Chapter B) be a canonical set of generators of $H_1(S)$, where g denotes the genus of S. The cutting produces the surfaces S^+ and S^- contained in ∂C^*. We assume S^3, and hence, C and C^* oriented; the induced orientation on S^- is supposed to induce on ∂S^- an orientation which coincides with that of $\mathfrak{k}$. The orientation of S^+ then induces the orientation of $-\mathfrak{k}$. The canonical curves $\{a_i\}$ become canonical curves $\{a_i^+\}$, $\{a_i^-\}$ on S^+, S^-. The space C^* is the complement of a handlebody of genus $2g$ in S^3 and there are $2g$ free generators

$\{s_k\}$ of $H_1(C^*)$ associated to $\{a_i\}$ by linking numbers in S^3 (as usual δ_{ik} denotes the Kronecker symbol):

$$\operatorname{lk}(a_i, s_k) = \delta_{ik}, \quad i, k = 1, \dots, 2g.$$

One sees easily that the s_k are determined by the a_i, if the above condition is imposed on their linking matrix: For $s'_j = \sum_k \alpha_{kj} s_k$ and $\delta_{ij} = \operatorname{lk}(a_i, s'_j)$, we get $\delta_{ij} = \operatorname{lk}(a_i, \sum_k \alpha_{kj} s_k) = \sum_k \alpha_{kj}\delta_{ik} = \alpha_{ij}$.

We are now going to free ourselves from the geometrically defined canonical bases $\{a_i\}$ of S and introduce a more general concept of a Seifert matrix $V = (v_{ik})$ (see Chapter 8).

13.2 Definition. Let $\{a_i \mid 1 \leqq i \leqq 2g\}$ be a basis of $H_1(S)$. A basis $\{s_i \mid 1 \leqq i \leqq 2g\}$ of $H_1(C^*)$ is called an *associated basis* with respect to $\{a_i\}$, if $\operatorname{lk}(a_i, s_k) = \delta_{ik}$. The matrix $V = (v_{ik})$ defined by the inclusion

$$i^- : S^- \to C^*, i^-_*(a^-_i) = \sum v_{ik} s_k$$

is called a *Seifert matrix*.

To abbreviate notations we use vectors $s = (s_k)$, $a = (a_i)$, $a^\pm = (a^\pm_i)$ etc. In Chapter 8 we have used special associated bases a and s derived from a band projection. For these we deduced $i^+_*(a^+) = V^T s$ from $i^-_*(a^-) = Vs$. Moreover, in this case $V - V^T = F$ represents the intersection matrix of the canonical basis $\{a_i\}$, if a suitable convention concerning the sign of the intersection numbers is agreed upon. The following proposition shows that these assertions remain true in the general case.

13.3 Proposition. *Let a, s be associated bases of $H_1(S)$, $H_1(C^*)$, respectively. If $i^-_*(a^-) = Vs$ then $i^+_*(a^+) = V^T s$. Moreover $V - V^T$ is the intersection matrix of the basis $a = (a_i)$.*

Proof. Let $\tilde{a}$, $\tilde{s}$ be the special bases of a band projection with $i^-_*(\tilde{a}^-) = \tilde{V}\tilde{s}$, and a, s another pair of associated bases, $a = C\tilde{a}$, $s = D\tilde{s}$, C, D unimodular $2g \times 2g$-matrices. From $\operatorname{lk}(a_i, s_k) = \operatorname{lk}(\tilde{a}_i, \tilde{s}_k) = \delta_{ik}$ we get $D = (C^{-1})^T$. We have $a^\pm = C\tilde{a}^\pm$; $i^-_*(a^-) = Vs$ implies $C\tilde{V}C^T s = i^-_*(a^-) = Vs$, and hence $V = C\tilde{V}C^T$. Now $i^+_*(a^+) = C\tilde{V}^T C^T s = V^T s$ follows. From this we get the transformation rule

$$C(V - V^T)C^T = \tilde{V} - \tilde{V}^T = F$$

which reveals $V - V^T$ as intersection matrix relative to the basis a. □

We shall use the following

13.4 Definition. Two symmetric $n \times n$-matrices M, M' over A are called *A-equivalent* if there is an A-unimodular matrix P – a matrix over A with $\det P$ a unit of A – with $M' = PMP^T$.

We use the term equivalent instead of $\mathbb{Z}$-equivalent.

13.5 Lemma (Trotter, Erle). *Let A be an integral domain with identity in which $\Delta(0)$ is a unity. ($\Delta(t)$ denotes the Alexander polynomial of a knot $\mathfrak{k}$). Every Seifert matrix V is A-equivalent to a matrix*

$$\begin{pmatrix} U & 0 \\ 0 & W \end{pmatrix}$$

where W is a $2m \times 2m$ integral matrix, $|W| = \det W \neq 0$, and U is of the form

$$U = \begin{pmatrix} 0 & & & & & & \\ -1 & 0 & & & & 0 & \\ 0 & * & 0 & & & & \\ \vdots & \vdots & -1 & 0 & & & \\ & & 0 & * & \ddots & & \\ & & \vdots & \vdots & & 0 & \\ 0 & * & 0 & * & \cdots & -1 & 0 \end{pmatrix}.$$

(*W is called a "reduced" Seifert matrix and may be empty.*)

Proof. If $|V| \neq 0$, V itself is reduced and nothing has to be proved. Let us assume $|V| = 0$. There are unimodular matrices Q and R such that QVR will have a first row of zeroes. The same holds for $QVQ^T = QVRR^{-1}Q^T$. Since $F = V - V^T$ is unimodular and skew-symmetric, so ist $QVQ^T - (QVQ^T)^T = QFQ^T$. Therefore its first column has a zero at the top and the remaining entries are relatively prime. But the first column of QFQ^T coincides with that of QVQ^T, because $(QVQ^T)^T$ has zero entries in its first column. So there is a unimodular R such that

$$RQVQ^T = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -1 & * & \cdots & * \\ 0 & \vdots & & \vdots \\ \vdots & & & \\ 0 & * & \cdots & * \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{pmatrix}.$$

To find R look for the element of smallest absolute value in the first column of QVQ^T. Subtract its row from other rows until a smaller element turns up in the first column. Since the elements of the first column are relatively prime one ends up with an element ± 1; the desired form is then easily reached. The operations on the rows can be realized by premultiplication by R. The matrix $RQVQ^TR^T$ has the same first row and column as $RQVQ^T$.

Similarly, for a suitable unimodular $\tilde{R}$,

$$RQVQ^TR^T\tilde{R}^T = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & * & * & \cdots & * \end{pmatrix}, \quad \tilde{R}^T = \begin{pmatrix} 1 & * & \cdots & * \\ & 1 & & \vdots \\ & & \ddots & * \\ & & & 1 \end{pmatrix}$$

and $\tilde{R}RQVQ^TR^T\tilde{R}^T$ is of the same form.

By repeating this process we obtain a matrix

$$\tilde{V} = \begin{pmatrix} 0 & & & & & \\ -1 & 0 & & & 0 & \\ 0 & * & 0 & & & \\ \vdots & \vdots & -1 & 0 & & \\ \vdots & \vdots & 0 & * & & \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & * & 0 & * & & \\ & & & & & W \end{pmatrix}$$

equivalent to V (over $\mathbb{Z}$), $|W| \neq 0$. For further simplification of $\tilde{V}$ we now make use of the assumption that $\Delta(0)$ is in A a unit. $|V^T - tV| = \Delta(t)$, $|\tilde{V}^T - t\tilde{V}|$ and $|W^T - tW|$ all represent the Alexander polynomial up to a factor $\pm t^\nu$. So $|W| = \Delta(0)$ is a unit of A. There is a unimodular P_1 over A with

$$\tilde{V}P_1 = \begin{pmatrix} 0 & & \\ -1 & 0 & \\ 0 & 0 & \\ \vdots & \vdots & W \\ 0 & 0 & \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & & \\ & & * & 1 & \\ & & \vdots & & \ddots \\ & & * & & & 1 \end{pmatrix}$$

where the column adjoining W has been replaced by zeroes, because it is a linear combination of the columns of W. Now

$$P_1^T \tilde{V} P_1 = \begin{pmatrix} 0 & & 0 & \cdots & 0 \\ -1 & 0 & * & \cdots & * \\ 0 & 0 & & & \\ \vdots & \vdots & & W & \\ 0 & 0 & & & \end{pmatrix}$$

Since the row over W contains -1, there is a unimodular P_2 with

$$P_1^T \tilde{V} P_1 P_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & & & \\ \vdots & \vdots & & W & \\ 0 & 0 & & & \end{pmatrix}.$$

and $P_2^T P_1^T \tilde{V} P_1 P_2$ is of the same type. The process can be repeated until the desired form is reached. $\square$

13.6 Corollary. *If A is an integral domain in which $\Delta(0)$ is a unit, then $(W^T - tW)$ is a presentation matrix of $H_1(C_\infty)$ as an $A(t)$-module, and $|W^T - tW| = \Delta(t)$. The A-module $H_1(C_\infty)$ is finitely generated and free and there is an A-basis of $H_1(C_\infty)$ such that the generating covering transformation $t = h_{j+1}h_j^{-1}$ (see 4.4) induces an isomorphism $t_*\colon H_1(C_\infty) \to H_1(C_\infty)$ which is represented by the matrix $W^{-1}W^T$.*

Proof. We may assume that as an $A(t)$-module $H_1(C_\infty)$ has a presentation matrix $(V^T - tV)$ where V is of the special form which can be achieved according to Lemma 13.5:

$$V^T - tV = \left(\begin{array}{c|c} U^T - tU & \\ \hline & W^T - tW \end{array}\right) \tag{1}$$

$$U^T - tU = \left(\begin{array}{cccccccc} 0 & -1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ t & 0 & 0 & * & \cdot & \cdot & \cdot & * \\ 0 & * & 0 & -1 & 0 & \cdot & \cdot & 0 \\ \vdots & \vdots & t & 0 & 0 & * & \cdot & \\ \vdots & \vdots & 0 & * & & & & \\ 0 & * & \cdot & \cdot & & & & \end{array}\right).$$

There is an equivalent presentation matrix in whose second column all entries but the first are zero, the first remaining -1. So the first row and second column can be omitted. In the remaining matrix the first row and the first column again may be omitted. This procedure can be continued until the presentation matrix takes the form $(W^T - tW)$, or, $(W^T W^{-1} - tE)$, since $|W| = \Delta(0)$ is a unit of A. This means that defining relations of $H_1(C_\infty)$ as an $A(t)$-module take the form: $W^T W^{-1} s = ts$, where $s = (s_i)$ are generators of $H_1(C_\infty)$. This proves the corollary. □

There is a distinguished generator $z \in H_2(S, \partial S) \cong \mathbb{Z}$ represented by an orientation of S which induces on ∂S the orientation of $\mathfrak{k}$. We shall now make use of cohomology to define a bilinear form. Since all homology groups $H_i(C_\infty)$, $H_i(C_\infty, \partial C_\infty)$, are torsion free, we have

$$H^i \cong \mathrm{Hom}_A(H_i, A) \cong H_i$$

for these spaces ([Franz 1965, Satz 17.6], [Spanier 1966, 5.5.3]). For every free basis $\{b_j\}$ of a group H_i there is a dual free basis $\{b^k\}$ of H^i defined by $\langle b^k, b_j \rangle = \delta_{kj}$, where the brackets denote the Kronecker product, that is $\langle b^k, b_j \rangle = b^k(b_j) \in A$ for $b^k \in \mathrm{Hom}_A(H_i, A)$. We use the cup-product [Hilton-Wylie 1960], [Stöcker-Zieschang 1985] to define

$$\beta\colon H^1(C_\infty, \partial C_\infty) \times H^1(C_\infty, \partial C_\infty) \to A, \quad (x, y) \mapsto \langle x \cup y, j_*(z) \rangle, \quad (2)$$

where $j\colon S \to C_\infty$ is the inclusion. (Here we write S instead of $S_0 \subset p^{-1}(S)$.) Now let $\{a_j \mid 1 \leqq j \leqq 2g\}$ and $\{s_i \mid 1 \leqq i \leqq 2g\}$ denote associated bases of $H_1(S)$ and $H_1(C_\infty, \partial C_\infty)$, respectively, $\mathrm{lk}(a_j, s_i) = \delta_{ji}$, such that j_* according to these bases is represented by a Seifert matrix

$$V = \left(\begin{array}{c|c} W & 0 \\ \hline 0 & U \end{array}\right), \quad W = (w_{ji})$$

where the reduced Seifert matrix W is $2m \times 2m$, $m \leqq g$. (See Lemma 13.5; observe that U and W are interchanged for technical reasons). From Corollary 13.6 it follows that $H_1(C_\infty, \partial C_\infty) \cong H_1(C_\infty)$ is already generated by $\{s_i \mid 1 \leqq i \leqq 2m\}$. It therefore suffices to consider the matrix $\begin{pmatrix} W \\ 0 \end{pmatrix}$ to describe the homomorphism $j_*\colon H_1(S) \to H_1(C_\infty, \partial C_\infty)$ with respect to the bases $\{a_j \mid 1 \leqq j \leqq 2g\}$ and $\{s_i \mid 1 \leqq i \leqq 2m\}$. The transpose $(w_{ij}) = (W^T 0)$ then describes the homomorphism

$$j^*\colon H^1(C_\infty, \partial C_\infty) \to H^1(S)$$

for the dual bases $\{s^j\}$, $\{a^i\}$, and we get from (2)

$$B = (\beta(s^i, s^k)) = (\langle s^i \cup s^k, j_*(z) \rangle) = (\langle j^*(s^i) \cup j^*(s^k), z \rangle). \tag{3}$$

We define another free basis $\{b^i \mid 1 \leqq i \leqq 2g\}$ of $H^1(S)$ by the Lefschetz-duality-isomorphism:

$$H^1(S) \xrightarrow{\cap z} H_1(S, \partial S),\ b^i \mapsto b^i \cap z = a_i.$$

The b^i connect z with the intersection matrix

$$V - V^T = (\mathrm{int}(a_i, a_k)) = (\langle b^i \cup b^k, z \rangle) = \Sigma.$$

On the other hand

$$\langle a^i \cup b^k, z \rangle = \langle a^i, b^k \cap z \rangle = \langle a^i, a_k \rangle = \delta_{ik}.$$

(See [Hilton-Wylie 1960, Theorem 4.4.13], [Stöcker-Zieschang 1985, Satz 15.4.1].)

The matrix L effecting the transformation $(a^i) = L(b^i)$ is

$$(\langle a^i \cup a^k, z \rangle) = L \cdot (\langle a^i \cup b^k, z \rangle) = L \cdot E = L.$$

Now $L = L\Sigma L^T$ or $(\Sigma^T)^{-1} = L$, and, by (1),

$$(W^T\ 0)\ L \begin{pmatrix} W \\ 0 \end{pmatrix} = (W^T 0)(\Sigma^T)^{-1} \begin{pmatrix} W \\ 0 \end{pmatrix}.$$

From

$$\Sigma = \left(\begin{array}{c|c} W - W^T & 0 \\ \hline 0 & U - U^T \end{array}\right)$$

and (3) it follows that

$$B = -W^T(W - W^T)^{-1}W. \tag{4}$$

13.7 Proposition. *The bilinear form $\beta\colon H^1(C_\infty, \partial C_\infty) \times H^1(C_\infty, \partial C_\infty) \to A$, $(x, y) \mapsto \langle x \cup y, j_*(z) \rangle$ can be represented by the matrix $-(W - W^T)^{-1}$, W is a reduced Seifert matrix, and β is non-degenerate.*

Proof. It remains to show that β is non-degenerate. But, see Lemma 13.5 and (1), $|V - V^T| = 1$ and $|U - U^T| = 1$; hence $|W - W^T| = 1$. □

We are now in a position to define an invariant quadratic form associated to a knot $\mathfrak{k}$. Let $t \colon C_\infty \to C_\infty$ denote the generator of the group of covering transformations which corresponds to a meridian linking the knot positively in the oriented S^3.

13.8 Proposition. *The bilinear form*

$$q \colon H^1(C_\infty, \partial C_\infty) \times H^1(C_\infty, \partial C_\infty) \to A, \quad q(x, y) = \langle\, x \cup t^* y + y \cup t^* x,\, j_*(z)\,\rangle$$

defines a quadratic form $q(x, x)$, which can be represented by the matrix $W + W^T$, where W, see 13.5*, is a reduced Seifert matrix of $\mathfrak{k}$. The quadratic form is non-degenerate.* $\Delta(0)$ *is required to be a unit in A.*

Proof. Remember that t_* is represented by $W^{-1}W^T$ with respect to the basis $\{s_i\}$, so t^* will be represented by $W(W^T)^{-1}$ relative to the dual basis $\{s^i\}$.

To calculate the matrix

$$Q = (q(s^i, s^k)) = (\langle\, j^*(s^i) \cup j^* t^*(s^k) + j^*(s^k) \cup j^* t^*(s^i), z\,\rangle)$$

we use $B = (\langle\, j^*(s^i) \cup j^*(s^k), z\,\rangle) = -W^T(W - W^T)^{-1}W$, see (3) and (4). We obtain

$$\begin{aligned} Q &= BW^{-1}W^T + W(W^T)^{-1}B^T \\ &= -W^T(W - W^T)^{-1}W^T + W((W - W^T)^{-1})^T W. \end{aligned}$$

Since $|W - W^T| = 1$, the matrices $(W - W^T)^{-1}$ and $W - W^T$ are equivalent, because there is only one skew symmetric form over $\mathbb{Z}$ with determinant $+1$; its normal form is F (see Appendix A.1). Let M be unimodular over $\mathbb{Z}$ with

$$\begin{aligned} &(W - W^T)^{-1} = M(W - W^T)M^T, \quad \text{or} \\ &(W - W^T)M(W - W^T)M^T = E. \end{aligned} \tag{5}$$

Now, $Q = -W^T M(W - W^T)M^T W^T + WM(W - W^T)M^T W$.

Using (5), we get

$$\begin{aligned} Q &= (E - WM(W - W^T)M^T)W^T + WM(W - W^T)M^T W \\ &= W^T + WM(W - W^T)M^T(W - W^T) = W^T + W. \end{aligned}$$

The quadratic form is non-degenerate, since $|W + W^T| \equiv |W - W^T| \equiv 1 \bmod 2$ □

Let us summarize the results of this section: Given a knot, we have proved, that $H_1(C_\infty, \partial C_\infty)$ is a free A-module, if $\Delta(0)$ is invertible in the integral domain A. By using the cup product, we defined a quadratic form q on

$$H^1(C_\infty, \partial C_\infty) \cong H_1(C_\infty,\ \partial C_\infty),$$

invariantly associated to the knot. The form can be computed from a Seifert matrix. q is known as Trotter's *quadratic form*.

In the course of our argument we used both, an orientation of S^3 and of the knot. Nevertheless, the quadratic form proves to be independent of the orientation of $\mathfrak{k}$. Clearly $j_*(z) = t_* j_*(z)$ in $H_1(C_\infty, \partial C_\infty)$, by the construction of C_∞, so that $q(x, y) = \langle\, x \cup (t^* - t^{*-1})y,\ j_*(z)\,\rangle$ is a equivalent definition of $q(x, y)$. Replacing z by $-z$ and t by t^{-1} does not change $q(x, y)$ (see Proposition 3.15). A reflection σ of S^3 which carries $\mathfrak{k}$ into its mirror image $\mathfrak{k}^*$ induces an isomorphism $\sigma^*\colon H^1(C_\infty, \partial C_\infty) \to H^1(C_\infty, \partial C_\infty)$. If $q_\mathfrak{k}$ und $q_{\mathfrak{k}^*}$ are the quadratic forms of $\mathfrak{k}$ and $\mathfrak{k}^*$, respectively, then $q_{\mathfrak{k}^*} = -q_\mathfrak{k}$, because $\sigma^* t^* = t^{*-1}\sigma^*$.

13.9 Proposition. *The quadratic form of a knot is the same as that of its inverse. The quadratic forms of $\mathfrak{k}$ and its mirror image $\mathfrak{k}^*$ are related by $q_{\mathfrak{k}^*} = -q_\mathfrak{k}$.* □

The quadratic form is easily seen to behave naturally with respect to the composition of knots (see 2.7). Let us assume that in A the leading coefficients of the Alexander polynomials of $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are invertible such that $q_{\mathfrak{k}_1}$ and $q_{\mathfrak{k}_2}$ are defined.

13.10 Proposition. $q_{(\mathfrak{k}_1 \# \mathfrak{k}_2)} = q_{\mathfrak{k}_1} \oplus q_{\mathfrak{k}_2}$

Proof. Obviously the Seifert matrix of $\mathfrak{k}_1 \# \mathfrak{k}_2$ has the form

$$V = \left(\begin{array}{c|c} V_1 & 0 \\ \hline 0 & V_2 \end{array}\right)$$

with V_i Seifert matrix of $\mathfrak{k}_i$, $i = 1, 2$. The same holds for the reduced Seifert matrices. □

Invariants of the quadratic form are, of course, invariants of the knot.

13.11 Definition (Signature). The signature σ of the quadratic form of a knot $\mathfrak{k}$ is called the *signature* $\sigma(\mathfrak{k})$ *of* $\mathfrak{k}$.

The signature of the quadratic form – the number of its positive eigenvalues minus the number of its negative eigenvalues – can be computed without much difficulty [Jones 1950, Theorem 4], see Appendix A.2. Obviously the signature of a quadratic form is an additive function with respect to the direct sum. Moreover the signature of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is zero.

13.12 Proposition. (a) *For any Seifert matrix V for $\mathfrak{k}$,* $\sigma(\mathfrak{k}) = \sigma(V + V^T)$.

(b) $\sigma(\mathfrak{k}_1 \# \mathfrak{k}_2) = \sigma(\mathfrak{k}_1) + \sigma(\mathfrak{k}_2)$.

(c) *If $\mathfrak{k}$ is amphicheiral,* $\sigma(\mathfrak{k}) = 0$.

Proof. We can replace V by an equivalent matrix of the form as obtained in Lemma 13.5. Then

$$V+V^T \sim \left(\begin{array}{cc|c|c} 0 & -1 & & * \cdots * \\ -1 & 0 & & 0 \cdots 0 \\ \hline & & \ddots & \\ \hline * & 0 & & \\ \vdots & \vdots & & W+W^T \\ * & 0 & & \end{array}\right) \sim \left(\begin{array}{cc|c|c} 0 & -1 & & \\ -1 & 0 & & 0 \\ \hline & & \ddots & \\ \hline & 0 & & W+W^T \end{array}\right).$$

□

B Computation of the Quadratic Form of a Knot

The computation of the quadratic form q of a given knot $\mathfrak{k}$ was based in the last paragraph on a Seifert matrix V which in turn relied on Seifert's band projection (see 8.2). Such a projection might not be easily obtainable from some given regular knot projection. Murasugi [1965] defined a knot matrix M over $\mathbb{Z}$, which can be read off any regular projection of a link. A link defines a class of *s-equivalent* matrices $\{M\}$ (see 13.34), and by symmetrizing, one obtains a class of *S-equivalent* matrices $\{M + M^T\}$ which can be described in the following way:

13.13 Definition. Two symmetric integral matrices M and M' are called *S-equivalent*, if there is a matrix

$$\left(\begin{array}{cccc} \begin{array}{cc|} 0 & 1 \\ 1 & 0 \\ \hline \end{array} & & & 0 \\ & \ddots & & \\ & & \begin{array}{|cc|} \hline 0 & 1 \\ 1 & 0 \\ \hline \end{array} & \\ 0 & & & M \end{array}\right)$$

A-equivalent (see 13.4) to M'. (Or, vice versa, exchanging M and M'.)

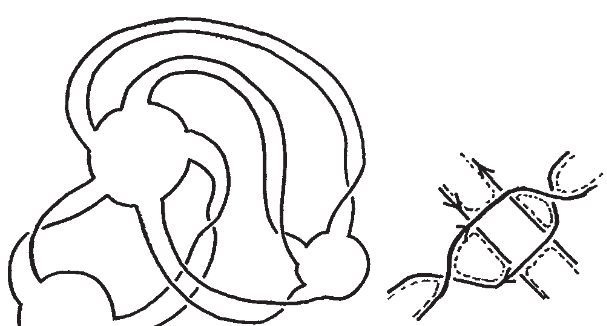
Murasugi [1965] proved that the class $\{M + M^T\}$ of S-equivalent symmetrized knot matrices is an invariant of the knot (or link). He thereby attaches a class of quadratic forms to a link.

Obviously, S-equivalent matrices have the same signature (see proof of 13.12), so the signature $\sigma\{M + M^T\}$ is defined and is a knot invariant. We shall prove: *If W is a reduced Seifert matrix of $\mathfrak{k}$, then $W + W^T \in \{M + M^T\}$.* This means that the quadratic form $q_{\mathfrak{k}}$ as defined in the first section of this chapter is a member of the class of quadratic forms represented by $M + M^T$, where M is Murasugi's knot matrix. Since the rule given by Murasugi to read off M from an arbitrary regular projection is rather complicated, we shall confine ourselves to so-called special projections, which hold a position between arbitrary projections and band projections. Any projection can be converted into a special one without much difficulty. We give a simple rule in (6) to read off the matrix M from a special projection.

13.14 Definition (Special projection). Let $p(\mathfrak{k})$ be a regular projection of a knot $\mathfrak{k}$ on $\mathbb{R}^2$. Choose a chessboard colouring (colours α and β) of the regions of $\mathbb{R}^2$ defined by $p(\mathfrak{k})$ such that the infinite region is an α-region (see 2.1). $p(\mathfrak{k})$ is called a *special projection* or *special diagram*, if the union of the β-regions is the image of a Seifert surface of $\mathfrak{k}$ under the projection p.

13.15 Proposition. *Every knot $\mathfrak{k}$ possesses a special projection.*

Proof. Starting from an arbitrary regular projection of $\mathfrak{k}$ we use Seifert's procedure (see 2.4) to construct an orientable surface S spanning $\mathfrak{k}$. We obtain S as a union of several disks spanning the Seifert circuits, and a couple of bands twisted by π, joining the disks, which may occur in layers over each other. There is an isotopy which places the disks separately into the projection plane $\mathbb{R}^2$, so that they do not meet each other or any bands, save those which are attached to them (Figure 13.1 (a)). By giving the



(a) (b)

Figure 13.1

overcrossing section at a band crossing a half-twist (Figure 13.1 (b)) it can be arranged, that only the type of crossing as shown in Figure 13.1 (b) occurs.

Now apply again Seifert's method. All Seifert circuits bound disjoint regions (β-regions) in $\mathbb{R}^2$. So they define a Seifert surface which – except in the neighbourhood of double points – consists of β-regions. □

It follows that the number of edges (arcs of $p(\mathfrak{k})$ joining double points) of every α-region in a special projection must be even. This also suffices to characterize a special projection, if the boundaries of β-regions are simple closed curves, that is, if at double points always different β-regions meet. It is easy to arrange that the boundaries of β- and α-regions are simple: in case they are not, a twist through π removes the double point which occurs twice in the boundary (Figure 13.2).

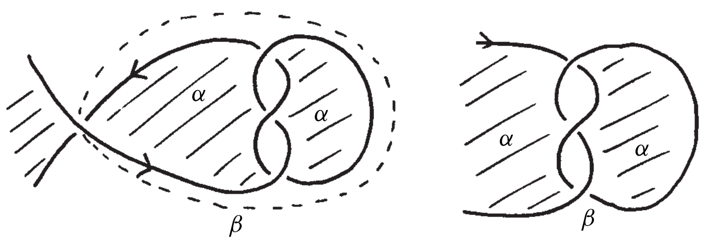


Figure 13.2

We now use a special projection to define associated bases $\{a_i\}$, $\{s_k\}$ of $H_1(S)$ and $H_1(C^*)$, respectively, and compute their Seifert matrix V. (It turns out that V is Murasugi's knot matrix M of the special diagram; see [Murasugi 1965, 3.3].) Let S be the Seifert surface of $\mathfrak{k}$ which projects onto the β-regions $\{\beta_j\}$ of a special projection. The special projection suggests a geometric free basis of $H_1(S)$. Choose simple closed curves a_i on S whose projections are the boundaries $\partial\alpha_i$ of the finite α-regions $\{\alpha_i \mid 1 \leqq i \leqq 2h\}$, oriented counterclockwise in the projection plane. (See Figure 13.3.) The number of finite α-regions is $2h$, where h is the genus of S. (We denote the infinite α-region by α_0.)

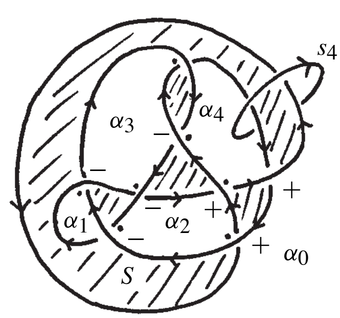


Figure 13.3

Now cut the knot complement C along S to obtain C^*. There is again a geometrically defined free basis $\{s_k \mid 1 \leqq k \leqq 2h\}$ of $H_1(C^*)$ associated to $\{a_i\}$ by linking numbers: $\mathrm{lk}(a_i, s_k) = \delta_{ik}$. The curve representing s_k pierces the projection plane once (from below) in a point belonging to α_k and once in α_0.

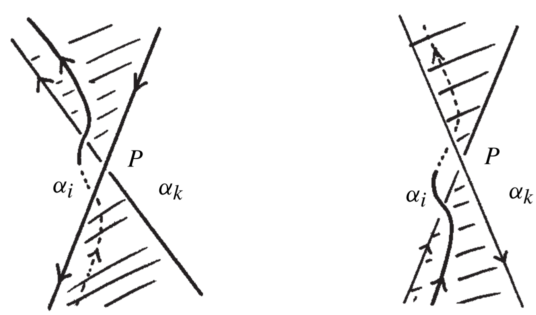
a_i splits up into a_i^+ and a_i^-. We move $i_*^+(a_i^+)$ by a small deformation away from S^+ and use the following convention to distinguish between $i_*^+(a_i^+)$ and $i_*^-(a_i^-)$. If in the neighbourhood of a double point P the curve $i_*^+(a_i^+)$ is directed as the parallel undercrossing edge of $\partial\alpha_i$, then $i_*^+(a_i^+)$ is supposed to run above the projection plane; otherwise it will run below. This arrangement is easily seen to be consistent in a special diagram.

In 2.3 we have defined the index $\theta(P)$ of a double point P. We need another function which takes care of the geometric situation at a double point with respect to the adjoining α-regions.

13.16 Definition (Index $\varepsilon_i(p)$). Let P be a double point in a special projection, $P \in \partial\alpha_i$. Then

$$\varepsilon_i(P) = \begin{cases} 1 & \text{if } \alpha_i \text{ is on the left of the underpassing arc at } P, \\ 0 & \text{if } \alpha_i \text{ is on the right,} \end{cases}$$

is called ε-*index of* P. (See Figure 13.4.)



$\theta(P) = +1$ $\quad\varepsilon_i(P) = 1\quad$ $\theta(P) = -1$

Figure 13.4

From this definition it follows that $\varepsilon_i(P)+\varepsilon_k(P) = 1$ for $P \in \partial\alpha_i \cap \partial\alpha_k$. Because of this symmetry it suffices to consider the two cases described in Figure 13.4.

We compute the Seifert matrix $V = (v_{ik})$, $i^+(a_i^+) = \Sigma v_{ik} s_k$:

$$v_{ii} = \textstyle\sum_{P\in\partial\alpha_i} \theta(P)\varepsilon_i(P), \quad v_{ik} = \textstyle\sum_{P\in\partial\alpha_i\cap\partial\alpha_k} \theta(P)\varepsilon_k(P) \tag{6}$$

This can be verified from our geometric construction using Figure 13.4. The formulas (6) coincide with Murasugi's definition of his knot matrix M [Murasugi 1965, Definition 3.3] in the case of a special projection. (A difference in sign is due to another choice of $\theta(P)$.)

The formulas (6) may be regarded as the definition of M; we do not give a definition of Murasug's knot matrix for arbitrary projections because it is rather intricate. The result of the consideration above can be formulated in the following way.

13.17 Proposition. *Let $p(\mathfrak{k})$ be a special diagram of $\mathfrak{k}$ with α-regions α_i, index functions $\theta(P)$ and $\varepsilon_i(P)$ according to* 2.3 *and* 13.16. *Then a Seifert matrix (v_{ik}) of $\mathfrak{k}$ is defined by* (6). (*The Seifert matrix coincides with Murasugi's knot matrix of $p(\mathfrak{k})$.*) □

13.18 Proposition. *If W is a reduced Seifert matrix then $(W + W^T)$ is contained in the class $\{M + M^T\}$ of S-equivalent matrices. The signature $\sigma(\mathfrak{k})$ coincides with the signature $\sigma(M + M^T)$ of* [Murasugi 1965].

Proof. If S is a Seifert surface which admits a special diagram as a projection the assertion follows from 13.13 and 13.17. Any Seifert surface S allows a band projection. By twists through π it can be arranged that the bands only cross as shown in Figure 13.1 (b). At each crossing we change S, as we did in the proof of 13.17, in order to get a spanning surface S' which projects onto the β-region of a special diagram. We then compare the band projections of S and S' and their Seifert matrices V and V'. It suffices to consider the case shown in Figure 13.5 (a).

It is not difficult to perform the local isotopy which carries Figure 13.5 (c) over to Figure 13.5 (d). The genus of the new surface is $g(S') = g(S) + 2$. Let $\{a_k\}$, $\{s_l\}$ be associated bases of $H_1(S)$, (see 13.2) and $H_1(C^*)$, and let V be their Seifert matrix. Substitute $\tilde{a}, a'_j, a''_j$ for $a_j \in \{a_k\}$ and $\tilde{s}, s'_j, s''_j$ for $s_j \in \{s_l\}$ to obtain associated bases relative to S'. The corresponding Seifert matrix V' is of the form

$$\begin{array}{c c}
 & \begin{array}{cccc} \tilde{s} & s'_j & s''_j & \end{array} \\
\begin{array}{c} \tilde{a} \\ a'_j \\ a''_j \\ \\ \end{array} & \begin{pmatrix} 0 & 1 & -1 & 0\dots \\ 0 & * & * & \dots. \\ 0 & * & * & \dots. \\ 0 & \vdots & \vdots & (v_{kl}) \end{pmatrix}
\end{array}.$$

Adding the s'_j-column to the s''_j-column and then the a'_j-row to the a''_j-row we get

$$\begin{array}{c c}
 & \begin{array}{ccccc} \tilde{s} & s'_j - s''_j & s''_j & & \end{array} \\
\begin{array}{r} \tilde{a} \\ a'_j \\ a_j = a'_j + a''_j \\ \\ \\ \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & * & * & * & \dots \\ 0 & * & & & \\ 0 & * & & V & \\ \vdots & \vdots & & & \end{pmatrix}
\end{array}.$$

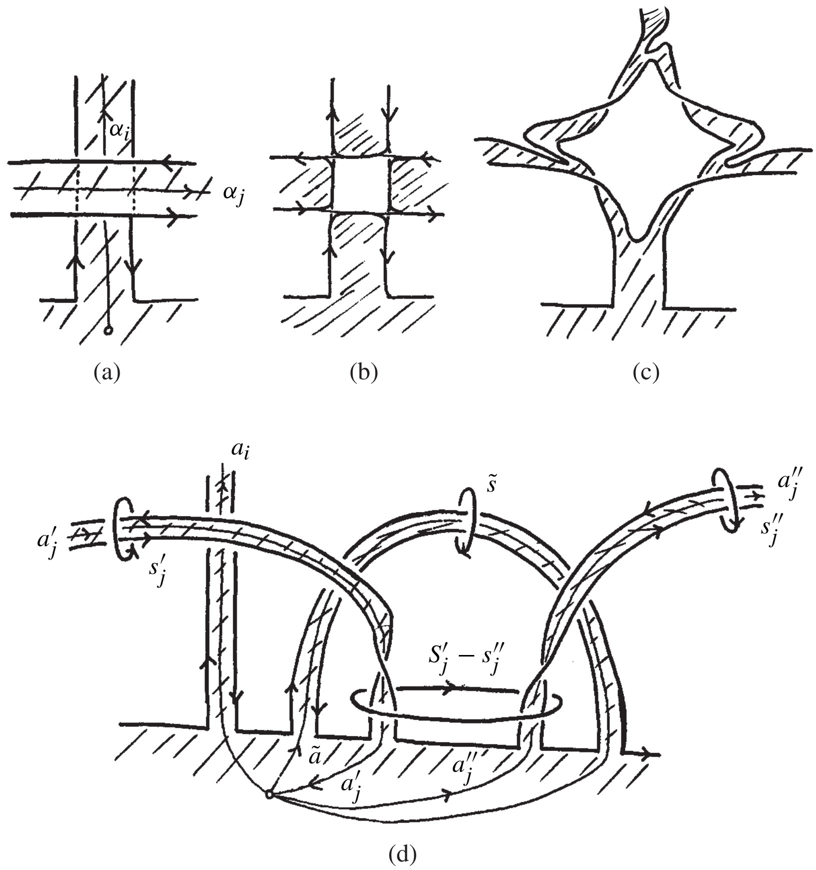


Figure 13.5

This follows from Figure 13.5 (d), because the overcrossings of s_j' and s_j'' add up to those of s_j, and $a_j = a_j' + a_j''$. Evidently, by adding multiples of the first row to the other rows the second column can be replaced by zeroes excepting the 1 on top. After these changes the bases remain associated. We have proved: $(V' + V'^T)$ and $(V + V^T)$ are S-equivalent (see Definition 13.13). The procedure can be repeated until a Seifert surface is reached which allows a special projection. (Observe: Twists in the bands do not hamper the process.) □

C Alternating Knots and Links

The concepts which have been developed in the preceding section provide a means to obtain certain results on alternating knots and links first proved in [Crowell 1959], [Murasugi 1958, 1958′, 1960, 1963]. R.H. Crowell's paper rests on a striking application of a graph theoretical result, the Bott–Mayberry matrix tree theorem [Bott-Mayberry 1954].

In 2.3 we defined the graph of a regular projection $p(\mathfrak{k})$ of a knot (or link) by assigning a vertex P_i to each α-region α_i and an edge to each double point; we call this graph the α-graph of $p(\mathfrak{k})$ and denote it by Γ_α. Its dual Γ_β is obtained by considering β-regions instead of α-regions.

We always assume the infinite region to be the α-region α_0. The following definition endows Γ_α and Γ_β with orientations and valuations.

13.19 Definition. Let Γ_α be the α-graph of $p(\mathfrak{k})$. The edge joining $P_i \in \alpha_i$ and $P_k \in \alpha_k$ assigned to the crossing Q_λ of $p(\mathfrak{k})$ is denoted by u_{ik}^λ. The orientation of u_{ik}^λ is determined by the characteristic $\eta(Q_\lambda)$ (see 3.4): The initial point of u_{ik}^λ is P_i resp. P_k for $\eta(Q_\lambda) = +1$ resp. $= -1$. (Loops ($P_i = P_k$) are oriented arbitrarily.) The oriented edge u_{ik}^λ obtains the *valuation* $f(u_{ik}^\lambda) = \theta(P)$. The edges v_{jl}^μ of the dual graph Γ_β are oriented in such a way that $\operatorname{int}(u_{ik}^\lambda, v_{jl}^\mu) = +1$ for every pair of dual edges with respect to a fixed orientation of the plane containing $p(\mathfrak{k})$. Now the valuation of Γ_β is defined by $f(v_{jl}^\mu) = -f(u_{ik}^\lambda)$. Denote the graphs with orientation and valuation by Γ_α^*, Γ_β^* respectively.

13.20. A Seifert matrix of a Seifert surface of $\mathfrak{k}$ which is composed of the β-regions of a special projection may now be interpreted in terms of Γ_α^*. Define a square matrix $H(\Gamma_\alpha^*) = (h_{ik})$ by

$$h_{ii} = \textstyle\sum_{j,\lambda} f(u_{ji}^\lambda), \quad h_{ik} = -\textstyle\sum_\lambda f(u_{ik}^\lambda),\ i \neq k. \tag{7}$$

Denote by H_{ii} the submatrix of H obtained by omitting the i-th row and column of H. From equations (6) and (7) we obtain (recall that the subscript 0 corresponds to the infinite region α_0)

13.21 Proposition. *Let $p(\mathfrak{k})$ be a special projection of a knot or link, Γ_α^* its α-graph, and H the graph matrix of Γ_α^*. Then H_{00} is a Seifert matrix of $\mathfrak{k}$ with respect to a Seifert surface which is projected onto the β-regions of $p(\mathfrak{k})$.* □

By a theorem in [Bott-Mayberry 1954], the principal minors $\det(H_{ii})$ of a graph matrix are connected with the number of *rooted trees* in a graph Γ; for definitions and the proof see Appendix A.3–A.5.

13.22 Theorem (Matrix tree theorem of Bott–Mayberry). *Let* Γ_α^* *be an oriented graph with vertices* P_i*, edges* u_{ik}^λ*, and a valuation* $f : \{u_{ik}^\lambda\} \mapsto \{1, -1\}$*. Then*

$$\det(H_{ii}) = \sum f(\mathrm{Tr}(i)), \tag{8}$$

where the sum is taken over the rooted trees $\mathrm{Tr}(i) \subset \Gamma_\alpha^*$ *with root* P_i*, and* $f(\mathrm{Tr}(i)) = \prod f(u_{jk}^\lambda)$*, the product taken over all* $u_{jk}^\lambda \in \mathrm{Tr}(i)$*.* □

13.23 Proposition. *The graphs* Γ_α^* *and* Γ_β^* *of a special alternating projection have the following properties* (*see Figure* 13.6).

(a) *Every region of* Γ_α^* *can be oriented such that the induced orientation on every edge in its boundary coincides with the orientation of the edge.*

(b) *No vertex of* Γ_β^* *is at the same time initial point and endpoint.*

(c) *The valuation is constant* (*we always choose* $f(u_{ik}^\lambda) = +1$).

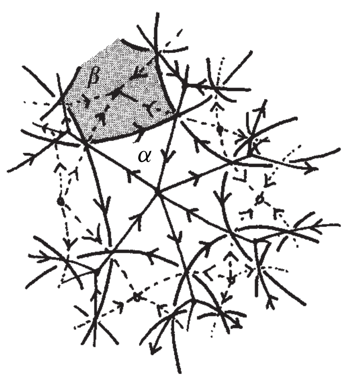


Figure 13.6

The *proof* of the assertion is left to the reader. It relies on geometric properties of special projections, see Figure 13.6, and the definitions 2.3 and 13.19. Note that the edges of Γ_α^* with P_i in their boundary, cyclically ordered, have P_i alternatingly as initial point and endpoint, and that the edges in the boundary of a region of Γ_β^* also alternate with respect to their orientation. □

13.24 Proposition. *Let* S *be the Seifert surface determined by the* β*-regions of a special alternating projection* $p(\mathfrak{k})$*, and* V *a Seifert matrix of* S*. Then* $\det V \neq 0$ *and* S *is of minimal genus. Furthermore,* $\det V = \pm 1$*, if and only if* $\deg P_i = \sum_k |h_{ik}| = 2$ *for* $i \neq 0$.

Proof. It follows from 13.23 (a) that every two vertices of Γ_α^* can be joined by a path in Γ_α^*. So there is at least one rooted tree for any root P_i. Since $f(u_{ik}^\lambda) = +1$, the number of P_i-rooted trees is by 13.22 equal to $\det(H_{ii}) > 0$. If $V = H_{00}$ is a $(m \times m)$-matrix then $\deg \Delta(t) = m$ in the case of a knot, and $\deg \nabla(t) = m - \mu + 1$ in the case of a μ-component link. It follows that $2h = m$ where h is the genus of S. Since $\deg \Delta(t) \leqq 2g$ resp. $\deg \nabla(t) + \mu - 1 \leqq 2g$ for the genus g of $\mathfrak{k}$, we get $g = h$, see 8.11, 9.18 and E 9.5.

To prove the last assertion we characterize the graphs Γ_α^* which admit only one P_0-rooted tree. We claim that for $i \neq 0$ one must have $\deg P_i = 2$. Suppose $\deg P_k \geqq 4$ for some $k \neq 0$, with $u_{ik}^\lambda \neq u_{jk}^{\lambda'}$, and u_{ik}^λ contained in a P_0-rooted tree T_0. Then $u_{jk}^{\lambda'} \notin T_0$, and there are two simple paths w_i, w_j in T_0 which intersect only in their common initial point P_l with endpoints P_i and P_j respectively, see Figure 13.7. Substitute $u_{jk}^{\lambda'}$ for u_{ik}^λ to obtain a different P_0-rooted tree.

Obviously every graph Γ_α^* with $\deg P_i = 2$ for all $i \neq 0$ has exactly one P_0-rooted tree. □

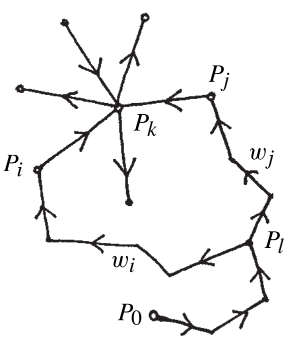


Figure 13.7

As an easy consequence one gets:

13.25 Proposition. *A knot or link $\mathfrak{k}$ with a special alternating projection is fibred, if and only if it is the product of torus knots or links $\mathfrak{k}_i = \mathfrak{t}(a_i, 2)$, $\mathfrak{k} = \mathfrak{k}_1 \# \cdots \# \mathfrak{k}_r$.*

Proof. See Figure 13.8. It follows from 13.24 that $\mathfrak{k}$ is of this form. By 4.11 and 7.19 we know that knots of this type are fibred. □

Proposition 13.25 was first proved in [Murasugi 1960].

13.26 Proposition. *Let $\mathfrak{k}$ be an alternating knot or link of multiplicity μ, and $p(\mathfrak{k})$ an alternating regular projection.*

(a) *The genus of the Seifert surface S obtained from the Seifert construction* 2.4 *is the genus $g(\mathfrak{k})$ of $\mathfrak{k}$ (genus and canonical genus coincide).*

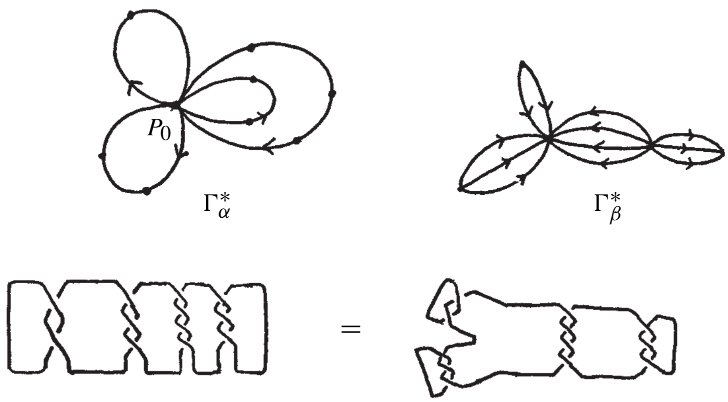


Figure 13.8

(b) deg $\Delta(t) = 2g$, *resp*. deg $\nabla(t) = 2g$.

(c) $\mathfrak{k}$ *is fibred if and only if* $|\Delta(0)| = 1$ *resp*. $|\nabla(0)| = 1$.

Proof. Consider the Seifert cycles of the alternating projection $p(\mathfrak{k})$. If a Seifert cycle contains another Seifert cycle in the projection of the disk it spans, it is called a *cycle of the second kind*, otherwise it is *of the first kind* [Murasugi 1960]. If there are no cycles of the second kind, the projection is special, see 2.4 and 13.14. Suppose there are cycles of the second kind; choose a cycle c bounding a disk $D \subset S^3$ such that $p(D)$ contains only cycles of the first kind. Place S in $\mathbb{R}^3$ in such a way that the part of $\mathfrak{k}$ which is projected on $p(D)$ is above a plane $E \supset D$, while the rest of $\mathfrak{k}$ is in the lower halfspace (Figure 13.9).

Cut S along D such that S splits into two surfaces S_1, S_2, contained in the upper resp. lower halfspace defined by E such that D is replaced by two disks D_1, D_2. The knots (or links) $\mathfrak{k}_1 = \partial S_1$, $\mathfrak{k}_2 = \partial S_2$ then possess alternating projections $p(\mathfrak{k}_1)$, $p(\mathfrak{k}_2)$, and $p(\mathfrak{k}_1)$ is special. One may obtain S back from S_1 and S_2 by identifying the disks D_1 and D_2. If $\mathfrak{k}$ results in this way from the components $\mathfrak{k}_1$ and $\mathfrak{k}_2$, we write $\mathfrak{k} = \mathfrak{k}_1 * \mathfrak{k}_2$ and call it $*$-product [Murasugi 1960]. (The reader is warned that the $*$-product does not depend merely on its factors $\mathfrak{k}_1$ and $\mathfrak{k}_2$.)

Let C^*, C_i^*, $1 \leqq i \leqq 2$, be obtained from the complements of $\mathfrak{k}$, $\mathfrak{k}_i$ by cutting along S, S_i, see 4.4. Choose a base point P on ∂D (Figure 13.9), then

$$\begin{aligned} \pi_1 C^* &\cong \pi_1 C_1^* * \pi_1 C_2^*, \qquad & \pi_1 S &\cong \pi_1 S_1 * \pi_1 S_2, \text{ resp.} \\ H_1(C^*) &\cong H_1(C_1^*) \oplus H_1(C_2^*), \qquad & H_1(S) &\cong H_1(S_1) \oplus H_1(S_2). \end{aligned} \tag{9}$$

It is evident that every alternating projection may be obtained by forming $*$-products of special alternating projections. We shall prove 13.26 by induction on the number of $*$-products needed to build up a given alternating projection $p(\mathfrak{k})$.

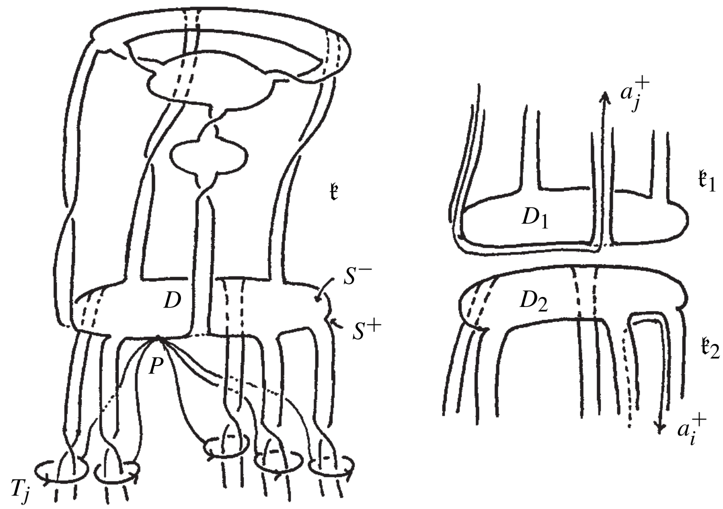


Figure 13.9

Proposition 13.25 proves the assertion if $p(\mathfrak{k})$ is special alternating. Suppose $\mathfrak{k} = \mathfrak{k}_1 * \mathfrak{k}_2$, $p(\mathfrak{k}_1)$ special alternating.

Let $i_1^\pm \colon S_1^\pm \to C_1^*, i_2^\pm \colon S_2^\pm \to C_2^*, i^\pm \colon S \to C^*$ denote the inclusions. If S^+ and S^- are chosen as indicated in Figure 13.9, the Seifert matrix V^+ associated with i_*^+ can be written in the form

$$V^+ = \begin{pmatrix} V_1^+ & \begin{matrix} * & \cdots & * \\ \vdots & & \vdots \\ * & \cdots & * \end{matrix} \\ \begin{matrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{matrix} & V_2^+ \end{pmatrix}$$

where V_1^+ and V_2^+ are Seifert matrices belonging to i_{1*}^+, i_{2*}^+. Assume (a) for $\mathfrak{k}_2$, S_2 as an induction hypothesis. By $|V^+| = |V_1^+| \cdot |V_2^+|$ property (a) follows for $\mathfrak{k}$, and (b) is a consequence of (a). To prove (c) let

$$w_1^{(1)} w_1^{(2)} w_2^{(1)} w_2^{(2)} \dots w_i^{(1)} w_i^{(2)}, \quad w_j^{(k)} \in \pi_1(C_k^*),\ 1 \leqq k \leqq 2,$$

be an element of $\pi_1(C_1^*) * \pi_1(C_2^*) \cong \pi_1(C^*)$. If $\mathfrak{k}_2$ is fibred, $i_{2\#}^+$ is an isomorphism. A closed curve $\omega_j^{(2)}$ in C^* representing $w_j^{(2)}$ is, therefore, homotopic rel P in C^* to a curve on S^+. Since $\mathfrak{k}_1$ is also fibred, a curve $\omega_j^{(1)}$ corresponding to a factor $w_j^{(1)}$ is homotopic to a closed curve composed of factors a_j^+ on S^+ and $T_j^{\pm 1}$, see Figure 13.9.

But the T_j can be treated as the curves $\omega_j^{(2)}$ and are homotopic to curves on S^+. Thus $i_\#^+$ is surjective; it is also injective, since S is of minimal genus [Neuwirth 1960]. □

This shows, together with Proposition 13.25, that a fibred alternating knot or link must be a $*$-product composed of factors

$$\mathfrak{k}_i = \mathfrak{t}(a_1, 2) \# \mathfrak{t}(a_2, 2) \# \mathfrak{t}(a_2, 2) \# \cdots \# \mathfrak{t}(a_r, 2).$$

There is a

13.27 Corollary. *The commutator subgroup of an alternating knot is either*

$$\mathfrak{G}' = \mathfrak{F}_{2g} \quad or \quad \mathfrak{G}' = \cdots * \mathfrak{F}_{2g} *_{\mathfrak{F}_{2g}} \mathfrak{F}_{2g} * \cdots$$

where g is the genus of the knot. C^* is a handlebody of genus $2g$ for a suitable Seifert surface.

Proof. The space C_1^* is a handlebody of genus $2g_1$, g_1 the genus of $\mathfrak{k}_1$. This follows by thickening the β-regions of $p(\mathfrak{k}_1)$. By the same inductive argument as used in the proof of 13.26 (see (9)) one can see that C^* is a handlebody of genus $2g$ obtained by identifying two disks D_1 and D_2 on the boundary of the handlebodies C_1^* and C_2^*. □

D Comparison of Different Concepts and Examples

In the Sections A and B we defined the quadratic form of a knot according to Trotter and Erle, and pointed out the connection to Murasugi's class of forms [Murasugi 1965]. Let us add now a few remarks on Goeritz's form. We shall give an example which shows that Goeritz's invariant is weaker than that of Trotter–Murasugi. Nevertheless, Goertiz's form is still of interest because it can be more easily computed than the other ones, and C.McA. Gordon and R.A. Litherland [Gordon-Litherland 1978] have shown that it can be used to compute the Trotter–Murasugi signature.

A regular knot projection is coloured as in 13.14. $\theta(P)$ is defined as in 2.3, see Figure 13.10. (Here we may assume again that at no point P the two α-regions coincide; if they do, define $\theta(P) = 0$ for such points.)

$$g_{ij} = \begin{cases} \sum\limits_{P \in \partial\alpha_i} \theta(P), & i = j \\ -\sum\limits_{P \in \partial\alpha_i \cap \partial\alpha_j} \theta(P), & i \neq j \end{cases} \tag{10}$$

then determines a symmetric $(n \times n)$-matrix $G = (g_{ij})$, where $\{\alpha_i \mid 1 \leqq i \leqq n\}$ are the finite α-regions. G is called *Goeritz matrix* and the quadratic form, defined by G, is called *Goeritz form*. (Observe that the orientation of the arcs of the projection do not

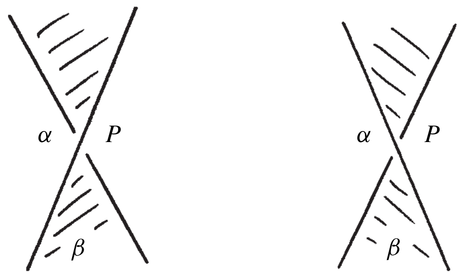


Figure 13.10

enter into the definition of the index $\theta(P)$, but that G changes its sign if $\mathfrak{k}$ is mirrored.) Transformations $G \mapsto LGL^T$ with unimodular L and the following matrix operation (and its inverse)

$$G \mapsto \begin{pmatrix} & & 0 \\ & G & \vdots \\ & & 0 \\ 0 \cdots & 0 & \pm 1 \end{pmatrix}$$

define a class of quadratic forms associated to the knot $\mathfrak{k}$ which Goeritz showed to be a knot invariant [Goeritz 1933]. A Goeritz matrix representing the quadratic form of a knot $\mathfrak{k}$ is denoted by $G(\mathfrak{k})$.

13.28 Proposition. *Let $p(\mathfrak{k})$ be a special diagram and V a Seifert matrix defined by* (6) (*see* 13.17). *Then $V + V^T = G(\mathfrak{k})$ is the Goeritz matrix of $p(\mathfrak{k})$.*

Proof. This is clear for elements $g_{ij}, i \neq j$, since $\varepsilon_i(P)+\varepsilon_j(P) = 1$ for $P \in \partial\alpha_i \cap \partial\alpha_j$. For $i = j$ it follows from the equality

$$v_{ii} = \sum_{P \in \partial\alpha_i} \theta(P)\varepsilon_i(P) = \sum \theta(P)(1 - \varepsilon_i(P));$$

the first sum describes the linking number of $i_*^+(a_i^+)$ with $\partial\alpha_i$, the second the linking number of $i_*^-(a_i^-)$ with $\partial\alpha_i$ which are the same for geometric reasons. (There is a ribbon $S^1 \times I \subset S^3$, $S^1 \times 0 = a_i^-$, $S^1 \times 1 = a_i^+$, $S^1 \times \frac{1}{2} = \partial\alpha_i$.) □

From this it follows that each Goeritz matrix G can be interpreted as presentation matrix of $H_1(\hat{C}_2)$ (see 8.21). H. Seifert [1936], M. Kneser and D. Puppe [1953] have investigated this connection and were able to show that the Goeritz matrix defines the linking pairing $H_1(\hat{C}_2) \times H_1(\hat{C}_2) \to \mathbb{Z}$.

Figure 13.11 (a) shows the trefoil's usual (minimal) diagram and 13.11 (b) a special diagram of it. The sign at a crossing point P denotes the sign of $\theta(P)$, a dot at P in an α-region α_i indicates $\varepsilon_i(P) = 1$ for $P \in \partial\alpha$. Thus we get $G_a = (-3)$ from Figure 13.11 (a) and

$$M + M^T = G_b = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

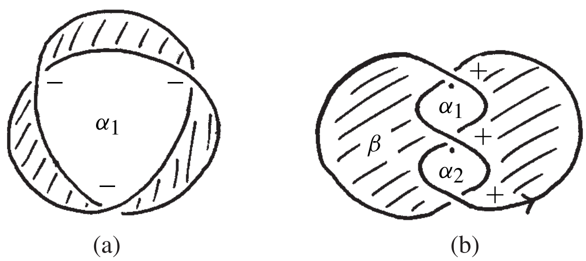


Figure 13.11

G_a and G_b can be transformed into each other by Goeritz moves which are described before 13.28. (It is necessary to use an extension by 3×3 matrices.) Figures 13.12 (a) and 13.12 (b) show a minimal and a special projection of the knot 8_{19}. Figure 13.12 (a)

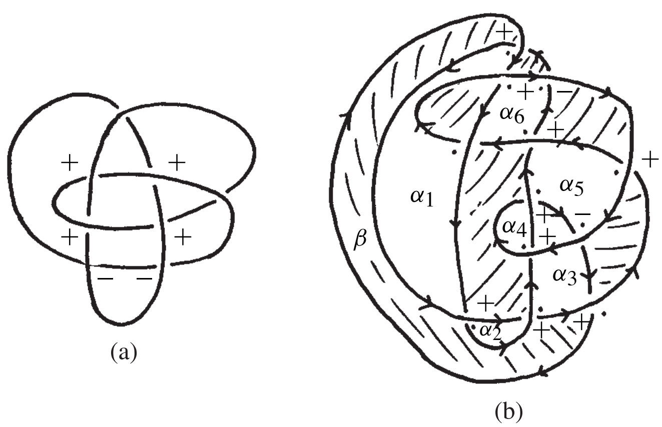


Figure 13.12

yields a Goeritz matrix

$$G = \begin{pmatrix} 0 & -1 & -1 & 0 \\ -1 & 1 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 1 \end{pmatrix} \sim (-3)$$

which is equivalent to that of a trefoil of Figure 13.11 (a). A Seifert matrix V can be read off Figure 13.12 (b):

$$V = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Since $|V| = 1$, V is already reduced, so its quadratic form $q_{\mathfrak{k}}$ is of rank 6, different to that of a trefoil which is of rank 2.

We finally demonstrate the advantage of using a suitable integral domain A instead of $\mathbb{Z}$. Figure 13.3 shows a special diagram of 8_{20}. Its Seifert matrix is

$$V = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 1 & 0 \end{pmatrix}, \quad |V| = 1,$$

$$V + V^T = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & -4 & 2 \\ 0 & -1 & 2 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & \multicolumn{2}{c}{0} \\ \multicolumn{2}{c}{0} & -2 & 3 \\ & & 3 & 0 \end{pmatrix}$$

$V + V^T$ is S-equivalent (see 13.13) to

$$\begin{pmatrix} -2 & 3 \\ 3 & 0 \end{pmatrix} = V' + V'^T, \quad V' = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}.$$

Using the construction of 8.7 one obtains a knot $\mathfrak{k}'$ with Seifert matrix V'. Thus over

$\mathbb{Z}_2$ there are different Trotter forms represented by

$$\begin{pmatrix} -2 & 3 \\ 3 & 0 \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & -2 & 3 \\ & & 3 & 0 \end{pmatrix}$$

associated to 8_{20} resp. $\mathfrak{k}'$, while their Murasugi matrices are equivalent. Moreover both knots have zero-signature, but over $\mathbb{Z}_3$ their forms prove that they are not amphicheiral.

13.29 Corollary. *The absolute value of the determinant of the quadratic form is an invariant of the knot. It is called the determinant of the knot. It can be expressed in several forms:*

$$|\det(M + M^T)| = |\det(W + W^T)| = |\det G| = |H_1(\hat{C}_2)| = |\Delta(-1)|.$$

Proof. See 8.11 and 8.20 □

In the case of alternating knots the determinant is a strong invariant; in fact, it can be used to classify alternating knots in a certain sense:

13.30 Proposition ([Bankwitz 1930], [Crowell 1989]). *The order* (*minimal number of crossings*) *with respect to regular alternating projections of a knot does not exceed its determinant.*

Proof. Let $p(\mathfrak{k})$ be a regular alternating projection of minimal order n. Consider the (unoriented) graph Γ_α of $p(\mathfrak{k})$. Since n is minimal, Γ_α does not contain any loops, and every edge of Γ_α is contained in a circuit, compare Figure 13.2. It follows from the Corollary to the Bott–Mayberry Theorem (Appendix A.4) that the determinant $\det G(\mathfrak{k})$ of $\mathfrak{k}$ is equal to the number of spanning trees of Γ_α. It remains to show that in a planar graph Γ_α with the aforesaid properties the number n of edges never exceeds the number of trees. One may reduce Γ_α by omitting points of order two and loops. If then Γ_α defines more than two regions on S^2 there exists an edge b in the boundary of two regions such that these two regions have no other edge in common. $(\Gamma_\alpha - b)$ then is a connected planar graph with no loops where every edge is in a circuit. Every tree of $(\Gamma_\alpha - b)$ is a tree of Γ_α. There is at least one tree more in Γ_α which contains b. □

The inequality $n \leqq \det G(\mathfrak{k})$ can be improved [Crowell 1959], see E 13.4.

Since there are only finitely many alternating knots with $\Delta(-1) = d$, there are a forteriori only finitely many such knots with the same Alexander polynomial. If $\Delta(-1) = \pm 1$ (in particular, if $\Delta(t) = 1$), the knot is either non-alternating or any alternating projection of it can be trivialized by twists of the type of Figure 13.2. Consider as an example the knot 6_1, see Figure 13.13. The Goeritz matrix is

$$(g_{ij}) = \begin{pmatrix} 5 & -3 & -2 \\ -3 & 4 & -1 \\ -2 & -1 & 3 \end{pmatrix}$$

One checks easily in Figure 13.13 that the graph has $11 = \Delta(-1) = \begin{vmatrix} 4 & -1 \\ -1 & 3 \end{vmatrix}$ maximal trees.

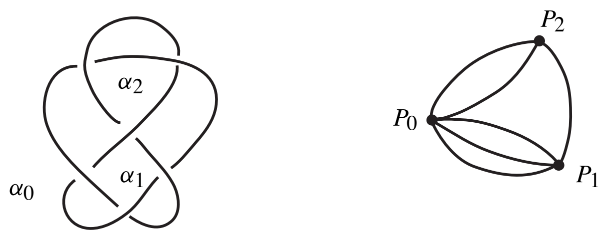


Figure 13.13

Proposition 13.30 of Bankwitz can also be used to show that certain knots are non-alternating, that is, do not possess any alternating projection. This is true for all non-trivial knots with trivial Alexander polynomial. Crowell was able to prove that most of the knots with less than ten crossings which are depicted in Reidemeister's table as non-alternating, really are non-alternating. If, for instance, 8_{19} were alternating, it would have a projection of order $\Delta(-1) = 3$ or less. But 8_{19} is non-trivial and different from 3_1 by its Alexander polynomial.

We now give a description of a result of Gordon and Litherland. In a special diagram the β-regions are bounded by Seifert circuits. If in a chessboard colouring of an arbitrary projection the Seifert circuits follow the boundaries of α-regions in the neighbourhood of a crossing P we call P *exceptional*, and by ν we denote the number $\nu = \sum \theta(P)$, where the sum is taken over the exceptional points of the projection. (The β-regions form an orientable Seifert surface if and only if there are no exceptional points.) Obviously the signature $\sigma(G)$ of a Goeritz matrix is no invariant in the class of equivalent Goeritz matrices. But in [Gordon-Litherland 1978] the following proposition is proved.

13.31 Proposition. *$\sigma(q_{\mathfrak{k}}) = \sigma(G) - \nu$, where ν is defined above.* □

The fact that $\sigma(G) - \nu$ is a knot invariant can be proved by the use of Reidemeister moves Ω_i (Exercise E 13.3).

Since the order of G will in most cases be considerably smaller than that of $M+M^T$, Proposition 13.29 affords a useful practical method for calculating $\sigma(q_{\mathfrak{k}})$. To compute the signature of any symmetric matrix over $\mathbb{Z}$ one can take one's choice from a varied spectrum of methods in numerics. The following proposition was used in [Murasugi 1965] and can be found in [Jones 1950]; we give a proof in Appendix A.2.

13.32 Proposition. *Let Q be a symmetric matrix of rank r over a field. There exists a chain of principal minors D_i, $i = 0, 1, \dots, r$ such that D_i is a principal minor of D_{i+1} and that no two consecutive determinants D_i, D_{i+1} vanish* ($D_0 = 1$). *For any such sequence of minors, $\sigma(Q) = \sum_{i=0}^{r-1} \operatorname{sign} D_i D_{i+1}$.* □

As an application consider the two projections of the trefoil 3_1 in Figure 13.11. The signature of the Goeritz matrix G_a of Figure 13.11 (a) is -1 and $\nu = 3$, Figure 13.11 (b) yields $\sigma(3_1) = 2$, hence $\sigma(G_a) - \nu = \sigma(q_{3_1})$.

13.33. *Proof of Proposition* 12.20**.** Let $\mathfrak{k}$ be a link of multiplicity μ, and S any Seifert surface spanning it. As in the case of a knot one may use S to construct the infinite cyclic covering C_∞ of $\mathfrak{k}$ corresponding to the normal subgroup $\mathfrak{N} = \ker \chi\varphi$ of 9.18. There is a band projection of $\mathfrak{k}$ (see 8.2), and $H_1(C_\infty)$ – as a $Z(t)$-module – is defined by a presentation matrix $(V^T - tV)$ where V is the Seifert matrix of the band projection. We show in 13.35 the result of [Kauffman 1981] that the (unique) Conway potential function $\nabla_{\mathfrak{k}}(t - t^{-1})$ is equal to $\det(tV - t^{-1}V^T)$ for any Seifert matrix V. □

To prove that $\det(tV - t^{-1}V^T)$ is a link invariant, we use a result of [Murasugi 1965].

13.34 Definition (s-equivalence). Two square *integral matrices* are *s-equivalent* if they are related by a finite chain of the following operations and their inverses:

$$\Lambda_1 : V \mapsto \quad L^T V L, \qquad L \text{ unimodular},$$

$$\Lambda_2 : V \mapsto \left(\begin{array}{cc|ccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & * & \cdots & * \\ \hline \vdots & \vdots & & & \\ & & & V & \\ 0 & 0 & & & \end{array}\right),$$

$$\Lambda_3 : V \mapsto \left(\begin{array}{cc|cc} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \hline 0 & * & & \\ \vdots & \vdots & V & \\ 0 & * & & \end{array}\right).$$

It is proved in [Murasugi 1965] that any two Murasugi knot matrices of isotopic links are s-equivalent. (This can be done by checking their invariance under Reidemeister moves Ω_i, see 1.13.) We showed in the proof of 13.18 that every Seifert matrix is s-equivalent to a Murasugi knot matrix. Hence, any two Seifert matrices of a link are s-equivalent.

13.35 Proposition. *The function* $\Omega_{\mathfrak{k}}(t) = \det(tV - t^{-1}V^T)$ *is the* (*unique*) *Conway potential function for any Seifert matrix* V.

Proof. By 8.11 and E 9.5,

$$\Omega_{\mathfrak{k}}(t) \doteq \Delta(t^2) \quad \text{for a knot,}$$
$$\Omega_{\mathfrak{k}}(t) \doteq (t^2-1)^{\mu-1}\nabla(t^2) \quad \text{for a link.}$$

Moreover $\Omega_{\mathfrak{k}}(1) = |V - V^T| = 1$. This proves 12.19 (1). For a split link $\Delta(t) = 0$ (see 9.17, 9.18). It remains to prove 12.19 (3). If $\mathfrak{k}_+$ is split so is $\mathfrak{k}_-$ and $\mathfrak{k}_0$, and all functions are zero. Figure 13.14 demonstrates the position of the Seifert surfaces S_+, S_-, S_0 in the region where a change occurs. (An orientation of a Seifert surface induces the orientation of the knot).

We may assume that the projection of $\mathfrak{k}_0$ is not split, because otherwise $\Omega_{\mathfrak{k}_0} = 0$, and $\mathfrak{k}_+, \mathfrak{k}_-$ are isotopic. If the projection of $\mathfrak{k}_+, \mathfrak{k}_-, \mathfrak{k}_0$ are all not split, then the change from $\mathfrak{k}_0$ to $\mathfrak{k}_+$ or $\mathfrak{k}_-$ adds a free generator a to $H_1(S_0)$: $H_1(S_+) \cong \langle a \rangle \oplus H_1(S_0) \cong H_1(S_-)$. Likewise $H_1(S^3 - S_\pm) \cong H_1(S^3 - S_0) \oplus \langle s \rangle$, see Figure 13.14.

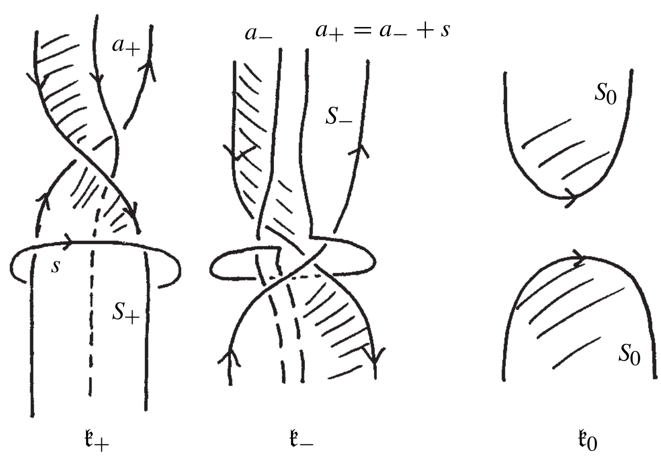


Figure 13.14

We denote by V_+, V_-, V_0 the Seifert matrices of $\mathfrak{k}_+, \mathfrak{k}_-, \mathfrak{k}_0$ which correspond to the connected Seifert surfaces obtained from the projections as described in 2.4. It follows that

$$V_+ = V_- + \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \qquad V_- = \begin{pmatrix} * & \cdots & \cdots & * \\ \vdots & & & \\ \vdots & & V_0 & \\ * & & & \end{pmatrix}$$

where the first column and first row correspond to the generators s and $a_\pm$. The rest is a simple calculation:

$$\Omega_{\mathfrak{k}_+}(t) - \Omega_{\mathfrak{k}_-}(t) = \left| tV_+ - t^{-1}V_+^T \right| - \left| tV_- - t^{-1}V_-^T \right|$$

$$= \begin{vmatrix} t - t^{-1} & * & * \\ 0 & \multicolumn{2}{c}{} \\ \vdots & \multicolumn{2}{c}{tV_0 - t^{-1}V_0^T} \\ 0 & & \end{vmatrix} = (t - t^{-1})\, \Omega_{\mathfrak{k}_0}(t).$$

□

Remark. It is possible to introduce a Conway potential function in μ variables corresponding to the Alexander polynomials of links rather than to the Hosokawa polynomial [Hartley 1982]. The function is defined as a certain normalized Alexander polynomial $\Delta(t_1^2, \ldots, t_n^2) \cdot t_1^{\mu_1} \ldots t_n^{\mu_n}$ where the μ_i are determined by curvature and linking numbers. Invariance is checked by considering Reidemeister moves.

E History and Sources

An invariant consisting of a class of quadratic forms was first defined by L. Goeritz [1933]. They yielded the Minkowski units, new knot invariants [Reidemeister 1932]. Further contributions were made by H. Seifert [1936], M. Kneser and D. Puppe [1953], K. Murasugi [1965], H.F. Trotter [1962], J. Milnor, D. Erle and others. Our exposition is based on [Erle 1969] and [Murasugi 1965], the quadratic form is that of Trotter [1962].

In [Gordon-Litherland 1978] a new quadratic form ws introduced which simultaneously generalized the forms of Trotter and Goeritz. As a by-product a simple way to compute the signature of a knot from a regular projection was obtained.

F Exercises

E 13.1. Compute the quadratic forms of Goeritz and Trotter and the signature of the knot 6_1, and the torus knots or links $\mathfrak{t}(2, b)$.

E 13.2. Characterize the 2×2 matrices which represent quadratic forms of knots.

E 13.3. Prove the invariance of $\sigma(G) - \nu$ (see 13.31) under Reidemeister moves.

E 13.4. [Crowell 1959] An alternating prime knot $\mathfrak{k}$ has a graph Γ_α with m vertices and k regions in S^2 such that the number of trees $\mathrm{Tr}(\Gamma_\alpha)$ satisfies the inequality $\det G(\mathfrak{k}) = \mathrm{Tr}(\Gamma_\alpha) \geqq 1 + (m-1)(k-1)$. Show that $8_{20}, 9_{42}, 9_{43}$ and 9_{46} are non-alternating knots.

Chapter 14

Representations of Knot Groups

Knot groups as abstract groups are rather complicated. Invariants which can be effectively calculated will, in general, be extracted from homomorphic images of knot groups.

We use the term representation in this chapter as a synonym for homomorphism, and we call two representations $\varphi, \psi : \mathfrak{G} \to \mathfrak{H}$ equivalent, if there is an automorphism $\alpha : \mathfrak{H} \to \mathfrak{H}$ with $\psi = \alpha\varphi$. There have been many contributions to the field of representations of knot groups in the past decades, and the material of this chapter comprises a selection from a special point of view – the simpler and more generally applicable types of representations.

The first section deals with metabelian (2-step metabelian) representations, the second with a class of 3-step metabelian representations which means that the third commutator group of the homomorphic image of the knot group vanishes. These representations yield an invariant derived from the peripheral system of the knot which is closely connected to linking numbers in coverings defined by the homomorphisms. These relations are studied in Section C. Section D contains some theorems on periodic knots, and its presence in this chapter is, perhaps, justified by the fact that a special metabelian representation in Section A of a geometric type helps to prove one of the theorems and makes it clearer.

A Metabelian Representations

14.1. Throughout this chapter we consider only knots of multiplicity $\mu = 1$. A knot group $\mathfrak{G}$ may then be written as a semidirect product $\mathfrak{G} = \mathfrak{Z} \ltimes \mathfrak{G}'$, where $\mathfrak{Z}$ is a free cyclic group generated by a distinguished generator t represented by a meridian of the knot $\mathfrak{k}$. An abelian homomorphic image of $\mathfrak{G}$ is always cyclic, and an *abelian representation of* $\mathfrak{G}$ *will*, hence, *be called trivial*. A group $\mathfrak{G}$ is called *k-step metabelian*, if its k-th commutator subgroup $\mathfrak{G}^{(k)}$ vanishes. ($\mathfrak{G}^{(k)}$ is inductively defined by $\mathfrak{G}^{(k)}$ = commutator subgroup of $\mathfrak{G}^{(k-1)}$, $\mathfrak{G} = \mathfrak{G}^{(0)}$.) The 1-step metabelian groups are the abelian groups, and 2-step metabelian groups are also called *metabelian*. It seems reasonable, therefore, to try to find metabelian representations as a first step. They turn out to be plentiful and useful.

Let $\varphi : \mathfrak{G} \to \mathfrak{H}$ be a surjective homomorphism onto a metabelian group $\mathfrak{H}$. Then $\varphi(\mathfrak{G}) = \mathfrak{H} = \varphi(\mathfrak{Z}) \ltimes \varphi(\mathfrak{G}')$ is a semidirect product and can be considered as a $\varphi(\mathfrak{Z})$-module. Since $\mathfrak{G}$ is trivialized by putting $t = 1$, the same holds for $\varphi(\mathfrak{G})$, if the

φ-image of t (also denoted by t) is made a relator. For the normal closure $\overline{\langle t \rangle}$ one has $\overline{\langle t \rangle} = \mathfrak{G}$ and $\overline{\langle t \rangle} = \varphi(\mathfrak{Z}) \ltimes \mathfrak{H}'$. The relations $ta = a$, $a \in \mathfrak{G}'$ trivialize $\mathfrak{G}'$; hence elements of the form $(t-1)a$, $a \in \mathfrak{H}'$ generate $\mathfrak{H}'$ as a $\mathbb{Z}(\mathfrak{Z})$-module: $\mathfrak{H}' = (t-1)\mathfrak{H}'$. This module is finitely generated and has an annulating polynomial of minimal degree coprime to the isomorphism $(t-1)\colon \mathfrak{H}' \to \mathfrak{H}'$.

14.2 Proposition. *Let $\varphi\colon \mathfrak{G} \to \mathfrak{H}$ be any nontrivial surjective metabelian representation of a knot group $\mathfrak{G} = \mathfrak{Z} \ltimes \mathfrak{G}'$, $\mathfrak{Z} = \langle t \rangle$, t a meridian. Then $\mathfrak{H} = \varphi(\mathfrak{Z}) \ltimes \mathfrak{H}'$ and $t - \mathrm{id}\colon \mathfrak{H}' \to \mathfrak{H}'$ is an isomorphism.* □

Since $\varphi(\mathfrak{G}') = \mathfrak{H}'$ is abelian the homomorphism φ factors through $\mathfrak{Z} \ltimes \mathfrak{G}'/\mathfrak{G}''$. If $\varphi(\mathfrak{Z}) = \mathfrak{Z}_n$ is finite, it factors through $\mathfrak{Z}_n \ltimes \mathfrak{G}'/\mathfrak{G}'_n$, $\mathfrak{G}_n = n\mathfrak{Z} \ltimes \mathfrak{G}'$, compare 8.19. The group $\mathfrak{G}'/\mathfrak{G}''$ is the first homology group of the infinite cyclic covering C_∞ of $\mathfrak{k}$, $\mathfrak{G}'/\mathfrak{G}'' = H_1(C_\infty)$ and may be regarded as a $\mathfrak{Z}$-module (Alexander module) where the operation is defined by that of the semidirect product. Likewise $\mathfrak{G}'/\mathfrak{G}'_n = H_1(\hat{C}_n)$ is the homology group of the n-fold cyclic branched covering of $\mathfrak{k}$, see 8.19 (c). The following proposition summarizes our result.

14.3 Proposition. *A metabelian representation of a knot group*

$$\varphi\colon \mathfrak{G} \to \mathfrak{Z} \ltimes \mathfrak{A}, \ \textit{respectively}\ \varphi_n\colon \mathfrak{G} \to \mathfrak{Z}_n \ltimes \mathfrak{A},\ \mathfrak{A}\ \textit{abelian},$$

factors through

$$\beta\colon \mathfrak{G} \to \mathfrak{Z} \ltimes H_1(C_\infty),\ \textit{respectively}\ \beta_n\colon \mathfrak{G} \to \mathfrak{Z}_n \ltimes H_1(\hat{C}_n),$$

mapping a meridian of $\mathfrak{k}$ onto a generator of $\mathfrak{Z}$ resp. $\mathfrak{Z}_n$. The group $\mathfrak{A}$ may be considered as a $\mathfrak{Z}$-module resp. $\mathfrak{Z}_n$-module. One has $\ker\beta = \mathfrak{G}''$, $\ker\beta_n = n\mathfrak{Z} \ltimes \mathfrak{G}'_n$. □

We give a simple example with a geometric background.

14.4 The groups of similarities. The replacing of the Alexander module $H_1(C_\infty) = \mathfrak{G}/\mathfrak{G}''$ by $H_1(C_\infty) \otimes_{\mathbb{Z}} \mathbb{C}$ suggests a metabelian representation of $\mathfrak{G}$ by linear mappings $\mathbb{C} \to \mathbb{C}$ of the complex plane. Starting from a Wirtinger presentation $\mathfrak{G} = \langle S_1, \dots, S_n \mid R_1, \dots, R_n \rangle$, a relation

$$S_k^{-1} S_i S_k S_{i+1}^{-1} = 1 \tag{1}$$

takes the form

$$-tu_k + tu_i + u_k - u_{i+1} = 0 \tag{2}$$

for $u_j = \beta(S_j S_1^{-1})$, $1 \leqq j \leqq n$. ($u \mapsto tu$, $u \in H_1(C_\infty)$ denotes the operation of a meridian.) The equations (2) form a system of linear equations with coefficients in $\mathbb{Z}(t)$. We may omit one equation (Corollary 3.6) and the variable $u_1 = 0$.

The determinant of the remaining $(n-1) \times (n-1)$ linear system equals the Alexander polynomial $\Delta_1(t)$, see 8.10, 9.11. Thus, by interpreting (2) as linear equations over $\mathbb{C}$, one obtains non-trivial solutions if and only if t takes the value of a root α of $\Delta_1(t)$. For suitable $z_i \in \mathbb{C}$ (z a complex variable)

$$S_i \mapsto \delta_\alpha(S_i) \colon z \mapsto \alpha(z - z_i) + z_i \tag{3}$$

maps $\mathfrak{G}$ into the group $\mathfrak{C}^+$ *of orientation preserving similarities of the plane* $\mathbb{C}$, since a Wirtinger relator (1) results in an equation (2) for $t = \alpha$, $u_i = z_i$. The representation δ_α is non-trivial (non-cyclic) if and only if $\Delta_1(\alpha) = 0$; it is metabelian because $\mathfrak{G}'$ is mapped into the group of translations. The class K of elements in $\mathfrak{G}$ conjugate to a meridian ($K =$ *Wirtinger class*) is mapped into the class K_α of conjugate similarities of $\mathfrak{C}^+$ characterized by α. (Note that $\alpha \neq 1$.)

14.5 Proposition. *There exists a non-trivial representation* $\delta_\alpha \colon (\mathfrak{G}, K) \to (\mathfrak{C}^+, K_\alpha)$ *if and only if* α *is a root of the Alexander polynomial* $\Delta_1(t)$*. When* α *and* α' *are roots of an* (*over* $\mathbb{Z}$) *irreducible factor of* $\Delta_1(t)$ *which does not occur in* $\Delta_2(t)$*, then any two representations* $\delta_\alpha, \delta'_{\alpha'}$ *are equivalent. In particular, any two such maps* δ_α*,* δ'_α *differ by an inner automorphism of* $\mathfrak{C}^+$*.*

Proof. The first assertion has been proved above. For α satisfying $\Delta_1(\alpha) = 0$, $\Delta_2(\alpha) \neq 0$ – that means that the system of linear equations has rank $n-2$ – there are indices i and k such that there is a unique non-trivial representation δ_α of the form (3) for any choice of a pair (z_i, z_k) of distinct complex numbers. Since $\mathfrak{C}^+$ is 2-transitive on $\mathbb{C}$ it follows that δ_α and δ'_α differ by an inner automorphism of $\mathfrak{C}^+$. Finally there is a Galois automorphism $\tau \colon \mathbb{Q}(\alpha) \to \mathbb{Q}(\alpha')$, if α and α' are roots of an irreducible factor of $\Delta_1(t)$. Put $\delta_{\alpha'}(S_i) \colon z \mapsto \alpha'(z - \tau(z_i)) + \tau(z_i)$ to obtain a representation equivalent to $\delta_\alpha(S_i) \colon z \mapsto \alpha(z - z_i) + z_i$. (In the special case $\alpha' = \bar{\alpha}$ a reflection may be used.) □

Remark. The complex numbers α for which there are non-trivial representations

$$\delta_\alpha \colon (\mathfrak{G}, k) \to (\mathfrak{C}^+, K_\alpha)$$

are invariants of $\mathfrak{G}$ in their own right. The Alexander polynomial $\Delta_1(t)$, though, is a stronger invariant, because it includes also the powers of its prime factors. This is, of course, exactly what is lost when the operation of t is replaced by complex multiplication by α: $(p(\alpha))^\nu \cdot a = 0$, $a \neq 0$ implies $p(\alpha) \cdot a = 0$, but $(p(t))^\nu \cdot a = 0$ does not imply $(p(t))^{\nu-1} \cdot a = 0$. (Compare [Burde 1967].)

Example. Figure 14.1 shows a class of knots (compare Figure 9.3, E 9.6) with Alexander polynomials of degree two. They necessarily have trivial second Alexander polynomials. Figure 14.2 shows the configuration of the fixed points z_i of $\delta_\alpha(S_i)$ for $m = 5$, $k = 3$. Then $\delta_\alpha(S_i)$ are rotations through α, $\cos\alpha = \frac{2k-1}{2k} = \frac{5}{6}$.

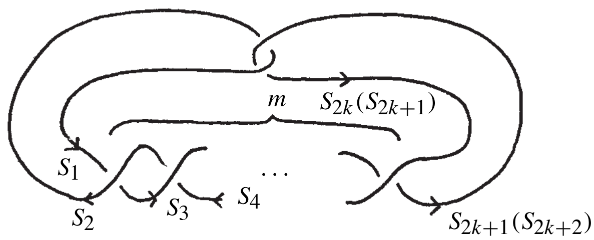


Figure 14.1

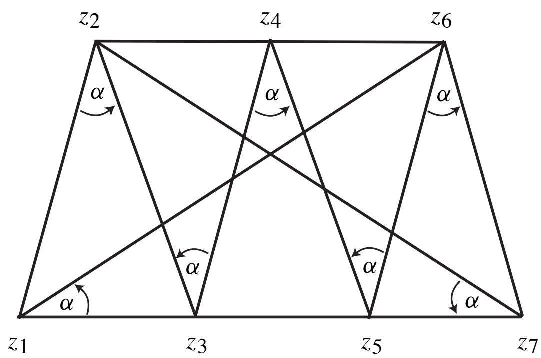


Figure 14.2

14.6 Metacyclic representations. A *representation* β^* of $\mathfrak{G}$ is called *metacyclic*, if $\beta^*(\mathfrak{G}') = \mathfrak{H}'$ is a cyclic group $\langle a \rangle \neq 1$:

$$\beta^*(\mathfrak{G}) = \langle t \rangle \ltimes \langle a \rangle.$$

The operation of t is denoted by $a \mapsto ta$. Putting

$$\beta^*(S_i) = (t, \nu_i a), \quad \nu_i \in \mathbb{Z},$$

transforms a set of Wirtinger relators (1) into a system of n equations in n variables ν_j:

$$-\nu_{i+1} + t\nu_i + (1-t)\nu_k = 0. \tag{4}$$

These equations are to be understood over $\mathbb{Z}$ if $\langle a \rangle$ is infinite, and as congruences modulo m if $\langle a \rangle \cong \mathfrak{Z}_m$.

In the first case β^* is trivial when $t = 1$. If $t = -1$, β^* must also be trivial, because the rank of (4) is $n-1$: Every $(n-1) \times (n-1)$ minor of its matrix is $\pm\Delta_1(-1) = \pm|H_1(\hat{C}_2)|$ which is an odd integer by 8.21, 13.19.

We may, therefore, confine ourselves to the finite case $\langle a \rangle = \mathfrak{Z}_m$.

14.7 Proposition (Fox). *A non-trivial metacyclic representation of a knot group is of the form*

$$\beta_m^*\colon \mathfrak{G} \to \mathfrak{Z} \ltimes \mathfrak{Z}_m,\ m > 1,$$

mapping a meridian onto a generator t of the cyclic group $\mathfrak{Z}$. The existence of a surjective homomorphism β_m^* implies $m|\Delta_1(k)$ for $k \in \mathbb{Z}$ with $ka = ta$, $a \in \mathfrak{Z}_m$.

For a prime p, $p|\Delta_1(k)$, $\gcd(k, p) = 1$, there exists a surjective representation β_p^*. If the rank of the system (4) *of congruences modulo p is $n-2$, all representations β_p^* are equivalent.*

Proof. If a surjective representation β_m^* exists, the system (4) admits a solution with $\nu_1 = 0$, $\gcd(\nu_2, \dots, \nu_n) = 1$. Let $Ax \equiv 0 \bmod m$ denote the system of congruences in matrix form obtained from (4) by omitting one equation and putting $\nu_1 = 0$. By multiplying Ax with the adjoint matrix A^* one obtains

$$A^*A \cdot x = (\det A) \cdot E \cdot x \equiv 0 \mod m.$$

This means $m|\Delta_1(k)$ since $\Delta_1(k) = \pm \det A$, see 9.11.

The rest of Proposition 14.7 follows from standard arguments of linear algebra, since (4) is a system of linear equations over a field $\mathbb{Z}_p$ if $m = p$. □

Remark. If m is not a prime, the existence of a surjective representation β_m^* does not follow from $m|\Delta_1(k)$. We shall give a counterexample in the case of a dihedral representation. By a chinese remainder argument, however, one can construct β_m^* for composite m, if m is square-free. One may obtain from β_m^* a homomorphism onto a finite group by mapping $\mathfrak{Z}$ onto $\mathfrak{Z}_r$, where r is a multiple of the order of the automorphism $t\colon a \mapsto ka$. As a special case we note

14.8 Dihedral representations. *There is a surjective homomorphism*

$$\gamma_p^*\colon \mathfrak{G} \to \mathfrak{Z}_2 \ltimes \mathfrak{Z}_p$$

onto the dihedral group $\mathfrak{Z}_2 \ltimes \mathfrak{Z}_p$ if and only if the prime p divides the order of $H_1(\hat{C}_2)$. If p does not divide the second torsion coefficient of $H_1(\hat{C}_2)$, then all representations γ_p^* are equivalent. (*See Appendix* A.6.) □

Since any such homomorphism γ_p^* must factor through $\mathfrak{Z}_2 \ltimes H_1(\hat{C}_2)$, see 14.3, the existence of dihedral representations $\mathfrak{G} \to \mathfrak{Z}_2 \ltimes \mathfrak{Z}_m$, $m||H_1(\hat{C}_2)|$, depends on the cyclic factors of $H_1(\hat{C}_2)$. If $H_1(\hat{C}_2)$ is not cyclic – for instance $H_1(\hat{C}_2) \cong \mathbb{Z}_{15} \oplus \mathbb{Z}_3$ for 8_{18} – there is no homomorphism onto $\mathfrak{Z}_2 \ltimes \mathfrak{Z}_{45}$, though $45|\Delta_1(-1)$.

The group $\gamma_p^*(\mathfrak{G})$ can be interpreted as the symmetry group of a regular p-gon in the euclidean plane. A meridian of the knot is mapped onto a reflection of the euclidean plane.

14.9 Example. Consider a Wirtinger presentation of the four-knot:

$$\mathfrak{G} = \langle\, S_1, S_2, S_3, S_4 \mid S_3S_1S_3^{-1}S_2^{-1},\ S_4^{-1}S_2S_4S_3^{-1},\ S_1S_3S_1^{-1}S_4^{-1},\ S_2^{-1}S_4S_2S_1^{-1} \,\rangle,$$

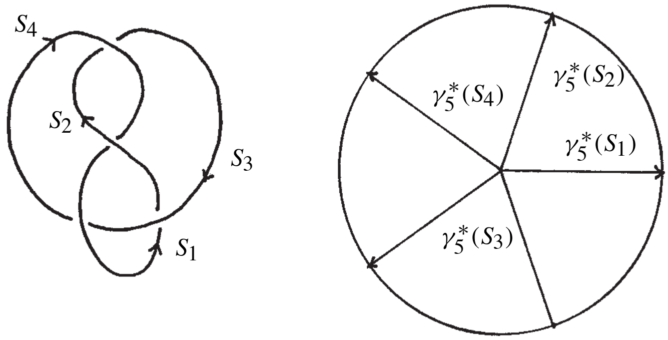


Figure 14.3

see Figure 14.3. One has $\Delta_1(-1) = 5 = p$, see 8.15 (b). The system (4) of congruences mod p takes the form

$$\left.\begin{array}{rrrrl} -\nu_1 & -\nu_2 & +2\nu_3 & & \equiv 0 \\ & -\nu_2 & -\nu_3 & +2\nu_4 & \equiv 0 \\ +2\nu_1 & & -\nu_3 & -\nu_4 & \equiv 0 \\ -\nu_1 & +2\nu_2 & & -\nu_4 & \equiv 0 \end{array}\right\} \bmod 5.$$

Putting $\nu_1 \equiv 0$, $\nu_2 \equiv 1$, one obtains $\nu_3 \equiv 3$, $\nu_4 \equiv 2 \bmod 5$. The relations of $\mathfrak{G}$ are easily verified in Figure 14.3.

Remark. Since $\Delta_1(-1)$ is always odd, only odd primes p occur.

B Homomorphisms of $\mathfrak{G}$ into the Group of Motions of the Euclidean Plane

We have interpreted the dihedral representations γ_p^* as homomorphisms of $\mathfrak{G}$ into the *group* $\mathfrak{B}$ *of motions of* E^2, and we studied a class of maps $\delta_\alpha\colon \mathfrak{G} \to \mathfrak{C}$ into the *group of similarities* $\mathfrak{C}$ *of the plane* E^2. It seems rather obvious to choose any other suitable conjugacy class in one of these well-known geometric groups as a candidate to map a Wirtinger class K onto. It would be especially interesting to obtain new non-metabelian representations, because metabelian representations necessarily map a longitude, see 3.12, onto units, and are, therefore, not adequate to exploit the peripheral system of the knot. We propose to "lift" the representation γ_p^* to a homomorphism $\gamma_p\colon \mathfrak{G} \to \mathfrak{B}$ which maps the Wirtinger class K into a class of glide-reflections. The representation γ_p will not be metabelian and will yield a useful tool in proving non-amphicheirality of knots. As above, *p is a prime*.

Let γ_p^* be a homomorphism of the knot group $\mathfrak{G}$ onto the dihedral group $\mathfrak{Z}_2 \ltimes \mathfrak{Z}_p$. There is a regular covering $q\colon R_p \to C$ corresponding to the normal subgroup $\ker \gamma_p^*$. One has $2\mathfrak{Z} \ltimes \mathfrak{G}' = \mathfrak{G}_2 \supset \ker \gamma_p^* \supset \mathfrak{G}''$ and $\mathfrak{G}_2/\ker \gamma_p^* \cong \mathfrak{Z}_p$. The space R_p is a p-fold cyclic covering of the 2-fold covering C_2 of C. For a meridian m and longitude ℓ of the knot $\mathfrak{k}$ we have: $m^2 \in \ker \gamma_p^*$, $\ell \in \mathfrak{G}'' \subset \ker \gamma_p^*$. The torus ∂C is covered by p tori T_i, $0 \leqq i \leqq p-1$, in R_p. There are distinguished canonical curves $\hat{m}_i$, $\hat{\ell}_i$ on T_i with $q(\hat{m}_i) = m^2$, $q(\hat{\ell}_i) = \ell$. By a theorem of H. Seifert [1932], the set $\{\hat{m}_i, \hat{\ell}_i\}$ of $2p$ curves contains a subset of p (> 2) linearly independent representatives of the Betti group of $H_1(R_p)$. From this it follows that the cyclic subgroup $\mathfrak{Z}_p \lhd \mathfrak{Z}_2 \ltimes \mathfrak{Z}_p$ of covering transformations operates non-trivially on the Betti group of $H_1(R_p)$. Now abelianize $\ker \beta_p^*$ and trivialize the torsion subgroup of $H_1(R_p) = \ker \gamma_p^*/(\ker \gamma_p^*)'$ to obtain a homomorphism of the knot group $\mathfrak{G}$ onto an extension $[\mathfrak{D}_p, \mathbb{Z}^q]$ of the Betti group $\mathbb{Z}^q$ of $H_1(R_p)$, $q \geqq p$, with factor group $\mathfrak{D}_p = \mathfrak{Z}_2 \ltimes \mathfrak{Z}_p$. The operation of $\mathfrak{D}_p$ on $\mathbb{Z}^q$ is the one induced by the covering transformations. We embed $\mathbb{Z}^q$ in a vector space $\mathbb{C}^q$ over the complex numbers and use a result of the theory of representations of finite groups: The dihedral group $\mathfrak{D}_p$ admits only irreducible representations of degree 1 and degree 2 over $\mathbb{C}$.

This follows from Burnside's formula and the fact that the degree must divide the order $2p$ of $\mathfrak{D}_p$. (See [van der Waerden 1955, §133].) Since $\mathfrak{Z}_p \lhd \mathfrak{D}_p$ operates non-trivially on $\mathbb{Z}^q$, the operation of $\mathfrak{D}_p$ on $\mathbb{C}^q$ contains at least one summand of degree 2. Such a representation has the form

$$\tau \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad a \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \tag{5}$$

with $\mathfrak{Z}_2 = \langle\, \tau \,\rangle$, $\mathfrak{Z}_p = \langle\, a \,\rangle$ and ζ a primitive p-th root of unity. (The representation is faithful, hence irreducible.)

This representation is equivalent to the following when $\mathbb{C}^2$ is replaced by $\mathbb{R}^4$:

$$\tau \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \alpha \mapsto \begin{pmatrix} \xi & -\eta & 0 & 0 \\ \eta & \xi & 0 & 0 \\ 0 & 0 & \xi & -\eta \\ 0 & 0 & \eta & \xi \end{pmatrix}, \quad \zeta = \xi + i\eta.$$

It splits into two identical summands. Introduce again a complex structure on each of the invariant subspaces $\mathbb{R}^2$; the operation of $\mathfrak{D}_p$ on each of them may then be described by:

$$\tau(z) = \bar{z}, \quad a(z) = \zeta z. \tag{6}$$

By this construction the knot group $\mathfrak{G}$ is mapped onto an extension of a finitely generated (additive) subgroup $\mathfrak{T} \subset \mathbb{C}$, $\mathfrak{T} \neq 0$, with factor group $\mathfrak{D}_p$ operating on $\mathfrak{T}$ according to (6). First consider the extension $[\mathfrak{Z}_p, \mathfrak{T}]$ and denote its elements by pairs (a^ν, z).

One has

$$(a,0)\big((a^{p-1},0)(a,0)\big) = (a,0)(1,w) = (a,w), \quad \text{for } w = a^p \in \mathfrak{T},$$

and

$$\big((a,0)(a^{p-1},0)\big)(a,0) = (1,w)(a,0) = (a,\zeta w).$$

It follows that $w = \zeta w$, $\zeta \neq 1$; hence, $w = 0$, and $[\mathfrak{Z}_p, \mathfrak{T}] = \mathfrak{Z}_p \ltimes \mathfrak{T}$. Similarly one may denote the elements of $[\mathfrak{D}_p, \mathfrak{T}] = [\mathfrak{Z}_2, \mathfrak{Z}_p \ltimes \mathfrak{T}]$ by triples (τ^ν, a^μ, z). Put $(\tau, 1, 0)^2 = (1, 1, v)$, $v \in \mathbb{C}$. Then

$$(\tau, 1, 2\bar{v}) = (\tau,1,0)^2(\tau,1,0) = (\tau,1,0)(\tau,1,0)^2 = (\tau,1,2v).$$

This proves $v = \bar{v} \in \mathbb{R}$.

We obtain a homomorphism $\gamma_p \colon \mathfrak{G} \to [\mathfrak{D}_p, \mathfrak{T}] \subset \mathfrak{B}$. Put

$$\begin{aligned} &(1,a,b): \ z \mapsto \zeta z + b, \quad \zeta \text{ a primitive } p\text{-th root of unity,} \\ &(\tau,1,0): \ z \mapsto \bar{z} + v. \end{aligned} \tag{7}$$

There are two distinct cases: $v \neq 0$ and $v = 0$. In the first case a Wirtinger generator is mapped onto a glide reflection whereas in the second case its image is a reflection. We may in the first case choose $v = 1$ by replacing a representation by an equivalent one.

14.10 Proposition. *For any dihedral representation $\gamma_p^* \colon \mathfrak{G} \to \mathfrak{Z}_2 \ltimes \mathfrak{Z}_p \subset \mathfrak{B}$ of a knot group $\mathfrak{G}$ into the group $\mathfrak{B}$ of motions of the plane there is a lifted representation $\gamma_p \colon \mathfrak{G} \to \mathfrak{B}$ such that $\gamma_p^* = \kappa \cdot \gamma_p$, $\kappa \colon \gamma_p(\mathfrak{G}) \to \gamma_p(\mathfrak{G})/\mathfrak{T}$, where $\mathfrak{T} \neq 0$ is the subgroup of translations in $\gamma_p(\mathfrak{G}) \subset \mathfrak{B}$ (p is a prime).*

An element of the Wirtinger class K is either mapped onto a glide reflection ($v = 1$) *or a reflection* ($v = 0$).

If γ_p^* is unique up to equivalence, that is, if p divides the first but not the second torsion coefficient of $H_1(\hat{C}_2)$, see 14.8, *the first case takes place and γ_p is determined up to equivalence.*

Proof. The existence of a lifted mapping γ_p has already been proved. We prove uniqueness by describing γ_p with the help of a system of linear equations which at the same time serves to carry out an effective calculation of the representation. Denote by $\mathbb{Q}(\zeta)$ the cyclotomic field over the rationals and by ζ_j a p-th root of unity. Put

$$\gamma_p^*(S_j) \colon \ z \mapsto \zeta_j^2 \bar{z} \tag{8}$$

$$\gamma_p(S_j) \colon \ z \mapsto \zeta_j^2 \bar{z} + b_j \tag{9}$$

for Wirtinger generators S_j of $\mathfrak{G} = \langle S_1, \ldots S_n \mid R_1, \ldots, R_n \rangle$. Equation (9) describes a reflection followed by the translation through

$$2\zeta_j v = \zeta_j^2 \bar{b}'_j + b'_j \tag{10}$$

in the direction of the fixed line. There are two cases: $v = 0$ or $v \neq 0$; in the latter case we normalize to $v = 1$. We prove that $v = 0$ cannot occur if the dihedral representation γ_p^* is unique up to equivalence, see 14.8. A Wirtinger relator

$$R_j = S_j S_i^{-\eta_j} S_k^{-1} S_i^{\eta_j}, \quad \eta_j = \pm 1, \tag{11}$$

yields

$$\zeta_i^2 = \zeta_j \zeta_k \tag{12}$$

under (8), and

$$-\bar{\zeta}_k b'_k - \bar{\zeta}_j b'_j + (\bar{\zeta}_k \zeta_i + \bar{\zeta}_j \zeta_i)\bar{\zeta}_i b'_i = 0 \tag{13}$$

under (9), if $v = 0$. Here we introduce the convention that on the right hand side of $\gamma_p(W_1 W_2) = \gamma_p(W_1)\gamma_p(W_2)$, $W_1, W_2 \in \mathfrak{G}$, the combination is carried out from right to left, as is usual in a group of motions, whereas in the fundamental group the combination $W_1 W_2$ is understood from left to right.

The linear equations (13) form a system over $\mathbb{Q}(\zeta)$ with real variables $x_j = \bar{\zeta}_j b'_j$ (use (10)). The rank of (13) is at least $n-2$, because the homomorphism $\psi : \mathbb{Q}(\zeta) \to \mathbb{Z}_p$, defined by $\psi(\zeta) = 1$, transforms (13) into the system of congruences mod p:

$$-\nu_k - \nu_j + 2\nu_i \equiv 0 \mod p \tag{14}$$

which has rank $= n - 2$ as γ_p^* is unique up to equivalence. (Compare 14.7 and (4), p. 252.) If there is a proper lift γ_p – that is $\mathfrak{T} \neq 0$ – the fixed lines g_i of $\gamma_p(S_i)$ cannot pass through one point or be parallel. But then there is a 3-dimensional manifold of such representations, obtained by conjugation with $\mathfrak{C}^+$, the orientation preserving group of similarities. This contradicts rank $\geqq n - 2$.

Remark. The non-existence of γ_p under our assumption $v = 0$ is a property of the Euclidean plane. In a hyperbolic plane where there are no similarities such lifts γ_p may exist.

We may assume that there is a lift γ_p of γ_p^* which maps Wirtinger generators on glide reflections with $v = 1$. Substitute

$$b'_j = \zeta_j b_j + \zeta_j. \tag{15}$$

Instead of (13) we get the following system of inhomogeneous linear equations

$$-b_k - b_j + (\bar{\zeta}_j \zeta_i + \bar{\zeta}_i \zeta_j) b_j = \eta_j (\bar{\zeta}_j \zeta_i - \bar{\zeta}_i \zeta_j). \tag{16}$$

(Observe that the equations (12) are valid.) We may again employ the homomorphism $\psi : \mathbb{Q}(\zeta) \to \mathbb{Z}_p$ to see that the rank of the homogeneous part of (16) is $n - 2$. Since conjugation by translations gives a 2-dimensional manifold of solutions, the rank of (16) is exactly equal to $n - 2$. □

For a given primitive p-th root of unity ζ and a suitable enumeration of the Wirtinger generators we may assume

$$\gamma_p(S_1)\colon z \mapsto \bar{z} + 1, \quad \gamma_p(S_2)\colon z \mapsto \zeta^2\bar{z} + \zeta.$$

This corresponds to putting $b_1 = b_2 = 0$. The fixed lines g_1 and g_2 of $\gamma_p(S_1)$ and $\gamma_p(S_2)$ meet in the origin and pass through 1 and ζ (Figure 14.4). A representation normalized in this way is completely determined up to the choice of ζ.

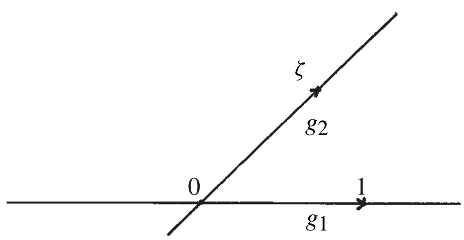


Figure 14.4

The main application of Proposition 14.10 is the exploitation of the peripheral system $(\mathfrak{G}, m, \ell)$ by a normalized representation γ_p. Let m be represented by S_1, then γ_p *maps the longitude ℓ onto a translation by $\lambda(\zeta)$:*

$$\gamma_p(\ell)\colon z \mapsto z + \lambda(\zeta),$$

since $\ell \in \mathfrak{G}'' \subset \ker \gamma_p^*$. The solutions b_j of (16) are elements of $\mathbb{Q}(\zeta)$. From $m \cdot \ell = \ell \cdot m$ it follows that $\lambda(\zeta) \in \mathbb{Q}(\zeta) \cap \mathbb{R}$.

14.11 Definition and Proposition. *Let $\mathfrak{G}(\mathbb{Q}(\zeta)|\,\mathbb{Q})$ be the Galois group of the extension $\mathbb{Q}(\zeta) \supset \mathbb{Q}$. The set $[\lambda(\zeta)] = \{\lambda(\tau(\zeta)) \mid \tau \in \mathfrak{G}(\mathbb{Q}(\zeta)|\,\mathbb{Q})\}$ is called the longitudinal invariant with respect to γ_p. It is an invariant of the knot.* □

14.12 Example. We want to lift the homomorphism γ_5^* of the group of the four-knot which we computed in 14.9. We had obtained $\zeta_1 = 1$, $\zeta_2 = \zeta$, $\zeta_3 = \zeta^3$, $\zeta_4 = \zeta^2$ for

$\gamma_5^*(S_j) = \zeta_j$, and we may put $\zeta = e^{2\pi i/5}$. The equations (16) are then

$$\begin{aligned}
-b_2 - b_1 + (\zeta^3 + \zeta^2)b_3 &= \zeta^3 - \zeta^2,\\
-b_3 - b_2 + (\zeta + \zeta^4)b_4 &= -(\zeta - \zeta^4),\\
-b_4 - b_3 + (\zeta^2 + \zeta^3)b_1 &= (\zeta^2 - \zeta^3),\\
-b_1 - b_4 + (\zeta^4 + \zeta)b_2 &= -(\zeta^4 - \zeta).
\end{aligned}$$

Putting $b_1 = b_2 = 0$ yields

$$b_3 = 1 + 2\zeta + 2\zeta^3, \quad b_4 = \zeta^4 - \zeta$$

and, using (15)

$$\begin{aligned}
&b_1' = 1, \quad b_2' = \zeta, \quad b_3' = -2(1+\zeta^2), \quad b_4' = \zeta + \zeta^2 - \zeta^3;\\
&\gamma_5(S_1)\colon z \mapsto \bar{z} + 1,\\
&\gamma_5(S_2)\colon z \mapsto \zeta^2\bar{z} + \zeta,\\
&\gamma_5(S_3)\colon z \mapsto \zeta\bar{z} - 2 - 2\zeta^2,\\
&\gamma_5(S_4)\colon z \mapsto \zeta^4\bar{z} + \zeta + \zeta^2 - \zeta^3.
\end{aligned}$$

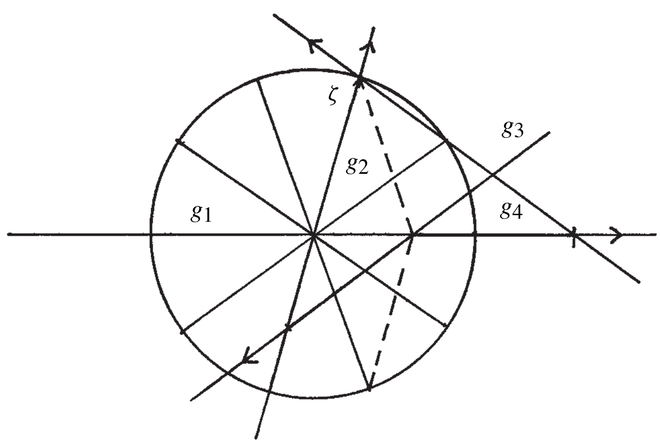


Figure 14.5

Figure 14.5 shows the configuration of the fixed lines g_j of the glide reflections $\gamma_5(S_j)$. One may verify the Wirtinger relations by well-known geometric properties of the regular pentagon. The longitude ℓ of $(\mathfrak{G}, m, \ell)$ with $m = S_1$ may be read off the projection drawn in Figure 14.3:

$$\ell = S_3^{-1} S_4 S_1^{-1} S_2.$$

One obtains

$$\gamma_5(\ell)\colon z \mapsto z + \lambda(\zeta), \quad \lambda(\zeta) = 2\big(\zeta + \zeta^{-1} - (\zeta^2 + \zeta^{-2})\big).$$

The class $[\lambda(\zeta)]$ contains only two different elements, $\lambda(\zeta)$ and $-\lambda(\zeta)$ which reflects the amphicheirality of the four-knot.

14.13 Proposition. *The invariant class* $[\lambda(\zeta)]$ *of an amphicheiral knot always contains* – $\lambda(\zeta)$ *if it contains* $\lambda(\zeta)$.

Proof. A conjugation by a rotation through π maps $(\gamma_p(m), \gamma_p(\ell))$ onto $(-\gamma_p(m), -\gamma_p(\ell))$. Hence, 3.19 implies that the group of an amphicheiral knot admits normalized representations γ_p and γ_p' with $\gamma_p(\ell^{-1}) = -\gamma_p(\ell) = \gamma_p'(\ell)$. □

Remark. The argument shows at the same time that the invariant $[\lambda(\zeta)]$ is no good at detecting that a knot is non-invertible. Similarly, γ_p is not strong enough to prove that a knot has Property P: a relation $\gamma_p(\ell^a) = \gamma_p(m)$, $a \neq 0$, would abelianize $\gamma_p(\mathfrak{G})$, and, hence, trivialize it.

The invariant has been computed and a table is contained in Appendix C, Table III.

Representations of the type γ_p have been defined for links [Hafer 1974], [Henninger 1978]. In [Hartley-Murasugi 1977] linking numbers in covering spaces were investigated in a more general setting which yielded the invariant $[\lambda(\zeta)]$ as a special case.

C Linkage in Coverings

The covering $q\colon R_p \to C$ of the complement C of a knot $\mathfrak{k}$ defined by $\ker \gamma_p^* \cong \pi_1 R_p$ is an invariant of $\mathfrak{k}$ as long as there is only one class of equivalent dihedral representations

$$\gamma_p^*\colon \pi_1(C) = \mathfrak{G} \to \mathfrak{D}_p = \mathfrak{Z}_2 \ltimes \mathfrak{Z}_p.$$

The same holds for the branched covering $\hat{q}\colon \hat{R}_p \to S^3$, obtained from R_p, with branching set $\mathfrak{k}$. *In the following p is a prime.*

The *linking numbers* $\mathrm{lk}(\hat{\mathfrak{k}}_i, \hat{\mathfrak{k}}_j)$ of the link $\hat{\mathfrak{k}} = \bigcup_{i=0}^{p-1} \hat{\mathfrak{k}}_i = \hat{q}^{-1}(\mathfrak{k})$ have been used since the beginning of knot theory to distinguish knots which could not be distinguished by their Alexander polynomials. $\ker \gamma_p^*$ is of the form $\langle t^2 \rangle \ltimes \mathfrak{N}$, t a meridian, and is contained in the subgroup $\langle t \rangle \ltimes \mathfrak{N} = \mathfrak{U} \subset \mathfrak{G}$ with $[\mathfrak{G} : \mathfrak{U}] = p$. The subgroup $\mathfrak{U}$ defines an irregular covering I_p, $\pi_1(I_p) \cong \mathfrak{U}$, and an associated branched covering $\hat{I}_p$ which was, in fact, used in [Reidemeister 1929, 1932] to study linking numbers. The regular covering $\hat{R}_p$ is a two-fold branched covering of $\hat{I}_p$, and its linking numbers $\mathrm{lk}(\hat{\mathfrak{k}}_i, \hat{\mathfrak{k}}_j)$ determine those in $\hat{I}_p$ [Hartley 1979]. We shall, therefore, confine ourselves mainly to $\hat{R}_p$.

Linking numbers exist for pairs of disjoint closed curves in $\hat{R}_p$ which represent elements of finite order in $H_1(\hat{R}_p)$ [Seifert-Threlfall 1934], [Stöcker-Zieschang 1985, 15.6].

In the preceding section we made use of a theorem in [Seifert 1932] which guarantees that there are at least p linearly independent free elements of $H_1(R_p)$ represented in the set $\{\hat{m}_0, \dots, \hat{m}_{p-1}, \hat{\ell}_0, \dots, \hat{\ell}_{p-1}\}$. To obtain more precise information, we now have to employ a certain amount of algebraic topology [Hartley-Murasugi 1977]. Consider a section of the exact homology sequence

$$\cdots \to H_2(\hat{R}_p, V; \mathbb{Q}) \xrightarrow{\partial_*} H_1(V; \mathbb{Q}) \xrightarrow{i_*} H_1(\hat{R}_p; \mathbb{Q}) \to \cdots$$

of the pair $(\hat{R}_p, V)$, where V is the union $V = \bigcup_{i=0}^{p-1} V_i$, $\partial V_i = T_i$, of the tubular neighbourhoods V_i of $\hat{\mathfrak{k}}_i$ in $\hat{R}_p$. As indicated, we use rational coefficients. The Lefschetz Duality Theorem [Stöcker-Zieschang 1985, 14.8.5] and excision yield isomorphisms

$$H^1(R_p; \mathbb{Q}) \cong H_2(R_p, \partial R_p; \mathbb{Q}) \cong H_2(\hat{R}_p, V; \mathbb{Q}).$$

One has [Stöcker-Zieschang 1985, 14.6.4 (b)]

$$\begin{aligned} &\Delta^*\colon H^1(R_p; \mathbb{Q}) \to H_2(\hat{R}_p, V; \mathbb{Q}), \\ &\langle z^1, z_1 \rangle = \operatorname{int}(z_2, z_1), \quad z_2 = \Delta^*(z^1), \quad z^1 \in H^1(R_p; \mathbb{Q}) \end{aligned}$$

where $\langle \, , \, \rangle$ denotes the Kronecker product. We claim that the surjective homomorphism

$$\partial_* \Delta^*\colon H^1(R_p; \mathbb{Q}) \to \ker i_*$$

is described by

$$z^1 \mapsto \sum_{i=0}^{p-1} \langle z^1, \hat{m}_i \rangle \hat{\ell}_i. \tag{1}$$

To prove (1) put

$$\partial_* \Delta^* z^1 = \partial_* z_2 = \sum_{j=0}^{p-1} a_j \hat{\ell}_j, \quad a_j \in \mathbb{Q}.$$

Let δ_i be a disk in T_i bounded by $\hat{m}_i = \partial \delta_i$. Then

$$\langle z^1, \hat{m}_i \rangle = \operatorname{int}(z_2, \partial \delta_i) = \operatorname{int}(\partial_* z_2, \delta_i) = \operatorname{int}\Big(\sum_{j=0}^{p-1} a_j \hat{\ell}_j, \delta_i \Big) = a_i.$$

Since $j_*\colon H_1(R_p; \mathbb{Q}) \to H_1(\hat{R}_p; \mathbb{Q})$, induced by the inclusion j, is surjective, $j^*\colon H^1(\hat{R}_p; \mathbb{Q}) \to H^1(R_p; \mathbb{Q})$ is injective. $j^*(H^1(\hat{R}_p))$ consists exactly of the homomorphisms $\varphi\colon H_1(R_p) \to \mathbb{Q}$ which factor through $H_1(\hat{R}_p;\mathbb{Q})$. But these constitute $\ker \partial_*\Delta^*$ by (1). Thus, one has

$$\dim\ker\partial_*\Delta^* = \dim H^1(\hat{R}_p) = \dim H_1(\hat{R}_p) \quad \text{and}$$
$$\dim \partial_*\Delta^*(H^1(R_p;\mathbb{Q})) = \dim\ker i_*.$$

14.14 Proposition (Hartley-Murasugi).

$$\dim H_1(R_p;\mathbb{Q}) = \dim H_1(\hat{R}_p;\mathbb{Q}) + \dim\ker i_*$$

where $i\colon V \to \hat{R}_p$ is the inclusion. □

It is now easy to prove that only two alternatives occur:

14.15 Proposition. *Either* (*case* 1) *all longitudes $\hat{\ell}_i$, $0 \leqq i \leqq p-1$ represent in $H_1(\hat{R}_p;\mathbb{Z})$ elements of finite order* (*linking numbers are defined*) *and the meridians $\hat{m}_i$, $0 \leqq i \leqq p-1$, generate a free abelian group of rank p in $H_1(R_p;\mathbb{Z})$, or* (*case* 2) *the longitudes ℓ_i generate a free abelian group of rank $p-1$ presented by $\langle\, \hat{\ell}_0,\dots,\hat{\ell}_{p-1} \mid \hat{\ell}_0+\hat{\ell}_1+\cdots+\hat{\ell}_{p-1}\,\rangle$, and the meridians $\hat{m}_i$ generate a free group of rank one in $H_1(R_p;\mathbb{Z})$; more precisely, $\hat{m}_i \sim \hat{m}_j$ in $H_1(R_p;\mathbb{Q})$ for any two meridians.*

Proof. A Seifert surface S of $\mathfrak{k} = \partial S$ lifts to a surface $\hat{S}$ with $\partial\hat{S} = \sum_{i=0}^{p-1}\hat{\ell}_i \sim 0$ in R_p or $\hat{R}_p$: the construction of C_2 (see 4.4) shows that S can be lifted to S_2 in C_2 resp. $\hat{C}_2$. The inclusion $i\colon S_2 \to \hat{C}_2$ induces an epimorphism $i_*\colon H_1(S_2) \to H_1(\hat{C}_2)$. This follows from $(a^- + a^+) = Fs$ (see 8.6) and $a^+ = ta^- = -a^-$ in the case of the twofold covering where $t=-1$ (see Remark on p. 120). Thus S_2 is covered in R_p by a connected surface $\hat{S}$ bounded by the longitudes $\hat{\ell}_i$. If the longitudes $\hat{\ell}_i$ satisfy in $H_1(\hat{R}_p)$ only relations $c\cdot\Sigma\hat{\ell}_i \sim 0$, $c\in\mathbb{Z}$, which are consequences of $\Sigma\hat{\ell}_i \sim 0$, we have $\dim(\ker i_*) = 1$ in Proposition 14.14. Hence the meridians $\hat{m}_i$ generate a free group of rank one in $H_1(R_p)$. There is a covering transformation of $R_p \to C_2$ which maps $\hat{m}_i$ onto $\hat{m}_j \sim r\hat{m}_i$, $i\neq j$, $r\in\mathbb{Q}$. From this one gets $r^p = 1$, thus $r=1$. This disposes of case 2. If the longitudes $\hat{\ell}_i$ are subject to a relation that is not a consequence of $\Sigma\hat{\ell}_i \sim 0$, then one may assume $\Sigma a_i\hat{\ell}_i \sim 0$, $\Sigma a_i \neq 0$. (If necessary, replace a_i by a_i+1.) Applying the cyclic group $\mathfrak{Z}_p$ of covering transformation to this relation yields a set of p relations forming a cyclic relation matrix. Such a cyclic determinant is always different from zero [Neiss 1962, §19.6]. Hence, the longitudes generate a finite group. In fact, since the $\hat{\ell}_i$ are permuted by the covering transformations their orders coincide; we denote it by $|\ell|$ = order of $\hat{\ell}_i$ in $H_1(\hat{R}_p)$. It follows that $\dim\ker i_* = p$, and by 14.14 that the meridians $\hat{m}_i$ generate a free group of rank p. □

14.16 Proposition. *If there is exactly one class of equivalent dihedral homomorphisms $\gamma_p^*\colon \mathfrak{G} \to \mathfrak{D}_p$,* (*$p$ divides the first torsion coefficient of $H_1(\hat{C}_2)$ but not the* *second), then the dihedral linking numbers* $\nu_{ij} = \mathrm{lk}(\hat{\mathfrak{k}}_i, \hat{\mathfrak{k}}_j)$ *are defined* (*case* 1). *The invariant* $[\lambda(\zeta)]$ (*see* 14.11) *associated to the lift* γ_p *of* γ_p^* (14.10) *then takes the form*

$$\lambda_j(\zeta) = 2\sum_{i=0}^{p-1} \nu_{ij}\zeta^i \quad \text{with} \nu_{ii} = -\sum_{j\neq i} \nu_{ij}. \tag{2}$$

(*Here we have put* $[\lambda(\zeta)] = \{\lambda_j(\zeta) \mid 1 \leqq j < p\}$. *Case* 1 *and case* 2 *refer to* 14.15.)

Proof. The occurrence of case 2 implies $\gamma_p(\hat{m}_i) = \gamma_p(\hat{m}_j)$ for all meridians $\hat{m}_i, \hat{m}_j$. But in the case of a representation γ_p, mapping Wirtinger generators on glide reflections, $\gamma_p(\hat{m}_i)$ and $\gamma_p(\hat{m}_j)$ will be translations in different directions for some i, j. Thus the Wirtinger class is mapped onto reflections, that is, $\gamma_p(\hat{m}_i) = 0$. This contradicts 14.10.

In case 1 the longitudes $\hat{\ell}_j$ are of finite order in $H_1(\hat{R}_p; \mathbb{Z})$. Since the covering transformations permute the $\hat{\ell}_j$, they all have the same order $|\hat{\ell}_j| = |\ell|$. Consider a section of the Mayer–Vietoris sequence:

$$\cdots \longrightarrow H_1(\partial V) \xrightarrow{\psi_*} H_1(R_p) \oplus H_1(V) \xrightarrow{\varphi_*} H_1(\hat{R}_p) \longrightarrow \cdots .$$

Since $\varphi_*(|\ell|\hat{\ell}_j, 0) = 0$, one has, for suitable $b_k, c_k \in \mathbb{Z}$,

$$(|\ell|\hat{\ell}_j, 0) = \psi_*\Big(\sum_{k=0}^{p-1} b_k\hat{m}_k + \sum_{k=0}^{p-1} c_k\hat{\ell}_k\Big) = \Big(\sum b_k\hat{m}_k + \sum c_k\hat{\ell}_k, -\sum c_k\hat{\ell}_k\Big).$$

This gives

$$|\ell|\hat{\ell}_j = \sum_{k=0}^{p-1} b_k\hat{m}_k \quad \text{and} \quad |\ell|\cdot \mathrm{lk}(\hat{\ell}_i, \hat{\ell}_j) = \mathrm{lk}(\hat{\ell}_i, \sum b_k\hat{m}_k) = b_i.$$

Since $\mathrm{lk}(\hat{\ell}_i, \hat{\ell}_j) = \mathrm{lk}(\hat{\mathfrak{k}}_i, \hat{\mathfrak{k}}_j)$, one has

$$\hat{\ell}_j = \Sigma\nu_{ij}\hat{m}_i. \tag{3}$$

The relation $\sum_{j=0}^{p-1} \hat{\ell}_j \sim 0$ yields $0 = \mathrm{lk}(\hat{\ell}_i, \Sigma\hat{\ell}_j) = \sum_j \nu_{ij}$. Formula (2) of 14.16 follows from $\gamma_p(\hat{m}_j)\colon z \mapsto z + 2\zeta^j$ for a suitable indexing after the choice of a primitive p-th root of unity ζ. □

Remark. Evidently any term $\sum_{i=0}^{p-1} a_i\zeta^i$, $a_i \in \mathbb{Q}$, can be uniquely normalized such that $\Sigma a_i = 0$ holds. In Table III the invariant $[\lambda(\zeta)]$ is listed, but a different normalization was chosen: $a_0 = 0$. One obtains from a sequence $\{a_1, \ldots, a_{p-1}\}$ in this table a set of linking numbers ν_{0j}, $0 < j \leqq p-1$, by the formula

$$2\nu_{0j} = a_j - \frac{1}{p}\sum_{k=1}^{p-1} a_k. \tag{4}$$

14.17. Linking numbers associated with the dihedral representations $\gamma_\alpha^*\colon \mathfrak{G} \to \mathfrak{Z}_2 \ltimes \mathfrak{Z}_\alpha$ for two bridge knots $\mathfrak{b}(\alpha,\beta)$ have been computed explicitly. In this case a unique lift γ_α always exists even if α is not a prime. The linking matrix is

$$\begin{pmatrix} -\sum \varepsilon_j & \varepsilon_1 & \dots & \varepsilon_{\alpha-1} \\ \varepsilon_{\alpha-1} & -\sum \varepsilon_j & \dots & \varepsilon_{\alpha-2} \\ \vdots & \vdots & & \vdots \\ \varepsilon_1 & \varepsilon_2 & \dots & -\sum \varepsilon_j \end{pmatrix} \tag{5}$$

with $\varepsilon_k = (-1)^{\left[\frac{k\beta}{\alpha}\right]}$, $[x]$ = integral part of x and $\sum = \sum_{i=1}^{\alpha-1}$ [Burde 1975].

The property $|\varepsilon_k| = 1$ affords a good test for two-bridged knots. "Most" of the knots with more than 2 bridges (see Table I) can be detected by this method, compare [Perko 1976].

A further property of dihedral linking numbers follows from the fact that $\lambda_i(\zeta)$ is a real number, $\lambda_i(\zeta) = \overline{\lambda_i(\zeta)}$. This gives.

$$\nu_{i,i-j} = \nu_{ij}, \quad i \neq j, \tag{6}$$

where $i - j$ is to be taken $\bmod p$. Furthermore,

$$\nu_{ij} = \nu_{ji} = \nu_{i+k,j+k}. \tag{7}$$

The first equation expresses a general symmetry of linking numbers, and the second one the cyclic p-symmetry of $\hat{R}_p$.

As mentioned at the beginning of this section, $\hat{R}_p$ is a two-fold branched covering of the irregular covering space $\hat{I}_p$ with one component $\hat{\mathfrak{k}}_j$ of $\hat{\mathfrak{k}} = q^{-1}(\mathfrak{k})$ as branching set in $\hat{R}_p$. (There are, indeed, p equivalent covering spaces $\hat{I}_p$ corresponding to p conjugate subgroups $\mathfrak{U}_j = \langle t_j \rangle \ltimes \mathfrak{K}$, depending on the choice of the meridian t_j resp. the component $\hat{\mathfrak{k}}_j$.) We choose $j = 0$. Let $\hat{q}\colon \hat{R}_p \to \hat{I}_p$ be the covering map. The link $\mathfrak{k}' = \hat{q}(\hat{\mathfrak{k}})$ consists of $\frac{p+1}{2}$ components $\hat{\mathfrak{k}}'_0 = \hat{q}(\hat{\mathfrak{k}}_0)$, $\mathfrak{k}'_j = \hat{q}(\hat{\mathfrak{k}}_j) = \hat{q}(\hat{\mathfrak{k}}_{-j})$, $0 < j \leqq \frac{p-1}{2}$. (Indices are read $\bmod p$.) Going back to the geometric definition of linking numbers by intersection numbers one gets for $\mu_{ij} = \operatorname{lk}(\mathfrak{k}'_i, \mathfrak{k}'_j)$,

$$\mu_{0j} = 2\nu_{0j}, \quad \mu_{ij} = \nu_{ij} + \nu_{i,-j}, \quad i \neq j. \tag{8}$$

This yields by (6) and (7) Perko's identities [Perko 1976]:

$$2\mu_{ij} = \mu_{0,i-j} + \mu_{0,i+j}, \quad \text{or} \quad \mu_{ij} = \nu_{0,i-j} + \nu_{0,i+j}. \tag{9}$$

As $\nu_{ij} = \pm 1$ for two bridge knots, $\mu_{ij} = \pm 2$ or 0 for these.

It follows from (7), (8) and 14.16 that the linking numbers ν_{ij}, the linking numbers μ_{ij}, and the invariant $[\lambda(\zeta)]$ determine each other. All information is already contained in the ordered set $\left\{\nu_{0j} \mid 1 \leqq j \leqq \frac{p-1}{2}\right\}$. Equation (8) shows that [Hartley-Murasugi 1977, Theorem 6.3] is a consequence of 14.16.

The theory developed in this section has been generalized in [Hartley 1983]. Many results carry over to metacyclic homomorphisms $\beta^*_{r,p}\colon \mathfrak{G} \to \mathfrak{Z}_r \ltimes \mathfrak{Z}_p$, see 14.7 and [Burde 1970]. The homomorphism $\beta^*_{r,p}$ can be lifted and the invariant $[\lambda(\zeta)]$ can be generalized to the metacyclic case. This invariant has a new quality, in that it can identify non-invertible knots which $[\lambda(\zeta)]$ cannot, as we pointed out at the end of Section B, [Hartley 1983′].

14.18 Examples (a) The four-knot is a two-bridge knot, $4_1 = \mathfrak{b}(5, 3)$. Thus

$$\nu_{0j} = (-1)^{\left[\frac{3j}{5}\right]}, \quad (\nu_{ij}) = \begin{pmatrix} 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & 1 & -1 & -1 \\ -1 & 1 & 0 & 1 & -1 \\ -1 & -1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 \end{pmatrix},$$

and

$$(\mu_{ij}) = \begin{pmatrix} * & 2 & -2 \\ 2 & * & 0 \\ -2 & 0 & * \end{pmatrix}.$$

The link $\mathfrak{k}' = \hat{q}^{-1}(4_1)$ in $\hat{I}_5 \cong S^3$ has been determined (Figure 14.6) in [Burde 1971]. (For the definition of $\hat{I}_p$ see the beginning of Section C.)

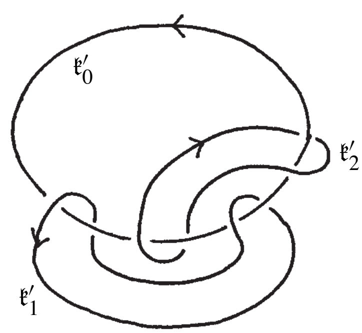


Figure 14.6

(b) As a second example consider the knot $7_4 = \mathfrak{b}(15, 11)$ and the irregular covering $\hat{I}_{15}$. Its linking matrix (μ_{ij}) is

$$(\mu_{ij}) = \begin{pmatrix} * & 2 & -2 & 2 & 2 & -2 & 2 & -2 \\ 2 & * & 2 & 0 & 0 & 2 & -2 & 0 \\ -2 & 2 & * & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & * & 0 & -2 & 2 & 0 \\ 2 & 0 & 0 & 0 & * & 2 & -2 & 2 \\ -2 & 2 & 0 & -2 & 2 & * & 2 & 0 \\ 2 & -2 & 0 & 2 & -2 & 2 & * & 0 \\ -2 & 0 & 0 & 0 & 2 & 0 & 0 & * \end{pmatrix}$$

by (9) and $\nu_{0j} = (-1)^{\left[\frac{11j}{15}\right]}, 0 < j < 15$.

The numbers $\frac{1}{2}\sum_{j \neq i} |\mu_{ij}| = v_i$, $0 \leqq i \leqq 7$, are 7, 4, 2, 3, 4, 5, 5, 2. (Compare [Reidemeister 1932, p. 69].)

In general an effective computation of linking numbers can be carried out in various ways. One may solve equations (14) and (16) in the proof of 14.10 and thereby determine γ_p^*, γ_p and $[\lambda(\zeta)]$. A more direct way is described in [Hartley-Murasugi 1977] using the Reidemeister–Schreier algorithm. See also [Perko 1974].

D Periodic Knots

Some knots show geometric symmetries – for instance torus knots. The term "geometric" implies "metric", a category into which topologists usually do not enter. Nevertheless, symmetries have been defined and considered in various ways [Fox 1962‴]. We shall, however, occupy ourselves with only one of the different versions of symmetry, the one most frequently investigated. It serves in this chapter as an application of the metabelian representation δ_α of the knot group introduced in 14.5 – in this section $\mathfrak{k}$ will always have one component.

A knot will be said to have period $q > 1$, if it can be represented by a curve in euclidean 3-space E^3 which is mapped onto itself by a rotation r of E^3 of order q. The axis h must not meet the knot. The positive solution of the Smith conjecture (see Appendix B.8) allows a topological definition of periodicity.

14.19 Definition. A *knot* $\mathfrak{k} \subset S^3$ *has period* $q > 1$, if there is an orientation preserving homeomorphism $S^3 \to S^3$ of order q with a set of fixed points $h \cong S^1$ disjoint from $\mathfrak{k}$ and mapping $\mathfrak{k}$ into itself.

Remark. The orientation of $\mathfrak{k}$ is not essential in this definition. A period of an unoriented knot automatically respects an orientation of the knot (E 14.7).

Suppose a knot $\mathfrak{k}$ has period q. We assume that a regular projection of $\mathfrak{k}$ onto a plane perpendicular to the axis of the rotation has period q with respect to a rotation of the plane (Figure 14.7). Denote by $E_q^3 = E^3/\mathfrak{Z}_q$ the Euclidean 3-space which is the quotient space of E^3 under the action of $\mathfrak{Z}_q = \langle r \rangle$. There is a cyclic branched covering $f^{(q)}\colon E^3 \to E_q^3$ with branching set h in E^3 and $f^{(q)}(\mathfrak{k}) = \mathfrak{k}^{(q)}$ a knot in E_q^3. We call $\mathfrak{k}^{(q)}$ the *factor knot* of $\mathfrak{k}$. It is obtained from $\mathfrak{k}$ in Figure 14.8 by identifying x_i and z_i.

One has $\lambda = \mathrm{lk}(\mathfrak{k}, h) = \mathrm{lk}(\mathfrak{k}^{(q)}, h^{(q)}) \neq 0$, $h^{(q)} = f^{(q)}(h)$. The equality of the linking numbers follows by looking at the intersection of $\mathfrak{k}$ resp. $\mathfrak{k}^{(q)}$ with half-planes in E^3 resp. E_q^3 spanning h resp. $h^{(q)}$. If $\lambda = \mathrm{lk}(\mathfrak{k}^{(q)}, h^{(q)}) = 0$, then $\mathfrak{k}^{(q)} \simeq 1$ in $\pi_1(E_q^3 - h^{(q)})$, and $\mathfrak{k} \subset E^3$ would consist of q components. By choosing a suitable direction of h we may assume $\lambda > 0$. Moreover, $\gcd(\lambda, q) = 1$, see E 14.8.

The symmetric projection (Figure 14.7) yields a symmetric Wirtinger presentation of the knot group of $\mathfrak{k}$ (see 3.4):

$$\mathfrak{G} = \langle x_i^{(0)}, y_k^{(0)}, z_i^{(0)}, x_i^{(1)}, y_k^{(1)}, z_i^{(1)}, \cdots \mid R_j^{(0)}, R_j^{(1)}, \ldots, x_i^{(l)} = z_i^{(l-1)}, \ldots \rangle, \qquad 1 \leqq i \leqq n, \quad 1 \leqq k \leqq m, \quad 1 \leqq j \leqq m+n. \tag{1}$$

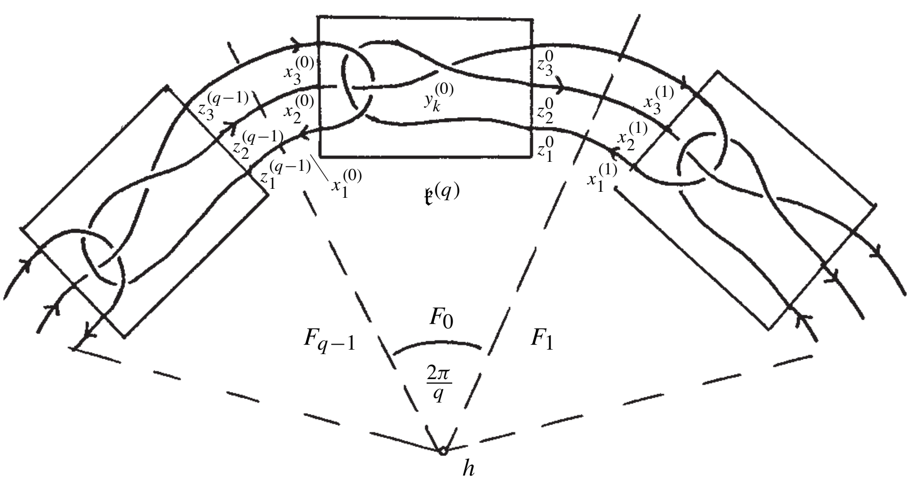


Figure 14.7

The arcs entering a fundamental domain F_0 of $\mathfrak{Z}_q$, a $2\pi/q$-sector, from the left side, correspond to generators $x_i^{(0)}$ and its images under the rotation r to generators $z_i^{(0)} = r_\#(x_i^{(0)})$. The remaining arcs in F_0 give rise to generators $y_k^{(0)}$. Double points in F_0 define relators R_j. The generators $x_i^{(l)}, y_k^{(l)}, z_i^{(l)}$, $0 \leqq l \leqq q-1$, correspond to the images of the arcs of $x_i^{(0)}, y_k^{(0)}, z_i^{(0)}$ under the rotation through $2\pi l/q$, and $R_j^{(l)} = R_j^{(0)}(x_i^{(l)}, y_k^{(l)}, z_i^{(l)})$.

The Jacobian of the Wirtinger presentation

$$\left(\frac{\partial R_j^{(l)}}{\partial(x_i^{(l)}, y_k^{(l)}, z_i^{(l)})}\right)^{\varphi\psi} = A(t), \quad \varphi(x_i^{(l)}) = \varphi(y_k^{(l)}) = \varphi(z_i^{(l)}) = t,$$

see 9.9, is of the following form:

$$\left(\begin{array}{c|c|c|c}
-E_n & E_n & & \\ \hline
\overline{A}(t) & & & \\ \hline
& -E_n & E_n & \\ \hline
& \overline{A}(t) & & \\ \hline
& & & \\ \hline
E_n & & & -E_n \\ \hline
& & & \overline{A}(t)
\end{array}\right) = A(t).$$

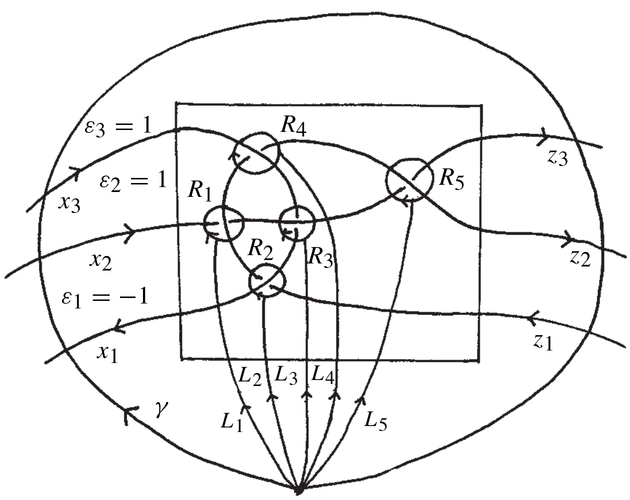


Figure 14.8

Here E_n is an $n \times n$ identity matrix, and $\bar{A}(t)$ is a $(n+m) \times (2n+m)$ matrix over $\mathbb{Z}(t)$. We rearrange rows and columns of $A(t)$ in such a way that the columns correspond to generators ordered in this way:

$$x_1^{(0)}, x_1^{(1)}, \dots, x_1^{(q-1)}, x_2^{(0)}, x_2^{(1)}, \dots, x_n^{(q-1)}, y_1^{(0)}, \dots, y_m^{(q-1)}, z_1^{(0)}, \dots, z_n^{(q-1)}.$$

The relators and rows have the following order:

$$x_1^{(1)}(z_1^{(0)})^{-1},\ x_1^{(2)}(z_1^{(1)})^{-1}, \dots, x_1^{(0)}(z_1^{(q-1)})^{-1}, \dots,\ R_1^{(0)}, R_1^{(1)}, \dots.$$

This gives a matrix

$$A^*(t) = \left(\begin{array}{ccccccc} Z_q & & & & -E_q & & \\ & Z_q & & & & \ddots & \\ & & Z_q & & & & \\ & & & \ddots & & -E_q & \\ & & & Z_q & & & -E_q \\ \hdashline & & & \overline{A}^*(t) & & & \end{array}\right).$$

Here $\bar{A}^*(t)$ is obtained from $\bar{A}(t)$ by replacing every element $a_{ik}(t)$ of $\bar{A}(t)$ by the $q \times q$ diagonal matrix

$$a_{ik}^{(q)}(t) = \begin{pmatrix} a_{ik}(t) & & & 0 \\ & a_{ik}(t) & & \\ & & \ddots & \\ 0 & & & a_{ik}(t) \end{pmatrix}.$$

The $q \times q$-matrix

$$Z_q = \begin{pmatrix} 0 & 1 & & & 0 \\ 0 & 0 & 1 & & \\ \vdots & & & \ddots & \\ & & & & 1 \\ 1 & & & & 0 \end{pmatrix}$$

is equivalent to the diagonal matrix

$$Z(\zeta) = W Z_q W^{-1} = \begin{pmatrix} 1 & & & & 0 \\ & \zeta & & & \\ & & \zeta^2 & & \\ & & & \ddots & \\ 0 & & & & \zeta^{q-1} \end{pmatrix}$$

over $\mathbb{C}$ where ζ is a primitive q-th root of unity (Exercise E 14.9). The matrix $\tilde{W}A^*(t)\tilde{W}^{-1}$ with

$$\tilde{W} = \left(\begin{array}{c|c|c|c} W & & & \\ \hline & W & & \\ \hline & & \ddots & \\ \hline & & & W \end{array}\right)$$

may be obtained from $A^*(t)$ by replacing the submatrices Z_q by $Z(\zeta)$. Returning to the original ordering of rows and columns as in $A(t)$, the matrix $\tilde{W}A^*(t)\tilde{W}^{-1}$ takes the form

$$A(t,\zeta) = \left(\begin{array}{c|c|c|c} A^{(q)}(t,1) & & & \\ \hline & A^{(q)}(t,\zeta) & & 0 \\ \hline & & \ddots & \\ \hline 0 & & & A^{(q)}(t,\zeta^{q-1}) \end{array}\right) \tag{2}$$

where

$$A^{(q)}(t,\zeta^\nu) = \left(\begin{array}{ccc|ccc} \zeta^\nu & & & -1 & & \\ & \ddots & & & \ddots & \\ & & \zeta^\nu & & & -1 \\ \hline \multicolumn{6}{c}{\overline{A}(t)} \end{array}\right)$$

$A(t,\zeta)$ is equivalent to $A(t)$ over $\mathbb{C}$, and $A^{(q)}(t,1)$ is a Jacobian of the factor knot $\mathfrak{k}^{(q)}$. We replace ζ^ν by a variable τ and prove:

14.20 Proposition. $\det(A^{(q)}(t,\tau)) = (\tau - 1)D(t,\tau)$ *with*

$$D(t,1) \doteq \varrho_\lambda(t)\Delta_1^{(q)}(t) \quad \textit{where } \varrho_\lambda(t) = 1 + t + \cdots + t^{\lambda-1},\ \lambda = \mathrm{lk}(h,\mathfrak{k}).$$

$\Delta_1^{(q)}(t)$ *is the Alexander polynomial of the factor knot* $\mathfrak{k}^{(q)}$.

Proof. Replace the first column of $A^{(q)}(t,\tau)$ by the sum of all columns and expand according to the first column:

$$\det(A^{(q)}(t,\tau)) = (\tau - 1)\cdot\sum_{i=1}^{n} D_i(t,\tau)$$

where $(-1)^{i+1}D_i(t,\tau)$ denotes the minor obtained from $A^{(q)}(t,\tau)$ by omitting the first column and i-th row. This proves the first assertion for $D(t,\tau)=\sum_{i=1}^{n} D_i(t,\tau)$. To prove the second one we show that the rows $\mathfrak{a}_l$ of the Jacobian

$$A^{(q)}(t,1)=\left(\begin{array}{cccc} 1 & & -1 & \\ & \ddots & & \ddots \\ & 1 & & -1 \\ \hline & & \overline{A}(t) & \end{array}\right)$$

of $\mathfrak{k}^{(q)}$ satisfies a special linear dependence

$$\sum_{l=1}^{2n+m}\alpha_l\mathfrak{a}_l=0 \quad \text{with} \sum_{l=1}^{n}\alpha_l=\varrho_\lambda(t).$$

(Compare 9.12 (b).) Denote by $\mathfrak{F}$ the free group generated by $\{X_i, Y_k, Z_i \mid 1 \leqq i \leqq n, 1 \leqq k \leqq m\}$, $\psi(X_i)=x_i^{(0)}$, $\psi(Y_k)=y_k^{(0)}$, $\psi(Z_i)=z_i^{(0)}$. There is an identity

$$\Big(\prod_{i=1}^{n}X_i^{\varepsilon_i}\Big)\Big(\prod_{i=1}^{n}Z_i^{\varepsilon_i}\Big)^{-1}=\prod_{j=1}^{n+m}L_jR_jL_j^{-1} \tag{3}$$

for $L_j \in \mathfrak{F}$, $\varepsilon_i=\pm 1$, and $R_j=R_j^{(0)}(X_i,Z_k,Z_i)$. This follows by the argument used in the proof of 3.6: The closed path γ in Figure 14.8 can be expressed by both sides of equation (3). From this we define:

$$\alpha_l=\frac{\partial}{X_l}\Big(\prod_{i=1}^{n}X_i^{\varepsilon_i}\Big)^{\varphi\psi}=\sum_{j=1}^{n+m}(L_j)^{\varphi\psi}\left(\frac{\partial R_j}{\partial X_l}\right)^{\varphi\psi},$$

hence

$$-\alpha_l=\sum_{j=1}^{n+m}(L_j)^{\varphi\psi}\left(\frac{\partial R_j}{\partial Z_l}\right)^{\varphi\psi}, \quad 1 \leqq l \leqq n,$$

$$0=\sum_{j=1}^{n+m}(L_j)^{\varphi\psi}\left(\frac{\partial R_j}{\partial Y_k}\right)^{\varphi\psi}, \quad 1 \leqq k \leqq m.$$

Putting $\alpha_{n+j}=-(L_j)^{\varphi\psi}$, $1 \leqq j \leqq n+m$, gives $\sum_{i=1}^{2n+m}\alpha_i\mathfrak{a}_i=0$. The fundamental formula 9.8 (c) yields

$$(t-1)\sum_{l=1}^{n}\alpha_l=\sum_{l=1}^{n}\frac{\partial}{\partial X_l}\Big(\prod_{i=1}^{n}X_i^{\varepsilon_i}\Big)^{\varphi\psi}(t-1)=\Big(\prod_{i=1}^{n}X_i^{\varepsilon_i}\Big)^{\varphi\psi}-1=t^\lambda-1,$$

hence $\sum_{l=1}^{n} \alpha_l = \varrho_\lambda(t)$. Now $D_1(t,1) \doteq \Delta_1^{(q)}(t)$, and $\alpha_i D_1(t,1) = D_i(t,1)$. The last equation is a consequence of $\Sigma \alpha_i \mathfrak{a}_i = 0$, compare 10.20. □

14.21 Proposition (Murasugi). *The Alexander polynomial $\Delta_1(t)$ of a knot $\mathfrak{k}$ with period q satisfies the equation*

$$\Delta_1(t) \doteq \Delta_1^{(q)}(t) \cdot \prod_{i=1}^{q-1} D(t, \zeta^i). \tag{4}$$

Here $D(t,\zeta)$ is an integral polynomial in two variables with

$$D(t,1) \doteq \varrho_\lambda(t)\, \Delta_1^{(q)}(t),$$

and ζ is a primitive q-th root of unity. $0 < \lambda = \mathrm{lk}(h, \mathfrak{k})$ is the linking number of $\mathfrak{k}$ with the axis h of rotation.

Proof. To determine the first elementary ideal of $A(t,\zeta)$, see (2), it suffices to consider the minors obtained from $A(t,\zeta)$ by omitting an i-th row and a j-th column, $1 \leqq i, j \leqq n$, because $\det(A^{(q)}(t,1)) = 0$. Proposition 14.21 follows from the fact that $A^{(q)}(t,1)$ is a Jacobian of $\mathfrak{k}^{(q)}$. □

14.22 Corollary (Murasugi's congruence).

$$\Delta_1(t) \mathrel{\dot\equiv} (\Delta_1^{(p^a)}(t))^{p^a} \cdot (\varrho_\lambda(t))^{p^a-1} \mod p \quad \textit{for } p^a | q,\ p \textit{ a prime.}$$

Proof. A knot $\mathfrak{k}$ with period q also has period p^a, $p^a|q$. Let $\mathcal{O}(p^a)$ denote the cyclotomic integers in $\mathbb{Q}(\zeta)$, ζ a p^a-th root of unity. There is a homomorphism

$$\Phi_p \colon \mathcal{O}(p^a) \to \mathbb{Z}_p, \quad \sum_{i=1}^{p^a} n_i \zeta^i \mapsto \sum_{i=1}^{p^a} [n_i] \mod p.$$

Extending Φ_p to the rings of polynomials over $\mathcal{O}(p^a)$ resp. $\mathbb{Z}_p$ yields the corollary. □

14.23 Proposition. *Let $\mathfrak{k}$ be a knot of period p^a and $\Delta_1(t) \not\equiv 1 \mod p$. Then $D(t,\zeta_i)$ is not a monomial for some p^a-th root of unity $\zeta_i \neq 1$. Any common root of $\Delta_1^{(p^a)}(t)$ and $D(t,\zeta_i)$ is also a root of $\Delta_2(t)$. If all roots of $D(t,\zeta_i)$ are roots of $\Delta_1^{(p^a)}(t)$, then $\lambda \equiv \pm 1 \mod p$.*

Proof. If $D(t,\zeta_i)$ is monomial, $1 \leqq i \leqq p^a$, then (4) yields $\Delta_1(t) = \Delta_1^{(p^a)}(t)$. Apply Φ_p to this equation and use 14.22 to obtain $\Delta_1^{(p^a)} \mathrel{\dot\equiv} 1 \mod p$ and $\lambda = 1$. From this it follows that $\Delta_1(t) \equiv 1 \mod p$.

Suppose now that $D(t, \zeta_i)$ and $\Delta_1^{(p^r)}(t)$ have a common root η. Transform $A(t, \zeta)$ over $\mathbb{Q}(\zeta)[t]$ into a diagonal matrix by replacing each block $A^{(q)}(t, \zeta_i)$, $0 \leqq i \leqq p^a$, see (2), by an equivalent diagonal block. Since $\det(A^q(t, 1)) = 0$, it follows that the second elementary ideal $E_2(t)$ vanishes for $t = \eta$; hence, $\Delta_2(\eta) = 0$.

If all roots of $D(t, \zeta_i)$ are roots of $\Delta_1^{(p^a)}(t)$, every prime factor $f(t)$ of $D(t, \zeta_i)$ is a prime factor of $\Delta_1^{(p^a)}(t)$ in $\mathbb{Q}(\zeta)[t]$.

Since $\Delta_1^{(p^a)}(1) = \pm 1$, it follows that $\Phi_p(f(1)) \equiv \pm 1 \bmod p$. But $|D(1, 1)| = \lambda$. To prove this consider $A^{(q)}(1, \tau)$. This matrix is associated to the knot projection, but it treats overcrossings in the same way as undercrossings. By a suitable choice of undercrossings and overcrossings one may replace $\mathfrak{k}$ by a closed braid of a simple

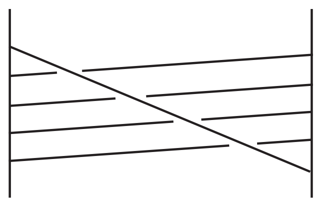


Figure 14.9

type (Figure 14.9) while preserving its symmetry. The elimination of variables does not alter $|\det A^{(q)}(1, \tau)|$. Finally $A^{(q)}(1, \tau)$ takes the form:

$$\begin{pmatrix} \tau E_\lambda & -E_\lambda \\ -E_\lambda & P_\lambda \end{pmatrix}$$

where E_λ is the $\lambda \times \lambda$-identity matrix and P_λ the representing matrix of a cyclic permutation of order λ. It follows that

$$\det(A^{(q)}(1, \tau)) = \pm \det(E_\lambda - \tau P_\lambda) = \pm(1 - \tau^\lambda),$$

because the characteristic polynomial of P_λ is $\pm(1 - \tau^\lambda)$. Proposition 14.20 then shows

$$D(1, \tau) = \pm(1 + \tau + \cdots + \tau^{\lambda - 1}) = \pm \varrho_\lambda(\tau) \quad \text{and} \quad |D(1, 1)| = \lambda. \qquad \square$$

14.24 Proposition. *Let $\mathfrak{k}$ be a knot of period p^a, $a \geqq 1$, p a prime. If $\Delta_1(t) \not\equiv 1$ mod p and $\Delta_2(t) = 1$, the splitting field $\mathbb{Q}(\Delta_1)$ of $\Delta_1(t)$ over the rationals $\mathbb{Q}$ contains the p^a-th roots of unity.*

Proof. By 14.20 and 14.21 (4) there is a root $\alpha \in \mathbb{C}$ of $\Delta_1(t)$ which is not a root of $\Delta_1^{(p^a)}(t)$. Thus, there exists a uniquely determined equivalence class of representations $\delta_\alpha \colon \mathfrak{G} \to \mathfrak{C}^+$ of the knot group $\mathfrak{G}$ of $\mathfrak{k}$ into the group of similarities $\mathfrak{C}^+$ of the plane, see 14.5. If $D(\alpha, \zeta_i) = 0$, the fixed points $b_j(S_j)$ of

$$\delta_\alpha(S_j) \colon z \mapsto \alpha(z - b_j) + b_j$$

assigned to Wirtinger generators S_j are solutions of a linear system of equations with coefficient matrix $\bar{A}(\alpha)$, satisfying $b_j(z_j) = \zeta_i b_j(x_j)$; for the notation see (1) on p. 267. Thus the configuration of fixed points b_j associated to the symmetric projection of Figure 14.7 also shows a cyclic symmetry; its order is that of ζ_i. All representations are equivalent under similarities, and all configurations of fixed points are, therefore, similar. Since the b_j are solutions of the system of linear equations (2) in 14.4 for $t = \alpha$, $u_j = b_j$, they may be assumed to be elements of $\mathbb{Q}(\alpha)$. It follows that

$$b_j(z_j)b_j^{-1}(x_j) = \zeta_i \in \mathbb{Q}(\alpha).$$

We claim that there exists a representation δ_α such that the automorphism

$$r_*(\alpha)\colon \delta_\alpha(\mathfrak{G}) \to \delta_\alpha(\mathfrak{G})$$

induced by the rotation r has order p^a. If $p^b, b < a$, were the maximal order occurring for any δ_α, all (non-trivial) representations δ_α would induce non-trivial representations of the knot group $\mathfrak{G}^{(p^{a-b})}$ of the factor knot $\mathfrak{k}^{(p^{a-b})}$. Then α would be a root of $\Delta_2(t)$ by 14.23, contradicting $\Delta_2(t) = 1$. □

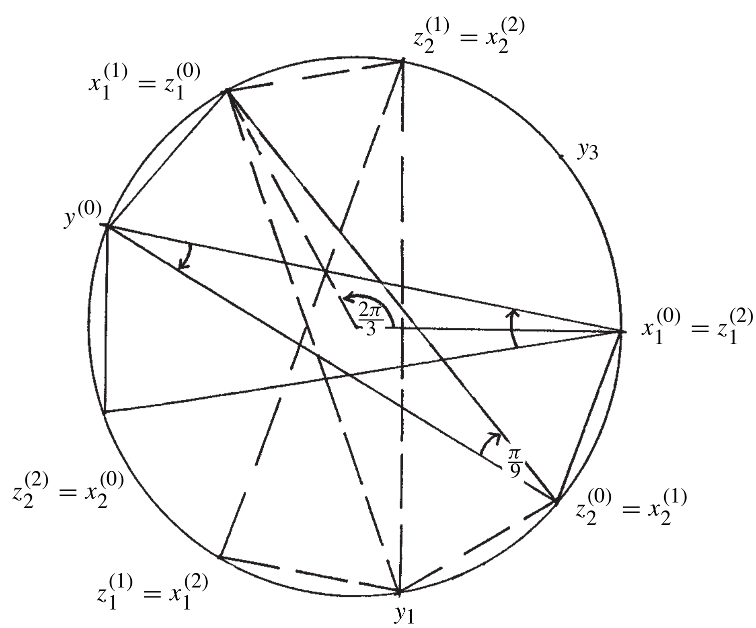


Figure 14.10

Figure 14.10 shows the fixed point configuration of the knot 9_1 as a knot of period three. One finds: $D(t,\tau) = t^3 + \tau$, $D(t,1) = \varrho_2(t) \cdot \Delta_1^{(3)}(t)$, $\Delta_1^{(3)}(t) = t^2 - t + 1$. For $\tau = e^{2\pi i/3}$, and $D(\alpha,\tau) = 0$, we get $\alpha = e^{-\pi i/9}$.

14.25 Corollary. *Let $\mathfrak{k}$ be a knot of period $q > 1$ with $\Delta_1(t) \neq 1$, $\Delta_2(t) = 1$. Then the splitting field of $\Delta_1(t)$ contains the q-th roots of unity or $\Delta_1(t) \equiv 1 \bmod p$ for some $p|q$.*

If $\mathfrak{k}$ is a non-trivial fibred knot of period q with $\Delta_2(t) = 1$, the splitting field of $\Delta_1(t)$ contains the q-th roots of unity [Trotter 1961]. □

The preceding proof contains additional information in the case of a prime period.

14.26 Corollary. *If $\mathfrak{k}$ is a knot of period p and $\Delta_1(\alpha) = 0$, $\Delta_1^{(p)}(\alpha) \neq 0$, then the p-th roots of unity are contained in $\mathbb{Q}(\alpha)$.*

Proof. There is a non-trivial representation δ_α of the knot group of $\mathfrak{k}$ with $b_j(z_j) = \zeta b_j(x_j)$, ζ a primitive p-th root of unity. □

As an application we prove

14.27 Proposition. *The periods of a torus knot $\mathfrak{t}(a, b)$ are the divisors of a and b.*

Proof. By 9.15

$$\Delta_1(t) = \frac{(t^{ab} - 1)(t - 1)}{(t^a - 1)(t^b - 1)}, \quad \Delta_2(t) = 1.$$

From Corollary 14.25 we know that a period q of $\mathfrak{t}(a, b)$ must be a divisor of ab. Suppose $p_1 p_2 | q$, $p_1 | a$, $p_2 | b$ for two prime numbers p_1, p_2, then $\mathfrak{t}(a, b)$ has periods p_1, p_2, and Corollary 14.22 gives

$$(t - 1)^\lambda (t^{a'b} - 1)^{p_1^c} \doteq\!\!\equiv (t^a - 1)(t^b - 1)[\varrho_\lambda(t)\Delta_1^{(p_1^c)}(t)]^{p_1^c} \mod p_1$$

with $a = p_1^c a'$, $\gcd(p_1, a') = 1$. Let ζ_0 be a primitive b-th root of unity. We have $\gcd(b, p_1) = 1$ and $\gcd(\lambda, p_1 p_2) = 1$, hence $p_2 \not| \lambda$. (See E 14.8.) The root ζ_0 has multiplicity s with $s \equiv 1 \mod p_1$ according to the right-hand side of the congruence, but since ζ_0 is not a λ-th root of unity, its multiplicity on the left-hand side ought to be $s \equiv 0 \mod p_1$. So there is no period q containing primes from both a and b.

It is evident that the divisors of a and b are actually periods of $\mathfrak{t}(a, b)$. □

There have been further contributions to this topic. In [Lüdicke 1978] the dihedral representations γ_p have been exploited. The periodicity of a knot is reflected in its invariant $[\lambda(\zeta)]$. In [Murasugi 1980] these results were generalized, completed and formulated in terms of linking numbers of coverings. In addition to that, certain conditions involving the Alexander polynomial and the signature of a knot have been proved when a knot is periodic [Gordon-Litherland-Murasugi 1981]. Together all these criteria suffice to determine the periods of knots with less than ten crossings, see Table I. In [Kodama-Sakuma 1992] and [Shawn-Weeks 1992] the complete information on periods and symmetry groups can be found up to 10 crossings. Many results on periodic knots carry over to links [Knigge 1981], [Sakuma 1981′, 1981″].

It follows from Murasugi's congruence in 14.22 that a knot of period p^a either has Alexander polynomial $\Delta_1(t) \equiv 1 \mod p$ or $\deg \Delta_1(t) \geqq p^a - 1$. Thus a knot with $\Delta_1(t) \not\equiv 1$ can have only finitely many prime periods. No limit could be obtained for periods p^a, if $\Delta_1(t) \equiv 1 \mod p$. A fibred knot has only finitely many periods, since its Alexander polynomial is of degree $2g$ with a leading coefficient ± 1. It has been proved in [Flapan 1983] that only the trivial knot admits infinitely many periods. A new proof of this theorem and a generalization to links was proved in [Hillman 1984]. The generalization reads: A link with infinitely many periods consists of μ trivial components spanned by disjoint disks.

14.28 Knots with $\deg \Delta_1(t) = 2$. Murasugi's congruence 14.22 shows that a knot with a quadratic Alexander polynomial can only have period three. Furthermore it follows from 14.22 that

$$\Delta_1(t) \equiv t^2 - t + 1 \mod 3.$$

Corollary 14.25 yields a further information: If $\mathfrak{k}$ has period three, its Alexander polynomial has the form

$$\Delta_1(t) = nt^2 + (1-2n)t + n, \quad n = 3m(m+1) + 1, \ m = 0, 1, \ldots,$$

see E 14.11.

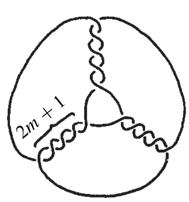


Figure 14.11

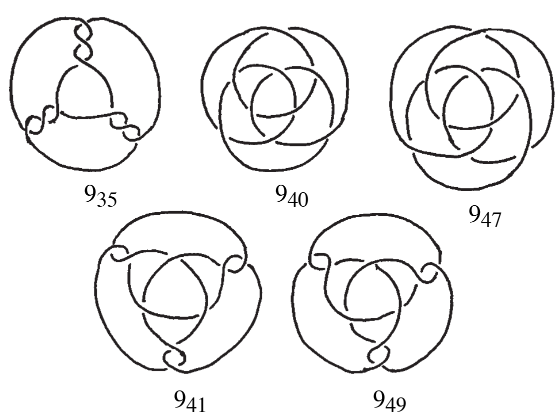


Figure 14.12

There are, in fact, symmetric knots which have these Alexander polynomials, the pretzel knots $\mathfrak{p}(2m+1, 2m+1, 2m+1)$, Figure 14.11. Their factor knot $\mathfrak{p}^{(3)}$ is trivial.

One obtains

$$\begin{aligned}&D(t,\tau) = (\tau + n(\tau-1))t + n(1-\tau) + 1,\\ &D(t,1) = 1+t,\ D(t,\tau) = 1+\tau, \quad \text{hence } \lambda = 2,\\ &D(t,\zeta)D(t,\zeta^{-1}) = \Delta_1(t), \quad \zeta \text{ a primitive third root of unity.}\end{aligned}$$

(We omit the calculations.) $\mathfrak{p}(1,1,1)$ is the trefoil, $\mathfrak{p}(3,3,3) = 9_{35}$.

14.29. The different criteria or a combination of them can be applied to exclude periods of given knots. As an example consider $\mathfrak{k} = 8_{11}$. Its polynomials are $\Delta_1(t) = (t^2-t+1)(2t^2-5t+2)$, $\Delta_2(t) = 1$. Murasugi's congruence excludes all periods different from three, but $\Delta_1(t) \doteq t^4+t^3+t+1 \equiv (1+t)^4 \bmod 3$, hence, $\lambda = 2$ and $\Delta_1^{(3)} \equiv 1 \bmod 3$ would satisfy the congruence. The splitting field of $\Delta_1(t)$ obviously contains the third roots of unity. The second factor $2t^2-5t+2$, though, has a splitting field contained in $\mathbb{R}$. By 14.25 and 14.26 this excludes a period three, since $2t^2-5t+2 \not\equiv 1 \bmod 3$.

Figure 14.12 shows symmetric versions of the knots of period three with less than ten crossings, 9_{35}, 9_{40}, 9_{41}, 9_{47}, 9_{49}. (The torus knots are omitted, $\mathfrak{t}(4,3) = 8_{19}$, $\mathfrak{t}(5,3) = 10_{124}$ and $\mathfrak{t}(2m+1,2)$, $1 \leqq m \leqq 4$.)

We conclude this section by showing that the condition $\Delta_2(t) = 1$ cannot be omitted. The 'rosette'-knot 8_{18} evidently has period four. The Alexander polynomials are $\Delta_1(t) = (1-t+t^2)^2(1-3t+t^2)$, $\Delta_2(t) = (1-t+t^2)$. One has $D(t,\tau) = \tau t^2 + (\tau^2-\tau+1)t + \tau$. It follows that $D(t,1) = 1+t+t^2 = \varrho_3(t)$, $\Delta_1^{(4)}(t) = 1$, $D(t,-1) = 1-3t+t^2$, $D(t,\pm i) = \pm i(1-t+t^2)$. The representations δ_α, $D(\alpha,i) = \Delta_2(\alpha) = 0$, are not unique. $1-3\beta+\beta^2 = 0$ yields unique representations with period 2. In fact, the splitting fields $\mathbb{Q}(\Delta_1(t))$ does not contain i. (See also [Trotter 1961].) Nevertheless, the condition $\Delta_2(t) = 1$ can be replaced by a more general one involving higher Alexander polynomials [Hillman 1983].

Remark. It is not clear whether the second condition $\Delta_1(t) \not\equiv 1 \bmod p$ in 14.23, 14.24 is necessary. The Alexander polynomials of the knots 9_{41} and 9_{49} (which have period three) satisfy $\Delta_1(t) \equiv 1 \bmod 3$, their splitting fields nevertheless contain the third roots of unity.

When looking at the material one may venture a conjecture: Let $M(\mathfrak{k})$ and $M(\mathfrak{k}^{(q)})$ denote the minimal numbers of crossings of a knot $\mathfrak{k}$ of period q and of its factor knot $\mathfrak{k}^{(q)}$. Then

$$M(\mathfrak{k}) \geqq q \cdot M(\mathfrak{k}^{(q)}).$$

E History and Sources

It seems to have been J.W. Alexander who first used homomorphic images of knot groups to obtain effectively calculable invariants, [Alexander 1928]. The groups

$\mathfrak{G}/\mathfrak{G}''$ resp. $\mathfrak{G}'/\mathfrak{G}''$, ancestral to all metabelian representations, have remained the most important source of knot invariants.

In [Reidemeister 1932] a representation of the group of alternating pretzel knots onto Fuchsian groups is used to classify these knots. This representation is not metabelian but, of course, is restricted to a rather special class of groups. It was repeatedly employed in the years to follow to produce counterexamples concerning properties which escape Alexander's invariants. By it, in [Seifert 1934], a pretzel knot with the same Alexander invariants as the trivial knot could be proved to be non-trivial – shattering all hopes of classifying knot types by these invariants. Trotter [1964] used it to show that non-invertible knots (pretzel knots) exist. The natural class of knots to which the method developed for pretzel knots can be extended is the class of Montesinos knots (Chapter 12).

R.H. Fox drew the attention to a special case of metabelian representations – the metacyclic ones. Here the image group could be chosen finite. (Compare also [Hartley 1979].) A lifting process of these representations obtained by abelianizing its kernel yielded a further class of non-metabelian representations [Burde 1967, 1970], [Hartley 1983].

A class of representations of fundamental importance in the theory of 3-manifolds was introduced by R. Riley. The image groups are discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$, and they can be understood as groups of orientation preserving motions of hyperbolic 3-space. The theory of these representations (Riley-reps), [Riley 1973, 1975, 1975′] has not been considered in this book – the same holds for homomorphisms onto the finite groups $\mathrm{PSL}(2,p)$ over a finite field $\mathbb{Z}_p$, see [Magnus-Peluso 1967], [Riley 1971], [Hartley-Murasugi 1978].

F Exercises

E 14.1. Show that the group of symmetries of a regular a-gon is the image of a dihedral representation γ_a^* of the knot group of the torus knot $\mathfrak{t}(a,2)$. Give an example of a torus knot that does not allow a dihedral representation.

E 14.2. Let $\delta_\alpha : \mathfrak{G} \to \mathfrak{C}^+$ be a representation into the group of similarities (see 14.4) of the group $\mathfrak{G}$ of a knot $\mathfrak{k}$, and $\{b_j\}$ the configuration of fixed points in $\mathbb{C}$ corresponding to Wirtinger generators S_j of a regular projection $p(\mathfrak{k})$. Show that one obtains a representation δ_α^* of $\mathfrak{k}^*$ with a fixed point configuration $\{b_j'\}$ resulting from $\{b_j\}$ by reflection in a line.

E 14.3. (a) Let $\mathfrak{k} = \mathfrak{k}_1 \# \mathfrak{k}_2$ be a product knot and $\Delta_1^{(1)}(t) \neq 1$, $\Delta_1^{(2)}(t) \neq 1$ be the Alexander polynomials of its summands. Show that there are non-equivalent representations δ_α for $\Delta_1^{(1)}(\alpha) = \Delta_1^{(2)}(\alpha) = 0$. Derive from this that $\Delta_2(\alpha) = 0$.

(b) Consider a regular knot projection $p(\mathfrak{k})$ and a second projection $p^*(\mathfrak{k})$ in the same plane E obtained from a mirror image $\mathfrak{k}^*$ reflected in a plane perpendicular to E.

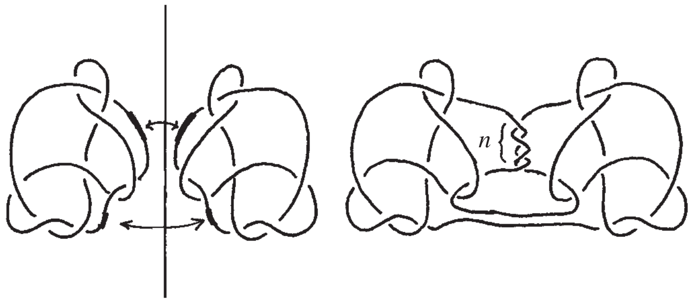


Figure 14.13

Join two corresponding arcs of $p(\mathfrak{k})$ and $p^*(\mathfrak{k})$ as shown in Figure 14.13 one with an n-twist and one without a twist – the resulting projection is that of a *symmetric union* $\mathfrak{k} \cup \mathfrak{k}^*$ of $\mathfrak{k}$ [Kinoshita-Terasaka 1957]. Show that a representation δ_α for $\mathfrak{k}$ can always be extended to a representation δ_α for the symmetric union, hence, that every root of the Alexander polynomial of $\mathfrak{k}$ is a root of that of $\mathfrak{k} \cup \mathfrak{k}^*$. (Use E 14.2.)

E 14.4. Compute the representations γ_a for torus knots $\mathfrak{t}(a, 2)$ that lift the dihedral representations γ_a^* of E 14.1, see 14.10. Show that $[\lambda(\zeta)] = \{2a\}$. Derive from this that $\mathfrak{t}(a, 2)$ # $\mathfrak{t}(a, 2)$ and $\mathfrak{t}(a, 2)$ # $\mathfrak{t}^*(a, 2)$ have non-homeomorphic complements but isomorphic groups.

E 14.5. (Henninger) Let $\gamma_p \colon \mathfrak{G} \to \mathfrak{B}$ be a normalized representation according to 14.10, $\gamma_p(S_1)\colon z \mapsto \bar{z} + 1$, $\gamma_p(S_2)\colon z \mapsto \zeta^2\bar{z} + \zeta$, with ζ a primitive p-th root of unity. Show that $\gamma_p(\mathfrak{G}) \cong \mathfrak{D}_p \ltimes \mathbb{Z}^{p-1}$. (Hint: use a translation of the plane by $2 \cdot \sum_{j=0}^{\frac{p-3}{2}} \zeta^{2j+1} + \sum_{j=1}^{\frac{p-1}{2}} \zeta^{2j}$.)

E 14.6. Compute the matrix (μ_{ij}) of linking numbers (see 14.10 (b)) of the irregular covering $\hat{I}_{15}$ of 9_2. Compare the invariants $\frac{1}{2} \sum_{j \neq i} |\mu_{ij}| = \nu_i$, $0 \leqq i \leqq 7$ with those of 7_4.

(Result: 7, 6, 5, 4, 4, 3, 2, 1, [Reidemeister 1932].)

E 14.7. If a knot has period q as an unoriented knot, it has period q as an oriented knot. Show that the axis of a rotation through π which maps $\mathfrak{k}$ onto $-\mathfrak{k}$ must meet $\mathfrak{k}$.

E 14.8. Let $\mathfrak{k}$ be a knot of period q and h the axis of the rotation. Prove that $\gcd(\mathrm{lk}(h, \mathfrak{k}), q) = 1$.

E 14.9. Produce a matrix W over $\mathbb{C}$ such that $WZ_qW^{-1} = Z(\zeta)$,

$$Z_q = \begin{pmatrix} 0 & 1 & \cdot & & \vdots \\ \vdots & 0 & 1 & & \vdots \\ \vdots & & & & \vdots \\ \vdots & & & & 1 \\ 1 & & & & 0 \end{pmatrix}, \quad Z(\zeta) = \begin{pmatrix} 1 & & & \\ & \zeta & & \\ & & \ddots & \\ & & & \zeta^{q-1} \end{pmatrix},$$

ζ a primitive q-th root of unity.

E 14.10. We call an *oriented tangle $\mathfrak{T}_n$ circular*, if its arcs have an even number of boundary points $X_1, \dots, X_n, Z_1, \dots, Z_n$ which can be joined pairwise (Figure 14.14) to give an oriented knot $\mathfrak{k}(\mathfrak{T}_n)$, inducing on $\mathfrak{T}_n$ the original orientation. A q-periodic knot $\mathfrak{k}$ may be obtained by joining q circular tangles $\mathfrak{T}_n$; the knot $\mathfrak{k}(\mathfrak{T}_n)$ is then the factor knot $\mathfrak{k}^{(q)} = \mathfrak{k}(\mathfrak{T}_n)$, see Figure 14.7. A circular tangle defines a polynomial $D(t, \tau)$, see 14.20.

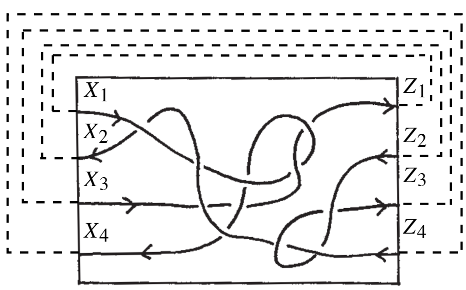


Figure 14.14

(a) Show $D(t + \tau) = t + \tau$ for the circular tangle $\mathfrak{T}_2$ with one crossing and compute $\Delta_1(t) = \prod_{i=1}^{q-1}(t + \zeta^i)$, ζ a primitive q-th root of unity, q odd. $\Delta_1(t)$ is the Alexander polynomial of $\mathfrak{t}(q, 2)$.

(b) Find all circular tangles with less than four crossings. Construct knots of period $\leqq 4$ by them.

E 14.11. If the Alexander polynomial $\Delta_1(t)$ of a periodic knot of period three is quadratic, it has the form

$$\Delta_1(t) = nt^2 + (1 - 2n)t + n, \quad n = 3m(m + 1) + 1, \ m = 0, 1, \dots.$$

Prove that the pretzel knot $\mathfrak{p}(2m + 1, 2m + 1, 2m + 1)$ has this polynomial as $\Delta_1(t)$. Hint: Compute $D(\tau, t)$.

E 14.12. [Lüdicke 1979]. Let $\mathfrak{k}$ be a knot with prime period q. Suppose there is a unique dihedral presentation $\gamma_p^*\colon \mathfrak{G} \to \mathfrak{Z}_2 \ltimes \mathfrak{Z}_p$ of its group, and $p \nmid \Delta_1^{(q)}(-1)$.

Then either $q = p$ or $q \mid p - 1$.

Chapter 15

Knots, Knot Manifolds, and Knot Groups

The long-standing problem concerning the correspondence between knots and their complements was solved in [Gordon-Luecke 1989]: "Knots are determined by their complements". The proof of the theorem is beyond the scope of this volume.

The main object of this chapter will be the relation between knot complements and their fundamental groups.

A consequence of the famous theorem of Waldhausen [1968] (see Appendix B.7) on sufficiently large irreducible 3-manifolds is that the complements of two knots are homeomorphic if there is an isomorphism between the fundamental groups preserving the peripheral group system. We study to what extent the assumption concerning the boundary is necessary.

In Part A we describe examples which show that there are links of two components which do not have Property P, see Definition 3.18, and that there are non-homeomorphic knot complements with isomorphic groups. In Part B we investigate Property P for knots. In Part C we discuss the relation between the complement and its fundamental group for prime knots and in Part D for composite knots.

A Examples

The following example of [Whitehead 1937] shows that, in general, the complement of a link does not characterize the link.

15.1 Proposition (Whitehead). *Let $\mathfrak{l}_n$, $n \in \mathbb{Z}$ denote the link consisting of a trivial knot $\mathfrak{k}$ and the n-twist knot $\mathfrak{d}_n$, see Figure* 15.1. *Then:*

(a) *The links $\mathfrak{l}_{2n}$ and $\mathfrak{l}_{2m}$ are not isotopic if $n \neq m$.*

(b) $S^3 - \mathfrak{l}_{2n} \cong S^3 - \mathfrak{l}_0$ *for all $n \in \mathbb{Z}$.*

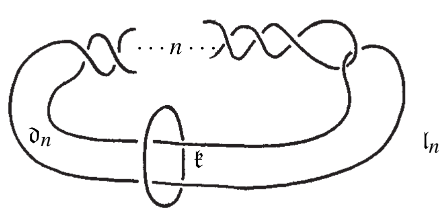


Figure 15.1

Proof. By E 9.6, the Alexander polynomial of $\mathfrak{d}_{2n}$ is $nt^2 + (1-2n)t + n$; hence, $\mathfrak{d}_{2n} = \mathfrak{d}_{2m}$ only if $n = m$.

To prove (b) take an unknotted solid torus V and the trivial doubled knot $\mathfrak{d}_0 \subset V$ "parallel" to the core of V. $W = \overline{S^3 - V}$ is a solid torus with core $\mathfrak{k}$ and $W - \mathfrak{k} \cong \partial W \times [0,1) = \partial V \times [0,1)$. Consider the following homeomorphism $V \to V$: cut V along a meridional disk, turn it $|n|$ times through 2π in the positive sense if $n > 0$, in the negative sense if $n < 0$ and glue the disks together again. This twist maps $\mathfrak{d}_0$ to $\mathfrak{d}_{2n}$. The map can be extended to $W - \mathfrak{k} \cong \partial V \times [0,1) = (\overline{S^3 - V}) - \mathfrak{k}$ to get the desired homeomorphism. □

For later use we determine from Figure 15.2 and 15.3 the group and peripheral system of the twist knots $\mathfrak{d}_n$, following [Bing-Martin 1971]. (See E 3.5.)

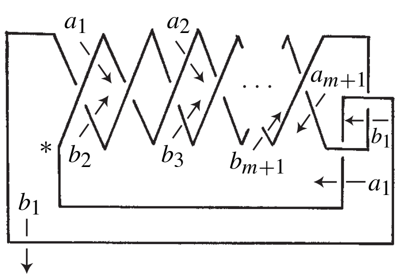


Figure 15.2

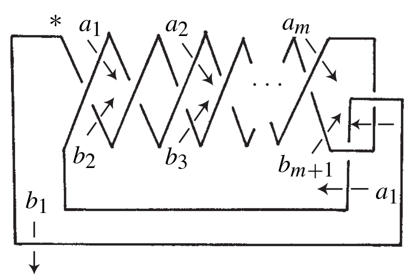


Figure 15.3

15.2 Lemma. *The twist knot $\mathfrak{d}_n$ has the following group $\mathfrak{T}_n$ and peripheral system.*

15.3 (a) $\mathfrak{T}_{2m} = \langle\, a, b \mid b^{-1}(a^{-1}b)^m a (a^{-1}b)^{-m} a (a^{-1}b)^m a^{-1} (a^{-1}b)^{-m} \,\rangle$, meridian a, longitude $(a^{-1}b)^m a^{-1} (a^{-1}b)^{-m} b^m (a^{-1}b)^{-1-m} a^{-1} (a^{-1}b)^m a^{2-m}$;

(b) $\mathfrak{T}_{2m-1} = \langle\, a, b \mid b^{-1}(a^{-1}b)^m b^{-1} (a^{-1}b)^{-m} a (a^{-1}b)^m b (a^{-1}b)^{-m} \,\rangle$, meridian b, longitude $(a^{-1}b)^{-m} b (a^{-1}b)^{2m-1} b (a^{-1}b)^{-m} b^{-2}$.

Proof. In Figure 15.2 we have drawn the Wirtinger generators and we obtain the defining relations (here $a = a_1, b = b_1$)

$$\begin{aligned}
b_2 &= a_1^{-1} b_1 a_1 = a^{-1} b a \\
a_2 &= b_2 a_1 b_2^{-1} = (a^{-1}b) a (a^{-1}b)^{-1} \\
b_3 &= a_2^{-1} b_2 a_2 = (a^{-1}b)^2 b (a^{-1}b)^{-2} \\
&\ \ \vdots \\
b_{m+1} &= a_m^{-1} b_m a_m = (a^{-1}b)^m b (a^{-1}b)^{-m}
\end{aligned}$$

$$a_{m+1} = b_{m+1}a_m b_{m+1}^{-1} = (a^{-1}b)^m a(a^{-1}b)^{-m}$$
$$b_1 = b = a_{m+1}a_1a_{m+1}^{-1} = (a^{-1}b)^m a(a^{-1}b)^{-m} a(a^{-1}b)^m a^{-1}(a^{-1}b)^{-m}$$

for $n = 2m$. For $n = 2m - 1$ the last two relations from above must be replaced by one relation

$$b = b_{m+1}^{-1} a b_{m+1} = (a^{-1}b)^m b^{-1}(a^{-1}b)^{-m} a(a^{-1}b)^m b(a^{-1}b)^{-m}$$

(see Figure 15.3).

For the calculation of the longitude we use the formulas

$$a_1 \dots a_m = b^m(a^{-1}b)^{-m} \quad \text{and} \quad b_m \dots b_2 = (a^{-1}b)^{m-1}a^{m-1}.$$

A longitude of $\mathfrak{d}_{2m}$ associated to the meridian a is given by

$$\begin{aligned} &a_{m+1}^{-1}a_1a_2\dots a_m b_1^{-1} b_{m+1}\dots b_2 a^{2-2m}\\ &\quad = (a^{-1}b)^m a^{-1}(a^{-1}b)^{-m} b^m (a^{-1}b)^{-m} b^{-1}(a^{-1}b)^m a^m a^{2-2m}\\ &\quad = a^m(a^{-1}b)^m a^{-1}(a^{-1}b)^{-2m-1}a^{-1}(a^{-1}b)^m a^{2-m}; \end{aligned}$$

for the last step we applied the defining relation from 15.2 (a) and replaced b by a conjugate of a. Since the longitude commutes with the meridian a we get the expression in 15.3 (a).

For $\mathfrak{d}_{2m-1}$ a longitude is given by

$$\begin{aligned} &a_1a_2\dots a_m b_1 b_m b_{m-1}\dots b_2 b_{m+1} b_1^{-1-2m}\\ &\quad = b^m(a^{-1}b)^{-m} b(a^{-1}b)^{m-1}a^{m-1}(a^{-1}b)^m b(a^{-1}b)^{-m} b^{-1-2m}\\ &\quad = b^m(a^{-1}b)^{-m} b(a^{-1}b)^{2m-1} b(a^{-1}b)^{-m} b^{-2-m}; \end{aligned}$$

here we used the relation from 15.3 (b). □

As we have pointed out in 3.15, the results of [Waldhausen 1968] imply that the peripheral system determines the knot up to isotopy and the complement up to orientation preserving homeomorphisms. A knot and its mirror image have homeomorphic complements; however, if the knot is not amphicheiral every homeomorphism of S^3 taking the knot onto its mirror image is orientation reversing. Using this, one can construct non-homeomorphic knot complements which have isomorphic groups:

15.4 Example ([Fox 1952]). The knots $\mathfrak{k}\#\mathfrak{k}^*$ and $\mathfrak{k}\#\mathfrak{k}$ where $\mathfrak{k}$ is a trefoil are known as the *square* and the *granny knot* (see Figure 15.4). They are different knots by Schubert's theorem on the uniqueness of the prime decomposition of knots, see Theorem 7.12, and their complements are not homeomorphic. This is a consequence of Theorem 15.11. The first proof of this fact was given by R.H. Fox [1952] who showed that the peripheral systems of the square and granny knots are different. We derive it from E 14.4: the longitudes ℓ and ℓ' are mapped by a normalized presentation γ_p, $p = 3$, onto $12 = 6 + 6$ resp. $0 = 6 - 6$, compare E 14.4 ([Fox 1952]). Their groups, though, are isomorphic by E 7.5.

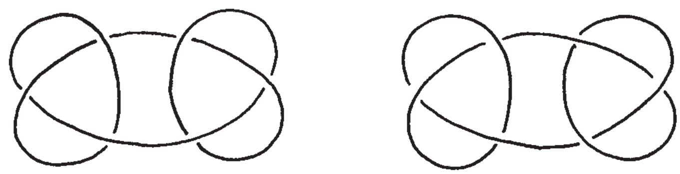


Figure 15.4

B Property P for Special Knots

For torus and twist knots suitable presentations of the groups provide a means to prove Property P. This method, however, reflects no geometric background. For product knots and satellite knots a nice geometric approach gives Property P. The results and methods of this section are mainly from [Bing-Martin 1971].

15.5 Definition. (a) The unoriented knots $\mathfrak{k}_1, \mathfrak{k}_2$ are of the *same knot type* if there is a homeomorphism $h\colon S^3 \to S^3$ with $h(\mathfrak{k}_1) = \mathfrak{k}_2$.

(b) Let $\mathfrak{k}$ be a non-trivial knot, V a neighbourhood of $\mathfrak{k}$, $C(\mathfrak{k}) = \overline{S^3 - V}$ the knot complement and m, ℓ meridian and longitude of $\mathfrak{k}$ on $\partial V = \partial C(\mathfrak{k})$. Then $C(\mathfrak{k})$ is called a *knot manifold*. For $\gcd(r, n) = 1$ let M denote the closed 3-manifold $C(\mathfrak{k}) \cup_f V'$ where V' is a solid torus with meridian m' and f an identifying homeomorphism $f\colon \partial V' \to \partial C(\mathfrak{k})$, $f(m') \sim rm + n\ell$ on $\partial C(\mathfrak{k})$. We say that M *is obtained from* S^3 *by (Dehn)-surgery on* $\mathfrak{k}$ and write $M = \text{srg}(S^3, \mathfrak{k}, r/n)$.

Thus $H_1(\text{srg}(S^3, \mathfrak{k}, r/n)) = \mathbb{Z}_{|r|}$. The knot $\mathfrak{k}$ has Property P, (compare Definition 3.18), if and only if $\pi_1(\text{srg}(S^3, \mathfrak{k}, 1/n)) = 1$ implies $n = 0$.

15.6 Proposition. *Torus knots have Property P.*

Proof. By 3.28,

$$\pi_1(\text{srg}(S^3, \mathfrak{t}(a, b), 1/n)) = \langle\, u, v \mid u^a v^{-b},\ u^c v^d (u^a (u^c v^d)^{-ab})^n \,\rangle,$$
$$|a|, |b| > 1,\ ad + bc = 1,$$

and we have to show that this group is trivial only for $n = 0$. By adding the relation u^a we obtain the factor group

$$\langle\, u, v \mid u^a, v^b, (u^c v^d)^{1-nab} \,\rangle = \langle\, \tilde{u}, \tilde{v} \mid \tilde{u}^a, \tilde{v}^b, (\tilde{u}\tilde{v})^{1-nab} \,\rangle$$

with $\tilde{u} = u^c$, $\tilde{v} = v^d$. For $n \neq 0$ this is a non-trivial triangle group, see [ZVC 1980, p. 124], since $|1 - nab| > 1$. □

In the proof of Property P for twist knots we construct homeomorphisms onto the so-called *Coxeter groups*, and in the next lemma we convince ourselves that the Coxeter groups are non-trivial.

15.7 Lemma ([Coxeter 1962]). *The Coxeter group*

$$\mathfrak{A} = \langle x, y \mid x^3, y^s, (xy)^3, (x^{-1}y)^r \rangle$$

is not trivial when $s, r \geqq 3$.

Proof. We assume that $3 \leqq s \leqq r$; otherwise replace x by x^{-1}. Introducing $t = xy$ and eliminating y gives $\mathfrak{A} = \langle t, x \mid x^3, t^3, (x^{-1}t)^s, (xt)^r \rangle$. We choose a complex number c such that

$$c\bar{c} = 4\cos^2\frac{\pi}{r} \quad \text{and} \quad c + \bar{c} = 4\cos^2\frac{\pi}{s} - 4\cos^2\frac{\pi}{r} - 1.$$

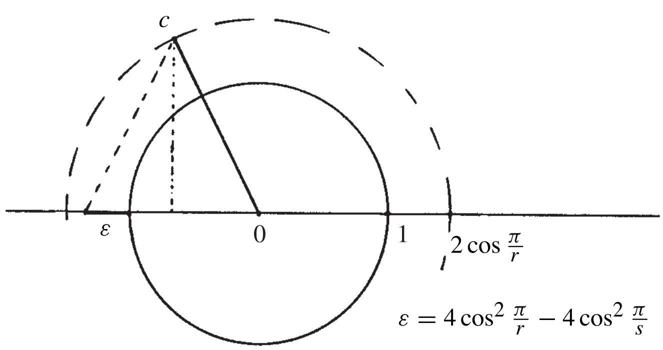


Figure 15.5

This choice is always possible if $r \geqq s \geqq 3$, see Figure 15.5. Let X, T be the following 3×3 matrices:

$$X = \begin{pmatrix} 1 & c & c+1 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 & 1 \\ 1+\bar{c} & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix}.$$

Then

$$XT = \begin{pmatrix} c\bar{c}-1 & c & 0 \\ -\bar{c} & -1 & 0 \\ 1+\bar{c} & 1 & 1 \end{pmatrix}, \quad X^{-1}T = \begin{pmatrix} c+\bar{c}+c\bar{c} & c+1 & c+1 \\ -1 & 0 & -1 \\ -\bar{c} & -1 & 0 \end{pmatrix}.$$

The characteristic polynomials are

$$
\begin{aligned}
p_X &= 1-\lambda^3, \quad p_T = 1-\lambda^3,\\
p_{XT} &= 1-(c\bar{c}-1)\lambda + (c\bar{c}-1)\lambda^2 - \lambda^3 = -(\lambda-1)(\lambda^2 - 2\lambda\cos^2\frac{\pi}{r}+1),\\
p_{X^{-1}T} &= 1-(c+\bar{c}+c\bar{c})\lambda + (c+\bar{c}+c\bar{c})\lambda^2 - \lambda^3\\
&= -(\lambda-1)(\lambda^2 - 2\lambda\cos\frac{2\pi}{s}+1).
\end{aligned}
$$

The roots of the last two polynomials are $1, e^{\pm 2\pi i/r}$ and $1, e^{\pm 2\pi i/s}$, respectively. This proves that $p_{XT} \mid \lambda^r - 1$ and $p_{X^{-1}T} \mid \lambda^s - 1$. Since a matrix annihilates its characteristic polynomial, see [van der Waerden 1955, § 118], it follows that X^3, T^3, $(XT)^r$ and $(X^{-1}T)^s$ are unit matrices. So X, T generate a non-trivial homomorphic image of $\mathfrak{A}$. □

15.8 Theorem (Bing–Martin). *The twist knot $\mathfrak{d}_n$, $n \neq 0, -1$, has Property P. In particular, the figure-eight knot $4_1 = \mathfrak{d}_2$ has Property P.*

Proof. We use the presentation 15.3 (a). Define $w = a^{-1}b$ and replace b by aw. Then

$$\mathfrak{T}_{2m} = \langle\, a, w \mid (aw)^{-1}w^m a w^{-m} a w^m a^{-1} w^{-m} \,\rangle \tag{1}$$

and, introducing $k = aw^{-m}$ instead of $a = kw^m$,

$$\mathfrak{T}_{2m} = \langle\, k, w \mid w^{-2m-1}k^{-1}w^m k^2 w^m k^{-1} \,\rangle. \tag{2}$$

The longitude is

$$
\begin{aligned}
\ell &= (a^{-1}b)^m a^{-1}(a^{-1}b)^{-m} b^m (a^{-1}b)^{-1-m} a^{-1}(a^{-1}b)^m a^{2-m}\\
&= w^m a^{-1} w^{-m}(aw)^m w^{-1-m} a^{-1} w^m a^{2-m}.
\end{aligned}
$$

By the relation in the presentation (1),

$$aw = w^m a w^{-m}\cdot a \cdot (w^m a w^{-m})^{-1};$$

hence, $(aw)^m = w^m a w^{-m}\cdot a^m \cdot w^m a^{-1} w^{-m}$ and

$$\ell = a^m w^m a^{-1} w^{-1-2m} a^{-1} w^m a^{2-m}.$$

Since ℓ commutes with the meridian a, the surgery on $\mathfrak{d}_{2m}$ gives an additional relation

$$(w^m a^{-1} w^{-1-2m} a^{-1} w^m a^2)^n a = 1,$$

or,

$$(k^{-1}w^{-1-3m}k^{-1}w^m k w^m k w^m)^n k w^m = 1.$$

Therefore

$$\begin{aligned}\mathfrak{H}_{2m,n} &= \pi_1(\mathrm{srg}(S^3, \mathfrak{d}_{2m}, 1/n)) \qquad (3)\\ &= \langle\, k, w \mid kw^{2m+1}k\cdot(w^m k^2 w^m)^{-1}, (k^{-1}w^{-1-3m}k^{-1}w^m k w^m k w^m)^n k w^m \,\rangle.\end{aligned}$$

We introduce in $\mathfrak{H}_{2m,n}$ the additional relations $w^{3m+1}=1, k^3=1$. Then the relations of (3) turn into $(kw^{-m})^3 = 1$, $(kw^m)^{3n+1} = 1$ and, with $v = k$ and $u = w^m$, the factor group has the presentation

$$\langle\, u, v \mid u^{3m+1}, v^3, (uv^{-1})^3, (uv)^{3n+1} \,\rangle.$$

By Lemma 15.7 this Coxeter group is not trivial if $m \neq 0$ and $|3n+1| > 2$. The latter condition is violated only if $n = 0, -1$.

For $n = -1$ the group is

$$\mathfrak{H}_{2m,-1} = \langle\, k, w \mid kw^{2m+1}k(w^m k^2 w^m)^{-1},\ w^{-m}k^{-1}w^{-m}kw^{3m+1}k \,\rangle.$$

By $w \mapsto y^{-6}$, $k \mapsto yx^{-1}$, we obtain an epimorphism of $\mathfrak{H}_{2m,-1}$ to the triangle group $\langle\, x, y \mid y^{6m+1}, x^3, (xy)^2 \,\rangle$ since

$$\begin{aligned}&yx^{-1}y^{-12m-6}yx^{-1}y^{6m}xy^{-1}xy^{-1}y^{6m} = yx^{-1}y^{-3}x^{-1}y^{-1}xy^{-1}xy^{-1}y^{-1}\\ &\quad= yx^{-1}y^{-2}x^2y^{-1}xy^{-2} = yx^{-1}y^{-1}x^2y^{-2} = yx^{-1}y^{-1}x^{-1}y^{-2} = yy^{-1} = 1,\end{aligned}$$

and

$$y^{6m}xy^{-1}y^{6m}yx^{-1}y^{-18m-6}yx^{-1} = y^{-1}xy^{-1}x^{-1}y^{-2}x^{-1} = y^{-1}x^2y^{-1}x^{-1} = 1.$$

The triangle group is not trivial, see [ZVC 1980, p. 124].

Next we consider $\mathfrak{d}_{2m-1}$. To achieve a more convenient presentation we define $w = a^{-1}b$ and replace a by bw^{-1}. Further we substitute $k = bw^{-m}$ and eliminate b by kw^m. Then we obtain from 15.3 (b)

$$\begin{aligned}\mathfrak{T}_{2m-1} &= \langle\, b, w \mid b^{-1}w^m b^{-1}w^{-m}bw^{-1+m}bw^{-m} \,\rangle\\ &= \langle\, k, w \mid w^{-1}k^{-2}w^{-m}kw^{-1+2m}k \,\rangle.\end{aligned}$$

The longitude is

$$\ell = w^{-m}bw^{2m-1}bw^{-m}b^{-2} = w^{-m}kw^{3m-1}kw^{-m}k^{-1}w^{-m}k^{-1}.$$

Thus

$$\begin{aligned}\mathfrak{H}_{2m-1,n} &= \pi_1(\mathrm{srg}(S^3, \mathfrak{d}_{2m-1}, 1/n)) \qquad (4)\\ &= \langle\, k, w \mid w^{-m}k^{-2}w^{-m}kw^{2m-1}k, (w^{-m}kw^{3m-1}kw^{-m}k^{-1}w^{-m}k^{-1})^n kw^m \,\rangle.\end{aligned}$$

Adding the relations w^{2m-1}, k^3 we obtain the group

$$\begin{aligned}&\langle\, k, w \mid k^3, w^{3m-1}, (kw^{-m})^3, (k^{-1}w^{-1})^{3n-1} \,\rangle\\ &\quad= \langle\, x, y \mid x^{3m-1}, y^3, (xy^{-1})^3, (xy)^{3n-1} \,\rangle\end{aligned}$$

with $x = w^{-m}$, $y = k^{-1}$.

By Lemma 15.7 this group is not trivial unless $|3m-1| \leqq 2$ or $|3n-1| \leqq 2$, that is, unless m or n is 0 or 1. For $m = 0$ we get the trivial knot and this case was excluded. In the case $m = 1$ the knot $\mathfrak{d}_1$ is the trefoil which has Property P by 15.6. So we may assume that $|3m-1| \geqq 3$. For $n = 1$

$$\mathfrak{H}_{2m-1,1} = \langle\, k, w \mid w^{-m}k^{-2}w^{-m}kw^{2m-1}k,\ w^{-m}kw^{3m-1}kw^{-m}k^{-1}\,\rangle.$$

The relations are the equations

$$kw^{2m-1}k = w^m k^2 w^m, \quad w^m k w^m = kw^{3m-1}k.$$

We rewrite the first as

$$(w^m k w^m)w^{-2m}(w^m k w^m) = kw^{2m-1}k$$

and substitute the second in this expression to obtain

$$kw^{-2m}k = w^{1-4m}, \quad kw^{3m-1}k = w^m k w^m.$$

Put $k = xw^m$. Now the defining equations are

$$xw^{-m}xw^{-m} = w^{1-6m}, \quad w^{1-6m} = w^{-m}xw^{-m}x^{-1}w^{-m}xw^{-m}.$$

Substituting the first in the second we obtain

$$w^{1-6m} = (xw^{-m})^2, \quad x^3 = (xw^{-m})^2.$$

Hence the non-trivial triangle group $\langle\, x, w \mid x^3, w^{6m-1}, (xw)^2\,\rangle$ is a homomorphic image of $\mathfrak{H}_{2m-1,1}$. □

Next we establish Property P for product knots. It is convenient to use a new view of the knot complement: one looks at the complement $C(\mathfrak{k})$ of a regular neighbourhood

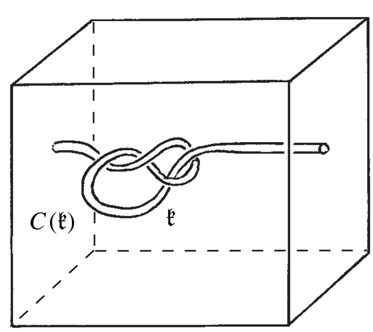


Figure 15.6

of the knot $\mathfrak{k}$ from the centre of a ball in the regular neighbourhood. Now $C(\mathfrak{k})$ looks like a ball with a knotted hole. Following [Bing-Martin 1971] we say that the complement of $\mathfrak{k}$ is a *cube with a* $\mathfrak{k}$*-knotted hole* or, simply, a cube with a (knotted) hole, see Figure 15.6. A cube with an unknotted hole is a solid torus. Suppose that W is a regular neighbourhood of a knot $\mathfrak{h}$ and $C(\mathfrak{k})$ a knotted hole, associated to the knot $\mathfrak{k}$, such that $C(\mathfrak{k}) \subset W$ and $C(\mathfrak{k}) \cap \partial W = \partial C(\mathfrak{k}) \cap \partial W$ is an annulus, then $(\overline{S^3 - W}) \cup C(\mathfrak{k})$ is the complement of $\mathfrak{k} \# \mathfrak{h}$, if the annulus is meridional with respect to $\mathfrak{h}$ and $\mathfrak{k}$, Figure 15.8.

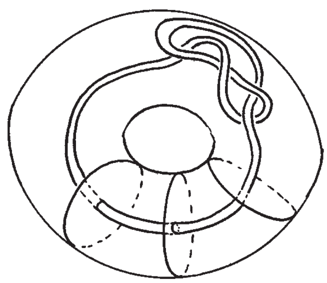


Figure 15.7

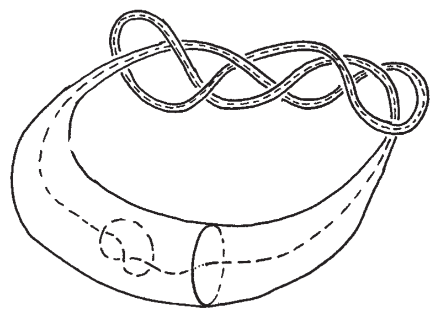


Figure 15.8

15.9 Lemma. *Let V be a homotopy solid torus, that is a 3-manifold with boundary a torus and infinite cyclic fundamental group. Suppose that K is a cube with a knotted hole in the interior of V. Then there is a homotopy 3-ball $B \subset V$ such that $K \subset B$. (B is a compact 3-manifold bounded by a sphere with trivial fundamental group).*

Proof. $\pi_1 V \cong \mathbb{Z}$ implies, as follows from the loop theorem (Appendix B.5), that there is a disk $D \subset V$ with $D \cap \partial V = \partial D$ and ∂D is not null-homologous on ∂V. By general position arguments we may assume that $D \cap \partial K$ consists of mutually disjoint simple closed curves and that, after suitable simplifications, each component of $D \cap \partial K$ is not homotopic to 0 on ∂K. Let γ be an innermost curve of the intersection on D and let D_0 be the subdisk of D bounded by γ. As K is a knotted cube, $\pi_1 \partial K \to \pi_1 K$ is injective; hence, $D_0 \subset \overline{V - K}$. By adding a regular neighbourhood of D_0 to K we obtain $B \supset K$, $\partial B = S^2$. So we may assume $D \cap \partial K = \emptyset$. Let U be a regular neighbourhood of D in V. Now $\overline{V - U}$ is a homotopy 3-ball containing K. □

15.10 Lemma. *Let V_1, V_2 be solid tori, $V_2 \subset \mathring{V}_1$ such that*

(a) *there is a meridional disk of V_1 whose intersection with V_2 is a meridional disk of V_2 and*

(b) *V_2 is not parallel to V_1, see Figure* 15.7.

Then the result of removing V_2 from V_1 and sewing it back differently is not a homotopy solid torus.

Proof. Let F be a meridional disk of V_1, that is $F \cap \partial V_1 = \partial F \not\simeq 0$ on ∂V_1, which intersects V_2 in a meridional disk of V_2. Let N be a regular neighbourhood of F in V_1. Then $K_1 = \overline{V_1 - (N \cup V_2)}$ is a cube with a knotted hole since V_2 is not parallel to V_1. Now $K_1 \cap \partial V_1$ is an annulus. We push this annulus slightly into the interior of V_1 and call the resulting cube with a knotted hole $\tilde{K}_1$.

Suppose that V_2 is removed from V_1 and a solid torus V_2' is sewn back differently; denote the resulting manifold by V_1'. Assume that V_1' is a homotopy solid torus. Then there is a disk $D \subset V_1'$ such that $D \cap \partial V_1' = D \cap \partial V_1 = \partial D$ and $\partial D \not\simeq 0$ on $\partial V_1'$. Since, by Lemma 15.9, $\tilde{K}_1$ lies in a homotopy 3-ball contained in V_1' we may assume that $D \cap \tilde{K}_1 = \emptyset$ and, hence, that also $D \cap K_1 = \emptyset$. This implies that $D \cap \partial V_1 = D \cap \partial V_1'$ is parallel to $F \cap \partial V_1$. Moreover, suppose that D and $\partial V_2' = \partial V_2$ are in general position so that $D \cap \partial V_2' = D \cap \partial V_2$ is a finite collection of mutually disjoint simple closed curves, none of which is contractible on ∂V_2. Now the complement of K_1 in $\overline{V_1 - V_2}$ is the Cartesian product of an annulus and an interval, and the boundary contains an annulus on ∂V_1 and another on $\partial V_2 = \partial V_2'$. Therefore each curve of $D \cap \partial V_2'$ is homotopic on $\partial V_2'$ to the simple closed curve $F \cap \partial V_2$ which is meridional in V_2. Let γ be an innermost curve of $D \cap \partial V_2'$ and $D_0 \subset D$ the disk bounded by γ, $D_0 \cap \partial V_2' = \gamma$. Since γ is a meridian of V_2 it is not a meridian of V_2'; hence, $D_0 \subset \overline{V_1' - V_2'} = \overline{V_1 - V_2}$, in fact $D_0 \subset \overline{V_1 - (V_2 - K_1)} \cong (S^1 \times I) \times I$ which contradicts the fact that γ represents the generator of the annulus $S^1 \times I$. Consequently, $D \cap \partial V_2 = \emptyset$ and $\partial D \simeq 0$ in $\overline{V_1 - (V_2 \cup K_1)}$, contradicting the fact that ∂D also represents the generator of $\pi_1(S^1 \times I)$. This shows that V_1' is not a homotopy solid torus. □

15.11 Theorem (Bing–Martin, Noga). *Product knots have Property P.*

Proof. Let $\mathfrak{k} = \mathfrak{k}_1 \# \mathfrak{k}_2$ be a product knot in S^3. We use the construction shown in Figure 7.2 and 15.8. Let V be a regular neighbourhood of $\mathfrak{k}_2$. Replace a segment of $\mathfrak{k}_2$ by $\mathfrak{k}_1$ such that $\mathfrak{k}_1 \subset V$, see Figure 15.8. Notice that $\overline{S^3 - V}$ is a cube with a $\mathfrak{k}_2$-knotted hole and, hence, it is not a homotopy solid torus.

Now let N be a regular neighbourhood of $\mathfrak{k}$, $N \subset \mathring{V}$, and let M result from S^3 by removing N and sewing it back differently. Lemma 15.10 implies that ∂V does not bound a homotopy solid torus in M. Thus $\pi_1 M$ is the free product of two groups amalgamated over $\pi_1(\partial V) \cong \mathbb{Z} \oplus \mathbb{Z}$ and therefore $\pi_1 M$ is not trivial. □

15.12 Theorem (Bing–Martin). *Let $\mathfrak{k} \subset S^3$ be a satellite, $\hat{\mathfrak{k}}$ its companion and $(\tilde{V}, \tilde{\mathfrak{k}})$ its pattern. Denote by m, ℓ; $\hat{m}, \hat{\ell}$; $\tilde{m}, \tilde{\ell}$ the meridian and longitude of $\mathfrak{k}$, $\hat{\mathfrak{k}}$, $\tilde{\mathfrak{k}}$ and by m_V, ℓ_V those of $\tilde{V}$. Then $\mathfrak{k}$ has Property P if*

(a) $\tilde{\mathfrak{k}}$ *has Property P, or*

(b) $\hat{\mathfrak{k}}$ *has Property P and* $q = \mathrm{lk}(m_V, \tilde{\mathfrak{k}}) \neq 0$.

Proof of 15.12 (a). (The proof for (b) will be given in 15.15.)

There is a homeomorphism $h\colon \tilde{V} \to \hat{V}$, $h(\tilde{\mathfrak{k}}) = \mathfrak{k}$. Let $\tilde{U}$ be a regular neighbourhood of $\tilde{\mathfrak{k}}$ in $\tilde{V}$. We remove $h(\tilde{U})$ from S^3 and sew it back differently to obtain a manifold M. If $\hat{\mathfrak{k}}$ is the trivial knot then h can be extended to a homeomorphism $S^3 \to S^3$ and it follows from assumption (a) that M is not simply connected.

So we may assume that $\hat{\mathfrak{k}}$ is a non-trivial knot. If the result W of a surgery on $\tilde{\mathfrak{k}}$ in $\tilde{V}$ does not yield a homotopy solid torus, then $h(\partial\tilde{V})$ divides M into two manifolds which are not homotopy solid tori. Since $\hat{\mathfrak{k}}$ is a knot, $\pi_1(h(\partial\tilde{V})) \to \pi_1(\overline{M - W}) = \pi_1(\overline{S^3 - h(\tilde{V})})$ is injective. When $\pi_1(h(\partial\tilde{V})) \to \pi_1 W$ has non-trivial kernel, there is a disk $D \subset W$, $\partial D \subset \partial W$, $\partial D \not\simeq 0$ in ∂W such that $X = \overline{W - U(D)}$ is bounded by a sphere, $U(D)$ being a regular neighbourhood of D in W. Now X cannot be a homotopy ball because W is not a homotopy solid torus. Therefore $\pi_1 M \neq 1$. If $\pi_1(h(\partial\tilde{V})) \to \pi_1 W$ is injective, $\pi_1 M$ is a free product with an amalgamation over $\pi_1(h(\partial\tilde{V})) \cong \mathbb{Z}^2$, hence non-trivial.

Finally, suppose that $\hat{\mathfrak{k}}$ is non-trivial and the sewing back of $h(\tilde{U})$ in $h(\tilde{V})$ yields a homotopy solid torus W. Then a meridian of W can be presented in the form $ph(m_V) + qh(\ell_V)$ where p, q are relatively prime integers. From $h(\ell_V) \sim 0$ in $\overline{S^3 - h(\tilde{V})}$ it follows that $H_1(M)$ is isomorphic to $\mathbb{Z}_{|p|}$ or $\mathbb{Z}$ (for $p = 0$). To see that $|p| \neq 1$, we perform the surgery on $\tilde{\mathfrak{k}}$ in $\tilde{V}$ which transforms $\tilde{V}$ into the manifold $\tilde{V}' = h^{-1}(W)$. (The new meridian defining the surgery represents $m_V^p \ell_V^q \in \pi_1(\partial\tilde{V})$.) Now $\tilde{V}' \cup \overline{S^3 - \tilde{V}}$ is obtained from S^3 by surgery on $\tilde{\mathfrak{k}}$. Since $\ell_V \simeq 1$ in $\overline{S^3 - \tilde{V}}$ the relation $m_V^p \ell_V^q \simeq 1$ is equivalent to $m_V^p \simeq 1$, and $|p| = 1$ implies that $\tilde{V}' \cup \overline{S^3 - \tilde{V}}$ is a homotopy sphere. Thus $|p| \neq 1$ because $\tilde{\mathfrak{k}}$ has Property P. □

15.13 Remark. The knot $h(\tilde{\mathfrak{k}})$ is a satellite and $(\tilde{V}, \tilde{\mathfrak{k}})$ is the pattern of $h(\tilde{\mathfrak{k}})$. The condition $h(\ell_V) \sim 0$ in $C(\hat{\mathfrak{k}})$ ensures that the mapping h does not unknot $\tilde{\mathfrak{k}}$; this could be done, for instance, with the twist knots $\mathfrak{d}_n$, $n \neq 0, -1$ when h removes the twists. As an example, using the definition of twisted doubled knots in E 9.6 and Theorem 15.8, we obtain

15.14 Corollary. *Doubled knots with q twists, $q \neq 0, -1$ have Property P.* □

15.15. *Proof of* 15.12 (b). We consider surgery along the knot $h(\tilde{\mathfrak{k}})$; for the definition of h see p. 291. Replace a tubular neighbourhood $\tilde{U} \subset \tilde{V}$ on $\tilde{\mathfrak{k}}$ by another solid torus $\tilde{T}$ using a gluing map $f\colon \partial\tilde{T} \to \partial\tilde{U}$. The manifold obtained is

$$M = (\overline{S^3 - \hat{V}}) \cup_h ((\overline{\tilde{V} - \tilde{U}}) \cup_f \tilde{T}).$$

Define $\hat{C} = C(\hat{\mathfrak{k}}) = \overline{S^3 - \hat{V}}$ and $X = (\overline{\tilde{V} - \tilde{U}}) \cup_f \tilde{T}$. Since $\hat{\mathfrak{k}}$ is non-trivial the inclusion $\partial\hat{C} \to \hat{C}$ defines a monomorphism $\pi_1(\partial\hat{C}) \to \pi_1\hat{C}$. If $\partial X \to X$ induces also a monomorphism $\pi_1(\partial X) \to \pi_1 X$, then $\pi_1 M$ is a free product with amalgamated subgroup $\pi_1(\partial\hat{C}) = \pi_1(\partial X) \cong \mathbb{Z}^2$.

Therefore, if M is a homotopy sphere, $\ker(\pi_1(\partial X) \to \pi_1 X) \neq 1$. By the loop theorem (Appendix B.5), there is a simple closed curve $\nu \subset \partial X$, ν not contractible on ∂X, which bounds a disk D in X, $\partial D \cap \partial X = \partial D = \nu$. Then $\nu \simeq \hat{m}^a \hat{\ell}^b$ on ∂X with $\gcd(a, b) = 1$; we may assume $a \geqq 0$.

If W is a regular neighbourhood of D in X, the boundary of $\overline{X - W}$ is a 2-sphere S^2 and

$$M = (C \cup W) \cup (\overline{X - W}),\ S^2 = (C \cup W) \cap (\overline{X - W}).$$

Therefore $\pi_1 M = \pi_1(C \cup W) * \pi_1(\overline{X - W})$. Thus $\pi_1(C \cup W) = 1$. Since by assumption 15.12 (b) $\hat{\mathfrak{k}}$ has Property P, it follows that ν must be the meridian $\hat{m}$ of $\hat{\mathfrak{k}}$ and $b = 0$ and $a = 1$; moreover, $\hat{m} = h(m_V)$ if m_V is a meridian of $\tilde{V}$.

Let $\tilde{m}$ be a meridian of the tubular neighbourhood $\tilde{U}$ of $\tilde{\mathfrak{k}}$. Then, for the meridian m_V of $\tilde{V}$

$$m_V \sim q\tilde{m} \quad \text{in } \overline{\tilde{V} - \tilde{U}} \text{ with } q = \mathrm{lk}(m_V, \tilde{k}). \tag{5}$$

Moreover, there is a longitude $\tilde{\ell}$ of $\tilde{U}$ such that

$$\tilde{\ell} \sim q\ell_V \quad \text{in } \overline{\tilde{V} - \tilde{U}}. \tag{6}$$

$\tilde{\ell}$ can be obtained from an arbitrary longitude $\tilde{\ell}_0$ as follows. There is a 2-chain c_2 in $\overline{\tilde{V} - \tilde{U}}$ – the intersection of $\overline{\tilde{V} - \tilde{U}}$ with a projecting cylinder of $\tilde{\ell}_0$ – such that

$$\partial c_2 = \tilde{\ell}_0 + \alpha\tilde{m} + \beta m_V + \gamma\ell_V.$$

Now

$$q = \mathrm{lk}(m_V, \tilde{\mathfrak{k}}) = \mathrm{lk}(m_V, \tilde{\ell}_0) = \mathrm{lk}(m_V, -\alpha\tilde{m} - \beta m_V - \gamma\ell_V) = -\gamma,$$

and

$$\tilde{\ell} = \tilde{\ell}_0 + (\alpha + \beta q)\tilde{m} = \tilde{\ell}_0 + \alpha\tilde{m} + \beta m_V \sim q\ell_V$$

in $\overline{\tilde{V} - \tilde{U}}$. (See E 15.1.)

For a meridian m_T of $\tilde{T}$ one has

$$m_T \sim \varrho\tilde{m} + \sigma\tilde{\ell} \quad \text{on } \partial\tilde{T} = \partial\tilde{U},\ \gcd(\varrho, \sigma) = 1. \tag{7}$$

Here $\varrho = \pm 1$ since we assume that the surgery along $\mathfrak{k}$ gives a homotopy sphere. The disk D is bounded by m_V. We assume that D is in general position with respect to $\partial\tilde{T}$ and that $D \cap \partial\tilde{T}$ does not contain curves that are contractible on $\partial\tilde{T}$; otherwise D can be altered to get fewer components of $\partial\tilde{T} \cap D$. This implies that $\partial\tilde{T} \cap D$ is a collection of disjoint meridians of $\tilde{T}$ and that $\partial\tilde{T} \cap D$ consists of parallel meridional disks, and, thus, for a suitable p

$$m_V \sim p\, m_T \quad \text{in } \overline{\tilde{V} - \tilde{U}}. \tag{8}$$

ℓ_V and $\tilde{m}$ are a basis of $H_1(\overline{\tilde{V} - \tilde{U}}) \cong \mathbb{Z}^2$. The formulas (5) - (8) imply

$$q\,\tilde{m} \sim m_V \sim p\,m_T \sim p\,\varrho\,\tilde{m} + p\,\sigma\,\tilde{\ell};$$

thus

$$p\,\sigma = 0,\ \ p\,\varrho = q,\ \text{ that is, since } q \neq 0,\ \sigma = 0,\ \varrho = \pm 1,\ \ p = \pm q.$$

So we may assume that $\varrho = 1$ and $p = 1$. But then $m_T = \tilde{m}$. □

15.16 Proposition. (a) (p, q)*-cable knots with* $2 \leqq |p|, |q|$ *have Property P.*

(b) *Let* $\mathfrak{k}$ *be a* $(\pm 1, q)$*-cable knot about the non-trivial knot* $\hat{\mathfrak{k}}$*. If* $|q| \geqq 3$ *then* $\mathfrak{k}$ *has Property P.* (*For the notation see* 15.20.)

Proof. The first statement is a consequence of 15.6 and 15.12 (a). For the proof of the second assertion, we consider the pattern $(\tilde{V}, \tilde{\mathfrak{k}})$. It can be constructed as follows. Let ϱ denote the rotation of the unit disk $\tilde{B}$ through the angle $2\pi/q$. Choose in $\tilde{B}$ a small disk $\tilde{D}_1$ with centre $\tilde{x}_1$ such that $\tilde{D}_1$ is disjoint to all its images $\varrho^j \tilde{D}_1$, $1 \leqq j \leqq q-1$. Then the pattern consists of the solid torus $\tilde{B} \times I/\varrho$, that is, the points $(\tilde{x}, 1)$ and $(\varrho(\tilde{x}), 0)$ are identified, and the knot $\tilde{\mathfrak{k}}$ consists of the arcs $\varrho^j(\tilde{x}_1) \times I$, $0 \leqq j < q$. A regular neighbourhood $\tilde{U}$ of $\tilde{\mathfrak{k}}$ is $\bigcup_{j=0}^{q-1}(\varrho^j(\tilde{D}_1) \times I)$, see Figure 15.9.

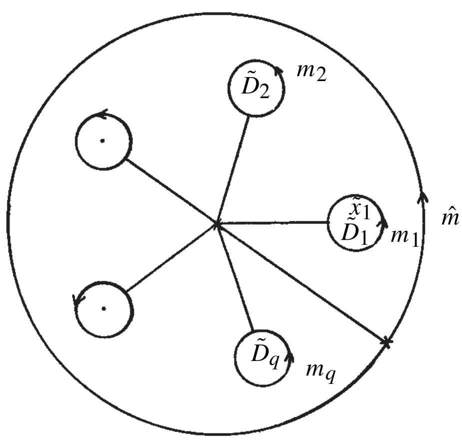


Figure 15.9

Then $C(\mathfrak{k}) = C(\hat{\mathfrak{k}}) \cup X$, $C(\hat{\mathfrak{k}}) \cap X = \partial C(\mathfrak{k}) \subset \partial X$, where X is homeomorphic to the pattern described above. Let $\hat{m}$ be a meridian of $\hat{\mathfrak{k}}$ ($\hat{m}$ is the image of $\partial \tilde{B}$) and $m_1, \ldots, m_q$ meridians of $\mathfrak{k}$ corresponding to $\partial \tilde{D}_1, \ldots, \partial \tilde{D}_q$. Let $\hat{\ell}$ be the longitude of $\hat{\mathfrak{k}}$. Then

$$\begin{aligned}\pi_1 X = \langle\, &\hat{m}, m_1, \ldots, m_q, \hat{\ell} \mid \hat{m}^{-1} \cdot m_1 \ldots m_q,\ [\hat{m}, \hat{\ell}],\\ &\hat{\ell}^{-1} m_j \hat{\ell} \cdot m_{j+1}^{-1}\ (1 \leqq j < q),\ \hat{\ell}^{-1} m_q \hat{\ell} \cdot (\hat{m}^{-1} m_1 \hat{m})^{-1} \,\rangle\\ = \langle\, &\hat{m}, m_1, \hat{\ell} \mid \hat{m}^{-1}(m_1\hat{\ell}^{-1})^q \hat{\ell}^q,\ \hat{\ell}^{-q} m_1 \hat{\ell}^q (\hat{m}^{-1} m_1 \hat{m})^{-1},\ [\hat{m}, \hat{\ell}] \,\rangle\\ = \langle\, &m_1, \hat{\ell} \mid [m_1, (m_1\hat{\ell}^{-1})^q],\ [(m_1\hat{\ell}^{-1})^q, \hat{\ell}] \,\rangle.\end{aligned}$$

Note that $m_1^{-q}(m_1\hat{\ell}^{-1})^q$ is a longitude of $\mathfrak{k}$.

Next we attach a solid torus W to $C(\mathfrak{k})$ such that the result is a homotopy sphere. The meridian of W has the form $m_1 \cdot (m_1^{-q}(m_1\hat{\ell}^{-1})^q)^n = m_1^{1-nq}(m_1\hat{\ell}^{-1})^{nq}$. If we show that $n = 0$ the assertion (b) is proved. We have

$$\pi_1(X \cup W) = \langle\, m_1, \hat{\ell} \mid [m_1, (m_1\hat{\ell}^{-1})^q],\ [(m_1\hat{\ell}^{-1})^q, \hat{\ell}],\ m_1^{1-nq}(m_1\hat{\ell}^{-1})^{nq} \,\rangle$$

and $\pi_1 M = \pi_1(C(\hat{\mathfrak{k}}) \cup X \cup W)$ is obtained by adding the relation $\hat{m} = (m_1\hat{\ell}^{-1})^q \hat{\ell}^q = 1$. Put $v = m_1\hat{\ell}^{-1}$ and replace $\hat{\ell}$ by $v^{-1}m_1$ to get

$$\pi_1 M = \langle\, m_1, v \mid [m_1, v^q],\ [v^q, v^{-1}m_1],\ m_1^{1-nq}v^{nq},\ v^q(v^{-1}m_1)^q \,\rangle.$$

Adding the relator $v^q = 1$ we obtain the group

$$\langle\, m_1, v \mid m_1^{1-nq},\ v^q,\ (v^{-1}m_1)^q \,\rangle$$

which must be trivial. Since $|q| \geqq 3$ this implies $1 - nq = \pm 1$, see [ZVC 1980, p. 122]; hence, $n = 0$. □

C Prime Knots and their Manifolds and Groups

In this section we discuss to what extent the group of a prime knot determines the knot manifold. For this we need some concepts from 3-dimensional topology.

15.17 Definition. (a) A submanifold $N \subset M$ is *properly embedded* if $\partial N = N \cap \partial M$.

(b) Let A be an annulus and $a \subset A$ a non-separating properly embedded arc, a so-called *spanning arc*. A mapping $f\colon (A, \partial A) \to (M, \partial M)$, M a 3-manifold, is called *essential* if $f_\#\colon \pi_1 A \to \pi_1 M$ is injective and if there is no relative homotopy $f_t\colon (A, \partial A) \to (M, \partial M)$ with $f_0 = f$, $f_1(a) \subset \partial M$. The *annulus* $f(A)$ is also called *essential*.

(c) The properly embedded surface $F \subset M$ is *boundary parallel* if there is an embedding $g\colon F \times I \to M$ such that

$$g(F \times \{0\}) = F \quad \text{and} \quad g((F \times \{1\} \cup (\partial F \times I)) \subset \partial M.$$

An annulus A is boundary parallel if and only if there is a solid torus $V \subset M$ such that $A \subset \partial V$, $\partial V - A \subset \partial M$ and the core of A is a longitude of V. (Proof as Exercise E 15.2).

To illustrate the notion of an essential annulus we give another characterizing condition and discuss two important examples.

15.18 Lemma. *Let A be a properly embedded incompressible annulus in a knot manifold C. Then A is boundary parallel if and only if the inclusion $i: A \to C$ is not essential.*

Proof. Clearly, if A is boundary parallel, then i is homotopic rel ∂A to a map into ∂C, thus not essential. If i is not essential then, since A is incompressible, that is $i_\#: \pi_1 A \to \pi_1 C$ is injective, a spanning arc a of A is homotopic to an arc $b \subset \partial C$. We may assume that b intersects ∂A transversally, intersects the two components of ∂A alternatingly and is simple; the last assumption is not restrictive since any arc on a torus with different endpoints can be deformed into a simple arc by a homotopy keeping the endpoints fixed. The annulus A decomposes C into two 3-manifolds C_1, C_2: $C = C_1 \cup C_2$, $A = C_1 \cap C_2$, such that $\partial C_j = (\partial C_j \cap \partial C) \cup A$ $(j = 1, 2)$ is a torus. We have

$$\pi_1 C = \pi_1 C_1 *_{\pi_1 A} \pi_1 C_2.$$

If $b \subset \partial C_j$ for some j then $b \cup a \subset \partial C_j$ is nullhomotopic in C_j, thus bounds a disk in C_j. This implies that C_j is a solid torus and ∂A consists of two longitudes of C_j. By the remark above, A is boundary parallel.

If b intersects ∂A more than twice then $b = b_1 \dots b_n$ where b_j and b_{j+1} are alternately contained in C_1 and C_2. The boundary points of each b_j are on different components of ∂A. By adding segments $c_j \subset A$ we obtain

$$b \simeq (b_1 c_1)(c_1^{-1} b_2 c_2)(c_2^{-1} \dots (c_{n-1}^{-1} b_n)$$

such that ab_1c_1, $c_1^{-1}b_2c_2, \dots, c_{n-1}^{-1}b_n$ are closed and are contained in ∂C_1 or ∂C_2. If in some C_j, ab_1c_1 is contractible or homotopic to a power c^p of the core of A we replace b by b_1c_1 or $b_1c_1c^{-p}$, respectively, and argue as above. If one of the $c_{k-1}^{-1}b_kc_k$ (c_n is the trivial arc) is contractible or homotopic to a curve in A in some C_j it can be eliminated and we obtain a simpler arc, taking the role of b. Thus we may assume that none of ab_1c_1, $c_1^{-1}b_2c_2, \dots, c_{n-1}^{-1}b_n$ is homotopic to a curve in A. Then the above product determines a word in $\pi_1 C$ where consecutive factors are alternatingly in $\pi_1 C_1$ and $\pi_1 C_2$ and none is in the amalgamated subgroup; thus the word has length n and represents a non-trivial element of $\pi_1 C$, see [ZVC 1980, 2.3.3], contradicting $ab \simeq 0$ in C. □

15.19 Proposition. *Let $C(\mathfrak{k}) = C(\mathfrak{k}_1) \cup C(\mathfrak{k}_2)$ be the knot manifold of a product knot $\mathfrak{k} = \mathfrak{k}_1 \# \mathfrak{k}_2$ with $A = C(\mathfrak{k}_1) \cap C(\mathfrak{k}_2)$ an annulus. If $\mathfrak{k}_1$ and $\mathfrak{k}_2$ are non-trivial, then A is essential in $C(\mathfrak{k})$.*

Proof. Otherwise, by 15.18, A and one of the annuli of $\partial C(\mathfrak{k})$, defined by ∂A bounds a solid torus which must be one of the $C(\mathfrak{k}_j)$. This is impossible since a knot with complement a solid torus is trivial, see 3.17. □

15.20 Example (Cable knots). Let W be a solid torus in S^3 with core $\mathfrak{k}$, m and ℓ meridian and longitude of W where $\ell \sim 0$ in $C(\mathfrak{k}) = \overline{S^3 - W}$. A simple closed curve $\mathfrak{c} \subset \partial W$, $\mathfrak{c} \sim pm + q\ell$ on ∂W, $|q| \geqq 2$ is called a *(p,q)-cable knot with core* $\mathfrak{k}$. (Compare 2.9.) Another description is the following: *Let V be a solid torus with core $\mathfrak{k}$ in S^3 and $C(\mathfrak{k}) \cap V = (\partial C(\mathfrak{k})) \cap (\partial V) = A$ an annulus the core of which is of type (p,q) on $\partial C(\mathfrak{k})$. Then $\partial(C(\mathfrak{k}) \cup V)$ is a torus and $U(\mathfrak{c}) = \overline{S^3 - (C(\mathfrak{k}) \cup V)}$ is a solid torus the core of which is a (p,q)-cable knot $\mathfrak{c}$ with core* $\mathfrak{k}$, see Figure 15.10. This follows from the fact that the core of $\overline{S^3 - (C(\mathfrak{k}) \cup V)}$ is isotopic in W to the core of A.

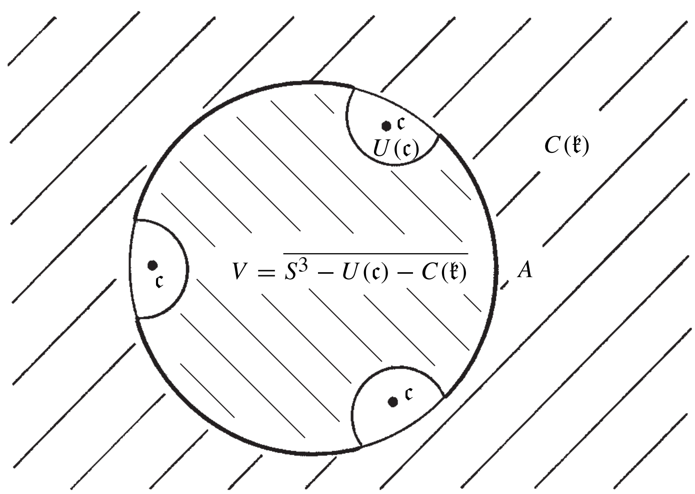


Figure 15.10

We will see that the annuli of 15.19, 15.20 are the prototypes of essential annuli in knot manifolds. To see this we need the following consequence of Feustel's Theorem [Feustel 1976, Theorem 10], which we cannot prove here.

15.21 Theorem (Feustel). *Let M and N be compact, connected, irreducible, boundary irreducible 3-manifolds. Suppose that ∂M is a torus and that M does not admit an essential embedding of an annulus. If $\varphi\colon \pi_1 M \to \pi_1 N$ is an isomorphism then there is a homeomorphism $h\colon M \to N$ with $h_\# = \varphi$.* □

We prove in 15.36 the following result of [Simon 1980′], without using Feustel's Theorem 15.21.

15.22 Theorem (Simon). *There are at most two cable knots with the same knot group.*

A consequence of 15.21 and 15.22 is the following

15.23 Corollary ([Simon 1980′]). *The complements of at most two prime knot types can have the same group.*

Proof. Suppose $\mathfrak{k}_0, \mathfrak{k}_1, \mathfrak{k}_2$ are prime knots whose groups are isomorphic to $\pi_1(C(\mathfrak{k}_0))$. If $\mathfrak{k}_j$ is not a cable knot then $C(\mathfrak{k}_j)$ does not contain essential annuli, see 15.26. Now Theorem 15.21 implies that the $C(\mathfrak{k}_j)$, $j = 0, 1\ 2$ are homeomorphic. So we may assume that $\mathfrak{k}_0, \mathfrak{k}_1, \mathfrak{k}_2$ are cable knots and the assertion follows from Theorem 15.22. □

It remains to prove 15.22 and 15.26.

15.24 Lemma ([Simon 1973, Lemma 2.1]). *Let C, W_0, W_1 be knot manifolds. $C = W_0 \cup (A \times [0, 1]) \cup W_1$, $W_0 \cap ((A \times [0, 1]) \cup W_1) = A \times \{0\}$, $W_1 \cap (W_0 \cup (A \times [0, 1])) = A \times \{1\}$, where A is an annulus, see Figure* 15.11. *Then either the components of ∂A bound disks in ∂C or the components bound meridional disks in $\overline{S^3 - C}$ and the groups $\pi_1 C$, $\pi_1 W_0$, $\pi_1 W_1$ are the normal closures of the images of $\pi_1 A$.*

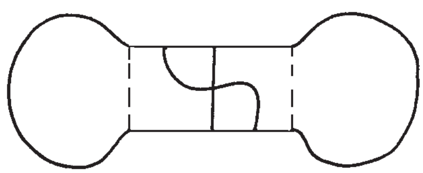


Figure 15.11

Proof. Since W_0 is a knot manifold, $\overline{S^3 - W_0}$ is a solid torus containing W_1. By Lemma 15.9, there is a 3-ball B such that $W_1 \subset \mathring{B} \subset B \subset S^3 - W_0$; so the 2-sphere $S^2 = \partial B$ separates W_0 and W_1 and therefore must intersect $A \times (0, 1)$. We may assume that $S^2 \cap (\partial A \times (0, 1))$ consists of a finite number of pairwise disjoint curves $\sigma_1, \ldots, \sigma_r$. If σ_i is innermost in S^2 then σ_i bounds a disk $D \subset S^2$ such that either $D \subset \overline{S^3 - C}$ or $D \subset A \times (0, 1)$.

If σ_i also bounds a disk $E \subset \partial A \times (0, 1)$ – which it necessarily does in the latter case – then the intersection line σ_i can be removed by an isotopy which replaces S^2 by a sphere S_1^2 still separating W_0 and W_1. It is impossible that all curves σ_j can be eliminated in this way, as $\partial A \times \{0\}$ and $\partial A \times \{1\}$ are separated by S^2. There exists a curve $\gamma \subset S^2 \cap (\partial A \times (0, 1))$ bounding a disk in $\overline{S^3 - C}$ which is not trivial on $\partial A \times (0, 1)$. So there are non-trivial curves γ_1, γ_2 on each component of $\partial A \times (0, 1)$ bounding disks in $\overline{S^3 - C}$. They are isotopic on $\partial A \times [0, 1]$ to the components of $\partial A \times \{0\}$, respectively, which, hence bound disks in $\overline{S^3 - C}$. □

15.25 Lemma. *Let C be a knot manifold in S^3, $C = W_0 \cup W_1$, where W_0 is a cube with a hole, W_1 is a solid torus, and $A = W_0 \cap W_1 = \partial W_0 \cap \partial W_1$ is an annulus. Denote by $\mathfrak{k}_C$ the core of the solid torus $\overline{S^3 - C}$. Assume that $\pi_1 A \to \pi_1 W_1$ is not surjective. Then $\mathfrak{k}_C$ is a (p, q)-cable of the core $\mathfrak{k}_0$ of $\overline{S^3 - W_0}$, $|q| \geqq 2$. If W_0 is a solid torus then $\mathfrak{k}_C$ is a torus knot.*

Proof. We may write $C = W_0 \cup_f W_1$ where f is an attaching map on A. This mapping f is uniquely determined up to isotopy by the choice of the core of A on ∂W_1, since $\overline{S^3 - C}$ is a solid torus. Hence, the core $\mathfrak{k}_C$ of $\overline{S^3 - C}$ is by 15.20 the (p, q)-cable of $\mathfrak{k}_0$. When $|q| = 1$ the homomorphism $\pi_1 A \to \pi_1 W_1$ is surjective. If $q = 0$, $\mathfrak{k}_C$ is trivial and C is not a knot manifold. In the special case where W_0 is a solid torus, $\mathfrak{k}_0$ is trivial and $\mathfrak{k}_C$ a torus knot. □

15.26 Lemma. *Let C be a knot manifold in S^3, and let A be an annulus in C, $\partial A \subset \partial C$, with the following properties:*

(a) *the components of ∂A do not bound disks in ∂C;*

(b) *A is not boundary parallel in C.*

Then a core of $\overline{S^3 - C}$ is either a product knot or a cable knot isotopic to each of the components of ∂A.

Proof. By (a), the components of ∂A bound annuli in ∂C. Hence, there are submanifolds X_1 and X_2 bounded by tori such that $C = X_1 \cup X_2$, $X_1 \cap X_2 = A$, and, by Alexander's theorem (Appendix B.2) X_i is either a knot manifold or a solid torus.

If X_1 and X_2 are both knot manifolds then, by Lemma 15.24, each component of ∂A bounds a meridional disk in $\overline{S^3 - C}$, and a core of $\overline{S^3 - C}$ is, by Definition 2.7, a product knot.

Suppose now that X_2 is a solid torus. There is a annulus $B \subset \partial C$ satisfying $A \cup B = \partial X_2$. If the homomorphism $\pi_1 A \to \pi_1 X_2$, induced by the inclusion, is not surjective, then, by Lemma 15.25, a core of $\overline{S^3 - C}$ is a cable knot. Now assume that $\pi_1 A \to \pi_1 X_2$ is surjective. Then a simple arc $\beta \subset B$ which leads from one component of ∂B to the other can be extended by a simple arc $\alpha \subset A$ to a simple closed curve $\mu \subset \partial X_2$ which is 0-homotopic in the solid torus X_2 and, hence, a meridian of X_2. Since μ intersects each component of ∂A in exactly one point it follows that A is boundary parallel, contradicting hypothesis (b). □

15.27 Lemma. *Let $\mathfrak{k}_1$ and $\mathfrak{k}$ be cable knots with complements $C(\mathfrak{k}_1)$ and $C(\mathfrak{k})$. Assume that $\mathfrak{k}$ is not a torus knot and that*

$$C(\mathfrak{k}) = X \cup V, \ A = X \cap V = \partial X \cap \partial V,$$

where X is a knot manifold, V a solid torus, and A an annulus. Let $\mathfrak{k}$ be a (p, q)-curve on a torus parallel to the boundary of $\overline{S^3 - X}$, $|q| \geqq 2$.

If $\pi_1 C(\mathfrak{k}_1) \cong \pi_1 C(\mathfrak{k})$ then there is a homotopy equivalence $f\colon C(\mathfrak{k}_1) \to C(\mathfrak{k})$ such that $f^{-1}(A)$ is an annulus.

15.28 Remark. We do not use the fact that $\mathfrak{k}_1$ and $\mathfrak{k}$ are cable knots in the first part of the proof including Claim 15.30. By Theorem 6.1 we know that $\mathfrak{k}_1$ is not a torus knot. The proof of Lemma 15.27 is quite long and of a technical nature. However, some of the intermediate steps have already been done in Chapter 5. The proof of Lemma 15.27 will be finished in 15.34.

Proof. Since $C(\mathfrak{k}_1)$ and $C(\mathfrak{k})$ are $K(\pi, 1)$-spaces any isomorphism $\pi_1 C(\mathfrak{k}_1) \xrightarrow{\cong} \pi_1 C(\mathfrak{k})$ is induced by a homotopy equivalence $g\colon C(\mathfrak{k}_1) \to C(\mathfrak{k})$, [Spanier 1966, 7.6.24], [Stöcker-Zieschang 1994, S. 459]. We may assume that g has the following properties:

(1) g is transversal with respect to A, that is, there is a neighbourhood $g^{-1}(A) \times [-1, 1] \subset C(\mathfrak{k}_1)$ of $g^{-1}(A) = g^{-1}(A) \times \{0\}$ and a neighbourhood $A \times [-1, 1]$ of A such that $g(x, t) = (g(x), t)$ for $x \in g^{-1}(A)$, $t \in [-1, 1]$.

(2) $g^{-1}(A)$ is a compact 2-manifold, properly imbedded and two-sided in $C(\mathfrak{k}_1)$.

(3) If A' is a component of $g^{-1}(A)$ then

$$\ker\left(\pi_j(A') \xrightarrow{g_\#} \pi_j(C(\mathfrak{k}))\right) = 0 \quad \text{for } j = 1, 2, .$$

These properties can be obtained by arguments similar to those used in 5.3; see also [Waldhausen 1968, p. 60].

Choose among all homotopy equivalences g that have the above properties one with minimal number n of components A_i of $g^{-1}(A)$.

15.29 Claim. *Each A_i is an annulus which separates $C(\mathfrak{k}_1)$ into a solid torus V_i and a knot manifold W_i, and $\pi_1 A_i \to \pi_1 V_i$ is not surjective.*

Proof. Since $\pi_2 C(\mathfrak{k}_1) = 0$ it follows from (3) that $\pi_2 A_i = 0$; moreover, since $\pi_1 A_i \to \pi_1 C(\mathfrak{k}_1)$ is injective and $g_\#\colon \pi_1 C(\mathfrak{k}_1) \to \pi_1 C(\mathfrak{k})$ is an isomorphism, $(g|A_i)_\#\colon \pi_1 A_i \to \pi_1 A$ is injective. This shows that $\pi_1 A_i$ is a subgroup of $\mathbb{Z}$, hence, trivial or isomorphic to $\mathbb{Z}$. Now A_i is an orientable compact connected surface and therefore either a disk, a sphere or an annulus. We will show that A_i is an annulus. $\pi_2 A_i = 0$ excludes spheres. If A_i is a disk then $\partial A_i \subset \partial C(\mathfrak{k}_1)$ is contractible in $C(\mathfrak{k}_1)$. If ∂A_i is not nullhomotopic on $\partial C(\mathfrak{k}_1)$ then $C(\mathfrak{k}_1)$ is a solid torus and $\mathfrak{k}_1$ is the trivial knot. But then $\pi_1 C(\mathfrak{k}_1) \cong \mathbb{Z}$ and this implies that $\mathfrak{k}$ is also unknotted, contradicting the assumption that it is a (p, q)-cable knot. Therefore ∂A_i also bounds a disk $D \subset \partial C(\mathfrak{k}_1)$ and $D \cup A_i$ is a 2-sphere that bounds a ball B in $C(\mathfrak{k}_1)$. Now $Q = \overline{C(\mathfrak{k}_1) - B}$ is homeomorphic to $C(\mathfrak{k}_1)$, $g|Q\colon Q \to C(\mathfrak{k})$ satisfies the conditions (1)–(3), and $(g|Q)^{-1}(A)$ has at most $(n-1)$ components. This proves that there is also a mapping $g'\colon C(\mathfrak{k}_1) \to C(\mathfrak{k})$ satisfying (1)–(3) with less components in $g'^{-1}(A)$ than in $g^{-1}(A)$, contradicting the minimality of n.

Thus we have proved that A_i is an annulus. Because of (3), ∂A_i is not nullhomotopic on $\partial C(\mathfrak{k}_1)$ and decomposes $\partial C(\mathfrak{k}_1)$ into two annuli, while A_i decomposes $C(\mathfrak{k}_1)$ into two submanifolds W_i, V_i which are bounded by tori and, thus, are either knot manifolds or solid tori.

If V_i and W_i are knot manifolds then, by 15.24, $\pi_1 C(\mathfrak{k}_1)/\overline{\pi_1 A_i} = 1$, where $\overline{\pi_1 A_i}$ denotes the normal closure of $\pi_1 A_i$ in $\pi_1 C(\mathfrak{k}_1)$, and so, since g is a homotopy equivalence

$$\pi_1 C(\mathfrak{k})/\overline{\pi_1 A} = 1.$$

This implies that each 1-cycle of $C(\mathfrak{k})$ is homologous to a cycle of A, that is $H_1(A) \to H_1(C(\mathfrak{k}))$ is surjective and, hence, an isomorphism. From the exact sequence

$$\begin{array}{ccccc} \cdots \to H_1(A) & \to & H_1(C(\mathfrak{k})) & \to & H_1(C(\mathfrak{k}), A) = 0 \\ \| \wr & & \| \wr & & \\ \mathbb{Z} & & \mathbb{Z} & & \\ t & \mapsto & \pm pqt & & \end{array}$$

it follows that $|pq| = 1$, a contradiction. (Prove in Exercise E 15.4 that $H_1(A) \to H_1(C(\mathfrak{k}))$ is defined by $t \mapsto \pm pqt$, where t denotes a generator of $\mathbb{Z}$.)

So we may assume that V_i is a solid torus. If $\pi_1 A_i \to \pi_1 V_i$ is surjective, that is $|q| = 1$, then g can be modified homotopically such that A_i disappears, i.e. we can find a neighbourhood U of V_i in $C(\mathfrak{k}_1)$ such that $U \cong A_i \times [-1, 1]$, $A_i \times \{-1\} = \overline{V_i \cap \partial C(\mathfrak{k}_1)}$, $A_i \times \{0\} = A_i$, $A_i \times [-1, 0] = V_i$, $U \cap g^{-1}(A) = A_i$. Then $Q = \overline{C(\mathfrak{k}_1) - U} \cong C(\mathfrak{k}_1)$ and $g|Q \colon Q \to C(\mathfrak{k})$ is a homotopy equivalence satisfying (1)–(3) and having fewer than n components in $g^{-1}(A)$; this defines a mapping $C(\mathfrak{k}_1) \to C(\mathfrak{k})$ with the same properties, contradicting the choice of g. Therefore $\pi_1 A_i \to \pi_1 V_i$ is not surjective.

W_i is not a solid torus, since $\mathfrak{k}_1$ is not a torus knot. □

15.30 Claim. *$W_1 \subset \cdots \subset W_n$, after a suitable enumeration of the annuli A_i.*

Proof. It suffices to show that for any two components A_1, A_2 either $W_1 \subset W_2$ or $W_2 \subset W_1$. Otherwise either (a) $W_2 \subset V_1$ or (b) $V_2 \subset W_1$.

Case (a). By 15.29, W_2 is a knot manifold which can be contracted slightly in order to be contained in the interior of the solid torus V_1. By Lemma 15.9, there is a 3-ball B such that $W_2 \subset \mathring{B} \subset B \subset V_1$; hence $A_2 \subset \partial W_2$ is contractible in $C(\mathfrak{k}_1)$, contradicting (3).

Case (b). Put $Y = W_1 \cap W_2$ and denote by $\mathfrak{k}_{W_i}$ the core of $\overline{S^3 - W_i}$. Since ∂Y consists of the two annuli A_1, A_2 and two parallel annuli on $\partial C(\mathfrak{k}_1)$ and since S^3 does not contain Klein bottles it follows that ∂Y is a torus, $W_2 = Y \cup V_1$, $A_1 = Y \cap V_1 = \partial Y \cap \partial V_1$ and $\pi_1 A_1 \to \pi_1 V_1$ is not surjective. When Y is a solid torus then $\mathfrak{k}_{W_2}$ is a non-trivial torus knot. When Y is a knot manifold then, by Lemma 15.25, $\mathfrak{k}_{W_2}$ is a cable about the core $\mathfrak{k}_Y$ of Y. The knot $\mathfrak{k}_{W_2}$ is non-trivial and parallel to each component of ∂A_1, see Lemma 15.26.

Since $A_2 = V_2 \cap W_2$ and $\pi_1 A_2 \to \pi_1 V_2$ is not surjective, Lemma 15.25 implies also that $C(\mathfrak{k}_1) = V_2 \cup W_2$ is the complement of an (iterated) cable knot of type (p', q') with $|q'| > 1$ about $\mathfrak{k}_{W_2}$. This implies for the genera that

$$g(\mathfrak{k}_1) \geqq \frac{(|q'|-1)(|p'|-1)}{2} + |q'|\, g(\mathfrak{k}_{W_2}), \tag{4}$$

see 2.10. However, $\mathfrak{k}_1$ is parallel to a component of ∂A_2, by 15.26, which bounds, together with a component of ∂A_1, an annulus; hence, the knots $\mathfrak{k}_1$ and $\mathfrak{k}_{W_2}$ are equivalent, contradicting (4) since $|q'| \geqq 2$. □

15.31 Claim. $(W_n \cap V_1, A_1, \dots, A_n)$ *is homeomorphic to* $(A_1 \times [1, n], A_1 \times \{1\}, \dots, A_1 \times \{n\})$.

Proof. $V_i \cap W_{i+1}$ is bounded by four annuli, hence by a torus. This shows that $V_i \cap W_{i+1}$ is either a knot manifold or a solid torus contained in the solid torus V_i. The first case is impossible by Lemma 15.9, since A_i is incompressible in $C(\mathfrak{k}_1)$. Now

$$V_i = (V_i \cap W_{i+1}) \cup V_{i+1}, \quad (V_i \cap W_{i+1}) \cap V_{i+1} = A_{i+1}$$

where V_i, V_{i+1}, $V_i \cap W_{i+1}$ are solid tori and A_{i+1} is incompressible. Therefore

$$\mathfrak{Z} \cong \pi_1 V_i = \pi_1(V_i \cap W_{i+1}) *_{\pi_1 A_{i+1}} \pi_1 V_{i+1}.$$

Since, by 15.29, $\pi_1 A_{i+1}$ is a proper subgroup of $\pi_1 V_{i+1}$ it follows that $\pi_1 A_{i+1} = \pi_1(V_i \cap W_{i+1})$. Since ∂A_i is parallel to ∂A_{i+1} which contains the generator of $\pi_1 A_{i+1}$ it follows that $\pi_1 A_i$ also generates $\pi_1(V_i \cap W_{i+1})$. Moreover, $A_i \cup A_{i+1} \subset \partial(V_i \cap W_{i+1})$ and $A_i \cap A_{i+1} = \emptyset$.

This means that

$$(V_i \cap W_{i+1}, A_i, A_{i+1}) \cong (A_i \times [i, i+1], A_1 \times \{i\}, A_1 \times \{i+1\}). \qquad □$$

15.32 Claim. $g|A_i$ *is homotopic to a homeomorphism.*

Proof. In the following commutative diagram all groups are isomorphic to $\mathbb{Z}$.

$$\begin{tikzcd}
H_1(A_i) \arrow[r, "j_{i*}"] \arrow[d, "(g|A_i)_*"'] & H_1(C(\mathfrak{k}_1)) \arrow[d, "g_*", "\cong"'] \\
H_1(A) \arrow[r, "j_*"'] & H_1(C(\mathfrak{k}))
\end{tikzcd}$$

where $j_i : A_i \hookrightarrow C(\mathfrak{k}_1)$ and $j : A \hookrightarrow C(\mathfrak{k})$ are the inclusions. As g is a homotopy equivalence, g_* is an isomorphism.

By Claim 15.29, A_i decomposes $C(\mathfrak{k}_1)$ into a knot manifold W_i and a solid torus V_i: $C(\mathfrak{k}_1) = W_i \cup V_i$, $A_i = W_i \cap V_i$, and by Lemma 15.26 a component b_i of ∂A_i is isotopic to $\mathfrak{k}_1$. The component b_i is, for suitable p', q', $|q'| \geqq 2$, a (p', q')-curve on $\partial(\overline{S^3 - W_i})$. For generators of the cyclic groups of the above diagram and for some $r \in \mathbb{Z}$ we obtain

$$\begin{tikzcd} z_i \arrow[r, mapsto, "j_{i*}"] \arrow[d, mapsto] & t'^{\pm|p'q'|} \arrow[d, mapsto, "g_*"] \\ z^r \arrow[r, mapsto] & t^{\pm r|pq|}; \end{tikzcd}$$

here we used the fact that a component of ∂A is a (p, q)-curve on $\partial(\overline{S^3 - X})$ (for the notations, see 15.27). Since g_* is an isomorphism, $g_*(t') = t$; hence, $|p'q'| = \pm r|pq|$. This implies that pq divides $p'q'$.

By a deep theorem of Schubert [1953, p. 253, Satz 5], $\mathfrak{k}_1$ determines the core $\mathfrak{k}_{W_i}$ and the numbers p', q'. Hence, since g is a homotopy equivalence, we may apply the above argument with the roles of $\mathfrak{k}_1$ and $\mathfrak{k}$ interchanged and obtain that $p'q'$ divides pq; thus $|r| = 1$.

This implies that $g|A_i \colon A_i \to A$ can be deformed into a homeomorphism. Since A_i and A are two-sided, g is homotopic to a mapping g' such that $g'|A_i \colon A_i \to A$ is a homeomorphism and g' coincides with g outside a small regular neighbourhood $U(A_i) \cong A_i \times [0, 1]$ of A_i. □

For the following, we assume that g has the property of 15.32 for all A_i.

15.33 Claim. $g^{-1}(A) \neq \emptyset$. *In fact, the number of components of* $g^{-1}(A)$ *is odd.*

Proof. By 15.31, V_n contains a core v_1 of V_1. Let δ be a path in V_1 from $x_1 \in A_1$ to v_1. Then $\pi_1 W_1$ and $\delta v_1 \delta^{-1}$ generate the group $\pi_1(C(\mathfrak{k}_1)) = \pi_1(W_1, x_1) *_{\pi_1(A_1, x_1)} \pi_1(V_1, x_1)$. Since g is transversal with respect to A, it follows that $\mathring{V}_i \cap \mathring{W}_{i+1}$ and $\mathring{V}_{i+1} \cap \mathring{W}_{i+2}$ are mapped to different sides of A; hence, *if the number of components of* $g^{-1}(A)$ *is even*, g maps W_1 and V_n – and hence v_1 – both into X or both into V. Since g is a homotopy equivalence, hence $g_\#$ an isomorphism, it follows that $\pi_1 C(\mathfrak{k})$ is isomorphic to a subgroup of $\pi_1 X$ or $\pi_1 V$, in fact, to $\pi_1 X$ or $\pi_1 V$, respectively. In the latter case $\pi_1 C(\mathfrak{k})$ is cyclic; hence, $\mathfrak{k}$ is the trivial knot, contradicting the assumption that $\mathfrak{k}$ is a cable knot. In the first case $\pi_1 C(\mathfrak{k})/\pi_1 X = 1$. This implies that $H_1(X) \to H_1(C(\mathfrak{k}))$ is isomorphic; hence $H_1(C(\mathfrak{k}), X) = 0$ as follows from the exact sequence

$$\begin{tikzcd}[column sep=small] H_1(X) \arrow[r] & H_1(C(\mathfrak{k})) \arrow[r] \arrow[dr] & H_1(C(\mathfrak{k}), X) \arrow[r] \arrow[dr] & H_0(X) \arrow[r, "\cong"] & H_0(C(\mathfrak{k})). \\ & & 0 \arrow[u] & 0 \arrow[u] & \end{tikzcd}$$

On the other hand by the excision theorem,

$$H_i(C(\mathfrak{k}), X) \cong H_i(V, A)$$

and

$$\begin{tikzcd}
H_2(V,A) \arrow[r] & H_1(A) \arrow[r] & H_1(V) \arrow[r] & H_1(V,A) \arrow[r] \arrow[dr] & H_0(A) \arrow[r, "\cong"] & H_0(V); \\
& z \longmapsto t^{\pm pq} & & & 0 \arrow[u] &
\end{tikzcd}$$

this implies $H_1(V, A) \cong \mathbb{Z}_{|pq|} \neq 0$. Thus the assumption that the number of components of $g^{-1}(A)$ is even was wrong. □

15.34 Claim. *The number n of components of $g^{-1}(A)$ is* 1. (*This finishes the proof of Lemma* 15.27.)

Proof. It will be shown that for $n > 1$ the mapping g can be homotopically deformed to reduce the number of components of $g^{-1}(A)$ by 2, contradicting the minimality of n; thus, by Claim 15.33, $n = 1$. The proof applies a variation of Stalling's technique of *binding ties* from [Stallings 1962] which was used in the original proof of Theorem 5.1, but in a more general setting.

Choose $x \in A$, $x_i \in A_i$ for $1 \leqq i \leqq n$ such that $g(x_i) = x$. There is a path α in $C(\mathfrak{k}_1)$ from x_1 to x_n with the following properties:

(1) $g(\alpha) \simeq 0$ in $C(\mathfrak{k})$;

(2) i) $\alpha = \alpha_1 \dots \alpha_r$ where

ii) $\mathring{\alpha}_j \subset C(\mathfrak{k}_1) - \bigcup\limits_{i=1}^{n} A_i$, $\partial \alpha_j \in \bigcup\limits_{i=1}^{n} A_i$ and

iii) α_j is either a loop with some x_i as basepoint or a path from x_i to $x_{i\pm 1}$.

A path with these properties can be obtained as follows: Let β be a path from x_1 to x_n. Then $[g \circ \beta] \in \pi_1(C(\mathfrak{k}), x)$ and, since g_* is an isomorphism, there is a loop $\delta \subset C(\mathfrak{k}_1)$ in the homotopy class $g_\#^{-1}[g \circ \beta] \in \pi_1(C(\mathfrak{k}_1), x_n)$. Then $\alpha = \beta\delta^{-1}$ has property (1). We can choose α transversal to $g^{-1}(A)$ and, since each A_i is connected, intersecting an A_i in x_i.

Assume that α is chosen such that the number r is minimal for all paths with the properties (1) and (2). In $\pi_1 C(\mathfrak{k}) = \pi_1 X *_{\pi_1 A} \pi_1 V$,

$$1 = [g \circ \alpha] = [g \circ \alpha_1] \dots [g \circ \alpha_r].$$

Since $g \circ \alpha_i$ and $g \circ \alpha_{i+1}$ are in different components X, V it follows that there is at least one α_j with $[g \circ \alpha_j] \in \pi_1 A$. A loop α_j from x_i to x_i in $V_i \cap W_{i+1}$ with $[g \circ \alpha_j] \in \pi_1 A$ can be pushed into $V_{i-1} \cap W_i$, contradicting the minimality of r. Therefore α_j connects x_i and x_{i+1}, for a suitable i.

By 15.31, $V_i \cap W_{i+1} \cong A_1 \times [i, i+1]$ and $A_i = A_1 \times \{i\}$, $A_{i+1} = A_1 \times \{i+1\}$, and therefore α_j is homotopic to an arc $\beta \subset \partial(V_i \cap W_{i+1})$ connecting x_i and x_{i+1}. Let γ be an arc in $\partial(V_i \cap W_{i+1})$ such that $\beta \cup \gamma$ is a meridian of the solid torus $V_i \cap W_{i+1}$

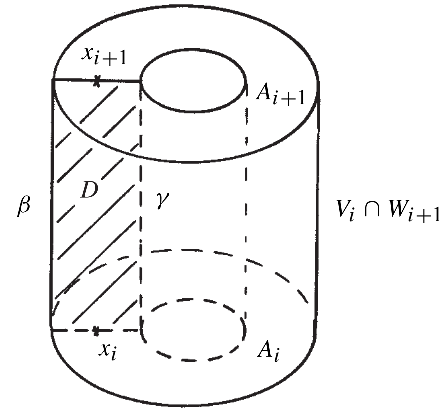


Figure 15.12

and bounds a disk D. We may assume that $\partial D \cap A_i$ and $\partial D \cap A_{i+1}$ are arcs connecting the boundary components, see Figure 15.12.

Let B^3 be the closure of the complement of a regular neighbourhood of $A_i \cup D \cup A_{i+1}$ in $V_i \cap W_{i+1}$; then B^3 is a 3-ball.

In the following we keep g fixed outside of a regular neighbourhood of $V_i \cap W_{i+1}$. Since $[g \circ \beta] \in \pi_1 A$ and $g \circ \beta \simeq g \circ \gamma$, g may be deformed such that $g(\beta) \subset A$ and $g(\gamma) \subset A$. Since A is incompressible in $C(\mathfrak{k})$ and $\pi_2 C(\mathfrak{k}) = 0$, g can be altered such that g maps D and also the small neighbourhood into A, that is, $g(V_i \cap W_{i+1} - B^3) \subset A$. Finally since $\pi_3 C(\mathfrak{k}) = 0$, we obtain $g(B^3) \subset A$; thus $g(V_i \cap W_{i+1}) \subset A$, and an additional slight adjustment eliminates both components A_i, A_{i+1} of $g^{-1}(A)$. $\square$

15.35 Lemma. *Let $\mathfrak{k}_1$ and $\mathfrak{k}$ be (p_1, q_1)- and (p, q)-cable knots about the cores $\mathfrak{h}_1$ and $\mathfrak{h}$ where $|q_1|$, $|q| \geqq 2$, and let $C(\mathfrak{k}) = C(\mathfrak{h}) \cup V$, $C(\mathfrak{h}) \cap V = \partial C(\mathfrak{h}) \cap \partial V = A$ an annulus. If $\pi_1 C(\mathfrak{k}_1) \cong \pi_1 C(\mathfrak{k})$ then*

(a) *there is a homeomorphism $F\colon C(\mathfrak{h}_1) \to C(\mathfrak{h})$ such that $A_1 = F^{-1}(A)$ defines a cable presentation of $\mathfrak{k}_1$, that is*

$$C(\mathfrak{k}_1) = \overline{C(\mathfrak{k}_1) - C(\mathfrak{h}_1)} \cup C(\mathfrak{h}_1),$$
$$\overline{C(\mathfrak{k}_1) - C(\mathfrak{h}_1)} \cap C(\mathfrak{h}_1) = \partial \overline{C(\mathfrak{k}_1) - C(\mathfrak{h}_1)} \cap \partial C(\mathfrak{h}_1) = A_1,$$

and

(b) $|p_1| = |p|$ *and* $|q_1| = |q|$.

Proof. We may assume that $\mathfrak{h}_1$ and $\mathfrak{h}$ are non-trivial, because otherwise $\mathfrak{k}_1$ and $\mathfrak{k}$ are torus knots and 15.35 follows from 15.6. We have $\pi_1 C(\mathfrak{k}) = \pi_1 C(\mathfrak{h}) *_{\pi_1 A} \pi_1 V$. Since $\pi_1 A \to \pi_1 V_1$ is not surjective (as $|q| \geqq 2$), the free product with amalgamation is not trivial. By Lemma 15.27, there is a homotopy equivalence $f\colon C(\mathfrak{k}_1) \to C(\mathfrak{k})$ such that $f^{-1}(A) = A_1$ is an annulus. Then A_1 decomposes $C(\mathfrak{k}_1)$ into a knot manifold X_1 and a solid torus V_1:

$$C(\mathfrak{k}_1) = X_1 \cup V_1, \quad X_1 \cap V_1 = \partial X_1 \cap \partial V_1 = A_1.$$

For any basepoint $a_1 \in A_1$,

$$\pi_1(C(\mathfrak{k}_1), a_1) = \pi_1(X_1, a_1) *_{\pi_1(A_1, a_1)} \pi_1(V_1, a_1).$$

Since $f^{-1}(A) = A_1$ consists of one component only, one of the groups $f_\#(\pi_1(X_1, a_1))$ and $f_\#(\pi_1(V_1, a_1))$ is contained in $\pi_1(C(\mathfrak{h}), f(a_1))$ and the other in $\pi_1(V, f(a_1))$. By assumption $C(\mathfrak{h})$ and X_1 are knot manifolds, V, V_1 solid tori and $f_\#$ is an isomorphism. From the solution of the word problem in free products with amalgamated subgroups, see [ZVC 1980, 2.3.3], it follows that

$$f_\#(\pi_1(X_1, a_1)) = \pi_1(C(\mathfrak{h}), f(a_1)) \quad \text{and} \quad f_\#(\pi_1(V_1, a_1)) = \pi_1(V, f(a_1)).$$

This implies

(1) $f(X_1) \subset C(\mathfrak{h})$, $f(V_1) \subset V$, and that $(f|X_1)_\#$ and $(f|V_1)_\#$ are isomorphisms and $f|X_1\colon X_1 \to C(\mathfrak{h})$ and $f|V_1\colon V_1 \to V$ are homotopy equivalences because all spaces are $K(\pi, 1)$.

For the proof of (b) we note that $(f|A_1)_\#\colon \pi_1 A_1 \to \pi_1 A$ is also an isomorphism.

Assume that $f|X_1$ is homotopic to a mapping $f_0\colon X_1 \to C(\mathfrak{h})$ such that $f_0(\partial X_1) \subset \partial C(\mathfrak{h})$ and $f_0|\partial A_1 = f|\partial A_1$. Then, by [Waldhausen 1968, Theorem 6.1], see Appendix B.7, there is a homotopy $f_t\colon (X_1, \partial X_1) \to (C(\mathfrak{h}), \partial C(\mathfrak{h}))$, $0 \leqq t \leqq 1$ such that f_1 is a homeomorphism; this proves (a).

To prove the above assumption on ∂X_1 we consider $B_1 = \partial X_1 \cap \partial C(\mathfrak{k}_1)$. Now $\partial B_1 = \partial A_1$. We have to show that $f|B_1\colon (B_1, \partial B_1) \to (C(\mathfrak{h}), \partial C(\mathfrak{h}))$ is not essential. Otherwise, by Lemma 15.18 there is a properly imbedded essential annulus $A' \subset C(\mathfrak{h})$ such that $\partial A' = \partial A$. The components of ∂A are (p, q)-curves on $\partial C(\mathfrak{h})$ and $(n, \pm 1)$-curves on $\partial C(k)$ for a suitable n; the last statement is a consequence of the fact that the components of ∂A are isotopic to $\mathfrak{k}$.

Since A' is essential, $C(\mathfrak{h})$ is either the complement of a cable knot or of a product knot, see Lemma 15.26. In the first case the components of $\partial A'$ are isotopic to the knot $\mathfrak{h}$; hence they are $(n', \pm 1)$-curves on $\partial C(\mathfrak{h})$. In the latter case they are $(\pm 1, 0)$-curves. Both cases contradict the fact $\partial A = \partial A'$ and the assumption $|q| \geqq 2$.

For the proof of (b), let m_1 and m be meridians on the boundaries ∂V_1, ∂V of the regular neighbourhoods V_1, V of $\mathfrak{h}_1$, $\mathfrak{h}$. In the proof of (a) we saw that there is a homotopy equivalence $f\colon C(\mathfrak{k}_1) \to C(\mathfrak{k})$ with $f(A_1) = A$. Let s_1 be a component of ∂A_1 and $s = f(s_1)$; consider s_1 and s as oriented curves. Then s_1 represents $\pm p_1 m_1$ in $H_1(X_1)$ and s represents $\pm pm$ in $H_1(C(\mathfrak{h}))$. The homotopy equivalence f induces an isomorphism $f_*\colon H_1(X_1) \to H_1(C(\mathfrak{h}))$ and $f_*(p_1 m_1) = pm$; hence, $|p_1| = |p|$.

By (1), $(f|V_1)_\#$ and $(f|A_1)_\#$ are isomorphisms, thus $f_*\colon H_1(V_1, A_1) \to H_1(V, A)$ is an isomorphism. Now $H_1(V_1, A_1) \cong \mathbb{Z}_{|q_1|}$ and $H_1(V, A) \cong \mathbb{Z}_{|q|}$ imply $|q_1| = |q|$. □

15.36. *Proof of Theorem* 15.22. Assume that $\mathfrak{k}_0, \mathfrak{k}_1, \mathfrak{k}_2$ are (p,q)-, (p_1,q_1)-, (p_2,q_2)-cables about $\mathfrak{h}_0, \mathfrak{h}_1, \mathfrak{h}_2$ with the same group. If $\mathfrak{h}_i$ is unknotted then $\mathfrak{k}_i$ is a torus knot and the equivalence of $\mathfrak{k}_0, \mathfrak{k}_1, \mathfrak{k}_2$ is a consequence of 15.6. Now we assume that $\mathfrak{h}_0, \mathfrak{h}_1, \mathfrak{h}_2$ are knotted. By Lemma 15.35, $C(\mathfrak{h}_i) \cong C(\mathfrak{h}_0)$, $|p_i| = |p|$, $|q_i| = |q|$ for $i = 1, 2$.

Let, for $i = 0, 1, 2$, an essential annulus A_i decompose $C(\mathfrak{k}_i)$ into a knot manifold $C(\mathfrak{h}_i)$ and a solid torus V_i; now the knot $\mathfrak{k}_i$ is parallel to each of the components of ∂A_i. Because of Lemma 15.35 there are homotopy equivalences

$$F_{ij} \colon C(\mathfrak{k}_i) \to C(\mathfrak{k}_j) \quad (i = 0, 1;\ j = 1, 2)$$

such that

$$\tilde F_{ij} = F_{ij}|C(\mathfrak{h}_i) \colon (C(\mathfrak{h}_i), A_i) \to (C(\mathfrak{h}_j), A_j))$$

are homeomorphisms.

It suffices to prove that $\tilde F_{01}$, $\tilde F_{12}$ or $\tilde F_{02} = \tilde F_{12} \circ \tilde F_{01}$ can be extended to a homeomorphism of S^3, because by [Schubert 1953, p. 253] cable knots are determined by their cores and winding numbers.

Let (m_i, ℓ_i) be meridian-longitude for $\mathfrak{h}_i$, $i = 0, 1, 2$; assume that they are oriented such that the components of ∂A_i are homologous to $pm_i + q\ell_i$ on $\partial C(\mathfrak{h}_i)$. There are numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \eta \in \{1, -1\}$ and $x, y \in \mathbb{Z}$ such the $\tilde F_{ij}|\partial C(\mathfrak{h}_i)$ are given by the following table.

$\tilde F_{01}$		$\tilde F_{12}$		$\tilde F_{02}$	
m_0	$\mapsto m_1^{\alpha}\ell_1^{x}$	m_1	$\mapsto m_2^{\gamma}\ell_2^{y}$	m_0	$\mapsto m_2^{\alpha\gamma}\ell_2^{\alpha y+\delta x}$
ℓ_0	$\mapsto \ell_1^{\beta}$	ℓ_1	$\mapsto \ell_2^{\delta}$	ℓ_0	$\mapsto \ell_2^{\beta\delta}$
$m_0^p\ell_0^q$	$\mapsto (m_1^p\ell_1^q)^{\varepsilon}$	$m_1^p\ell_1^q$	$\mapsto (m_2^p\ell_2^q)^{\eta}$	$m_0^p\ell_0^q$	$\mapsto (m_2^p\ell_2^q)^{\varepsilon\eta}$;

The last row is a consequence of the fact that the $\tilde F_{ij} \colon A_i \to A_j$ are homeomorphisms.

If some m_i is mapped to $m_j^{\pm 1} = m_j^{\pm 1}\ell^0$ then the homeomorphism $\tilde F_{ij}$ can be extended to S^3 and this finishes the proof. Hence, we will show that one of the exponents x, y and $\alpha y + \delta x$ vanishes. Assume that $x \neq 0 \neq y$. Now

$$\begin{aligned}
&(m_1^p\ell_1^q)^{\varepsilon} = \tilde F_{01}(m_0^p\ell_0^q) = m_1^{\alpha p}\ell_1^{\beta q + xp}\\
\Longrightarrow\quad &\varepsilon p = \alpha p,\ \varepsilon q = \beta q + xp;\\
\Longrightarrow\quad &\varepsilon = \alpha,\ xp = (\alpha - \beta)q.
\end{aligned}$$

Now $p \neq 0 \neq x$ implies $\alpha \neq \beta$, and $|\alpha| = |\beta| = 1$ gives $\alpha = -\beta$. Therefore $xp = 2\alpha q$ and $x = \frac{2\alpha q}{p}$. The same arguments for $\tilde F_{12}$ imply that $\delta = -\gamma$ and $y = \frac{2\gamma q}{p}$. Therefore

$$\alpha y + \delta x = \alpha\frac{2\gamma q}{p} - \gamma\frac{2\alpha q}{p} = 0.$$

□

D Groups of Product Knots

Next we consider problems for product knots similar to those in Part C. The situation is in some sense simpler, as product knots have Property P, see 15.11; hence, product knots with homeomorphic complements are of the same type. However, the groups of two product knots of different type may be isomorphic as we have shown in 15.4. We will prove that there are no other possibilities than those described in Example 15.4.

15.37 Lemma. *Let $\mathfrak{k}_1$ and $\mathfrak{k}_2$ be knots with $\pi_1 C(\mathfrak{k}_1) \cong \pi_1 C(\mathfrak{k}_2)$. Then both knots are prime or both are product knots.*

Proof. Assume that $\mathfrak{k}_2$ is a product knot. Then there is a properly embedded incompressible annulus $A \subset C(\mathfrak{k}_2)$ such that $C(\mathfrak{k}_2) = X' \cup X''$, $A = X' \cap X''$ where X' and X'' are knot manifolds. Since $\pi_n C(\mathfrak{k}_i) = 0$ for $i = 1, 2, n \geqq 2$ there is a homotopy equivalence $f\colon C(\mathfrak{k}_1) \to C(\mathfrak{k}_2)$. By Claim 15.29, see Remark 15.28, we may assume that the components of $f^{-1}(A)$ are incompressible, properly embedded annuli which are not boundary parallel in $C(\mathfrak{k}_1)$. Now $f^{-1}(A) = \emptyset$ is impossible, since, otherwise, $f_\#(\pi_1 C(\mathfrak{k}_1)) \subset \pi_1 X'$ or $f_\#(\pi_1 C(\mathfrak{k}_1)) \subset \pi_1 X''$, contradicting the assumption that $\pi_1 X'$ and $\pi_1 X''$ are proper subgroups of $\pi_1 C(\mathfrak{k}_2)$ and that $f_\#$ is an isomorphism. By Lemma 15.26, $C(\mathfrak{k}_1)$ is the complement of a product knot or a cable knot. In the first case the assertion is proved. In the latter case, $\pi_1 C(\mathfrak{k}_2) = \pi_1 C(\mathfrak{k}_1)$ is the group of a cable knot and, thus, applying the arguments of 15.29 to $C(\mathfrak{k}_2)$ and the inverse homotopy equivalence, it follows that $C(\mathfrak{k}_2)$ is also the complement of a cable knot. Since products knots have Property P (Theorem 15.11), we conclude that $\mathfrak{k}_2$ is a cable knot, contradicting the fact that cable knots are prime, see [Schubert 1953, p. 250, Satz 4]. □

15.38 Theorem ([Feustel-Whitten 1978]). *Let $\mathfrak{k} = \mathfrak{k}_1 \# \cdots \# \mathfrak{k}_m$ and $\mathfrak{h} = \mathfrak{h}_1 \# \cdots \# \mathfrak{h}_n$ be knots in S^3, where the $\mathfrak{k}_i$ and $\mathfrak{h}_j$ are prime and $n > 1$. If $\pi_1(S^3 - \mathfrak{k}) \cong \pi_1(S^3 - \mathfrak{h})$ then $\mathfrak{k}$ is a product knot, $m = n$ and there is a permutation σ such that $\mathfrak{k}_j$ and $\mathfrak{h}_{\sigma(j)}$ are of the same type.*

Proof. By Lemma 15.37, $\mathfrak{k}$ is also a product knot, i.e. $m > 1$. Let A be a properly embedded annulus in $C(\mathfrak{h}) = X' \cup X''$, $A = X' \cap X''$ where X' and X'' are knot manifolds. As in the proof above we conclude that there is a homotopy equivalence $f\colon C(\mathfrak{k}) \to C(\mathfrak{h})$ such that $f^{-1}(A)$ consists of disjoint incompressible, properly embedded essential annuli. Let A_1 be a component of $f^{-1}(A)$. In the following commutative diagram all groups are isomorphic to $\mathbb{Z}$.

$$\begin{tikzcd} H_1(A_1) \arrow[r, "j_{1*}"] \arrow[d, "(f|A_1)_*"'] & H_1(C(\mathfrak{k}_1)) \arrow[d, "f_*"] \\ H_1(A) \arrow[r, "j_*"'] & H_1(C(\mathfrak{k})); \end{tikzcd}$$

where $j_1 \colon A_1 \to C(\mathfrak{k})$, $j \colon A \to C(\mathfrak{h})$ are the inclusions. As f is a homotopy equivalence, f_* is an isomorphism. Since $C(\mathfrak{k})$ and $C(\mathfrak{h})$ are complements of product knots the components of ∂A_1 and ∂A bound disks in $\overline{S^3 - C(\mathfrak{k})}$ and $\overline{S^3 - C(\mathfrak{h})}$, respectively, see Lemma 15.24. The boundaries of these disks are generators of $H_1(C(\mathfrak{k}))$ and $H_1(C(\mathfrak{h}))$; hence, j_{1*} and j_* are isomorphisms. This proves that $(f|A_1)_*$ is an isomorphism and, consequently, that $f|A_1 \colon A_1 \to A$ is a homotopy equivalence homotopic to a homeomorphism. Since f is transversal with respect to A, see (1) in the proof of 15.27, there is a neighbourhood $A \times [0, 1) \subset C(\mathfrak{h})$ such that the homotopy $f|A_1$ can be extended to a homotopic deformation of f which is constant outside of $A \times [0, 1)$. By the same arguments as in the proof of 15.34 one concludes that in addition f can be chosen such that $A_1 = f^{-1}(A)$ is connected. The annulus A_1 decomposes $C(\mathfrak{k})$ into two subspaces Y', Y'' of S^3 bounded by tori, which are mapped to X' and X'', respectively: $f(Y') \subset X'$, $f(Y'') \subset X''$. It follows that $(f|Y')_\#$ and $(f|Y'')_\#$ are isomorphisms. This proves that Y' and Y'' are knot manifolds. Therefore $\mathfrak{k} = \mathfrak{k}' \# \mathfrak{k}''$ and $\mathfrak{h} = \mathfrak{h}' \# \mathfrak{h}''$ where $\mathfrak{k}'$ and $\mathfrak{h}'$ have isomorphic groups. This isomorphism maps meridional elements to meridional elements, since they are realized by the components of ∂A_1 and ∂A. The same is true for $\mathfrak{k}''$ and $\mathfrak{h}''$.

Assume that $\mathfrak{h}'$ and, hence, $\mathfrak{k}'$ are prime knots. Then $\partial(C(\mathfrak{k}')) = B_1 \cup A_1$ where B_1 is an annulus. If $f(B_1)$ is essential then there is a properly embedded essential annulus in $C(\mathfrak{k}')$. One has $\partial B_1 = \partial A_1$ and $f(\partial B_1) = \partial A$. Now ∂A bounds meridional disks in $\overline{S^3 - C(\mathfrak{h})}$ and therefore also in $\overline{S^3 - C(\mathfrak{h}')}$; this contradicts the assumption that $\mathfrak{h}'$ is prime. Therefore $f(B_1)$ is not essential and thus $f|B_1$ is homotopic to a mapping with image in $\partial C(\mathfrak{h}')$ – by a homotopy constant on $\partial B_1 = \partial A_1$. This homotopy can be extended to a homotopy of f which is constant on A_1. Finally one obtains a homotopy equivalence $(Y', \partial Y') \to (X', \partial X')$ which preserves meridians. By Corollary 6.5 of [Waldhausen 1968], $Y' \cong X'$, where the homeomorphism maps meridians to meridians and, thus, can be extended to S^3, see 3.15. This proves that $\mathfrak{k}'$ and $\mathfrak{h}'$ are of the same knot type.

Now the theorem follows from the uniqueness of the prime factor decomposition of knots. □

In fact, we have proved more than claimed in Theorem 15.38:

15.39 Proposition. *Under the assumptions of Theorem* 15.38*, there is a system of pairwise disjoint, properly embedded annuli* $A_1, \dots, A_{n-1}$ *in* $C(\mathfrak{k})$ *and a homeomorphism* $f \colon C(\mathfrak{k}) \to C(\mathfrak{h})$ *such that* $\{A_1, \dots, A_{n-1}\}$ *decomposes* $C(\mathfrak{k})$ *into the knot manifolds* $C(\mathfrak{k}_1), \dots, C(\mathfrak{k}_n)$ *and* $\{f(A_1), \dots, f(A_{n-1})\}$ *decomposes* $C(\mathfrak{h})$ *into* $C(\mathfrak{h}_{\sigma(1)}), \dots, C(\mathfrak{h}_{\sigma(n)})$.

Since product knots have the Property P, the system of homologous meridians $(m(\mathfrak{k}_1), \dots, m(\mathfrak{k}_n))$ *is mapped, for a fixed* $\varepsilon \in \{1, -1\}$*, onto the system of homologous meridians* $(m(\mathfrak{h}_{\sigma(1)})^\varepsilon, \dots, m(\mathfrak{h}_{\sigma(n)})^\varepsilon)$. □

15.40 Proposition ([Simon 1980′]). *If $\mathfrak{G}$ is the group of a knot with n prime factors ($n \geqq 2$), then $\mathfrak{G}$ is the group of at most 2^{n-1} knots of mutually different knot types. Moreover, when the n prime factors are of mutually different knot types and when each of them is non-invertible and non-amphicheiral, then $\mathfrak{G}$ is the group of exactly 2^{n-1} knots of mutually different types and of 2^{n-1} knot manifolds.*

Proof. By Theorem 3.15, an oriented knot $\mathfrak{k}$ is determined up to isotopy by the peripheral system $(\mathfrak{G}, m, \ell)$ and we use this system now to denote the knot. Clearly (proof as E 15.5, see also 3.19),

$$-\mathfrak{k} = (\mathfrak{G}, m, \ell^{-1}), \quad \mathfrak{k}^* = (\mathfrak{G}, m^{-1}, \ell), \quad -\mathfrak{k}^* = (\mathfrak{G}, m^{-1}, \ell^{-1}),$$

and

$$\mathfrak{k}_1 \# \mathfrak{k}_2 = (\mathfrak{G}_1 *_{m_1=m_2} \mathfrak{G}_2, m_1, \ell_1\ell_2).$$

Let $\mathfrak{k} = \mathfrak{k}_1 \# \cdots \# \mathfrak{k}_n$, $n \geqq 2$. By 15.11, $\mathfrak{k}$ has Property P; hence, on $\partial C(\mathfrak{k})$ the meridian is uniquely determined up to isotopy and reversing the orientation. It is

$$(\mathfrak{G}, m, \ell) = (\mathfrak{G}_1, m_1, \ell_1) \# \cdots \# (\mathfrak{G}_n, m_n, \ell_n) = (\mathfrak{G}_1 \underset{m_1=\cdots=m_n}{* \cdots *} \mathfrak{G}_n, m_1, \ell_1\ell_2 \ldots \ell_n).$$

Suppose $\mathfrak{h}$ is a knot whose group is isomorphic to $\mathfrak{G}$. Now the above remark and 15.39 imply that

$$\mathfrak{h} = (\mathfrak{G}_1, m_1^{\varepsilon}, \ell_1^{\delta_1}) \# \cdots \# (\mathfrak{G}_n, m_n^{\varepsilon}, \ell_n^{\delta_n}) = (\mathfrak{G}_1 \underset{m_1=\cdots=m_n}{* \cdots *} \mathfrak{G}_n, m_1^{\varepsilon}, \ell_1^{\delta_1} \ldots \ell_n^{\delta_n}).$$

Corresponding to the choices of $\varepsilon, \delta_1, \ldots, \delta_n$ there are 2^{n+1} choices for $\mathfrak{h}$. Therefore $\mathfrak{h}$ represents one of, possible, 2^{n+1} oriented isotopy types and $\frac{1}{4}2^{n+1}$ knot types.

Clearly, this number is attained for knots with the properties mentioned in the second assertion of the proposition. □

If prime knots are indeed determined by their groups, then the hypothesis $n \geqq 2$ in 15.40 is unnecessary.

E History and Sources

The theorem of F. Waldhausen [1967] on sufficiently large irreducible 3-manifolds, see Appendix B.7, implies that the peripheral group system determines the knot complement. Then the question arises to what extend the knot group characterizes the knot type. The difficulty of this problem becomes obvious by the example of J.H.C. Whitehead [1937] of different links with homeomorphic complements, see 15.1. First results were obtained by D. Noga [1967] who proved Property P for product knots, and by

R.H. Bing and J.M. Martin [1971] who showed it for the four-knot, twist knots, product knots again and for satellites. The 2-bridge knots have Property P by [Takahaski 1981].

A first final answer was given by C. Gordon and J. Luecke [1989] proving that the knot complement determines the knot type.

The Annulus and the Torus Theorem [1974] of C.D. Feustel [1972, 1976], [Cannon-Feustel 1976] gave strong tools to approach the problem of to what extend the group determines the complement. The results of J. Simon [1970, 1973, 1976′, 1980′], W. Whitten [1974], [Feustel-Whitten 1978], K. Johannson [1979], Whitten [1985], Culler-Shalen [1985], and C.McA. Gordon and J. Luecke combine to give a positive answer to the question: Is the complement of a prime knot determined by its group?

The status of Property P is – according to [Culler-Gordon-Luecke-Shalen 1987] – that there are at most two possibilities to obtain a homotopy sphere by Dehn-fillings of a knot complement.

F Exercises

E 15.1. Use Lemma 2.11 to prove that $h^{-1}(\ell) = \pm\tilde{\ell}$ satisfies equation (6) in 15.15.

E 15.2. Let M be a 3-manifold, $V \subset M$ a solid torus, $\overline{\partial V \cap \mathring{M}} = A$ an annulus such that the core of A is a longitude of V. Then A is boundary parallel.

E 15.3. Show that both descriptions in 15.20 define the same knot.

E 15.4. Let $\mathfrak{k}$ be a (p, q)-cable knot and let A be an annulus, defining $\mathfrak{k}$ as cable. Then $\mathbb{Z} \cong H_1(A) \to H_1(C(\mathfrak{k})) \cong \mathbb{Z}$ is defined by $t \mapsto \pm pqt$, where t is the generator of $\mathbb{Z}$.

E 15.5. Let $\mathfrak{k} = (\mathfrak{G}, m, \ell)$ and $\mathfrak{k}_i = (\mathfrak{G}_i, m_i, \ell_i)$. Prove that $-\mathfrak{k} = (\mathfrak{G}, m, \ell^{-1})$, $\mathfrak{k}^* = (\mathfrak{G}, m^{-1}, \ell)$, $-\mathfrak{k}^* = (\mathfrak{G}, m^{-1}, \ell^{-1})$, and $\mathfrak{k}_1 \# \mathfrak{k}_2 = (\mathfrak{G}_1 *_{m_1=m_2} \mathfrak{G}_2, m_1, \ell_1\ell_2)$.

Chapter 16

The 2-variable skein polynomial

In 12.18 we mentioned the Conway polynomial as an invariant closely connected with the Alexander polynomial. It can be computed by using the *skein relations*, Figure 12.19, and hence is called a skein invariant. Shortly after the discovery of the famous Jones polynomial several authors independently contributed to a new invariant for oriented knots and links, a Laurent polynomial $P(z, v)$ in two variables which also is a skein invariant and which comprises both, the Jones and the Alexander–Conway polynomials. It has become known as the HOMFLY polynomial after its main contributors: Hoste, Ocneano, Millet, Floyd, Lickorish, and Yetter.

A Construction of a trace function on a Hecke algebra

In the following the HOMFLY polynomial is established via representations of the braid groups $\mathfrak{B}_n$ into a Hecke algebra using Markov's theorem, see 10.22. We follow Jones [1987] and Morton [1988].

16.1 On the symmetric group. The symmetric group $\mathfrak{S}_n$ admits a presentation

$$\begin{aligned}\mathfrak{S}_n = \langle\, \tau_1, \dots, \tau_{n-1} \mid\ & \tau_i^2 = 1 \text{ for } 1 \le i \le n-1,\\ & \tau_i\tau_j = \tau_j\tau_i \text{ for } 1 \le i < j-1 \le n-2,\\ & \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1} \text{ for } 1 \le i \le n-2\,\rangle,\end{aligned}$$

where τ_i is the transposition $(i, i+1)$. We write the group operation in $\mathfrak{S}_n$ from left to right; for example, the product of the transpositions $(1,2) \times (2,3) = (1,2)(2,3)$ is the cycle $(1,3,2)$.

We identify $\mathfrak{S}_{n-1}$ with the subgroup of $\mathfrak{S}_n$ of permutations leaving n fixed.

Every permutation $\pi \in \mathfrak{S}_n$ can be written as a word in the generators τ_i in many ways; we choose a unique representative $b_\pi(\tau_i)$ for each π in the following. If $\pi(n) = j$ we put

$$b_\pi(\tau_i) = (j, j+1)(j+1, j+2)\dots(n-1, n)\cdot b_{\pi'}(\tau_i) \quad \text{with } \pi' \in \mathfrak{S}_{n-1},$$

see Figure 16.1. The words $\mathcal{W}_n = \{b_\pi(\tau_i) \mid \pi \in \mathfrak{S}_n\}$ satisfy the "Schreier" condition which means that, if $b_\pi(\tau_i) = w(\tau_i)\,\tau_k$, then $w(\tau_i) = b_{\pi\cdot\tau_k^{-1}}(\tau_i)$, and b_{id} is the empty word. Furthermore the $b_\pi(\tau_i)$ are of minimal length, and the generator τ_{n-1} occurs at most once in each $b_\pi(\tau_i) \in \mathcal{W}_n$; both assertions are evident in Figure 16.1. Figure 16.2 shows for the cycle $(1\,3\,2\,5)$ the representative $b_\pi(\tau_i) = \tau_2\tau_3\tau_4\tau_3\tau_1\tau_2\tau_1$.

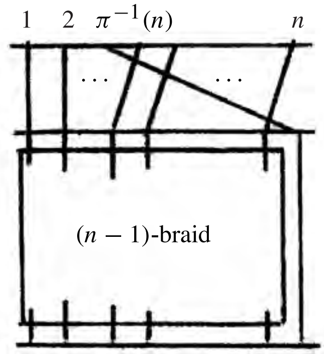


Figure 16.1

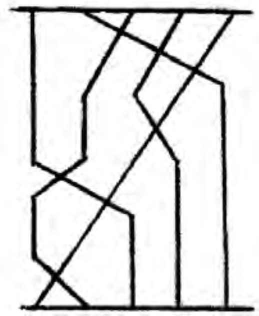


Figure 16.2

16.2 Definition. The following presentation

$$\begin{aligned}\hat{\mathfrak{S}}_n = \langle\, \hat\tau_1, \hat\tau_2, \dots, \hat\tau_{n-1} \mid \hat\tau_i\hat\tau_j = \hat\tau_j\hat\tau_i \text{ for } 1 \le i < j-1 \le n-2, \\ \hat\tau_i\hat\tau_{i+1}\hat\tau_i = \hat\tau_{i+1}\hat\tau_i\hat\tau_{i+1} \text{ for } 1 \le i \le n-2 \,\rangle\end{aligned}$$

defines a semigroup $\hat{\mathfrak{S}}_n$.

The elements of $\hat{\mathfrak{S}}_n$ are the classes of words defined by the following equivalence relation $\hat{=}$: two words $w(\hat\tau_i)$ and $w'(\hat\tau_i)$ are equivalent, $w(\hat\tau_i) \mathrel{\hat{=}} w'(\hat\tau_i)$, if and only if they are connected by a chain of substitutions

$$\hat\tau_i\hat\tau_j \mapsto \hat\tau_j\hat\tau_i, \quad \hat\tau_i\hat\tau_{i+1}\hat\tau_i \mapsto \hat\tau_{i+1}\hat\tau_i\,\hat\tau_{i+1}$$

employing the relations of 16.2. (The building of inverses is not permitted.)

There is a canonical homomorphism $\kappa\colon \hat{\mathfrak{S}}_n \to \mathfrak{S}_n$, $\kappa(\hat\tau_i) = \tau_i$; we write $\hat b_\pi = b_\pi(\hat\tau_i)$ and $\hat{\mathcal{W}}_n = \{\hat b_\pi \mid \pi \in \mathfrak{S}_n\}$.

Two cases occur in forming a product $\hat b_\pi \cdot \hat\tau_k$: either $\hat b_\pi\hat\tau_k \mathrel{\hat{=}} \hat b_\varrho$, the class of $\hat b_\pi\hat\tau_k$ contains a representative $\hat b_\varrho \in \hat{\mathcal{W}}_n$ (case α), or not (case β). Case α occurs when the strings crossing at τ_k do not cross in b_π (Figure 16.3), $\varrho = \pi\tau_k$. In case β they do, and Figure 16.4 shows

$$\hat b_\pi\,\hat\tau_k \mathrel{\hat{=}} \hat b_\varrho\,\hat\tau_k^2, \quad \varrho\,\tau_k = \pi.$$

We note down the result in

16.3 Lemma.

$$\hat b_\pi \cdot \hat\tau_k \mathrel{\hat{=}} \begin{cases} \hat b_\varrho, & \varrho = \pi\tau_k, \quad \textit{case } \alpha, \\ \hat b_\varrho\hat\tau_k^2, & \varrho\,\tau_k = \pi, \quad \textit{case } \beta. \end{cases} \tag{1}$$

□

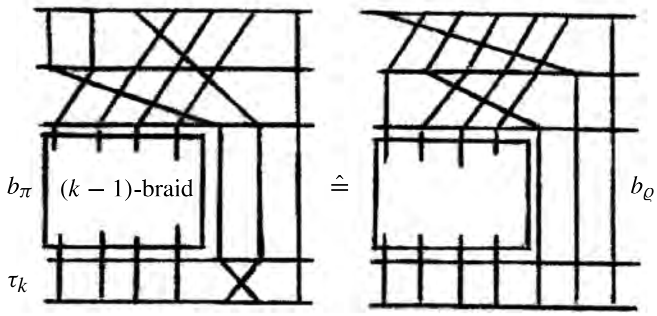


Figure 16.3

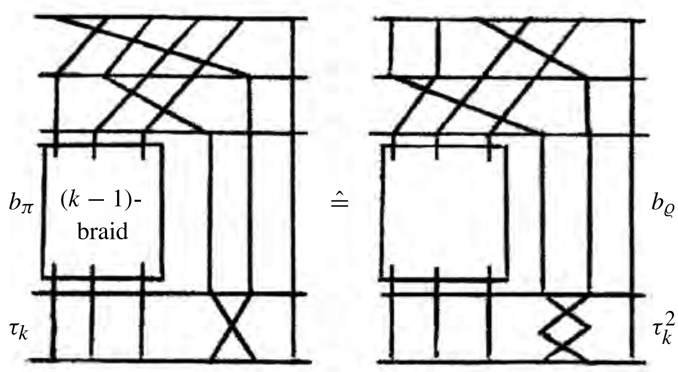


Figure 16.4

16.4 Construction of a Hecke algebra. Next we construct a special algebra, a so-called *Hecke algebra*. We define a free module M_n of rank $n!$ over a unitary commutative ring $R \ni 1$ using the $n!$ words of $\mathcal{W}_n$. We replace the generators $\hat{\tau}_i$ by c_i, $1 \leq i \leq n-1$ and write $w(c_i) = w'(c_i)$ iff $\hat{w}(\hat{\tau}_i) \mathrel{\hat{=}} \hat{w}'(\hat{\tau}_i)$. Let M_n be the free R-module with basis $\mathcal{W}_n(c_i) = \{b_\pi(c_i) \mid \pi \in S_n\}$. Note that $\mathcal{W}_n(c_i) \ni c_j = b_{\tau_j}(c_i)$, $1 \leq j \leq n-1$. We introduce an associative product in M_n which transforms M_n into an R-algebra $H_n(z)$ of rank $n!$.

16.5 Definition. We put $c_k^2 = zc_k + 1$, $1 \leq k \leq n-1$ for some fixed element $z \in R$. Then (1) takes the form

$$b_\pi(c_i) \cdot c_k = \begin{cases} b_{\pi\tau_k}(c_i), & \text{case } \alpha, \\ z\, b_\pi(c_i) + b_\varrho(c_i), \quad \varrho\,\tau_k = \pi, & \text{case } \beta. \end{cases} \tag{2}$$

By iteration, (2) defines a product for the elements of the basis $\mathcal{W}_n(c_i)$ and, thus, a product on M_n by distributivity. It remains to prove associativity for the product on $\mathcal{W}_n(c_i)$.

16.6 Lemma. *The product defined in* 16.5 *is associative on* $\mathcal{W}_n(c_i)$.

Proof. Given a word $w(c_i)$ we apply the rule (2) from left to right (product algorithm) to obtain an element

$$\sum_j \gamma_j \, b_{\pi_j}(c_i) = \overline{w(c_i)} \in M_n, \quad \gamma_i \in R.$$

One has $\overline{b_\pi(c_i)} = b_\pi(c_i)$ by the Schreier property. We prove

$$(b_1 b_2)\, b_3 = b_1(b_2 b_3), \quad b_j \in \mathcal{W}_n(c_i),$$

by induction on $|b_1| + |b_2| + |b_3|$ where $|b_i|$ denotes the length of b_i. We may assume $|b_i| \geq 1$. Applying the product algorithm on the left side let case β occur for the first time for some c_k in b_2. (It cannot happen in b_1 since $\overline{b}_1 = b_1$.) We have

$$b_2 = b_2' b_2'' \quad \text{and} \quad \overline{b_1 b_2'} = \sum \gamma_j \, b_{\pi_j}(c_i), \quad |b_{\pi_j}| < |b_1| + |b_2'|. \tag{3}$$

We stop the product algorithm at this point and get:

$$(b_1 b_2)\, b_3 = \Big(\sum \gamma_j \, b_{\pi_j}(c_i) \cdot b_2''\Big) b_3.$$

On the right side we have

$$b_1\big((b_2' b_2'') b_3\big) = b_1\big(b_2'(b_2'' b_3)\big)$$

by induction. Applying the algorithm and stopping at the same c_k we obtain:

$$b_1(b_2 b_3) = \Big(\sum \gamma_j \, b_{\pi_j}(c_i)\Big)(b_2'' b_3).$$

Using the distributivity and the induction hypothesis, compare (3), we get the desired equality.

If the case β occurs for the first time in b_3 at c_k when applying the algorithm, then we have $\overline{b_1 b_2} = b_1 b_2$. Since the strings meeting in τ_k have not met in $b_1(\tau_i) \cdot b_2(\tau_i)$, they have not met in $b_2(\tau_i)$. So case β cannot have occurred when the algorithm is applied to $b_2 b_3$ at an earlier time. Now the same argument applies as in the first case. If case β does not occur at all, equality is trivial. □

The module M_n has become an R-algebra of rank $n!$, a so called *Hecke algebra*; we denote it by $H_n(z)$.

16.7 Proposition. *Let R be a commutative unitary ring, and $z \in R$. An algebra generated by elements $\{c_i \mid 1 \le i \le n-1\}$ and defined by the relations*

$$\begin{aligned} c_i c_{i+1} c_i &= c_{i+1} c_i\, c_{i+1}, && 1 \le i \le n-2, \\ c_i c_j &= c_j c_i, && 1 \le i < j-1 \le n-2, \\ c_i^2 &= z c_i + 1, && 1 \le i \le n-1 \end{aligned}$$

is isomorphic to the Hecke algebra $H_n(z)$.

The proof follows from the construction above. □

16.8 Remark. One has $(c_j - z)c_j = c_j^2 - zc_j = 1$; hence, $c_j^{-1} = c_j - z$.

We choose $R = \mathbb{Z}[z^{\pm 1}, v^{\pm 1}]$ to be the 2-variable ring of Laurent polynomials and denote by $H_n(z, v) = H_n$ the Hecke algebra with respect to $R = \mathbb{Z}[z^{\pm 1}, v^{\pm 1}]$. Next we define a representation of the braid group $\mathfrak{B}_n$:

$$\varrho_v \colon , \mathfrak{B}_n \to H_n, \quad \varrho_v(\sigma_j) = v c_j, \quad 1 \le j \le n-1,$$

see 10.3. There are natural inclusions

$$H_{n-1} \hookrightarrow H_n, \quad \mathcal{W}_{n-1}(c_i) \hookrightarrow \mathcal{W}_n(c_i),$$

and we define

$$H = \bigcup_{n=1}^{\infty} H_n, \quad \mathcal{W}(c_i) = \bigcup_{n=1}^{\infty} \mathcal{W}_n(c_i), \quad H_1 = R.$$

For the following definition we use temporarily the ring $R = \mathbb{Z}[z^{\pm 1}, v^{\pm 1}, T]$ adding a further variable T.

16.9 Definition (Trace). A function $\operatorname{tr}\colon H_n \to \mathbb{Z}[z^{\pm 1}, v^{\pm 1}, T]$ is called a *trace on H* if it satisfies the following conditions for all $n \in \mathbb{N}$.

(α) $\operatorname{tr}\Big(\sum_{\pi \in S_n} \alpha_\pi b_\pi\Big) = \sum_{\pi \in S_n} \alpha_\pi \operatorname{tr}(b_\pi)$ where $\alpha_\pi \in \mathbb{Z}[z^{\pm 1}, v^{\pm 1}]$ (linearity);

(β) $\operatorname{tr}(ba) = \operatorname{tr}(ab)$ for $a, b \in H_n$;

(γ) $\operatorname{tr}(1) = 1$;

(δ) $\operatorname{tr}(x c_{n-1}) = T \cdot \operatorname{tr}(x)$ for $x \in H_{n-1}$.

16.10 Lemma. *There is a unique trace on H.*

Proof. It suffices to show that a trace on H_n can be uniquely extended to a trace on H_{n+1}. From (β) and (δ) we get

$$\mathrm{tr}(xc_n y) = \mathrm{tr}(yxc_n) = T \cdot \mathrm{tr}(yx) = T \cdot \mathrm{tr}(xy) \quad \text{for } x, y \in H_n.$$

The basic elements of H_{n+1} which do not belong to H_n are of the form $xc_n y$ with $x, y \in H_n$; this follows from the remark in 16.1 that τ_n appears only once in b_π. So we must define the extension of the trace by

$$\mathrm{tr}(xc_n y) = T \cdot \mathrm{tr}(xy) \quad \text{for } xc_n y \in \mathcal{W}_{n+1}(c_i) \setminus \mathcal{W}_n(c_i).$$

We have to show that the linear extension of this definition to H_{n+1} is in fact a trace. Condition (α) is the linearity which is valid by definition. We first prove $\mathrm{tr}(xc_n y) = T \cdot \mathrm{tr}(xy)$ for arbitrary $x, y \in H_n$. An element $\xi \in \mathcal{W}_n$ has the form $\xi = c_j c_{j+1} \dots c_{n-1} \cdot \xi', \xi' \in \mathcal{W}_{n-1}(c_i)$. Now

$$\xi\, c_n\, y = c_j c_{j+1} \dots c_{n-1}\, \xi'\, c_n y = c_j c_{j+1} \dots c_{n-1} c_n\, \xi'\, y$$

by the braid relation $\xi' c_n = c_n \xi'$. Put $\xi' y = \sum \beta_j \eta_j,\ \eta_j \in \mathcal{W}_n$; by the linearity ($\alpha$):

$$\mathrm{tr}(\xi c_n y) = \sum \beta_j \mathrm{tr}(c_1 \dots c_n \eta_j) = \sum \beta_j \cdot T \cdot \mathrm{tr}(c_1 \dots c_{n-1} \eta_j)$$

since $c_1 \dots c_n \eta_j$ is in the basis of H_{n+1}. It follows

$$\mathrm{tr}(\xi c_n y) = T \cdot \mathrm{tr}(c_1 \dots c_{n-1} \xi' y) = T \cdot \mathrm{tr}(\xi y).$$

Since x is a linear combination of elements like ξ from above, we obtain by (α)

$$\mathrm{tr}(xc_n y) = T \cdot \mathrm{tr}(xy) \quad \text{for } x, y \in H_n.$$

This implies (δ).

It remains to prove property (β). A basis element $b_{n+1} \in \mathcal{W}_{n+1}(c_i)$ is of the form $b_{n+1} = xc_n y,\ x = c_1 \dots c_{n-1},\ y \in H_n$. For $k < n$ we have

$$\mathrm{tr}(c_k \cdot x\, c_n\, y) = T \cdot \mathrm{tr}(c_k\, x\, y) = T \cdot \mathrm{tr}(x\, y\, c_k) = \mathrm{tr}(x\, c_n\, y \cdot c_k)$$

by induction. To prove $T(b_{n+1} \cdot b'_{n+1}) = T(b'_{n+1} \cdot b_{n+1})$ – which implies (β) by (α) – we now need only to prove $T(b_{n+1} \cdot c_n) = T(c_n \cdot b_{n+1})$.

Case 1: If $b_n = x\, c_n\, y$ with $x, y \in H_{n-1}$ then $b_n c_n = c_n b_n$.

Case 2: If $x = ac_{n-1}b$ with $a, b, y \in H_{n-1}$ then

$$\begin{aligned}
\mathrm{tr}(c_n \cdot ac_{n-1}bc_n y) &= \mathrm{tr}(ac_n c_{n-1} c_n by) = \mathrm{tr}(ac_{n-1} c_n c_{n-1} by) \\
&= T \cdot \mathrm{tr}(ac_{n-1}^2 by) = T \cdot \mathrm{tr}\,(a(zc_{n-1} + 1)by) \\
&= z \cdot T \cdot \mathrm{tr}(ac_{n-1}by) + T \cdot \mathrm{tr}(aby) = (zT^2 + T)\mathrm{tr}(aby); \\
\mathrm{tr}(ac_{n-1}bc_n y \cdot c_n) &= \mathrm{tr}(ac_{n-1}bc_n^2 y) = \mathrm{tr}\,(ac_{n-1}b(zc_n + 1)y) \\
&= z\mathrm{tr}(ac_{n-1}bc_n y) + \mathrm{tr}(ac_{n-1}by) \\
&= z \cdot T \cdot \mathrm{tr}(ac_{n-1}by) + T \cdot \mathrm{tr}(aby) = (zT^2 + T)\mathrm{tr}(aby).
\end{aligned}$$

Case 3: The case $x = cc_{n-1}d$ with $x, c, d \in H_{n-1}$ can be dealt with analogously.
Case 4: Let $x = ac_{n-1}b$, $y = dc_{n-1}e$ with $a, b, d, e \in H_{n-1}$. Then

$$\begin{aligned}
\operatorname{tr}(c_n \cdot ac_{n-1}b \cdot c_n \cdot dc_{n-1}e) &= T \cdot \operatorname{tr}(ac_{n-1}^2 b \cdot dc_{n-1}e) \\
&= T \cdot z \cdot \operatorname{tr}(ac_{n-1}b \cdot dc_{n-1}e) + T^2 \cdot \operatorname{tr}(abde); \\
\operatorname{tr}(ac_{n-1}b \cdot c_n \cdot dc_{n-1}e \cdot c_n) &= T \cdot \operatorname{tr}(ac_{n-1}bdc_{n-1}^2 e) \\
&= T \cdot z \cdot \operatorname{tr}(ac_{n-1}bdc_{n-1}e) + T^2 \cdot \operatorname{tr}(abde). \qquad \square
\end{aligned}$$

We deduce from $c_n^{-1} = c_n - z$

16.11 Remark. $\operatorname{tr}(xc_n^{-1}) = \operatorname{tr}(xc_n) - z \cdot \operatorname{tr}(x) = (T - z) \cdot \operatorname{tr}(x), \ \forall x \in H_n$.

B The HOMFLY polynomial

Consider the representation

$$\varrho_v : , \mathfrak{B}_n \to H_n, \quad \varrho_v(\sigma_i) \mapsto v\, c_i,$$

where the Hecke algebra H_n is understood over $\mathbb{Z}[z^{\pm 1}, v^{\pm 1}, T]$. We put

$$P_{\mathfrak{z}_n} = k_n \cdot \operatorname{tr}(\varrho_v(\mathfrak{z}_n)), \quad \mathfrak{z}_n \in \mathfrak{B}_n,$$

for some $k_n \in \mathbb{Z}[z^{\pm 1}, v^{\pm 1}, T]$ which is still to be determined. Property (β) in Definition 16.9 of the trace implies that $P_{\mathfrak{z}_n} \in \mathbb{Z}[z^{\pm 1}, v^{\pm 1}, T]$ is invariant under conjugation of $\mathfrak{z}_n$ in $\mathfrak{B}_n$, and is, hence, a polynomial $P_{\hat{\mathfrak{z}}_n}$ assigned to the closed braid $\hat{\mathfrak{z}}_n$. To turn $P_{\mathfrak{z}_n} \in \mathbb{Z}[z^{\pm 1}, v^{\pm 1}, T]$ to an invariant of the link represented by $\hat{\mathfrak{z}}_n$, we have to check the effect of a Markov move $\mathfrak{z}_n \mapsto \mathfrak{z}_n \sigma_n^{\pm 1}$ on $\hat{\mathfrak{z}}_n$, see 10.21, 10.22. We postulate:

$$k_n \cdot \operatorname{tr}(\varrho_v(\mathfrak{z}_n)) = k_{n+1} \operatorname{tr}(\varrho_v(\mathfrak{z}_n \sigma_n)).$$

It follows $k_n = k_{n+1} \cdot v \cdot T$ since

$$\operatorname{tr}(\varrho_v(\mathfrak{z}_n \sigma_n)) = v \cdot \operatorname{tr}(\varrho_v(\mathfrak{z}_n \cdot c_n)) = v \cdot T \cdot \operatorname{tr}(\varrho_v(\mathfrak{z}_n)).$$

Another condition follows in the second case:

$$k_{n+1} \operatorname{tr}(\varrho_v(\mathfrak{z}_n \sigma_n^{-1})) = k_{n+1} v^{-1} \operatorname{tr}(\varrho_v(\mathfrak{z}_n) c_n^{-1}) = k_{n+1} v^{-1} (T - z) \cdot \operatorname{tr}(\varrho_v(z_n))$$

(see Remark 16.7); hence $k_n = k_{n+1} \cdot v^{-1}(T - z)$. We solve $v^{-1}(T - v) = vT$ in the quotient field of $\mathbb{Z}[z^{\pm 1}, v^{\pm 1}, T]$ by $T = \frac{zv^{-1}}{v^{-1} - v}$ and define inductively

$$k_{n+1} = k_n \cdot \frac{1}{v \cdot T} = k_n \cdot z^{-1}(v^{-1} - v), \ k_1 = 1 \quad \Longrightarrow \quad k_n = \frac{(v^{-1} - v)^{n-1}}{z^{n-1}}.$$

16.12 Remark. The extension $H_n \subset H_{n+1}$ introduces the factor $T = \frac{zv^{-1}}{v^{-1}-v}$, but the denominator $v^{-1} - v$ is eliminated by the factor $k_{n+1}k_n^{-1} = z^{-1}(v^{-1} - v)$ such that $P_{\mathfrak{z}_n}(z, v)$ is indeed a Laurent polynomial in z and v.

From the above considerations we obtain the first part of the following

16.13 Theorem and Definition. *The Laurent polynomial*

$$P_{\mathfrak{z}_n}(z, v) = \frac{(v^{-1} - v)^{n-1}}{z^{n-1}} \cdot \operatorname{tr}(\varrho(z_n))$$

associated to a braid $\mathfrak{z}_n \in \mathfrak{Z}_n$ is an invariant of the oriented link $\mathfrak{l}$ represented by the closed braid $\hat{\mathfrak{z}}_n$.

$P_{\mathfrak{l}}(z, v) = P_{\mathfrak{z}_n}(z, v)$ is called the 2*-variable skein polynomial or* HOMFLY *polynomial of the oriented link $\mathfrak{l}$.*

The trivial braid with n strings represents the trivial link with n strings; its polynomial is $\frac{(v^{-1}-v)^{n-1}}{z^{n-1}}$.

To prove the last statement observe that $\varrho(\mathfrak{z}_n) = 1$ for the trivial braid $\mathfrak{z}_n$, the empty word in σ_i. As a special case we have $P_{\mathfrak{z}_n}(z, v) = 1$ for $\hat{\mathfrak{z}}_n$ the trivial knot. □

16.14 Definition. For an oriented link $\mathfrak{k}$ the smallest number n for which $\mathfrak{k}$ is isotopic to some $\hat{\mathfrak{z}}_n$ is called the *braid index* $\beta(\mathfrak{k}) = n$ of $\mathfrak{k}$.

The following proposition gives a lower bound for the braid index $\beta(\mathfrak{k})$ in terms of the HOMFLY polynomial $P_{\mathfrak{k}}(z, v)$ of $\mathfrak{k}$. Write

$$\begin{aligned} P_{\mathfrak{k}}(z, v) = a_m(z)v^m + \cdots + a_n(z)v^n, \quad & a_j(z) \in \mathbb{Z}[z, z^{-1}], \\ & m \le n,\ m, n \in \mathbb{Z},\ \text{and } a_m \neq 0 \neq a_n. \end{aligned}$$

By $\operatorname{Sp}_v(P_{\mathfrak{k}}(z, v)) = n - m$ we denote the "v-span" of $P_{\mathfrak{k}}(z, v)$.

16.15 Proposition.

$$\beta(\mathfrak{k}) \ge 1 + \frac{1}{2}\operatorname{Sp}_v\left(P_{\mathfrak{k}}(z, v)\right).$$

Proof. Suppose that $\mathfrak{k}$ is isotopic to $\hat{\mathfrak{z}}_n$. From Definition 16.9 (δ) it follows by induction that the trace of an element of H_n is a polynomial in T of degree at most $n - 1$. Hence, for $\mathfrak{z}_n = \prod_{j=0}^{n-1} \sigma_{i_j}^{\varepsilon_{i_j}}$ we obtain

$$\begin{aligned}
&\varrho_v(\mathfrak{z}_n) = v^k \cdot \sum_{j=0}^{n-1} c_{i_j}^{\varepsilon_j} \quad \text{with } k = \sum \varepsilon_j \\
\Longrightarrow \quad & \operatorname{tr}(\varrho_v(\mathfrak{z}_n)) = v^k \cdot \sum_{j=0}^{n-1} a_j(z) T^j \quad \text{where } T^j = \frac{(v^{-1}-v)^{j-1}}{z^{j-1}} \\
\Longrightarrow \quad & P_{\mathfrak{z}_n}(z,v) = \frac{(v^{-1}-v)^{n-1}}{z^{n-1}} \cdot \operatorname{tr}(\varrho_v(\mathfrak{z}_n)) \\
& = v^k \cdot \sum_{j=0}^{n-1} a_j(z) \cdot z^{-n+j-1}(v^{-1}-v)^{n-2j-1}, \\
& \operatorname{Sp}_v(P_{\mathfrak{z}_n}) \le 2(n-1).
\end{aligned}$$

□

16.16 Example. $6_1,\ 7_2,\ 7_4$ have braid index 4.

Let $\mathfrak{k}_+$ be a diagram of an oriented link. We focus on a crossing and denote by $\mathfrak{k}_-$ resp. $\mathfrak{k}_0$ the projections which are altered in the way depicted in Figure 16.5, but are unchanged otherwise.

16.17 Proposition. *Let $\mathfrak{k}_+, \mathfrak{k}_-, \mathfrak{k}_0$ be link projections related as in Figure* 16.5. *Then there is the skein relation*

$$v^{-1} P_{\mathfrak{k}_+} - v P_{\mathfrak{k}_-} = z P_{\mathfrak{k}_0}.$$

There exists an algorithm to calculate $P_{\mathfrak{k}}$ for an arbitrary link $\mathfrak{k}$ given by a projection.

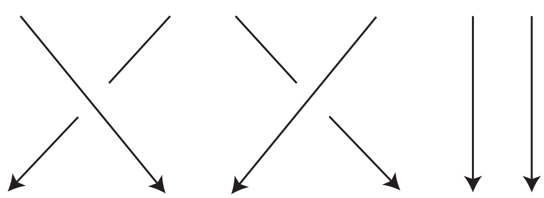


Figure 16.5

Proof. The braiding process which turns an arbitrary link projection into that of a closed braid as described in 2.14 can be executed in such a way that a neighbourhood of any chosen crossing point of the projection is kept fixed. Furthermore, the representing braid $\mathfrak{z}_n$ can suitably be chosen such that $\mathfrak{k}_+ = \mathfrak{z}_n\sigma_i$, $\mathfrak{k}_- = \mathfrak{z}_n\sigma_i^{-1}$ and $\mathfrak{k}_0 = \mathfrak{z}_n$. Now,

$$\begin{aligned}
v^{-1} P_{\mathfrak{k}_+} - v P_{\mathfrak{k}_-} &= v^{-1} k_n \operatorname{tr}\big(\varrho_v(\mathfrak{z}_n\sigma_i)\big) - v k_n \operatorname{tr}\big(\varrho_v(\mathfrak{z}_n\sigma_i^{-1})\big) \\
&= k_n z \operatorname{tr}\big(\varrho_v(\mathfrak{z}_n)\big) = z P_{\mathfrak{k}_0}
\end{aligned}$$

since

$$v^{-1}\varrho_v(\mathfrak{z}_n\sigma_i) - v\varrho_v(\mathfrak{z}_n\sigma_i^{-1}) = \varrho_v(\mathfrak{z}_n)(v^{-1}\varrho_v(\sigma_i) - v\varrho_v(\sigma_i^{-1}))$$

and

$$\varrho_v(\mathfrak{z}_n)(c_i - c_i^{-1}) = \varrho_v(\mathfrak{z}_n) \cdot z. \qquad \square$$

16.18 Remark. The skein relation permits to calculate each of the polynomials $P_{\mathfrak{k}_+}$, $P_{\mathfrak{k}_-}$, $P_{\mathfrak{k}_0}$ from the remaining two. By changing overcrossings into undercrossings or vice versa any link projection can be turned into the projection of an unlink. This implies that the skein relation supplies an algorithm for the computation of $P_{\mathfrak{k}}$. The process is illustrate in Figure 16.6: each vertex of the "skein-tree" (Figure 16.6 (b)) represents a projection; the root at the top represents the projection of $\mathfrak{k}$, the terminal points represent unlinks. Starting with the polynomials of these one can work one's way upwards to compute $P_{\mathfrak{k}}$. The procedure is of exponential time complexity.

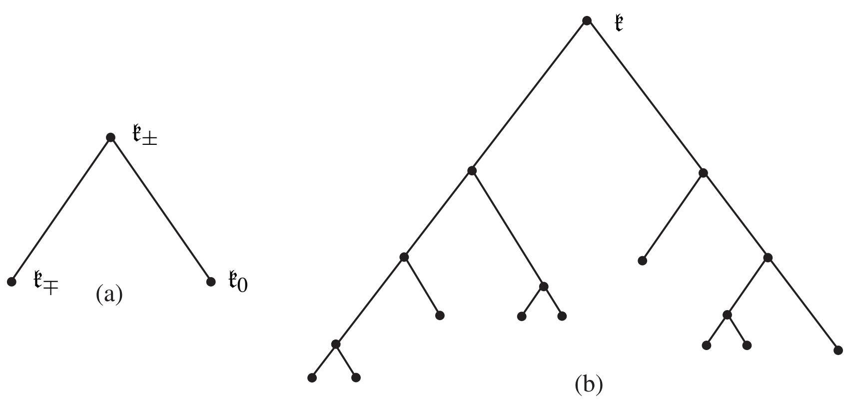


Figure 16.6

16.19 Proposition. *Let $-\mathfrak{k}$ resp. $\mathfrak{k}^*$ denote the inverted resp. mirrored knot, and # resp. $\sqcup$ the product resp. the disjoint union. Then:*

(a) $P_{\mathfrak{k}}(z, v) = P_{-\mathfrak{k}}(z, v)$;

(b) $P_{\mathfrak{k}}(z, v) = P_{\mathfrak{k}^*}(z, -v^{-1})$

(c) $P_{\mathfrak{k}_1 \# \mathfrak{k}_2}(z, v) = P_{\mathfrak{k}_1}(z, v) \cdot P_{\mathfrak{k}_2}(z, v)$;

(d) $P_{\mathfrak{k}_1 \sqcup \mathfrak{k}_2}(z, v) = z^{-1}(v^{-1} - v)P_{\mathfrak{k}_1}(z, v) \cdot P_{\mathfrak{k}_2}(z, v)$.

Proof. (a) Changing $\mathfrak{k}$ into $-\mathfrak{k}$ allows to use the same skein-tree.

(b) If $\mathfrak{k}$ is replaced by $\mathfrak{k}^*$, we can still use the same skein-tree, and at each vertex the associated projection is also replaced by its mirror image. The skein relation

$$v^{-1} P_{\mathfrak{k}_+}(z, v) - v\, P_{\mathfrak{k}_-}(z, v) = z\, P_{\mathfrak{k}_0}(z, v)$$

remains valid if v is changed into $-v^{-1}$:

$$-v\, P_{\mathfrak{k}_+}(z, -v^{-1}) + v^{-1} P_{\mathfrak{k}_-}(z, -v^{-1}) = z\, P_{\mathfrak{k}_0}(z, -v^{-1}),$$

but

$$P_{\mathfrak{k}_+}(z, -v^{-1}) = P_{\mathfrak{k}_-^*}(z, v), \;\; P_{\mathfrak{k}_-}(z, -v^{-1}) = P_{\mathfrak{k}_+^*}(z, v), \;\; P_{\mathfrak{k}_0}(z, -v^{-1}) = P_{\mathfrak{k}_0^*}(z, v),$$

and $z^{-(n-1)} \cdot (v^{-1} - v)^{n-1}$ is invariant under the substitution $v \mapsto -v^{-1}$.

The formulae (c) and (d) for the product knot and a split union easily follow by similar arguments. □

16.20 Example. We calculate the HOMFLY polynomials of the trefoil and its mirror image; using this invariant they are shown to be different, a result first obtained in [Dehn 1914]. Let us call $\mathfrak{b}(3, 1) = 3_1^+$ (see Figure 12.6) the *right-handed* trefoil and $\mathfrak{b}(3, -1) = 3_1^-$ the *left-handed* one. Figure 16.7 describes the skein tree starting with

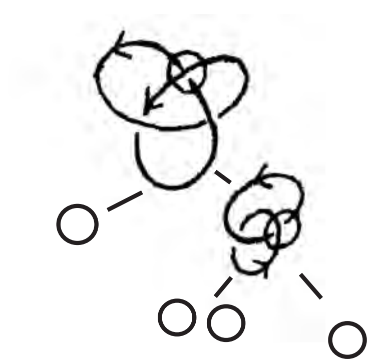


Figure 16.7

$\mathfrak{k} = \mathfrak{k}_+ = 3_1^+$. The crossing where the skein relation is applied is distinguished by a circle. One has:

$$v^{-1} P_{\mathfrak{k}_+} - v\, P_{\mathfrak{k}_-} = v^{-1} P_{\mathfrak{k}} - v = z\, P_{\mathfrak{k}_0}$$

and

$$v^{-1} P_{\mathfrak{k}_{0+}} - v\, P_{\mathfrak{k}_{0-}} = v^{-1} P_{\mathfrak{k}_0} - v = z\, P_{\mathfrak{k}_{00}};$$

hence

$$P_{\mathfrak{k}_0} = v^2 P_{\mathfrak{k}_{\circ\circ}} + vz.$$

Using $P_{\mathfrak{k}_{\circ\circ}} = z^{-1}(v^{-1} - v)$ from 16.15 the first equation gives:

$$P_{3_1^+}(z, v) = -v^4 + 2v^2 + z^2v^2.$$

By Proposition 16.19 we have

$$P_{3_1^-}(z, v) = -v^{-4} + 2v^{-2} + z^2v^{-2},$$

and, hence, $3_1^+ \neq 3_1^-$. (For an exercise do a calculation of $P_{3_1^-}(z, v)$ using a skein-tree.)

We give a second computation of $P_{3_1^+}(z, v)$ using the definition in 16.13:

$$P_{3_1^+}(z, v) = z^{-1}(v^{-1} - v)\,\mathrm{tr}\big(\varrho(\sigma_1^3)\big).$$

Here $n = 2$, and $3_1^+ = b(3, 1) = \hat{\sigma}_1^3$, see Figure 12.6. We have $\varrho(\sigma_1^3) = v^3 \cdot c_1^3$. Applying $c_1^2 = vzc_1 + 1$ twice we get $c_1^3 = (z^2 + 1)c_1 + z$. By 16.9 (α)

$$P_{3_1^+}(z, v) = z^{-1}(v^{-1} - v) \cdot v^3\big((z^2 + 1)\mathrm{tr}(c_1) + z\big) = v^2z^2 + 2v^2 - v^4,$$

using (δ): $\mathrm{tr}(c_1) = T = zv^{-1}(v^{-1} - v)$.

The HOMFLY polynomial $P(z, v)$ contains as special cases the Alexander–Conway polynomial and the Jones polynomial.

16.21 Theorem.

$$P\big(t^{\frac{1}{2}} - t^{-\frac{1}{2}}, 1\big) = \Delta(t) = \textit{Alexander–Conway polynomial}$$
$$P\big(t^{\frac{1}{2}} - t^{-\frac{1}{2}}, t\big) = \nabla(t) = \textit{Jones polynomial}.$$

Proof. In the first case we obtain the skein relation of the Alexander–Conway polynomial,

$$\Delta_{\mathfrak{k}_+}(t) - \Delta_{\mathfrak{k}_-}(t) = \Delta_{\mathfrak{k}_0}(t)$$

for $P\big((t^{\frac{1}{2}} - t^{-\frac{1}{2}}), 1\big)$, and since both sides are equal to 1 for the trivial knot, equality must hold. In the second case we obtain the skein relation

$$t^{-1}\nabla_{\mathfrak{k}_+}\big(t^{\frac{1}{2}}\big) + t^{-1}\nabla_{\mathfrak{k}_-}\big(t^{\frac{1}{2}}\big) = \big(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\big)\nabla_{\mathfrak{k}_0}\big(t^{\frac{1}{2}}\big)$$

of the Jones polynomial $\nabla(t)$.

16.22 Remark. For a one component link (knot), $\nabla\big(t^{\frac{1}{2}}\big)$ is in fact a polynomial in t.

C History and Sources

The discovery of a new knot polynomial by V.F.R. Jones in 1985 ([Jones 1985, 1987]) which can distinguish mirror images of knots had the makings of a sensation. The immediate success in proving long-standing conjectures of Tait as an application added to its fame. In the following many authors (*H*oste, *O*cneano, *M*illet, *F*loyd, *L*ickorish, *Y*etter, and Conway, Kauffman, Prytycki, Traczyk etc.) combined to study new and old (Alexander-) polynomials under the view of skein theory; as a result the 2-variable skein polynomial (HOMFLY) was established which comprises both, old and new knot polynomials.

D Exercises

E 16.1. Prove Proposition 16.19 using the Definition 16.13 of $P(z, v)$.

E 16.2. Compute the HOMFLY polynomial for the Borromean link, see Example 9.19 (b) and Figure 9.2.

E 16.3. Prove that $6_1,\ 7_2,\ 7_4,\ 7_6,\ 7_7$ are the only knots with less than eight crossings whose braid index exceeds 3.

Appendix A
Algebraic Theorems

A.1 Theorem. *Let Q be a $n \times n$ skew symmetric matrix ($Q = -Q^T$) over the integers $\mathbb{Z}$. Then there is an integral unimodular matrix L such that*

$$L^T Q L = \begin{pmatrix} 0 & a_1 & & & & & & & & \\ -a_1 & 0 & & & & & & & & \\ & & 0 & a_2 & & & & & & \\ & & -a_2 & 0 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & 0 & a_s & & & \\ & & & & & -a_s & 0 & & & \\ & & & & & & & 0 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 0 \end{pmatrix}$$

with $a_1|a_2|\dots|a_s$.

Proof. Let $\mathfrak{M}$ denote the module of $2n$-columns with integral coefficients: $\mathfrak{M} \cong \mathbb{Z}^{2n}$. Every $\mathfrak{x}_1 \in \mathfrak{M}$ defines a principal ideal

$$\{\mathfrak{x}_1^T Q \mathfrak{y} \mid \mathfrak{y} \in \mathfrak{M}\} = (a_1) \subset \mathbb{Z}.$$

We may choose $a_1 > 0$ if $Q \neq 0$. So there is a vector $\mathfrak{y}_1 \in \mathfrak{M}$ such that $\mathfrak{x}_1^T Q \mathfrak{y}_1 = a_1$; hence, $\eta_1^T Q \mathfrak{x}_1 = -a_1$. If follows that a_1 also generates the ideal defined by $\mathfrak{y}_1$. Let $\mathfrak{x}_1$ be chosen in such a way that $a_1 > 0$ is minimal.

Put

$$\mathfrak{M}_1 = \{\mathfrak{u} \mid \mathfrak{x}_1^T Q u = \mathfrak{y}_1^T Q \mathfrak{u} = 0\}.$$

We prove that

$$\mathfrak{M} = \mathbb{Z}\mathfrak{x}_1 \oplus \mathbb{Z}\mathfrak{y}_1 \oplus \mathfrak{M}_1;$$

in particular, $\mathfrak{M}_1 \cong \mathbb{Z}^{2n-2}$.

Consider $\mathfrak{z} \in \mathfrak{M}$ and define $\alpha, \beta \in \mathbb{Z}$ by

$$\mathfrak{x}_1^T Q \mathfrak{z} = \beta a_1, \ \mathfrak{y}_1^T Q \mathfrak{z} = \alpha a_1.$$

Then

$$\begin{aligned} \mathfrak{x}_1^T Q(\mathfrak{z} - \beta \mathfrak{y}_1 - \alpha \mathfrak{x}_1) &= \beta a_1 - \beta a_1 - 0 = 0 \\ \mathfrak{y}_1^T Q(\mathfrak{z} - \beta \mathfrak{y}_1 - \alpha \mathfrak{x}_1) &= \alpha a_1 - 0 - \alpha a_1 = 0; \end{aligned}$$

note that $Q^T = -Q$ implies that $\mathfrak{x}^T Q \mathfrak{x} = 0$. Now $\mathfrak{z} - \beta\mathfrak{y}_1 - \alpha\mathfrak{x}_1 \in \mathfrak{M}_1$ and $\mathfrak{x}_1$ and $\mathfrak{y}_1$ generate a module isomorphic to $\mathbb{Z}^2$. From

$$\mathfrak{x}_1^T Q(\xi\mathfrak{x}_1 + \eta\mathfrak{y}_1) = \eta a_1, \quad \mathfrak{y}_1^T Q(\xi\mathfrak{x}_1 + \eta\mathfrak{y}_1) = -\xi a_1$$

it follows that $\xi\mathfrak{x}_1 + \eta\mathfrak{y}_1 \in \mathfrak{M}_1$ implies that $\xi = \eta = 0$. Thus $\mathfrak{M} = \mathbb{Z}\mathfrak{x}_1 \oplus \mathbb{Z}\mathfrak{y}_1 \oplus \mathfrak{M}_1$.

The skew-symmetric form Q induces on $\mathfrak{M}_1$ a skew-symmetric form Q'. As an induction hypothesis we may assume that there is a basis $\mathfrak{x}_2, \mathfrak{y}_2, \dots, \mathfrak{x}_n, \mathfrak{y}_n$ of $\mathfrak{M}_1$ such that Q' is represented by a matrix as desired.

To prove $1 \leqq a_1|a_2|\dots|a_s$, we may assume by induction $1 \leqq a_2|a_3|\dots|a_s$ already to be true. If $1 \leqq d = \gcd(a_1, a_2)$ and $d = ba_1 + ca_2$ then

$$(b\mathfrak{x}_1 + c\mathfrak{x}_2)^T Q(\mathfrak{y}_1 + \mathfrak{y}_2) = ba_1 + ca_2 = d.$$

Hence, by the minimality of a_1: $d = a_1$. □

A.2 Theorem ([Jones 1950]). *Let $Q_n = (q_{ik})$ be a symmetric $n \times n$ matrix over $\mathbb{R}$, and $p(Q_n)$ the number of its positive, $q(Q_n)$ the number of its negative eigenvalues, then $\sigma(Q_n) = p(Q_n) - q(Q_n)$ is called the signature of Q_n. There is a sequence of principal minors $D_0 = 1, D_1, D_2, \dots$ such that D_i is a principal minor of D_{i+1} and no two consecutive D_i, D_{i+1} are both singular for $i <$ rank Q_n. For any such (admissible) sequence*

$$\sigma(Q_n) = \sum_{i=0}^{n-1} \operatorname{sign}(D_i D_{i+1}). \tag{1}$$

Proof. The rank r of Q_n is the number of non-vanishing eigenvalues λ_i of Q_n; it is, at the same time, the maximal index i for which a non-singular principal minor exists – this follows from the fact that Q_n is equivalent to a diagonal matrix containing the eigenvalues λ_i in its diagonal. We may, therefore, assume $r = n$ and $D_i = \lambda_1 \dots \lambda_i$, $D_n \neq 0$.

The proof is by induction on n. Assume first that we have chosen a sequence $D_0, D_1, \dots$ with a non-singular minor D_{n-1}. (It will be admissible by induction.) We may suppose that $D_{n-1} = \det Q_{n-1}$ where Q_{n-1} is the submatrix of Q_n consisting of its first $n-1$ rows and columns. Now sign $(D_{n-1}D_n) = \operatorname{sign} \lambda_n$, and (1) follows by induction.

Suppose we choose a sequence with $D_{n-1} = 0$. Then $D_{n-2} \neq 0$, and, since we have $D_n \neq 0$, we obtain an admissible sequence for Q_n. There is a transformation $B_n^T Q_n B_n = Q_n'$ with

$$B_n = \left(\begin{array}{ccc|c} & & & 0 \\ & B_{n-1} & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right), \quad B_{n-1} \in \mathrm{SO}(n-1, \mathbb{R})$$

which takes Q_{n-1} into diagonal form

$$Q'_{n-1} = \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_k & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}, \quad \lambda_i \neq 0.$$

By a further transformation

$$C_n^T Q'_n C_n = Q''_n, \quad C_n = \left(\begin{array}{c|c|c} E_k & & \begin{matrix} t_1 \\ \vdots \\ t_k \end{matrix} \\ \hline & B_{n-k-1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline & & 1 \end{array}\right),$$

$$t_i \in \mathbb{R},\ B_{n-k-1} \in SO(n-k-1, \mathbb{R})$$

one can achieve the following form

$$Q''_n = \left(\begin{array}{cc|cc|c} \lambda_1 & & & & 0 \\ & \lambda_k & & & 0 \\ \hline & & 0 & & 0 \\ & & & 0 & \alpha \\ \hline 0 \cdots & 0 & 0 \cdots & \alpha & \beta \end{array}\right).$$

Since $D_n \neq 0$ it follows that $k = n-2$. Thus there exists an admissible sequence, and we can use the induction hypothesis for $n-2$. Now

$$\sigma \begin{pmatrix} 0 & \alpha \\ \alpha & \beta \end{pmatrix} = 0, \quad \text{and} \quad \sigma(Q_n) = \sigma \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{n-2} \end{pmatrix}.$$

The same result is obtained by (1) if $D_{n-1} = 0$. $\square$

Let Γ be a finite oriented graph with vertices $\{P_i \mid 1 \leqq i \leqq n\}$ and oriented edges $\{u_{ij}^{\lambda}\}$, such that P_i is the initial point and P_j the terminal point of u_{ij}^{λ}. (For the basic terminology see [Berge 1970]). By a *rooted tree* (root P_1) we mean a subgraph of $n-1$ edges such that every point P_k is terminal point of a path with initial point P_1.

Let a_{ij} denote the number of edges with initial point P_i and terminal point P_j.

A.3 Theorem (Bott–Mayberry). *Let Γ be a finite oriented graph without loops $(a_{ii}=0)$. The principal minor H_{ii} of the graph matrix*

$$H(\Gamma) = \begin{pmatrix} (\sum\limits_{k\neq 1} a_{k1}) & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & (\sum\limits_{k\neq 2} a_{k2}) & -a_{23} & \cdots & -a_{2n} \\ \vdots & \vdots & & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & & (\sum\limits_{k\neq n} a_{kn}) \end{pmatrix}$$

is equal to the number of rooted trees with root P_i.

Proof. The principal $(n-1)\times(n-1)$-minor H_{ii} is the determinant of the submatrix obtained from $H(\Gamma)$ by omitting the i-th row and column. We need a

Lemma. *A graph C (without loops) with n vertices and $n-1$ edges is a rooted tree, root P_i, if $H_{ii}(C)=1$; otherwise $H_{ii}(C)=0$.*

Proof of the lemma. Suppose C is a rooted tree with root P_1. One has $\sum_{k\neq j} a_{kj} = 1$ for $j \neq 1$, because there is just one edge in C with terminal point P_j. If the indexing of vertices is chosen in such a way that indices increase along any path in C, then H_{11} has the form

$$H_{11} = \begin{vmatrix} 1 & * & & * \\ 0 & 1 & & \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & & 1 \end{vmatrix} = 1$$

To prove the converse it suffices to show that C is connected, if $H_{11}\neq 0$. Assuming this, use the fact that every point P_j, $j\neq 1$, must be a terminal point of C, otherwise the j-th column would consist of zeroes, contradicting $H_{11}\neq 0$. There is, therefore, an unoriented spanning tree in the (unoriented) graph C. The graph C coincides with this tree, since a spanning tree has $n-1$ edges. It must be a tree, rooted in P_1, because every vertex P_j, $j \neq 1$, is a terminal point.

The rest is proved by induction on n. We assume that C is not connected. Then we may arrange the indexing such that H_{11} is of the form:

$$H_{11} = \begin{pmatrix} B' & 0 \\ 0 & B'' \end{pmatrix}, \quad \det B' \neq 0,\ \det B'' \neq 0.$$

By the induction hypothesis we know that the subgraphs Γ' resp. Γ'' each containing P_1 and the vertices associated with the rows of B' resp. B'' – together with all edges of C joining these points – are P_1-rooted trees. This contradicts the assumption that C is not connected. □

We return to the proof of the main theorem. One may consider H_{11} as a multilinear function in the $n-1$ column vectors $\mathfrak{a}_j$, $j = 2, \ldots, n$ of the matrix (a_{ij}), $i \neq j$. This is true, since the diagonal elements $\sum_{k \neq j} a_{kj}$ are themselves linear functions. Let $\mathfrak{e}_i$ denote a column vector with an i-th coordinate equal to one, and the other coordinates equal to zero. Then

$$H_{11}(\mathfrak{a}_2, \ldots, \mathfrak{a}_n) = \sum_{1 \leqq k_2, \ldots, k_n \leqq n} a_{k_2 2} \ldots a_{k_n n}\, H_{11}(\mathfrak{e}_{k_2}, \ldots, \mathfrak{e}_{k_n}) \tag{1}$$

with

$$\mathfrak{a}_i = \sum_{k_i=1}^{n} a_{k_i i}\, \mathfrak{e}_{k_i}.$$

By the lemma $H_{11}(\mathfrak{e}_{k_2}, \ldots, \mathfrak{e}_{k_n}) = 1$ if and only if the $n-1$ edges $u_{k_2 2}, u_{k_3 3}, \ldots, u_{k_n n}$ form a P_1-rooted tree. Any such tree is to be counted $a_{k_2 2} \ldots a_{k_n n}$ times. □

Two corollaries follow easily.

A.4 Corollary. *Let Γ be an unoriented finite graph without loops, and let b_{ij} the number of edges joining P_i and P_j. A principal minor H_{ii} of*

$$\begin{pmatrix} \sum_{k\neq 1} b_{k1} & -b_{12} & -b_{13} & \cdots \\ -b_{21} & \sum_{k\neq 2} b_{k2} & \cdots & \\ \vdots & & & \vdots \\ \vdots & & & \sum_{k\neq n} b_{kn} \end{pmatrix}$$

gives the number of spanning trees of Γ, independent of i.

Proof. Replace every unoriented edge of Γ by a pair of edges with opposite directions, and apply Theorem A.3. $\square$

A.5 Corollary. *Let* Γ *be a finite oriented loopless graph with a valuation* $f\colon \{u_{ij}^{\lambda}\} \to \{1, -1\}$ *on edges. Then the principal minor* H_{ii} *of* $(f(a_{ij}))$, $f(a_{ij}) = \sum_{\lambda} f(u_{ij}^{\lambda})$, *satisfies the following equation:*

$$H_{ii} = \sum f(\mathrm{Tr}(i))$$

where the sum is to be taken over all P_i*-rooted trees* $\mathrm{Tr}(i)$, *and where*

$$f(\mathrm{Tr}(i)) = \prod_{u_{kj}^{\lambda} \in \mathrm{Tr}(i)} f(u_{kj}^{\lambda}).$$

Proof. The proof of Theorem A.3 applies; it is only necessary to replace a_{ij} by $f(a_{ij})$. $\square$

For other proofs and generalizations see [Bott-Mayberry 1954]. We add a well-known theorem without giving a proof. For a proof see [Bourbaki, Algèbre Chap. 7].

A.6 Theorem. *Let* M *be a finitely generated module over a principal ideal domain* R. *Then*

$$M \cong M_{\varepsilon_1} \oplus \cdots \oplus M_{\varepsilon_r} \oplus M_{\beta}$$

where M_{β} *is a free module of rank* β *and* $M_{\varepsilon_i} = \langle\, a \mid \varepsilon_i a \,\rangle$ *is a cyclic module generated by an element* a *and defined by* $\varepsilon_i a = 0$, $\varepsilon_i \in R$. *The* ε_i *are not units of* R, *different from zero, and form a chain of divisors* $\varepsilon_i \mid \varepsilon_{i+1}$, $1 \leqq i \leqq r$. *They are called the elementary divisors of* M*; the rank* β *of the free part of* M *is called the Betti number of* M.

The Betti numbers β *and* β' *of finitely generated modules* M *and* M' *coincide and their elementary divisors are pairwise associated,* $\varepsilon_i' = \alpha_i \varepsilon_i$, α_i *a unit of* R, *if and only if* M *and* M' *are isomorphic.* $\square$

Remark. If M is a finitely presented module over an abelian ring A with unit element, the theorem is not true. Nevertheless the elementary ideals of its presentation matrix are invariants of M.

In the special case $R = \mathbb{Z}$ the theorem applies to finitely generated abelian groups. The elementary divisors form a chain $T_1|T_2|\ldots|T_r$ of positive integers > 1, the orders of the cyclic summands. T_r is called the *first*, T_{r-1} the *second torsion number*, etc. of the abelian group.

Appendix B

Theorems of 3-dimensional Topology

This section contains a collection of theorems in the field of 3-dimensional manifolds which have been frequently used in this book. In each case a source is given where a proof may be found.

B.1 Theorem (Alexander). *Let S^2 be a semilinearly embedded 2-sphere in S^3. There is a semilinear homeomorphism $h\colon S^3 \to S^3$ mapping S^2 onto the boundary $\partial[\sigma^3]$ of a* 3-*simplex σ^3.* □

[Alexander 1924′], [Graeub 1950].

B.2 Theorem (Alexander). *Let T be a semilinearly embedded torus in S^3. Then $S^3 - T$ consists of two components X_1 and X_2, $\bar{X}_1 \cup \bar{X}_2 = S^3$, $\bar{X}_1 \cap \bar{X}_2 = T$, and at least one of the subcomplexes $\bar{X}_1$, $\bar{X}_2$ is a torus.* □

[Alexander 1924′], [Schubert 1953].

B.3 Theorem (Seifert–van Kampen). (a) *Let X be a connected polyhedron and X_1, X_2 connected subpolyhedra with $X = X_1 \cup X_2$ and $X_1 \cap X_2$ a (non-empty) connected subpolyhedron. Suppose*

$$\begin{aligned} \pi_1(X_1, P) &= \langle S_1, \dots, S_n \mid R_1, \dots, R_m \rangle, \\ \pi_1(X_2, P) &= \langle T_1, \dots, T_k \mid N_1, \dots, N_\ell \rangle \end{aligned}$$

with respect to a base point $P \in X_1 \cap X_2$. A set $\{v_j \mid 1 \leqq j \leqq r\}$ of generating loops of $\pi_1(X_2 \cap X_2, P)$ determines sets $\{V_{1j}(S_i)\}$ and $\{V_{2j}(T_i)\}$ respectively of elements in $\pi_1(X_1, P)$ or $\pi_1(X_2, P)$ respectively. Then

$$\pi_1(X, P) = \langle S_1, \dots, S_n, T_1, \dots, T_k \mid R_1, \dots, R_m, N_1, \dots, N_\ell, V_{11}V_{21}^{-1}, \dots, V_{1r}V_{2r}^{-1} \rangle.$$

(b) *Let X_1, X_2 be disjoint connected homeomorphic subpolyhedra of a connected polyhedron X, and denote by $\bar{X} = X/h$ the polyhedron which results from identifying X_1 and X_2 via the homeomorphism $h\colon X_1 \to X_2$. For a base point $P \in X_1$ and its image $\bar{P}$ under the identification a presentation of $\pi_1(\bar{X}; \bar{P})$ is obtained from one of $\pi_1(X; P)$ by adding a generator S and the defining relations $ST_iS^{-1} = h_\#(T_i)$, $1 \leqq i \leqq r$ where $\{T_i \mid 1 \leqq i \leqq r\}$ generate $\pi_1(X_1; P)$.* □

For a proof see [ZVC 1980, 2.8.2]. A topological version of B.3 (a) is valid when X, X_1,, X_2, $X_1 \cap X_2$, are path-connected and X_1, X_2 are open, [Crowell-Fox 1963], [Massay 1967], [Stöcker-Zieschang 1988, 5.3.11]. A topological version of B.3 (b) may be obtained if X. X_1, X_2 are path-connected, X_1, X_2 are closed, and if the identifying homeomorphism can be extended to a collaring.

B.4 Theorem (Generalized Dehn's lemma). *Let $h\colon S(0,r) \to M$ be a simplicial immersion of an orientable compact surface $S(0,r)$ of genus* 0 *with r boundary components into the* 3*-manifold M with no singularities on the boundary $\partial h(S(0,r)) = \{C_1, C_2, \dots, C_r\}$, C_i a closed curve. Suppose that the normal closure $\langle \overline{C_1, \dots, C_r} \rangle$ in $\pi_1(M)$ is contained in the subgroup $\hat{\pi}_1(M) \subset \pi_1(M)$ of orientation preserving paths. Then there is a non-singular disk $S(0,q)$ embedded in M with $\partial S(0,q)$ a non-vacuous subset of $\{C_1, \dots, C_r\}$.* □

[Shapiro-Whitehead 1958], [Hempel 1976], [Rolfsen 1976], [Jaco 1977].

Remark. Theorem B.4 was proved by Shapiro and Whitehead. The original lemma of Dehn with $r = 1$ $(= q)$ was formulated by M. Dehn in 1910 but proved only in 1957 by Papakyriakopoulos [1957′].

B.5 Theorem (Generalized loop theorem). *Let M be a* 3*-manifold and let B be a component of its boundary. If there are elements in* $\ker(\pi_1 B \to \pi_1 M)$ *which are not contained in a given normal subgroup $\mathfrak{N}$ of $\pi_1(B)$ then there is a simple loop C on B such that C bounds a non-singular disk in M and* $[C] \notin \mathfrak{N}$. □

[Papakyriakopoulos 1957], [Stallings 1959], [Rolfsen 1967], [Hempel 1976], [Jaco 1977].

Remark. The proof is given in the second reference. The original version of the loop theorem ($\mathfrak{N} = 1$) was first formulated and proved by Papakyriakopoulos. Another generalization analogous to the Shapiro–Whitehead version of Dehn's Lemma was proved in [Waldhausen 1967].

B.6 Theorem (Sphere theorem). *Let M be an orientable* 3*-manifold and $\mathfrak{N}$ a $\pi_1 M$-invariant subgroup of $\pi_2 M$.* ($\mathfrak{N}$ *is $\pi_1 M$-invariant if the operation of $\pi_1 M$ on $\pi_2 M$ maps $\mathfrak{N}$ onto itself.) Then there is an embedding $g\colon S^2 \to M$ such that* $[g] \notin \mathfrak{N}$. □

[Papakyriakopoulos 1957], [Hempel 1976], [Jaco 1977].

This triad of Papakyriakopoulos theorems started a new era in 3-dimensional topology. The next impulse came from W. Haken and F. Waldhausen:

A surface F is *properly embedded* in a 3-manifold M if $\partial F = F \cap \partial M$. A 2-sphere ($F = S^2$) is called *incompressible in M*, if it does not bound a 3-ball in M, and a surface $F \neq S^2$ is called *incompressible*, if there is no disk $D \subset M$ with $D \cap F = \partial D$, and ∂D not contractible in F. A manifold is *sufficiently large* when it contains a properly embedded 2-sided incompressible surface.

B.7 Theorem (Waldhausen). *Let M, N be sufficiently large irreducible* 3*-manifolds not containing* 2*-sided projective planes. If there is an isomorphism*

$$f_\#\colon (\pi_1 M, \pi_1 \partial M) \to (\pi_1 N, \pi_1 \partial N)$$

between the peripheral group systems, then there is a boundary preserving map $f\colon (M, \partial M) \to (N, \partial N)$ inducing $f_\#$. Either f is homotopic to a homeomorphism of M to N or M is a twisted I-bundle over a closed surface and N is a product bundle over a homeomorphic surface. □

[Waldhausen 1967], [Hempel 1976].

Remark. The Waldhausen theorem states for a large class of manifolds what has long been known of surfaces: there is a natural isomorphism between the mapping class group of M and the group of automorphisms of $\pi_1(M)$ modulo inner automorphisms.

Evidently Theorem B.7 applies to knot complements $C = M$. A Seifert surface of minimal genus is a properly embedded incompressible surface in C.

B.8 Theorem (Smith conjecture). *A simplicial orientation preserving map $h\colon S^3 \to S^3$ of period q is conjugate to a rotation.* □

A conference on the Smith conjecture was held in 1979 at Columbia University in New York, the proceedings of which are recorded in [Morgan-Bass 1984] and contain a proof. The case $q = 2$ is due to Waldhausen, a proof is given in [Waldhausen 1969].

Appendix C
Tables

The following Table I lists certain invariants of knots up to ten crossings. The identification (first column) follows [Rolfsen 1976] but takes into account that there is a duplication ($10_{161} = 10_{162}$) in his table which was detected by Perko. For each crossing number alternating knots are grouped in front, a star indicates the first non-alternating knot in each order.

The first column ($\Delta_1(t)$, $\Delta_2(t)$) contains the Alexander polynomials , factorized into irreducible polynomials. The polynomials $\Delta_k(t)$, $k > 2$, are always trivial. (See Chapter 8.) Alexander polynomials of links or of knots with eleven crossings are to be found in [Rolfsen 1976], [Conway 1970] and [Perko 1980].

The second column (T) gives the torsion numbers of the first homology group $H_1(\bar{C}_2)$ of the two-fold branched covering of the knot. The numbers are $T_r, T_{r-1}, \ldots$ where $T_1|T_2|\ldots|T_r$ is the chain of elementary divisors of $H_1(\hat{C}_2)$. (See Chapter 9.) For torsion numbers of cyclic coverings of order $n > 2$, see [Metha 1980]. Torsion numbers for $n = 3$ (knots with less than ten crossings) are listed in [Reidemeister 1932].

The column (σ) records the signature of the knot. (See Chapter 13.)

The column (q) states the periods of the knot; a question-mark indicates that a certain period is possible but has not been verified. (See Chapter 14 D.)

The column headed α, β contains Schubert's notation of the knot as a two-bridged knot. (The first number α always coincides with T_r.) Where no entry appears the bridge number is three. (See Chapter 12.)

The column (s) contains complete information about symmetries in Conway's notation. (See Chapter 2.)

	amphicheiral	non-amphicheiral
invertible	f	r
non-invertible	i	n

It has been checked that up to ten crossings the genus of a knot always equals half the degree of its Alexander polynomial.

Acknowledgement: The Alexander polynomials, the signature and most of the periods have been computed by U. Lüdicke. Periods up to nine crossings were taken from [Murasugi 1980]. Symmetries and 2-bridge numbers (α, β) were copied from [Conway 1967] and compared with other results on amphicheirality and invertibility [Hartley 1980]. Periods and symmetries have been corrected and brought up to date using [Kawauchi 1996].

Table I

$\mathfrak{k}$	$\Delta_1(t)$	$\Delta_2(t)$	T	σ	q	α, β	s
3_1	$t^2 - t + 1$		3	2	2, 3	3, 1	r
4_1	$t^2 - 3t + 1$		5	0	2	5, 2	f
5_1	$t^4 - t^3 + t^2 - t + 1$		5	4	2, 5	5, 1	r
5_2	$2t^2 - 3t + 2$		7	2	2	7, 3	r
6_1	$2t^2 - 5t + 2$		9	0	2	9, 4	r
6_2	$t^4 - 3t^3 + 3t^2 - 3t + 1$		11	2	2	11, 4	r
6_3	$t^4 - 3t^3 + 5t^2 - 3t + 1$		13	0	2	13, 5	f
7_1	$t^6 - t^5 + t^4 - t^3 + t^2 - t + 1$		7	6	2, 7	7, 1	r
7_2	$3t^2 - 5t + 3$		11	2	2	11, 5	r
7_3	$2t^4 - 3t^3 + 3t^2 - 3t + 2$		13	4	2	13, 4	r
7_4	$4t^2 - 7t + 4$		15	2	2	15, 4	r
7_5	$2t^4 - 4t^3 + 5t^2 - 4t + 2$		17	4	2	17, 7	r
7_6	$t^4 - 5t^3 + 7t^2 - 5t + 1$		19	2	2	19, 7	r
7_7	$t^4 - 5t^3 + 9t^2 - 5t + 1$		21	0	2	21, 8	r
8_1	$3t^2 - 7t + 3$		13	0	2	13, 6	r
8_2	$t^6 - 3t^5 + 3t^4 - 3t^3 + 3t^2 - 3t + 1$		17	4	2	17, 6	r
8_3	$4t^2 - 9t + 4$		17	0	2	17, 4	f
8_4	$2t^4 - 5t^3 + 5t^2 - 5t + 2$		19	2	2	19, 5	r
8_5	$(t^2 - t + 1)(-t^4 + 2t^3 - t^2 + 2t - 1)$		21	4	2		r
8_6	$2t^4 - 6t^3 + 7t^2 - 6t + 2$		23	2	2	23, 10	r
8_7	$t^6 - 3t^5 + 5t^4 - 5t^3 + 5t^2 - 3t + 1$		23	2	2	23, 9	r
8_8	$2t^4 - 6t^3 + 9t^2 - 6t + 2$		25	0	2	25, 9	r
8_9	$t^6 - 3t^5 + 5t^4 - 7t^3 + 5t^2 - 3t + 1$		25	0	2	25, 7	f
8_{10}	$(t^2 - t + 1)^3$		27	2			r
8_{11}	$(2t^2 - 5t + 2)(t^2 - t + 1)$		27	2	2	27, 10	r
8_{12}	$t^4 - 7t^3 + 13t^2 - 7t + 1$		29	0	2	29, 12	f
8_{13}	$2t^4 - 7t^3 + 11t^2 - 7t + 2$		29	0	2	29, 11	r
8_{14}	$2t^4 - 8t^3 + 11t^2 - 8t + 2$		31	2	2	31, 12	r
8_{15}	$(t^2 - t + 1)(3t^2 - 5t + 3)$		33	4	2		r
8_{16}	$t^6 - 4t^5 + 8t^4 - 9t^3 + 8t^2 - 4t + 1$		35	2			r
8_{17}	$t^6 - 4t^5 + 8t^4 - 11t^3 + 8t^2 - 4t + 1$		37	0			i
8_{18}	$(t^2 - t + 1)(t^2 - 3t + 1)$,	$t^2 - t + 1$	15,3	0	2, 4		f
8_{19}	$(t^2 - t + 1)(t^4 - t^2 + 1)$		3	6	2,3,4		r
8_{20}	$(t^2 - t + 1)$		9	0			r
8_{21}	$(t^2 - t + 1)(t^2 - 3t + 1)$		15	2	2		r
9_1	$(t^2 - t + 1)(t^6 - t^3 + 1)$		9	8	2,3,9	9, 1	r
9_2	$4t^2 - 7t + 4$		15	2	2	15, 7	r
9_3	$2t^6 - 3t^5 + 3t^4 - 3t^3 + 3t^2 - 3t + 2$		19	6	2	19, 6	r
9_4	$3t^4 - 5t^3 + 5t^2 - 5t + 3$		21	4	2	21, 5	r
9_5	$6t^2 - 11t + 6$		23	2	2	23, 6	r
9_6	$(t^2 - t + 1)(-2t^4 + 2t^3 - t^2 + 2t - 2)$		27	6	2	27, 5	r
9_7	$3t^4 - 7t^3 + 9t^2 - 7t + 3$		29	4	2	29, 13	r

$\mathfrak{k}$	$\Delta_1(t)$	$\Delta_2(t)$	T	σ	q	α, β	s
9_8	$2t^4 - 8t^3 + 11t^2 - 8t + 2$		31	2	2	31, 11	r
9_9	$2t^6 - 4t^5 + 6t^4 - 7t^3 + 6t^2 - 4t + 2$		31	6	2	31, 9	r
9_{10}	$4t^4 - 8t^3 + 9t^2 - 8t + 4$		33	4	2	33, 10	r
9_{11}	$t^6 - 5t^5 + 7t^4 - 7t^3 + 7t^2 - 5t + 1$		33	4	2	33, 14	r
9_{12}	$(t^2 - 3t + 1)(2t^2 - 3t + 2)$		35	2	2	35, 13	r
9_{13}	$4t^4 - 9t^3 + 11t^2 - 9t + 4$		37	4	2	37, 10	r
9_{14}	$2t^4 - 9t^3 + 15t^2 - 9t + 2$		37	0	2	37, 14	r
9_{15}	$2t^4 - 10t^3 + 15t^2 - 10t + 2$		39	2	2	39, 16	r
9_{16}	$(t^2 - t + 1)(-2t^4 + 3t^3 - 3t^2 + 3t - 2)$		39	6	2		r
9_{17}	$t^6 - 5t^5 + 9t^4 - 9t^3 + 9t^2 - 5t + 1$		39	2	2	39, 14	r
9_{18}	$4t^4 - 10t^3 + 13t^2 - 10t + 4$		41	4	2	41, 17	r
9_{19}	$2t^4 - 10t^3 + 17t^2 - 10t + 2$		41	0	2	41, 16	r
9_{20}	$t^6 - 5t^5 + 9t^4 - 11t^3 + 9t^2 - 5t + 1$		41	4	2	41, 15	r
9_{21}	$2t^4 - 11t^3 + 17t^2 - 11t + 2$		43	2	2	43, 18	r
9_{22}	$t^6 - 5t^5 + 10t^4 - 11t^3 + 10t^2 - 5t + 1$		43	2			r
9_{23}	$(t^2 - t + 1)(4t^2 - 7t + 4)$		47	2	2	47, 18	r
9_{24}	$(t^2 - t + 1)^2(t^2 - 3t + 1)$		45	0			r
9_{25}	$3t^4 - 12t^3 + 17t^2 - 12t + 3$		47	2			r
9_{26}	$t^6 - 5t^5 + 11t^4 - 13t^3 + 11t^2 - 5t + 1$		45	4	2	45, 19	r
9_{27}	$t^6 - 5t^5 + 11t^4 - 15t^3 + 11t^2 - 5t + 1$		49	0	2	49, 19	r
9_{28}	$(t^2 - t + 1)(-t^4 + 4t^3 - 7t^2 + 4t - 1)$		51	2	2		r
9_{29}	$(t^2 - t + 1)(-t^4 + 4t^3 - 7t^2 + 4t - 1)$		51	2			r
9_{30}	$t^6 - 5t^5 + 12t^4 - 17t^3 + 12t^2 - 5t + 1$		53	0			r
9_{31}	$t^6 - 5t^5 + 13t^4 - 17t^3 + 13t^2 - 5t + 1$		55	2	2	55, 21	r
9_{32}	$t^6 - 6t^5 + 14t^4 - 17t^3 + 14t^2 - 6t + 1$		59	2			n
9_{33}	$t^6 - 6t^5 + 14t^4 - 19t^3 + 14t^2 - 6t + 1$		61	0			n
9_{34}	$t^6 - 6t^5 + 16t^4 - 23t^3 + 16t^2 - 6t + 1$		69	0			r
9_{35}	$7t^2 - 13t + 7$		9,3	2	2, 3		r
9_{36}	$t^6 - 5t^5 + 8t^4 - 9t^3 + 8t^2 - 5t + 1$		37	4			r
9_{37}	$(t^2 - 3t + 1)(2t^2 - 5t + 2)$		15,3	0	2		r
9_{38}	$(t^2 - t + 1)(5t^2 - 9t + 5)$		57	4			r
9_{39}	$(t^2 - 3t + 1)(3t^2 - 5t + 3)$		55	2			r
9_{40}	$(t^2 - t + 1)(t^2 - 3t + 1)2$	$t^2 - 3t + 1$	15,5	2	2, 3		r
9_{41}	$3t^4 - 12t^3 + 19t^2 - 12t + 3$		7,7	0	3		r
9_{42}	$t^4 - 2t^3 + t^2 - 2t + 1$		7	2			r
9_{43}	$t^6 - 3t^5 + 2t^4 - t^3 + 2t^2 - 3t + 1$		13	4			r
9_{44}	$t^4 - 4t^3 + 7t^2 - 4t + 1$		17	0			r
9_{45}	$t^4 - 6t^3 + 9t^2 - 6t + 1$		23	2			r
9_{46}	$2t^2 - 5t + 2$		3,3	0	2		r
9_{47}	$t^6 - 4t^5 + 6t^4 - 5t^3 + 6t^2 - 4t + 1$		9,3	2	3		r
9_{48}	$t^4 - 7t^3 + 11t^2 - 7t + 1$		9,3	2	2		r
9_{49}	$3t^4 - 6t^3 + 7t^2 - 6t + 3$		5,5	4	3		r
10_1	$4t^2 - 91 + 4$		17	0	2	17, 8	r
10_2	$t^8 - 3t^7 + 3t^6 - 3t^5 + 3t^4 - 3t^3 + 3t^2 - 3t + 1$		23	6	2	23, 8	r

$\mathfrak{k}$	$\Delta_1(t)$ $\Delta_2(t)$	T	σ	q	α, β	s
10_3	$6t^2 - 13t + 6$	25	0	2, 3	25, 6	r
10_4	$3t^4 - 7t^3 + 7t^2 - 7t + 3$	27	2	2	27, 7	r
10_5	$(t^2 - t + 1)(t^6 - 2t^5 + 2t^4 - t^3 + 2t^2 - 2t + 1)$	33	4	2	33, 13	r
10_6	$2t^6 - 6t^5 + 7t^4 - 7t^3 + 7t^2 - 6t + 2$	37	4	2	37, 16	r
10_7	$3t^4 - 11t^3 + 15t^2 - 11t + 3$	43	2	2	43, 16	r
10_8	$2t^6 - 5t^5 + 5t^4 - 5t^3 + 5t^2 - 5t + 2$	29	4	2	29, 6	r
10_9	$(t^2 - t + 1)(t^6 - 2t^5 + 2t^4 - 3t^3 + 2t^2 - 2t + 1)$	39	2	2	39, 11	r
10_{10}	$3t^4 - 11t^3 + 17t^2 - 11t + 3$	45	0	2	45, 17	r
10_{11}	$4t^4 - 11t^3 + 13t^2 - 11t + 4$	43	2	2	43, 13	r
10_{12}	$2t^6 - 6t^5 + 10t^4 - 11t^3 + 10t^2 - 6t + 2$	47	2	2	47, 17	r
10_{13}	$2t^4 - 13t^3 + 23t^2 - 13t + 2$	53	0	2	53, 22	r
10_{14}	$2t^6 - 8t^5 + 12t^4 - 13t^3 + 12t^2 - 8t + 2$	57	4	2	57, 22	r
10_{15}	$2t^6 - 6t^5 + 9t^4 - 9t^3 + 9t^2 - 6t + 2$	43	2	2	43, 19	r
10_{16}	$4t^4 - 12t^3 + 15t^2 - 12t + 4$	47	2	2	47, 14	r
10_{17}	$t^8 - 3t^7 + 5t^6 - 7t^5 + 9t^4 - 7t^3 + 5t^2 - 3t + 1$	41	0	2	41, 9	f
10_{18}	$4t^4 - 14t^3 + 19t^2 - 14t + 4$	55	2	2	55, 23	r
10_{19}	$2t^6 - 7t^5 + 11t^4 - 11t^3 + 11t^2 - 7t + 2$	51	2	2	51, 14	r
10_{20}	$3t^4 - 9t^3 + 11t^2 - 9t + 3$	35	2	2,	35, 16	r
10_{21}	$(2t^2 - 5t + 2)(-t^4 + t^3 - t^2 + t - 1)$	45	4	2	45, 16	r
10_{22}	$2t^6 - 6t^5 + 10t^4 - 13t^3 + 10t^2 - 6t + 2$	49	0	2	49, 13	r
10_{23}	$2t^6 - 7t^5 - 13t^4 - 15t^3 + 13t^2 - 7t + 2$	59	2	2	59, 23	r
10_{24}	$4t^4 - 14t^3 + 19t^2 - 14t + 4$	55	2	2	55, 24	r
10_{25}	$2t^6 - 8t^5 + 14t^4 - 17t^3 + 14t^2 - 8t + 2$	65	4	2	65, 24	r
10_{26}	$2t^6 - 7t^5 + 13t^4 - 17t^3 + 13t^2 - 7t + 2$	61	0	2	61, 17	r
10_{27}	$2t^6 - 8t^5 + 16t^4 - 19t^3 + 16t^2 - 8t + 2$	71	2	2	71, 27	r
10_{28}	$4t^4 - 13t^3 + 19t^2 - 13t + 4$	53	0	2	53, 19	r
10_{29}	$t^6 - 7t^5 + 15t^4 - 17t^3 + 15t^2 - 7t + 1$	63	2	2	63, 26	r
10_{30}	$4t^4 - 17t^3 + 25t^2 - 17t + 4$	67	2	2	67, 26	r
10_{31}	$4t^4 - 14t^3 + 21t^2 - 14t + 4$	57	0	2	57, 25	r
10_{32}	$(t^2 - t + 1)(-2t^4 + 6t^3 - 7t^2 + 6t - 2)$	69	0	2	69, 29	r
10_{33}	$4t^4 - 16t^3 + 25t^2 - 16t + 4$	65	0	2	65, 18	f
10_{34}	$3t^4 - 9t^3 + 13t^2 - 9t + 3$	37	0	2	37, 13	r
10_{35}	$2t^4 - 12t^3 + 21t^2 - 12t + 2$	49	0	2	49, 20	r
10_{36}	$3t^4 - 13t^3 + 19t^2 - 13t + 3$	51	2	2	51, 20	r
10_{37}	$4t^4 - 13t^3 + 19t^2 - 13t + 4$	53	0	2	53, 23	f
10_{38}	$4t^4 - 15t^3 + 21t^2 - 15t + 4$	59	2	2	59, 25	r
10_{39}	$2t^6 - 8t^5 + 13t^4 - 15t^3 + 13t^2 - 8t + 2$	61	4	2	61, 22	r
10_{40}	$(t^2 - t + 1)(-2t^4 + 6t^3 - 9t^2 + 6t - 2)$	75	2	2	75, 29	r
10_{41}	$t^6 - 7t^5 + 17t^4 - 21t^3 + 17t^2 - 7t + 1$	71	2	2	71, 26	r
10_{42}	$t^6 - 7t^5 + 19t^4 - 27t^3 + 19t^2 - 7t + 1$	81	0	2	81, 31	r
10_{43}	$t^6 - 7t^5 + 17t^4 - 23t^3 + 17t^2 - 7t + 1$	73	0	2	73, 27	f
10_{44}	$t^6 - 7t^5 + 19t^4 - 25t^3 + 19t^2 - 7t + 1$	79	2	2	79, 30	r
10_{45}	$t^6 - 7t^5 + 21t^4 - 31t^3 + 21t^2 - 7t + 1$	89	0	2	89, 34	f
10_{46}	$t^8 - 3t^7 + 46 - 5t^5 + 5t^4 - 5t^3 + 4t^2 - 3t + 1$	31	6			r

$\mathfrak{k}$	$\Delta_1(t)$ $\Delta_2(t)$	T	σ	q	α, β	s
10_{47}	$t^8 - 3t^7 + 6t^6 - 7t^5 + 7t^4 - 7t^3 + 6t^2 - 3t + 1$	41	4			r
10_{48}	$t^8 - 3t^7 + 6t^6 - 9t^5 + 11t^4 - 9t^3 + 6t^2 - 3t + 1$	49	0			r
10_{49}	$3t^6 - 8t^5 + 12t^4 - 13t^3 + 12t^2 - 8t + 3$	59	6			r
10_{50}	$2t^6 - 7t^5 + 11t^4 - 13t^3 + 11t^2 - 7t + 2$	53	4			r
10_{51}	$2t^6 - 7t^5 + 15t^4 - 19t^3 + 15t^2 - 7t + 2$	67	2			r
10_{52}	$2t^6 - 7t^5 + 13t^4 - 15t^3 + 13t^2 - 7t + 2$	59	2			r
10_{53}	$6t^4 - 18t^3 + 25t^2 - 181 + 6$	73	4			r
10_{54}	$2t^6 - 6t^5 + 10t^4 - 11t^3 + 10, 2 - 6t + 2$	47	2			r
10_{55}	$5t^4 - 15t^3 + 21t^2 - 15t + 5$	61	4			r
10_{56}	$2t^6 - 8t^5 + 14t^4 - 17t^3 + 14t^2 - 8t + 2$	65	4			r
10_{57}	$2t^6 - 8t^5 + 18t^4 - 23t^3 + 18t^2 - 8t + 2$	79	2			r
10_{58}	$(t^2 - 3t + 1)(3t^2 - 7t + 3)$	65	0	2		r
10_{59}	$(t^2 - t + 1)(12 - 3t + 1)2$	75	2			r
10_{60}	$(t^2 - 3t + 1)(-t^4 + 4t^3 - 7t^2 + 4t - 1)$	85	0	2		r
10_{61}	$(t^2 - t + 1)(-2t^4 + 3t^3 - t^2 + 3t - 2)$	33	4	2		r
10_{62}	$(t^2 - t + 1)2(t^4 - t^3 + t^2 - t + 1)$	45	4			r
10_{63}	$(t^2 - t + 1)(5t^2 - 9t + 5)$	57	4	2		r
10_{64}	$(t^2 - t + 1)(t^6 - 2t^5 + 3t^4 - 5t^3 + 3t^2 - 2t + 1)$	51	2	2		r
10_{65}	$(t^2 - t + 1)2(-2t^2 + 3t - 2)$	63	2			r
10_{66}	$(12 - t + 1)(-3t^4 + 6t^3 - 7t^2 + 6t - 3)$	75	6	2		r
10_{67}	$(2t^2 - 3t + 2)(2t^2 - 5t + 2)$	63	2	2		n
10_{68}	$4t^4 - 14t^3 + 21t^2 - 14t + 4$	57	0	2		r
10_{69}	$t^6 - 7t^5 + 21t^4 - 29t^3 + 21t^2 - 7t + 1$	87	2	2		r
10_{70}	$t^6 - 7t^5 + 16t^4 - 19t^3 + 16t^2 - 7t + 1$	67	2			r
10_{71}	$t^6 - 7t^5 + 18, 4 - 25t^3 + 18t^2 - 7t + 1$	77	0			r
10_{72}	$2t^6 - 9t^5 + 16t^4 - 19t^3 + 16t^2 - 9t + 2$	73	4			r
10_{73}	$t^6 - 7t^5 + 20t^4 - 27t^3 + 20t^2 - 7t + 1$	83	2			r
10_{74}	$(2t^2 - 3t + 2)(2t^2 - 5t + 2)$	21,3	2	2		r
10_{75}	$t^6 - 7t^5 + 19, 4 - 27t^3 + 19t^2 - 7t + 1$	27,3	0	2		r
10_{76}	$(12 - t + 1)(-2t^4 + 5t^3 - 5t^2 + 5t - 2)$	57	4	2		r
10_{77}	$(t^2 - t + 1)2(-2t^2 + 3t - 2)$	63	2			r
10_{78}	$(t^2 - t + 1)(-t^4 + 6t^3 - 9t^2 + 6t - 1)$	69	4	2		r
10_{79}	$t^8 - 3t^7 + 7t^6 - 12t^5 + 15t^4 - 12t^3 + 7t^2 - 3t + 1$	61	0			i
10_{80}	$3t^6 - 9t^5 + 15t^4 - 17t^3 + 15t^2 - 9t + 3$	71	6			n
10_{81}	$t^6 - 8t^5 + 20t^4 - 27t^3 + 20t^2 - 8t + 1$	85	0			
10_{82}	$(t^2 - t + 1)2(t^4 - 2t^3 + t^2 - 2t + 1)$	63	2			n
10_{83}	$2t^6 - 9t^5 + 19t^4 - 25t^3 + 19t^2 - 9t + 2$	83	2			n
10_{84}	$(t^2 - t + 1)(-2t^4 + 7t^3 - 11t^2 + 7t - 2)$	87	2			n
10_{85}	$(t^2 - t + 1)(t^6 - 3t^5 + 4t^4 - 3t^3 + 4t^2 - 3t + 1)$	57	4			n
10_{86}	$2t^6 - 9t^5 + 19t^4 - 23t^3 + 19t^2 - 9t + 2$	85	0			n
10_{87}	$(t^2 - t + 1)2(-2t^2 + 5t - 2)$	81	0			n
10_{88}	$t^6 - 8t^5 + 24t^4 - 35t^3 + 24t^2 - 8t + 1$	101	0			i
10_{89}	$t^6 - 8t^5 + 24t^4 - 33t^3 + 24t^2 - 8t + 1$	99	2			r
10_{90}	$2t^6 - 8t^5 + 17t^4 - 23t^3 + 17t^2 - 8t + 2$	77	0			n

$\mathfrak{k}$	$\Delta_1(t)$ $\Delta_2(t)$	T	σ	q	α, β	s
10_{91}	$t^8 - 4t^7 + 9t^6 - 14t^5 + 17t^4 - 14t^3 + 9t^2 - 4t + 1$	73	0			n
10_{92}	$2t^6 - 10t^5 + 20t^4 - 25t^3 + 20t^2 - 10t + 2$	89				n
10_{93}	$2t^6 - 8t^5 + 15t^4 - 17t^3 + 15t^2 - 8t + 2$	67				n
10_{94}	$t^8 - 4t^7 + 9t^6 - 14t^5 + 15t^4 - 14t^3 + 9t^2 - 4t + 1$	71				n
10_{95}	$(2t^2 - 3t + 2)(-t^4 + 3t^3 - 5t^2 + 3t - 1)$	91				n
10_{96}	$t^6 - 7t^5 + 22t^4 - 33t^3 + 22t^2 - 7t + 1$	93	0			r
10_{97}	$5t^4 - 22t^3 + 33t^2 - 22t + 5$	87				r
10_{98}	$(t^2 - t + 1)^2(-2t^2 + 5t - 2),$ $(t^2 - t + 1)$	27,3	4	2		n
10_{99}	$(t^2 - t + 1)^4$ $(t^2 - t + 1)^2$	9,9	0			f
10_{100}	$(t^4 - t^3 + t^2 - t + 1)(t^4 - 3t^3 + 5t^2 - 3t + 1)$	65	4			r
10_{101}	$7t^2 - 21t^3 + 29t^2 - 21t + 7$	85	4			r
10_{102}	$2t^6 - 8t^5 + 16t^4 - 21t^3 + 16t^2 - 8t + 2$	73	0			n
10_{103}	$(t^2 - t + 1)(-2t^4 + 6t^3 - 9t^2 + 6t - 2)$	15,5	2			r
10_{104}	$t^8 - 4t^7 + 9t^6 - 15t^5 + 19t^4 - 15t^3 + 9t^2 - 4t + 1$	77	0			r
10_{105}	$t^6 - 8t^5 + 22t^4 - 29t^3 + 22t^2 - 8t + 1$	91	2			r
10_{106}	$(t^2 - t + 1)(t^6 - 3t^5 + 5t^4 - 7t^3 + 5t^2 - 3t + 1)$	75	2			n
10_{107}	$t^6 - 8t^5 + 22t^4 - 31t^3 + 22t^2 - 8t + 1$	93	0			n
10_{108}	$2t^6 - 8t^5 + 14t^4 - 15t^3 + 14t^2 - 8t + 2$	63	2			r
10_{109}	$t^8 - 4t^7 + 10t^6 - 17t^5 + 21t^4 - 17t^3 + 10t^2 - 4t + 1$	85	0			i
10_{110}	$t^6 - 8t^5 + 20t^4 - 25t^3 + 20t^2 - 81 + 1$	83	2			n
10_{111}	$(2t^2 - 3t + 2)(-t^4 + 3t^3 - 3t^2 + 3t - 1)$	77	4			r
10_{112}	$(t^2 - t + 1)(16 - 4t^5 + 6t^4 - 7t^3 + 6t^2 - 4t + 1)$	87	2			r
10_{113}	$(t^2 - t + 1)(-2t^4 + 9t^3 - 15t^2 + 9t - 2)$	111	2			r
10_{114}	$(t^2 - t + 1)(-2t^4 + 8t^3 - 11t^2 + 8t - 2)$	93	0			r
10_{115}	$t^6 - 9t^5 + 26t^4 - 37t^3 + 26t^2 - 9t + 1$	109	0			i
10_{116}	$t^8 - 5t^7 + 12t^6 - 19t^5 + 21t^4 - 19t^3 + 12t^2 - 5t + 1$	95	2			r
10_{117}	$2t^6 - 10t^5 + 24t^4 - 31t^3 + 24t^2 - 10t + 2$	103	2			n
10_{118}	$t^8 - 5t^7 + 12t^6 - 19t^5 + 23t^4 - 19t^3 + 12t^2 - 5t + 1$	97	0			i
10_{119}	$2t^6 - 10t^5 + 23t^4 - 31t^3 + 23t^2 - l0t + 2$	101	0			n
10_{120}	$(2t^2 - 3t + 2)(4t^2 - 7t + 4)$	105	4	2		r
10_{121}	$2t^6 - 11t^5 + 27t^4 - 35t^3 + 27t^2 - 11t + 2$	115	2			r
10_{122}	$(t^2 - t + 1)(t^2 - 3t + 1)(-2t^2 + 3t - 2)$	105	0	2		r
10_{123}	$(t^4 - 3t^3 + 3t^2 - 3t + 1)^2,$ $t^4 - 3t^3 + 3t^2 - 3t + 1$	11,11	0	5		f
10_{124}	$t^8 - t^7 + t^5 - t^4 + t^3 - t + 1$		8	3, 5		r
10_{125}	$t^6 - 2t^5 + 2t^4 - t^3 + 2t^2 - 2t + 1$	11	2			r
10_{126}	$16 - 2t^5 + 4t^4 - 5t^3 + 4t^2 - 2t + 1$	19	2			r
10_{127}	$t^6 - 4t^5 + 6t^4 - 7t^3 + 6t^2 - 4t + 1$	29	4			r
10_{128}	$2t^6 - 3t^5 + t^4 + t^3 + t^2 - 3t + 2$	11	6			r
10_{129}	$2t^4 - 6t^3 + 9t^2 - 6t + 2$	25	0			r
10_{130}	$2t^4 - 4t^3 + 5t^2 - 4t + 2$	17	0			r
10_{131}	$2t^4 - 8t^3 + 11t^2 - 8t + 2$	31	2			r
10_{132}	$t^4 - t^3 + t^2 - t + 1$	5	0			r
10_{133}	$t^4 - 5t^3 + 7t^2 - 5t + 1$	19	2			r
10_{134}	$2t^6 - 4t^5 + 4t^4 - 3t^3 + 4t^2 - 4t + 2$	23	6			r

$\mathfrak{k}$	$\Delta_1(t)$ $\Delta_2(t)$	T	σ	q	α, β	s
10_{135}	$3t^4 - 9t^3 + 13t^2 - 9t + 3$	37	0			r
10_{136}	$(t^2 - t + 1)(t^2 - 3t + 1)$	15	2	2		r
10_{137}	$(t^2 - 3t + 1)^2$	25	0			r
10_{138}	$(t^2 - 3t + 1)(-t^4 + 2t^3 - t^2 + 2t - 1)$	35	2	2		r
10_{139}	$(12 - t + 1)(-t^6 + t^4 - t^3 + t^2 - 1)$	3	6	2		r
10_{140}	$(t^2 - t + 1)^2$	9	0			r
10_{141}	$(t^2 - t + 1)(-t^4 + 2t^3 - t^2 + 2t - 1$	21	0	2		
10_{142}	$(t^2 - t + 1)(-2t^4 + t^3 + t^2 + t - 2)$	15	6	2		
10_{143}	$(t^2 - t + 1)^3$	27	2			
10_{144}	$(t^2 - t + 1)(3t^2 - 7t + 3)$	39	2	2		
10_{145}	$t^4 + t^3 - 3t^2 + t + 1$	3	2	2		
10_{146}	$2t^4 - 8t^3 + 13t^2 - 8t + 2$	33	0	2		
10_{147}	$(t^2 - t + 1)(2t^2 - 5t + 2)$	27	2	2		
10_{148}	$t^6 - 3t^5 + 7t^4 - 9t^3 + 7t^2 - 3t + 1$	31	2			
10_{149}	$t^6 - 5t^5 + 9t^4 - 11t^3 + 9t^2 - 5t + 1$	41	4			
10_{150}	$t^6 - 4t^5 + 6t^4 - 7t^3 + 6t^2 - 4t + 1$	29	4			
10_{151}	$t^6 - 45 + 10t^4 - 13t^3 + 10t^2 - 4t + 1$	43	2			
10_{152}	$t^8 - t^7 - t^6 + 45 - 5t^4 + 4t^3 - t^2 - t + 1$	11	6			
10_{153}	$t^6 - t^5 - t^4 + 3t^3 - t^2 - t + 1$	0	2			
10_{154}	$t^6 - 4t^4 + 7t^3 - 4t^2 + 1$	13	4			
10_{155}	$t^6 - 3t^5 + 5t^4 - 7t^3 + 5t^2 - 3t + 1$	5,5	0			
10_{156}	$t^6 - 45 + 8t^4 - 9t^3 + 8t^2 - 4t + 1$	35	2			
10_{157}	$t^6 - 6t^5 + 11t^4 - 13t^3 + 11t^2 - 6t + 1$	7,7	4			
10_{158}	$t^6 - 4t^5 + 10t^4 - 15t^3 + 10t^2 - 4t + 1$	45	0			
10_{159}	$(t^2 - t + 1)(-t^4 + 3t^3 - 5t^2 + 3t - 1)$	39	2			
10_{160}	$t^6 - 4t^5 + 4t^4 - 3t^3 + 4t^2 - 4t + 1$	21	4			
10_{161}	$t^6 - 2t^4 + 3t^3 - 2t^2 + 1$	5	4			
10_{162}	$3t^4 - 9t^3 + 11t^2 - 9t + 3$	35	2			
10_{163}	$(t^2 - t + 1)(-t^4 + 4t^3 - 7t^2 + 4t - 1)$	51	2			
10_{164}	$3t^4 - 11t^3 + 17t^2 - 11t + 3$	45	0			
10_{165}	$2t^4 - 10t^3 + 15t^2 - 10t + 2$	39	2			

Table II gives non-singular Seifert matrices of knots up to ten crossings, computed by U. Lüdicke. $2m$ is the number of rows; the entries run through successive rows, $x + y$ resp. $x - y$ means that the entry $+y$ resp. $-y$ has to be repeated x times. As an example

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

is the Seifert matrix of 5_1 according to the table. (See Chapter 13.)

Table II

3_1	$m=1$	1 −1 0 1
4_1	$m=1$	−1 0 2+1
5_1	$m=2$	1 0 −1 2+0 1 0 −1 2+0 1 0 −1 2+0 1
5_2	$m=1$	2 −2 −1 2
6_1	$m=1$	−1 0 1 2
6_2	$m=2$	1 0 −1 2+0 1 3+0 −1 1 0 −1 0 1 −1
6_3	$m=2$	1 1 −1 3+0 1 2+0 1 2−1 0 −1 2+1 −1
7_1	$m=3$	2+0 −1 3+0 1 2+0 −1 3+0 12+0 −1 3+0 1 2+0 −1 3+0 1 2+0 −1 3+0 1
7_2	$m=1$	3 −3 −2 3
7_3	$m=2$	−2 0 2+1 0 −1 2+0 2 0 −2 2+0 1 0 −1
7_4	$m=1$	−2 0 1 −2
7_5	$m=2$	2 0 2−1 0 1 4+0 1 3−1 0 2
7_6	$m=2$	1 0 −1 2+0 1 −1 0 −1 0 2 0 1 −1 0 −1
7_7	$m=2$	1 3+0 −1 1 2+0 1 2−1 2+0 1 0 −1
8_1	$m=1$	−1 0 1 3
8_2	$m=3$	2+0 −1 3+0 1 2+0 −1 3+0 1 5+0 −1 1 2+0 −1 3+0 1 0 −1 2+0 1 0 −1
8_3	$m=1$	−2 0 1 2
8_4	$m=2$	1 0 −1 2+0 1 3+0 −1 1 2+0 1 −1 −2
8_5	$m=3$	−1 6+0 −1 0 1 4+0 −1 0 1 3+0 1 −1 2+0 1 3+0 −1 0 1 2+0 −1 0 1
8_6	$m=2$	2 0 −2 2+0 1 2+0 2−1 2 0 −1 0 1 −1
8_7	$m=3$	−1 2+0 1 3+0 −1 4+0 1 0 −1 4+0 1 0 −1 6+0 1 −1 1 0 −1 2+0
8_8	$m=2$	1 1 −1 3+0 1 2+0 −1 1 −2 2+1 −1 2 −2
8_9	$m=3$	1 −1 6+0 −1 1 3+0 1 0 −1 3+0 1 0 −1 1 6+0 1 −1 3+0 −1 0 1
8_{10}	$m=3$	1 −1 5+0 1 4+0 −1 0 −1 2+0 1 3+0 −1 4+0 1 0 −1 4+0 1 0 −1
8_{11}	$m=2$	2 0 −2 0 −1 1 2+0 −1 0 2 0 1 −1 0 −1
8_{12}	$m=2$	−2 1 2+0 1 −1 2+0 1 −1 1 2+0 1 0 1
8_{13}	$m=2$	1 −1 3+0 1 4+0 −2 1 −1 0 2 −2
8_{14}	$m=2$	−1 3+0 2+1 4+0 1 −1 0 −1 0 2
8_{15}	$m=2$	2 2−1 2+0 2 −1 0 −1 0 2 −1 0 −1 0 1
8_{16}	$m=3$	−1 1 5+0 −1 4+0 1 −1 1 0 −1 2+0 1 0 1 2+0 −1 3+0 −1 3+0 −1 0 1
8_{17}	$m=3$	−1 1 5+0 −1 4+0 1 0 −1 3+0 1 −1 0 1 3+0 1 −1 0 1 −1 3+0 −1 0 1
8_{18}	$m=3$	1 0 −1 4+0 1 5+0 −1 1 3+0 −1 0 1 −1 3+0 1 −1 1 −1 2+0 −1 2+0 1 −1

8_{19}	$m = 3$	−1 6+0 −1 1 3+0 1 0 −1 3+0 −1 0 1 −1 0 2+1 −1 2+0 −1 5+0 1 −1
8_{20}	$m = 2$	1 3+0 1 0 1 0 −1 0 4−1 2+0
8_{21}	$m = 2$	1 2+0 −1 0 1 −1 0 2−1 1 0 −2 −1 2+0
9_1	$m = 4$	3+0 −1 4+0 1 3+0 −1 4+0 1 3+0 −1 4+0 1 3+0 −1 4+0 1 3+0 −1 4+0 1 3+0 −1 4+0 1 3+0 −1 4+0 1
9_2	$m = 1$	4 −4 −3 4
9_3	$m = 3$	−1 3+0 1 2+0 −1 3+0 1 2+0 −1 3+0 7 2+0 −2 3+0 1 2+0 −1 3+0 1 2+0 −1
9_4	$m = 2$	3 0 −1 −2 0 1 0 −1 2+0 1 0 −3 2+0 3
9_5	$m = 1$	−2 0 1 −3
9_6	$m = 3$	2+0 −1 3+0 1 2+0 −1 3+0 2 −1 0 −1 2+0 −1 2 2+0 −1 3+0 1 2+0 −1 3+0 1
9_7	$m = 2$	3 0 −1 −2 0 1 4+0 1 −1 −2 −1 0 3
9_8	$m = 2$	−2 4+0 2 3−1 0 1 0 1 −1 0 1
9_9	$m = 3$	2 0 2−1 3+0 1 2+0 −1 0 −1 0 2 2+0 −1 3+0 1 2+0 −1 3+0 1 2+0 −1 3+0
9_{10}	$m = 2$	−2 2+0 2 0 −2 0 2+1 0 −1 0 1 2 0 −3
9_{11}	$m = 3$	−1 2+0 1 3+0 −1 2+0 1 3+0 −1 3+0 2+1 0 −2 4+0 1 0 −1 0 1 −1 3+0 1
9_{12}	$m = 2$	2 0 −2 2+0 1 −1 0 −2 0 3 0 1 −1 0 −1
9_{13}	$m = 2$	−1 2+0 1 0 −2 0 1 0 1 −2 1 0 1 2 −3
9_{14}	$m = 2$	1 3+0 −1 1 2+0 1 −1 −2 2+0 1 0 −1
9_{15}	$m = 2$	−1 3+0 2+1 4+0 −1 2+0 −1 1 −2
9_{16}	$m = 3$	−2 0 1 2+0 1 0 −1 0 1 3+0 1 −2 3+0 1 2+0 −1 2+0 1 3+0 −1 3+0 1 2+0 −1
9_{17}	$m = 3$	−2 1 4+0 1 −1 6+0 1 2+0 −1 1 −1 0 1 3+0 1 0 −1 1
9_{18}	$m = 2$	0 −1 3+0 −1 1 2 2+0 −1 0 2 2−1 2+0 1 −1 −2 −1 0 3
9_{19}	$m = 2$	2 −2 2+0 −1 2 2+0 1 0 −2 1 0 −1 1 −1
9_{20}	$m = 3$	1 2+0 −1 3+0 1 2+0 −1 3+0 1 −1 3+0 −1 0 2 4+0 −1 0 1 0 1 −1 3+0 −1
9_{21}	$m = 2$	1 −2 2+1 0 2 −3 3+0 1 −1 3+0 2+1
9_{22}	$m = 3$	−1 2+0 1 3+0 −1 1 3+0 1 0 −1 6+0 −1 2+0 1 −1 2+0 1 −1 0 1 −1 0 −1 2
9_{23}	$m = 2$	1 −1 3+0 3 2−1 0 −1 2 3−1 0 2
9_{24}	$m = 3$	1 5+0 1 −1 0 1 4+0 −1 5+0 1 −1 6+0 1 −1 0 −1 3+0 1
9_{25}	$m = 2$	2 −1 2+0 −1 2 −1 3+0 1 0 −1 1 0 −1
9_{26}	$m = 3$	−1 6+0 −1 0 1 4+0 −1 1 2+0 1 0 1 −2 2+0 1 0 −1 0 1 3+0 1 2−1 1
9_{27}	$m = 3$	1 −1 5+0 2 −1 3+0 −1 0 1 3+0 −1 1 0 −1 0 1 3+0 1 −1 0 1 4+0 −1

9_{28}	$m=3$	$1\ \ -1\ \ 5+0\ \ 1\ \ 6+0\ \ -1\ \ 3+0\ \ -1\ \ 0\ \ 1\ \ -1\ \ 6+0\ \ 1\ \ -1\ \ 3+0\ \ -1$ $0\ \ 1$
9_{29}	$m=3$	$-1\ \ 6+0\ \ 1\ \ 0\ \ -1\ \ 4+0\ \ 1\ \ 0\ \ -1\ \ 4+0\ \ 1\ \ 2+0\ \ 1\ \ -1\ \ 2+0\ \ 1$ $2+0\ \ 1\ \ -1\ \ 2+0\ \ -1$
9_{30}	$m=3$	$1\ \ -1\ \ 5+0\ \ 2\ \ -1\ \ 4+0\ \ -1\ \ 1\ \ 6+0\ \ -1\ \ 0\ \ 1\ \ -1\ \ 1\ \ 0\ \ 1\ \ -1\ \ 0$ $1\ \ 0\ \ -1\ \ 2+0\ \ -1$
9_{31}	$m=3$	$1\ \ -1\ \ 5+0\ \ 1\ \ 6+0\ \ 1\ \ -1\ \ 5+0\ \ 1\ \ 3+0\ \ 1\ \ -1\ \ 0\ \ -1$ $2+0\ \ -1\ \ 2+0\ \ 1\ \ -1$
9_{32}	$m=3$	$-1\ \ 6+0\ \ -1\ \ 0\ \ 1\ \ 4+0\ \ -1\ \ 5+0\ \ 1\ \ -1\ \ 2+0\ \ 1\ \ -1\ \ 2+0\ \ 1$ $2+0\ \ 0\ \ 2-1\ \ 1$
9_{33}	$m=3$	$1\ \ -1\ \ 5+0\ \ 1\ \ -1\ \ 3+0\ \ -1\ \ 0\ \ 2\ \ 3+0\ \ -1\ \ 1\ \ 0\ \ -1\ \ 0\ \ 1\ \ 3+0\ \ 1$ $-1\ \ 0\ \ 1\ \ 4+0\ \ -1$
9_{34}	$m=3$	$1\ \ -1\ \ 5+0\ \ 1\ \ 4+0\ \ 1\ \ 0\ \ -1\ \ 0\ \ 1\ \ 4+0\ \ -1\ \ 3+0\ \ -1\ \ 0\ \ 1\ \ -1$ $3+0\ \ 1\ \ -1\ \ 0\ \ 1$
9_{35}	$m=1$	$3\ \ -2\ \ -1\ \ 3$
9_{36}	$m=3$	$-1\ \ 6+0\ \ -1\ \ 2+0\ \ 1\ \ 3+0\ \ -1\ \ 4+0\ \ 1\ \ 0\ \ -1\ \ 4+0\ \ 1\ \ 0\ \ -1\ \ 0$ $1\ \ -1\ \ 3+0\ \ 1$
9_{37}	$m=2$	$2\ \ 3+0\ \ -1\ \ 1\ \ 2+0\ \ 1\ \ 2-1\ \ 2+0\ \ 1\ \ 0\ \ -1$
9_{38}	$m=2$	$2\ \ 0\ \ 2-1\ \ 0\ \ 2\ \ -1\ \ 2+0\ \ -1\ \ 2\ \ 3-1\ \ 0\ \ 2$
9_{39}	$m=2$	$1\ \ 4+0\ \ -2\ \ 0\ \ 1\ \ 0\ \ 1\ \ -2\ \ 1\ \ -1\ \ 2+1\ \ -2$
9_{40}	$m=3$	$1\ \ -1\ \ 5+0\ \ 1\ \ 4+0\ \ 1\ \ 0\ \ -1\ \ 1\ \ 3+0\ \ -1\ \ 0\ \ -1\ \ 4+0\ \ 1\ \ -1\ \ 1$ $4+0\ \ 1\ \ -1\ \ 1$
9_{41}	$m=2$	$1\ \ -1\ \ 3+0\ \ 1\ \ -1\ \ 2+0\ \ 1\ \ 0\ \ 2\ \ -1\ \ 0\ \ 2-1\ \ 2$
9_{42}	$m=2$	$3+0\ \ -1\ \ 0\ \ -1\ \ 2+0\ \ -1\ \ 3+0\ \ -1\ \ 2+1\ \ 0$
9_{43}	$m=3$	$-1\ \ 5+0\ \ 1\ \ 0\ \ -1\ \ 4+0\ \ -1\ \ 1\ \ 6+0\ \ -1\ \ 0\ \ 1\ \ -1\ \ 1\ \ 0\ \ 1\ \ -1\ \ 0$ $1\ \ 0\ \ -1\ \ 2+0\ \ -1$
9_{44}	$m=2$	$1\ \ 1\ \ 6+0\ \ 2+1\ \ 2-1\ \ 0\ \ -1\ \ 1\ \ 2+0$
9_{45}	$m=2$	$1\ \ 2\ \ -1\ \ 1\ \ 0\ \ -1\ \ 3+0\ \ 2+1\ \ 2+0\ \ -1\ \ 1\ \ -2\ \ 1$
9_{46}	$m=1$	$3\ \ -2\ \ -1\ \ 0$
9_{47}	$m=3$	$-1\ \ 6+0\ \ 1\ \ 0\ \ 1\ \ -1\ \ 2+0\ \ -1\ \ 1\ \ 0\ \ 1\ \ -1\ \ 3+0\ \ -1\ \ 6+0\ \ -1\ \ 1$
		$-1\ \ 2+0\ \ 1\ \ 0\ \ -1$
9_{48}	$m=2$	$1\ \ -1\ \ 4+0\ \ -1\ \ 0\ \ 1\ \ 0\ \ 1\ \ -1\ \ 1\ \ -1\ \ 2+1\ \ -2$
9_{49}	$m=2$	$1\ \ -1\ \ 4+0\ \ -2\ \ 0\ \ 1\ \ 0\ \ 1\ \ -2\ \ 1\ \ -1\ \ 2+1\ \ -2$
10_1	$m=1$	$1\ \ 4\ \ 0\ \ 1\ \ -1$
10_2	$m=4$	$-1\ \ 8+0\ \ 1\ \ 3+0\ \ -1\ \ 4+0\ \ 1\ \ 3+0\ \ -1\ \ 4+0\ \ 1\ \ 3+0\ \ -1\ \ 1$ $3+0\ \ 1\ \ 3+0\ \ -1\ \ 3+0\ \ -1\ \ 1\ \ 3+0\ \ -1\ \ 4+0\ \ 1\ \ 3+0\ \ -1$ $4+0\ \ 1$
10_3	$m=1$	$1\ \ -2\ \ 0\ \ 1\ \ 3$
10_4	$m=2$	$1\ \ -1\ \ 3+0\ \ 1\ \ 2+0\ \ -1\ \ 0\ \ 1\ \ 0\ \ 1\ \ -1\ \ 0\ \ -3$
10_5	$m=4$	$-1\ \ 3+0\ \ 1\ \ 4+0\ \ -1\ \ 3+0\ \ 1\ \ 4+0\ \ -1\ \ 5+0\ \ 1\ \ 2+0\ \ -1\ \ 5+0$ $1\ \ 2+0\ \ -1\ \ 5+0\ \ 1\ \ 2+0\ \ -1\ \ 8+0\ \ 1\ \ -1\ \ 1\ \ 2+0\ \ -1\ \ 3+0\ \ 1$
10_6	$m=3$	$2\ \ 2+0\ \ -2\ \ 3+0\ \ 1\ \ 2+0\ \ -1\ \ 3+0\ \ 1\ \ 3+0\ \ -1\ \ 0\ \ -1\ \ 2\ \ 2+0$ $-1\ \ 3+0\ \ 1\ \ 0\ \ -1\ \ 2+0\ \ 1\ \ 0\ \ -1$

10_7	$m=2$	1 3 0 −3 2+0 1 2+0 −2 −1 3 2+0 1 2−1
10_8	$m=3$	−2 5+0 2+1 6+0 1 0 −1 4+0 1 0 −1 0 −1 2+0 1 3+0 −1 2+0 1
10_9	$m=4$	−1 3+0 1 4+0 −1 8+0 −1 1 4+0 1 2+0 −1 5+0 1 2+0 −1 3+0 1 2+0 −1 0 1 8+0 1 −1 5+0 −1 0 1
10_{10}	$m=2$	1 −1 3+0 1 4+0 −3 2 −1 0 3 −3
10_{11}	$m=2$	2 2−1 0 −2 2 4+0 1 0 1 −1 0 −2
10_{12}	$m=3$	−2 0 2+1 3+0 −1 4+0 2 0 −2 4+0 1 0 −1 6+0 1 −1 1 0 1 2+0 1
10_{13}	$m=2$	1 −1 1 2+0 1 −2 2+0 1 −1 2 −2 0 1 −2 3
10_{14}	$m=3$	2 0 2−1 3+0 1 2+0 −1 0 −1 0 2 5+0 −1 1 2+0 −1 3+0 1 0 −1 2+0 1 0 −1
10_{15}	$m=3$	1 −1 2+0 1 3+0 −1 4+0 1 0 −1 4+0 1 0 −1 6+0 1 −1 1 0 −1 2+0 2
10_{16}	$m=2$	1 −1 1 3+0 −2 2+0 1 0 −1 0 1 −1 0 2
10_{17}	$m=4$	−1 2+0 1 5+0 −1 6+0 1 0 −1 6+0 1 0 −1 8+0 1 0 −1 6+0 1 0 −1 1 0 −1 3+0 1 5+0 −1 2+0 1
10_{18}	$m=2$	2 0 −1 0 −1 1 4+0 1 0 1 −1 0 −2
10_{19}	$m=3$	1 0 −1 4+0 1 0 −1 4+0 1 3+0 −1 2+0 1 6+0 −2 0 −1 0 1 0 1 −1
10_{20}	$m=2$	3 −3 2+0 −2 3 2+0 −1 0 1 0 1 −1 0 −1
10_{21}	$m=3$	1 3+0 −1 2+0 1 6+0 2 2−1 2+0 −1 0 1 4+0 −2 0 2 3+0 1 −1 0 −1
10_{22}	$m=3$	−2 0 1 4+0 −1 1 3+0 2 0 −2 3+0 1 0 −1 1 6+0 1 −1 3+0 −1 0 1
10_{23}	$m=3$	−1 2+0 1 3+0 −1 4+0 1 0 −2 4+0 1 0 −1 6+0 1 −1 1 0 1 2+0 1
10_{24}	$m=2$	2 0 −2 2+0 2 −1 0 −1 −2 3 2+0 1 2−1
10_{25}	$m=3$	2 2+0 2−1 2+0 1 6+0 1 0 −1 3+0 −1 1 2+0 2−1 2+0 2 0 −1 2+0 1 0 −1
10_{26}	$m=3$	1 0 −1 4+0 1 5+0 −1 1 3+0 −1 0 1 −1 1 0 1 0 −1 0 −2 4+0 1 0 −1
10_{27}	$m=3$	−1 5+0 1 −1 6+0 1 0 −1 4+0 2 3−1 1 0 −1 2 3+0 −1 2+0
10_{28}	$m=2$	1 −2 3+0 1 −2 3+0 2+1 −1 0 −1 0 1
10_{29}	$m=3$	1 2+0 −1 3+0 1 6+0 1 −1 3+0 2−1 2 3+0 1 −1 0 −1 3+0 1 −1 0 −1
10_{30}	$m=2$	2 −2 2+0 −1 3 −1 3+0 1 2+0 −1 1 −1
10_{31}	$m=2$	−1 1 3+0 −2 2+0 1 −1 2 −2 −1 1 −1 2
10_{32}	$m=3$	−1 1 5+0 −1 4+0 1 0 −1 3+0 1 −1 0 1 0 2−1 1 0 −1 2 −1 4+0 −1 2
10_{33}	$m=2$	2 −2 2+0 −1 2 3+0 1 −1 2+0 −1 1 −2
10_{34}	$m=2$	1 −1 3+0 1 2+0 −1 1 −3 3+0 1 −1

10_{35}	$m=2$	$1\ \ -1\ \ 2+0\ \ -1\ \ 2\ \ 2+0\ \ -1\ \ 0\ \ -2\ \ 0\ \ 1\ \ -1\ \ 0\ \ -1$
10_{36}	$m=2$	$1\ \ -1\ \ 3+0\ \ 3\ \ -1\ \ 3+0\ \ 1\ \ 2+0\ \ -1\ \ 1\ \ -1$
10_{37}	$m=2$	$2\ \ -2\ \ 2+0\ \ -1\ \ 2\ \ 2+0\ \ -1\ \ 1\ \ -2\ \ 3+0\ \ 1\ \ -1$
10_{38}	$m=2$	$-1\ \ 3+0\ \ -1\ \ 3\ \ 2-1\ \ 1\ \ -2\ \ 2\ \ 4+0\ \ 1$
10_{39}	$m=3$	$-1\ \ 5+0\ \ 1\ \ 2\ \ -1\ \ 4+0\ \ -1\ \ 2\ \ 0\ \ -1\ \ 4+0\ \ 1\ \ 0\ \ -1\ \ 0\ \ -1\ \ 2+0$ $1\ \ 3+0\ \ -1\ \ 2+0\ \ 1$
10_{40}	$m=3$	$1\ \ 5+0\ \ -1\ \ 6+0\ \ -2\ \ 2+0\ \ 2+1\ \ -1\ \ 0\ \ -1\ \ 0\ \ 1\ \ 2+0\ \ 7\ \ 0\ \ -1$ $0\ \ -1\ \ 2+1\ \ 0\ \ 1\ \ -2$
10_{41}	$m=3$	$1\ \ 0\ \ -1\ \ 3+0\ \ -1\ \ 2\ \ 6+0\ \ 1\ \ -1\ \ 3+0\ \ -1\ \ 0\ \ 1\ \ 2+0\ \ 1\ \ -1$ $2+0\ \ -1\ \ 1\ \ -1\ \ 1\ \ 0\ \ 2+1\ \ -2$
10_{42}	$m=3$	$1\ \ 5+0\ \ -1\ \ 1\ \ 6+0\ \ 1\ \ 2+0\ \ 1\ \ -1\ \ 1\ \ 0\ \ -1\ \ 0\ \ 2+1\ \ -1\ \ 0\ \ 1$ $-2\ \ 5+0\ \ 1\ \ -1$
10_{43}	$m=3$	$1\ \ 0\ \ -1\ \ 4+0\ \ 1\ \ -1\ \ 3+0\ \ -1\ \ 0\ \ 2\ \ 6+0\ \ -1\ \ 2+0\ \ 1\ \ -1\ \ 2+0$ $-1\ \ 0\ \ -1\ \ 0\ \ 2+1\ \ 0\ \ -1$
10_{44}	$m=3$	$1\ \ 2+0\ \ -1\ \ 2+0\ \ -1\ \ 2\ \ -1\ \ 5+0\ \ 1\ \ 4+0\ \ -1\ \ 0\ \ 1\ \ 2+0\ \ 1\ \ -1$ $2+0\ \ -1\ \ 3+0\ \ 1\ \ -1\ \ 0\ \ -1$
10_{45}	$m=3$	$1\ \ 0\ \ -1\ \ 3+0\ \ -1\ \ 2\ \ 5+0\ \ -1\ \ 1\ \ 3+0\ \ 1\ \ -1\ \ 0\ \ -1\ \ 0\ \ 1\ \ 0\ \ 1$ $-1\ \ 0\ \ -1\ \ 1\ \ -1\ \ 3+1\ \ 0\ \ -2$
10_{46}	$m=4$	$-1\ \ 5+0\ \ 1\ \ 2+0\ \ -1\ \ 8+0\ \ -1\ \ 0\ \ 1\ \ 6+0\ \ -1\ \ 0\ \ 1\ \ 5+0\ \ 1\ \ -1$ $3+0\ \ 1\ \ 4+0\ \ -1\ \ 3+0\ \ 1\ \ 4+0\ \ -1\ \ 0\ \ 1\ \ 3+0\ \ -1\ \ 2+0\ \ 1$
10_{47}	$m=4$	$1\ \ -1\ \ 7+0\ \ 1\ \ 6+0\ \ -1\ \ 0\ \ -1\ \ 3+0\ \ 1\ \ 4+0\ \ -1\ \ 3+0\ \ 1\ \ 4+0$ $-1\ \ 5+0\ \ 1\ \ 2+0\ \ -1\ \ 5+0\ \ 1\ \ 2+0\ \ -1\ \ 5+0\ \ 1\ \ 2+0\ \ -1$
10_{48}	$m=4$	$1\ \ 0\ \ -1\ \ 6+0\ \ 1\ \ 0\ \ -1\ \ 6+0\ \ 1\ \ 5+0\ \ -1\ \ 2+0\ \ 1\ \ 8+0\ \ -1$ $2+0\ \ 1\ \ 5+0\ \ -1\ \ 4+0\ \ -1\ \ 0\ \ 1\ \ 0\ \ -1\ \ 3+0\ \ 1\ \ 2+0\ \ 1\ \ 0\ \ -1$
10_{49}	$m=3$	$1\ \ 0\ \ -1\ \ 4+0\ \ 1\ \ 0\ \ -1\ \ 4+0\ \ 2\ \ 0\ \ 3-1\ \ 2+0\ \ 1\ \ 3+0\ \ -1\ \ 2+0$ $2\ \ -1\ \ 2+0\ \ -1\ \ 2+0\ \ 2$
10_{50}	$m=3$	$-1\ \ 6+0\ \ -2\ \ 0\ \ 2\ \ 4+0\ \ -1\ \ 0\ \ 1\ \ 2+0\ \ 2+1\ \ -2\ \ 2+0\ \ 1\ \ 3+0$ $-1\ \ 0\ \ 1\ \ 2+0\ \ -1\ \ 0\ \ 1$
10_{51}	$m=3$	$1\ \ -1\ \ 5+0\ \ 1\ \ 4+0\ \ -1\ \ 1\ \ -1\ \ 2+0\ \ 1\ \ 3+0\ \ -2\ \ 4+0\ \ 1\ \ 0\ \ -1$ $4+0\ \ 1\ \ 0\ \ -1$
10_{52}	$m=3$	$2\ \ -2\ \ 4+0\ \ -1\ \ 2\ \ 6+0\ \ -1\ \ 2+0\ \ 1\ \ 3+0\ \ -1\ \ 3+0\ \ -1\ \ 1\ \ 0$ $-1\ \ 2+0\ \ 1\ \ 0\ \ 1\ \ 0\ \ -1$
10_{53}	$m=2$	$1\ \ 2\ \ -2\ \ 2+0\ \ -1\ \ 3\ \ 3-1\ \ 0\ \ 2\ \ -1\ \ 0\ \ -1\ \ 0\ \ 2$
10_{54}	$m=3$	$2\ \ -1\ \ 5+0\ \ 1\ \ 6+0\ \ -1\ \ 2+0\ \ 1\ \ 3+0\ \ -1\ \ 2+0\ \ 1\ \ 0\ \ 1\ \ 0\ \ -1\ \ 0$ $-1\ \ 2+0\ \ 1\ \ 0\ \ -1$
10_{55}	$m=2$	$2\ \ -1\ \ 3+0\ \ 1\ \ -1\ \ 0\ \ 3\ \ -1\ \ 3+0\ \ 1$
10_{56}	$m=3$	$-2\ \ 1\ \ 0\ \ 1\ \ 2+0\ \ 1\ \ -2\ \ 2+0\ \ 1\ \ 0\ \ 1\ \ 0\ \ -1\ \ 4+0\ \ 1\ \ 0\ \ -1\ \ 6+0$ $-1\ \ 2+0\ \ 1\ \ -1\ \ 2+0\ \ 1$
10_{57}	$m=3$	$1\ \ -1\ \ 5+0\ \ 1\ \ 4+0\ \ -1\ \ 1\ \ -1\ \ 2+0\ \ 2+1\ \ -1\ \ 0\ \ -2\ \ 0\ \ 1\ \ 3+0$ $1\ \ -1\ \ 4+0\ \ 2+1\ \ -2$
10_{58}	$m=2$	$1\ \ 2\ \ -1\ \ 2+0\ \ -1\ \ 2\ \ 2+0\ \ 1\ \ 0\ \ -2\ \ 1\ \ 0\ \ -1\ \ 1\ \ -1$
10_{59}	$m=3$	$-2\ \ 1\ \ 5+0\ \ -1\ \ 0\ \ 1\ \ 2+0\ \ 1\ \ 0\ \ -1\ \ 5+0\ \ 1\ \ -1\ \ 2+0\ \ 1\ \ -1$ $2+0\ \ 1\ \ 2\ \ -1\ \ 1\ \ 0\ \ 2-1\ \ 2$

10_{60}	$m=3$	$1\ \ -1\ \ 4+0\ \ -1\ \ 2\ \ 6+0\ \ 1\ \ 3+0\ \ -1\ \ 2+0\ \ -1\ \ 1\ \ 0\ \ 1\ \ -1\ \ 2+0$ $-1\ \ 1\ \ 0\ \ 1\ \ -1\ \ 2+0\ \ -1$
10_{61}	$m=3$	$-1\ \ 2+0\ \ 1\ \ 3+0\ \ -1\ \ 2+0\ \ 1\ \ 3+0\ \ -1\ \ 4+0\ \ 1\ \ 0\ \ -1\ \ 4+0\ \ 1$ $0\ \ -1\ \ 3+0\ \ 1\ \ -1\ \ 0\ \ 2$
10_{62}	$m=4$	$-1\ \ 3+0\ \ 1\ \ 4+0\ \ -1\ \ 3+0\ \ 1\ \ 4+0\ \ -1\ \ 5+0\ \ 1\ \ 2+0\ \ -1\ \ 5+0$ $1\ \ 2+0\ \ -1\ \ 5+0\ \ 1\ \ 2+0\ \ -1\ \ 8+0\ \ 1\ \ -1\ \ 1\ \ 0\ \ -1\ \ 4+0\ \ 1$
10_{63}	$m=2$	$3\ \ 0\ \ -1\ \ -2\ \ 0\ \ 1\ \ 0\ \ -1\ \ 2+0\ \ 1\ \ 0\ \ -2\ \ 2+0\ \ 3$
10_{64}	$m=4$	$-1\ \ 8+0\ \ -1\ \ 0\ \ 1\ \ 6+0\ \ -1\ \ 0\ \ 1\ \ 5+0\ \ 1\ \ -1\ \ 4+0\ \ 1\ \ 3+0\ \ -1$ $8+0\ \ 1\ \ 0\ \ -1\ \ 1\ \ 2+0\ \ -1\ \ 2+0\ \ 1\ \ 0\ \ -1\ \ 2+0\ \ 1\ \ 2+0\ \ -1\ \ 1$
10_{65}	$m=3$	$-1\ \ 2+0\ \ 1\ \ 3+0\ \ -2\ \ 0\ \ 1\ \ 2+0\ \ 1\ \ 0\ \ -1\ \ 4+0\ \ 2\ \ 0\ \ -2\ \ 2+0$ $-1\ \ 1\ \ 2+0\ \ 1\ \ -1\ \ 1\ \ -1\ \ 3+0\ \ 1$
10_{66}	$m=3$	$2\ \ 2+0\ \ -1\ \ 0\ \ -1\ \ 0\ \ 1\ \ 2+0\ \ -1\ \ 3+0\ \ 1\ \ 2+0\ \ -1\ \ 3+0\ \ 1$ $2+0\ \ -1\ \ 3+0\ \ 1\ \ 2+0\ \ -1\ \ 3+0\ \ 2$
10_{67}	$m=2$	$-1\ \ 3+0\ \ 1\ \ 2\ \ 0\ \ -1\ \ 2+0\ \ 1\ \ -1\ \ 0\ \ -2\ \ 0\ \ 3$
10_{68}	$m=2$	$-1\ \ 3+0\ \ 1\ \ -1\ \ 4+0\ \ 3\ \ -2\ \ -1\ \ 0\ \ -1\ \ 2$
10_{69}	$m=3$	$-1\ \ 2+0\ \ 1\ \ 3+0\ \ -1\ \ 0\ \ 1\ \ 4+0\ \ -1\ \ 3+0\ \ 3+1\ \ -3\ \ 2+0\ \ 1\ \ -1$ $2+0\ \ 1\ \ 2+0\ \ 1\ \ -1\ \ 0\ \ -1\ \ 1$
10_{70}	$m=3$	$1\ \ 6+0\ \ 1\ \ 0\ \ 1\ \ -1\ \ 0\ \ -1\ \ 0\ \ -2\ \ 2+1\ \ 0\ \ 1\ \ 2+0\ \ -1\ \ 4+0\ \ 1\ \ 0$ $-1\ \ 3+0\ \ 1\ \ 2+0\ \ -1$
10_{71}	$m=3$	$-1\ \ 6+0\ \ -1\ \ 1\ \ 3+0\ \ 2+1\ \ -2\ \ 4+0\ \ -1\ \ 1\ \ 2\ \ -1\ \ 0\ \ -1\ \ 1\ \ 0$ $-1\ \ 1\ \ 0\ \ 1\ \ 0\ \ 2-1\ \ 0\ \ 1$
10_{72}	$m=3$	$-1\ \ 2+0\ \ 1\ \ 3+0\ \ -1\ \ 4+0\ \ 1\ \ 0\ \ -1\ \ 4+0\ \ 1\ \ 0\ \ -2\ \ 1\ \ 5+0\ \ -1$ $4+0\ \ 1\ \ -1$
10_{73}	$m=3$	$1\ \ -1\ \ 5+0\ \ 2-1\ \ 3+0-1\ \ 0\ \ 1\ \ 6+0\ \ 1\ \ 0\ \ 1\ \ -1\ \ 3+0\ \ -1\ \ 0\ \ 1$ $3+0\ \ 1\ \ -1$
10_{74}	$m=2$	$-1\ \ 3+0\ \ 2+1\ \ 0\ \ 3-1\ \ 2\ \ 4+0\ \ 2$
10_{75}	$m=3$	$-1\ \ 1\ \ 4+0\ \ 1\ \ -3\ \ 1\ \ 4+0\ \ 1\ \ -1\ \ 3+0\ \ 1\ \ -1\ \ 0\ \ 1\ \ 3+0\ \ 1\ \ 2-1$ $1\ \ 3+0\ \ 1\ \ 0\ \ -1\ \ 1$
10_{76}	$m=3$	$-2\ \ 2+0\ \ 1\ \ 3+0\ \ -1\ \ 0\ \ 1\ \ 4+0\ \ -1\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 1\ \ -2\ \ 2+0\ \ 1$ $3+0\ \ -1\ \ 0\ \ 1\ \ 2+0\ \ -1\ \ 0\ \ 1$
10_{77}	$m=3$	$1\ \ -1\ \ 5+0\ \ 1\ \ 6+0\ \ -1\ \ 2+0\ \ 1\ \ 3+0\ \ -1\ \ 4+0\ \ 1\ \ 0\ \ -1\ \ 0\ \ 1$ $-1\ \ 0\ \ 1\ \ 0\ \ -2$
10_{78}	$m=3$	$2\ \ -1\ \ 5+0\ \ 1\ \ -1\ \ 3+0\ \ -1\ \ 0\ \ 1\ \ 6+0\ \ 1\ \ -1\ \ 5+0\ \ 1\ \ 3+0\ \ 1$ $-1\ \ 0\ \ -1$
10_{79}	$m=4$	$1\ \ 0\ \ -1\ \ 6+0\ \ 1\ \ 0\ \ -1\ \ 6+0\ \ 1\ \ 5+0\ \ -1\ \ 2+0\ \ 1\ \ 4+0\ \ -1\ \ 1$ $2+0\ \ -1\ \ 2+0\ \ 1\ \ 5+0\ \ -1\ \ 6+0\ \ 1\ \ 0\ \ -1\ \ 6+0\ \ 1\ \ 0\ \ -1$
10_{80}	$m=3$	$2\ \ 0\ \ 2-1\ \ 3+0\ \ 1\ \ 0\ \ -1\ \ 3+0\ \ -1\ \ 2\ \ 0\ \ -1\ \ 0\ \ -1\ \ 2+0\ \ 2\ \ 2+0$ $-1\ \ 3+0\ \ 1\ \ 3+0\ \ -1\ \ 2+0\ \ 1$
10_{81}	$m=3$	$-1\ \ 6+0\ \ -1\ \ 5+0\ \ 1\ \ -1\ \ 6+0\ \ 2\ \ 3-1\ \ 1\ \ 2+0\ \ 1\ \ 0\ \ 1\ \ -1\ \ 0$ $-1\ \ 0$
10_{82}	$m=4$	$-1\ \ 8+0\ \ -1\ \ 1\ \ 5+0\ \ 1\ \ 0\ \ -1\ \ 8+0\ \ 1\ \ 2+0\ \ -1\ \ 5+0\ \ 1\ \ 2+0$ $-1\ \ 0\ \ 1\ \ -1\ \ 2+0\ \ 1\ \ 2+0\ \ -1\ \ 0\ \ 1\ \ 2+0\ \ -1\ \ 1\ \ 4+0\ \ -1$ $3+0\ \ 1$

10_{83}	$m=3$	$-2\ \ 2+0\ \ 2\ \ 3+0\ \ -1\ \ 4+0\ \ 1\ \ 0\ \ -1\ \ 3+0\ \ 2+1\ \ 0\ \ -2\ \ 3+0\ \ 1$ $-1\ \ 0\ \ 1\ \ -1\ \ 0\ \ -1\ \ 0\ \ 1\ \ 0\ \ 1$
10_{84}	$m=3$	$1\ \ -1\ \ 5+0\ \ 1\ \ 4+0\ \ -1\ \ 1\ \ -1\ \ 5+0\ \ 1\ \ -1\ \ 3+0\ \ -1\ \ 0\ \ 1\ \ -2$ $5+0\ \ 1\ \ -1$
10_{85}	$m=4$	$1\ \ 2+0\ \ -1\ \ 5+0\ \ 1\ \ 8+0\ \ 1\ \ 0\ \ -1\ \ 6+0\ \ 1\ \ 0\ \ -1\ \ 3+0$ $-1\ \ 2+0\ \ 15+0\ \ -1\ \ 2+0\ \ 1\ \ 5+0\ \ 1\ \ -1\ \ 0\ \ -1\ \ 4+0$ $-1\ \ 0\ \ 2+1\ \ -1$
10_{86}	$m=3$	$-1\ \ 0\ \ 1\ \ 4+0\ \ -2\ \ 1\ \ 4+0\ \ 2\ \ -2\ \ 3+0\ \ 1\ \ 0\ \ -1\ \ 1\ \ 6+0\ \ 1\ \ -1$ $2+0\ \ -1\ \ 0\ \ 1$
10_{87}	$m=3$	$-1\ \ 0\ \ 1\ \ 4+0\ \ -1\ \ 5+0\ \ 1\ \ -2\ \ 3+0\ \ 1\ \ 0\ \ -1\ \ 1\ \ 6+0\ \ 1\ \ -1$ $2+0\ \ 1\ \ -1\ \ 0\ \ 1$
10_{88}	$m=3$	$1\ \ 5+0\ \ 1\ \ -2\ \ 0\ \ 1\ \ 2+0\ \ -1\ \ 1\ \ -1\ \ 4+0\ \ 1\ \ 0\ \ -1\ \ 4+0\ \ 1\ \ -1\ \ 1$ $4+0\ \ 1\ \ -1\ \ 1$
10_{89}	$m=3$	$-1\ \ 1\ \ 2+0\ \ 1\ \ 2+0\ \ -1\ \ 3+0\ \ -1\ \ 1\ \ -1\ \ 1\ \ 4+0\ \ 1\ \ 0\ \ 1\ \ 6+0\ \ 1$ $-1\ \ 5+0\ \ 1$
10_{90}	$m=3$	$1\ \ 0\ \ -1\ \ 4+0\ \ 1\ \ 5+0\ \ -1\ \ 1\ \ 3+0\ \ -1\ \ 0\ \ 1\ \ -2\ \ 0\ \ 1\ \ 4+0\ \ -1\ \ 0$ $1\ \ -1\ \ 2+0\ \ 1\ \ -1$
10_{91}	$m=4$	$-1\ \ 1\ \ 7+0\ \ -1\ \ 0\ \ 1\ \ 6+0\ \ -1\ \ 7+0\ \ 1\ \ -1\ \ 4+0\ \ 1\ \ -1\ \ 2+0\ \ 1$ $0\ \ -1\ \ 6+0\ \ 1\ \ 0\ \ 2-1\ \ 0\ \ 1\ \ 3+0\ \ 1\ \ 5+0\ \ -1\ \ 2+0\ \ 1$
10_{92}	$m=3$	$-1\ \ 3+0\ \ 1\ \ 2+0\ \ -1\ \ 0\ \ 1\ \ 4+0\ \ -1\ \ 3+0\ \ 1\ \ 2+0\ \ -2\ \ 1\ \ 2+0$ $2\ \ -1\ \ 0\ \ -2\ \ 0\ \ 1\ \ 0\ \ -1\ \ 2+0\ \ 1$
10_{93}	$m=3$	$-1\ \ 5+0\ \ 1\ \ -2\ \ 6+0\ \ 1\ \ -1\ \ 3+0\ \ -1\ \ 0\ \ 1\ \ 0\ \ 2-1\ \ 1\ \ -1\ \ 0\ \ 1\ \ 0$ $1\ \ 4+0\ \ 1$
10_{94}	$m=4$	$1\ \ 0\ \ -1\ \ 5+0\ \ -1\ \ 1\ \ 8+0\ \ 1\ \ 8+0\ \ -1\ \ 4+0\ \ 1\ \ -1\ \ 2+0\ \ -1$ $0\ \ 1\ \ 2+0\ \ 1\ \ -1\ \ 2+0\ \ -1\ \ 0\ \ 1\ \ 5+0\ \ 1\ \ -1\ \ 0\ \ -1\ \ 0\ \ 2+1$ $3+0\ \ -1$
10_{95}	$m=3$	$1\ \ -1\ \ 5+0\ \ 1\ \ 6+0\ \ -2\ \ 1\ \ 0\ \ 1\ \ 2+0\ \ 1\ \ -2\ \ 2+0\ \ -1\ \ 2+1\ \ 0$ $-1\ \ 0\ \ 1\ \ -1\ \ 0\ \ 1\ \ 0\ \ -1$
10_{96}	$m=3$	$-1\ \ 6+0\ \ 1\ \ 6+0\ \ 1\ \ -1\ \ 2+0\ \ 1\ \ -1\ \ 0\ \ 7\ \ 3+0\ \ 1\ \ 0\ \ -1\ \ -2\ \ 1$ $2+0\ \ -1\ \ 2+1\ \ -1$
10_{97}	$m=2$	$1\ \ 1\ \ 3+0\ \ -1\ \ -2\ \ 1\ \ 0\ \ 1\ \ 0\ \ -2\ \ 1\ \ 0\ \ 2+1\ \ -2$
10_{98}	$m=3$	$-1\ \ 6+0\ \ 2\ \ 2+0\ \ 2-1\ \ 1\ \ 0\ \ 1\ \ 6+0\ \ 1\ \ -1\ \ 2+0\ \ 2-1\ \ 0\ \ 2\ \ 0\ \ -1$ $2+0\ \ -1\ \ 0\ \ 1$
10_{99}	$m=4$	$-1\ \ 8+0\ \ -1\ \ 1\ \ 7+0\ \ -1\ \ 1\ \ 4+0\ \ 1\ \ 2+0\ \ -1\ \ 4+0\ \ 1\ \ -1$ $2+0\ \ 1\ \ 0\ \ -1\ \ 2+0\ \ 1\ \ -1\ \ 2+0\ \ 1\ \ 0\ \ 2-1\ \ 0\ \ 1\ \ 3+0\ \ 1\ \ 5+0$ $-1\ \ 2+0\ \ 1$
10_{100}	$m=4$	$-1\ \ 1\ \ 7+0\ \ -1\ \ 6+0\ \ 1\ \ -1\ \ 1\ \ 2+0\ \ -1\ \ 3+0\ \ 1\ \ 0\ \ 1\ \ 3+0\ \ 2-1$ $3+0\ \ 1\ \ 0\ \ -1\ \ 6+0\ \ 1\ \ 5+0\ \ -1\ \ 2+0\ \ 1\ \ 3+0\ \ -1\ \ 4+0\ \ 1$
10_{101}	$m=2$	$1\ \ -3\ \ 0\ \ 3+1\ \ -1\ \ 3+0\ \ 1\ \ -2\ \ 0\ \ 1\ \ 0\ \ 1\ \ -2$
10_{102}	$m=3$	$1\ \ 0\ \ -1\ \ 4+0\ \ 1\ \ 5+0\ \ -1\ \ 1\ \ 3+0\ \ -1\ \ 0\ \ 1\ \ -2\ \ 0\ \ 2\ \ 4+0\ \ -1\ \ 0$ $1\ \ -1\ \ 0\ \ 2+1\ \ -2$
10_{103}	$m=3$	$-1\ \ 5+0\ \ 1\ \ -1\ \ 6+0\ \ 1\ \ -1\ \ 3+0\ \ -1\ \ 0\ \ 2\ \ 0\ \ -2\ \ -1\ \ 1\ \ -1\ \ 0\ \ 1$ $0\ \ 1\ \ 2+0\ \ -1\ \ 0\ \ 2$

10_{104}	$m=4$	1 7+0 −1 1 8+0 1 −1 5+0 −1 0 1 4+0 −1 2+0 1 −1 0 1 0 1 −1 3+0 −1 7+0 1 −1 5+0 1 2+0 −1
10_{105}	$m=3$	2 −1 4+0 −1 1 4+0 1 2−1 0 1 2+0 1 0 −1 0 1 3+0 1 −2 5+0 1 −1
10_{106}	$m=4$	1 8+0 1 −1 5+0 −1 0 1 5+0 −1 2+0 −1 1 3+0 1 0 −1 0 −1 2+0 1 5+0 −1 5+0 1 2+0 −1 6+0 1 0 −1
10_{107}	$m=3$	1 6+0 1 −1 5+0 1 6+0 −2 0 1 −1 2+0 1 −1 0 1 −1 0 1 1 0 −1
10_{108}	$m=3$	−1 6+0 −1 1 5+0 −1 1 2+0 1 2+0 −1 2+0 −1 0 1 0 2 −1 0 1 −1 2+0 1
10_{109}	$m=4$	−1 2+0 1 5+0 −1 6+0 1 0 −1 6+0 1 0 −1 8+0 1 0 −1 6+0 1 0 −1 1 2+0 −1 2+0 1 4+0 1 −1 2+0 1
10_{110}	$m=3$	−1 6+0 −1 6+0 2 2−1 4+0 1 2+0 −1 1 −1 0 1 0 1 0 −1 2+0 1
10_{111}	$m=3$	1 5+0 2−1 0 1 4+0 −1 0 1 3+0 1 −2 0 2+1 3+0 −1 2+0 1 1 0 1 0 −2
10_{112}	$m=4$	−1 8+0 −1 1 5+0 1 0 −1 6+0 1 −1 1 0 −1 6+0 1 0 −1 2+0 −1 3+0 1 0 −1 1 5+0 1 5+0 −1 2+0 1
10_{113}	$m=3$	1 −1 5+0 1 6+0 −2 0 1 3+0 1 −1 2+0 −1 1 0 1 −2 2+1 0 1 2+0 −1
10_{114}	$m=3$	1 0 −1 4+0 2 −1 4+0 −2 2 3+0 −1 0 1 −1 3+0 1 −1 1 −1 2+0 −1 2+0 1 −1
10_{115}	$m=3$	−1 6+0 −1 1 5+0 −1 3+0 1 −1 0 1 0 −1 0 1 −1 0 1 −1 3+0 −1 0 2
10_{116}	$m=4$	−1 1 7+0 −1 1 7+0 −1 5+0 1 −1 0 1 3+0 −1 −1 0 1 0 −1 3+0 1 2+0 1 5+0 −1 2+0 1 0 −1 4+0 −1 0 1 0 1
10_{117}	$m=3$	1 −1 5+0 1 6+0 −2 0 2+1 −1 2+1 −1 2+0 1 3+0 −1 2+0 −1 2+1 0 −2
10_{118}	$m=4$	−1 2+0 1 5+0 −1 6+0 1 0 −1 6+0 1 0 −1 5+0 1 −1 0 1 2+0 −1 5+0 1 −1 0 1 2+0 −1 2+0 1 2+0 −1 0 1 0 −1 0 1
10_{119}	$m=3$	1 0 −1 4+0 1 5+0 −1 1 6+0 −2 1 2+0 1 −1 1 −2 1 −1 0 2+1 0 −1
10_{120}	$m=2$	1 2 2−1 2+0 2 −1 0 −1 0 2 −1 0 −1 0 2
10_{121}	$m=3$	−1 5+0 1 −1 6+0 2 −1 2+0 −1 1 0 1 0 −1 1 0 −1 0 1 2+0 −1 3+0 1
10_{122}	$m=3$	−1 1 5+0 −2 1 3+0 2+1 −2 3+0 1 −1 0 1 −1 0 −1 0 1 0 1 −1 2+0 −1 2+0 1
10_{123}	$m=4$	−1 1 7+0 −1 8+0 −1 1 4+0 1 2+0 −1 4+0 1 −1 2+0 1 2+0 −1 2+0 1 −1 0 1 3+0 1 2+0 −1 0 1 0 −1 2+0 1 0 −1 0 1
10_{124}	$m=4$	−1 5+0 1 2+0 −1 8+0 −1 0 1 6+0 −1 0 1 5+0 1 −1 13+0 1 4+0 −1 3+0 1 4+0 −1 0 −1 3+0 1 2+0 −1

10_{125}	$m=3$	1 5+0 2−1 6+0 2 −1 4+0 −1 5+0 −1 0 1 0 1 2+0 −1 0 −1
10_{126}	$m=3$	1 1 5+0 −1 1 4+0 1 0 1 6+0 −1 4+0 1 −1 0 1 3+0 2+1 0
10_{127}	$m=3$	0 −1 0 1 3+0 2 0 2−1 3+0 1 2+0 −1 1 5+0 −1 3+0 1 2+0 −1 3+0 1
10_{128}	$m=3$	−1 1 5+0 −1 4+0 −1 0 −2 0 1 0 1 −1 0 −1 0 1 0 1 2 0 −2 5+0 1 −1
10_{129}	$m=2$	2 0 −1 2 2+1 2+0 −1 3+0 1 2−1 0
10_{130}	$m=2$	0 −1 2+0 −1 0 2 0 1 2 2+0 2−1 2 −1
10_{131}	$m=2$	0 −1 2+0 −1 2+0 2 0 2+1 0 1 2 2+0
10_{132}	$m=2$	3+0 −1 1 3+0 −1 0 1 2+0 −1 2+0
10_{133}	$m=2$	1 4+0 1 2+0 −1 0 1 0 2−1 1 −1
10_{134}	$m=3$	−1 5+0 1 −1 6+0 −2 0 1 3+0 1 −1 2+0 −1 1 0 1 −2
10_{135}	$m=2$	1 −1 3+0 −1 1 3+0 2+1 0 2−2 −1 −3
10_{136}	$m=2$	1 −1 3+0 1 −1 2+0 1 −1 0 2+1 0 1 0
10_{137}	$m=2$	1 1 −1 2+0 −1 1 −1 0 −1 1 −1 1 2 −3 0 −2
10_{138}	$m=3$	1 −1 4+0 −1 7+0 1 3+0 −1 2+0 −1 1 0 1 −1 2+0 −1 1 0 1 −1 2+0 −1
10_{139}	$m=4$	−1 2+0 1 5+0 −1 6+0 1 0 −1 6+0 1 0 −1 8+0 −1 8+0 −1 0 1 0 −1 0 1 0 1 −1 0 −1 1 2+0 1 2+0 −1
10_{140}	$m=2$	1 2+1 2+0 1 4+0 2+1 0 1 −1 0 −1
10_{141}	$m=3$	1 6+0 −1 4+0 1 2+0 −1 0 −1 2+1 0 −1 2+0 2+1 2+0 1 3+0 2−1 2+0
10_{142}	$m=3$	−1 6+0 −1 0 1 4+0 −1 0 1 3+0 1 −1 2+0 1 3+0 −1 0 −1 2+0 1 0 −2
10_{143}	$m=3$	3+0 −1 0 1 −1 1 6+0 1 3+0 1 −1 2+1 2+0 2 −1 2+1 −1 0 −1 1 2−1 1 0
10_{144}	$m=2$	2−1 2 2+0 2 0 −1 2 2−1 0 −1 2+0 1
10_{145}	$m=2$	1 0 1 0 −1 1 2+0 1 −1 2+1 2 −3 0 2
10_{146}	$m=2$	−1 3+0 2−1 0 2+1 −1 2 −1 0 2 −1 0
10_{147}	$m=2$	−1 1 3+0 −1 2 0 1 2 −4 2+0 1 −2 −1
10_{148}	$m=3$	−1 6+0 1 6+0 1 4+0 −1 2+1 2+0 −1 2+0 −1 0 1 2 0 −1 2+1 −2
10_{149}	$m=3$	4+0 −1 0 −1 1 4+0 1 0 1 5+0 −1 1 2+0 −1 9+0 −1 1 2+1 0 1 2+0 −1
10_{150}	$m=3$	−1 0 1 3+0 1 −1 0 −1 3+0 1 −1 5+0 −1 1 2+0 −1 2+0 1 −1 2+1 0 −1 2+0 −1
10_{151}	$m=3$	−1 6+0 −1 5+0 1 −1 7+0 −1 1 −1 1 2+0 1 0 1 −1 0 1 0 −1
10_{152}	$m=4$	1 2+0 −1 5+0 1−1 7+0 1 6+0 −1 0 1 4+0 1 −1 2+0 1 0 −1 6+0 1 0 −1 6+0 1 5+0 −1 2+0 1

10_{153}	$m=3$	$-2\ \ 2+0\ \ 2+1\ \ 2+0\ \ 7\ \ 0\ \ -1\ \ 2+0\ \ 1\ \ -1\ \ 4+0\ \ 1\ \ 7+0\ \ 1\ \ 0$ $-1\ \ 3+0\ \ -1\ \ 2+0\ \ 1$
10_{154}	$m=3$	$-1\ \ 6+0\ \ -1\ \ 5+0\ \ 1\ \ -1\ \ 6+0\ \ -2\ \ 0\ \ 1\ \ -1\ \ 1\ \ 0\ \ 1\ \ -1\ \ 0\ \ 1\ \ -1$ $0\ \ 1\ \ 0\ \ -1$
10_{155}	$m=3$	$4+0\ \ -1\ \ 2+0\ \ -1\ \ 4+0\ \ -1\ \ 5+0\ \ -1\ \ 3+1\ \ 2+0\ \ 3-1\ \ -2$ $5+0\ \ -1\ \ 2\ \ 1$
10_{156}	$m=3$	$1\ \ 6+0\ \ 1\ \ 6+0\ \ -1\ \ 0\ \ 1\ \ 3+0\ \ 1\ \ 3+0\ \ 1\ \ -1\ \ 0\ \ 1\ \ -1\ \ 2+0\ \ 1$ $0\ \ -1\ \ 0\ \ 1$
10_{157}	$m=3$	$-1\ \ 6+0\ \ -1\ \ 4\ \ 0\ \ -1\ \ 1\ \ -1\ \ 1\ \ 6+0\ \ -1\ \ 1\ \ -2\ \ 1\ \ -1\ \ 1\ \ -3\ \ 2$ $-1\ \ 1\ \ 0\ \ 2+1\ \ 0$
10_{158}	$m=3$	$-1\ \ 1\ \ 5+0\ \ -1\ \ 4+0\ \ -1\ \ 0\ \ -1\ \ 0\ \ 1\ \ 0\ \ 1\ \ -1\ \ 0\ \ 1\ \ 3+0\ \ 1\ \ 2\ \ 0$ $2-1\ \ 3+0\ \ -1\ \ 0\ \ 1$
10_{159}	$m=3$	$1\ \ 5+0\ \ 2-1\ \ 6+0\ \ 7\ \ 3+0\ \ -2\ \ -1\ \ 0\ \ 1\ \ 0\ \ 3-1\ \ 1\ \ -1\ \ 3+0\ \ -1$ $1\ \ -2\ \ -1\ \ 0$
10_{160}	$m=3$	$1\ \ -1\ \ 5+0\ \ -1\ \ 5+0\ \ 1\ \ -1\ \ 3+0\ \ -1\ \ 1\ \ 0\ \ -1\ \ 3+0\ \ -1\ \ 2+1$ $-1\ \ 3+0\ \ -1\ \ 0\ \ 1\ \ -1$
10_{161}	$m=3$	$1\ \ 6+0\ \ 1\ \ 4+0\ \ 2-1\ \ 1\ \ 3+0\ \ 1\ \ 0\ \ -1\ \ 1\ \ 3+0\ \ 1\ \ 0\ \ -1\ \ 1\ \ 2+0$ $-2\ \ 2+1\ \ -1\ \ 1$
10_{162}	$m=2$	$-1\ \ 3+0\ \ -1\ \ 1\ \ 2+0\ \ -1\ \ 2+1\ \ 0\ \ 2\ \ -2\ \ -1\ \ 3$
10_{163}	$m=3$	$1\ \ 0\ \ -1\ \ 4+0\ \ -1\ \ 2\ \ 4+0\ \ 1\ \ -1\ \ 3+0\ \ -1\ \ 0\ \ 1\ \ -1\ \ 3+0\ \ 1\ \ -1$ $1\ \ -1\ \ 2+0\ \ -1\ \ 2+0\ \ 1\ \ -1$
10_{164}	$m=2$	$-1\ \ 5+0\ \ 1\ \ 0\ \ -1\ \ 2+1\ \ 3\ \ -2\ \ 1\ \ 5\ \ 6$
10_{165}	$m=2$	$-2\ \ 0\ \ 1\ \ 0\ \ 1\ \ 2+0\ \ -1\ \ 0\ \ 1\ \ -2\ \ 2+0\ \ -1\ \ 1\ \ 0$

Table III contains the invariant $\lambda(\zeta)$ computed by G. Wenzel and U. Lüdicke. It is given for prime numbers p with $p|T_r,\ p\not| T_{r-1}$ (see Table I), ζ is a primitive p-th root of unity. (Compare 14.11.)

The sequences printed are $a_1, a_2, \dots, a_{\frac{p-1}{2}}$ computed for the knot indicated and its mirror image where $\lambda(\zeta) = \sum_{k=1}^{p-1} a_k\zeta^k$, $a_k = a_{p-k}$. From the class $[\lambda(\zeta)]$ always a lexicographically first (and unique) member was chosen. If the two sequences do not coincide the knot is shown to be non-amphicheiral by this invariant.

The following formulae allow to compute the linking number ν_{ij} and μ_{ij} of the regular and irregular dihedral branched coverings $\hat{R}_p$ and $\hat{I}_p$, see Section 14 C:

$$2\nu_{0j} = a_j - \frac{1}{p}\sum_{k=1}^{p-1} a_k,$$

$$\mu_{ij} = 2\nu_{0j}, \quad \mu_{ij} = \nu_{0,i-j} + \nu_{0,i+j}.$$

A blank in the table indicates that either no admissible prime p exists or that no result was obtained due to computer overflow. Table III contains $\lambda(\zeta)$ for knots with less than ten crossings. It was computed, though, for knots with ten crossings, but the material seemed to be too voluminous to be included here.

Further invariants are available under:

`http://www.pims.math.ca/knotplot/`

`http://dowker.math.utk.edu/knotscape.html`

Table III

Knot: 3_1 $p = 3$					
6					
-6					
Knot: 4_1 $p = 5$					
-2	2				
-2	2				
Knot: 5_1 $p = 5$					
10	10				
-10	-10				
Knot: 5_2 $p = 7$					
2	6	6			
-6	-6	-2			
Knot: 6_1 $p = 3$					
-6					
6					
Knot: 6_2 $p = 11$					
2	6	2	6	6	
-6	-6	-6	-2	-2	
Knot: 6_3 $p = 13$					
-2	-2	2	-2	2	2
-2	-2	2	-2	2	2
Knot: 7_1 $p = 7$					
14	14	14			
-14	-14	-14			
Knot: 7_2 $p = 11$					
2	2	6	6	6	
-6	-6	-2	-6		

Knot: 7_3 $p = 13$ -10 -10 -10 -10 -6 -6 6 10 6 10 10 10
Knot: 7_4 $p = 3$ 6 -6
Knot: 7_4 $p = 5$ -14 -6 6 14
Knot: 7_5 $p = 17$ 6 6 10 10 6 10 10 10 -10 -10 -10 -10 -6 -10 -6 -6
Knot: 7_6 $p = 19$ 2 2 2 6 6 6 2 6 6 -6 -6 -6 -2 -6 -6 -2 -2 -2
Knot: 7_7 $p = 3$ -18 18
Knot: 7_7 $p = 7$ 2 2 10 -10 -2 -2
Knot: 8_1 $p = 13$ -2 -2 -2 2 2 2 -2 -2 2 2 -2 2
Knot: 8_2 $p = 17$ 10 6 10 6 10 10 10 -10 -10 -10 -10 -6 -6 -6
Knot: 8_3 $p = 17$ -2 -2 -2 2 2 -2 2 2 -2 -2 -2 2 2 -2 2 2
Knot: 8_4 $p = 19$ 2 2 6 2 6 6 2 6 6 -6 -6 -6 -6 -2 -6 -2 -2 -2

Knot: 8_5 $p = 3$

-12

12

Knot: 8_5 $p = 7$

$-6 \quad 6 \quad 14$

$-14 \quad 6 \quad -6$

Knot: 8_6 $p = 23$

$2 \quad 2 \quad 2 \quad 6 \quad 6 \quad 2 \quad 2 \quad 6 \quad 6 \quad 6 \quad 6$

$-6 \quad -6 \quad -6 \quad -6 \quad -2 \quad -6 \quad -2 \quad -2 \quad -6 \quad -2 \quad -2$

Knot: 8_7 $p = 23$

$-6 \quad -6 \quad -6 \quad -6 \quad -2 \quad -6 \quad -2 \quad -6 \quad -2 \quad -2 \quad -2$

$2 \quad 2 \quad 6 \quad 2 \quad 2 \quad 6 \quad 6 \quad 2 \quad 6 \quad 6 \quad 6$

Knot: 8_8 $p = 5$

$6 \quad 14$

$-14 \quad -6$

Knot: 8_9

$-2 \quad 2$

$-2 \quad 2$

Knot: 8_{10} $p = 3$

0

0

Knot: 8_{11} $p = 3$

-18

18

Knot: 8_{12} $p = 29$

$-2 \quad -2 \quad -2 \quad -2 \quad 2 \quad -2 \quad 2 \quad -2 \quad 2 \quad 2 \quad -2 \quad 2 \quad 2 \quad 2$

$-2 \quad -2 \quad -2 \quad -2 \quad 2 \quad -2 \quad 2 \quad -2 \quad 2 \quad 2 \quad -2 \quad 2 \quad 2 \quad 2$

Knot: 8_{13} $p = 29$

$-2 \quad -2 \quad -2 \quad -2 \quad -2 \quad 2 \quad 2 \quad 2 \quad 2 \quad -2 \quad -2 \quad 2 \quad 2 \quad 2$

$-2 \quad -2 \quad -2 \quad 2 \quad -2 \quad -2 \quad -2 \quad 2 \quad 2 \quad -2 \quad 2 \quad 2 \quad 2 \quad 2$

Knot: 8_{14} $p = 31$

$2 \quad 2 \quad 2 \quad 6 \quad 2 \quad 2 \quad 6 \quad 6 \quad 2 \quad 2 \quad 6 \quad 6 \quad 6 \quad 6$

$-6 \quad -6 \quad -6 \quad -6 \quad -6 \quad -6 \quad -2 \quad -2 \quad -2 \quad -2 \quad -6 \quad -6 \quad -2 \quad -2$

6

-2

Knot: 8_{15} $p = 3$

12
−12

Knot: 8_{15} $p = 11$

10	18	22	30	30
−30	−22	−10	−30	−18

Knot: 8_{16} $p = 5$

18	42
−42	−18

Knot: 8_{16} $p = 7$

−42	−2	−26
2	26	42

Knot: 8_{17} $p = 37$

−6	6	−2	−2	−2	6	2	−6	2	−2	6	−6	2	2	6	−6	−2
−6	6	−2	−2	−2	6	2	−6	2	−2	6	−6	2	2	6	−6	−2
2																
2																

Knot: 8_{18} $p = 5$

−4	4
−4	4

Knot: 8_{19} $p = 3$

12
12

Knot: 8_{20} $p = 3$

0
0

Knot: 8_{21} $p = 3$

12
−12

Knot: 8_{21} $p = 5$

14	26
−26	−14

Knot: 9_{1} $p = 3$

18
−18

Knot: 9_2 $p = 3$

6
−6

Knot: 9_2 $p = 5$

−2	2
−2	2

Knot: 9_3 $p = 19$

−14	−14	−14	−14	−14	−14	−10	−10	−10
10	14	10	14	10	14	14	14	14

Knot: 9_4 $p = 3$

6
−6

Knot: 9_4 $p = 7$

10	14	18
−18	−10	−14

Knot: 9_6 $p = 3$

18
−18

Knot: 9_7 $p = 29$

6	6	6	6	10	10	10	6	6	10	10	10
−10	−10	−10	−10	−6	−10	−10	−10	−6	−6	−6	−10
10	10										
−6	−6										

Knot: 9_8 $p = 31$

2	2	2	2	6	6	2	2	6	6	6	6	2	6
−6	−6	−6	−6	−6	−2	−2	−6	−6	−6	−2	−2	−2	−2
6													
−2													

Knot: 9_9 $p = 31$

10	10	10	14	14	10	10	14	14	14	10
−14	−14	−14	−14	−14	−14	−10	−14	−14	−10	−10
14	14	14	14							
−14	−10	−10	−10							

Knot: 9_{10} $p = 3$

18
−18

Knot: 9_{10} $p = 11$

−18	−10	−18	−10	−10
10	10	10	18	18

Knot: 9_{11} $p = 3$

−18
18

Knot: 9_{11} $p = 11$

−10	−6	−6	−2	2
−2	10	2	6	6

Knot: 9_{12} $p = 5$

14	26
−26	−14

Knot: 9_{12} $p = 7$

2	2	10
−10	−2	−2

Knot: 9_{13} $p = 37$

−10	−10	−10	−10	−10	−10	−6	−10	−6	−10	−6	−6
6	6	6	6	10	10	6	6	6	10	10	10
−10	−6	−10	−6	−6	−6	−6					
6	10	10	10	10	10	2					

Knot: 9_{14} $p = 37$

−2	−2	−2	2	−2	−2	−2	2	2	−2	−2	2	2	−2	2	2
−2	−2	−2	−2	−2	−2	−2	2	2	2	2	2	−2	−2	2	2
2	2														
2	2														

Knot: 9_{15} $p = 13$

18
18

Knot: 9_{16} $p = 13$

−14	−2	−10	−6	−10	−10
2	6	10	10	10	14

Knot: 9_{16} $p = 3$

−12
12

Knot: 9_{16} $p = 13$

−10	−10	−2	2	10	10
−10	2	10	−10	10	−2

Knot: 9_{17} $p = 3$

−30
30

Knot: 9_{17} $p = 13$

6	6	6	6	14	14
−14	−6	−14	−6	−6	−6

Knot: 9_{18} $p = 41$

6	6	6	10	6	6	6	10	10	6	6	10	10
−10	−10	−10	−10	−10	−10	−10	−10	−6	−6	−6	−6	−6
6	10	10	10	10	10	10	−6					
−10	−10	−10	−6	−6	−6	−6	2					

Knot: 9_{19} $p = 41$

−2	−2	−2	−2	2	−2	−2	2	−2	2	−2	−2	2
−2	−2	−2	−2	−2	−2	2	−2	2	−2	2	2	2
−2	2	2	2	2	2	2						
2	−2	2	−2	2	2	2						

Knot: 9_{20} $p = 41$

6	6	6	6	6	10	10	10	6	6	6	10	10
−10	−10	−10	−10	−10	−6	−10	−10	−10	−10	−6	−6	−6
10	10	10	6	10	10	10						
−10	−10	−6	−6	−6	−6	−6						

Knot: 9_{21} $p = 43$

−6	−6	−6	−6	−6	−6	−2	−6	
2	2	2	2	6	2	2	2	
−2	−2	−6	−2	−6	−2	−2	−2	−2
6	6	6	6	2	6	6	6	6
2	−6	−2	−6					
6	6	2	2					

Knot: 9_{22} $p = 43$

−2	−2	−2	−2	2	2	2	6	10	10	14	14	18
−26	−2	−18	−10	−10	−22	2	−26	2	−26	−2	−18	−14
18	22	22	22	26	26	26	26					
−6	−22	2	−26	2	−22	−2	−14					

Knot: 9_{23} $p = 3$

18
−18

Knot: 9_{23} $p = 5$

−18	−2
2	18

Knot: 9_{24} $p = 3$

0
0

Knot: 9_{24} $p = 5$

−2	2
−2	2

Knot: 9_{25} $p = 47$

6	6	6	6	10	10	10	14	14	18	18	22
−34	−14	−18	−30	−6	−34	−14	−18	−30	−6	−34	−10
22	22	26	30	30	30	34	34	34	34	34	
−22	−30	−6	−34	−10	−22	−26	−6	−34	−10	−22	

Knot: 9_{26} $p = 47$

−6	−6	−6	−6	−2	−6	−6	−6	−6	−2	−2	−6	−6
2	2	2	2	2	2	6	2	6	2	6	2	6
−6	−2	−2	−2	−6	−2	−2	−2	−2	−2			
6	6	6	2	6	2	6	6	6	6			

Knot: 9_{27} $p = 7$

−6	−6	−2
2	6	6

Knot: 9_{28} $p = 3$

12
−12

Knot: 9_{28} $p = 17$

6	6	14	14	18	26	26	26
−26	−18	−6	−26	−14	−6	−26	−14

Knot: 9_{29} $p = 3$

30
−30

Knot: 9_{29} $p = 17$

−10	6	−10	6	−6	6	2	6
−6	−6	−6	−6	−2	6	10	10

Knot: 9_{30} $p = 53$

2	2	2	2	2	6	6	6	10	10	14	14
30	−10	−14	−26	−2	−30	−2	−22	−18	−6	−30	−2
14	18	18	22	22	26	26	26	26	30	30	30
−26	10	−14	−26	−2	−30	−6	−22	−18	−6	−30	−2
30	30										
−26	−14										

Knot: 9_{31} $p = 5$

22	38
−38	−22

Knot: 9_{31} $p = 11$

−14	2	−6	2	−6
−2	−2	6	6	14

Knot: 9_{32} $p = 59$

−10	−10	−6	−6	−2	−2	2	−2	−6	−6	−6	−2	−2	−2
−2	2	6	6	10	2	6	6	6	6	−2	6	6	2
2	−2	−2	2	2	−2	−6	−2	−10	−10	−6	−6		
2	2	10	2	−2	2	6	10	2	6	10	2		
−6	−6	−6											
2	−2	6											

Knot: 9_{33} $p = 61$

2	−2	2	2	2	−2	2	2	2	6	−2	−2
2	−2	6	2	6	2	2	2	6	−6	2	−2
2	6	−2	6	6	−2						
2	−6	−2	−2	−2	−2						

Knot: 9_{34} $p = 3$

$-42/5$
$42/5$

Knot: 9_{34} $p = 23$

$-1498/277$	$-1022/277$	$206/277$	$-366/277$	$-250/277$	$-214/277$
$-206/277$	$214/277$	$526/277$	$438/277$	$146/277$	$250/277$
$-802/277$	$-146/277$	$-526/277$	$-878/277$	$-438/277$	
$1022/277$	$1498/277$	$366/277$	$802/277$	$878/277$	

Knot: 9_{35}

Knot: 9_{36} $p = 37$

-6	-6	-6	-2	-2	-2	2	6	6	10	10	14	18
-22	-2	-6	-18	6	-22	2	-6	-18	6	-22	2	-10
18	22	22	22	22								
-14	6	-22	2	-10								

Knot: 9_{37} $p = 5$

-2	2
-2	2

Knot: 9_{38} $p = 3$

42
-42

Knot: 9_{38} $p = 19$

-6	6	-6	10	-2	10	6	10	10
-10	-10	-6	6	6	2	-10	-10	-6

Knot: 9_{39} $p = 5$

-22	-18
18	22

Knot: 9_{39} $p = 11$

$-242/23$	$-142/23$	$-82/23$	$-190/23$	$-70/23$
$70/23$	$242/23$	$190/23$	$142/23$	$82/23$

Knot: 9_{40} $p = 3$

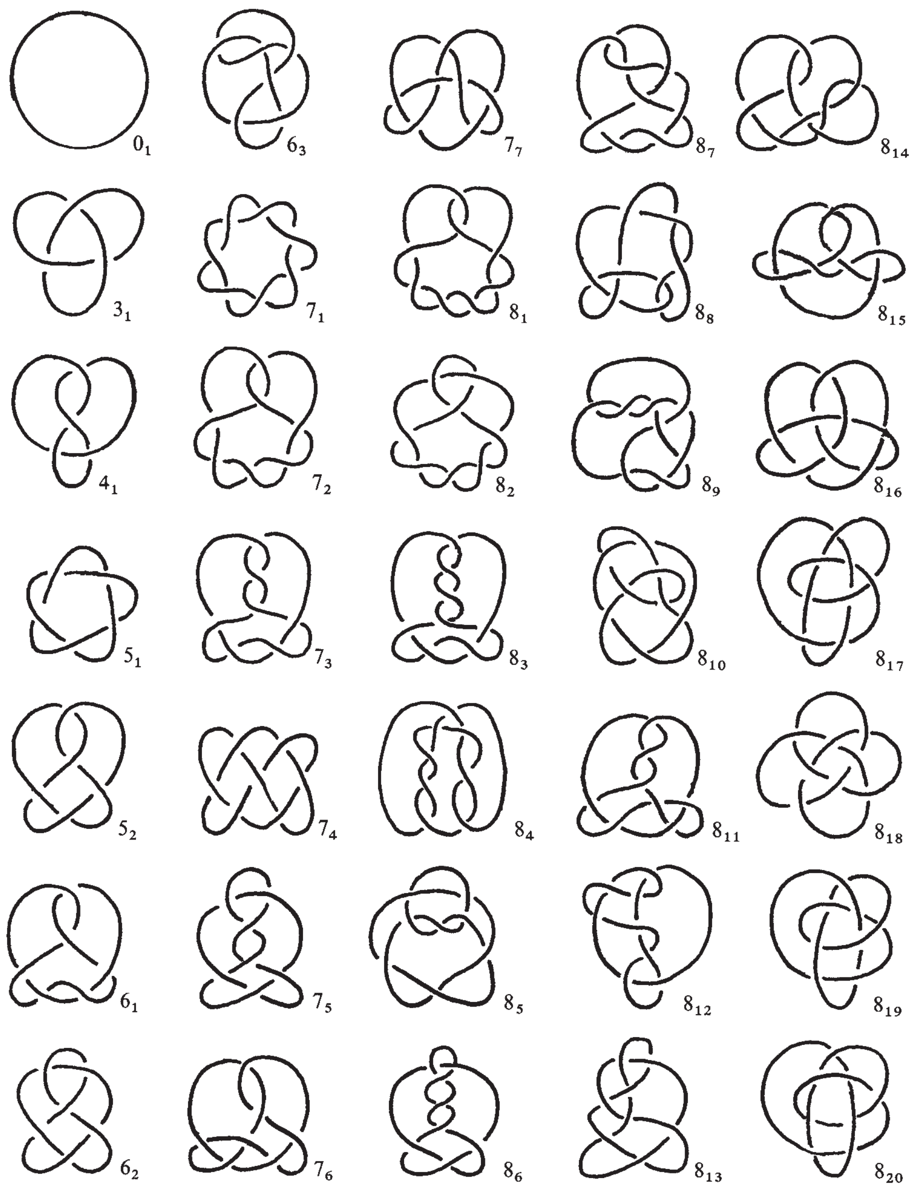
12
-12

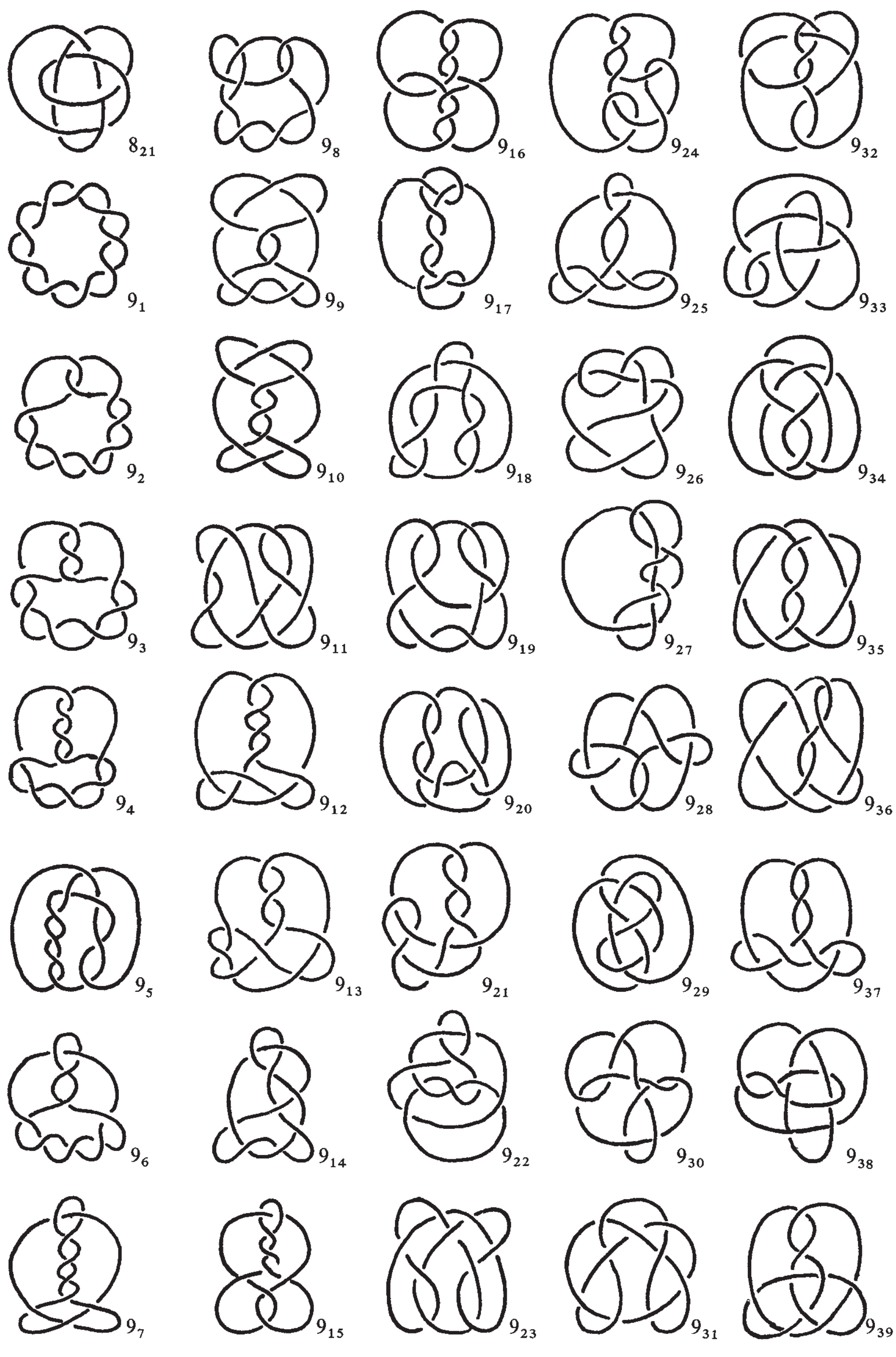
Knot: 9_{41}

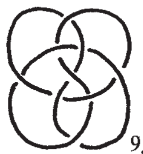
Knot: 9_{42} $p = 7$										
2	14	26								
−26	−2	−14								
Knot: 9_{43} $p = 13$										
−6	−2	6	14	18	22					
−22	6	−18	2	−14	−6					
Knot: 9_{44} $p = 17$										
2	6	10	14	18	26	30	30			
−30	−10	−14	−26	−2	−30	−6	−18			
Knot: 9_{45} $p = 23$										
6	6	10	14	18	22	26	30	30	34	34
−34	−10	−26	−22	−14	−30	−6	−34	−6	−30	−18
Knot: 9_{46}										
Knot: 9_{47}										
Knot: 9_{48}										

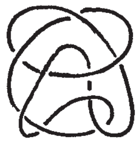
Appendix D

Knot Projections 0_1–9_{49}




8_{21}
9_1
9_2
9_3
9_4
9_5
9_6
9_7
9_8
9_9
9_{10}
9_{11}
9_{12}
9_{13}
9_{14}
9_{15}
9_{16}
9_{17}
9_{18}
9_{19}
9_{20}
9_{21}
9_{22}
9_{23}
9_{24}
9_{25}
9_{26}
9_{27}
9_{28}
9_{29}
9_{30}
9_{31}
9_{32}
9_{33}
9_{34}
9_{35}
9_{36}
9_{37}
9_{38}
9_{39}


9_{40}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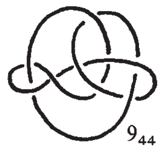
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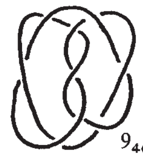
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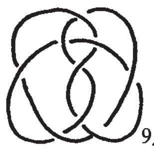
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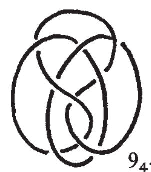
9_{43}


9_{45}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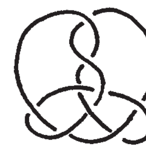
9_4

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In addition to the usual data each title contains one or more code numbers indicating the particular fields the paper belongs to (e.g., <K16>, knot groups; <M>, 3-dimensional topology).

A complete list of these code numbers and the corresponding fields is given on page 507.

In addition on pages 508–548 there are listed, for each code, all items in the bibliography having this code number.

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List of Code Numbers

General classification

- A algebraic topology
- B geometric topology
- F surface theory, Fuchsian groups
- G combinatorial group theory
- K knot theory
- M 3-dimensional topology
- X further fields

Knot Theory

- K11 books, survey articles
- K12 general theory of knots and links
- K13 tables
- K14 elementary geometric constructions, knotting numbers etc.
- K15 surfaces spanned by knots, genus of knots, surfaces in the complement
- K16 knot groups
- K17 companion, product, prime, satellite and cable knots, etc.
- K18 fibred knots
- K19 Property P and related problems, knot complements
- K20 knots and 3-manifolds: branched coverings
- K21 knots and 3-manifolds: surgery
- K22 knots and periodic maps of 3-manifolds
- K23 symmetries of knots and links
- K24 knot cobordisms, concordance
- K25 Alexander module
- K26 Alexander polynomial, Conway polynomial
- K27 quadratic forms and signatures of knots, braids and links
- K28 representations of knot and braid groups
- K29 algorithmic questions, calculation of knot invariants, determination of numbers of special knots
- K30 bridges, 2-bridge knots, bridge number, tunnels, tunnel number, tangles
- K31 alternating knots
- K32 algebraic knots and links
- K33 slice knots and links
- K34 singularities and knots and links
- K35 further special knots
- K36 Jones and HOMFLY polynomials, Conway function, Kauffman brackets and polynomials, skein method, A-polynomial
- K37 knots and physics, chemistry or biology, quantum groups
- K38 differential geometric properties of knots (curvature, integral invariants)
- K40 braids, braid groups
- K45 singular knots, Vassiliev invariants, Fiedler invariants
- K50 links (special articles on links with more than one component)
- K55 wild knots
- K59 properties of 1-knots not classified above
- K60 higher dimensional knots
- K61 $S^2 \subset S^4$

List of Authors According to Codes

K11 books, survey articles

Adams 1994
Aigner–Seidel 1995
Aitchison 1989
Alexander 1932
Atiyah 1996
Bing 1983
Birman 1974, 1991, 1991$'$, 1992, 1993
Boltyanskij–Efremovich 1982
Burde 1993
Burde–Zieschang 1985
Cappell 1992
Crowell–Fox 1963
Eisenbud–Neumann 1985
Epple 1999, 1999$'$
Eudave-Muños 1987
Farmer–Stanford 1996
Fiedler 2001
Fox 1952$'$, 1962, 1962$''$
van der Geer 1999
Gilbert–Porter 1994
Gordon 1978, 1999
Gramain 1991$'$
Hacon 1985
Hansen 1989
de la Harpe 1988
Hemion 1992
Hillman 1981$'$, 1989
Jablan 1999
Jones 1993, 2000
Kaiser 1997
Kauffman 1987, 1989$^{\mathrm{IV}}$, 1992$'$, 1997$'''$, 2000
Kawauchi 1996, 1996$''$
Kervaire–Weber 1978
Kirby 1978$'$, 1997
Lee 1996
Levine 1975, 1980
Lickorish 1997
Lickorish–Millett 1988
Lin 1979
Livingston 1993
Magnus 1973
Manzoli Neto 1987
Moran 1983
Morgan–Bass 1984
Morton 1988$'$, 2000
Murakami 1991
Murasugi 1974, 1991, 1996
Neuwirth 1965, 1974
Ohtsuki 1997
Papakyriakopoulos 1958
Prasolov–Sosinskii 1993, 1997
Prieto 2000
Przytycki 1995, 1995$'''$, 1998$'$
Reidemeister 1932
Rolfsen 1976
Rolin 1989
Rudolph 1985, 1989
Scharlemann 1992, 1998
Seifert 1936$'$
Seifert–Threlfall 1950
Simon 1988
Sossinsky 1999, 2000
Stillwell 1980
Sumners 1987$'$, 1990, 1992
Suzuki 1994
Thistlethwaite 1985
Thurston 1982
Tietze 1942
Traczyk 1995
Traldi 1988
Vassiliev 1990

Zhang 1991′
Zieschang 1993
Zieschang–Chernavsky 1963

K12 general theory of knots and links

Akbuiut–King 1981
Alexander 1923′, 1928
Ashley 1944
Banchoff 1976
Bing–Klee 1964
Birman 1976
Bothe 1974, 1981
Brauner 1928
Brunn 1892, 1892′, 1897
Călugăreanu 1961, 1961′, 1961″, 1970, 1970′
Caudron 1981
Clark 1978′
Conway 1970
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Edwards 1962
Eilenberg 1936
Farber 1981″, 1981$^{\mathrm{IV}}$
Fenn–Rourke 1979
Ghrist 1995
Ghrist–Holmes–Sullivan 1997
Ghrist–Young 1998
Giller 1982
Gilmer 1982
Gluck 1963
Goodrick 1969, 1970
Grot–Rovelli 1996
Habegger 1997
Hanner 1983
Hartley 1983″
Hotz 1959, 1960
Joyce 1982
Kauffman 1981
Kinoshita 1986
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Kurlin 1999, 1999′
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Martin 1974
Matveev 1981, 1982′
Milnor 1957
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Moran 1983
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Murasugi 1961″
Nanyes 1994
Nejinskii 1976
Neuzil 1973
Penney 1972
Quách 1983, 1983′
Reidemeister 1926′, 1960
Rolfsen 1972
Schmid 1963
Shibuya 1974
Simon 1973
Skora 1991
Smythe 1967′
Stoel 1962
Svetlov 2001
Tait 1898
Terasaka 1960′
Treybig 1968, 1971, 1971′
Tsau 2001
Viro 2001
Weber 1982
Wendt 1937
Yajima–Kinoshita 1957

K13 tables

Adams–Hildebrand–Weeks 1991
Alexander–Briggs 1927
Arnold–Au–Candy–Erdener–Fan–Flynn–Muir–
–Wu–Hoste 1994
Ashley 1944

Burde–Zieschang 1985
Caudron 1981
Doll–Hoste 1991
Dunfield 2001
Gabai 1986′
Hartley 1983′
Hoste–Thistlethwaite–Weeks 1998
Jaeger 1993
Kanenobu–Sumi 1992
Perko 1979
Rolfsen 1976
Simon 1987
Tait 1898
Thistlethwaite 1985

K14 elementary geometric constructions, knotting numbers etc.

A'Campo 1998
Adams–Brennan–Greilsheimer–Woo 1997
Aida 1992, 1992′
Ait Nouh–Yasuhara 2001
Askitas 1998, 1998′, 1999, 1999′, 1999″, 2000
Bae–Park 2000
Bernhard 1994
Bleiler 1984′
Bozhüyük 1993
Bridgeman 1996
Bruschi 1996
Călugăreanu 1962, 1962′, 1965, 1967
Callahan–Dean–Weeks 1999
Calvo–Millett 1998
Chen 2000
Cromwell 1995, 1998
Cromwell–Nutt 1996
Darcy–Sumners 2000
Dimovski 1988
Dowker–Thistlethwaite 1982, 1983
Dynnikov 1998, 1999, 2000, 2001
El Naschie 1998
Epstein–Gunn 1991. 1991′ 1992
Fiedler 1994
Fukuhara–Matsumoto–Saeki 1991
Furstenberg–Schneider 1998
Gilmer–Litherland 1986
Harris–Quenell 1999
Hass–Lagarias 2001
Hayashi–Wada 1993
Henry–Weeks 1992
Hirasawa 2000′
Hoste–Kidwell 1990
Hoste–Nakanishi–Taniyama 1990
Hotz 1959, 1960
Janse van Rensburg 1996
Jurisic 1996
Kauffman 1987′, 1997′
Kawamura 1998, 2000
Kinoshita 1957, 1958
Kobayashi 1989
Kobayashi 1989″
Kouno–Sakamoto–Niki–Sekiya 1996
Krishnamurtz–Sen 1973
Kurlin 2001
Kurpita–Murasugi 1998′
Lackenby 1997′, 1998
Liang–Mislo Flapan 1998
Lickorish 1985, 1986′
Lines 1996
Maehara–Oshiro 1999
Makanin 1988
Manturov 2000′, 2000″
Mathieu 1992
Meissen 1998
Menasco 1994
Millett 2000
Miyazaki–Yasuhara 1994, 1997
Miyazawa 2000′
Montesinos–Amilibia 1997
Morton 1986″
Murakami 1990, 1993, 1987
Murasugi 1993, 1994, 1994′
Nagura 1999
Nakamura–Nakanishi–Uchida 1998
Nakanishi 1983, 1983′, 1994, 1997

Nakanishi–Shibuya 2000
Nanyes 1995
Negami 1984
Neuwirth 1984
Ohyama 1990, 1992, 1994
Ouyang 1996
Pannwitz 1933
Petronio 1997
Przytycki 1993, 1998
Randell 1998
Randrup–Røgen 1997
Reidemeister 1926′
Rice 1968
Rudolph 1992, 1992′
Saeki 1999
Scharlemann 1985, 1993
Scharlemann–Thompson 1988
Schmid 1963
Shibuya 1989$^{\mathrm{IV}}$, 1996
Silver–Williams 1998, 2000
Sossinskij 1992
Stoimenov 2003
Sumners–Whittington 1988
Tanaka 1998′
Taniyama–Yasuhara 1994
Thompson 1989
Torisu 1998
Trace 1983
Traczyk 1999
Trautwein 1998
Turner 1985
Uchida 1993
Vappereau 1995
Vogel 1990
Wada 1993
Wendt 1937
Yajima–Kinoshita 1957
Yamamoto 1987

K15 surfaces spanned by knots, genus of knots, surfaces in the complement

Aitchison–Rubinstein 1997
Aitchison–Matsumoto–Rubinstein 1998
Akaho 1999
Alford 1970
Almgren–Thurston 1977
Bar-Natan–Fulman–Kauffman 1998
Birman–Menasco 1994
Brittenham 1999
Burde 1984
Călugăreanu 1970′
Carter–Saito 1997
Cervantes–Fenn 1988
Chalcraft 1992
Crowell 1959
Culler–Shalen 1984, 1999
Domergue–Mathie Vincent 1986
Eisner 1977
Eudave-Muños 1997′, 1999, 2000
Farber 1983
Fenley 1998
Finkelstein 1998
Finkelstein–Moriah 1999, 2000
Floyd–Hatcher 1988
Francis 1983
Francis–Collins 1992
Frankl–Pontrjagin 1930
Fukuhara 1992
Funcke 1978
Gabai 1983, 1983′, 1984, 1984′, 1986, 1986″
Gauld 1993
Gilmer 1982
Goda 1993
Goodrick 1969
Greene–Wiest 2001
Gustafson 1994
Gutiérez 1974
Habiro 2000
Han 1995, 1997, 2001, 1997′
Hatcher–Thurston 1985
Hayashi 1993
Hillman 1980′
Hirasawa 1995

Hudson 1993
Hughes 1998
Ichihara 2001
Kakimizu 1991, 1992, 1992′
Kanenobu 1992
Kauffman 1983, 1983′
Kinoshita 1962
Ko–Lee 1997
Kobayashi–Kobayashi 1996
Kobayashi 1989, 1989′, 1989‴
Kyle 1955
Laudenbach 1979
Livingston 1982
Lopez 1992, 1993
Lustig–Moriah 1999
Lyon 1971, 1972, 1974, 1974′
Menasco–Thistlethwaite 1992
Moriah 1987, 1998
Morimoto 1986, 1994′
Murakami–Yasuhara 2000
Murasugi 1958, 1958′
Naik 1994
Nakagawa 1976′
Nakanishi 1981
Neuwirth 1960
Ng 1998′
Oertel 1984
Ohtsuki 1994
Ozawa 2000, 2001
Patton 1995
Polyak 1997
Rudolph 1983′, 1985, 2001
Sakuma 1994
Scharlemann–Thompson 1988
Schubert–Soltsien 1964
Seifert 1934, 1950
Shibuya 1985′, 1989′
Shibuya–Yasuhara 2001
Shimokawa 1998
Stoimenow 1999, 2001′
Suzuki 1969
Swarup 1973, 1974
Tanaka 1998
Taniyama–Yasuhara 1994
Taylor 1979
Thompson 1989′
Thurston 1986
Torres 1951
Trotter 1975
Tsau 1994
Vogel 1990
Whitten 1973
Wu–Zhao 1993′
Wu 2001′
Xu 1992
Zulli 1997

K16 knot groups

Altin–Bozhüyük 1996
Anick 1987
Appel 1974
Appe–Schupp 1972
Aumann 1956
Bankwitz 1930′
Birman 1973, 1979
Boileau–Zieschang 1983, 1985
Boileau–Rost–Zieschang 1988
Bozhüyük 1990, 1993
Brown–Crowell 1965, 1966
Brunner 1992
Burau 1936′
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Burde–Zieschang 1966
Burde–Murasugi 1970
Burger 1950
Călugăreanu 1968, 1969, 1973
Chang = Jiang 1972, 1973, 1974
Chen 1952
Christensen 1998
Collins 1978
Collins–Zieschang 1990
Cooper 1996

Cossey–De Meskin 1971
Crowell 1963, 1970
Culler–Shalen 1984
Dane 1993$'$
Doyle 1973
Dugopolski 1982, 1985
Dynnikov 2000$'$
Edmonds–Livingstone 1984
Fox–Torres 1954
Funcke 1975
González-Acuña 1975
González-Acuña–Montesinos 1978
González-Acuña–Whitten 1987
Gordon–Luecke 1989
Gutiérez 1971
Gutiérez 1973$'$
Hain 1985
Hausmann–Kervaire 1978$'$
Higman 1948
Hilden–Tejad Toro 2002
Hirasawa–Sakuma 1997
Johnsgard 1997
Johnson 1980
Johnson–Livingston 1989
Johnson–Kim–Song 1995
Jones 1995
Joyce 1982
Kakimizu 1989
Kanenobu 1980
Kuhn 1994
Labute 1989
Lambert 1970
Le Ty Kuok Tkhang 1993
Lien 1986, 1987
Livingston 1987, 1995
Lomonaco 1981
Lozano 1987
Maeda 1977, 1977$'$, 1978
Magnus–Peluso 1967
Markl 1993
Massey–Traldi 1986, 1981
Mathieu–Vincent 1975
Mayland 1972, 1975, 1974, 1975$'$
Mayland–Murasugi 1976
Milnor 1954
Moran 1995
Murasugi 1961, 1963, 1965$'$, 1971$'$, 1974, 1974$'$, 1977
Neuwirth 1960, 1961, 1961$'$, 1963,1963$'$, 1963$'''$, 1965, 1974
Newman–Whitehead 1937
Ng 1998
Niblo–Wise 2001
Norwood 1982
Papakyriakopoulos 1955, 1957$'$
Penna–Rasetti–Spera 1991
Plans 1957
Rapaport 1960, 1975
Ratcliffe 1981, 1983
Reidemeister 1926, 1928, 1929, 1932
Rosebrock 1994
Rost–Zieschang 1984
Samuelsson 1996
Scharlemann–Thompson 1989
Schaufele 1966, 1967$'$
Sela 1993
Silver 1991, 1992
Silver–Williams 2000$'$
Simon 1976, 1980
Stebe 1968
Takeuchi 1990
Tipp 1989
Torres–Fox 1954
Traldi 1989$'$
Traldi–Sakuma 1983
Trotter 1962
Tsau 1985, 1988
Weinbaum 1971
Whittemore 1973
Whitten 1974$'$, 1987
Yajima 1962, 1969, 1970
Yoshikawa 1997
Zieschang 1963, 1977, 1995

K17 companion, product, prime, satellite and cable knots etc.

Abchir 1996
Azram 1994
Bayer-Fluckiger 1985
Bing–Martin 1971
Bleiler 1983, 1984, 1985′, 1990
Bothe 1981
Burau 1933′, 1934
Clark 1982
Cromwell 1991, 1991′
Domergue–Mathieu–Vincent 1986
El-Rifai 1999
Eudave-Muños 1988, 1989, 1990
Fintushel–Stern 1980
Fukumoto–Shinohara 1997
Gabai 1987′
Gilmer 1997
Gomez-Larrañage 1982
Gompf–Miyazaki 1995
Goodman-Strauss 1997
Gordon 1983
Gramain 1991
Hammer 1963
Hashizume 1958
Hashizum–Hosokawa 1958
Hayashi 1999 1999′
Hayashi–Matsuda–Ozawa 1999
Hayashi–Motegi 1997
Hayashi–Shimokawa 1998′
He 1998
Hillman 1984′
Ikeda 1993, 1994
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Kanenobu 1981′, 1983, 1987
Kearton 1979′, 1979″, 1983
Kinoshita–Terasaka 1957
Kirby–Lickorish 1979
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Lambert 1969
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Lickorish 1981
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Litherland 1979, 1979″
Livingston 1981, 1987, 1987′, 1988
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Malesic 1995
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Milnor 1962
Miyazaki 1989
Montesinos 1976
Moran 1995
Moriah 1998
Morimoto 1995′, 1997, 2000′
Morimoto–Schultens 2000
Morton 1979
Morton–Traczyk 1988
Morton–Strickland 1992
Motegi 1993′, 1997
Motegi–Shibuya 1992
Murakami 1989
Nakagawa–Nakanishi 1981
Nakanishi 1980′, 1983, 1996
Nakanishi–Suzuki 1987
Noga 1967
Norwood 1982
Nutt 1999
Ozawa 1999
Przytycki 1989′
Quách 1983, 1983′
Ramadevi–Govindaraja Kaul 1995
Rassai–Newcomb 1989
Rieck–Sedgwick 2002
Rudolph 1990
Ryder 1996
Sakai 1958, 1983
Sakuma 1981″
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Seifert 1949
Shibuya 1982, 1983, 1985, 1989^{V}
Short 1985
Simon 1970, 1976
Soltsien 1965
Soma 1987
Suetsugu 1996
Sullivan 1993, 1994, 2000
Swarup 1986
Teragaito 1992, 1995, 1999
Terasaka 1960
Thompson 1987
Torisu 1999
Whitehead 1937
Whitten 1981
Williams 1983, 1992
Wu 1990
Yano 1985
Zhang 1991

K18 fibred knots

Birman 1979
Birman–Williams 1983′
Boden 1997′
Boden–Nicas 2000
Bonahon 1983
Burde 1966, 1969, 1984, 1985
Burde–Zieschang 1967
Casson–Gordon 1983
Durfee 1974
Edmonds–Livingston 1983
Farber 1981‴
Fenn 1989
Francis 1983
Funcke 1978
Gabai 1983, 1986‴
Gaeta 1993
Goda–Hirasaw Yamamoto 2001
Goldsmith 1975
Goodman–Tavares 1984
Gordon–Montesinos 1986
Harer 1982, 1983
Hirasawa 2000′
Kakimizu 1989
Kanenobu 1981, 1984′
Kobayashi 1989″, 1992
Lackenby 1997′
Melvin–Morton 1986
Miyazaki 1994, 1998
Montesinos-Amilibia–Morton 1991
Morimoto 1989
Morton 1977, 1978, 1983′
Murasugi 1971, 1974′
Neuwirth 1961′, 1963, 1963′, 1965
Qiu 2000
Quách 1979
Quách–Weber 1979
Ratcliffe 1983
Silver 1992, 1995
Simon 1976″
Soma 1984, 1984′
Stallings 1962, 1978
Stoimenow 1999′, 2000‴
Whitten 1972‴, 1972′
Wu 1987
Yamamoto 1983, 2000
Yoshikawa 1981

K19 Property P and related problems, knot complements

Bing–Martin 1971
Bleiler–Scharlemann 1986, 1988
Bleiler 1985
Brittenham 1999
Culler–Gordon–Luecke–Shalen 1985, 1987
Delman–Roberts 1999
Domergue–Mathieu 1990, 1991
Feustel–Whitten 1978
Fox 1952
Gabai 1986′
Gordon 1990

Gordon–Luecke 1989, 1989′
Hempel 1964
Kirby–Melwin 1978
Kuga 1993
Litherland 1979′, 1980
Luft–Zhang 1994
Mangum–Stanford 2001
Marumoto 1977
Mathieu–Domergue 1988
Mayland 1977
Menasco 1984
Nakagawa 1975
Noga 1967
Parry 1990
Riley 1974′
Rong 1993
Sakai 1997
Scharlemann 1984, 1986
Short 1985
Simon 1970, 1971, 1973, 1976′, 1980′
Stein 1989
Swarup 1980, 1986
Takahashi 1981
Thompson 1987, 1989
Tsau 1986, 1988
Wang–Wu 1993
Whitten 1974, 1974′, 1976, 1986, 1987
Yokoyama 1977
Zhang 1993
Zieschang 1988

K20 knots and 3-manifolds: branched coverings

Hilden–Lozano–Montesinos-Amilibia 1992′
Andersson 1995
Baker 1987, 1991, 2001
Bandieri–Kim–Mulazzani 1999
Bankwitz 1930″
Bayer-Fluckiger–Kearton–Wilson 1989
Bedient 1984
Beltrami–Cromwell 1998
Boileau–Flapan 1995
Boileau–Zieschang 1983
Boileau–Zimmermann 1989
Bonahon 1979
Bozhüyük 1978, 1982, 1985
Brunner 1997
Burde 1971, 1988
Burde–Murasugi 1970
Casali 1987
Casali–Grasselli 1989.
Cavicchioli–Ruin Spaggiari 1999
Chumillas Checa 1986
Clark 1983
Cochran–Lickorish 1986
Collin 2000
Cooper 1982
Costa 1985
Davidow 1992
Davis 1995
Dellomo 1986
Derevnin–Kim 1998
Durfee–Kauffman 1975
Edmonds–Livingstone 1984
Fox 1956, 1960, 1972
Garoufalidis 1999
Giffen 1967′
Goeritz 1934
González-Acuña 1991
González-Acuña–Ramírez 1996
Gordon 1971
Gordon 1972
Gordon–Heil 1972
Hansen 1998
Harikae–Uchida 1993
Harou 2001
Hartley–Murasugi 1977, 1978
Hempel 1984, 1990
Hilden 1976
Hilden–Lozano–Montesinos 1983, 1985, 1983′, 1985′, 1987, 1988
Hilden–Montesinos–Thickstun 1976
Hillman 1981$^{\mathrm{IV}}$, 1993

Hillman–Sakuma 1997
Hirsch–Neumann 1975
Hodgson–Rubinstein 1985
Hosokawa–Kinoshita 1960
Hosokawa–Nakanishi 1986
Ishibe 1997
Jones 1994
Kanenobu 1981′
Kauffman 1974, 1987
Kawauchi 1987
Kawauchi–Matumuto 1980
Kim 2001
Kinoshita 1957, 1957′, 1958, 1958′, 1967, 1985
Ko–Smolinsky 1992
Kojima 1986, 1997
Kopuzlu 1997
Kopuzlu–Bozhüyük 1996
Kuono 1983
Kwak–Lee–Sohn 1999
Kyle 1954, 1959
Lines 1996
Livingston 1982′, 1995′
Lozano 1983, 1987
Lozano–Safont 1989
Maeda–Murasugi 1983
Mayberry–Murasugi 1982
Mecchia 2001
Mecchia–Reni 2000, 2001
Milnor 1968′, 1975
Minkus 1982
Moishezon 1981
Montesinos 1974, 1975
Montesinos–Whitten 1986
Montesinos 1973, 1973′, 1976′, 1979, 1980, 1983
Montesinos-Amilibia–Morton 1991
Morgan–Bass 1984
Motter 1976
Mulazzani–Piergallini 1998
Mullins 1993
Murasugi 1974, 1980, 1985, 1985′, 1985″, 1986′, 1987″
Núnez 1998
Ocken 1990
Paoluzzi 1999
Perko 1974, 1976
Plans 1953
Przyticki 1988″
Reidemeister 1929
Reni 1997
Reni 2000, 2000′
Reni–Vesnin 2001
Reni–Zimmermann 2001, 2001′
Reyner 1970
Riley 1989, 1990
Rong 1991
Safont 1990
Sakuma 1979, 1981, 1982, 1995
Seifert 1933′, 1936
Shinohar–Sumners 1972
Simon 1998, 1976‴
Soma 1984′
Strickland 1984
Sumners 1972, 1974
Takahashi 1977, 1978
Takahashi–Ochiai 1982
Tayama 2000, 2000′
Traldi 1985
Trotter 1962
Turaev 1988$^{\mathrm{IV}}$
Viro 1972, 1973, 1973″, 1976
del Val–Weber 1990
Wang–Wu 1993
Weber 1979, 1984
Zieschang 1984, 1988
Zimmermann 1990′, 1995, 1997, 1997′, 1998

K21 knots and 3-manifolds: surgery

Aitchison–Rubinstein 1992
Akiyoshi 1997
Alexander 1920
Altintas 1998, 1998′
Aravinda–Farrell–Roushon 1997
Bailey–Rolfsen 1977

Baker 1991
Berge 1991
Birman–Kanenobu 1988
Blanchet–Habegge–Masbau–Vogel 1992
Bleiler 1985, 1990
Bleiler–Hodgson–Weeks 1999
Bleiler–Litherland 1989
Bleiler–Scharlemann 1986
Boileau–Gonzales-Acuña–Montesinos 1987
Boyer 1998, 2002
Boyer–Zhang 1994, 1996
Brakes 1980′
Brittenham 1998
Brittenham–Wu 2001
Brody 1960
Callahan 1997
Cantwell–Conlon 1993
Caudron 1982
Clark 1978, 1980, 1982
Culler–Gordon–Luecke–Shalen 1985, 1987
Delman 1995
Domergue 1997
Domergue–Short 1985, 1987
Eudave-Muños 1997
Fintushel–Stern 1980
Flapan 1985′
Fox 1962′
Freedman–Lin 1989
Frohman–Long 1992
Fukuhara 1983, 1984, 1985
Furusawa–Sakuma 1983
Gabai 1989
Gabai–Kazez 1990
González-Acuña 1970
González-Acuña–Short 1986
Gordon 1983, 1991, 1998, 1999
Gordon–Luecke 1989, 1989, 1995, 2000
Gordon–Zhang 2000
Hayashi 1999″
Hayashi–Motegi 1997′
Hempel 1962
Hirasawa–Shimokawa 2000
Hodgson–Meyerhoff–Weeks 1992
Hoffman 1998
Hoste 1986′, 1997
Hoste–Przytycki 1997
Ichihara 2001
Ikeda 1994
Ishii 2000
Iwase 1988
Kalliongis–Tsau 1990
Kanenobu–Murakami 1986
Kauffman 1974
Kirk 1993
Kopuzlu–Bozhüyük 1996
Kuga 1988
Lackenby 1997
Lambert 1977′
Leininger 2002
Levine–Orr 2000
Levine 1987
Li–Chariya 1997
Lickorish 1962, 1977, 1992
Litherland 1979′, 1979″, 1980
Livingston 1982′
Lopez 1992
Lu 1992
Luecke 1995
Manchón 1999
Maruyama 1987
Masataka 2001
Matignon 1997
Matsuda–Ozawa–Shimokawa 2002
Matveev 1982
Mayland 2000
Menasco–Zhang 2001
Miyazaki–Motegi 2000, 2000′
Montesinos 1975
Morgan–Sullivan 1974
Morton–Strickland 1992
Moser 1974, 1971
Motegi 1988

K22 knots and periodic maps of 3-manifolds

K23 symmetries of knots and links

Boileau 1979, 1985
Boileau–Zimmermann 1987
Boileau–Gonzales-Acuña–Montesinos 1987
van Buskirk 1983
Cerf 1997
Cochran 1970
Coray–Michel 1983
Dasbach–Hougardy 1996
Davis–Livingston 1991, 1991′
Edmonds 1984
Eudave-Muñoz 1986
Flapan 1985′, 1986, 1986′, 1987
Furusawa–Sakuma 1983
Goldsmith 1975
Grünbaum–Shepard 1985
Hartley 1981, 1983′
Hartley–Kawauchi 1979
Hayashi–Shimokawa 1998′
Henry–Weeks 1992
Hillman 1986, 1986′
Jiang–Lin–Wang–Wu 2002
Kawauchi 1979
Kirk–Livingston 1999′
Kiziloglu 1998
Kodama–Sakuma 1992
Kwak–Lee–Sohn 1999
Li 1998
Liang–Cerf–Mislow 1996
Liang–Mislow–Flapan 1998
Litherland 1984
Livingston 1983
Luft–Zhang 1994
McPherson 1971
Menasco–Thistlethwaite 1991′
Miyazawa 1992
Montesinos–Whitten 1986
Murasugi 1962
Murasugi–Pryztycki 1997
Osborne 1981
Pizer 1984′
Przytycki 1994′
Ramadevi–Govindaraja–Kaul 1994
Riley 1989′
Sakai 1983′
Sakuma 1981′, 1981″, 1986, 1986′, 1987, 1989
Traczyk 1990
Trotter 1964
Tsau 1986
Wang–Zhou 1992
Whitten 1972″, 1970′, 1971, 1972, 1981, 1970, 1969′
Yokota 1991, 1991″
Zimmermann 1990, 1991

K24 knot cobordism, concordance

Bellis 1998
Cappell–Shaneson 1980
Casson–Gordon 1986
Cochran 1984′, 1985, 1985′, 1989, 1990, 1991, 1992
Cochran–Gompf 1988
Cochran–Orr 1990, 1993, 1994
Coray–Michel 1983
Endo 1995
Fox–Milnor 1966
Freyd–Yetter 1992
Garoufalidis–Levine 2001
Giffen 1979
Gilmer 1992, 1993′, 1996, 1984
Goldsmith 1978, 1979
Gompf 1986, 1989
Gompf–Miyazaki 1995
Gordon 1981′
Gutiérez 1973
Hatcher–Oertel 1989
Hillman 1985
Jiang = Chang 1981
Jin–Kim 2002
Kaiser 1992
Kanenobu 1986″
Kauffman 1974″
Kawauchi 1978, 1980
Kawauchi–Murakam Sugishita 1983
Kearton 1975″, 1975‴, 1981, 1989

Kervaire 1971
Kirby–Lickorish 1979
Kirk–Livingston 1999′, 2001
Ko 1987, 1989
Kojima–Yamasaki 1979
Le Dimet - 1987
Levine 1969, 1989, 1989′
Lin 1991
Lines 1979
Lines–Weber 1983
Lipson 1990
Litherland 1984
Livingston 1981, 1983, 1990, 2001
Livingston–Naik 1999
Milnor–Fox 1966
Miyazaki 1990, 1998
Morita 1988
Murakami 1985′
Murakami–Sugishita 1984
Myers 1983
Naik 1996, 1997
Nakagawa–Nakanishi 1981
Nakanishi 1980′, 1981, 1981′, 1986
Orr 1989
Papadima 1997
Robertello 1965
Rolfsen 1972, 1985
Sato 1981
Scharlemann 1977
Shibuya 1980, 1983, 1984,1986
Smolinsky 1989′
Soma 1984, 1983
Stoltzfus 1978, 1979, 1977
Sturm Beiss 1990
Tristram 1969
Vogt 1978
Yamada 1987

K25 Alexander module

Bailey 1977
Bankwitz 1930
Blanchfield–Fox 1951
Burde 1969
Burger 1950
Chen 1951, 1952, 1952′
Cochran–Crowell 1970
Cochran 1990′
Crowell 1961, 1963, 1964, 1965, 1971
Crowell–Strauss 1969
Farber 1991, 1983, 1981′, 1981
Fox 1970, I960′
Fox–Torres 1954
Fox–Smythe 1964
Gamst 1967
Gutiérez 1972′
Hillman 1978, 1978′, 1981′, 1981″, 1981‴, 1982
Hitt–Silver 1991
Holmes–Smythe 1966
Kadokami 1997
Kanenobu 1979, 1981, 1986
Kawauchi 1984
Kearton 1975‴, 1978, 1989
Kearton–Wilson 1997
Kidwell 1986
Kinoshita 1958, 1986, 1972, 1973
Laufer 1971
Levine 1971, 1975, 1977, 1982, 1980,1983′
Libgober 1983, 2002
Litherland 1989
Massey 1980′
McPherson 1969, 1973
Mehta 1980
Miyazaki 1990
Moran 1981
Murakami 1990, 1993
Murakami–Sakai 1993
Murasugi 1963, 1969, 1974, 1984, 1985^{V}
Nakanishi 1981′
Pizer 1984, 1985, 1988, 1988
Puppe 1952
Reidemeister 1932
Reidemeister–Schumann 1934

Rice 1971
Sakuma 1979
Sato 1981, 1978
Seifert 1933′, 1934, 1936
Shinohara–Sumners 1972
Smythe 1967
Sumners 1971, 1972, 1974
Traldi 1980, 1982, 1982′, 1983, 1983′, 1983″, 1984, 1985
Trotter 1962, 1973
Turaev 1981,1986, 1988‴, 1990
Weber 1978
Yamamoto 1978
Yoshikawa 1991
de Oliveira Barros–Manzoli Neto 1998

K26 Alexander polynomial, Conway polynomial

Alaniya 1994
Andersson 1995
Artal Bartolo–Cassou-Noguès 2000
Austin–Rolfsen 1999
Birman 1985
Blanchfield 1957
Boyer–Lines 1992
Burde 1966, 1985
van Buskirk 1983, 1985
Cattaneo 1997
Cattaneo–Cotta-Ramusin–Martellini 1995
Cromwell 1991
Dane 1985, 1993
Dasbach–Mangum 2001
Davis–Livingston 1991, 1991′
Degtyarev 1994
Deguchi 1994
Dynnikov 1997
Farber 1992
Feigelstock 1985
Foo–Wong 1991
Fox 1958′
Frohman–Nicas 1990
Fukuhara 1985
Fukumoto–Shinohara 1997
Gómez–Sierra 1993
Goldman–Kauffman 1993
Goldschmidt 1990
Gutiérez 1974
Hacon 1985
Hartley 1979′, 1983
Hartley–Kawauchi 1979
Hillman 1977, 1981, 1982, 1983, 1995
Hironaka 2001
Hitt–Silver 1991
Hongler 1999
Hosokawa 1958
Hoste–Kidwell 1990
Jaeger–Kauffma Saleur 1994
Jeong–Park 2002
Jin 1988
Jones–Przytycki 1998
Jonish–Millett 1991
Kanenobu 1984, 1989′
Kauffman 1981, 1983, 1983′, 1987, 1989″
Kauffman–Saleur 1991, 1992
Kawauchi 1978, 1996′
Kidwell 1978, 1978′, 1979, 1982, 1986
Kinoshita 1958″, 1959, 1961, 1980
Kirk–Livingston 1999, 1999′
Kitano 1996
Kondu 1979
Kulikov 1994
Lamm 2000
Lee 1990, 1996, 1998′
Levine 1965, 1966, 1967, 1987′, 1997, 1999
Li 1995
Libgober 1980′
Lickorish 1986
Lickorish–Millett 1987
Lieberum 2000′
Lin 2001
Links–Zhang 1994
Livingston 1987′
Loeser–Vaquié 1990

Majid–Rodríguez-Plaza 1993
Masataka 2001
Michels–Wiegel 1986
Miyazawa 1994
Mizuma 2002
Morton 1978, 1983′, 1984, 1995, 1999′
Morton–Short 1987
Murakami 1986, 1999, 1992, 1992′, 1992″, 1993′
Murasugi 1958″, 1961″, 1966, 1970, 1971, 1983, 1985‴, 1988′
Naik 1994, 1997′
Nakagawa 1976, 1978, 1986, 1991
Nakagawa–Nakanishi 1981
Nakanishi 1980, 1980′, 1991, 1997
Nakanishi–Suketa 1996
Neumann 1999
Penne 1998
Quách 1979
Quách–Weber 1979
de Rham 1967
Riley 1972′
Rolfsen 1975, 1975′
Rong 2002
Rosso 1999
Rozansky–Saleur 1994
Sakai 1977, 1983′
Sakuma 1981′
Saleur 1992, 1992′
Seifert 1950
Shibuya 1983, 1988
Stoimenow 2000‴
Sumners–Woods 1977
Suzuki 1969, 1984
Terasaka 1959, 1960
Threlfall 1949
Torres 1953
Torres–Fox 1954
Traldi 1984′
Trotter 1961
Turaev 1975, 1976, 1981, 1986, 1988$^{\mathrm{V}}$
Wada 1994
Wenzel 1979
Wu 2001
Yamamoto 1983, 1984
Zieschang 1993

K27 quadratic forms and signatures of knot, braids and links

Andrews–Dristy 1964
Bayer 1983′
Erle 1969, 1969′, 1999
Fukuhara 1992
Garoufalidis 1999
Gilmer 1993
Goeritz 1933, 1934′
Goldschmidt 1990
Gordon–Litherland 1978, 1979′
Gordon–Litherland–Murasugi 1981
Greene–Wiest 1998
Kauffman–Taylor 1976
Kawauchi 1977, 1999
Kearton 1979, 1979″, 1981, 2000
Kervaire 1985
Kneser–Puppe 1953
Kyle 1954, 1959
Li 1995′
Litherland 1979‴
Murakami 1986
Murasugi 1965, 1965″, 1965‴, 1970′, 1974, 1983
Perko 1976
Puppe 1952
Rudolph 1982
Shibuya 1983, 1989″
Shinohara 1971, 1976
Traczyk 1988
Trotter 1973
Turaev 1981
Wu 1986, 1988

K28 representations of knot and braid groups

Abdelghani 1998
Akutsu–Wadati 1988
Altschuler 1996

de Wit–Link–Kauffman 1999
Zhang 1992

K29 algorithmic questions, calculation of knot invariants, determination of numbers of special knots

Aneziris 1997, 1999
Appel 1974
Appel–Schupp 1972
Bae–Park 1996
Birman–Rampichini–Boldi–Vigna 20002
Birman–Hirsch 1998
Bleiler 1984′
Boileau–Weber 1983
Brinkmann–Schleimer 2001
Calvo 1997
Conway 1970
Conway–Gordon 1975
Deguchi 1994′
Dowker–Thistlethwaite 1982, 1983
Dugopolski 1982
Ewing–Millett 1991, 1997
Funcke 1978
Gillette–van Buskirk 1968
Hammer 1963
Hartley 1983″
Hass 1998
Homma–Ochiai 1978
Hoste–Thistlethwait Weeks 1998
Jacobsen–Zinn-Justin 2002
Jaeger–Vertigan–Welsh 1990
Johannson 1986
Kobayashi–Kiyoshi 1987
Kohn 1998
Krishnamurthz–Sen 1973
Ladegaillerie 1976
Li–Chariya 1997
Ligocki–Sethian 1994
Matveev 1981, 2001
Meissen 1998
Nencka 1997
Ochiai 1990
Penney 1972
Rabin 1958′
Randell 1994, 1998
Schubert 1961
Schubert–Soltsien 1964
Shalashov 1998
Soltsien 1965
Stanford 1997
Stoimenow 2000′
Sundberg–Thistlethwaite 1998
Treybig 1971, 1971′
Weinbaum 1971
Welsh 1992, 1993, 1993′, 1994
Yamamoto 1982
Zinn-Justin–Zuber 2000

K30 bridges, 2–bridge knots, bridge number, tunnels, tunnel number, tangles

Adams 1995
Adams–Reid 1996
Akiyoshi–Yoshida 1999
Akiyoshi–Sakuma–Wada–Yamashita 2000, 2000′
Ammann 1982
Asano–Marumuto–Yanagana 1981
Bankwitz 1935
Bankwitz–Schumann 1934
Bing–Martin 1971
Birman 1973, 1976
Bleiler 1990, 1994, 1998
Bleiler–Eudave-Muños 1990
Bleiler–Moriah 1988
Boileau–Zieschang 1985
Boileau–Lustig–Moriah 1994
Bozhüyük 1978
Brittenham 1998
Brittenham–Wu 2001
Burde 1975, 1985, 1988, 1990, 1997
Cavicchioli–Ruini 1994
Cavicchioli–Ruini–Spaggiari 1999
Cromwell 1998
Darcy–Sumners 2000

Delman 1995
Doll 1992
Emert–Ernst 2000
Ernst 1996, 1997
Eudave-Muños 1999, 2000
Eudave-Muñoz–Luecke 1999
Eudave-Muñoz–Uchida 1996
Floyd–Hatcher 1988
Fogel 1994
Fujii 1996
Fukuhama–Ozawa–Teragaito 1999
Fukuhara 1993, 1994
Funcke 1975, 1978
Gabai 1990
Gabai–Kazez 1990
Goda 1997
Goda–Ozaw Teragaito 1999
Goda–Scharleman Thompson 2000
Goda–Teragaito 1999
Goodrick 1972
Gordon–Reid 1995
Hachimori 2000
Hagiwara 1994
Hartley 1979′
Hatcher–Thurston 1985
Hayashi 1999
Hayashi–Shimokawa 1998
Heath–Kobayashi 1997
Heusener 1994′
Hilden–Tejad Toro 2002
Hodgson–Rubinstein 1985
Hoffman 1998
Hoidn 2000
Jones 1993
Kanenobu 1988, 1989′
Kanenobu–Murakami 1986
Kanenobu–Sumi 1993
Kanenobu–Miyazawa 1992
Kobayashi 1990, 1994, 1999′, 2001
Kohn 1991, 1998
Kohno 1994, 1994′
Kosuda 2000
Kuhn 1996
Lustig–Moriah 1993
Mathieu 1992
Mayland 1974, 1977
McCabe 1998
Mecchia–Zimmermann 2000
Mecchia–Reni 2000
Minkus 1982
Mizuma 2002
Morikawa 1981, 1982
Morimoto 1993, 1994, 1994″, 1995, 1995′, 1997, 2000, 2000′
Morimoto–Schultens 2000
Morimoto–Sakuma–Yokota 1996, 1996′
Murasugi 1961′, 1971, 1974, 1983, 1988″
Naik 1996
Nakabo 2000, 2000′, 2002
Nakagawa 1981, 1998
Nakanishi 1996′
Nakanishi–Suketa 1996
Negami 1984
Negami–Okita 1985
Ochiai 1991
Ohtsuki 1994
Otal 1982, 1985
Ozawa 1997
Perko 1976, 1982
Reni–Vesnin 2001
Riley 1982, 1984, 1992
Sakuma 1998, 1999
Scharlemann–Schultens 1999
Schubert 1954, 1956
Schultens 2000
Shimokawa 1998′
Shinohara 1976
Stoimenow 2000″, 2000‴
Takeuchi 1990
Taniyama 1991
Teragaito 1989
Thompson 1997
Tipp 1989

K31 alternating knots

K32 algebraic knots and links

Le Dũng Trans 1972
Levine 1997
Libgober 1980′, 1983
Livingston 1999, 2002
Livingston–Melvin 1983
Long 1984
Michel 1983
Mishra 1999
Muramoto–Nagase 1992
Murasugi 1985‴
Neumann 1987, 1999′
Neumann–Le Van Thanh 1993
Okamoto 1998
Orevkov 1999, 2000
Perron 1982
Quách 1983′
Rudolph 1983
Stephan 1997, 1999
Stipsicz–Szabó 1994
Sumners–Woods 1977
Wu 1996′
Yamamoto 1984, 1987
Zariski 1935

K33 slice knots and links

Akbulut 1977
Boyer 1985
Casson–Gordon 1978
Cha–Ko 1999
Cochran 1984
Endo 1995
Fintushel–Stern 1985
Flapan 1986
Fox 1973
Freedman 1985, 1988
Freedman–Lin 1989
Gilmer 1982, 1983
Gordon 1975
Hass 1983
Igusa–Orr 2001
Jiang 1981
Kanenobu 1987
Kaplan 1982
Kawauchi–Shibuya–Suzuki 1983
Kearton 1981
Kirby–Melwin 1978
Kirk–Livingston 1999
Letsche 2000
Levine 1983
Lickorish 1979
Livingston 1999, 2002
Long 1984
Montesinos 1986
Murakami–Sugishita 1984
Murasugi 1965‴
Nakagawa 1976, 1978
Ogasa 1998
Ruberman 1983
Rudolph 1982, 1997, 1998
Sakuma 1999
Satoh 1998
Shibuya 1980
Trace 1986
Yasuhara 1991, 1992
Zeeman 1965

K34 singularities and knots and links

A'Campo 1973, 1999
Askitas 1998
Baird 2001
Brauner 1928
Brieskorn 1970
Durfee 1974, 1975
Ehlers–Neuman Scherk 1987
Fox–Milnor 1957, 1966
Hirzebruch–Mayer 1968
Kahler 1929
Kauffman–Neumann 1977
Milnor 1968
Milnor–Fox 1966
Neumann 1989
Neumann–Wahl 1990

K35 further special knots

Saito 1983
Sakai 1991
Sakuma 1986, 1987, 1990
Scharlemann 1977
Shibuya 1989^{V}, 1996
Simon 1976‴
Smythe 1967′
Soma 1983
Stephan 1996
Stoimenow 2002
Sullivan 1994
Swarup 1980
Takahashi 1981
Tanaka 1998′
Teragaito 1993, 1997
Terasaka 1959
Thistlethwaite 1988
Torisu 1996, 1996′
Traczyk 1988, 1990′
Trotter 1964
Tuler 1981
Turaev 1985, 1988^{IV}
Turner 1986
Uchida 1992, 2000′
Ue 1983
Waddington 1996
Wenzel 1978, 1979
Whittemore 1973
Whitten 1981
Williams 1983
Yamamoto 1982, 1986, 1978
Yasuda 1992, 1994
Yasuhara 1991
Yokota 2000
Zieschang 1963′, 1984
Zimmermann 1997′, 1998

K36 Jones and HOMFLY polynomials, Conway function, Kauffman brackets and polynomials, skein method, A–polynomial

Adams–Brock–Bugbee–Comar–Faigin–
–Huston–Joseph–Pesikoff 1992
Aitchison 1989
Akutsu–Deguchi–Ohtsuki 1992
Akutsu–Deguchi–Wadati 1987, 1987′, 1988, 1988″
Al-Rubaee 1991
Andersen–Turaev 2001
Anstee–Przytyck Rolfson 1989
Atiyah 1990, 1990′, 1990″, 1989
Backofen 1996
Bae 2000
Bae–Kim–Park 1998
Balteanu 1993
Bar-Natan 2002
Bar-Natan–Garoufalidis 1996
Barrett 1999
Beliakova 1999
Benevenuti 1994
Berger–Stassen 1999
Bigelow 2002
Birman 1985, 1991, 1991′
Birman–Wenzl 1989
Birman–Kanenobu 1988
Blanchet–Habegger–Masbaum–Vogel 1995
Boden 1997
Bonacina–Martellini–Nelson 1991
Boyer–Lines 1992
Bradford 1990
Brandt–Lickorish–Millett 1986
Broda 1993
Bullock 1995, 1997, 1998, 1999
Bullock–Frohman–Kania-Bartoszynska 1998, 1999
Bullock–Przytycki 2000
Burri 1997
Carpentier 2000
Chalcraft 1992
Chang–Shrock 2001
Chbili 1997, 1997′
Cheng–Ge–Liu–Xue 1992
Chmutov–Goryunov 1996, 1997
Chmutov–Goryunov–Murakami 2000
Christensen–Rosebrock 1996
Cochran 1985′

Connes 1986

Cotta-Ramusino–Rinaldi 1991′, 1992, 1996

Cromwell 1989, 1993

Cromwell–Morton 1992

Dasbach–Hougardy 1997

Deguchi 1994

Deguchi–Akutsu 1990

Deguchi–Akutsu–Wadati 1988

tom Dieck 1994

Doll–Hoste 1991

Donaldson–Thomas 1990

Drobotukhina 1991, 1994

El Naschie 1999

Eliashou–Kauffman–Thistlethwaite 2003

Fiedler 1991, 2001

Föhlich–King 1989

Franks–Williams 1987

Freyd–Yetter–Hoste–Lickorish–
–Millet–Ocneanu 1985

Freyd

Frohman–Gelca–Lofaro 2002

Fuchs–Tabachnikov 1997

Fujii 1996

Fujii 1999

Fukumoto–Shinohara 1997

Gaeta 1992

Gambini–Pullin 1997

Ge–Wan Xu Wu 1989

Geck–Lambropoulou 1997

Gelca 2002

Gilmer 1993

Gilmer–Zhong 2001

Goldman–Kauffman 1997

Goryunov–Hill 1999

Griego 1996′

Grosberg–Nechaev 1992

Guadagnini 1992

Guadagnini–Martellini–Mintchev 1989, 1992

Gusarov 1991

Häring-Oldenburg 2001

Habiro 2000′

Hacon 1985

Han–Li 2000

Hara 1993

Hara–Tan Yamamoto 1999

de la Harpe 1994

de la Harpe–Kervaire–Weber 1986

Hayashi 1990

Hendriks 1988

Hitt–Silver 1991

Hoste 1985, 1986

Hoste–Przytycki 1989, 1992, 1995, 1997

Inoue–Kaneto 1994

Iwase–Kiyoshi 1987

Jaeger 1988, 1989, 1990, 1991, 1992, 1997

Jaeger–Vertigan–Welsh 1990

Jeong–Park 2002

Jin–Kim–Ko 1992

Jin–Rolfsen 1991

Jin–Lee 2002

Jones 1985, 1987, 1989, 1989′, 1990′, 1991, 1992, 1992′, 1992″, 1992‴, 1992^{IV}, 1993

Jones–Rolfsen 1994

Jonish–Millett 1991

Kadison 1994

Kaiser 1992,r 1994

Kalfagianni 2000

Kalfagianni–Lin 1999

Kamada 1997

Kanenobu 1986, 1986′, 1989, 1989′, 1990, 1991, 1992, 1995, 1997, 2000′

Kanenobu–Sumi 1992

Kanenobu–Miyazawa 1998, 1999

Kaneto 2000

Kauffman 1987, 1987′, 1988, 1988′, 1988″, 1989, 1989′, 1989‴, 1989^{IV}, 1990, 1990″, 1990‴, 1990^{IV}, 1991, 1991′, 1992′, 1992″, 1994, 1994′, 1995, 1997, 1998″, 1999, 1999′

Kauffman–Radfor Sawin 1998

Kauffman–Vogel 1992

Kawauchi 1994

Khovanov 2000

Kidwell–Stanford 2001

Kim 1999

King 1992
Kirby–Melvin 1991
Kneissler 1999′
Ko–Smolinsky 1991
Ko–Lee 1989
Kobayashi 1987, 1988
Kobayashi–Kodama 1988
Kobayashi–Kurakami–Murakami 1988
Kohno 1990, 1994′
Kosuda 1997, 2000
Kosuda–Murakami 1992
Krebes 1999
Kuperberg 1994
Kurpita–Murasugi 1992, 1995, 1998
Labastida–Pérez , 2000
Labastida–Marino 1995
Lackenby 1996
Lamaugarny 1991
Lambropoulou 1994, 1999
Landvoy 1998
Lawrence 1993
Le Dimet 1989
Le–Murakami 1995, 1996′
Li 1995
Li–Li 1994
Li–Ge 1991
Lickorish 1986, 1987, 1988, 1988′, 1989, 1991, 1992, 1993, 1997′
Lickorish–Lipson 1987
Lickorish–Millett 1986, 1986′, 1987, 1988, 1988′
Lickorish–Rong 1998
Lickorish–Thistlethwaite 1988
Lieberum 2000″
Lin–Wang 2001
Links–Gould 1992
Links–Gould–Zhang 2000
Lipson 1986, 1988, 1990, 1992
Liu 1999
Lofaro 1999
Lück 1997
Ma–Zhao 1989
Majid 1990
Mazurovskij 1989
McRobie–Thompson 1993
Melvin–Morton 1995
Meyer 1992
Millett 1992
Miyauchi 1987
Miyazawa 1997, 1998, 2000‴
Morton 1986, 1986′, 1988, 1993, 1993′, 1995, 2002
Morton–Aiston 1997
Morton–Beltrami 1998
Morton–Cromwell 1996
Morton–Hadji 2002
Morton–Short 1987, 1990
Morton–Strickland 1991
Morton–Traczyk 1988
Mullins 1993
Murakami 1986′, 1987, 1987′, 1989, 1990′, 1991, 1994, 2001
Murakami–Murakami 2001
Murasugi 1986, 1987, 1987′, 1988, 1991, 1991′, 1992, 1993, 2000
Murasugi–Przytycki 1989
Nakabo 2000, 2000′, 2002
Nikitin 1995
Nutt 1997
Ochiai–Murakami 1994
Ohyama 2000
Okada 1990
Okamoto 2000
Piunikhin 1995′
Prasolov–Sosinskii 1997
Przytycki 1988, 1988′, 1989, 1989′, 1993, 1994, 1995′, 1995″, 1998″, 1999
Przytycka–Przytycki 1993
Przytycki–Sikora 1998, 2000
Przyticki–Traczyk 1987, 1987′
Radford 1994
Reshetikhin Y–Turaev 1991
Roberts 1994
Rolfsen 1993, 1994
Rong 1991, 1997
Rozansky 1994, 1996, 1996′, 1997, 1998

K37 knots and physics, chemistry or biology, quantum groups

Courture–Ge–Lee–Schmeing 1990
Couture–Lee–Schmeing 1990
Crane 1991
Darcy–Sumners 1998
Date–Jimbo–Miki–Miwa 1992
Deguchi 1990, 1990′
Deguchi–Akutsu 1990
Deguchi–Tsurusaki 1997
Deguchi–Wadati 1994
Deguchi–Wadati–Akutsu 1988, 1988′, 1989
Devi–Govindaraja–Kaul 1993
Diao–Ernst–Janse van Rensburg 1997, 1997′, 1998, 1998′
Djemai 1996
Doebner–Groth 1997
Donaldson–Thomas 1990
Dreyer ju 1996
Drinfeld 1985, 1986
El Naschie 1999, 1999′
El-Misiery 1993
El-Rifai–Hegazi–Ahmed 1998
El-Rifai–El-Massri 1999
Ennes–Ramallo– Sanchez de Santo–Ramadevi 1998
Evans–Berger 1922
Ferguson 1993
Fiedler 1993
Fintushel–Stern 1998
Flapan 1998
Fröhlich–King 1989
Fröhlich–Gabbiani 1990
Frohman–Nicas 1990
Fukuhara 1988
Gaeta 1992
Gambini–Griego–Pullin 1998
Gambini–Pullin 1996, 1996
Garoufalidis 1998
Gauss 1833
Ge–Li–Xue 1990
Ge–Pia–Wang–Xue 1990
Ge–Wang–Xue–Wu 1989
Ge–Xue 1991
Gelca 1997, 1997′
Gilmer 1997
Goldman–Kauffman 1993, 1997
Gordon 1990
Gould 1995
Gould–Links–Zhang 1996
Gould–Tsohantjis–Bracken 1993
Goussarov 1998
Graña 2002
Green 1998
Griego 1996, 1996′
Grosberg 1998
Grzeszczuk–Huan Kauffman 1998
Guadagnini 1990, 1993
Guadagnini–Martellini–Mintchev 1989, 1990, 1990′,
1992
Guilarte 1990
Häring-Oldenburg 1997
Habegger–Orr 1999
de la Harpe–Jones 1993
Hasslacher–Meyer 1990
Hennings 1991
Hikami 2001
Hirshfeld–Sassenberg 1996
Hirshfeld–Sassenberg–Klöker 1997
Horowitz–Srednicki 1990
Jaeger 1996
Jones 1992, 1989, 1990, 1990′, 1991′, 1992″, 1992‴
Kashaev 1995, 1997, 1999
Kassel 1993, 1995
Kassel–Rosso–Turaev 1997
Kassel–Turaev 1998
Kauffman 1987′, 1988′, 1989, 1989″, 1989‴, 1989^{IV}, 1990, 1990′, 1990‴, 1991, 1991′, 1992, 1992′, 1992″, 1993, 1994, 1994′, 1994″, 1995, 1996, 1997″, 1998′, 1998″, 1998‴, 1999, 1999′
Kauffman–Huang–Greszczuk 1998
Kauffman–Lin 1991
Kauffman–Saleur 1991, 1992
Kaul 1994
Kaul–Govindarajan 1992, 1993
Kholodenko–Rolfsen 1996

Kirby–Melvin 1991
Kirk 1993
Ko–Smolinsky 1991
Kohno 1990, 1988, 1989, 1994$'$
Kontsevich 1992
Kreimer 1997, 1998, 2000
Kuperberg 1994
Kusner–Sullivan 1998
Küük 1997
Labastida–Marino 1995
Le 2000
Lee 1990, 1996
Lescop 2002
Li 1995, 1993
Lickorish 1992, 1993$'$, 1997$'$, 2000
Lin 1998
Links–Gould–Zhang 1993
Müller-Nedebock–Edwards 1999
Majid 1990
Majid–Rodríguez-Plaza 1993
Michels–Wiegel 1986, 1989
Millett 1992
Miyazawa–Okamoto 1997
Moffatt 1998
Morton 1993, 1993$'$
Morton–Ryder 1998
Mullins 1993
Murakami 2000
Murakami–Ohtsuki 1996
Murakami–Ohtsuki–Okada 1992
Murakami 1992$'$, 1993$'$
Murphy–Sen 1991
Niemi 1998, 1998
O'Hara 1998, 1999
Ohtsuki 1993, 1995
Okubo 1994
Pant–Wu 1997
Piunikhin 1993
Poénaru–Tanasi 1997
Prieto 2000
Pullin 1993
Ramadevi–Govindarajan–Kaul 1994, 1994$'$, 1995
Ranada–Trueba 1995
Reshetikhin 1991, 1989
Reshetikhin–Turaev 1990, 1990$'$, 1991
Robertson 1989
Rong 1994
Rosso–Jones 1993
Rozansky 1994, 1996, 1996$'$, 1997$'$
Rozansky–Saleur 1994
Saleur 1992
Sawin 1996$'$
Schücker 1991
Scharlemann 1992
Sen–Murphy 1989
Sikora 2000
Simon 1987, 1992, 1998, 1998$'$
Stasiak 2000
Stasiak–Dubochet–Katritch–Pieranski 1998
Sumners 1988
Tabor–Klapper 1994, 1994$'$
Takata 1992
Tesi–Janse van Rensburg–Orlandini–Whittington 1998
Turaev 1988, 1989$'$, 1989$''$, 1992, 1994, 1994$'$
Uberti–Janse van Rensburg–Orlandini–Tesi–Whittington 1998
Vogel 1996
Wadati–Deguchi–Akutsu 1990, 1992
Wadati–Akutsu–Deguchi 1990
Weber 1995
Welsh 1993
Welsh–Merino 2000
Wenzl 1990
Witten 1986, 1988, 1989, 1989$'$, 1989$''$, 1990, 1990$'$, 1994
Wu 1992, 1993, 1994
Wu–Pant–King 1994, 1995
Wu–Yamagishi 1990
Yamada 1995
Yamagishi–Ge–Wu 1990
Yetter 1992, 2001
Yokota 1995

K38 differential geometric properties of knots (curvature, integral invariants)

K40 braids, braid groups

Collins–Zieschang 1990
Couture–Lee–Schmeing 1990
Cowan 1974
Cromwell 1993′
Dahm 1962
Date–Jimbo–Miki–Miwa 1992
Deguchi 1990
Dehornoy 1995, 1999, 1999′, 2000
Deligne 1972
tom Dieck 1997, 1997′, 1998
Donaldson–Thomas 1990
Dubrovina–Dubrovin 2001
Dyer 1980
Dyer–Grossman 1981
Erle 1999
Eudave-Muñoz 1992
Fadell 1962
Fadell–van Buskirk 1961, 1962
Fadell–Neuwirth 1962
Fenn 1997
Fenn–Keyman 2000
Fenn–Keyman–Rourke 1998
Fenn–Jim–Rimányi 2001
Fenn–Rimányi–Rourke 1993, 1997
Fenn–Rolfsen–Zhu 1996
Finkelstein 1998
Fox–Neuwirth 1962
Fröhlich 1936
Fröhlich–King 1989
Franks–Williams 1987
Frenkel 1988
Fuks 1970
Garside 1969
Gassner 1961
Geck–Lambropoulou 1997
Gemein 1997, 2001
Gillette–van Buskirk 1968
Giordino–de la Harpe 1991
Goldberg 1973
Goldsmith 1974
Goldschmidt–Jones 1989
Gorin–Lin 1969, 1969′
Goryunov 1978, 1981
Gurso 1984
Habegger–Lin 1990
Hansen 1994
Hartley 1980
Henninger 1978
Hilden 1975
Humphries 1991, 1992, 1994, 1997
Husch 1969
Járai 1999
Jacquemard 1990
Jiang 1984, 1985
Jones 1985, 1987
Jones–Rolfsen 1994
Kamada 1999
Kamada–Matsumoto 2000
Kaminski 1996
Kanenobu 1989′
Kang–Lee 1997
Kaul 1994
Keever 1994
Kidwell 1982
Klassen 1970
Kneissler 1997, 1999′
Ko–Smolinsky 1992, 1992′
Ko–Lee 1997
Kohno 1987, 1988, 1989, 1989′, 1996, 1997, 2000
Krammer 2000
Kurpita–Murasugi 1998, 1998′
Labruere 1997
Ladegaillerie 1976
Lambropoulou–Rourke 1997
Le Dimet 1989
Lee–Park 1997
Lehrer 1988
Levine 1999′
Levinson 1973, 1975
Lin 1972, 1974, 1979
Lipschutz 1961, 1963
Long 1989, 1989′

Los 1994
Ma–Zhao 1989
Maclachlan 1978
Magnus 1972, 1973
Magnus–Peluso 1969
Makanin 1968, 1971, 1987, 1989
Manturov 2002
Markoff = Markov 1936, 1945
McRobie–Thompson 1993
Menasco 1994, 2001
Merkov 1999
Moishezon 1981, 1983
Moody 1991
Moran 1983, 1995
Morton 1985, 1979, 1983, 1984, 1986″, 1999′
Morton–Rampichini 2000
Mostow 1987
Mulazzani–Piergallini 1998
Mullins 1996
Murasugi 1974, 1982, 1991′
Murasugi–Kurpita 1999
Murasugi–Przytycki 1993
Murasugi–Thomas 1972
Natiello–Solari 1994
Nencka 1998, 1999
Neukirch 1981
Newman 1942
Ng–Stanford 1999
Nutt 1999
Ochiai 1978
Ohyama 1993
Orevkov 2000
Penne 1995
Platt 1988
Prasolov–Sosinskii 1997
Rolfsen–Wiest 2001
Rudolph 1982, 1983, 1983′, 1993, 1998
Scott 1970
Shalashov 1998
Shepperd 1962
Shibuya 1988
Simon 1998
Sinde 1975, 1977
Skora 1992
Smythe 1979
Song–Los 2002
Stanford 1996
Stoimenow 1999″
Stysnev 1978
Sullivan 1997
Thislethwaite 1991
Thomas 1971, 1975, 1975′
Thomas–Paley 1974
Traczyk 1998, 1998′
Turaev 2002
Vainshtein 1978
Vassiliev 1998′
Vershinin 1997, 1998″
Viro 1972
Vogel 1990
Wada 1992
Wajnryb 1988
Weinberg 1939
Wenzl 1990, 1993
Westbury 1997
Williams 1988
Wright 2000
Xu 1992
Yamada 1987
Yetter 1988, 1992
Zhu 1997′
Zieschang–Vogt–Coldewey 1970, 1980 1988
Zinno 2002

K45 singular knots, Vassiliev invariants, Fiedler invariants

Aicardi 1995, 1996
Akhmet'ev–Repovs 1998
Akhmet'ev–Maleshich–Repovs 2001
Altschuler 1996
Altschuler–Freidel 1995, 1995′, 1997
Alvarez–Labastida 1996
Alvarez–Labastida–Pérez 1997

Arnold 1994
Baez 1992
Bar-Natan 1995, 1995′, 1995″, 1996, 1997
Bar-Natan–Garoufalidis 1996
Bar-Natan–Garoufalidis–Rozansky–Thurston 2000
Bar-Natan–Stoimenow 1997
Bar-Natan–Thurston 2002
Berger–Stassen 2000
Birman 1994
Birman–Lin 1993
Bott–Taubes 1994
Burri 1997
Cartier 1993
Chbili 1997′
Chmutov–Duzhin 1994, 1999, 2001
Chmutov–Duzhin–Lando 1994, 1994′, 1994″, 1994‴
Chmutov–Varchenko 1997
Dasbach 1997, 1998, 2000
Dean 1994
Deguchi 1994′
Deguchi–Tsurusaki 1994
Duzhin–Chmutov 1999
Dynnikov 1997
Eisermann 2000, 2000′
Fenn 1994, 1997
Fenn–Keyman 2000
Fiedler 2001
Fiedler–Stoimenow 2000
Gambini–Griego–Pullin 1998
Goryunov 1997, 1999
Goussarov–Polya Viro 2000
Greenwood–Lin 1999
Gusarov 1993, 1995
Habegger–Masbaum 2000
Habiro 2000
Hirshfeld–Sassenberg–Klöker 1997
Jeong–Park 2002
Jin–Lee 2002
Jones 1992
Kalfagianni 1998′
Kamada 1999
Kanenobu 1997
Kanenobu–Miyazawa 1998
Kanenobu–Miyazawa–Tani 1998
Kauffman 1994″, 1995, 1998‴, 1999′
Kauffman–Saito–Sawin 1997
Kirk–Livingston 1997
Kneissler 1997, 1999
Kofman–Lin 2003
Kohno 1994″, 1996, 1997, 2000
Kontsevich 1992, 1993
Kricker 1997
Kricker–Spence 1997
Kricker–Spence–Aitchison 1997
Kuperberg 1996
Labastida–Pérez 2000
Lambropoulou 2000
Lando 1997
Lannes 1993
Le 1999
Le–Murakami 1995, 1995′, 1996, 1996′, 1997
Lescop 2002
Lieberum 1999, 2000, 2000″
Lin 1994, 1997, 1998
Manturov 1998, 2002′
Mellor 1999
Merkov 1999, 1999′
Miyazawa 2000, 2000′
Mohnke 1994
Morton 1999
Murakami 1994, 1996, 1997, 1998, 2000
Ng–Stanford 1999
Noble–Welsh 1999
Ohyama 1995, 1997, 2002
Ohyama–Yamada 2002
Ohyama–Tsukamoto 1999
Okamoto 1997, 1998
Papi–Procesi 1998
Piunikhin 1993, 1994, 1995, 1995′
Plachta 2000
Polyak–Viro 1994
Prasolov–Sosinskii 1997

Przytycki 1994′
Przytycki–Sikora 2002
Randell 1998
Rong 1997
Røgen 1999
Sawin 1996
Shimada 1998
Shumakovitch 1997
Sossinsky 1997
Stanford 1994, 1996, 1996′, 1997
Stoel 1962
Stoimenow 1998, 1988′, 1999, 1999′, 1999″, 2000, 2000$^{\mathrm{IV}}$, 2001, 2003
Suetsugu 1996
Tchernov 1998, 2003
Trapp 1994
Tsukamoto 2000
Tyurina 1999, 1999′
Vaintrob 1994
Vasil'ev 1992, 1994
Vassiliev 1987, 1988, 1988′, 1990, 1993, 1995, 1996, 1997, 1998, 1998′, 1999, 1999′, 1999″, 2001, 2001′
Vershinin 1998, 1998′, 1999
Vogel 1993
Willerton 1996, 1998, 1998′, 2000
de Wit–Links–Kauffman 1999′
Wu–Zhao 1993
Yetter 1998, 2001
Zhu 1998

K50 links (special articles on links with more than one component)

Adams 1986, 1996
Akbulut 1977
Baker 1992
Beiss 1990
Berger 1990
Birman–Menasco 1990, 1992″, 1994
Brown–Crowell 1966
Brown 1962
Burau 1934, 1934′, 1936′
Burde–Murasugi 1970
Carter–Saito 1996
Casali–Grasselli 1989.
Cervantes–Fenn 1988
Chang–Lee–Park 2000
Chen 1952′
Clark 1978
Cochran 1970, 1990, 1990′, 1991, 1992
Cochran–Levine 1991
Cochran–Orr 1999
Cooper–Lickorish 1999
Cromwell–Beltrami–Rampichini 1998
Crowell 1959′
Crowell–Strauss 1969
Debrunner 1961
Dimovski 1993
Domergue–Mathieu 1991
Eliashou–Kauffman–Thistlethwaite 2003
Farber 1991
Fenn 1989
Freedman 1986
Gabai 1986″, 1986‴
Garoufalidis 1999
Gilmer–Livingston 1992
Gordon–Luecke 1994
Gusarov 1995
Gutiérez 1972″, 1974
Habegger–Lin 1990, 1998
Hain 1985
Hansen 1998
Hara–Tani–Yamamoto 1999
Hennings 1991
Higman 1948
Hillman 1977, 1978, 1981′, 1985
Hodgson–Meyerhof Weeks 1992
Hongler 1999
Honma–Saeki 1994
Hosokawa 1958
Hoste 1984
Hoste–Nakanishi–Taniyama 1990
Howie–Short 1985

Hughes 1993, 1998
Ikeda 1992
Jin 1988, 1991
Jin–Rolfsen 1991
Kadokami–Yasuhara 2000
Kaiser 1992, 1991′, 1997
Kaminski 1996
Kanenobu 1979, 1981, 1984′, 1985, 1986″
Kanenobu–Miyazawa 1999
Kawauchi 1980, 1984, 1987
Kidwell 1978, 1978′, 1979
Knigge 1981
Kobayashi 1999
Kobayashi–Kodama 1988
Kohn 1993, 1991
Kojima–Yamasaki 1979
Krushkal 1998
Kuhn 1994
Labute 1989
Lambert 1970
Langevin–Michel 1985
Le Dimet 1989
Lee-Park-Seo 2001
Levine 1967, 1982, 1983′, 1987, 1987′, 1988, 1988′, 1988″
Levinson 1973
Li 1998
Liang–Mislow 1994
Lin 1996
Lindström–Zetterström 1991
Lu 1992
Massey 1998
Massey–Traldi 1986, 1981
Mayberry–Murasugi 1982
McPherson 1971
Menasco 1983
Milnor 1954
Miyazawa 1992, 1994′
Morimoto 1994
Murakami–Nakanishi 1989
Muramoto–Nagase 1992
Murasugi 1962, 1963, 1966, 1970, 1970′, 1984, 1985′, 1985$^{\mathrm{IV}}$, 1988′
Nakagawa 1976, 1978, 1981
Nakanishi 1981′, 1983, 1990′, 1990″
Negami–Okita 1985
Nejinskii 1976
Ng 1998′
Ogasa 1998
Okamoto 1998
Orr 1989
Platt 1986
Przytycki 1988′, 1990
Rolfsen 1972, 1974, 1991
Sato 1981
Schmid 1963
Schrijver 1993
Schubert 1961
Shibuya 1977, 1983, 1984, 1986, 1987, 1989′, 1989″, 1989$^{\mathrm{V}}$, 1992, 2000
Smythe 1966, 1970, 1970′
Tayama 2000,2000′
Sturm Beiss 1990
Thistlethwaite 1988″
Traldi 1982, 1982′, 1983, 1983′, 1984, 1985, 1989′, 2000
Traldi–Sakuma 1983
Tuler 1981
Turaev 1988‴, 1988$^{\mathrm{V}}$, 1990
Uchida 1991
Vappereau - 1995
Whitten 1972″, 1970′, 1971, 1972, 1970, 1969, 1969′
Wong 1992
Wu–Tian 1995
Wu 1986, 1988
Yamada 1987′
Yamamoto 1983
Yano 1984, 1985

K55 wild knots

Alford 1962
Antoine 1921
Bing 1956, 1983

Blankinship 1951
Blankinship–Fox 1950
Borsuk 1947
Bothe 1981
Brode 1981
Doyle 1973
Fox 1949
Fox–Artin 1948
Fox–Harrold 1962
Harrold 1981
Kakimizu 1987
Kinoshita 1962′
McPherson 1969, 1970, 1971″, 1973, 1973′, 1973″
Milnor 1957, 1964
Nanyes 1991
Newman–Whitehead 1937
Shilepsky 1973
Sikkema 1972
Tietze 1908
Whitehead 1935, 1935′

K59 properties of 1-knots not classified above

A'Campo 1998′
Abchir 1996
Adams 1986, 1989
Aicardi 1995
Akiyoshi–Sakuma–Wada–Yamashita 2000
Akutsu–Wadati 1987, 1988
Andersson 1995
Aravinda–Farrel Roushon 1997
Armentrout 1994
Atiyah 1989
Bar-Natan 1995‴
Bennequin 1983
Birman–Williams 1983, 1983′
Birman–Menasco 1990
Blanchfield–Fox 1951
Bleiler 1985′
Bogle–Hearst–Jones–Stoilov 1994
Boileau–Rost–Zieschang 1986
Bothe 1981′
Boyer–Lines 1992
Boyer–Mattman–Zhang 1997
Bozhüyük 1993
Brunner–Lee 1994
Buck 1994
Buck–Orloff 1993, 1995
Buck–Simon 1993, 1997
Călugăreanu 1962
Călugăreanu 1962′
Callahan–Reid 1998
Calvo 2001
Cantwell–Conlon 1991
Carter–Elhamdadi–Saito 2002
Carter–Jelsovsky–Kamanda–Langford–Saito 2000
Chaves–Weber 1994
Chekanov 2002
Christensen 1998
Churchard–Spring 1988, 1990
Clark–Schneider 1984
Cochran 1985, 1985′
Cochran–Ruberman 1989
Cochran–Lickorish 1986
Collin 1997, 2000
Cooper–Long 1993
Cowan 1974
Dasbach–Mangum 2001
Deguchi 1990
Deguchi–Tsurusaki 1994, 1998
Diao 1994, 1994′,1995
Diao–Ernst 1998
Diao–Pippenger–Sumners 1994
Dimovski 1988
Dreyer 1996
Drobotukhina 1991, 1991″
Dynnikov 1998
El-Rifai–Ahmed 1995
Emert–Ernst 2000
Ernst 1996, 1997
Ernst–Sumners 1987, 1990
Eudave-Muños 1988, 1986
Farber 1992

Kohno 1994″
Kouno–Shibuya 1991
Kuiper 1987
Kurlin 2001
Kurpita–Murasugi 1997
Kusner–Sullivan 1997, 1998
Lackenby 1996
Lambert 1977
Langer–Singer 1984
Lannes 1985
Laudenbach 1979
Laurence–Stredulinsky 2000
Lazarev 1992
Le Dimet 1987
Lee 1992, 1998
Levine 1987, 1989″
Li 1999, 1999′, 2000, 2001
Lickorish 1985, 1986′
Lien 1987
Lin–Wang 2001
Litherland 1989
Livingston 1986
Lozano–Przytycki 1985
Lu 1992
Ma 1990
Malesic 1995
Manchón 1999
Mandelbaum–Moishezon 1983
Manturov 2000, 2000′, 2000″, 2000‴
Marumoto–Nakanishi 1991
Mattman 2000
Matveev 1982
Menasco 1985
Miles 1994
Mitchell–Przytycki–Repovs 1989
Miyazaki 1989
Miyazaki–Yasuhara 1994
Miyazawa 1995
Mohnke 2001
Monastyrsky–Nechaev 1997
Moriah 1991
Morimoto 1989
Morimoto–Sakuma 1991
Morishita 2001
Morton 1986
Mostovoy 2002
Murakami 1985, 1985′
Murakami–Yasuhara 1995
Murakami–Ohtsuk Okada 1992
Murasugi 1985^{IV}, 1985^{V}, 1986′, 1989
Myers 1982
Nakagawa 1991
Nakanishi 1981, 1992
Nakauchi 1993
Nanyes 1991, 1997
Negami 1991
Neumann–Wahl 1990
Norman 1969
O'Hara 1992, 1994, 1997
Ochiai 1990
Ohyama 1990
Ohyama–Ogushi 1990
Okada 1990
Onda 2000
Orlandini–Tesi–Janse van Rensburg–Whittington 1996, 1998
Otsuki 1996
Ouyang 1996
Ozawa 1999′
Papadima 1997
Park–Seo 2000
Perez 1984
Petronio 1992
Pieranski 1998
Pippinger 1989
Polyak 1997
Prishlyak 1997
Przytycki 1990, 1991, 1997, 1989″
Przytycki–Sikora 2002
Przytycki–Traczyk 1987
Ricca 1993
Riley 1975″, 1979, 1983, 1990
Ruberman 1987

Sakuma 1991
Sallenave 1999
Sawollek 1999
Scharlemann 1984, 1985
Scharlemann–Thompson 1988, 1991
Schmitt 1997
Shaw–Wang 1994
Shibuya 1987, 1989
Shimokawa 1999
Shinohara–Ueda 2000
Shukla 1994
Silver 1995
Silver–Williams 1998
Simon 1994
Smythe 1967
Soteros 1998
Stanford 2000
Stein 1989, 1990
Stillwell 1979
Stoimenow 2000^{IV}, 2002
Sullivan 1993, 1994′, 1998
Sumners 1987, 1988″
Sumners–Whittington 1988
Suzuki 1974, 1984′
Tanaka 1999
Taniyama 1989, 1989′
Thompson 1994
Traldi 1996
Trautwein 1998
Traynor 1998
Turaev 1981, 1986, 1991
Uchida 1990
Varopoulos 1985
Vassiliev 1996
Vogel 1989
Wadati–Tsuru 1986
Walba 1987
Wang 1995
Weber–Pajitnov–Rudolph 2002
Welsh 1992
Wilkinson 1991
Williams 1998
Woodard 1991
Yamada 1987
Yamamoto 1982, 1990
Yano 1988
Yoshikawa 1991
Zhang 1991′
Zieschang 1988
tom Dieck 1994
van de Griend 1992

K60 higher dimensional knots

Akbulut–King 1981
Anderson 1983
Bayer 1980, 1980′, 1983′
Bayer-Fluckiger 1983, 1985
Bayer-Fluckiger–Kearton–Wilson 1989
Bayer-Fluckinger–Stoltzfus 1983
Bayer–Hillman–Kearton 1981
Bayer–Michel 1979
César de Sá 1979
Cappell–Shaneson 1976
Cochran 1984″, 1985
Cochran–Orr 1990
Dror 1975
Dunwoody–Fenn 1987
Durfee–Lawson 1972
Durfee–Kauffman 1975
Dyer–Vasques 1973
Eckmann 1976
Epstein 1960
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