THE GENERATING INTEGRAL AND THE CANONICAL MASLOV OPERATOR IN THE WKB METHOD

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This paper contains a new approach to the results obtained by V. P. Maslov [1] on quasiclassical asymptotic forms (the WKB method).

§1. QUASICLASSICAL ASYMPTOTIC FORMS

1. The WKB Method. The asymptotic behavior of the solutions of differential equations having a small parameter h in the derivatives frequently can be formulated in the form

$$\sum_{k\geq 0}\left(\frac{h}{i}\right)^k u_k \exp\frac{i}{h}S, \tag{1.1}$$

where S is a real-valued function, u_k is a complex-valued function, and i is the imaginary unit. Under these conditions the functions S and u_k are determined by formal substitution of (1.1) into the equation and a comparison of the coefficients of the powers of h. In other words, it is assumed that (1.1) is a formal solution of the differential equation. This method of formulating the asymptotic forms is usually called the WKB method.

In quantum mechanics the asymptotic form described above is similarly called quasiclassical for both the nonstationary

$$ih\frac{\partial\psi}{\partial t}=\left[\frac{1}{2}\left(\frac{h}{i}\frac{\partial}{\partial\xi}\right)^2+v(t,\ \xi)\right]\psi,\qquad \psi=\psi(h,\ t,\ \xi), \tag{1.2}$$

and the stationary

$$E\psi=\left[\frac{1}{2}\left(\frac{h}{i}\frac{\partial}{\partial\xi}\right)^2+v(\xi)\right]\psi,\qquad \psi=\psi(h,\ E,\ \xi), \tag{1.3}$$

Schrödinger equations, where t, $E\in\mathbf{R}$, $\xi\in\mathbf{R}^n$. This is connected with the fact that the functions S and u_k satisfy equations that are formulated in terms of the corresponding classical dynamic systems in the phase states $M=\mathbf{R}^n\oplus\mathbf{R}^n$ generated by the Hamilton function

$$H=H(t,\ x)=\frac{1}{2}p^2+v(t,q),\qquad x=\{q,p\}\in M. \tag{1.4}$$

For (1.2) the function S satisfies the Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t}+H\left(t,\left\{q,\frac{\partial S}{\partial q}\right\}\right)=0, \tag{1.5}$$

and for (1.3) it satisfies the equation for "curtailed action." As far as the coefficients u_k are concerned, they obey a recursion system of ordinary total differential equations with respect to the trajectories of the

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dynamic system. Under certain definite conditions such asymptotic representations can be used to trace the transition from quantum-mechanical dynamics to classical dynamics for $h \to 0$.

In this paper we introduce and investigate a class of asymptotic representations which is broader than the expansion (1.1). The necessity of broadening the class of asymptotic representations is dictated by the well-known difficulties which the conventional approach encounters. For a stationary equation these difficulties are manifested in the appearance of inflection points, caustics, etc. Their nonstationary equivalent is noninvariance of formal solutions of the form (1.1) with respect to dynamics. Let us dwell on this in greater detail.

Assume that we are dealing with the formal solution of Eq. (1.2) which has the form (1.1) and becomes

$$\sum_{k\geqslant 0}\left(\frac{h}{i}\right)^k \overset{0}{u}_k \exp\frac{i}{h}S^0 \tag{1.6}$$

for t = 0. Under these conditions the Hamilton-Jacobi equation (1.5) is complemented by the initial condition

$$S(t,\ \xi)|_{t=0} = S^0(\xi). \tag{1.7}$$

It is well known that the Hamilton-Jacobi equation is equivalent (with an accuracy up to terms S which depend only on t) to a situation in which the manifold

$$\Gamma_t = \left\{\left\{q,\ \frac{\partial S(t,\ q)}{\partial q}\right\}\middle|\, q \in \mathbf{R}^n\right\} \tag{1.8}$$

moves in the space M due to the effect of the diffeomorphism m_t of this space, which is generated by the canonical system

$$J\dot{x} = \frac{\partial H}{\partial x},\quad J = \begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}, \tag{1.9}$$

where I is an identical transformation of $\mathbf{R}^n$, so that

$$\Gamma_t = m_t\Gamma_0. \tag{1.10}$$

Note that the function S: $\mathbf{R}^n \to \mathbf{R}$ is restored from a manifold in the form

$$\left\{\left\{q, \frac{\partial S(q)}{\partial q}\right\}\middle|\, q \in \mathbf{R}^n\right\} \tag{1.11}$$

with an accuracy of up to a constant term. We examine the manifold

$$\Gamma^0 = \left\{\left\{q,\ \frac{\partial S^0(q)}{\partial q}\right\}\middle|\, q \in \mathbf{R}^n\right\}. \tag{1.12}$$

From the above it follows that Eq. (1.5) with the initial conditions (1.7) has a unique solution (only for those t ($t_1 < t < t_2$, $t_1 < 0$, $t_2 > 0$) for which the manifold $m_t\Gamma^0$ remains uniquely projectable onto the plane Q, $Q = \mathbf{R}^n \oplus 0$; in other words, it is a solution for those t for which $m_t\Gamma^0$ preserves the representation

$$\{\{q,\ f(q)\}\,|\, q \in \mathbf{R}^n\} \tag{1.13}$$

with a certain f: $\mathbf{R}^n \to \mathbf{R}^n$. In this case $f = \partial S/\partial q$. We could verify the fact that for these same t the recursion system of equations for the coefficients u_k, complemented by the initial conditions $u_k|_{t=0} = \overset{0}{u}_k$, is easily solved.

Assume the expansion (1.6) is the asymptotic form of a certain function $\psi^0 = \psi^0(h, \xi)$ for $h \to 0$. We examine the solution of the Cauchy problem defined by Eq. (1.2) and the initial condition

$$\psi(h, t, \xi)|_{t=0} = \psi^0(h, \xi). \tag{1.14}$$

Under well-known assumptions the formal solution formulated above will be the asymptotic form of the exact solution $\psi(h, t, \xi)$. What will be the asymptotic form of this solution for $t \notin (t_1, t_2)$?

The class of formal expansions introduced below is invariant relative to dynamics and can be used for the asymptotic representation of the solutions of the Cauchy problem (1.2), (1.4) for all t, $t \in \mathbf{R}$. Invariance relative to dynamics means that the formal solution belonging to this class and having an initial condition from this same class exists for all t. Of course, it is assumed that the diffeomorphism m_t exists for all t.

2. The Content of the Paper. The manifolds of the form (1.11) which were involved in the previous subsection form a subclass of a certain special class of n-dimensional manifolds in M—so-called Lagrange manifolds; an n-dimensional manifold Γ in M is called a Lagrange manifold if the contraction of the differential form Γ on $\omega = {}^1/_2(p\,dq - q\,dp)$ is closed. A general Lagrange manifold has the form (1.11) only when it is uniquely projected onto Q.

The function S can be characterized by stipulating the Lagrange manifold Γ, which is uniquely projected onto Q, and the primitive Σ form of $\sigma = \omega + {}^1/_2\, d(qp)$ on it:

$$S(q) = \Sigma(\{q, p\}), \quad \{q, p\} \in \Gamma, \quad q \in \mathbf{R}^n. \tag{1.15}$$

The coefficients u_k: $\mathbf{R}^n \to \mathbf{C}$ of the asymptotic representation (1.1) can be treated as functions on Γ. Thereby the representation (1.1) becomes the set $\{\Gamma, \Sigma, v\}$, where v is a formal series of functions on Γ. The generalization of the asymptotic representations which is examined here consists in the fact that the representations are juxtaposed with an arbitrary Lagrange manifold which is no longer necessarily projected uniquely onto Q.

We begin by studying asymptotic representations that correspond to Lagrange manifolds which are uniquely projected onto a certain arbitrary Lagrange plane Λ (i.e., onto a certain linear Lagrange manifold). The general form of the Lagrange plane is

$$\Lambda = g^{-1}Q, \tag{1.16}$$

where g is a transformation from the group G of linear (inhomogeneous) canonical transformations of M. The quantization of the space M generates the unitary representation V B $L_2(\mathbf{R}^n)$ of the group of transformations G in Λ. It is natural to choose the formal expressions

$$\mathbf{V}(g)\varphi, \quad \varphi = \sum_{k \geqslant 0} \left(\frac{h}{i}\right)^k u_k \exp \frac{i}{h} S \tag{1.17}$$

as the asymptotic representations which correspond to Lagrange manifolds that are uniquely projectable onto Λ. This can be supported by the argument that in choosing the plane Q to play the role of the configuration plane in M, the quantum-mechanical state represented by the element ψ, $\psi \in L_2(\mathbf{R}^n)$ will be represented by the element $\mathbf{V}^{-1}(g)\psi$.

On the next step finite or infinite sums of expressions of the form (1.17), which are connected with the arbitrary Lagrange manifold Γ and the primitive Ω (or Σ) form of ω (or σ) on it, are introduced for the asymptotic representation of the function $\mathbf{R}^n \to \mathbf{C}$. It turns out that such asymptotic representations already have the property of invariance relative to dynamics. We designate these representations by the letter Ψ. Representations of the type Ψ play a dual role in our analysis. On the one hand they are included as formal solutions of equations of the type (1.1) or (1.2), regardless of asymptotic applications. In this connection it is necessary to develop a certain formal calculus, and, in particular, it is necessary to define linear operations on Ψ, as well as differentiation $ih(d/dt)$ and the action of an operator of the Schrödinger type. On the other hand, Ψ must generate a sequence of functions Ψ^n: $\mathbf{R}^n \to \mathbf{C}$, N = 0, 1, 2, ..., which are used for asymptotic representation in the same sense as the functions $\sum_{k=0}^{N} \left(\frac{h}{i}\right)^k u_k \exp \frac{i}{h} S$ are used in the classical WKB method. The result is the use of the formal solutions for the asymptotic representation of the exact solutions.

In our analysis the center of gravity is concentrated on formal construction, and the asymptotic applications are touched on only in passing.

Different sums of expressions of the form (1.17) can generate the same asymptotic representation Ψ. It turns out that Ψ (with an accuracy of up to natural identity) can be brought to a one-to-one relationship with the sets $\{\Gamma, \Omega, \mu\}$, where $\mu = \sum_{k \geq 0} \left(\frac{h}{i}\right)^k$, and μ_k are smooth complex-valued measures on Γ.* The transition $\Psi \longleftrightarrow \{\Gamma, \Omega, \mu\}$ is accomplished by means of the symbolic generating integral

$$\Psi(\xi) = \int_{\Gamma} \mu(dx) K_{\langle\Gamma, \Omega\rangle}(\xi, x), \tag{1.18}$$

where $K_{<\Gamma, \Omega>}$ is a certain universal kernel. We arrive at this integral by approximating Γ by means of tangential Lagrange planes Λ_α at certain points x_α of the manifold Γ and representing Ψ by means of a sum of the form $\sum_\alpha V(g_\alpha)\varphi_\alpha$, where the carrier φ_α is localized in the vicinity of x_α in a definite sense. The integral (1.18) originates as a result of the natural transition in the limit in this construction. Using such an integral, it is possible to describe the basic operations on Ψ rather simply.

This paper originated during a study of the papers by V. P. Maslov, who was the first to overcome the shortcomings of the conventional approach. The Maslov presentation was formulated on the basis of Eqs. (1.17) in which g was reduced merely to a change in the roles played by certain components of the coordinate vector q and the momentum vector p. An examination of arbitrary g immediately led to a convenient representation of Ψ by means of the generating integral. The canonical operator used by Maslov was, of course, essentially equivalent to the generating integral (if we examine only the leading terms of the asymptotic representations, as is done by Maslov). However, the generating integral has the advantages residing in the fact that its definition is explicitly invariant and does not include such a concept as the index of a curve on a Lagrangian manifold (the Maslov index). Note similarly that by virtue of the transition from the Lagrange manifolds themselves (on which the primitive form of Ψ may not exist) to their covering manifolds (on this see §2 for greater detail) we can examine asymptotic representations which are generated by arbitrary Lagrange manifolds and not solely by manifolds which satisfy the "quantization conditions" that occupy an important position in the Maslov construction. Besides, these conditions originate automatically if asymptotic applications to stationary equations of the type (1.3) are examined (see §4).

We describe the plan of our subsequent presentation. In §2 we have collected the necessary information from classical and quantum mechanics, and we have also given a new formula for the Maslov index. The last section of this paper has points in common with the papers by V. I. Arnol'd [2] and D. B. Fuks [3] which were devoted to a clarification of the topological nature of the Maslov index. §3 is central: here we formulate the generating integral and clarify its relationship to the Maslov canonical operator. In §4 we examine the Cauchy problem for an equation of the Schrödinger type

$$ih\frac{d\Psi}{dt} = \mathcal{H}\Psi \tag{1.19}$$

and discuss the asymptotic expansions. A general description of the class of operators that can play the role of the operator $\mathcal{H}$ under these conditions is given.

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§2. THE PHASE SPACE AND QUANTIZATION

1. The phase space. The unitary space C^l, which is treated as a real space, is called the phase space M. The points of M will be designated by x and a. The real and imaginary parts of the scalar products $(\cdot, \cdot) + i[\cdot, \cdot]$ in C^n define a Hermitian and simplectic structure on M. The complex structure is specified by the operator J which corresponds to multiplication by i in C^n; under these conditions, $[\cdot, \cdot] = (\cdot, J\cdot)$.

* The primitive form Ω is introduced instead of Σ merely for convenience, and instead of μ we could have examined $v = \sum_{k \geq 0} \left(\frac{h}{i}\right)^k v_k$, where v_k: $\Gamma \to C$ and $v_k = (d\mu_k/ds)$, and s is an element of area on Γ.

We examine the differential form $\omega = {}^1/_2\,[x, dx]$ on M. The n-dimensional submanifold Γ is called Lagrangian in M if the form $\omega_{|\Gamma}$ is closed. The linear Lagrange manifold Λ is called a Lagrange plane. The subspace Λ is a Lagrange plane only when the form [·, ·] is nullified on it. The set of Lagrange planes is designated by **Λ**, and the set of Lagrange subspaces is designated by $\mathbf{\Lambda}_0$.

We fix Q, Q∈**Λ**. Q is assumed to be a Euclidian space with a scalar product qp = (q, p), q, p ∈Q. The space M can be treated as the direct sum of two copies of the space Q, and under these conditions the identification of x∈M and the pair {q, p}, q, p∈Q, is given by the formula x = q + Jp. The letters q and p will always designate the components of the pair x = {q, p}.

2. The Group G. The diffeomorphism m of the space M is called canonical if it preserves the form dω. The diffeomorphism m will be canonical only when dm∈**Sp**(M), where dm is the differential of m, and **Sp**(M) is the simpletic group of M (i.e., the group of nondegenerate linear transformations of M which preserve [·, ·]). The canonical diffeomorphism converts a Lagrange manifold into a Lagrange manifold.

We examine the universal covering group $\widehat{\mathbf{Sp}}$(M) of the group **Sp**(M). Its elements will be designated by A, and $\mathring{A}$ will designate their canonical projections onto **Sp**(M). The elements $\widehat{\mathbf{Sp}}$(M) are naturally parametrized by the triplet $\{\vartheta, \delta, \rho\}$, where ϑ, δ, ρ are linear transformations of Q, and ϑ and δ are symmetrical transformations. In these terms, $\mathring{A} = \exp J\Theta \exp J\{0, \delta\}\exp\{\rho, -{}^t\rho\}$, where $\Theta = \{\vartheta, \vartheta\}$ and {·, ·} are quasi-diagonal block matrices of the 2 × 2 type which define the transformations of M and correspond to the expansion M = Q + Q. $\mathring{A}$ is used to determine 2JΘ uniquely. After Θ has been fixed, δ and ρ are found in a unique manner.

Assume G is the half-line product of the linear group of the space M and $\widehat{\mathbf{Sp}}$(M). The elements of G will be designated by the letter g. They are pairs g = {*a*, A}, where *a*∈M. The group G generates the group of transformations $\mathring{G}$ of the space M, which operate according to the formula $gx = \mathring{g}x = a + \mathring{A}x$, and is a universal covering group for $\mathring{G}$. The group G is none other than the group of linear (inhomogeneous) canonical diffeomorphisms of M.

The general form of the Lagrange plane is: Λ = gQ, where g∈G. The set $\mathbf{\Lambda}_0$ of Lagrange subspaces can be interpreted as the homogeneous space of the group **Sp**(M), and it can easily be established that each $\Lambda \in \mathbf{\Lambda}_0$ is representable in the form Λ = (exp JΘ)Q, where exp 2JΘ is uniquely defined according to Λ.

3. A Lagrange Pair. Assume Γ is a connected Lagrange manifold. In our subsequent analysis E will designate the universal covering space of the manifold Γ. We similarly introduce the covering space E(ω) whose characteristic subgroup is the normal divisor χ(ω) in the group $\pi_1(\Gamma)$ which is formed by the classes of loops having the property $\int_\gamma \omega = 0$. On E and E(ω) there exists an original Ω: E → **R** of the form ω; here the original takes on different values at different points in each layer of the space E(ω). Each of the spaces E or E(ω) has its advantages from the standpoint of the subsequent analysis. The advantages of E(ω) are connected with the uniqueness of a juxtaposition of the type Ψ ⟷ {E(ω), Ω, μ} (see §1 and, for greater detail, §3), and the entire analysis could be formulated on the basis of E(ω). However, certain formulations are simpler if we use E. For us it is convenient to assume that E is equivalent to E(ω) in the first stage of the analysis right up to part 2 of §4. Later on this assumption is dropped, and we use E only.

The aggregate <Γ, Ω> is called a Lagrange pair. Assume that the representation τ: E → G has the property τx = {x, A(x)}, where A(x)Q is parallel to the plane which is tangential to E at the point X. The aggregate <Γ, Ω, τ> is called a Lagrange triplet.

Assume <Γ, Ω, τ> is a Lagrange triplet and g = {*a*, A} ∈G. We will interpret g<Γ, Ω, τ> as a Lagrange triplet <gΓ, Ωg, gτ>, where $\Omega g(x) = \Omega(g^{-1}x) + {}^1/_2[a, x]$, x∈gΓ. We define g<Γ, Ω> analogously.

The n-dimensional submanifold Γ in M is called uniquely projectable onto Q* if it has the form {{q, *f*(q)} | q∈D}, where *f*: D → Q and D is an open set in Q. If D is singly connected, then the submanifold is Lagrangian only when a function S: D → **R** exists which is such that $f = \partial S/\partial q$. Under these conditions <Γ, Ω>, where $\Omega(x) = S(q) - {}^1/_2\, q\,[\partial S(q)/\partial q]$ and x = {q, . . .} ∈Γ, is a Lagrange pair.

4. Quantization. The quantization of M is defined (see, for example, [4]) as the representation K of the space M in the set of selfconjugate operators of the Hilbert space $\mathfrak{H}$ which has the following property: the

* We will no longer use the definition (1.13).

unitary operators $W(x) = \exp(i/h)K(x)$ form a projective representation of the linear group of the space M in such a way that $W(x_1)W(x_2) = \exp(i/2h)[x_1, x_2]W(x_1 + x_2)$, where h is a stipulated constant, $h \in \Delta = (0, b)$. All irreducible representations of quantization are unitarily equivalent; here the operators which establish such equivalence are defined with an accuracy of up to a complex factor c, $|c| = 1$.

Schrödinger quantization of the phase space M is that quantization which is such that $\mathfrak{H} = L_2(Q)$, and the operators K(x) are stipulated by the differential expression $(K(x)f)(\xi) = [(q\xi) + (h/i)p(\partial/\partial\xi)]f(\xi)$, $\xi \in Q$. This quantization is irreducible.

The action of the group G is naturally defined in terms of the quantization G: $K \to gK = AK + aE$, where $(AK)(x) = K({}^tAx)$ and $(aE)(x) = (a, x)E$. It is easy to show that gK is an irreducible quantization if the quantization K is irreducible. Therefore, unitary operators V(g) exist which have the property $KV(g) = V(g)gK$. They are defined with an accuracy of up to the factor c, $|c| = 1$. It is clear that these operators form a unitary projective representation of the group G.

5. Explicit Formulas. We introduce the unitary operators

$$V(a) = \exp\frac{-i}{h} K(Ja), \qquad V(A) = \exp\frac{i}{2h}[\ln A K, K]. \tag{2.1}$$

Here $[BK, K] = \sum_{p=1}^{2n} (BK)(e_p)(JK)(e_p)$, where $\{e_p\}$ is an orthonormalized basis in M, $JB = (JB)^*$ and $*$ designates Hermitian conjugacy in the process of making M complex. We fix the operators V(A) by means of the normalization conditions $V(e) = E$ and the continuity condition. Then the operators $V(g) = V(a)V(A)$ satisfy the relationship $V(g_1)V(g_2) = \exp(i/2h)[a_1, A_1 a_2]V(g_1 g_2)$. If $A = \{\vartheta, \delta, \rho\}$, then $V(A) = V^{(1)}(\vartheta)V^{(2)}(\delta)V^{(3)}(\rho)$. In the Schrödinger representation we have

$$(V(a)f)(\xi) = \exp\frac{i}{2h} qp \exp\frac{i}{h} p(\xi - q) f(\xi - q), \qquad a = q + Jp; \tag{2.2}$$

$$(V^{(3)}(\rho)f)(\xi) = |\det{}^{1/2} r| f(r^{-1}\xi), \qquad r = e^{\rho}; \tag{2.3}$$

here $V^{(1)}(\vartheta)V^{(2)}(\delta)$ is an integral operator whose kernel is equal to

$$\lim_{\varepsilon \downarrow 0} \left[\det \frac{2\pi h}{i}(\cos\vartheta_\varepsilon\delta_\varepsilon + \sin\vartheta_\varepsilon)\right]^{-1/2} \exp\left\{\left(-\frac{i}{2h}\right)\left[\xi(-\sin\vartheta_\varepsilon\delta_\varepsilon - \cos\vartheta_\varepsilon)(\cos\vartheta_\varepsilon\delta_\varepsilon + \sin\vartheta_\varepsilon)^{-1}\xi \right.\right.$$
$$\left.\left. + \xi'(\cos\vartheta_\varepsilon\delta_\varepsilon + \sin\vartheta_\varepsilon)^{-1}\cos\vartheta_\varepsilon\xi' - 2\xi'(\cos\vartheta_\varepsilon\delta_\varepsilon + \sin\vartheta_\varepsilon)^{-1}\xi\right]\right\}, \tag{2.4}$$

where $\vartheta_\varepsilon = \vartheta + i\varepsilon$; $\delta_\varepsilon = \delta + i\varepsilon$; $V^{(1,2)}(0) = E$.

In the presence of degeneracy of $\cos\vartheta\delta + \sin\vartheta$, this expression defines a generalized function. The ambiguities are eliminated by the continuity and normalization conditions.

The explicit formulas given in this subsection should be assumed known in quantum mechanics. Regrettably, the author has not been able to locate the papers where they have been developed in the form which we require, and therefore we make several comments on their proof here. Replacing the element g by the one-parameter subgroup g_t in the defining relationship $KV(g) = V(g)gK$ and differentiating with respect to t, we can go over to the following equivalent equation $KG = GK + g_0K$ for the generating operator G of the group V(gt): $V(gt) = \exp Gt$. By finding G in the form of a quadratic functional of the operators K and using the definition of K, we arrive at Eqs. (2.1) for the operator V(g). The relationship

$$V(g_1)V(g_2) = \exp\frac{i}{2h}[a_1, A_1 a_2]V(g_1 g_2)$$

is obtained further on by means of a direct check. Equations (2.2) and (2.3) are obvious. In order to clarify (2.4), we examine the group V(gt) and the equation $(d/dt)V(gt) = GV(gt)$, $V(g_0) = E$ once more. In the Schrödinger representation G is a differentiable operator of the second order having coefficients which are quadratic in the independent variables. The kernel of the operator V(gt), (i.e., the Green's function of the reduced problem) can be found in the form $\exp\{\xi A(t)\xi + \xi B(t)\xi' + \xi' C(t)\xi' + D(t)\}$ due to the latter fact. The substitution of this expression into the equation yields a system of ordinary differential equations for the matrices A, B, C and for D, which are easy to solve explicitly.

6. The Maslov Index. The set θ of symmetrical transformations ϑ of the space Q is a universal covering space for Λ_0 (the set of Lagrange subspaces). The projection is stipulated by the equation $\Lambda = (\exp \cdot J\Theta)Q$. We examine the function $v_\varepsilon(\vartheta) = \det^{-1/2}\cos\vartheta_\varepsilon \times |\det^{1/2}\cos\vartheta_\varepsilon|$, which is fixed by the continuity requirements and the normalization condition $v_\varepsilon(0) = 1$, on the set θ. It is easy to see that the product $\lim_{\varepsilon\downarrow 0} v_\varepsilon^{-1}(\vartheta)V^{(1)}(\vartheta)$ is fixed on each layer.

We examine the form $\varkappa_\varepsilon = (2/\pi i)\,d\ln v_\varepsilon$ on θ. It is a form on Λ_0. We examine the singular form $\varkappa = \lim_{\varepsilon\downarrow 0}\varkappa_\varepsilon$ on Λ_0. Assume γ is an oriented curve on Λ_0 with a beginning Λ_1 and an end Λ_2. Assume that the Lagrange planes Λ_1 and Λ_2 are uniquely projected onto Q. The index indγ of the curve γ is called the whole number $\mathrm{ind}\gamma = \int_\gamma \varkappa$. The indices of closed curves evidently define a certain class of integer cohomologies on Λ_0—the characteristic Maslov-Arnold class [2, 3]. The Arnold formula derives immediately from our definition: ind of the closed curve γ is equal to the degree of the representation $\varepsilon : \gamma \to S^1$, where ε is the contraction on γ of the representation $\Lambda_0 \to S^1$ stipulated by the formula $\det\exp 2i\vartheta$; here $\Lambda = (\exp J\Theta)Q$.

The oriented curve γ induces the oriented curve γ' in Λ_0 on the Lagrange manifold Γ. The index indγ' is called the Maslov index of the curve γ. We likewise designate it indγ. Analogously, the curve γ induces the curve γ' in Λ_0 on the group G. The index of the latter curve is likewise called ind γ of the curve γ.

7. The Dynamics. Assume m_t, $t \in \mathbf{R}$, is a family of canonical diffeomorphisms which are stipulated by the equation $Jx = \partial\chi/\partial x$, where $\chi : \mathbf{R}\times M \to \mathbf{R}$. We examine the differential form $\omega - \chi\,dt$ on $\mathbf{R}\times M$. Its contraction on $\bigcup_{-\infty<t<\infty} E_t$, where $E_t = m_t E$, E is the covering space for the Lagrange manifold Γ, is a closed form. $\Omega^{(t)}$ will designate the original of this form.

We examine the differential $dm : \mathbf{R}\times M \to \widehat{\mathrm{Sp}}(M)$, where dm is assumed fixed by the normalization condition $dm_{0,x} = e$ and the continuity condition. Here $dm_{t,x}$ is the value of dm at the point $\{t, x\}$. The trajectories $m_t a$, $a \in M$, and $g = \{a, A\} \in G$ corresponds to a path in the group $m_t g = \{m_t a, dm_{t, m_t a}A\}$. In general, its index coincides with the Morse index of the trajectory $m_t a$.

We will agree to designate the total derivative with respect to the trajectories m_t of the dynamic system by means of a dot or the symbol d/dt. The relationship

$$ih\frac{d}{dt}\left(\exp\frac{i}{h}\,\Omega^{(t)}\mathbf{V}(m_t g)\right) = \left\{\chi + \left(\mathrm{K} - x,\ \frac{\partial\chi}{\partial x}\right) + \frac{1}{2}\left(\mathrm{K} - x,\ \frac{\partial^2\chi}{\partial x^2}(\mathrm{K} - x)\right)\right\}\exp\frac{i}{h}\,\Omega^{(t)}\mathbf{V}(m_t g) \tag{2.5}$$

is valid. Operations on K are defined by analogy with the technique used on p. 186. The proof is obtained by direct calculation.

If $\langle\Gamma, \Omega, \tau\rangle$ is a Lagrange triplet, then we will agree to use $m_t\langle\Gamma, \Omega, \tau\rangle$ to define $\langle m_t\Gamma, \Omega^{(t)}, m_t\tau\rangle$, where $\Omega^{(t)}|_{t=0} = \Omega$ and $(m_t\tau)x = m_t(\tau x)$. We examine a function in E which is stipulated by the equation

$$K = K_{\langle\Gamma,\Omega,\tau\rangle}(x) = \exp\frac{i}{h}\,\Omega^{(t)}\mathbf{V}(\tau x)\,\delta, \tag{2.6}$$

where δ is a delta-function on Q, and **V** are operators which are connected with Schrödinger quantization. Note that $V(g)\delta$ in general defines a generalized function on Q for fixed g. It turns out that the following relationship is valid:

$$K_{\langle\Gamma,\Omega,\tau\rangle}(x) = \mathbf{V}(g^{-1})\,K_{g\langle\Gamma,\Omega,\tau\rangle}(gx). \tag{2.7}$$

§3. THE GENERATING INTEGRAL

In this and the succeeding sections series of the form $\sum_{k\geq 0}\left(\frac{h}{i}\right)^k u_k$ are assumed to be formal power series in h/i, $h \in \Delta$. Expressions of the form $\sum_{k\geq 0}\left(\frac{h}{i}\right)^k u_k$, where $u_k = \sum_{l\geq 0}\left(\frac{h}{i}\right)^l u_{k,l}$, should be understood to mean $\sum_{k\geq 0}\left(\frac{h}{i}\right)^k\sum_{l=0}^{k}u_{k-l,l}$. Expressions of the type $D(\xi\,|\,f,\Gamma)$ will designate linear differential operators which operate on the variable ξ and have coefficients which depend on the functions or geometric objects f and Γ in the finite vicinity of the point ξ.

1. The Expressions $V\varphi$. We introduce the formal expressions

$$V(g)u\exp\frac{i}{h}S, \tag{3.1}$$

i.e., the sets $\{g, u, S\}$, where 1) $g\in G$, 2) $u=\sum_{k\geq 0}\left(\frac{h}{i}\right)^k u_k$, $u_k: Q\to C$, and $\operatorname{supp} u=\overline{\bigcup_{k\geq 0}\operatorname{supp} u_k}$ is a compact, 3) $S\in C^\infty(\operatorname{supp} u)$, i.e., $S:\operatorname{supp} u\to \mathbf{R}$ and S can be continued on the open set U, $\operatorname{supp} u\subset U$. Expression (3.1) will be written in abridged form as $V\varphi$; under these conditions φ will similarly designate $u\exp(i/h)S$ and will symbolize the set $\{u, S\}$.

Assume S_U is the continuation of S on U. S_U can be connected with the Lagrange manifold Γ_{S_U} (see p. 185) which is uniquely projectable onto Q. The subset of Γ_{S_U} which lies above supp u is designated by Γ_S. If for any S_U the Lagrange manifold $g\Gamma_{S_U}$ is uniquely projectable onto Q we say that $g\Gamma_S$ is unambiguously projectable onto Q.

The symbol

$$S.\,P.\int_Q (2\pi h e^{-i\frac{\pi}{2}})^{-\frac{n}{2}}u\exp\frac{i}{h}f\,d\xi, \tag{3.2}$$

where u has been described above and $f:U\to\mathbf{R}$, where U, $\operatorname{supp} u\subset U$, is an open set and, finally, f has a unique nondegenerate critical point ξ_S on U, will be defined as the formal expression

$$\left[\sum_{k\geq 0}\left(\frac{h}{i}\right)^k D_k(\xi_s\,|\,f)u\right]\exp\frac{i}{h}f(\xi_s), \tag{3.3}$$

which appears if the procedure of the stationary phase method (see, for example, [5]) is applied to the integral symbol in (3.2).

We use an explicit equation for the operator V(g) in Schrödinger quantization. Then the expression $V\varphi$ can be connected with a symbolic integral of the form (3.2). The function f has a unique nondegenerate critical point under these conditions when, and only when, $g\Gamma_S$ is uniquely projectable onto Q.

When this condition is satisfied, the symbol S. P. $V\varphi$, which defines an expression of the form $\varphi_1 = u_1\exp(i/h)S_1$, has meaning.

The equivalence relation $V_1\varphi_1 = V_2\varphi_2$ is established by the formula $\varphi_1 = S.\,P.\,(V_1^{-1}V_2)\varphi_2$. This definition is correct.

2. The ψ Classes. We will agree to designate the equivalence classes which have been introduced above by the letter ψ. Each class ψ can be associated with a pair $\langle\Gamma_\psi, \Omega_\psi\rangle$. Here Ω_ψ is the original of the ω on the compact Γ_ψ. Assume the class ψ contains the expression $V(g)u\exp(i/h)S$. We examine the Lagrange pair $\langle\Gamma_{S_U}, \Omega_S\rangle$ (see p. 185). The pair $\langle\Gamma_\psi, \Omega_\psi\rangle$ is the contraction of the Lagrange pair $g\langle\Gamma_{S_U}, \Omega_{S_U}\rangle$ on $g\Gamma_S$. This contraction is independent of the choice of the representative of the class ψ.

We examine the representation $P_{V_\varphi}:\Gamma_\psi\to\operatorname{supp} u$ which is stipulated by the formula $P_{V_\varphi}=\pi_S\circ g^{-1}$, where π_S is the orthogonal projection of Γ_S onto Q. The locality property is in effect here: if $V_1\varphi_1 = V_2\varphi_2$, then

$$u_1\circ P_{V_1\varphi_1}=\left(\sum_{k\geq 0}\left(\frac{h}{i}\right)^k L_k(\cdot\,|\,P_{V_2\varphi_2}\Gamma_\psi)u_2\right)\circ P_{V_2\varphi_2}. \tag{3.4}$$

3. The Linear Space $\mathcal{L}(\Gamma, \Omega)$. We will say that the class ψ is subordinate to the Lagrange pair $\langle\Gamma, \Omega\rangle$ if the pair $\langle\Gamma_\psi, \Omega_\psi\rangle$ is a contraction of the Lagrange pair $\langle\Gamma, \Omega\rangle$ on Γ_ψ. Under these conditions Γ_ψ can be treated as a subset of E.

Vectors of the linear space $\mathcal{L}$ (Γ, Ω) are formal sums

$$\Psi = \sum_{\alpha \in I} \psi_\alpha \tag{3.5}$$

of a certain set I of classes ψ_α which are subordinate to the pair $\langle\Gamma, \Omega\rangle$ for the condition that each point $x \in E$ has a vicinity U(x) that intersects with only a finite number of Γ_{ψ_α}.

Linear operations in $\mathcal{L}$ (Γ, Ω) are defined in the obvious manner. We define a zero vector. Assume $\Psi \in \mathcal{L}$ (Γ, Ω). We fix the point x, $x \in E$, and examine its vicinity U(x). It can be assumed that U(x) is uniquely projectable onto a certain Lagrange plane Λ. Assume U_0 is a vicinity of x such that $\bar{U}_0 \subset U(x)$. We introduce the corresponding truncating function η. We place $\eta_\alpha = \eta \circ P_{V_\alpha \varphi_\alpha}^{-1}$. We examine the expressions $V(g_\alpha)(u_\alpha \eta_\alpha)\exp(i/h)S_\alpha$, where $V(g_\alpha)u_\alpha \exp(i/h)S_\alpha$ are included in the class ψ_α. Assume $g\Lambda = Q$, $g \in G$. We introduce the notation: $I(x) = \{\alpha \mid U(x) \cap \Gamma_{\psi_\alpha} \neq \emptyset\}$. For $\alpha \in I(x)$ we have

$$V(g_\alpha)(u_\alpha \eta_\alpha)\exp\frac{i}{h}S_\alpha = V(g)u^\alpha \exp\frac{i}{h}S.$$

We form the expression $V\varphi = V(g)\left(\sum_{\alpha \in I(x)} u^\alpha\right)\exp(i/h)S$. The vector Ψ is assumed to be a zero vector if $\left(\sum_{\alpha \in I(x)} u^\alpha\right)(P_{V\varphi}x) = 0$, and thus for all x we have $x \in E$.

4. The Generating Integral. In preparing for the subsequent definitions we will begin with certain symbolic transformations. We introduce the basic substitution

$$T\mu = T_{\langle\Gamma,\Omega,\tau\rangle}\mu \tag{3.6}$$

for the congruence described below: $T: \{\mu\} \rightarrow \mathcal{L}$ (Γ, Ω), where $\langle\Gamma, \Omega, \tau\rangle$ is a Lagrange triplet, and $\mu = \sum_{k\geq 0}\left(\frac{h}{i}\right)^k \mu_k$ and μ_k are smooth complex-valued measures on E. The basic substitution will be interpreted in greater detail by the symbol

$$T_{\langle\Gamma,\Omega,\tau\rangle}\mu = \widetilde{S.P.}\int_E \mu(dx) K_{\langle\Gamma,\Omega,\tau\rangle}(x). \tag{3.7}$$

We will return to the role played by $\widetilde{S.P.}$ later, and the expression for K was described in § 2. Equation (2.7) leads to the symbolic equation

$$T_{\langle\Gamma,\Omega,\tau\rangle}\mu = V(g^{-1})T_{g\langle\Gamma,\Omega,\tau\rangle}\mu_g, \tag{3.8}$$

where $g \in G$ and $\mu_g(\gamma) = \mu(g^{-1}\gamma)$, $\gamma \subset gE$.

THEOREM 1. It is possible to establish a one-to-one relationship between the vectors Ψ of the space $\mathcal{L}$ (Γ, Ω) and the measures μ; for this relationship the linear operations on Ψ become ordinary linear operations on μ.

We will conduct the proof by simultaneously constructing a specific form of the congruence T. This form will be assumed throughout the subsequent analysis. We will show how the measure μ is used to construct the vector Ψ. The reversibility of this construction will be obvious. We introduce a locally finite cover $\{E_\alpha\}_{\alpha \in I}$ and its subordinate partition of unity $\{\eta_\alpha\}$ on E. We impose condition 1: each E_α is uniquely projectable onto a certain Lagrange plane Λ_α. Assume $g_\alpha \in G$ is such that $g_\alpha Q = \Lambda_\alpha$. We impose condition 2: the operator $\delta_\alpha(x_\alpha) + \tan \vartheta_\alpha(x_\alpha)$, where ϑ_α and δ_α correspond to the element $g_\alpha^{-1}\tau x = \tau_\alpha x_\alpha = \{x_\alpha, A_\alpha(x_\alpha)\}$, $x \in E_\alpha$, of the group G, is not degenerate on E_α.

The symbol $T\mu$ is naturally juxtaposed with the symbolic sum $\sum_\alpha T(\eta_\alpha \mu)$ which is represented as follows in accordance with (3.8): $\sum_\alpha V(g_\alpha)T_{g_\alpha^{-1}\langle\Gamma,\Omega,\tau\rangle}(\eta_\alpha\mu)_{g_\alpha^{-1}}$. The integral (3.7) which is associated with $(T_{g_\alpha^{-1}\langle\Gamma,\Omega,\tau\rangle}(\eta_\alpha\mu)_{g_\alpha^{-1}})(\xi)$ has the form of the integral in (3.2). The point $x_\alpha = \{\xi, \ldots\} \in g_\alpha^{-1}E_\alpha$ is a nondegenerate critical point of the corresponding function f under these conditions. S. $\widetilde{P}$. in the symbol (3.7) designates the procedure of the stationary phase method with respect to this point. Thus, $T_{g_\alpha^{-1}\langle\Gamma,\Omega,\tau\rangle}\cdot(\eta_\alpha\mu)_{g_\alpha^{-1}}\to\varphi_\alpha=u_\alpha\exp(i/h)S_\alpha$ and the symbol $T_{\langle\Gamma,\Omega,\tau\rangle}\mu$ is juxtaposed with the vector (3.5), where the class ψ_α contains the expression $V(g_\alpha)\varphi_\alpha$.

We will agree to call $T\mu$ the generating integral for the vector Ψ, and we write $\Psi = T_\mu$.

5. The Connection with the Canonical Operator. Note that $T\mu$ reduces to one class ψ which contains the expression of the form $V(g)\varphi$ when, and only when, the Lagrange manifold $g^{-1}\Gamma$ is uniquely projectable onto Q. In particular, when g is equal to unity the expression for φ is $\varphi = u\exp(i/h)S$, where

$$S(\xi)=\Omega(x_\xi)+\frac{1}{2}\xi p,\quad x_\xi=\{\xi,p\}\in\Gamma$$

$$u_0(\xi)=\frac{d\mu_0}{ds}\,|\det{}^{-1/2}r^{-1}\cos\vartheta\,|\,|_{x=x_\xi}\exp\frac{i}{2}\pi k;$$

here, in turn, s is an element of area on Γ, $r = e^\rho$; ϑ and ρ are parameters of τx. Finally, $k = \operatorname{ind}\gamma$, where γ is the projection onto Λ_0 of the curve on θ which connects $\vartheta = 0$ and $\vartheta(x)$, where x is arbitrary. Thus, the index is included in the expression $V(g)\varphi$.

We will show how the canonical Maslov operation, which can be used for the asymptotic description of the higher order terms, can be described in our terminology.

Each class ψ contains an expression of the form $V^{(1)}(\mathring{\vartheta})u\exp(i/h)S$; here the eigenvalues $\mathring{\vartheta}$ can be assumed equal either to zero or to $\pi/2$. We examine the Lagrange pair $\langle\Gamma,\Omega\rangle$ and the function $v: E\to C$. We introduce the Lagrange triplet $\langle\Gamma,\Omega,\tau\rangle$ and the measure $\mu(\gamma)=\mu_0(\gamma)=\int v\,|\det{}^{1/2}r\,|\,s(dx)$. We introduce the vector $T_{\langle\Gamma,\Omega,\tau\rangle}\mu$ and represent it from the form (3.5), while in each class ψ_α we choose a representative of the form $V^{(1)}(\mathring{\vartheta}_\alpha)u_\alpha\exp(i/h)S_\alpha$. The canonical Maslov operator is defined as the representation of $\{\langle\Gamma,\Omega\rangle, v\}$ in a function: $Q\to C$ of the form $\sum_\alpha V^{(1)}(\mathring{\vartheta}_\alpha)u_\alpha\exp(i/h)S_\alpha$. It is assumed that v is finite and $\{\alpha\,|\,\operatorname{supp} v\cap E_\alpha\neq\emptyset\}$ is finite.

§ 4. APPLICATIONS OF THE GENERATING INTEGRAL

In subsections 1 and 2 we examined the Cauchy problem for the formal equation

$$ih\frac{d}{dt}\Psi(t)=\mathcal{H}(t)\Psi(t) \tag{4.1}$$

having the initial condition $\Psi(0)=\Psi\in\mathcal{L}(\Gamma,\Omega)$. In this connection a certain class of linear operators in the spaces $\mathcal{L}(\Gamma,\Omega)$ is defined in subsection 1, while the expression $ih(d/dt)\Psi(t)$ is defined in subsection 2. Subsection 3 discusses asymptotic applications.

1. The Quasiclassical Operator. We define linear operators of a special form in the spaces $\mathcal{L}(\Gamma,\Omega)$; these operators will be called quasiclassical. The quasiclassical operator $\mathcal{H}$ is stipulated by the Hamilton function $H=\sum_{k\geq 0}(h/i)^k H_k$, $H_k: M\to C$. If $H = H_0$ and $\Psi = T_{\langle\Gamma,\Omega,\tau\rangle}\mu$, then

$$\mathcal{H}\Psi=T_{\langle\Gamma,\Omega,\tau\rangle}\hat{H}(\Gamma,\tau)\mu, \tag{4.2}$$

where

$$\hat{H}(\Gamma,\tau)\mu=\sum_{l\geq 0}\left(\frac{h}{i}\right)^l D_l(x\,|\,H_0,\Gamma,\tau)\mu, \tag{4.3}$$

D_k depends linearly on H_0. For general H it is necessary to place

$$\hat{H}\mu=\sum_{k\geq 0}\left(\frac{h}{i}\right)^k\sum_{l\geq 0}\left(\frac{h}{i}\right)^l D_l(x\,|\,H_k,\Gamma,\tau)\mu \tag{4.4}$$

in (4.3).

We describe a construction which leads to the explicit form of the expressions for D_l. It is sufficient to assume $H = H_0$ and $\mu=\mu_0$. We give the chain of symbolic transformations:

$$\mathcal{H}\Psi \sim \widetilde{S.P.}\int_E \mu(dx)\sum_{k\geq 0}\frac{1}{k!}\left(\mathrm{K}-x,\ \frac{\partial}{\partial y}\right)^k H(y)|_{y=x} K_{\langle\Gamma,\Omega,\tau\rangle}(x)\ \sim \sum_{k\geq 0}\widetilde{S.P.}\int_E \mu(dx) KF_k. \tag{4.5}$$

The definition of F_k is clear. It is easy to see that F_k is a polynomial in h and ξ, which depends linearly on H and on the derivatives of H at the point x. Each term can be interpreted as an expression of the form $T\mu$. For this purpose it is necessary to use the transformations from § 2 which transform (T_μ there and the given expressions here) this term into $\Psi = \sum_\alpha \psi_\alpha$; then we represent the results in the form $T\mu$. In the end we obtain

$$\widetilde{S.P.}\int_E \mu(dx) KF_k \sim T_{\langle\Gamma,\Omega,\tau\rangle}\left(\sum_{l\geq E\left(\frac{k+1}{2}\right)}\left(\frac{h}{i}\right)^l d_{k,l}(x\,|\,H,\ \Gamma,\ \tau)\mu\right). \tag{4.6}$$

Here E(t), $t \in \mathbf{R}$, is the integer part of t. In the definition (4.3) we should place $D_0 = H$, $D_l = \sum_{k=1}^{2l} d_{k,l}$, $l \geq 1$.

Now we will free ourselves from the assumption, which was adopted in § 2, that E is equivalent to $E(\omega)$. By referring to the previous content of this paper we note that this assumption is manifested solely in the vanishing of the transformation (which is inverse with respect to T) of the measured μ on E into Ψ. The Lagrange triplet should be stipulated on E. However, the transformation which is the inverse of T is required only in the present subsection in formulating the differential operators D_l. In view of their local character, it is clear that the resulting equations can also be extended to the general case as a definition. An analogous comment applies to the very beginning of the next subsection.

We will change the notation. We will agree to use Ψ to designate the symbols $T_{\langle\Gamma,\Omega,\tau\rangle}\mu$ (i.e., ensembles of Lagrange triplets and measures). We will define $\mathcal{L}(\Gamma,\Omega)$ to be the linear space of these symbols for fixed $\langle\Gamma,\Omega,\tau\rangle$ that is generated by conventional linear operations on μ. The space introduced in subsection 3 of § 3 now differs from $\mathcal{L}(\Gamma,\Omega)$. We will designate it by $\mathcal{L}_T(\Gamma,\Omega)$. The element of $\mathcal{L}_T(\Gamma,\Omega)$ corresponding to Ψ, $\Psi \in \mathcal{L}(\Gamma,\Omega)$ is designated by Ψ_T.

2. The Cauchy Problem. We examine the Cauchy problem (4.1). We define the operation ih(d/dt). We assume that the dependence of $\Psi(t)$ on t has the form $\Psi(t) = T_{m_t\langle\Gamma,\Omega,\tau\rangle}$, where m_t are diffeomorphisms described in § 2. Basing ourselves on Eq. (2.5), we arrive at an expression of the type (4.5) for $ih(d/dt)\Psi(t)$; the interpretation of this expression makes the following definition a natural one:

$$ih\frac{d}{dt}\Psi(t) = T_{m_t\langle\Gamma,\Omega,\tau\rangle}\left\{ih\frac{d\dot{\mu}_t}{d\mu_t} + \chi + \sum_{k\geq 1}\left(\frac{h}{i}\right)^k\sum_{j=1}^{2} d_{j,k}(\cdot\,|\,\chi,\ \Gamma,\ \tau)\right\}\mu_t. \tag{4.7}$$

Returning to Eq. (4.1), we assume that $H_0 = \chi$. Equation (4.1) is then equivalent to the equation

$$ih\dot{\mu}_t + H_0\mu_t + \sum_{k\geq 1}\left(\frac{h}{i}\right)^k\left[\sum_{j=1}^{2} d_{j,k}(\cdot\,|\,H_0,\ \ldots) - \sum_{l\geq 0}\left(\frac{h}{i}\right)^l D_l(\cdot\,|\,H_0,\ \ldots)\right]\mu_t = 0, \tag{4.8}$$

which reduces to the system of recurrent equations

$$(\dot{\mu}_t)_k + H_1(\mu_t)_k = N_k((\mu_t)_i,\ i<k),\qquad k = 0,\ 1,\ 2,\ \ldots, \tag{4.9}$$

after certain cancellations; here $N_0 = 0$. In the Cauchy problem these equations are supplemented by the initial conditions which combine with them to define μ_t uniquely. The Cauchy problem has been solved.

The choice of τ in the Lagrange triplet $\langle\Gamma,\Omega,\tau\rangle$ which defines the congruence $\mu \to \Psi_T$ is arbitrary to a considerable degree. In particular, it is always possible to stipulate K in canonical form $K = K^{(1)} = \exp\cdot(i/h)\Omega^{(t)}V(x)V^{(1)}(\vartheta)\delta$. The transition from K to $K^{(1)}$ corresponds to a local linear transformation of the corresponding measures and μ and $\mu^{(1)}$; under these conditions $\mu_0 = \mu_0^{(1)}\,|\det^{1/2}r\,|$. If such a transition is carried out in the solution of the Cauchy problem on the assumption that $H_1 = 0$, then it turns out that

$$(\mu_t^{(1)})_0(m_t dx)/(\mu_0^{(1)})_0(dx) = [s_0(dx)/s_t(m_t dx)]^{1/2}, \tag{4.10}$$

where s_t is an element of area on Γ_t. A generating integral with a kernel of the form $K^{(1)}$ was described in [6]. Equation (4.10) establishes a connection between the solution of the Cauchy problem given there and the solution derived in the present paper.

3. Asymptotic Applications. For the expression $V\varphi$ we will agree to define $V\varphi^N$ as the element of $L_2(Q)$ stipulated by the equation

$$V\varphi^N = V(g)u^N \exp\frac{i}{h}S, \quad u^N = \sum_{k=0}^{N}\left(\frac{h}{i}\right)^k u_k. \tag{4.11}$$

It is easy to prove that from $V_1\varphi_1 = V_2\varphi_2$ it follows that $V_1\varphi_1^N - V_2\varphi_2^N = O(h^{N+1})$. We will assume that $O(h^k)$, $k = 0, 1, 2, \ldots$ defines an element of $L_2(Q)$ such that $h^{-k}O(h^k)$ is bounded, $h \in \Delta$; the quantity ψ^N defines the class of functions $V\varphi^N$, where $V\varphi$ belongs to the class ψ.

We use $\mathring{\mathscr{L}}(\Gamma,\Omega)$ to designate the subset of elements of $\mathscr{L}(\Gamma,\Omega)$ having finite "measures" μ. Assume $\Psi \in \mathring{\mathscr{L}}(\Gamma,\Omega)$ and $\Psi^N = \sum_\alpha \psi_\alpha$. The set I will be assumed finite (this is possible). We assume $\Psi_T^N = \sum_\alpha \psi_\alpha^N$.

We will say that Ψ^α, $\Psi \in \mathring{\mathscr{L}}(\Gamma,\Omega)$ is an asymptotic expansion of the elements ψ_h, $\psi_h \in L_2(Q)$, if $\psi_h - \Psi_T^N = O(h^{N+1})$, $N = 0, 1, 2, \ldots$ We use the notation $\psi_h \sim \Psi$. We will say that the linear operator H in $L_2(Q)$, which is dependent on h, generates a quasiclassical operator $\mathscr{H}$, if for any natural N and any $\Psi \in \mathring{\mathscr{L}}(\Gamma,\Omega)$ the following conditions are satisfied: 1) Ψ_T^N belongs to the definition domain of H; 2) $H\Psi_T^N = (\mathscr{H}\Psi)_T^N + O\cdot(h^{N+1})$.

It is possible to indicate simple effective conditions for which H is an operator that generates a quasiclassical operator. These conditions, in particular, are satisfied by the Schrödinger operator for simple assumptions concerning d; here the corresponding Hamilton function is given by Eq. (1.4).

Assume $H = H(t)$ depends on t. We examine the Cauchy problem in $L_2(Q)$:

$$ih\frac{d}{dt}\psi(t) = H(t)\psi(t) + f(t), \qquad \psi(0) = \psi. \tag{4.12}$$

We assume that: 1) the operator H(t) generates a quasiclassical operator $\mathscr{H}(t)$; 2) the problem (4.12) is solvable and $\|\psi(t)\| \le C(t)[\|\psi\| + \sup_{0\le\tau\le t}\|f(\tau)\|]$, where C(t) is independent of h. The substantive conditions for the validity of 2) can be found in [7-9].

THEOREM 2. If H satisfies the conditions 1), 2), and $f = 0$, and the relationships $\psi \sim \Psi$, $\Psi \in \mathring{\mathscr{L}}(\Gamma,\Omega)$ are valid, then $\psi(t) \sim \Psi(t)$, where $\Psi(t)$ is the solution of the problem (4.1).

The proof is obvious if we make allowance for the fact that $ih(d/dt)\Psi_T^N = (ih[d/dt]\Psi)_T^N + O(h^{N+1})$. An analogous result in terms of the canonical operator is contained in the papers by Maslov.

In conclusion, several remarks are in order on the use of the generating integral in investigating the asymptotic behavior of the eigenelements of the operator which generates the quasiclassical operator. It turns out that every closed compact Lagrange manifold Γ which is invariant with respect to the dynamic system m_t and has the defined stability property can be connected with an element Ψ, $\Psi \in \mathscr{L}(\Gamma,\Omega)$, which approaches the eigenfunction of the operator H asymptotically on a certain sequence h_n, $n = 0, 1, 2, \ldots$, $h_n \to 0$, $n \to \infty$. For $h_n \to 0$ this eigenfunction is concentrated on Γ in a well-known sense. The sequence h_n is determined by the characteristic Maslov-Arnold class of the manifold Γ. The modification of the generating integral makes it possible to obtain an analogous result for manifolds of lower dimensionality (for example, for stable closed one-dimensional orbits). The role played by the stability conditions in this group of problems was discovered in special cases in [10, 11], etc.

LITERATURE CITED

1. V. P. Maslov, Perturbation Theory and Asymptotic Methods [in Russian], Izd. MGU, Moscow (1965); Appendix to: Introduction to the Method of Phase Integrals [Russian translation], Mir, Moscow (1965).
2. V. I. Arnol'd, "On the characteristic class which is included in the quantization condition," Funktsional' Analiz i Ego Prilozhen., 1, No. 1, 1-14 (1967).

3. D. B. Fuks, "On the characteristic Maslov-Arnol'd classes," Dokl. Akad. Nauk SSSR, 178, No. 2, 303-306 (1968).
4. I. Siegal, Mathematical Problems of Relativistic Physics [Russian translation], Mir, Moscow (1968).
5. M. V. Fedoryuk, "The stationary phase method for multidimensional integrals," Zh. Vychisl. Matem. i Matem. Fiz., 2, No. 1, 145-150 (1962).
6. V. S. Buslaev, "Invariant descriptions of the canonical Maslov operator," Dokl. Akad. Nauk SSSR, 184, No. 1, 59-62 (1969).
7. O. A. Layzhenskaya, "On the solution of nonstationary operator equations," Matem. Sb., 39, 491-524 (1956).
8. S. G. Krein, Linear Differential Equations in Banach Space [in Russian], Nauka, Moscow (1967).
9. J. L. Lions, Equations Differentielles Operationneles et Problemes aux Limites, Springer (1961).
10. V. S. Buldyrev, "Shortwave asymptotic behavior of the eigenfunctions of the Helmholtz operator," Dokl. Akad. Nauk SSSR, 163, No. 4, 853-56 (1965).
11. V. M. Babich and V. F. Lazutkin, On Eigenfunctions Which Are Concentrated near a Closed Geodesic, Problems in Mathematical Physics, No. 2 [in Russian], Izd. LGU, Leningrad (1967), pp. 15-25.