THE MASLOV COCYCLE, SMOOTH STRUCTURES, AND REAL-ANALYTIC COMPLETE INTEGRABILITY

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Abstract. This paper proves two main results. First, it is shown that if Σ is a smooth manifold homeomorphic to the standard n-torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$ and H is a real-analytically completely integrable convex hamiltonian on $T^*\Sigma$, then Σ is diffeomorphic to $\mathbf{T}^n$. Second, it is proven that for some topological 7-manifolds, the cotangent bundle of each smooth structure admits a real-analytically completely integrable riemannian metric hamiltonian.

1. Introduction. One of the most intriguing facts of differential topology is that a topological manifold may admit several distinct smooth structures. An important smooth invariant of a smooth manifold is the cotangent bundle, so a smooth dynamical system on the cotangent bundle ought, in principle, to reflect the smooth structure. In the present note, it is shown that the existence of a real-analytically integrable convex hamiltonian on the cotangent bundle is a nontrivial smooth invariant.

1.1. Complete integrability. The cotangent bundle of a smooth manifold Σ admits a canonical symplectic form $\omega = \sum \mathrm{d}y_i \wedge \mathrm{d}x^i$, where x^i are coordinates on Σ and y_i are the induced fibre coordinates. A symplectic form permits one to define a Poisson algebra structure on $C^\infty(T^*\Sigma)$ and consequently each smooth function $H\colon\ T^*\Sigma \to \mathbf{R}$ induces a hamiltonian vector field X_H defined by

$$(1) \qquad X_H = \{H,\ \} \qquad \Longrightarrow X_H = \begin{cases} \dot{x}^i &= \frac{\partial H}{\partial y_i}, \\ \dot{y}_i &= -\frac{\partial H}{\partial x^i}. \end{cases}$$

A first integral of the hamiltonian vector field X_H is a smooth function F which Poisson commutes with H: $\{H,F\} = 0$. If X_H has $n = \dim\Sigma$ functionally independent first integrals $F_1,\ldots,F_n$, and the first integrals pairwise Poisson commute, then the compact regular level sets $\{F_1 = c_1,\ldots,\ \ F_n = c_n\}$ are n-dimensional lagrangian tori and the flow of X_H is translation-type. In this case, one says that X_H is *completely integrable*; if the first integrals are real-analytic, one says that X_H is real-analytically completely integrable.

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1.2. Geometric semisimplicity. Let us abstract the notion of complete integrability. A smooth flow $\varphi\colon\ M \times \mathbf{R} \to M$ is *integrable* if there is an open, dense subset $R \subset M$ that is covered by angle-action charts which conjugate φ to a translation-type flow on the tori of $\mathbf{T}^k \times \mathbf{R}^l$. There is an open dense subset $L \subset R$ fibred by φ-invariant tori; let $f\colon\ L \to B$ be the induced smooth quotient map and let $\Gamma = M - L$ be the *singular set*. If Γ is a tamely-embedded polyhedron, then φ is said to be *k-semisimple* with respect to (f, L, B), or just semisimple [12]. Of most interest is when φ is a hamiltonian flow on a cotangent bundle or possibly a regular iso-energy surface.

Definition 1. (cf. [32], [12]) A hamiltonian flow is *geometrically semisimple* if it is semisimple with respect to (f, L, B) and f is a lagrangian fibration. It is *finitely geometrically semisimple* if, in addition, each component of B has a finite fundamental group.

In this case, the lagrangian-ness of the fibres of f implies that φ is completely integrable, so geometric semisimplicity is a topologically-tame type of complete integrability. Taimanov [32] introduced a related notion of geometric simplicity, see Sections 2.2–2.3 of [12] for further discussion. If φ is real-analytically completely integrable, then the triangulability of real-analytic sets implies that φ is finitely geometrically semisimple; in fact, in this case B may be taken to be a disjoint union of open balls. On the other hand, geometric semisimplicity is a weaker property than real-analytic complete integrability [12]. A basic question is:

Question A. What are the obstructions to the existence of a geometrically semisimple (resp. semisimple, completely integrable) flow?

1.3. Main results. Recall that a topological n-torus is a topological manifold that is homeomorphic to the standard n-torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$. An exotic n-torus is a topological n-torus that is not diffeomorphic to $\mathbf{T}^n$. Exotic n-tori may be constructed by connect summing with exotic spheres, but not all arise this way.

THEOREM 1. *If Σ is an exotic n-torus, then there are no* finitely geometrically semisimple *convex hamiltonians on $T^*\Sigma$. In particular, there are no real-analytically completely integrable convex hamiltonians on $T^*\Sigma$.*

The obstruction here is the *smooth* structure of the configuration space. This is the first result that shows that a smooth invariant may preclude real-analytic complete integrability; as such, it prompts several questions.

Question B. If Σ is an exotic torus, does there exist a completely integrable convex hamiltonian on $T^*\Sigma$?

Theorem 5 in Section 5 strengthens Theorem 1 by showing that there are no completely integrable riemannian metrics on an exotic torus that are completely integrable via a geodesic equivalence, see also [25, Theorem 8]. This suggests that the answer to Question B may be *no*.

It is important to note that the definitiveness of Theorem 1 is not general and may be atypical. The Gromoll-Meyer exotic 7-sphere is a biquotient Sp(2)//Sp(1) and so it inherits a submersion metric from the bi-invariant metric on Sp(2) [20]. Paternain and Spatzier [29] proved the real-analytic complete integrability of the geodesic flow of this submersion metric. On the other hand, the remaining 12 unoriented diffeomorphism classes of the 7-sphere are not known to possess such geodesic flows.

Question C. Do all exotic 7-spheres admit a real-analytically completely integrable convex hamiltonian?

And, more generally:

Question D. What are the smooth obstructions to the existence of a geometrically semisimple convex hamiltonian?

While the present paper does not answer Questions B–D, it is able to answer the question for some classes of topological 7-manifolds with more than one smooth structure. A *Witten-Kreck-Stolz space* $M_{k,l}$ is the smooth 7-manifold obtained by quotienting $S^5 \times S^3$ by the action of U_1 given by the representation $z \mapsto z^k \cdot I \oplus z^l \cdot I\colon\ \mathrm{U}_1 \to \mathrm{U}_3 \oplus \mathrm{U}_2$, where k and l are coprime integers. Kreck and Stolz showed that $M_{k,l}$ has a maximum of 28 smooth structures; and, with modest conditions on k and l, this maximum is attained and each smooth structure is represented by some $M_{k',l'}$ [21]. This paper uses the work of Mykytyuk and Panasyuk [27] to show that:

THEOREM 2. *There is a real-analytically completely integrable convex hamiltonian on the cotangent bundle of each Witten-Kreck-Stolz space. In particular, if* $l = 0 \bmod 4$, $l = 0, 3, 4 \bmod 7$, $l \neq 0$ *and* $\gcd(k, l) = 1$, *then each one of the* 28 *diffeomorphism classes of* $M_{k,l}$ *is the configuration space of a real-analytically completely integrable convex hamiltonian.*

The convex hamiltonian in all cases may be taken to be the hamiltonian induced by the round metrics on S^5 and S^3. For each Witten-Kreck-Stolz space, there is an S^1 fibre bundle $S^1 \hookrightarrow M_{k,l} \to \mathbf{C}P^2 \times \mathbf{C}P^1$. In [10] (resp. [11]) Bolsinov and Jovanović prove, *inter alia*, the real-analytic noncommutative (resp. complete) integrability of the geodesic flows of certain homogeneous metrics on $\mathbf{C}P^2 \times \mathbf{C}P^1$, see especially [11, Remark 3.4].

The present paper finishes by proving a similar result for the Eschenburg and Aloff-Wallach 7-manifolds. These manifolds are obtained through a quotient of SU_3 by a subgroup $V \cong \mathrm{U}_1$ of the maximal torus of $\mathrm{SU}_3 \times \mathrm{SU}_3$. The existence of real-analytically completely integrable geodesic flows on some special Eschenburg spaces was proven by Paternain & Spatzier and Bazaikin [29], [4].

The results of the present paper extend their work. Kruggel [22] has obtained a complete list of invariants that classify the smooth structures on most Eschenburg spaces. It is unknown if each topological Eschenburg space admits the maximum 28 smooth structures and each smooth structure is itself an Eschenburg space. Numerical computations [15], [14] suggest this may be true for some families of Eschenburg spaces.

1.3.1. Related work. Bialy and Polterovich [7, Theorem 1.1] prove that if $F \subset T^*\mathbf{T}^2$ is an essential lagrangian torus that is invariant under a convex hamiltonian flow, and without periodic points, then the natural map $F \to \mathbf{T}^2$ is a diffeomorphism. Theorem 1 is based on a generalization of their theorem to higher dimensions, see Proposition 4.3 and Remark III below.

Taimanov [32] has proven that if a compact manifold Σ admits a real-analytically completely integrable geodesic flow, then $\pi_1(\Sigma)$ is almost abelian of rank at most $\dim\Sigma$; $\dim H^1(\Sigma;\mathbf{Q}) \le \dim\Sigma$; and there is an injection $H^*(\mathbf{T}^b;\mathbf{Q}) \hookrightarrow H^*(\Sigma;\mathbf{Q})$ where $b = \dim H^1(\Sigma;\mathbf{Q})$. These constraints are ineffective for exotic tori.

In [31], Rudnev and Ten assume that a geodesic flow is completely integrable with a nondegenerate first-integral map on an n-dimensional compact manifold with first Betti number equal to n. Nondegeneracy means, amongst other things, that the singular set is stratified by the rank of the first integral map and each stratum is a symplectic submanifold on which the system is completely integrable. From these hypotheses, they deduce that there is a lagrangian torus $F \subset T^*\Sigma$ such that the natural map ρ (Figure 1) is a *homeomorphism*. Theorem 2 of [31] states that ρ is a diffeomorphism, but this is mistaken. It is shown only that ρ is a $1-1$ smooth map, hence by invariance of domain, a homeomorphism. To prove that ρ is a diffeomorphism one must prove that the Maslov cocycle of F vanishes, or something equivalent. This is the first difficulty in proving theorem 1.

It should also be noted that either real-analyticity or nondegeneracy is a very restrictive hypothesis on the first-integral map. In [13], there is an example of a geometrically semisimple geodesic flow on $T^*(\mathbf{T}^2 \times S^2)$ which is not completely integrable with real-analytic (resp. nondegenerate) first integrals, nor is it approximable by a real-analytically (resp. nondegenerately) completely integrable system.

1.3.2. Technical clarifications of Theorem 1. One might also enquire if there is a convex hamiltonian H which enjoys an energy level $H^{-1}(c)$ which is geometrically semisimple. If the sub-level $H^{-1}((-\infty, c])$ contains the zero section of $T^*\Sigma$, then the answer is also *no*. Presumably, the answer may change if the sub-level set does not contain the zero section, but this is an open question (cf. [31]). The conditions on the fibration $f\colon L \to B$ in the definition of geometric semisimplicity may also be weakened somewhat and Theorem 1 continues to hold: one may require only that Γ satisfy condition (FI2) in Definition 9 of [12] in place of being a tamely-embedded polyhedron.

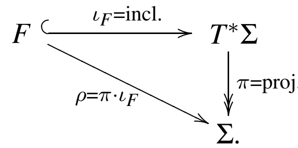

Figure 1.

1.3.3. A sketch of the proofs. If $g\colon\ X \times \mathbf{R} \to X$ is a flow, a point $x \in X$ is *nonwandering* if, for any neighbourhood U of x, $g_t(U) \cap U$ is nonempty for some $t > 1$. The set of nonwandering points for g is denoted by $\Omega(g)$ [28]. It is proven that

THEOREM 3. (cf. Theorem 4) *Let Σ be a smooth manifold and $H\colon\ T^*\Sigma \to \mathbf{R}$ a convex hamiltonian with complete hamiltonian flow φ. If $F \subset H^{-1}(c)$ is a lagrangian submanifold whose Maslov cocycle vanishes and $\Omega(\varphi|F) = F$, then ρ (figure 1) is a smooth covering map. In particular, if F is a torus, then Σ is finitely* smoothly *covered by a torus.*

This theorem is certainly known to experts, see [8] or [28, Section 2.5] and references therein. What makes this theorem crucial for the present note is that it provides a mechanism whereby the smooth structure of Σ enters: if, under the hypotheses of Theorem 1 one can prove that there must exist a lagrangian standard n-torus $F \subset T^*\Sigma$ with vanishing Maslov cocycle, and one can show that the degree of ρ must be ± 1, then one has obtained a proof of the theorem. This is done in Sections 3 and 4. Section 2 recalls the definition and properties of the Maslov cocycle. Section 5 deals with projectively equivalent metrics on exotic tori and Section 6 proves Theorem 2 and related results.

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2. The Maslov cocycle. Let us recall the definition and construction of the Maslov cocycle as it was introduced and developed in [2], [17], [3]. An interpretation of the Maslov cocycle as an obstruction class (a primary difference) is also recalled.

2.1. The Grassmannian of lagrangian planes in $\mathbf{C}^n$. Let H be the standard hermitian inner product on $\mathbf{C}^n$: $H(z, w) = \sum_{i=1}^n z_i \bar{w}_i$. This hermitian product is the sum of two real quadratic forms on $\mathbf{R}^{2n} = \mathbf{C}^n$, the real symmetric part is a euclidean inner product g and the real skew-symmetric part is a symplectic inner product ω. The identity $H(z, w) = g(z, w) + i\omega(z, w)$ shows that a real subspace $V \subset \mathbf{C}^n$ is a real inner product space rel. H iff $\omega|V = 0$. A real inner-product subspace V of n-dimensions is called a lagrangian subspace; it is clear that V is lagrangian iff there is a basis $v_1, \ldots, v_n$ of V such that $v = [v_1 \cdots v_n]$ is a unitary matrix. This shows that the set of all lagrangian subspaces of $\mathbf{C}^n$ is the homogeneous space $\Lambda_n = \mathrm{U}_n/\mathrm{O}_n$. It is a standard exercise that every $2n$-dimensional symplectic vector space is isomorphic to $(\mathbf{R}^{2n}, \omega)$.

The map $u \mapsto \det u^2 = \exp(2\pi i\theta)$ induces a submersion $\det^2\colon\ \Lambda_n \to \mathrm{U}_1$. Let $\mu_o = \mathrm{d}\theta$, the standard U_1-invariant 1-form on U_1.

Definition 2. The *Maslov cocycle* μ is the pullback of $\mathrm{d}\theta$ by $\det^2$. The Poincaré dual of μ is the *Maslov cycle*

$$\mathfrak{m} = \{\lambda \in \Lambda_n\colon\ \lambda \cap i\mathbf{R}^n \neq 0\},$$

the set of lagrangian planes with a nontrivial intersection with the plane $i\mathbf{R}^n$.

The coorientation of $\mathfrak{m}$ is defined by declaring that the closed curve $c\colon\ [0,\pi] \to \Lambda_n$, $c\colon\ t \mapsto e^{it}\mathbf{R} \oplus i\mathbf{R}^{n-1}$ crosses $\mathfrak{m}$ positively at $t = \pi/2$. It is straightforward to see that $\langle \mu, c\rangle = +1$, also.

2.2. The bundle of lagrangian planes. If $(E, w) \to M$ is a symplectic vector bundle, then E admits a complex structure J and a hermitian inner product H such that (E_x, J_x, H_x, w_x) is isomorphic to $(\mathbf{C}^n, i, H, \omega)$ for all $x \in M$. The associated bundle $\Lambda(E) \to M$ of lagrangian planes is naturally defined. Let $r, s\colon\ M \to \Lambda(E)$ be sections. The primary obstruction to the existence of a homotopy between r and s is a cohomology class $d \in H^1(M; \{\pi_1(\Lambda_n)\})$, called the primary difference. In general, the primary difference lies in a cohomology group with twisted coefficients; because $B\mathrm{U}_n$ is simply connected, the coefficients are untwisted in $H^1(M; \{\pi_1(\Lambda_n)\})$. One may identify $\pi_1(\Lambda_n)$ with the integers by choosing the standard generator of $\pi_1(\Lambda_n)$ to be the closed curve $c\colon\ [0,\pi] \to \Lambda_n$, $c\colon\ t \mapsto e^{it}\mathbf{R} \oplus i\mathbf{R}^{n-1}$. With this convention, the primary difference of r and s is a cohomology class $d \in H^1(M; \mathbf{Z})$.

When $M = \mathrm{U}_1$ and E is a $\mathbf{C}^n$-vector bundle over M, then the simple connectedness of $B\mathrm{U}_n$ implies that $E = M \times \mathbf{C}^n$ and $\Lambda(E) = M \times \Lambda_n$. If $r, s\colon\ M \to \Lambda(E)$ are sections, then one has the cohomology classes $\mu_r = r^*(1 \times \mu)$ and $\mu_s = s^*(1 \times \mu)$ and the primary difference equals

$$d = \mu_r - \mu_s.$$

Both μ_r and μ_s must be multiples of the generator $\mu_o \in H^1(\mathrm{U}_1; \mathbf{Z})$. It is clear that this multiple is the degree of the composite maps

$$U_1 \xrightarrow{r,s} \Lambda_n \xrightarrow{\det^2} U_1.$$

By naturality, this characterizes the primary difference in all cases: one can unnaturally trivialize the vector bundle $E \to M$ over the 1-skeleton of M and apply the preceding to determine the primary difference.

The Poincaré dual to the primary difference d is a codim-1 cycle that is henceforth denoted by $\mathfrak{m}_\zeta$. By transversality, one may assume that r and s are transversal sections $M \to \Lambda(E)$; the set

$$\mathfrak{m}_\zeta = \{m \in M\colon\ r(m) \cap s(m) \neq 0\}$$

is a smooth cycle that one may justifiably call a relative Maslov cycle.

The primary difference has a second, very important, interpretation. Since s and $J \circ s$ are homotopic, everywhere transverse sections, d is also the primary obstruction to r being homotopic to a section that is everywhere transverse to s.

Remark I. (1) (E, J, s) is a complex vector bundle with a real sub-bundle s. Homotopy classes of sections of $\Lambda(E)$ therefore classify the inequivalent real forms of E. These triples are classified, up to isomorphism, by the set of homotopy classes of maps $[M, \mathrm{U}_n/\mathrm{O}_n]$ modulo the image of $[\Sigma^1 M, B\mathrm{U}_n]$ under a connecting map [24, pp. 97–99]. (2) We have used the fact that a symplectic vector bundle E admits a nonnatural complex structure J. The set of such complex structures is contractible, so the nonnaturality does not affect homotopy invariants. (3) In [16], and in the preprint of the present paper, one finds the statement that the existence of a section of $\Lambda(E)$ implies its triviality. As pointed out by Leonardo Macarini, this is incorrect. Indeed, here is a simple example which shows that $\Lambda(E)$ may be nontrivial and have sections. According to [24, pp. 97–99], if $\Lambda(E)$ is trivial, then E is a trivial complex vector bundle. In our example, $\Lambda(E)$ has a section but E is nontrivial, so $\Lambda(E)$ must be nontrivial. Let $f\colon\ S^3 \to \mathrm{SO}_3$ be the canonical projection map, let $j\colon\ \mathrm{SO}_3 \to \mathrm{U}_3$ be the natural embedding. Let $E \to S^4$ be the $\mathbf{C}^3$-vector bundle over S^4 whose clutching function is jf. E is a nontrivial $\mathbf{C}^3$ bundle since $jf \in \pi_3(\mathrm{U}_3)$ is not trivial. Of course, E admits a global lagrangian sub-bundle, so $\Lambda(E)$ is nontrivial.

2.3. Cotangent bundles. Let us specialize the constructions above. $E := T(T^*\Sigma)$ is a symplectic vector bundle over $T^*\Sigma$ with the canonical symplectic form $w = \mathrm{d}p \wedge \mathrm{d}q$. The footpoint projection $\pi\colon\ T^*\Sigma \to \Sigma$ induces the *vertical sub-bundle* $V = \ker(\mathrm{d}\pi)$ of E. The fibres of V are lagrangian planes and the map $s(\theta) = V_\theta$ is a section of the lagrangian grassmannian bundle $\Lambda(E) \xrightarrow{\Pi} T^*\Sigma$. Let M be a submanifold of $T^*\Sigma$ that is fibred by compact lagrangian submanifolds, so that Figure 2 obtains where $\iota_\bullet$ is an inclusion map, f is a fibre-bundle map

$$\begin{array}{ccccc} F & \overset{\iota_F}{\hookrightarrow} & M & \overset{\iota_M}{\hookrightarrow} & T^*\Sigma \\ & & \downarrow{\scriptstyle f} & & \\ & & B. & & \end{array}$$

Figure 2.

whose fibres are lagrangian submanifolds and F is a typical fibre. This hypothesis includes the possibility that M itself is a lagrangian submanifold. Since the fibres of f are lagrangian submanifolds, for each $\theta \in M$ the tangent space to the fibre of f at θ is a lagrangian plane in E_θ. Let $E_M = E|M$ and define a section $r\colon M \to \Lambda(E_M)$ by

$$r(\theta) = \ker \mathrm{d}_\theta f, \qquad \forall\, \theta \in M.$$

The discussion of the relative Maslov cocycle and cycle from the previous section applies to the present construction: one can compute the primary difference d between the section r and the vertical section $s|M$ of $\Lambda(E_M)$. As above, the relative Maslov cycle $\mathfrak{m}_\zeta$ is the set of points where $\ker \mathrm{d}f$ and $\ker \mathrm{d}\pi$ have a nontrivial intersection. If F_θ denotes the fibre of f through θ, then

$$\mathfrak{m}_\zeta = \{\theta \in M\colon\ \operatorname{rank} \mathrm{d}_\theta \pi | F_\theta < \dim \Sigma\}.$$

Since F_θ and Σ are both n-dimensional manifolds, $\mathfrak{m}_\zeta$ is the set of θ where $\pi|F_\theta\colon\ F_\theta \to \Sigma$ fails to be a local diffeomorphism.

3. A nonvanishing Maslov cocycle. This section continues with the notation of the previous. Let us state the main result of this section. Recall that $\Omega(\bullet)$ is the nonwandering set of the flow $\bullet$.

THEOREM 4. *If* $\mathfrak{m}_\zeta \cap \Omega(\varphi|F) \neq \emptyset$*, then* $\mathfrak{m}_\zeta \cap F$ *is a nontorsion codimension-1 cycle on* F.

Note that this theorem requires only that $\mathfrak{m}_\zeta$ intersect the chain recurrent set of $\varphi|F$, similar to [8]. However, since F will generally be a Liouville torus for this paper, $\varphi|F$ will satisfy the somewhat stronger hypothesis here. The proof has been included for the sake of completeness. The basic underlying fact is that solution curves of convex hamiltonian systems cross the Maslov cycle positively, as pointed out by Duistermaat [17].

Proof. Let $\theta \in \mathfrak{m}_\zeta \cap \Omega(\varphi|F)$. By the convexity of H, there is an $s > 0$ such that

$$t \in [-s, s] \quad \text{and} \quad \varphi_t(\theta) \in \mathfrak{m}_\zeta \qquad \Longrightarrow \qquad t = 0.$$

Since the nonwandering set $\Omega(\varphi|F)$ is invariant, the points $\theta^\pm = \varphi_{\pm s}(\theta)$ are nonwandering. Let $U^\pm$ be neighbourhoods of $\theta^\pm$ that are disjoint from $\mathfrak{m}_\zeta$. Since the points are nonwandering and on the same orbit, there is a point $\theta' \in U^+$ and a $T > 1$ such that $\varphi_T(\theta') \in U^-$.

Let γ be the curve in F obtained by concatenating the orbit segment $\varphi_t(\theta)\colon\ t \in [-s, s]$, followed by an arc in U^+ joining θ^+ to θ', followed by the orbit segment $\varphi_t(\theta')\colon\ t \in [0, T]$, followed by a segment joining $\varphi_T(\theta')$ to θ^- in U^-. Since the segments of γ in $U^{\pm}$ are disjoint from $\mathfrak{m}_\zeta$, and the remaining segments are φ-orbit segments, convexity implies that

$$\#(\gamma, \mathfrak{m}_\zeta) \geq 0.$$

Since $\theta \in \gamma \cap \mathfrak{m}_\zeta$, the intersection number is positive. □

Remark II. If F is a Liouville torus of a completely integrable convex hamiltonian, then the Liouville-Arnold theorem implies that $\Omega(\varphi|F) = F$. Therefore, Theorem 4 implies that the Maslov cocycle $\iota_F^*(d)$, if nonzero, represents a non-torsion cohomology class in $H^1(F)$.

COROLLARY. *Assume $\Omega(\varphi|F) = F$. Then $\mathfrak{m}_\zeta \cap F$ is a trivial cycle iff $\iota_F^*(d)$ is a trivial cocycle iff the map $\rho = \pi \circ \iota_F$ in Figure 1 is a local diffeomorphism.*

4. Exotic tori and. . . .

4.1. Geometric semisimplicity. A *topological* n-torus is a smooth manifold that is homeomorphic to the *standard* n-torus $\mathbf{T}^n = \mathbf{R}^n/\mathbf{Z}^n$. An exotic n-torus is a topological n-torus that is not diffeomorphic to $\mathbf{T}^n$.

PROPOSITION 4.1. *Let Σ be a topological n-torus. If $H\colon\ T^*\Sigma \to \mathbf{R}$ is a finitely geometrically semisimple convex hamiltonian, then there is a lagrangian torus $F \subset T^*\Sigma$ such that the map ρ*

(*Figure* 1)
$$\begin{array}{ccc} F & \xrightarrow{\iota_F=\text{incl.}} & T^*\Sigma \\ & \underset{\rho=\pi\cdot\iota_F}{\searrow} & \downarrow{\scriptstyle \pi=\text{proj.}} \\ & & \Sigma. \end{array}$$

has a nonzero degree.

Proof. Let $f\colon\ L \to B$ be a lagrangian fibration, invariant under the hamiltonian flow of H, such that $T^*\Sigma$ is the disjoint union of L and a closed, nowhere dense, tamely-embedded polyhedral singular set Γ. Define the natural map ξ by

$$L \xrightarrow{\iota_L=\text{incl.}} T^*\Sigma \xrightarrow{\pi} \Sigma, \qquad \xi = \pi\cdot\iota_L\colon L \to \Sigma. \tag{2}$$

By [12, Lemma 15], and the fact that both F and Σ are topological n-tori, there is a component L_i of L such that $(\xi \cdot \iota_{L_i})_*\colon\ \pi_1(L_i) \to \pi_1(\Sigma)$ is almost surjective. Let us drop the subscript i in the following discussion; equivalently, let us assume that $L_i = L$.

By hypothesis, $\pi_1(B)$ is finite. The homotopy long exact sequence for the fibration (f, L, B) yields

$$\cdots \longrightarrow \pi_2(B) \xrightarrow{\partial_*} \pi_1(F) \xrightarrow{\iota_{F,L,*}} \pi_1(L) \longrightarrow \pi_1(B) \xrightarrow{f_*} 1,$$

so $\pi_1(L)$ contains the finite-index subgroup $\pi_1(F)/\partial_* \pi_2(B)$. One concludes that, since $\rho = \xi|F$, the image of $\pi_1(F)$ under the map $\rho_*\colon \pi_1(F) \to \pi_1(\Sigma)$ is a finite index subgroup. Since both F and Σ are topological n-tori, ρ has a nonzero degree. □

We continue with the hypotheses of Proposition 4.1.

Proposition 4.2. *If* $\deg \rho \neq 0$ *(Figure 1), then* ρ *is a local diffeomorphism.*

(It is known that there is a unique PL structure on the topological n-torus, a fact that is used without further reference.)

Proof. (cf. [34]) It is known that the smooth structure of the topological n-torus Σ is determined by a unique cohomology class σ contained in the cohomology group $\bigoplus_{i \leq n} H^i(\Sigma; \Gamma_i)$, where $\Gamma_i = \pi_i(PL/O)$ is the group of homotopy classes of maps from S^i into the classifying space of stable PL-structures modulo smooth structures; equivalently, Γ_i is the group of smooth structures on the topological i-sphere for $i \geq 7$ and 0 for $i < 7$ [36, p. 236]. This correspondence is natural with respect to local diffeomorphisms, so if $p\colon \Sigma' \to \Sigma$ is a local diffeomorphism and Σ (resp. Σ') is a topological n-torus whose smooth structure is determined by the cohomology class σ (resp. σ'), then $p^*\sigma = \sigma'$.

Since the cohomology class σ lies in a finite group, it has finite order. Therefore, if $p\colon \Sigma' \to \Sigma$ is a finite covering whose degree divides the order of σ, then $\sigma' = p^*\sigma$ must vanish. Thus Σ' is diffeomorphic to the standard n-torus. It is clear that such coverings p exist, so let us choose one such covering.

From Figure 1, one gets the pullback diagram where F' is a connected component of $P^{-1}(F)$ and ϕ is the covering map induced by P, see Figure 3. Because F is diffeomorphic to the standard n-torus and F' is a finite covering of F, F' is also diffeomorphic to the standard n-torus–this follows from the above-mentioned classification of smooth structures and their naturality under coverings.

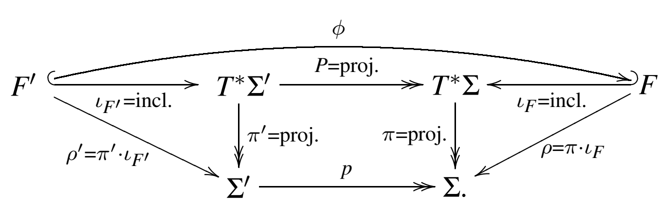


Figure 3. The pullback diagram of Figure 1.

Since $\deg\rho$ is nonzero by hypothesis and $\deg\rho'\cdot\deg p = \deg\phi\cdot\deg\rho$, the degree of ρ' is nonzero. The map $\iota_{F'}$ is an embedding by naturality. Therefore, $\iota_{F'}$ is a lagrangian embedding of the standard n-torus F' into the cotangent bundle of the standard n-torus $T^*\Sigma'$ such that the induced map ρ' has a nonzero degree. Viterbo [34, Corollary 3] proved that in this case the Maslov cocycle $\iota_{F'}^*(d')$ is cohomologically trivial. By the remark following Theorem 4, the Maslov cycle $\mathfrak{m}_{\zeta'}\cap F'$ is therefore empty. Thus ρ' is a local diffeomorphism. The commutativity of Figure 3 shows that this forces ρ to be a local diffeomorphism. □

Proposition 4.3. *The degree of* ρ *is* ±1. *Hence* ρ *is a diffeomorphism of the standard n-torus F with* Σ*.*

Proof. Let λ ("= p · dq") be the Liouville 1-form of $T^*\Sigma$ and define

$$\alpha = \iota_F^*(\lambda). \tag{3}$$

Since F is a lagrangian manifold, α is a closed 1-form on F.

Let Δ be the deck transformation group of the covering map ρ. Let $\wedge^k(F)$ be the vector space of smooth k-forms on F. Δ acts linearly on $\wedge^k(F)$ via pullback; let $\wedge^k(F)^\Delta$ denote the fixed-point set of Δ's action on $\wedge^k(F)$. It is a well-known fact that

$$\frac{\ker \mathrm{d}|\wedge^k(F)^\Delta}{\mathrm{im}\,\mathrm{d}|\wedge^{k+1}(F)^\Delta} \overset{\rho^*}{\cong} H^k_{\text{de Rham}}(F/\Delta) = H^k_{\text{de Rham}}(\Sigma).$$

Since F and Σ are topological n-tori, their cohomology groups are isomorphic. From these facts, there is a decomposition

$$\alpha = \alpha_0 + \alpha_1, \tag{4}$$

where $\alpha_0 \in \wedge^1(F)^\Delta$ is cohomologous to α and therefore $\alpha_1 = \mathrm{d}h$ is exact.

Observe that for all $x \in F$ and $\gamma \in \Delta$

$$\alpha_x - \gamma^*\alpha_{\gamma(x)} = \mathrm{d}h^\gamma_x,$$

where $h^\gamma(x) = h(x) - h(\gamma(x))$ is a smooth function. Since F is compact, there is a critical point $x = x_\gamma$ of h^γ, so that

$$\alpha_x - \gamma^*\alpha_{\gamma(x)} = 0. \tag{5}$$

To complete the proof, it is claimed that equation (5) implies that $\gamma = 1$. This proves that the deck transformation group Δ is trivial, whence ρ is a diffeomorphism.

To prove the claim, recall that the Liouville 1-form λ at $\theta \in T^*\Sigma$ is equal to the 1-form $(\mathrm{d}_\theta\pi)^*\,\theta \in T^*_\theta(T^*\Sigma)$. Therefore, for each $x \in F$,

$$\alpha_x = (\mathrm{d}_x\iota_F)^* \cdot \left(\mathrm{d}_{\iota_F(x)}\pi\right)^* (\iota_F(x)) = (\mathrm{d}_x\rho)^* (x), \tag{6}$$

where in the second step the identity $\rho = \pi \circ \iota_F$ has been used and the innocuous inclusion map dropped. Equation (6) implies that for all $\gamma \in \Delta$

$$\gamma^*\alpha_{\gamma(x)} = (\mathrm{d}_x\gamma)^* \cdot \left(\mathrm{d}_{\gamma(x)}\rho\right)^* (\gamma(x)) = (\mathrm{d}_x\rho)^* (\gamma(x)), \tag{7}$$

where $\rho \circ \gamma = \rho$ and the fact that $x, \gamma(x) \in T^*_{\rho(x)}\Sigma$ has been used. Therefore

$$\alpha_x - \gamma^*\alpha_{\gamma(x)} = (\mathrm{d}_x\rho)^* (x - \gamma(x))\,. \tag{8}$$

Since ρ is a local diffeomorphism, equation (8) shows that $\alpha_x - \gamma^*\alpha_{\gamma(x)} = 0$ iff $x = \gamma(x)$. Since Δ acts freely on F, equation (5) therefore implies that all elements of Δ are 1. □

Proof of Theorem 1. Proposition 4.3 proves Theorem 1. □

Remark III. Lalonde and Sikorav [23, p. 19] ask if the map ρ in figure 1 has $\deg\rho = \pm 1$ or possibly just $\neq 0$, when F is an exact lagrangian submanifold (i.e. when the 1-form α in equation (6) is exact). They prove that $\deg\rho = \pm 1$ and the Maslov class $\iota_F^*(d)$ vanishes when $F = \Sigma = \mathbf{T}^n$. In a similar vein, Bialy and Polterovich [6] prove that an invariant lagrangian 2-torus F contained in the unit co-sphere bundle of the 2-torus has $\deg\rho = \pm 1$ iff F is the disjoint union of lifts of globally minimizing unit-speed geodesics on the 2-torus. These results are sharpened in [7]. In [30], Polterovich proved that if F is an exact lagrangian torus in $T^*\mathbf{T}^n$, then F is a graph of a closed 1-form; see also [8, Corollary II] and [9, Theorem 1.1]. Propositions 4.2–4.3 may be viewed as an extension of each of these results.

5. Projectively equivalent metrics.

5.0.1. Preamble. Let Σ be a smooth n-dimensional manifold and let $\mathbf{g}, \bar{\mathbf{g}}$ be smooth riemannian metrics on Σ. These metrics are said to be projectively equivalent if their geodesics coincide as unparameterized curves. Projective equivalence is related to complete integrability in the following manner.

Define a $\bar{\mathbf{g}}$-self-adjoint $(1,1)$ tensor field G by

$$\mathsf{G} = \left(\frac{\det(\bar{\mathbf{g}})}{\det(\mathbf{g})}\right)^{\frac{1}{n+1}} \times \bar{\mathbf{g}}^{-1} \cdot \mathbf{g}, \tag{9}$$

where one views the metrics as self-adjoint bundle isomorphisms $T\Sigma \to T^*\Sigma$.

At each point $x \in \Sigma$, G has n real eigenvalues, and one can define continuous functions λ_i by declaring $\lambda_i(x)$ to be the i-th largest eigenvalue of G at x. The metrics are said to be strictly nonproportional at x if G has n distinct eigenvalues there. The functions λ_i are smooth in a neighbourhood of such an x.

Define a polynomial family of $(1, 1)$ tensor fields by

$$\mathsf{S}_\tau = \mathrm{adj}(\mathsf{G} - \tau), \tag{10}$$

where adj(•) is the classical adjoint matrix and τ is a real number. From these tensor fields, one obtains functions

$$I_\tau(x, v) = \langle \mathbf{g} \cdot \mathsf{S}_\tau \cdot v, v \rangle, \qquad \forall (x, v) \in T\Sigma. \tag{11}$$

Let $J_\tau = I_\tau \cdot \mathbf{g}^{-1}$ be the pullback of these functions to $T^*\Sigma$. Note that the lagrangian of the riemannian metric of $\bar{\mathbf{g}}$ is I_0, while that of $\mathbf{g}$ equals $\lim_{\tau\to\infty} \tau^{-n+1} I_\tau$.

THEOREM. (Topalov-Matveev 1998) *The family $\{J_\tau\}_{\tau\in\mathbf{R}}$ is a Poisson commuting family. If there exists a point $x \in M$ where G has n distinct eigenvalues, then the geodesic flow of $\mathbf{g}$ is completely integrable.*

This theorem, along with the theorem of Levi-Civita which establishes a normal form for the metrics in the neighbourhood of a regular point, suffice to prove the following:

THEOREM 5. *If Σ is a topological n-torus and $\mathbf{g}, \bar{\mathbf{g}}$ are projectively equivalent metrics that are strictly nonproportional at a single point, then Σ is diffeomorphic to the standard n-torus.*

Proof. From the discussion in Proposition 4.2, there is a finite covering $p\colon \Sigma' \to \Sigma$ where Σ' is diffeomorphic to the standard n-torus. The metrics $p^*\mathbf{g}, p^*\bar{\mathbf{g}}$ are also projectively equivalent and strictly nonproportional at some point.

Say that $\mathbf{g}_m, \bar{\mathbf{g}}_m$ are 'model' metrics on the standard torus $\mathbf{T}^n = \mathbf{R}/\mathbf{Z} \times \cdots \times \mathbf{R}/\mathbf{Z}$ if

$$\begin{aligned} \mathbf{g}_m &= \sum_{i=1}^{n} \Pi_i \, \mathrm{d}x_i^2, \\ \bar{\mathbf{g}}_m &= \sum_{i=1}^{n} \rho_i \Pi_i \, \mathrm{d}x_i^2, \quad \text{where} \\ \Pi_i &= (-1)^{n-i-1} \prod_{j\neq i} (\lambda_i - \lambda_j), \qquad \rho_i^{-1} = \lambda_i \cdot \lambda_1 \cdots \lambda_n, \end{aligned} \tag{12}$$

and $\lambda_i = \lambda_i(x_i)$ is a function of the i-th coordinate alone and

$$i < j \quad \Longrightarrow \quad \lambda_i(x) < \lambda_j(y) \quad \forall x, y. \tag{13}$$

By [25, Theorem 7], there is a diffeomorphism $h\colon \Sigma' \to \mathbf{T}^n$ which is an isometry of $p^*\mathbf{g}, p^*\bar{\mathbf{g}}$ with model metrics $\mathbf{g}_m, \bar{\mathbf{g}}_m$. Henceforth, it is assumed without loss of generality that h is the identity, $\Sigma' = \mathbf{T}^n$ and $p^*\mathbf{g} = \mathbf{g}_m, p^*\bar{\mathbf{g}} = \bar{\mathbf{g}}_m$.

In the coordinate system on Σ', one computes that

$$\text{(14)} \qquad \begin{aligned} \mathsf{S}_\tau &= \sum_{i=1}^n \mu_i(\tau) \frac{\partial}{\partial x_i} \otimes \mathrm{d}x_i, \qquad \mu_i(\tau) = \mu_i(\tau; x) = \prod_{j \neq i} (\lambda_j - \tau), \\ J_\tau &= \sum_{i=1}^n \mu_i(\tau)\, \Pi_i^{-1}\, y_i^2, \end{aligned}$$

where (x_i, y_i) are canonical coordinates on $T^*\Sigma'$. The hamiltonian vector field X_{J_τ} equals

$$\text{(15)} \qquad X_{J_\tau} = \begin{cases} \dot{x}_i &= \frac{2\mu_i y_i}{\Pi_i}, \\ \dot{y}_i &= \lambda_i' \sum_{j\neq i} \left(\frac{\lambda_j - \tau}{(\lambda_i - \tau)(\lambda_j - \lambda_i)}\right) \cdot \frac{\mu_j y_j^2}{\Pi_j}. \end{cases}$$

Define $\mathsf{H}_{(x,y)} \subset T_x\Sigma'$ to be the subspace spanned by the projection of the tangent vectors $X_{J_\tau}(x, y)$ to the base. If one chooses real numbers $t_1 < \cdots < t_n$, then it is clear that

$$\text{(16)} \qquad 0 \neq \det \begin{bmatrix} \mu_1(t_1) & \cdots & \mu_n(t_1) \\ \vdots & \ddots & \vdots \\ \mu_1(t_n) & \cdots & \mu_n(t_n) \end{bmatrix} \times \prod_{i=1}^n \frac{y_i}{\Pi_i} \quad \Longrightarrow \quad \dim \mathsf{H}_{(x,y)} = n.$$

Equation (13) implies that for all $x \in \Sigma'$, the eigenvalues λ_i are pairwise distinct, so the theory of Lagrange interpolation shows that the polynomials $\mu_i(\tau)$ are linearly independent, whence the determinant on the left is nowhere zero. To complete the proof of the theorem, it therefore suffices to show that there is a lagrangian torus $F' \subset T^*\Sigma'$ such that $y_1 \cdots y_n$ does not vanish on F'.

The image of the first-integral map. Let V be the vector space of polynomials in τ of degree at most $n - 1$. The map $(x, y) \mapsto J_\tau(x, y)$ defines a smooth map $J\colon T^*\Sigma' \to \mathsf{V}$. The following claim is essential to describe the image of J.

Claim 5.1. Let $x \in \Sigma'$ and $t_i = \lambda_i(x)$. If $\tau_i \in [t_i, t_{i+1}]$ for all i, then there exists $a_i \geq 0$ with $\sum_{i=1}^n a_i = 1$, such that

$$\text{(17)} \qquad p(\tau) = \sum_{i=1}^n a_i \mu_i(\tau; x)$$

vanishes at τ_i for all i. If $\tau_i \in (t_i, t_{i+1})$ for all i, then $a_i > 0$ for all i.

Check. Let $\tau_1 \leq \tau_2 \leq \cdots \leq \tau_{n-1}$ be real numbers and define

$$\sigma(\tau) := \prod_{i=1}^{n-1} (\tau_i - \tau). \tag{18}$$

A computation shows that if $p(\tau)$ in equation (17) vanishes at τ_i for all i, then the coefficients a_i may be written as

$$a_i = \frac{\sigma(t_i)}{\mu_i(t_i)} \qquad \forall i = 1, \ldots, n, \tag{19}$$

where $\sum a_i = 1$. Comparison of equations (18,14) shows that $a_i \geq 0$ (resp. $a_i > 0$) for all i iff $\tau_i \in [t_i, t_{i+1}]$ (resp. $\tau_i \in (t_i, t_{i+1})$) for all i.

To continue the mainline of the proof: Let $\underline{\lambda}_i$ (resp. $\bar{\lambda}_i$) be the maximum (resp. minimum) value attained by λ_i. Since the eigenvalues λ_i are everywhere distinct by (13), there exists $\tau_i \in (\underline{\lambda}_i, \bar{\lambda}_{i+1})$ for all i. Since $\lambda_i(x) \leq \underline{\lambda}_i < \tau_i < \bar{\lambda}_{i+1} \leq \lambda_{i+1}(x)$ for all $x \in \Sigma'$ and i, the claim establishes that for all $x \in \Sigma'$, there is a $y \in T^*_x\Sigma'$ such that $J_\tau(x, y) = p(\tau)$ where p has roots $\tau_1, \ldots, \tau_{n-1}$. Moreover, the coefficients $a_i = y_i^2/\Pi_i$ are everywhere nonzero by the same claim. From (16), one sees that the canonical projection map $\rho'\colon F' \to \Sigma'$ is a local diffeomorphism, where $F' = J^{-1}(p)$. Proposition 4.3 implies that ρ' is a diffeomorphism. Therefore, the map ρ in figure 3 is a diffeomorphism, and Σ is therefore diffeomorphic to the standard n-torus. □

Remark IV. Claim 5.1 implies that the image of the first-integral map J contains the polynomials of the form

$$q(\tau) = a \times \prod_{i=1}^{n-1} (\tau_i - \tau), \qquad a > 0,\ \tau_i \in (\bar{\lambda}_i, \underline{\lambda}_{i+1}). \tag{20}$$

A connected component of the pre-image $J^{-1}(q)$ of such a polynomial is a regular lagrangian torus whose projection to the base Σ' is a diffeomorphism. If one of the roots τ_i of q lie in $[\bar{\lambda}_i, \underline{\lambda}_i]$, and q is a regular value of J, then the Maslov cycle of each component of $J^{-1}(q)$ is nontrivial and one can see from the claim that the projection of $J^{-1}(q)$ does not cover the base.

Additionally, the claim shows that if $\tau_i \in [\underline{\lambda}_i, \bar{\lambda}_{i+1}]$ for all $i = 1, \ldots, n-1$, then there exists some $(x, y) \in T^*\Sigma'$ such that $J_\tau(x, y) = q(\tau)$. This shows that the image of J is the set of all polynomials

$$q(\tau) = a \times \prod_{i=1}^{n-1} (\tau_i - \tau), \quad \text{such that } a \geq 0, \quad \tau_i \in [\underline{\lambda}_i, \bar{\lambda}_{i+1}] \tag{21}$$
$$\text{and } \tau_1 \leq \tau_2 \leq \cdots \leq \tau_{n-1}.$$

In [25, Theorem 8], Matveev proves Theorem 5 in a different manner. The model metric's Levi-Civita coordinate system on Σ' induces an integral affine structure on Σ whose holonomy group is the orthogonal group. Therefore Σ is diffeomorphic to the standard n-torus by the second Bieberbach theorem. [33, Section 9] contains a theorem similar to Theorem 5, also.

6. Exotic smooth 7-manifolds.

6.1. Witten-Kreck-Stolz manifolds. Let k, l be coprime integers that are both nonzero. The action of U_1 on $S^5 \times S^3$ by

$$\forall z \in S^1, x \in S^5, y \in S^3:\ z \cdot (x, y) = (z^k \cdot x, z^l \cdot y) \tag{22}$$

is free. Let $M_{k,l}$ be the orbit space of this action; it is a compact simply connected 7-manifold. Equivalently, let $G = U_3 \times U_2$ and let $U \subset G$ be the subgroup isomorphic to $U_2 \times U_1^2$ defined by

$$U = \left\{ z^k \cdot \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} \oplus z^l \cdot \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix} :\ a \in U_2,\ z, w \in U_1 \right\}. \tag{23}$$

The manifold $M_{k,l}$ is G-equivariantly diffeomorphic to the homogeneous space G/U.

These manifolds have been studied by Witten in the context of Kaluza-Klein theory and by Wang and Ziller, who constructed Einstein metrics on each manifold with positive scalar curvature [38], [37]. In [21], Kreck and Stolz classify the manifolds up to homeomorphism and diffeomorphism. As a consequence of the triviality of $H^3(M_{k,l}; \mathbf{Z}_2)$, it is known from smoothing theory that if $M_{k',l'}$ is homeomorphic to $M_{k,l}$, then the former is diffeomorphic to the latter connect-summed with an exotic 7-sphere. There are, therefore, at most 28 oriented diffeomorphism classes within any homeomorphism class. Combining Corollary D and the Remark preceding it in [21], we have

THEOREM. (Kreck-Stolz 1988) *If* $l = 0 \bmod 4$, $l = 0, 3, 4 \bmod 7$, $l \neq 0$ *and* $(k, l) = 1$, *then the homeomorphism class of* $M_{k,l}$ *has* 28 *diffeomorphism classes. Each diffeomorphism class is represented by an* $M_{k',l'}$ *for suitable* k', l'.

The simplest example satisfying the hypotheses of the theorem is the manifold $M_{1,4}$. By theorem B of [21], $M_{k',l'}$ is homeomorphic to $M_{1,4}$ iff $l' = \pm 4$ and $k' = 1 \bmod 32$; on the other hand, $M_{k',l'}$ is diffeomorphic to $M_{1,4}$ iff $l' = \pm 4$ and $k' = 1 \bmod 28 \times 32$. Thus $M_{32t+1,4}$ enumerates all diffeomorphism classes of $M_{1,4}$ for $t = 0, \ldots, 27$.

On the other hand, Mykytyuk and Panasyuk have studied the integrability of the canonical quadratic hamiltonian on homogeneous spaces. Theorem 3.10 of [27] implies:

THEOREM. (Mykytyuk-Panasyuk 2004) *Let G be a compact reductive Lie group. Let $K \subset G$ be the stabilizer of some element $a \in \mathfrak{g}^*$. If $U \subset K$ contains the identity component of $[K, K]$, then the quadratic hamiltonian on $T^*(G/U)$ induced by a bi-invariant metric on G is completely integrable with real-analytic integrals.*

To apply this theorem to the Witten-Kreck-Stolz manifolds, let

$$K = \left\{ u \cdot \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} \oplus v \cdot \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix} : \ u, v, w \in \mathrm{U}_1, \quad a \in \mathrm{SU}_2 \right\},$$

which is easily seen to be the stabilizer subgroup under the coadjoint action of the element

$$\begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \in \mathrm{u}_3^* \oplus \mathrm{u}_2^*.$$

Since the subgroup U, defined in equation (23) above, contains $[K, K]$ and $U \subset K$, the Mykytyuk-Panasyuk theorem is applicable to each Witten-Kreck-Stolz manifold. This proves:

THEOREM 6. *Each homogeneous space $M_{k,l}$ has a real-analytically completely integrable convex hamiltonian on its cotangent bundle. In particular, if k and l satisfy the conditions of Theorem 6.1, then each diffeomorphism class of manifolds homeomorphic to $M_{k,l}$ has such a real-analytically integrable convex hamiltonian.*

Remark V. There is a more pedestrian approach to proving Theorem 6 which uses the above description of $M_{k,l}$ as the quotient of $S^5 \times S^3$ by a subgroup of the torus $V \cong \mathrm{U}_1 \subset \mathrm{U}_1 \times \mathrm{U}_1$ (equation (22)). Introduce the notation

$$\begin{aligned} T^*S^5 &= \left\{ (\mathbf{x}, \mathbf{y}) \in \mathbf{C}^3 \times \mathbf{C}^3 : \ \mathbf{x}^*\mathbf{x} = 1, \mathrm{Re}\, \mathbf{y}^*\mathbf{x} = 0 \right\}, \\ T^*S^3 &= \left\{ (\mathbf{w}, \mathbf{z}) \in \mathbf{C}^2 \times \mathbf{C}^2 : \ \mathbf{w}^*\mathbf{w} = 1, \mathrm{Re}\, \mathbf{z}^*\mathbf{w} = 0 \right\}, \end{aligned} \tag{24}$$

where $\mathbf{x}^*$ denotes the hermitian transpose of the vector $\mathbf{x}$ (this use of * is confined to the present remark). The momentum map of the $G = \mathrm{U}_3 \times \mathrm{U}_2$ action on $T^*(S^5 \times S^3)$ equals, in these coordinates

$$\Psi_G = \frac{1}{2} \times \left(\mathbf{x}\mathbf{y}^* - \mathbf{y}\mathbf{x}^* \right) \oplus \frac{1}{2} \times \left(\mathbf{w}\mathbf{z}^* - \mathbf{z}\mathbf{w}^* \right), \tag{25}$$

$$\Psi_G : \ T^*(S^5 \times S^3) \to \mathrm{u}_3 \oplus \mathrm{u}_2,$$

while the momentum map of the subgroup V equals

$$\Psi_V = k \times \mathbf{y}^*\mathbf{x} + l \times \mathbf{z}^*\mathbf{w}, \tag{26}$$

where we identify the Lie algebra of V with $i\mathbf{R}$. Define $\pi_{a,b}\colon\ \mathrm{u}_a \to \mathrm{u}_b$ to be the orthogonal projection onto the subalgebra $\mathrm{u}_b \subset \mathrm{u}_a$, which we embed in the standard way in the lower right-hand corner. Define the following functions for $\xi \oplus \eta \in \mathrm{u}_3 \oplus \mathrm{u}_2$ by

$$\begin{cases} f_1 = -i \cdot \pi_{3,1}(\xi), & f_2 = -i \cdot \operatorname{Trace} \pi_{3,2}(\xi), \\ f_3 = -i \operatorname{Trace} \xi, & f_4 = \frac{1}{2} \cdot \operatorname{Trace} \pi_{3,2}(\xi)^2, \\ f_5 = \frac{1}{2} \cdot \operatorname{Trace} \xi^2, & f_6 = -i \cdot \pi_{2,1}(\eta), \\ f_7 = -i \operatorname{Trace} \eta, & f_8 = \frac{1}{2} \cdot \operatorname{Trace} \eta^2, \end{cases} \tag{27}$$

and define to be $H_a = f_a \circ \Psi_G$. The functions H_a are all in involution, a fact that is easily confirmed by the observation that $\pi_{a,b}$ is a Poisson map (it is the transpose of the inclusion $\iota_{a,b}\colon\ \mathrm{u}_b \to \mathrm{u}_a$), and the functions being pulled-back are Casimirs. A simple computation shows that the eight hamiltonians are functionally independent at $(\mathbf{x}_o, \mathbf{y}_o) = (2/9, 1/9, 2/9, i, -4i, i)$ and $(\mathbf{w}_o, \mathbf{z}_o) = (3/5, 4/5, 4i, -3i)$. One also notes that $H = H_5 + H_8$ is a convex hamiltonian that equals $-\frac{1}{4} \times \left(|\mathbf{y}|^2 + |\mathbf{z}|^2 - (\mathbf{y}^*\mathbf{x})^2 - (\mathbf{z}^*\mathbf{w})^2\right)$ and that $H' = H_5 + H_8 + \frac{1}{4}\left(H_3^2 + H_7^2\right) = -\frac{1}{4} \times \left(|\mathbf{y}|^2 + |\mathbf{z}|^2\right)$.

One observes that

$$\Psi_V = ik \times H_3 + il \times H_7, \tag{28}$$

and that $T^*M_{k,l}$ is canonically symplectomorphic to $\Psi_V^{-1}(0)/V$. It is observed that the point $(\mathbf{x}_o, \mathbf{y}_o, \mathbf{w}_o, \mathbf{z}_o)$ at which the eight hamiltonians are functionally independent lies in $\Psi_V^{-1}(0)$. Therefore, since $kl \neq 0$, the seven hamiltonians H_a, $a \neq 3$, descend to $T^*M_{k,l}$ to give seven functionally independent real-analytic hamiltonians that are in involution. The convex hamiltonian $H = H_5 + H_8$ is the desired real-analytically integrable convex hamiltonian on $T^*M_{k,l}$. This provides an alternative, more computational, proof of Theorem 6.

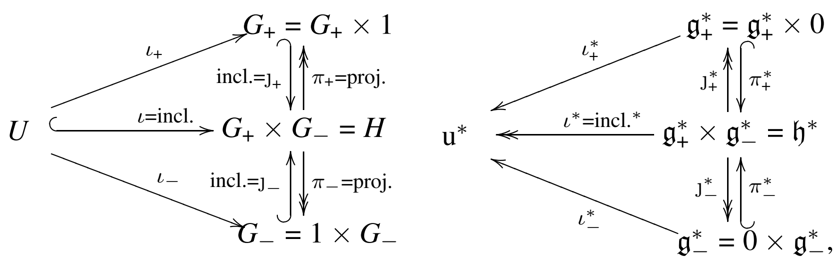
6.2. A Lemma on integrability. A second class of well-known 7-manifolds are the Eschenburg spaces, which include the well-known Aloff-Wallach spaces [19], [1]. In the next subsection, these spaces are described more completely. This subsection proves a lemma about the existence of real-analytically completely integrable convex hamiltonians on spaces constructed like the Eschenburg spaces. This lemma is new and possibly of independent interest.

Let G be a compact Lie group and let $G \times G$ act on G by the convention

$$\forall h = (h_1, h_2) \in G \times G, g \in G\colon\ h \cdot g = h_1 g h_2^{-1}.$$

This can be understood as a left action on G by the group $H = G_+ \times G_-$, where $G_{+/-}$ is G equipped with left/right multiplication respectively.

Let $U \subset G_+ \times G_-$ be a closed subgroup that acts freely on G. There are naturally induced maps



where notation is abused and the induced map on Lie algebras is denoted by the same symbol.

For a subgroup S of H, let $\Psi_S\colon\ T^*G \to \mathfrak{s}^*$ be the momentum map of S's action. This momentum map is the composition

$$T^*G \xrightarrow{\Psi_H} \mathfrak{h}^* \xrightarrow{\iota_S^*} \mathfrak{s}^*, \qquad \Psi_S = \iota_S^* \circ \Psi_H,$$

where $\iota_S\colon\ \mathfrak{s} \to \mathfrak{h}$ is the inclusion. Let us identify $T^*G = G \times \mathfrak{g}^*$ via left-translation. The momentum map Ψ_{G_+} of G's left action (resp. Ψ_{G_-}, right action) on T^*G is equal, in this trivialization, to

$$\forall g \in G, x \in \mathfrak{g}^*\colon\ \Psi_{G_+}(g,x) = \mathrm{Ad}^*_{g^{-1}}x, \qquad \Psi_{G_-}(g,x) = x.$$

In particular,

$$\Psi_H = \Psi_{G_+} \oplus -\Psi_{G_-}.$$

The key property of the momentum map is its equivariance

$$\forall s \in S, P \in T^*G\colon\ \Psi_S(s \cdot P) = \mathrm{Ad}^*_{s^{-1}}\Psi_S(P).$$

Let $\Sigma = G/U$, the quotient of G by the free action of U. The cotangent bundle of Σ is known to be symplectomorphic to the quotient $\Psi_U^{-1}(0)/U$. Therefore, if $f_1, \ldots, f_n$ is a collection of $n = \dim \Sigma$ analytic functions on T^*G that are U-invariant, in involution, and functionally independent along $\Psi_S^{-1}(0)$, then they induce a real-analytically completely integrable system on $T^*\Sigma$. In our case, we want f_1 to be a convex hamiltonian; the natural choice is to have f_1 be the natural bi-invariant metric on T^*G induced by the Cartan-Killing form.

To construct the requisite integrals, it is necessary to digress somewhat.

6.2.1. Poisson algebras. Let M be a smooth manifold. A Poisson structure on M is a Lie algebra structure $\{,\}$ on $C^\infty(M)$ that makes $C^\infty(M)$ into a Lie algebra of derivations of $C^\infty(M)$. Let $\mathfrak{a} \subset C^\infty(M)$ be a subset — generally, a subalgebra — and denote by

$$\text{(29)} \qquad \mathrm{d}\mathfrak{a}_m = \operatorname{span}\{\mathrm{d}_m f\colon\ f \in \mathfrak{a}\}, \qquad Z(\mathfrak{a}) = \{f \in \mathfrak{a}\colon\ \{f,g\} \equiv 0 \ \ \forall g \in \mathfrak{a}\}.$$

The subset $Z(\mathfrak{a})$ is the centre of $\mathfrak{a}$: all functions in $\mathfrak{a}$ are integrals of the hamiltonian vector fields $X_f(\bullet) = \{f, \bullet\}\colon\ f \in \mathfrak{a}$. Let

$$\text{(30)} \qquad \begin{aligned} \operatorname{ddim}(\mathfrak{a})_m &= \dim \mathrm{d}\mathfrak{a}_m, \\ \operatorname{drank}(\mathfrak{a})_m &= \dim \operatorname{span}\{X_f(m) = \{f,\cdot\}_m\colon\ f \in Z(\mathfrak{a})\}, \end{aligned}$$

be the differential dimension and the differential rank of $\mathfrak{a}$ at the point $m \in M$. We denote by $\operatorname{ddim}(\mathfrak{a})$ the maximum of $\{\operatorname{ddim}(\mathfrak{a})_m\colon\ m \in M\}$ and by $\operatorname{drank}(\mathfrak{a})$ the maximum of $\{\operatorname{drank}(\mathfrak{a})_m\colon\ \operatorname{ddim}(\mathfrak{a})_m = \operatorname{ddim}(\mathfrak{a})\}$. If $\mathfrak{a}$ contains a proper function, then $\operatorname{drank}(\mathfrak{a}) + \operatorname{ddim}(\mathfrak{a}) = \dim M$ implies that the flow of any hamiltonian in the centre of $\mathfrak{a}$ is integrable.

When $M = \mathfrak{h}^*$ equipped with the canonical Poisson bracket, the coadjoint orbits are symplectic leaves of the Poisson bracket. If a subset $\mathfrak{a} \subset \mathrm{C}^\omega(\mathfrak{h}^*)$ satisfies

$$\text{(31)} \qquad \operatorname{ddim}(\mathfrak{a})_x = \frac{1}{2}(\dim H + \operatorname{rank} H), \qquad \operatorname{drank}(\mathfrak{a})_x = \frac{1}{2}(\dim H - \operatorname{rank} H),$$

for some regular $x \in \mathfrak{h}^*$, then $\mathfrak{a}|\mathcal{O}_y$ defines a real-analytically completely integrable system for all y in an open real-analytic subset of $\mathfrak{h}^*$. If $\mathfrak{a}$ satisfies the conditions (31), then $\mathfrak{a}$ will be said to be completely integrable.

The centre of $\mathrm{C}^\omega(\mathfrak{h}^*)$ is the set of real-analytic Casimirs; these are functions that are constant on each coadjoint orbit. It is a classic theorem of Cartan's that $Z(\mathrm{C}^\omega(\mathfrak{h}^*))$ is finitely generated by polynomials when $\mathfrak{h}$ is semisimple. A technique that generates a completely integrable algebra $\mathfrak{a}$, that was discovered by Mischenko and Fomenko [26] and is related to Lax representations, is the argument-shift technique.

THEOREM. (Mischenko-Fomenko 1978) *Let $a \in \mathfrak{h}^*$ and define $\mathfrak{a} \subset \mathrm{C}^\omega(\mathfrak{h}^*)$ by*

$$\text{(32)} \qquad \mathfrak{a} := \{f\colon\ \exists\lambda \in \mathbf{R}, g \in Z(\mathrm{C}^\omega(\mathfrak{h}^*))\ s.t.\ f(x) = g(x + \lambda \cdot a)\}.$$

The algebra $\mathfrak{a}$ is abelian and if a is a regular element, then $\mathfrak{a}$ is completely integrable.

Note that $\mathfrak{a}$ always contains the Casimirs, and in particular, the Cartan-Killing form of $\mathfrak{h}^*$.

6.2.2. The pull-back algebra. Let $\mathfrak{a}$ be a completely integrable subalgebra of $C^{\omega}(\mathfrak{h}^*)$ and let $\mathfrak{b} = \Psi_H^* \mathfrak{a} \subset C^{\omega}(T^*G)$ be the algebra pulled-back to T^*G. Since the momentum map Ψ_H is Poisson, $\mathfrak{b}$ is also abelian. We would like to compute $\operatorname{drank}(\mathfrak{b})$ and $\operatorname{ddim}(\mathfrak{b})$.

LEMMA 6.1. *The differential rank and dimension of* $\mathfrak{b}$ *equal* $\dim G$. *Therefore,* $\mathfrak{b}$ *defines a real-analytic completely integrable subalgebra of* $C^{\omega}(T^*G)$.

Proof. Let $f \in C^{\omega}(\mathfrak{h}^*)$ and let $F = f \circ \Psi_H$ be the pull-back of f by the momentum map. The chain rule shows that for $P \in T^*G$

$$\mathrm{d}_P F = \mathrm{d}_x f \cdot \mathrm{d}_P \Psi_H, \tag{33}$$

where $x = \Psi_H(P)$. With the left-trivialization of T^*G, one can write $P = (g, \mu)$ and a tangent vector $v \in T_P(T^*G)$ equals $\xi \oplus \eta \in \mathfrak{g} \oplus \mathfrak{g}^*$. With this notation, and denoting $\alpha_\pm = \pi_\pm(\mathrm{d}_x f)$ and $x_\pm = \jmath_\pm^*(x)$, one sees that $\mathrm{d}_P F \cdot v = 0$ iff

$$0 = \mathrm{d}_x f \cdot \mathrm{d}_P \Psi_H \cdot v = \left\langle \alpha_+, -\mathrm{ad}_\xi^* x_+ + \mathrm{Ad}_{g^{-1}}^* \eta \right\rangle - \langle \alpha_-, \eta \rangle . \tag{34}$$

If $\mathrm{d}_P F = 0$, then equation (34) vanishes for all v. With $\eta = 0$, this shows that α_+ vanishes on $\mathrm{ad}_{\mathfrak{g}}^* x_+$, which is the tangent space to G's coadjoint orbit through x_+. From this, equation (34) yields that

$$\mathrm{d}_P F = 0 \qquad \Longleftrightarrow \qquad \alpha_- = \mathrm{Ad}_{g^{-1}} \alpha_+ \text{ and } \alpha_+|_{\mathrm{ad}_{\mathfrak{g}}^* x_+} = 0, \tag{35}$$

and the second condition implies that $\alpha_-|_{\mathrm{ad}_{\mathfrak{g}}^* x_-} = 0$, too.

Assume that the point x_+ is a regular element in $\mathfrak{g}_+^*$ (since $x_- = -\mu$ and $x_+ = \mathrm{Ad}_{g^{-1}}^* \mu$, one can equally assume that μ is a regular point). In this case, the annihilator of $\mathrm{ad}_{\mathfrak{g}}^* x_\pm$ is spanned by the derivatives of the Casimirs of $\mathfrak{g}_\pm^*$. Therefore, there is a Casimir ϕ of $\mathfrak{g}_+^*$ such that $\alpha_+ = \mathrm{d}_{x_+} \phi$. The right-hand side of (35) along with the fact that $x_- = -\mathrm{Ad}_g^* x_+$ implies that $\alpha_- = -\mathrm{d}_{x_-} \phi$. Therefore

$$\mathrm{d}_P F = 0 \qquad \Longleftrightarrow \qquad \exists \phi \in Z(C^{\omega}(\mathfrak{g}^*)) \ \text{ s.t. } \ \mathrm{d}_x f = \mathrm{d}_{x_+} \phi \oplus -\mathrm{d}_{x_-} \phi. \tag{36}$$

This implies that $\mathfrak{a}$ contains the 1-jets of all functions $F \in \Psi_H^* C^{\omega}(\mathfrak{h}^*)$ with $\mathrm{d}_P F = 0$. Therefore, the dimension of $\mathrm{d}_P \Psi_H^* \mathfrak{a}$ is equal to the dimension of $\mathrm{d}_x \mathfrak{a}$ minus the dimension of $\ker \mathrm{d}_P \Psi_H^*$. The latter dimension equals $\operatorname{rank} \mathfrak{g}$ by (36). Therefore,

$$\operatorname{ddim}(\Psi_H^* \mathfrak{a})_P = \frac{1}{2} \times (\dim H + \operatorname{rank} H) - \operatorname{rank} G = \dim G. \tag{37}$$

Since $\mathfrak{b} = \Psi_H^* \mathfrak{a}$ is abelian, this proves the lemma. □

THEOREM 7. *Let $T \subset G$ be a maximal torus and assume that $U \subset T \times T$ acts freely on G. Then, there are completely integrable algebras $\mathfrak{a} \subset C^{\omega}(\mathfrak{h}^*)$ such that $\mathfrak{b} = \Psi_H^* \mathfrak{a}$ induces a real-analytic, completely integrable convex hamiltonian on $T^*(G/U)$.*

Proof. $T^*(G/U)$ is canonically symplectomorphic to $\Psi_U^{-1}(0)/U$, so it suffices to find an algebra $\mathfrak{a}$ such that $\mathfrak{a}$ is Ad_U^*-invariant and $\Psi_U^{-1}(0)$ contains a regular point for $\mathfrak{b}$.

To achieve Ad_U^*-invariance of $\mathfrak{a}$, let the a in the Mischenko-Fomenko construction be chosen to lie in $\mathfrak{t}^* \oplus \mathfrak{t}^*$. Since $T \times T$ is a maximal torus containing U, there are regular elements $a \in \mathfrak{t}^* \oplus \mathfrak{t}^*$ and these are stabilized by Ad_U^*. The equivariance of Ψ_H implies that $\mathfrak{b}$ is invariant under the action of U. This implies that $\mathfrak{b}$ and $\Psi_U^* C^{\omega}(\mathfrak{u}^*)$ Poisson commute; it also implies that $\mathfrak{b}|\Psi_U^{-1}(0)$ induces a commutative Poisson algebra $\tilde{\mathfrak{b}}$ of real-analytic functions on $T^*(G/U)$.

The proof of Lemma 6.1 shows that $P = (g, \mu) \in T^*G$ is regular for $\mathfrak{b}$ if μ is a regular point in $\mathfrak{g}^*$. Since U is contained in a maximal torus $T \times T$, the simple form of Ψ_H shows that the image of $\Psi_H|\Psi_U^{-1}(0) \cap T_1^*G$ contains the subspace of vectors $x = \eta \oplus -\eta$ such that $\eta \in \mathfrak{t}^{\perp}$. Since $\mathfrak{t}^{\perp}$ contains regular elements, one concludes that there are regular points for $\mathfrak{b}$ in $\Psi_U^{-1}(0)$.

Since $\mathfrak{b}$ and $\Psi_U^* C^{\omega}(\mathfrak{u}^*)$ Poisson commute and each is real-analytic, there is no loss in replacing $\mathfrak{b}$ with the algebra $\mathfrak{b} + \Psi_U^* C^{\omega}(\mathfrak{u}^*)$. To avoid a proliferation of notation, let $\mathfrak{b}$ denote this expanded algebra. If $P \in \Psi_U^{-1}(0)$ is a regular point for $\mathfrak{b}$, then we conclude that

$$\begin{aligned} \mathrm{d}_{U\cdot P}\tilde{\mathfrak{b}} &\cong \mathrm{d}_P \mathfrak{b} / \mathrm{d}_P \Psi_U^* C^{\omega}(\mathfrak{u}^*) \\ \Longrightarrow \quad \mathrm{ddim}(\tilde{\mathfrak{b}})_{U\cdot P} &= \mathrm{ddim}(\mathfrak{b}) - \dim U = \dim G/U, \end{aligned} \tag{38}$$

since the action of U is free and $\mathfrak{b}$ is completely integrable. Therefore $\tilde{\mathfrak{b}}$ is a real-analytic, completely integrable algebra on $T^*(G/U)$. Since $\mathfrak{a}$ contains the Casimirs of $\mathfrak{h}^*$, the Cartan-Killing form in $\mathfrak{a}$ induces a riemannian metric hamiltonian in $\tilde{\mathfrak{b}}$. This is the convex hamiltonian that was sought. □

6.3. Aloff-Wallach and Eschenburg spaces. Recall the definition of an Eschenburg space [19]: let $U \cong U_1$ be a subgroup of $SU_3 \times SU_3$ such that the natural action of U on SU_3 defined by

$$\forall u = (u_1, u_2) \in U, g \in SU_3: \; u \cdot g = u_1 g u_2^{-1}$$

is free. U can be characterized in terms of 4 integers–k, l, p, q–as

$$U = \left\{ \mathrm{diag}(z^k, z^l, z^{-k-l}) \oplus \mathrm{diag}(z^p, z^q, z^{-p-q}) \colon \; z \in S^1 \right\}$$

and k, l, p, q satisfy

$$\begin{aligned}&\gcd(k-p, l-q), \quad \gcd(k-p, k+p+q), \quad \gcd(k+p+q, l-p),\\ &\gcd(k-q, l-p), \quad \gcd(k-q, k+p+q), \quad \text{and} \ \gcd(k+p+q, l-q) \quad \text{equal } 1.\end{aligned}$$

Such 4-tuples of integers are called admissible. Let k, l, p, q be an admissible quartet and let $U = U_{klpq}$ be such a group, $M = M_{klpq} = \mathrm{SU}_3/U$. The manifold M_{klpq} is an Eschenburg space. When $k = l = 0$, one has an Aloff-Wallach manifold [1].

Let κ be a bi-invariant metric on SU_3 and let κ_M be the submersion metric on M induced by κ. Let H and H_M be the induced fibre-quadratic hamiltonians on $T^*\mathrm{SU}_3$ and T^*M, respectively.

THEOREM 8. *The hamiltonian H_M is real-analytically completely integrable for any admissible quartet k, l, p, q.*

Proof. This is a simple corollary of the Theorem 7. □

Remark VI. Here is a sketch of a pedestrian proof of Theorem 8. The functions f_1,f_2,f_4,f_5 defined in equation (27) are independent and in involution on su_3; combined with $f_9 = \det \xi$, one obtains a completely integrable algebra of functions on su_3. The functions f_1,f_2 generate the coadjoint action of the maximal torus of diagonal matrices. Therefore, the functions $F_{i,\pm} = f_i \circ \Psi_{G_\pm}$ with $G = \mathrm{SU}_3$ yield a completely integrable algebra on T^*G that is invariant under the coadjoint action of the maximal torus in $G_+ \times G_-$ consisting of diagonal matrices.

Paternain and Spatzier [29] proved the integrability of H_M on Eschenburg spaces $M_{1,-1,2m,2m}$, using integrals like those in the above paragraph along with some involved computations. Bazaikin [4, Section 5] proved the integrability of a submersion geodesic flow on an Eschenburg space M when $M = H\backslash G/K$ where $G = \mathrm{U}_3 \oplus \mathrm{U}_2 \oplus \mathrm{U}_1$, $H = \mathrm{U}_2 \oplus \mathrm{U}_1$ and K is isomorphic to $\mathrm{U}_1 \oplus \mathrm{U}_1$ – these Eschenburg spaces are positively curved, in addition to having this special bi-quotient structure. Theorem 8 generalizes each of these results.

Kruggel [22] has obtained a homeomorphism and diffeomorphism classification of Eschenburg spaces that satisfy his condition C. Condition C implies that the Eschenburg space is cobordant to a union of 3 lens spaces. The second integral cohomology group of the Eschenburg space $M = M_{klpq}$ is infinite cyclic with generator u; the fourth integral cohomology group is finite cyclic of order r with generator u^2. The first Pontryagin class $p_1(M) = p_1 \cdot u^2$ and u has a self-linking number $-s^{-1}/r \in \mathbf{Q}/\mathbf{Z}$, where s^{-1} is the inverse to s in $\mathbf{Z}_r$. The final two invariants, the Kreck-Stolz invariants $s_1, s_2 \in \mathbf{Q}/\mathbf{Z}$ are invariants of M that Kruggel showed are calculable in terms of the eta-invariants of the cobounding lens spaces. Each invariant is expressible in terms of the parameters k, l, p and q, but the formulae for s_1, s_2 involve transcendental functions. The invariants $r \in \mathbf{Z}$, $p_1, s \in \mathbf{Z}_r$ and $s_2 \in \mathbf{Q}/\mathbf{Z}$ determine the oriented homeomorphism type of M_{klpq}, while r, p_1, s, s_1 and s_2 determine the oriented diffeomorphism type of M_{klpq}. Because $H^3(M; \mathbf{Z}_2) = 0$, there are a maximum of 28 oriented smooth structures

k	l	p	q	s_1 mod 1	k	l	p	q	s_1 mod 1
−29	10	−28	6	1	−21	−6	−18	−10	0.5
−38	−29	−66	22	0.964286	−5	−5	−6	−4	0.464286
−54	9	−52	4	0.928571	−13	2	−8	−6	0.428571
−17	−17	−18	−16	0.892857	−14	−5	−16	−2	0.392857
−6	−3	−8	0	0.857143	−9	−6	−12	−2	0.357143
−17	−14	−22	−8	0.821429	−38	−11	−48	8	0.321429
−14	−5	−18	2	0.785714	−22	−19	−40	12	0.285714
−1	−1	−2	0	0.75	−22	−1	−14	−12	0.25
−33	−6	−42	20	0.714286	−25	−1	−22	−6	0.214286
−46	−13	−32	−30	0.678571	−54	−9	−68	30	0.178571
−22	5	−20	0	0.642857	−39	−6	−32	−16	0.142857
−13	2	−14	6	0.607143	−29	−14	−32	−10	0.107143
−38	−11	−40	−8	0.571429	−11	1	−12	4	0.071429
−22	−1	−26	12	0.535714	−9	−9	−10	−8	0.035714

Figure 4. Representative Eschenburg spaces M_{klpq} in the oriented homeomorphism class with $r = 1$, $s = 0$, $p_1 = 0$ and $s_2 = 0.25$ mod 1. The Kreck-Stolz invariant s_1 has been rounded to six decimal places. See [14] for details of the computations.

on a topological Eschenburg space. In figure 4 one sees the results of a numerical search for these 28 oriented smooth structures on the oriented topological Eschenburg space $M_{-1,-1,-2,0}$. It is notable that, up to six decimal places, the Kreck-Stolz invariant s_1 equals $i/28$ for $i = 28, \ldots, 1$.

7. Conclusion. In [18], Dullin, Robbins, Waalkens, Creagh and Tanner demonstrate that a cohomologically nonvanishing Maslov cocycle constrains the monodromy of a completely integrable system. Specifically, they show that the cohomology class of the Maslov cocycle, if nonzero on a lagrangian torus, is a common eigenvector of the monodromy group of the lagrangian fibration. Their work *assumes* the nontriviality of the Maslov cocycle, and ends with the question: *Does the Maslov cocycle of an invariant torus of a natural mechanical hamiltonian on* $T^*\mathbf{R}^n$ *always vanish?*

The answer to their question is *yes* and was proven by Viterbo in the work cited above [34]. Viterbo proved that if $F \subset T^*\mathbf{R}^n$ is a lagrangian torus, then there is a cycle on F whose Maslov index is an even integer between 2 and $n+1$ inclusive. The proof uses Conley-Zehnder theory, and the strength of the result is the constraint on how far the Maslov cocycle may be from primitive. This is used to prove the above-cited result on the vanishing of the Maslov cocycle when ρ has a nonzero degree.

There are several natural questions that arise from the note of Dullin, *et al.*. First, there are higher-dimensional Maslov cocycles that measure the higher singularities of ρ; the cohomological nontriviality of these cocycles further constrains the monodromy of a completely integrable system. What is it possible to say about their nonvanishing? Second, if $F \subset T^*\Sigma$ is a lagrangian torus whose Maslov class $\iota_F^*(d)$ is cohomologically nontrivial, must this Maslov class be close to primitive?

That is, how far does Viterbo's results generalize? We note that Viterbo himself has obtained one generalization [35] and that Bialy [5] has shown in two degrees of freedom that the Maslov class is twice a primitive element in many cases.

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REFERENCES

[1] S. Aloff and N. R. Wallach, An infinite family of distinct 7-manifolds admitting positively curved Riemannian structures, *Bull. Amer. Math. Soc.* **81** (1975), 93–97.

[2] V. I. Arnold, On a characteristic class entering into conditions of quantization, *Funct. Anal. Appl.* **1** (1967), 1–13.

[3] ———, Sturm theorems and symplectic geometry, *Funct. Anal. Appl.* **19**(4) (1985), 1–10.

[4] Ya. V. Bazaikin, Double quotients of Lie groups with integrable geodesic flows, *Siberian Math. J.* **41**(3) (2000), 419–432.

[5] M. Bialy, On the number of caustics for invariant tori of Hamiltonian systems with two degrees of freedom, *Ergodic Theory Dynam. Systems* **11**(2) (1991), 273–278.

[6] M. L. Bialy and L. V. Polterovich, Geodesic flows on the two-dimensional torus and "commensurability-incommensurability" phase transitions, *Funktsional. Anal. i Prilozhen.* **20**(4) (1986), 9–16, 96.

[7] ———, Lagrangian singularities of invariant tori of Hamiltonian systems with two degrees of freedom, *Invent. Math.* **97**(2) (1989), 291–303.

[8] ———, Hamiltonian diffeomorphisms and Lagrangian distributions, *Geom. Funct. Anal.* **2**(2) (1992), 173–210.

[9] ———, Hamiltonian systems, Lagrangian tori and Birkhoff's theorem, *Math. Ann.* **292**(4) (1992), 619–627.

[10] A. V. Bolsinov and B. Jovanović, Non-commutative integrability, moment map and geodesic flows, *Ann. Global Anal. Geom.* **23**(4) (2003), 305–322.

[11] ———, Complete involutive algebras of functions on cotangent bundles of homogeneous spaces, *Math. Z.* **246**(1–2) (2004), 213–236.

[12] L. T. Butler, Invariant fibrations of geodesic flows, *Topology* **44**(4) (2005), 769–789.

[13] ———, Geometry and real-analytic integrability, *Regul. Chaotic Dyn.* **11**(3) (2006), 363–369.

[14] ———, Smooth structures on Eschenburg spaces: numerical computations; preprint.

[15] T. Chinburg, C. Escher, and W. Ziller, Topological properties of Eschenburg spaces and 3-Sasakian manifolds, *Math. Ann.* **339**(1) (2007), 3–20.

[16] G. Contreras, J.-M. Gambaudo, R. Iturriaga, and G. P. Paternain, The asymptotic Maslov index and its applications, *Ergodic Theory Dynam. Systems* **23**(5) (2003), 1415–1443.

[17] J. J. Duistermaat, On the Morse index in variational calculus, *Adv. Math.* **21**(2) (1976), 173–195.

[18] H. R. Dullin, J. M. Robbins, H. Waalkens, S. C. Creagh, and G. Tanner, Maslov indices and monodromy, *J. Phys. A* **38**(24) (2005), L443–L447.

[19] J.-H. Eschenburg, New examples of manifolds with strictly positive curvature, *Invent. Math.* **66**(3) (1982), 469–480.

[20] D. Gromoll and W. Meyer, An exotic sphere with nonnegative sectional curvature, *Ann. of Math. (2)* **100** (1974), 401–406.

[21] M. Kreck and S. Stolz, A diffeomorphism classification of 7-dimensional homogeneous Einstein manifolds with $SU(3) \times SU(2) \times U(1)$-symmetry, *Ann. of Math. (2)* **127**(2) (1988), 373–388.

[22] B. Kruggel, Homeomorphism and diffeomorphism classification of Eschenburg spaces, *Quart. J. Math.* **56**(4) (2005), 553–577.

[23] F. Lalonde and J.-C. Sikorav, Sous-variétés lagrangiennes et lagrangiennes exactes des fibrés cotangents, *Comment. Math. Helv.* **66**(1) (1991), 18–33.

[24] G. Luke and A. S. Mishchenko, *Vector Bundles and their Applications*, *Mathematics and its Applications*, vol. 447, Kluwer Academic Publishers, Dordrecht, 1998.

[25] V. S. Matveev, Projectively equivalent metrics on the torus, *Differential Geom. Appl.* **20**(3) (2004), 251–265.

[26] A. S. Miščenko and A. T. Fomenko, Euler equation on finite-dimensional Lie groups, *Izv. Akad. Nauk SSSR Ser. Mat.* **42**(2) (1978), 396–415, 471.

[27] I. V. Mykytyuk and A. Panasyuk, Bi-Poisson structures and integrability of geodesic flows on homogeneous spaces, *Transform. Groups* **9**(3) (2004), 289–308.

[28] G. P. Paternain, *Geodesic Flows*, *Progr. Math.*, vol. 180, Birkhäuser Boston Inc., Boston, MA, 1999.

[29] G. P. Paternain and R. J. Spatzier, New examples of manifolds with completely integrable geodesic flows, *Adv. Math.* **108**(2) (1994), 346–366.

[30] L. Polterovich, Monotone Lagrange submanifolds of linear spaces and the Maslov class in cotangent bundles, *Math. Z.* **207**(2) (1991), 217–222.

[31] M. Rudnev and V. Ten, Integrability versus topology of configuration manifolds and domains of possible motions, *Arch. Math. (Basel)* **86**(1) (2006), 90–96.

[32] I. A. Taimanov, Topological obstructions to the integrability of geodesic flows on nonsimply connected manifolds, *Izv. Akad. Nauk SSSR Ser. Mat.* **51**(2) (1987), 429–435, 448.

[33] P. Topalov and V. S. Matveev, Geodesic equivalence via integrability, *Geom. Dedicata* **96** (2003), 91–115.

[34] C. Viterbo, A new obstruction to embedding Lagrangian tori, *Invent. Math.* **100**(2) (1990), 301–320.

[35] ———, Exact Lagrange submanifolds, periodic orbits and the cohomology of free loop spaces, *J. Differential Geom.* **47**(3) (1997), 420–468.

[36] C. T. C. Wall, *Surgery on Compact Manifolds*, *Math. Surveys Monogr.*, 2nd ed., vol. 69, American Mathematical Society, Providence, RI, 1999; edited and with a foreword by A. A. Ranicki.

[37] M. Wang and W. Ziller, Einstein metrics with positive scalar curvature, *Curvature and Topology of Riemannian Manifolds (Katata, 1985)*, *Lecture Notes in Math.*, vol. 1201, Springer-Verlag, Berlin, 1986, pp. 319–336.

[38] E. Witten, Search for a realistic Kaluza-Klein theory, *Nuclear Phys. B* **186**(3) (1981), 412–428.