On a Theorem of Hermite and Hurwitz

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ABSTRACT

The Hermite-Hurwitz theorem computes the degree, over $\mathbb{R}$, of a real rational function f in terms of the signature of an associated quadratic form—known today as the Hankel matrix of f. This formula, which Hermite was led to by his work on the problem of representing integers as sums of squares, gave rise to striking applications in the theory of equations and in the stability theory of ordinary differential equations. In this paper, this theorem and various generalizations to the matrix-valued case are discussed and described in terms of signature formulae. These include its relation to stability theory and the matrix Hermite-Hurwitz theorem of Bitmead–Anderson as applied to questions of circuit synthesis. This also includes a global form of Hörmander's signature formula for the Maslov index of a rational loop in a Lagrangian Grassmannian, due to Byrnes and Duncan, and applications to the topology of spaces of rational matrix-valued functions, following the work of Brockett, Byrnes, and Duncan. This includes, in particular, a topological proof of the matrix Hermite-Hurwitz theorem.

INTRODUCTION

The Hermite-Hurwitz theorem computes the degree, over $\mathbb{R}$, of a rational function f in terms of the signature of an associated quadratic form—known today as the Hankel matrix of f. Hermite was led to his discovery by his work in number theory, specifically the question of representing integers as sums of squares. In the course of this work, he recognized that many classical problems—such as counting the number of roots of a polynomial in a given domain—which were solvable in terms of the Cauchy formulae could be expressed in a far more computable form, viz. with a winding number

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replaced by the signature of a quadratic form fashioned out of the problem at hand. Hurwitz extended and applied this theorem, giving a solution to a problem raised by Maxwell: to give explicit criteria, in terms of the coefficients of the characteristic polynomial, for a linear differential system to be asymptotically stable, i.e., to find explicit inequalities in the characteristic coefficients of A which are satisfied if and only if all eigenvalues of A have negative real parts. It is worth remarking that the quadratic form constructed by Hurwitz is positive definite if, and only if, the system is asymptotically stable, and in this case is a Lyapunov function for the differential system. It is my contention that the Hermite-Hurwitz theorem is a far more central theorem than is presently appreciated. In this paper, illustrations of the role which the Hermite-Hurwitz theorem and several of its generalizations play in linear algebra in topology, in differential equations, and in the theory of circuits and systems will be given in support of this contention. The reader is referred to the papers [1], [3], [4], [7]–[9], [11]–[15], [18], [19], [25], [33], [34], the references cited therein, and the original work [22, 24] for further interpretations of this basic and beautiful theorem.

In Section 1, I present the Hermite-Hurwitz theorem together with some relevant facts concerning rational functions and Hankel matrices. The elegant proof of this theorem, which is presented in Section 2, is due to R. W. Brockett [7], and to my knowledge this is the first point in the literature where it is recognized explicitly that the Hermite-Hurwitz theorem can be interpreted as a statement about the topology of spaces of rational functions. In the third section, I present Hurwitz's application of this theorem to the study of the stability of differential equations on $\mathbb{R}^n$. Indeed, using the fact—noted by Parks [33]—that the Hankel form is in fact a Lyapunov function, one can also prove the Poincaré-Lyapunov theorem as a corollary to Hurwitz's calculation.

The remainder of the paper deals with the matrix Cauchy index—which is an extension of the notion of degree to matrix-valued functions—as it arises in circuit synthesis, as an invariant (the Maslov index) of Lagrangian loops, and as it relates to the topology of matrix-valued rational functions. Indeed, in Section 4 three circuit synthesis problems are stated, and, following Bitmead and Anderson [3–4], the matrix Cauchy index arises as a natural tool for the characterization of impedance matrices of the circuits which arise in these synthesis problems. Thus, various forms of the matrix Hermite-Hurwitz theorem [4] give "testable" characterizations of these impedance matrices. In this section, a topological proof of a special case—sufficient for the characterization of lossless networks—of the matrix Hermite-Hurwitz theorem is given, and in Section 6 a complete proof the "symmetric" Hermite-Hurwitz theorem is given. This proof is due to T. E. Duncan and me, and is based on topological methods; see also [14].

Rational functions on the real line can of course be generalized in many ways. One rather natural generalization from the point of view of circuit theory, suggested by Hermann and Martin [20, 21], leads to the study of rational maps

$$f: S^1 \to LG(m, 2m),$$

i.e. the study of rational loops in a Lagrangian Grassmann manifold. In Section 5, a formula is proved which identifies the Maslov index of the loop f with the matrix Cauchy index of f, regarded as a symmetric matrix-valued function. This identity generalizes the interpretation of the winding number of a rational map

$$f: S^1 \to S^1 \simeq LG(1,2)$$

as the Cauchy index of f and is a global form, suitably generalized, of a formula of Hormander [23] for the local contributions to the Maslov index. This identification, also due to T. E. Duncan and me, is yet another motivation for deriving a simple algebraic expression for the matrix Cauchy index, and as corollary to the matrix Hermite-Hurwitz theorem (Section 6) and the computations of Section 5, one obtains the "topological Hermite-Hurwitz theorem"

$$\text{Maslov}\,\text{Ind}(f) = \text{sign}(\text{Hankel}(f))$$

for Lagrangian loops. This assertion, as in the classical case, has an interpretation in terms of the topology of spaces of $m \times m$ symmetric transfer functions of fixed degree and thus in terms of the global properties of symmetric linear systems (see [14]).

1. THE HERMITE-HURWITZ THEOREM

That is where I have stopped in the study of this beautiful and great discovery of Mr. Cauchy. I had been led to this study in great part by research into arithmetical questions which, since the year 1847, have called my attention to quadratic forms composed of a sum of squares formed from the roots of the same equation. In addition I have found a true satisfaction in applying these forms to the magnificent theorems of Mr. Sturm and Mr. Cauchy, which open a new era in modern algebra.

—C. Hermite, on the Hermite-Hurwitz theorem [22, §6]

This beautiful, but not very well-known, theorem lies in the early work by Cauchy, Hermite, Hurwitz, Kronecker, Sturm, and others on the qualitative theory of vector fields in the plane (and in $\mathbb{R}^n$) on the one hand, and in the elimination theory of two or more polynomials, on the other hand. We can begin with the question: When is a strictly proper meromorphic function f on $\mathbf{C}$ rational? Here, strictly proper means that f is meromorphic at ∞, and vanishes there.

One approach to this question, due to Kronecker [30], is to fashion the infinite (Hankel) matrix

$$\mathcal{H}_f = \left[l_{i+j-1}\right]_{i,j=1}^{\infty} \tag{1.1}$$

from the Laurent coefficients of

$$f(z) = \sum_{i=1}^{\infty} l_i z^{-i}. \tag{1.2}$$

Of course, if f is in fact rational, say

$$f(z) = n(z)/d(z), \tag{1.3}$$

then by multiplying each side of (1.2) by

$$d(z) = d_0 + \cdots + d_{n-1}z^{n-1} + z^n$$

one obtains a recurrence relation of length n among the Laurent coefficients l_i, for $i \geqslant n$. Explicitly, comparing the coefficients of z^{-j} in the equation

$$n(z) = \left(\sum_{i=1}^{\infty} l_i z^{-i}\right) d(z) \tag{1.3'}$$

yields the recurrence relation

$$0 = l_j d_0 + \cdots + l_{j+n-1}d_{n-1} + l_{j+n}. \tag{1.4}$$

In terms of the Hankel matrix $\mathcal{H}_f$, (1.4) asserts that the $(n+j)$th column of $\mathcal{H}_f$ is linearly dependent on the preceding n columns. In particular,

$$\operatorname{rank}(\mathcal{H}_f) < \infty. \tag{1.5}$$

Moreover, if $n(z), d(z)$ have no common factors, then (1.5) may be sharpened to

$$\operatorname{rank}(\mathcal{H}_f) = \deg(d(z)) = \deg(f(z)). \tag{1.5'}$$

Kronecker's theorem asserts that the converse holds, viz.

$$f \text{ is rational} \quad \Leftrightarrow \quad \operatorname{rank}(\mathcal{H}_f) < \infty.$$

This is proved by retracing the steps outlined above, and makes use of the following observation: Suppose

$$\operatorname{rank}(\mathcal{H}_f) = n.$$

Construct the truncated Hankel matrix

$$\mathcal{H}_f' = [l_{i+j-1}]_{i,j=1}^{n},$$

and note that, from the form of the Hankel matrices, one has

$$n = \operatorname{rank}(\mathcal{H}_f') = \operatorname{rank}\left([l_{i+j-1}]_{i,j=1}^{\infty}\right) = \cdots = \operatorname{rank}(\mathcal{H}_f). \tag{1.6}$$

From (1.6), one can construct a unique recurrence relation of the form (1.4); i.e., one can solve the linear equations

$$\begin{bmatrix} l_{n+1} \\ \vdots \\ l_{2n} \end{bmatrix} = -d_{n-1}\begin{bmatrix} l_n \\ \vdots \\ l_{2n-1} \end{bmatrix} - \cdots - d_0 \begin{bmatrix} l_1 \\ l_n \end{bmatrix} \tag{1.7}$$

among the columns of $\mathcal{H}_f'$. The coefficients are unique, so that one obtains $d(z)$ from the data $(\mathcal{H}_f', l_{2n})$ and finally one obtains $n(z)$ from (1.3′). The pair $n(z), d(z)$ are coprime, for if there were a common factor, then a recurrence relation of length $\leqslant n-1$ would exist among the l_i's, contradicting (1.6).

We can express Kronecker's theorem in a form which we shall find useful: Here $k = \mathbb{R}$ or $\mathbf{C}$. We define

$\operatorname{Rat}(n; k) = \{$strictly proper rational functions f, defined over k, having degree $n\}$,

$\operatorname{Hank}(n; k) = \{n \times n$ Hankel matrices, defined over k, of rank $n\}$.

If $f: k^N \to k$ is a polynomial, then $V(f)$ wil denote the zero set of f. For example, we shall consider the polynomial

$$\mathrm{Res}: k^{2N} \to k$$

whose value at a point $(n_0, \ldots, n_{N-1}, d_0, \ldots, d_{N-1})$ is given by the resultant, $\mathrm{Res}(n, d)$, of the polynomials

$$n(z) = n_0 + \cdots + n_{N-1} z^{N-1}, \qquad d(z) = d_0 + \cdots + d_{N-1} z^{N-1} + z^N.$$

Recall that

$$\mathrm{Res}(n, d) = 0 \quad \Leftrightarrow \quad n(z), d(z) \text{ have a common factor.}$$

In this notation, we can consider the open dense subspaces

$$\mathrm{Rat}(n; k) = k^{2n} - V(\mathrm{Res})$$

$$\mathrm{Hank}(n; k) = k^{2n-1} - V(\det)$$

as smooth manifolds. On the one hand, by Cauchy's integral formula the Laurent coefficients (l_i) are continuous functions of the coefficients of f. On the other hand, (1.7) and (1.3′) show that the coefficients of f are continuous functions of the coefficients $(l_i)_{i=1}^{2n}$. Thus, we have

THEOREM 1.1 (Kronecker). *The Laurent map* $\mathcal{L}(f) = (l_1, \ldots, l_{2n})$ *is a homeomorphism*

$$\mathcal{L}: \mathrm{Rat}(n, k) \to \mathrm{Hank}(n; k) \times k.$$

Now, each strictly proper rational function $f(z)$ extends to a holomorphic map

$$f: S^2 \to S^2, \qquad f(\infty) = 0, \tag{1.8}$$

and conversely. One can also phrase Kronecker's observation as giving a formula for the (Hopf) degree, $\deg_{\mathbf{C}}(f)$, in terms of algebraic data. That is, since $\deg_{\mathbf{C}}(f) = \deg(d(z))$,

$$\deg_{\mathbf{C}}(f) = \mathrm{rank}(\mathcal{H}_f). \tag{1.9}$$

The Hermite-Hurwitz theorem [22, 24] is concerned with the case when f is real. As above, f gives rise to a mapping

$$f: S^1 \to S^1, \qquad f(\infty) = 0, \tag{1.10}$$

and one might ask for a calculation of the winding number, $\deg_{\mathbf{R}}(f)$, in terms of the data $(\mathcal{H}'_f, l_{2n})$. Since f is real, $\mathcal{H}'_f$ is a real symmetric matrix and therefore possesses a second numeric invariant, viz. its signature, $\operatorname{sign}(\mathcal{H}'_f)$. Now, by Kronecker's Theorem, the identity

$$\operatorname{sign}(\mathcal{H}'_f) \equiv \operatorname{rank}(\mathcal{H}'_f) \equiv \deg_{\mathbf{C}}(f) \equiv \deg_{\mathbf{R}}(f) \bmod 2 \tag{1.11}$$

holds in the integers modulo 2. In 1856, Hermite proved that the identity

$$\operatorname{sign}(\mathcal{H}'_f) = \deg_{\mathbf{R}}(f) \tag{1.12}$$

holds in the integers for generic f, i.e. for f lying in an open, dense subset of $\operatorname{Rat}(n;\mathbb{R})$. From this statement, (1.12) follows for all $f \in \operatorname{Rat}(n;\mathbb{R})$ by a general position argument. Explicitly, thinking of $\mathcal{H}'_f$ as a continuous symmetric matrix-valued function of $f \in \operatorname{Rat}(n;\mathbb{R})$—as in Theorem 1.1—note that, since $\mathcal{H}'_f$ has constant rank, the signature of $\mathcal{H}'_f$ is constant on the connected components of $\operatorname{Rat}(n;\mathbb{R})$. In particular, the left-hand side of (1.12) is a continuous function of f. But the right-hand side is easily seen to be continuous as well, so by Hermite's calculation these functions agree on all of $\operatorname{Rat}(n;\mathbb{R})$. Hurwitz [15] proved the general theorem in 1894 by different techniques, which are still of interest at present.

THEOREM 1.2 (Hermite, Hurwitz). *For any real, strictly proper rational function f,*

$$\operatorname{sign}(\mathcal{H}'_f) = \deg_{\mathbf{R}}(f). \tag{1.12}$$

In the nineteenth century $\deg_{\mathbf{R}}(f)$ was, of course, expressed in a different way:

DEFINITION 1.3 (Cauchy). The local index of a real, rational f at a real pole x_0 is $+1$ if f changes from $-\infty$ to $+\infty$, -1 if the opposite occurs, and 0 if f has a pole of even order at x_0. The index of f, $C(f)$, is the sum of the local indices.

$C(f)$, which is the winding number $\deg_{\mathbf{R}}(f)$ of the map f in (1.10), was defined by Cauchy in [16]. In Part I of that work, he uses the Cauchy index to compute the number of real roots of a real polynomial (generalizing, among other things, Descartes's rule of signs), the number of negative real roots, and related questions. In Part II, he uses the Cauchy index to define and evaluate the index, at an equilibrium point, of a nondegenerate, polynomial vector field on the plane. This was later extended to the case $n \geqslant 2$ by Kronecker [29], who introduced the notion of the "characteristic" of a system of equations as a generalization of the Cauchy index of a plane vector field.

In the next section, we shall give a modern topological proof, following Brockett, of the Hermite-Hurwitz theorem.

2. THE HERMITE-HURWITZ THEOREM AND THE TOPOLOGY OF SPACES OF RATIONAL FUNCTIONS

Recall the statement of the

HERMITE-HURWITZ THEOREM. *For any real, strictly proper rational function f,*

$$\operatorname{sign}(\mathcal{H}_f') = \deg_{\mathbf{R}}(f). \tag{1.12}$$

Proof (Brockett [7]). From the general position argument sketched in Section 1, it suffices to check the identity (1.12) once on each component of $\mathrm{Rat}(n;\mathbb{R})$. Note that, from (1.11), both $\operatorname{sign}(\mathcal{H}_f')$ and $\deg_{\mathbf{R}}(f)$ can take on only $n+1$ values on $\mathrm{Rat}(n;\mathbb{R})$. We shall first determine the number of path components:

THEOREM 2.1 (Brockett). *$\mathrm{Rat}(n;\mathbb{R})$ has $n+1$ path components $\mathrm{Rat}(p,q)$, where $p+q=n$, $p \geqslant 0$, and $q \geqslant 0$. Furthermore $g \in \mathrm{Rat}(p,q)$ if, and only if,*

$$\deg_{\mathbf{R}}(g) = p - q.$$

Proof of Theorem 2.1 [7]. We may begin by considering the problem of deforming a rational function f with distinct poles, say

$$f_0(z) = \sum_{i=1}^{m} \frac{r_i}{z - z_i} + F_0(z), \qquad z_i \in \mathbb{R},$$

where $F_0(z)$ has only complex poles. By deforming z_j to z_{j+1} and then deforming the quadratic contribution .

$$\frac{az+b}{(z-z_{j+1})^2}$$

first to

$$\frac{az+b}{z^2}$$

and then to

$$\frac{az+b}{z^2+d},$$

where z^2+d has distinct, pure imaginary roots, we can deform $f_0(z)$ to another real rational function $f_1(z) \in \mathrm{Rat}(n;\mathbb{R})$ for which

$$f_1(z) = \sum_{i=1}^{q'} \frac{r_i}{z+i} + \sum_{j=1}^{p'} \frac{\tilde{r}_j}{z-j} + F_0(z),$$

where $r_i < 0$ and $\tilde{r}_j > 0$. Thus $f_1(z)$ has the property that all the real residues which are negative correspond to negative real poles, and all the real residues which are positive correspond to positive real poles. Note that

$$\deg_{\mathbf{R}}(f_0) = \deg_{\mathbf{R}}(f_1) = p' - q',$$

which follows from examining the behavior of the graph of $f_1(z)$ at real poles. Next, $F_0(z)$ may be deformed to an $F_1(z)$ which has purely imaginary poles, occurring of course in conjugate pairs. Taking the pair which is closest to the origin, which give rise to a contribution of the form

$$\frac{az+b}{z^2+d},$$

one can reverse the process used above, allowing d to tend to 0 and then splitting this multiplicity-2 contribution to one of the form

$$\frac{r}{s+\varepsilon} + \frac{\tilde{r}}{s-\varepsilon}$$

where $\tilde{r}$, $\varepsilon > 0$ and $r < 0$. In this way, we can deform f_1, and hence f_0, to the rational function

$$f_2(z) = \sum_{i=1}^{q} \frac{-i}{s+i} + \sum_{j=1}^{p} \frac{j}{s-j}, \tag{2.1}$$

where

$$\deg(f_2) = p - q = p' - q' = \deg(f_0).$$

Hence, $\mathrm{Rat}(p,q)$ is path-connected. ■

It now remains to check the identity

$$\mathrm{sign}(\mathcal{H}'_f) = C(f)$$

once on each path component $\mathrm{Rat}(p,q)$. Thus, we consider

$$f_{p,q}(s) = \sum_{i=1}^{q} \frac{-1}{s+1} + \sum_{j=1}^{p} \frac{+1}{s-j}.$$

By definition,

$$C(f_{p,q}) = p - q,$$

so that $f_{p,q} \in \mathrm{Rat}(p,q)$.

LEMMA 2.2. $\mathrm{sign}(\mathcal{H}'_{f_{p,q}}) = p - q.$

This lemma follows from a straightforward but tedious computation. A more elegant system-theoretic proof, based on circuit synthesis, can also be given in the context of realization theory (see Fuhrmann [18], this issue, for more details). Explicitly, any strictly proper real rational function $g(s)$ of degree n may be factored (or realized) matricially as

$$g(s) = c(sI - A)^{-1}b, \tag{2.2}$$

where $b = e_1$, A is the $n \times n$ companion matrix

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -d_n \\ 1 & 0 & \cdots & 0 & -d_n \\ 0 & 1 & \cdots & 0 & -d_n \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -d_n \end{bmatrix}$$

of the unique monic degree n denominator $d(s)=s^n+d_1s^{n-1}+\cdots+d_n$, and $c=[l_1,\dots,l_n]$ is the vector of the first n Laurent coefficients of $g(s)$. Since (A,b,c) determines $g(s)$, one can ask for a computation of $\operatorname{sign}(\mathcal{H}_g')$ in terms of (A,b,c).

For this purpose, we shall need the main theorem of realization theory (see e.g. [6], [26], [35]), which we shall state for a $p\times m$ matrix $G(s)$ of rational functions.

DEFINITION 2.3 (Kalman). Any triple (A,B,C) consisting of an $l\times l$ matrix A, an $l\times m$ matrix B, and a $p\times l$ matrix C which satisfies

$$C(sI-A)^{-1}B=G(s) \tag{2.3}$$

is said to be an l-dimensional realization of $G(s)$. The minimum such $l\in\mathbb{N}\cup(\infty)$ is referred to as the McMillan degree of $G(s)$, and any realization of this dimension is said to be minimal.

For scalar rational functions $g(s)$, we have constructed above a minimal realization of dimension equal to $\deg_{\mathrm{C}}(g)$, while from the identity (2.2) and the fact that the resolvent of A has poles at the eigenvalues of A it is also clear that this realization is minimal; i.e., for scalar $g(s)$ one knows

$$\text{McMillan}\deg(g)=\deg_{\mathrm{C}}(g)=\operatorname{rank}(\mathcal{H}_g). \tag{2.4}$$

THEOREM 2.4 (Kalman). *Every strictly proper rational matrix-valued function $G(s)$ admits a finite-dimensional realization. Furthermore, if (A_i,B_i,C_i), $i=1,2$, are two minimal realizations (of dimension n), then there is a unique $T\in GL(n,\mathbb{R})$ such that*

$$\begin{aligned} TA_1&=A_2T,\\ TB_1&=B_2,\\ C_1&=C_2T. \end{aligned} \tag{2.5}$$

Again, for scalar $g(s)$ the first assertion is trivial, while the second and less trivial assertion follows from a dynamical system-theoretic interpretation of realizations (see e.g., [6], [24], [35]).

We illustrate Theorem 2.4 by finding a formula for $\operatorname{sign}(\mathcal{H}_g')$ in terms of a realization: suppose $g(s)$ is a strictly proper, real rational function of McMillan degree n and that (A,b,c) is a minimal realization of $g(s)$. Then

(A^t, c^t, b^t) satisfies

$$b^t(sI - A^t)^{-1}c^t = c(sI - A)^{-1}b = g(s)$$

and is therefore also a minimal realization of $g(s)$. By Theorem 2.4, there exists a $T \in \mathrm{GL}(n, \mathbb{R})$ satisfying

$$\begin{aligned} TA &= A^tT, \\ Tb &= c^t, \\ c &= b^tT. \end{aligned} \tag{2.6}$$

Transposing (2.6), and taking into account the uniqueness of T, one has

$$T = T^t. \tag{2.7}$$

We shall now compute $\mathcal{H}_g$ in two ways. First,

$$g(s) = c(sI - A)^{-1}b = \sum_{i=1}^{\infty} \frac{cA^{i-1}b}{s^i},$$

yielding the entries $l_i = cA^{i-1}b$ of $\mathcal{H}_g$ in terms of (A, b, c). Thus, one may write

$$\begin{bmatrix} c \\ cA \\ \vdots \\ cA^r \\ \vdots \end{bmatrix} [b, Ab, \ldots, A^sb, \ldots] = \mathcal{H}_g.$$

Alternatively, one may write

$$[b, Ab, \ldots, A^rb, \ldots]^t T [b, Ab, \ldots, A^rb, \ldots] = \mathcal{H}_g, \tag{2.8}$$

where T is the unique symmetric matrix satisfying (2.6). It is well known [6, 24] that any minimal realization of $g(s)$ satisfies the controllability condition

$$\operatorname{rank}[b, Ab, \ldots, A^{n-1}b] = n. \tag{2.9}$$

Therefore, (2.8) reduces to

$$[b, Ab, \ldots, A^{n-1}b]^t T [b, Ab, \ldots, A^{n-1}b] = \mathcal{H}'_g,$$

and, in light of (2.9), we have

$$\operatorname{sign}(T) = \operatorname{sign}(\mathcal{H}'_g). \tag{2.10}$$

As an application of (2.10) we have

Proof of Lemma 2.2. If $f_{p,q}(s)$ is as defined above, then

$$A = \operatorname{diag}[-q, \ldots, -1, 1, \ldots, p], \qquad c = [1, \ldots, 1],$$

$$b = \begin{bmatrix} -1 \\ \vdots \\ -1 \\ +1 \\ \vdots \\ +1 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} q \\ \left.\vphantom{\begin{matrix} a \\ b \\ c \end{matrix}}\right\} p \end{matrix}$$

is a minimal realization of $f_{p,q}(s)$. It is trivial to check that if

$$I_{p,q} = \operatorname{diag}[\underbrace{-1, \ldots, -1}_{q}, \underbrace{1, \ldots, 1}_{p}],$$

then

$$\begin{aligned} I_{p,q} A &= A^t I_{p,q}, \\ I_{p,q} b &= c^t, \qquad c = b^t I_{p,q}. \end{aligned} \tag{2.11}$$

By uniqueness then, the matrix T in (2.6) and (2.10) is $I_{p,q}$, and therefore

$$\operatorname{sign}(\mathcal{H}'_{f_{p,q}}) = \operatorname{sign}(I_{p,q}) = p - q. \qquad \blacksquare$$

REMARK. This elegant calculation is part of the general theory of internally symmetric realizations. Briefly, realizations satisfying (2.11) were first studied systematically, for $G(s)$ an impedance matrix of a linear reciprocal RLC circuit, by Youla and Tissi [37] under the title of "network symmetric" realizations. The observation that such realizations exist for any rational symmetric (in particular, any scalar) matrix-valued function was made in [11] and has been the starting point for various algebraic and geometric investigations of transfer functions $G(s)$ possessing external symmetries. A systematic study of various external, and the corresponding internal, system-theoretic symmetries was made in [36], where the term "internally symmetric realization" was coined. The existence of internally symmetric realizations for systems defined over the integers and over a polynomial ring has been studied from an arithmetic point of view in [12] and [14]. External and internal symmetries have been studied and classified from a Lie-theoretic point of view in [8] and [9], and from a polynomial model point of view in [18]; in both treatments new forms and proofs of the Hermite-Hurwitz theorem are derived. The geometry of externally symmetric transfer functions has been studied in [6], [13]–[15], in particular stressing topological formulations of the Hermite-Hurwitz and of Kronecker's theorem.

The relationship between the Hermite-Hurwitz theorem and the topology of spaces of rational functions can, in fact, be pushed much further. First of all, Brockett's theorem can itself be proved following some observations made by Hermite, and again by Hurwitz (especially §8 of [24]). Explicitly, in [22] Hermite first considers the problem of determining the number N_+ of roots of

$$P_f(z) = n(z) + id(z) = 0, \tag{2.12}$$

which lie in the upper half plane $H_+ \subset \mathbf{C}$. Here $n(z)$ and $d(z)$ are real and coprime with

$$f(z) = n(z)/d(z) \tag{2.12$'$}$$

strictly proper, $\deg_{\mathbf{C}}(f) = n$. In §5 of [22], he remarks that N_+ is given by the formula

$$N_+ = \tfrac{1}{2}\left[\deg_{\mathbf{C}}(f) + \deg_{\mathbf{R}}(f)\right], \tag{2.13}$$

a statement which he attributes to Sturm, but derives from Theorem 1.2. There is a similar formula, after replacing z by $\bar{z}$, for the number N_- of roots in H_-, viz.

$$N_- = \tfrac{1}{2}\left[\deg_{\mathbf{C}}(f) - \deg_{\mathbf{C}}(f)\right]. \tag{2.13$'$}$$

This also follows from (2.13) upon observing that $P_f(z)$ cannot have any real roots, since $(n,d)=1$. Indeed, by the same reasoning no root in H_- can conjugate to a root in H_+. From these observations, the connectivity of $\mathrm{Rat}(p,q)$ follows from an easy divisor argument. That is, the (divisors of) roots $\mathcal{D}_f$ of P_f determine P_f uniquely as that polynomial of degree n vanishing on $\mathcal{D}_f$ and having leading coefficient i. Now, to say $f \in \mathrm{Rat}(p,q)$ is to say

$$\mathcal{D}_f = \mathcal{D}_1 + \mathcal{D}_2, \qquad \mathcal{D}_1 \cap \overline{\mathcal{D}}_2 = \varnothing, \tag{2.14}$$

where $\mathcal{D}_1$ ($\mathcal{D}_2$) consists of p unordered points in H_+ (q unordered points in H_-). Given $f, \tilde{f} \in \mathrm{Rat}(p,q)$, it is of course clear that $\mathcal{D}_f$ can be deformed to $\mathcal{D}_{\tilde{f}}$ along a path of divisors satisfying (2.14).

Thus, Brockett's theorem follows from the Hermite-Hurwitz theorem, and in this way the latter can be interpreted as a statement about the topology of spaces of rational functions. Explicitly, the Hermite-Hurwitz theorem calculates the number of components of $\mathrm{Rat}(n;\mathbb{R})$, or (what is the same) the rank of the cohomology space $H^0(\mathrm{Rat}(n);\mathbb{Z}_2)$. For interpretations of the Hermite-Hurwitz theorem in the higher cohomology $H^i(\mathrm{Rat}(n);\mathbb{Z}_2)$, the reader is referred to [14].

3. THE ROUTH-HURWITZ THEORY: ASYMPTOTIC STABILITY OF LINEAR DIFFERENTIAL SYSTEMS

I propose at present, without entering into any details of mechanism, to direct the attention of engineers and mathematicians to the dynamical theory of such governors.

It will be seen that the motion of a machine with its governor consists in general of a uniform motion, combined with a disturbance which may be expressed as the sum of several component motions. These components may be of four different kinds:—

1. The disturbance may continually increase.
2. It may continually diminish.
3. It may be an oscillation of continually increasing amplitude.
4. It may be an oscillation of continually decreasing amplitude.

The first and third cases are evidently inconsistent with the stability of the motion; and the second and fourth alone are admissible in a good governor. This condition is mathematically equivalent to the condition that all the possible roots, and all the possible parts of the impossible roots, of a certain equation shall be negative.

I have not been able completely to determine these conditions for equations of a higher degree than the third; but I hope that the subject will obtain the attention of mathematicians.

—J. C. Maxwell [31]

The dynamical system

$$\frac{dx}{dt} = Fx, \qquad x \in \mathbb{R}^n \tag{3.1}$$

is said to be asymptotically stable at 0 just in case for any initial condition $x_0 \in \mathbb{R}^n$, the solution x_t of (3.1) tends to 0 as $t \to \infty$. Stability is an old topic in differential equations, dating back to Newton's investigation of the stability of systems governed by an inverse qth-power law, e.g. gravitational attraction. In 1868, J. C. Maxwell published a study of the local asymptotic stability about 0 of certain nonlinear 3rd-order differential equations on $\mathbb{R}$, which were models of various closed-loop feedback systems (see [31]). Maxwell knew that local asymptotic stability of the nonlinear system ought to be determined by the (global) asymptotic stability of the linearized system

$$\frac{d^3x}{dt^3} + p_2\frac{d^2x}{dt^2} + p_1\frac{dx}{dt} + p_0x = 0, \tag{3.2}$$

and he knew that this in turn was determined by the roots of the characteristic equation

$$p(D) = D^3 + p_2D^2 + p_1D + p_0 = 0, \tag{3.2$'$}$$

where D may be thought of as an independent variable. That is, (3.2) is asymptotically stable if, and only if, the roots of (3.2′) lie in the left half plane. Clearly, the conditions $p_i > 0$ are necessary, and Maxwell found that adding the condition

$$p_1p_2 - p_0 > 0$$

gave necessary and sufficient conditions, in terms of the coefficients p_i, for $p(D)$ to have all of its roots in the left half plane. In [31] and at a meeting of the London Mathematical Society in 1868, Maxwell posed the following problem: to determine the conditions on the coefficients of a (monic) polynomial $P(s)$ of degree n which characterize those polynomials having all roots in the left half plane. Following a suggestion made by Clifford at the meeting, Routh solved Maxwell's problem, giving a set of $n(n+1)/2$ inequalities in the p_i's. These inequalities define an open region—the *domain of stability*—in the space $\mathbb{R}^n$ of such polynomials. The domain of stability was also characterized by Hurwitz [24], and this is the treatment we shall follow here.

First, note that we can recast the problem in a more modern framework: by introducing the new "phase" variables

$$x_1 = x, \qquad x_2 = \frac{dx}{dt}, \ldots, \qquad x_n = \frac{d^{n-1}x}{dt^{n-1}}$$

the nth-order equations which normally arise may be regarded as 1st-order equations (3.1) on $\mathbb{R}^n$. Now we may ask for the construction of *universal polynomials* in the coefficients of the right-hand side which decide the asymptotic stability of (3.1). As we shall see in Theorem 3.1, these polynomials may be constructed from the Hermite-Hurwitz theorem.

We thus consider linear systems

$$\frac{dx}{dt} = Fx \tag{3.3}$$

with characteristic polynomial

$$p(s) = \chi_F(s). \tag{3.3'}$$

Following Hurwitz, we assume that $p(s)$ has no pure imaginary zeros, and we denote by L and R the number of roots of p in the left half plane and right half plane, respectively. By contour integration, the change in $(1/2\pi)\arg(p(-is))$, as s varies from $-\infty$ to $+\infty$, yields $L-R$. By trigonometry, this is the Cauchy index of the rational function

$$f(s) = v(s)/u(s),$$

where v and u are defined as

$$cp(-is) = u(s) + iv(s)$$

and c is a complex constant rendering v/u strictly proper. Therefore,

$$L - R = \operatorname{sign}(\mathcal{H}_{v/u}). \tag{3.4}$$

Note that u, v, and the entries of $\mathcal{H}_{v/u}$ are easily obtainable from the coefficients of $p(s)$. Moreover, one has a criterion for $R=0$:

THEOREM 3.1 (Hurwitz). *$p(s)$ lies in the domain of stability if, and only if, the quadratic form $\mathcal{H}_{v/u}$ is positive definite.*

This theorem leads to n polynomial inequalities defining the domain of stability. Recall that, if

$$Q=(q_{ij})=Q^t, \qquad i,j=1,\ldots,n,$$

is a symmetric matrix, then the number of negative eigenvalues of Q is given by the Jacobi algorithm (see [19]): Let

$$Q_1=q_{11},\ldots,Q_n=Q$$

be the sequence of the $j\times j$ principal submatrices, and let

$$P_1=q_{11},\ldots,\ P_n=\det Q \tag{3.5}$$

be the corresponding minors. Provided no two successive P_i's vanish, the number of negative eigenvalues of Q is equal to the number of changes of sign in the sequence (3.5). Moreover, Q is positive definite if, and only if, all the terms P_i are positive.

Now, in the case at hand, we may define the n functions

$$\mathcal{F}_i(p_0,\ldots,p_{n-1})=P_i(\mathcal{H}_{v/u}). \tag{3.6}$$

It is not hard to see that the $\mathcal{F}_i$ are polynomials in the p_j. One therefore can settle Maxwell's problem: the domain of stability is defined by the equations

$$p(s)\in\mathfrak{D} \quad\Leftrightarrow\quad \mathcal{F}_i(p)>0,\quad i=1,\ldots,n. \tag{3.7}$$

There is another interpretation of Hurwitz's theorem, which is quite intriguing. $\mathcal{H}_{v/u}$ defines a function on the "phase" space $\mathbb{R}^n$ of the differential equation (3.4), viz.

$$L(x)=x^t\mathcal{H}_{v/u}x\geqslant 0,$$

where of course F and hence $\mathcal{H}_{v/u}$ is constant. That the quadratic form L is positive definite implies, for example, that the level surfaces

$$L(x)=c \qquad \text{for} \quad c>0 \tag{3.8}$$

are ellipsoids which are concentric about 0, decreasing as $c\to 0$. It is our claim that, in fact, $L(x)$ is a Lyapunov function for the system (3.4), i.e., along trajectories x_t of (3.4) one has

$$\frac{d}{dt}L(x_t) < 0 \qquad \text{for} \quad x_0 \neq 0. \tag{3.9}$$

Geometrically, (3.8) asserts that x_t crosses the ellipsoid from the outside to the inside and that $c \to 0$ as $t \to \infty$, and hence $x_t \to 0$. It is of course a well-known theorem of Lyapunov that such functions always exist provided (3.1) is asymptotically stable.

LEMMA 3.2. *If $\mathcal{H}_{v/u}$ is positive definite, then L is a Lyapunov function for* (3.4).

Proof. In the linear case, it is not hard to construct some quadratic form which is in fact a Lyapunov function, in the asymptotically stable case—which we are in, by Hurwitz' theorem. Let L_0 be such a form. Then by the spectral theorem, there exists a $T \in \mathrm{SO}(n)$ such that

$$L_0 = T^t L T. \tag{3.10}$$

To say

$$\frac{d}{dt}L_0(x_t) < 0$$

is to say the tangent vector to x_t at $x_{t_0} \in L_0^{-1}(c)$ points inward at x_{t_0}. After an orientation-preserving orthogonal transformation, the tangent vector to Tx_t remains not tangent to the level surface $Tx_{t_0} \in L^{-1}(c)$ and indeed points inward at x_{t_0}. Since each trajectory of the transformed system

$$\frac{dx}{dt} = (TFT^t)x \tag{3.4}$$

has the form Tx_t, where x_t is a trajectory of (3.4), Equation (3.9) holds for all trajectories. ∎

REMARK 1. That the "second method" of Lyapunov can be used to derive inequalities defining the domain of stability, and that the results of Hermite and of Hurwitz can be used to construct Lyapunov functions, is rather well known among control theorists, and the reader should consult [33], [34] for further information.

Remark 2. This derivation of the polynomial inequalities defining the domain of stability reposes upon the classical application of the Hermite-Hurwitz theorem, which asserts that to test if a polynomial has its roots in a given region of $\mathbf{C}$, one may construct a related quadratic form and calculate its signature. Such an assignment of a quadratic form to a polynomial arises in amazingly diverse settings. Purely algebraic methods have been derived by Kalman [25] and by Djaferis and Mitter [38]. Another, rather surprising result in this direction comes from Weyl's criterion for the compactness of a semisimple Lie group, viz. that its Killing form must be negative definite. Explicitly, in [9] R. W. Brockett continues his study of the real Lie algebras which arise (in a canonical way) from real rational functions—the particular Lie algebra which arises can be determined from symmetry properties of the rational function. For the case at hand, the rational function $f(s)=v(s)/u(s)$ constructed from $p(s)$ as in Hurwitz's theorem satisfies

$$f(s)=f^*(-s).$$

According to Brockett [9, Theorem 1], such an f gives rise to a Lie algebra $\mathrm{su}(p,q)$, where $p-q$ is the Cauchy index of f. In particular, $p(s)$ has all of its roots in either the left or the right half plane if, and only if, its associated Lie algebra has a compact form—that is, if and only if the Killing form is nonnegative definite.

4. THE MATRIX CAUCHY INDEX: A CHARACTERIZATION OF THE IMPEDANCE MATRICES OF LOSSLESS NETWORKS

An important problem in circuit synthesis is to characterize the impedance matrices of LC circuits among rational matrix-valued functions of a complex variable. In the scalar-input, scalar-output setting this is done rather elegantly in terms of the Cauchy index, and therefore in a very "testable" way in terms of the Hankel matrix. In order to obtain similar criteria for multichannel circuits, Bitmead and Anderson [3, 4] were led to define the matrix Cauchy index and then to prove a matrix version of the Hermite-Hurwitz theorem [4]. In this section, we will sketch the circuit-theoretic background of the problem considered in [4], state the generalized Hermite-Hurwitz theorem, and give an easy topological proof—in the fashion of Section 2—for a special case of this theorem which nevertheless suffices to characterize the impedances of RC circuits.

Suppose the differential equations

$$\frac{dx}{dt} = Ax + Bu, \qquad x(0) = x_0 \in \mathbb{R}^n, \qquad u(t) \in \mathbb{R}^m,$$
$$y = Cx, \qquad y(t) \in \mathbb{R}^m \tag{4.1}$$

model an RLC circuit in an initial state x_0, driven by m current sources, where $u_i(t)$ is an applied current and $y_i(t)$ is the resulting voltage across the i-th current source (e.g., a battery). The circuit, or rather the system (4.1), is said to be *passive* provided there exists a positive definite form L such that the *dissipation inequality*

$$x(t)^t Lx(t) - x_0^t Lx_0 \leqslant \int_0^t u(\tau)y(\tau)\,d\tau \tag{4.2}$$

holds for all x_0 and all t. If equality holds in (4.2), then the system is said to be *lossless*.

One can state the condition of passivity in terms of a property of the weighting pattern for (4.1) or in terms of the Laplace transform of the weighting pattern. Since (4.1) is constant-coefficient, we make contact with the study of properties of rational function; that is, assuming $x_0 = 0$, the Laplace transform of (4.1) is given by

$$\hat{y}(s) = \left[C(sI - A)^{-1}B\right]\hat{u}(s)$$
$$= R(s)\hat{u}(s), \tag{4.3}$$

which of course, is an analogue of Ohm's Law. The function $R(s)$ is referred to as the *impedance* of the circuit, and it is natural and important to ask which $m \times m$ matrix-valued, rational functions $R(s)$, vanishing at ∞, arise as the *impedance* of an RLC circuit driven by m current sources. Here, we may have to allow rational functions having a pole of finite order at ∞, but for the sake of exposition we will consider only strictly proper rational functions.

Now, by Theorem 2.4 each such rational function $R(s)$ can be factored as

$$R(s) = C(sI - A)^{-1}B. \tag{4.4}$$

Equivalently, every $m \times m$ rational, matrix-valued function vanishing at ∞ arises as the Laplace transform of a system of differential equations (4.1). Of course, there exists a formula for the minimal dimension n for which such a factorization is possible which is a generalization of Kronecker's theorem, viz. the identity

$$n = \operatorname{rank}\left(\left[L_{i+j-1}\right]_{i,j-1}^{\infty}\right),$$

where the block matrices L_i are defined as the coefficients

$$R(s) = \sum_{i=1}^{\infty} L_i s^{-i}, \tag{4.4'}$$

Indeed, this is not surprising in light of the identities

$$L_i = CA^{i-1}B$$

and the Cayley-Hamilton theorem. We shall always assume that the factorization (4.4) is minimal in this sense and write $\deg_C(R) = n$. This implies, for example, that the poles of $R(s)$ coincide with the eigenvalues of A, each set counted with multiplicity.

In particular, taking $u(t), y(t)$ in $L^2[0,\infty)$ in the dissipation inequality (4.2) and using the positive definiteness of the (storage) function L, one sees as in Section 3 that

(1) the poles of an impedance $R(s)$ lie in the closed left half plane [one may also show that $R(s)$ must satisfy the additional constraints];
(2) any poles $s_0 = i\omega$ of any entry of $R(s)$ are simple, for ω real;
(3) with the exception of such poles,

$$R(i\omega) + R^t(-i\omega) \geqslant 0 \qquad \text{for} \quad \omega \text{ real;}$$

(4) the residue of R at a pole $s_0 = i\omega$ is Hermitian nonnegative definite.

Again, however, we have excluded the possibility that $R(s)$ has a pole at ∞. The general case includes the possibility that $R(s)$ has a simple pole of finite order at ∞, with a symmetric nonnegative definite residue.

Furthermore, if the system is lossless, then the inequalities should be changed to equalities—for example, all poles of $R(s)$ must be simple and pure imaginary, and the residues at such poles are Hermitian nonnegative definite. Such an $R(s)$ is said to be *lossless positive real*, and it is known that any lossless positive real function is the impedance of a lossless LC network, perhaps containing ideal transformers and gyrators. Thus, it is of considerable interest to give an efficient method for deciding whether a rational matrix-valued function satisfying

$$R(s)+R'(-s)=0 \tag{4.5}$$

is lossless positive real.

It is also of interest in circuit theory to have "testable" criteria for the impedances of RC and of RL networks. We collect these problems in the following list.

Problem 4.1. Characterize the impedance matrices of lossless LC networks, perhaps containing ideal transformers and gyrators.

Problem 4.2. Characterize the impedance matrices of RC networks, possibly containing ideal transformers.

Problem 4.3. Characterize the impedance matrices of RL networks, possibly containing ideal transformers.

Naturally, characterizations of such networks have long been known in classical circuit synthesis (see [32]). These characterizations, however, have been in terms of the analytic character of the impedance matrices $R(s)$—e.g. the location of and the behavior near the poles of $R(s)$. Fortunately, such behavior is often encoded in the value taken on by a suitable matrix Cauchy index, and one of the major contributions in [4] was the evaluation, algebraically, of these Cauchy indices as the signatures of associated (block) Hankel matrices.

Indeed, in the scalar case, the conditions (1), (2), (4) can be checked using Cauchy's index $C(R)$, as in Definition 1.4. For replacing the rational function $R(s)$ by

$$g(\omega)=iR(i\omega) \tag{4.6}$$

one can see that g is real for real ω, by (4.5). Moreover, for each real pole ω of g, the local contribution to $C(g)$ will be ± 1 depending on the sign of the residue. Therefore:

Proposition 4.4.

$$R \textit{ is lossless positive real} \quad \Leftrightarrow \quad C(g)=n. \tag{4.7}$$

Moreover, by the Hermite-Hurwitz theorem, (4.7) can be decided by (universal) polynomials in the coefficients of R viz.

COROLLARY 4.5.

$$R \text{ is lossless positive real} \quad \Leftrightarrow \quad \mathcal{H}_g > 0. \tag{4.7'}$$

In order to generalize this criterion to the matrix case, Anderson and Bitmead [4] were led to define, for rational real symmetric (or Hermitian) matrix-valued functions (which we shall assume vanish at ∞):

DEFINITION 4.6 (Matrix Cauchy index). The local index of $G(s)$ at a real pole $s = s_0$ is the number of eigenvalues of $G(s)$ which jump from $-\infty$ to $+\infty$ minus the number which jump from $+\infty$ to $-\infty$ as s goes through s_0. The (matrix) Cauchy index, $C(G)$, is the sum of the local indices at all real poles.

Setting

$$G(\omega) = iR(i\omega), \tag{4.8}$$

one obtains the identity

$$G(\omega) = \overline{G(\omega)}^{t}$$

from (4.5). Thus, the generalization of Proposition 4.4 for the impedance of circuits driven by more than one current source is given by

THEOREM 4.7 [14]. *$R(s)$ is lossless positive real if, and only if, each entry of $R(s)$ has simple poles and*

$$C(G) = \deg_{\mathbf{C}}(R). \tag{4.9}$$

In order to calculate $C(G)$, Anderson and Bitmead prove a Hermitian matrix version of the Hermite-Hurwitz theorem. We prefer to state this Hermitian result in a notation following [8]. Thus, we form the symmetric matrix

$$\mathcal{J}_R = \left[(-1)^{i+j} L_{i+j-1}\right]$$

from the Laurent coefficients (4.4′) and refer to sign($\mathcal{J}_R$) as the alternating Cauchy index of R. Then, from the Anderson-Bitmead-Hermite-Hurwitz theorem and the observation of Theorem 4.2, one obtains [4]:

THEOREM 4.8. *$R(s)$ is lossless positive real if, and only if, $R(s)$ has only simple poles and has alternating Cauchy index n; i.e., if and only if $\mathcal{J}_R \geqslant 0$.*

This theorem also follows from Lie-theoretic considerations [8]. That is, symmetric lossless functions $R(s)$ satisfy

$$TR(s) = R^t(-s)T, \tag{4.10}$$

where T is a skew form. Brockett [8] has shown that (4.10) holds if, and only if, the matrix Lie algebra generated by $\langle A, BC\rangle$—where (A, B, C) is a minimal realization of $G(s)$—leaves invariant a nondegenerate symmetric form with signature = sign($\mathcal{J}_R$). The interpretation of $\mathcal{J}_R \geqslant 0$ in Theorem 4.8 then follows from earlier work by Bitmead and Anderson [3].

One can, alternatively, offer a simple topological proof of Theorem 4.8:

Proof. Denote the set of degree-n, $m \times m$ lossless positive real functions by $\mathcal{L}(n;m)$. As in Kronecker's theorem, we may think of $\mathcal{L}(n;m)$ as a subspace of the manifold $\mathcal{H}^n_{m,m}$ via the correspondence

$$R \to \left(\left[L_{i+j-1}\right]^n_{i,j=1}, L_{2n}\right).$$

CLAIM. *$\mathcal{L}(n,m)$ is path-connected.*

Proof. If R has distinct poles, then to say $R \in \mathcal{L}(n,m)$ is to say that R admits a partial-fraction expansion

$$R(s) = \sum_{j=1}^{n} \frac{R_j}{s - i\omega_j}, \qquad \operatorname{rank}(R_j) = 1,$$

where $\omega_j \in \mathbb{R}$ with $\omega_j < \omega_{j+1}$, and

$$\bar{R}_j^t = R_j \geqslant 0.$$

Now, given $R(s)$, $\tilde{R}(s) \in \mathcal{L}(n,m)$, one can deform the poles of $R(s)$—as an ordered set in the imaginary axis—continuously to the poles of $R(s)$. Next, since the space of rank-1, Hermitian positive semidefinite matrices is connected, one can deform the residues of $R(s)$—again as an ordered set—to the residues of $R(s)$. Thus, a dense open subset of $\mathcal{L}(n, m)$ is connected. ■

Since $\mathcal{L}(n, m)$ is connected and $\operatorname{rank}(\mathfrak{H}_R)$ is constant on $\mathcal{L}(n, m)$, $\operatorname{sign}(\mathfrak{H}_R)$ is also constant on $\mathcal{L}(n, m)$. On the other hand, $C(G)$ is constant on $\mathcal{L}(n, m)$ by Theorem 4.7 and the definition of $\mathcal{L}(n, m)$. Therefore, it suffices to check the identity

$$C(G) = \operatorname{sign}(\mathfrak{H}_R)$$

for any particular choice of R, which is a straightforward external symmetry argument, as in Section 2, and will be omitted. ■

Theorem 4.8 of course can be viewed as a special case of a matrix form of a Hermite-Hurwitz theorem. Let $\mathcal{M}(n, m)$ denote the set of real rational $m \times m$ matrix-valued functions R, vanishing at ∞, and satisfying

$$R(s) + R(-s)' = 0 \tag{4.11}$$

for $s \in \mathbf{C}$. Then, as before,

$$G(\omega) = iR(i\omega) \tag{4.11'}$$

is Hermitian for real ω and therefore has a matrix Cauchy index, which we denote by $\operatorname{Ind}(R)$. If $\mathfrak{H}_R$ is the alternating Hankel matrix defined above, then the theorem which is asserted in [2] and alluded to above is

THEOREM 4.9 (The lossless Hermite-Hurwitz theorem). *For* $R \in \mathcal{M}(n, m)$, *if* $C_H(R) = C(G)$ *then*

$$C_H(R) = \operatorname{sign}(\mathfrak{H}_R) \tag{4.12}.$$

This was also proved by a somewhat tedious connectivity argument in an unpublished manuscript [15]. The special case which we have proved above, viz. that Theorem 4.9 holds whenever $C_H(R) = \deg_{\mathbf{C}}(\mathbb{R})$, suffices to answer Problem 4.1. Problems 4.2 and 4.3 admit a solution in terms of the matrix Hermite-Hurwitz theorem [4] for symmetric $R(s)$, for which we shall sketch a topological proof [14] in Section 6, and in terms of certain Lie algebras [8] which are determined by rational matrix-valued functions. By the matrix Hermite-Hurwitz theorem, we of course mean the identity

$$C(R) = \operatorname{sign}(\mathcal{H}_R), \tag{4.13}$$

where $\mathcal{H}_R$ is the block Hankel matrix constructed from the Laurent coefficients (4.4′) of R. Then, following Bitmead and Anderson [4], one can reinterpret classical circuit synthesis conditions [32] for realizability of $R(s)$ in terms of the Cauchy index of R, and via (4.13) obtain the following "testable" criterion, thereby settling Problems 4.2–4.3. As before, for simplicity, we have excluded the case that $R(s)$ has a pole at ∞, but remark that the nonstrictly proper case can be handled within the same framework. Thus, as corollaries to the matrix Hermite-Hurwitz theorem (see Corollary 6.2) we have

THEOREM 4.10 [14]. *The strictly proper, real rational symmetric matrix-valued function $R(s)$ is the impedance of an RC network, perhaps containing ideal transformers, if and only if*

$$\operatorname{sign}(\mathcal{H}_R) = \deg_{\mathbf{C}}(R);$$

i.e., if and only if $\mathcal{H}_R \geqslant 0$.

THEOREM 4.11 [4]. *The strictly proper, real rational symmetric matrix-valued function $R(s)$ is the impedance of an RL network, possibly containing ideal transformers, if and only if*

$$\operatorname{sign}(\mathcal{H}_R) = -\deg_{\mathbf{C}}(R);$$

i.e., if and only if $\mathcal{H}_R \leqslant 0$.

The reader is referred to [18] for an alternative proof of the matrix Hermite-Hurwitz theorem using "polynomial model" methods, and to [14] for a topological proof, which we shall sketch in Section 6.

5. THE MATRIX CAUCHY INDEX AND THE MASLOV INDEX OF A RATIONAL LAGRANGIAN LOOP

In the next two sections we shall show how a natural generalization of the topological form of the Hermite-Hurwitz theorem gives a formula for computing the Maslov index of a 1-cycle of Lagrangian planes in $\mathbb{R}^{2m}$. This index was discovered in different contexts by Keller and by Maslov, and analyzed by Arnol'd and by Hormander in somewhat more geometric settings. In this section we shall identify the Maslov index with the matrix Cauchy index, and in the next section we give a topological proof of the matrix Hermite-Hurwitz theorem. Thus, as we have seen in Section 4, this index also turns up in the circuit-theory literature. Our treatment follows a recent paper [14] by the author and T. E. Duncan.

To begin with, consider $\mathbb{R}^m$ with the standard inner product (x, y). On $\mathbb{R}^m \oplus \mathbb{R}^m$ there is then a natural skew form

$$\langle (x,y),(x',y') \rangle = (x,y') - (x',y). \tag{5.1}$$

If $V \subset \mathbb{R}^{2m}$ is a subspace which is isotropic for $\langle\ ,\ \rangle$, i.e.

$$\langle v, w \rangle = 0 \qquad \text{for all} \quad v, w \in V,$$

then $\dim V \leqslant m$, since the skewform $\langle\ ,\ \rangle$ is nondegenerate. If $\dim V = m$, then V is said to be a *maximal isotropic*, or a *Lagrangian*, subspace. For example, if

$$T: \mathbb{R}^m \to \mathbb{R}^m$$

is symmetric,

$$\operatorname{graph}(T) = \{(x, Tx) : x \in \mathbb{R}^m\}$$

is a Lagrangian subspace, since to say

$$\langle (x,Tx),(x',Tx') \rangle = (x,Tx') - (x',Tx) = 0 \tag{5.2}$$

for all $x \in \mathbb{R}^m$ is to say T is self-adjoint. Thus, the vector space of symmetric matrices can be identified with a subset of the set $\mathrm{LG}(m, 2m)$ of Lagrangian subspaces in $\mathbb{R}^{2m}$.

Example. If $m = 1$, every line l is Lagrangian, since for nonzero $v, w \in l$ we have $v = \alpha w$ and therefore

$$\langle v, w \rangle = \alpha \langle w, w \rangle = -\alpha \langle w, w \rangle = 0.$$

Alternatively, every 1×1 matrix is symmetric. Indeed, consider $l \subset \mathbb{R}^2$ as in Figure 1. Each line l except one—the y-axis—is complementary to the y-axis and is therefore the graph of a linear function

$$y = mx.$$

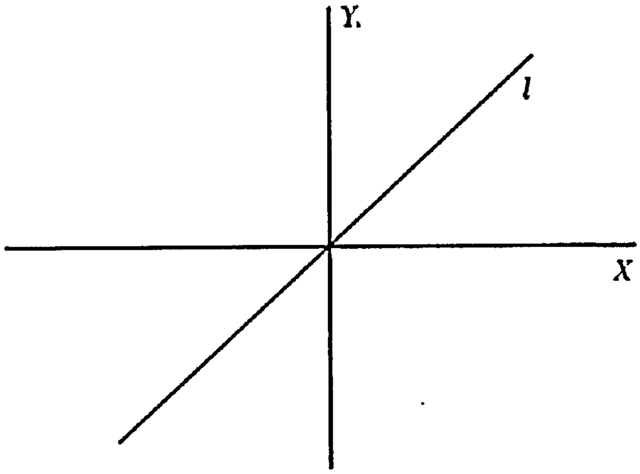

FIG. 1.

In this sense LG(1,2) is the one-point compactification of the space $\mathbb{R}$ of 1×1 symmetric matrices, i.e.

$$\mathrm{LG}(1,2)\simeq S^1. \tag{5.3}$$

In the same way, $\mathrm{LG}(m,2m)$ may be thought of as a compactification of the space of symmetric matrices. For as in Figure 1, almost every m-plane V is complementary to the m-plane Y and is therefore the graph of a linear function

$$y=Tx,$$

and according to (5.2) V is Lagrangian if, and only if, T is symmetric. If V intersects Y nontrivially, V is clearly the limit of a sequence of m planes V_n (which can be taken to be Lagrangian if V is) which are complementary to Y. In this sense, the compact space (in fact, manifold) of Lagrangian planes contains the symmetric matrices as an open dense subspace. More precisely, set

$$\sigma(Y)=\{V\in \mathrm{LG}(m,2m):\dim(V\cap Y)\geqslant 1\}. \tag{5.4}$$

Then

$$\mathrm{LG}(m,2m)-\sigma(Y)\simeq \mathbb{R}^{m(m+1)/2} \tag{5.5}$$

may be naturally identified with the space of symmetric matrices. The subspace $\sigma(Y)$ is referred to as the Maslov cycle.

Now, the connection with network and system theory lies in a seminal paper written by Hermann and Martin [20] (see also [21]). According to them, each strictly proper rational $p \times m$ matrix-valued function $G(s)$ gives rise to a map

$$G: S^2 \to \mathrm{Grass}(m, m+p)$$

from the Riemann sphere to the Grassmannian manifold of m-planes in $(m+p)$-space. Furthermore [21], if G is $m \times m$ symmetric, then G gives rise to a map

$$G: S^1 \to \mathrm{LG}(m, 2m)$$

of the equator S^1 of real points on S^2 to the subspace of Lagrangian planes.

Explicitly, if $\{s_1, \ldots, s_l\} \subset \mathbb{R}$ are the real poles of G, then the neat observation in [20] is that the correspondence

$$s \mapsto \mathrm{graph}(G(s)), \qquad s \in \mathbb{R} - \{s_1, \ldots, s_n\},$$

is an assignment to each such s of an m-plane in $\mathbb{R}^{2m}$. Since $G(s) = G(s)^t$, graph$(G(s))$ is a Lagrangian plane, and in this way we obtain the (symmetric) Hermann-Martin map, which we still denote by G,

$$\begin{aligned} G: \mathbb{R} - \{s_1, \ldots, s_l\} &\to \{\text{symmetric matrices}\} \\ &\simeq \mathrm{LG}(m, 2m) - \sigma(Y). \end{aligned}$$

Since $G(\infty) = 0$ and graph(0) is a Lagrangian subspace complementary to Y, we may extend G to a map defined at ∞:

$$G: S^1 - \{s_1, \ldots, s_l\} \to \mathrm{LG}(m, 2m) - \sigma(Y).$$

It is then elementary to check that G has removable singularities at $\{s_1, \ldots, s_l\}$ and therefore extends to the (Lagrangian) Hermann-Martin map

$$G: S^1 \to \mathrm{LG}(m, 2m).$$

That G extends to S^1 can also be seen by noting that

$$\mathrm{graph}(G(s)) = \text{column span}\begin{bmatrix} G(s) \\ I \end{bmatrix} = \text{column span}\begin{bmatrix} N(s) \\ D(s) \end{bmatrix},$$

where $G(s)=N(s)D(s)^{-1}$ is a factorization of $G(s)$ into coprime polynomial matrices [17, 35]. Such a factorization exists, of course, since the ring $k[s]^{m\times m}$ of $m\times m$ matrix polynomials in s is both left and right principal.

We can now proceed to the major results. Suppose

$$F: S^1 \to \mathrm{LG}(m,2m), \qquad F(\infty)=[X] \tag{5.6}$$

is a (continuous) closed curve. Since

$$\pi_1(\mathrm{LG}(m,2m))=\mathbb{Z} \tag{5.7}$$

canonically, just as in the case $m=1$, each F gives rise to an integer—which is known as its (Arnol'd)-Maslov index. The problem we shall consider is, in analogy with the Hermite-Hurwitz theorem, that of computing

$$\mathrm{Ind}(F)\in\pi_1(\mathrm{LG}(m,2m))$$

in terms of algebraic data. Now an isomorphism (5.7) may be defined by associating to F the intersection number of the 1-cycle (or curve) $F(S^1)$ with the codimension-1 cycle $\sigma(Y)$. If $\theta_1,\dots,\theta_n$ denote the points of intersection, then for $\theta\neq\theta_i$, $i=1,\dots,n$ $F(\theta)$ is a symmetric matrix T_θ in light of (5.5). Moreover the mapping

$$\begin{gathered} S^1-\{\theta_1,\dots,\theta_n\}\to\{\text{symmetric matrices}\},\\ \theta\mapsto T_\theta \end{gathered} \tag{5.8}$$

is continuous. By applying the Cayley transform to T_θ, we can regard F as a periodic matrix-valued function. By the Stone-Weierstrass theorem, one knows that F can be uniformly approximated by a G for which (5.8) is a finite Laurent polynomial

$$T_\theta=\sum_{i=-N}^{M} L_i\theta^i, \tag{5.9}$$

where the L_i satisfy

$$L_i=L_i^t.$$

Supposing F satisfies the base-point condition

$$F(\infty)=[X],$$

which asserts that $F(\infty)=0$, the zero matrix, we may take

$$L_i=0 \qquad \text{for} \quad i \geqslant 0.$$

Furthermore, since G can be taken sufficiently close to F,

$$\operatorname{Ind}(G)=\operatorname{Ind}(F),$$

and therefore it is enough to compute Ind(G).

THEOREM 5.1 [14]. *For any real rational symmetric* $G(s)$,

$$\operatorname{Maslov}\operatorname{Ind}(G)=\operatorname{Cauchy}\operatorname{Ind}(G).$$

Proof. The Maslov index of G can be computed from a general formula for the local contributions to Ind(G) regarded as an intersection number. If $s_0^- < s_0 < s_0^+$ are real points sufficiently close to an s_0 for which

$$G(s_0)\in\sigma(Y)\subset \mathrm{LG}(m,2m),$$

then, according to Hörmander [14, 3.3.4], the local intersection number at s_0 of $G(s^+)$ with $\sigma(Y)$ is given by

$$\operatorname{Ind}_{s_0}(G)=\frac{\operatorname{sgn}G(s_0^+)-\operatorname{sgn}G(s_0^-)}{2} \tag{5.10}$$

under very general conditions on G. Explicitly, if $\sigma(X)$ is the hypersurface in LG(m,2m) defined by X, then to say $G(s)\in\sigma(X)$ is to say $\det G(s)=0$. Unless

$$\det G(s)\equiv 0,$$

we can therefore assume that

$$\det G(s_0^-)\neq 0, \qquad \det G(s_0^+)\neq 0,$$

or that the Lagrangian planes

$$\operatorname{graph} G(s_0^-), \operatorname{graph} G(s_0^+)$$

are transverse to X. Now, the image of the interval $I = [s_0^-, s_0^+]$ under G is a path in $\mathrm{LG}(m, 2m)$, and Hörmander's calculation is valid whenever the path $G(I)$ remains transverse to X, i.e. under the condition

$$G(I) \subset \mathrm{LG}(m, 2m) - \sigma(X).$$

Since the condition $\det G(s) = 0$ is algebraic in s, one may choose s_0^-, s_0^+ so that

$$\det G(s) \neq 0 \qquad \text{for all} \quad s \in I, \quad s \neq s_0.$$

That is, $G(I) \cap \sigma(X)$ is either empty or consists of the singleton $\{G(s_0)\}$. Moreover, it follows that

$$G(I) \cap \sigma(X) = \{G(s_0)\}$$

if, and only if, the matrix-valued function $G(s)$ has a zero at s_0—in addition to the pole. This phenomenon cannot, of course, occur when $G(s)$ is scalar with numerator and denominator coprime. Indeed, in the scalar case the theorem is precisely the definition given by Cauchy.

Recall that the matrix Cauchy index (Definition 4.1) is computed as the sum, over real poles s_0, of a local index. Upon traversing such an s_0, the local index is calculated as the number of eigenvalues of $G(s)$ which change from $-\infty$ to $+\infty$ minus the number of eigenvalues which change from $+\infty$ to $-\infty$. If s_0 is not a zero of $G(s)$, then no eigenvalue of $G(s)$ can approach 0, so that this local index coincides with (5.10). On the other hand, if s_0 is a zero of $G(s)$, a negative (or positive) eigenvalue could deform through 0 to a positive (or negative) eigenvalue, in this case making a contribution to (5.10) but leaving the local index unchanged.

Thus, if the zeros and poles of $G(s)$ do not coincide,

$$\operatorname{Cauchy Ind}(G) = \operatorname{Maslov Ind}(G).$$

We claim that this identity holds for all G, but as the remarks above show, one cannot use (5.10) to prove this statement. Here, we shall follow Arnol'd [2].

In general, composition with the Cayley transform induces a map

$$G: S^1 \to U(m),$$

defined via

$$G(s) = (I - iG(s))(I + iG(s))^{-1} \tag{5.11}$$

and therefore leads to the invariant

$$[G] \in \pi_1(U(m)) \simeq \mathbb{Z}.$$

Now, $[G]$ may be computed as

$$[G] = \deg_{\mathbf{R}}(\det G(s)) \tag{5.12}$$

and, denoting by $\lambda_j(s)$ the algebraic function of s satisfying

$$\det(\lambda I - G(s)) = 0,$$

(5.12) yields

$$[G] = \deg\left(\prod_{j=1}^{m} (1 - i\lambda_j(s))(1 + i\lambda_j(s))^{-1}\right). \tag{5.12'}$$

On the other hand, we claim

$$\deg\left(\sum_{j=1}^{m} (1 - i\lambda_j(s))(1 + i\lambda_j(s))^{-1}\right) = \sum_{G(s_0)=\infty} \text{Cauchy}\operatorname{Ind}_{s_0}(G). \tag{5.13}$$

Now, the left-hand side of (5.13) calculates the degree of a product of algebraic functions

$$g_j(s) = (1 - i\lambda_j(s))(1 + i\lambda_j(s))^{-1}$$

which take values θ in $U(1)$ for s real. And the degree in (5.13) is computed with respect to the base point $\theta = e^{i\pi}$ in $U(1)$. Thus, the left-hand side is the sum of the degrees of the algebraic functions $g_j(s)$. With these conventions, suppose the pole s_0 occurs also as a zero of $G(s)$, i.e., some $\lambda_j(s)$ vanishes at s_0 while some other eigenfunction takes on infinite values. If we consider such

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a branch, then on the one hand as s goes through s_0, $\lambda_j(s)$ vanishes and hence makes no contribution to the degree of (its Cayley transform) $g_j(s)$. On the other hand, $\lambda_j(s)$ makes no contribution to the local Cauchy index of $G(s)$ at $s = s_0$, by definition.

Therefore,

$$[G] = \operatorname{Cauchy Ind}(G),$$

where the right-hand side is understood as the matrix Cauchy index, as in Definition 4.6.

Just as in the scalar case, one has a map

$$\pi : U(m) \to \mathrm{LG}(m,2m) \simeq U(m)/O(m),$$

and from the homotopy exact sequence of this fibration one obtains [2, Corollary 3.4.3]

$$\pi^* : \pi_1(U(m)) \simeq \pi_1(\mathrm{LG}(m,2m)).$$

Indeed,

$$\begin{aligned}\operatorname{Ind}(G) &= \deg_{\mathbf{R}}\left(\det{}^2 G(s)\right)\\ &= \deg_{\mathbf{R}}\left(\det\left[\{I - iG(s)\}\{I + iG(s)\}^{-1}\right]\right).\end{aligned}$$

This identity, together with (5.11), proves the fundamental identity (5.10). ■

6. THE MATRIX HERMITE-HURWITZ THEOREM AND THE TOPOLOGY OF MATRIX-VALUED RATIONAL FUNCTIONS

In this section, we sketch a proof of the topological version of the matrix Hermite-Hurwitz theorem, i.e.

THEOREM 6.1 [14].

$$\operatorname{Ind}(G) = \operatorname{sign}(\mathcal{H}_G'), \tag{6.1}$$

where $\mathcal{H}_G'$ is the (truncated) block Hankel matrix $\mathcal{H}_G' = [L_{i+j-1}]_{i,j=1}^{n}$.

REMARK. In Theorem 6.1, n can either be taken to be

$$\operatorname{rank}\left[L_{i+j-1}\right]_{i,j=1} < \infty,$$

which is finite (since G is rational), or, in analogy with Kronecker's Theorem, to be $\deg_{\mathbf{C}}(G)$, where the Hermann-Martin map

$$G: S^2 \to \operatorname{Grass}_{\mathbf{C}}(m, 2m)$$

is defined by

$$G(\theta) = \operatorname{graph}(G(\theta))$$

for any complex θ.

Thus, combining Theorem 6.1 with Theorem 5.1, we obtain the Anderson-Bitmead-Hermite-Hurwitz theorem:

COROLLARY 6.2 [14]. *For any real symmetric, matrix-valued rational function $R(s)$,*

$$C(R) = \operatorname{sign}(\mathcal{H}_R).$$

The remainder of this section is devoted to a proof of Theorem 6.1—in the context of the topology of spaces of rational *matrix-valued* functions.

Denote by $\operatorname{Rat}(n; m)$ the set of rational real $m \times m$ symmetric matrix-valued functions of s which vanish at ∞ and have degree n;

$$\operatorname{rank}\left(\left[L_{i+j-1}\right]_{i,j=1}^{\infty}\right) = n.$$

THEOREM 6.3 (14]. *$\operatorname{Rat}[n; m)$ is naturally a smooth manifold.*

Proof. As in Kronecker's theorem, we may consider the bijection

$$g \mapsto \left(\left[L_{i+j-1}\right]_{i,j=1}^{n}, L_{2n}\right). \tag{6.2}$$

We first show that the set of block Hankel matrices

$$\mathcal{H}_{m,m}^{n} = \left\{\mathcal{H}_g = \left[L_{i+j-1}\right]_{i,j=1}^{n+1} : \operatorname{rank} \mathcal{H}_g = n\right\}$$

is a smooth manifold. Recall that, by elementary linear algebra, the space $\mathfrak{M}^n_{m,m}$ of $m(n+1)\times m(n+1)$ real matrices of rank n is the orbit in k^N, $N=m^2(n+1)^2$ of the matrix

$$T_n=\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$$

under the action of $\mathrm{GL}(m(n+1),k)\times\mathrm{GL}(m(n+1),k)$, where

$$(S,T)M=SMT^{-1}.$$

Thus, the space $\mathfrak{M}^n_{m,m}$ is a homogeneous space for $\mathrm{GL}(m(n+1),k)\times \mathrm{GL}(m(n+1),k)$ and hence a smooth manifold. Let us write an element $M\in\mathfrak{M}^n_{m,m}$ in block form:

$$\begin{bmatrix} M_{11} & \cdots & M_{1,n+1} \\ \vdots & & \vdots \\ M_{n+1,1} & \cdots & M_{n+1,n+1} \end{bmatrix}.$$

There is a finite group

$$\mathcal{G}=\mathcal{G}_2\times\mathcal{G}_3\times\cdots\times\mathcal{G}_{n+1}\times\mathcal{G}_{n+2}\times\cdots\times\mathcal{G}_{2n-3}$$

which acts on $\mathfrak{M}^n_{m,m}$ in a natural way, viz.

the generator of $\mathcal{G}_2$ acting on M interchanges $M_{1,2}$ with $M_{2,1}$,

the generators of $\mathcal{G}_3$ acting on M sends $M_{1,3}$ to $M_{2,2}$, $M_{2,2}$ to $M_{3,1}$, and $M_{3,1}$ to $M_{1,3}$,

etc.

The idea is that the fixed-point set of $\mathcal{G}$ acting on $\mathfrak{M}^n_{m,m}$ is precisely $\mathcal{H}^n_{m,m}$. Now, by a theorem of Bochner [5], the fixed-point set for a compact group acting on a smooth manifold is always a smooth submanifold. Thus, $\mathcal{H}^n_{m,m}$ is a smooth submanifold of $\mathfrak{M}^n_{m,m}$.

But, to say G is symmetric, i.e. $G\in\mathrm{Rat}(n,m)$, is to say the block matrix $[L_{i+j-1}]$ is symmetric. Therefore, again by Bochner's theorem,

$$\mathrm{Rat}(n;m)\subset\mathcal{H}^n_{m,m}$$

is a smooth submanifold, as it is the fixed point set of the involution $M\mapsto M^t$. ∎

Next, consider the disjoint subspaces: for $p+q=n$, $p \geqslant 0$ and $q \geqslant 0$,

$$\mathrm{Rat}(p,q;m) \subset \mathrm{Rat}(n;m), \tag{6.3}$$

where $G \in \mathrm{Rat}(p,q;m)$ if, and only if, the Maslov index

$$\mathrm{Ind}(G) = p - q. \tag{6.4}$$

THEOREM 6.4 [14]. $\mathrm{Rat}(n;m) = \bigcup \mathrm{Rat}(p,q;m)$ *is a decomposition of* $\mathrm{Rat}(n;m)$ *into connected open submanifolds.*

From Theorem 6.4, the formula (6.1) follows as in Section 2: Ind(G) is constant on $\mathrm{Rat}(p,q;m)$ by definition, and $\mathrm{sign}(\mathcal{H}'_G)$ is constant on path components, since $\mathrm{rank}(\mathcal{H}'_G) = n$ is constant on $\mathrm{Rat}(n;m)$, by definition. Therefore, it is sufficient to check (6.1) once on each component. Consider, for p and q fixed,

$$G(s) = \left(\sum_{i=1}^{q} \frac{1}{s-i} - \sum_{j=1}^{p} \frac{1}{s-j} \right) E_{11} = f(s) E_{11}, \tag{6.5}$$

where $f(s)$ is a scalar rational function having Cauchy index $p-q$. It is clear that

$$\mathcal{H}'_G \sim \begin{bmatrix} \mathcal{H}'_f & 0 \\ 0 & 0 \end{bmatrix}$$

so that

$$\mathrm{sign}(\mathcal{H}'_G) = p - q$$

by Lemma 2.2.

On the other hand, for $g(s)$ defined in (3.13) we claim that for any $s_0 \in \{1,\ldots,n\}$, an interval $I = [s_0^-, s_0^+]$ can be chosen such that

$$g(I) \cap \sigma(X) = \varnothing,$$

and hence Hörmander's formula (5.10) applies, yielding

$$\mathrm{Ind}(g) = \sum_{g(s_0) = \infty} \frac{\mathrm{sgn}\, g(s_0^+) - \mathrm{sgn}\, g(s_0^-)}{2}.$$

Thus, assuming the claim, the identity follows. As for the claim,

$$g(s) = f(s)E_{11},$$

so that the zeros and poles of $g(s)$ are precisely those of $f(s)$. Since $f(s)$ is scalar, a zero of $g(s)$ cannot coincide with a pole.

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