TANGENT BUNDLE OF A MANIFOLD AND ITS HOMOTOPY TYPE

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Abstract

There is a homotopy equivalence $\varphi: M \longrightarrow M'$ between closed smooth manifolds of an odd dimension such that $\varphi^* TM'$, TM are stably isomorphic but not isomorphic to each other.

1. *Results*

Let $\varphi: M \longrightarrow M'$ be a homotopy equivalence between closed smooth manifolds of dimension n such that the tangent bundle TM and the pull-back $\varphi^* TM'$ are stably isomorphic to each other. Then are $\varphi^* TM'$, TM isomorphic to each other?

It makes the question more interesting that there is an invariant [**5**, **8**], when n is odd, which seems to depend only on the homotopy type of the manifolds. In fact, once Dupont [**5**] announced that it was the case indeed, only to realise later, together with Sutherland, that his proof had a gap and it was still an open problem. In this paper, we will answer the question in the negative.

Let ξ be a vector bundle over M of rank n which is stably inverse to ν^k, the normal bundle of M for a smooth embedding into S^{n+k}, $k \geqslant n+2$.

If n is even, the Euler characteristic in its generalised form can be used to prove that stable isomorphism between TM and $\varphi^* TM'$ implies isomorphism.

Therefore, assume that n is odd. Note that there are at most two isomorphism classes of vector bundles ξ of rank n over M which are stably isomorphic to the tangent bundle (cf. [**4**]). We consider ξ together with a *trivialisation*, $\theta: \varepsilon_M^{n+k} \longrightarrow \xi + \nu$, from the trivial vector bundle $\varepsilon_M^{n+k} = \varepsilon^{n+k}$ to the Whitney sum $\xi + \nu$.

Note that an invariant $b(\zeta, \vartheta)$ is defined in [**3**] for a pair (ζ, ϑ) consisting of an $(n-1)$-sphere fibration ζ over M and a trivialisation $\vartheta: \varepsilon_M^{n+k} \longrightarrow \zeta + S\nu$, presuming a normal invariant $c: S^{n+k} \longrightarrow T(\nu)$. Here $\varepsilon_M^{n+k} = \varepsilon^{n+k}$ denotes the trivial $(n+k-1)$-sphere fibration and $S\nu$ denotes the sphere bundle of ν. We will write $b_c^f(\zeta, \vartheta)$ to denote this invariant, in effect, regarding c as a variable.

Then, we set $b_c(\xi, \theta) = b_c^f(S\xi, S\theta)$.

Note that this definition does not use the universal vector bundle, unlike Sutherland's [**8**]. However, using naturality of the functional Steenrod square, one may easily show that $b_c(\xi, \theta)$ above is $b(\xi, \theta)$ in Sutherland's sense if $c: S^{n+k} \longrightarrow T(\nu)$ is chosen as the collapse map coming from the embedding $M \hookrightarrow S^{n+k}$.

We say that two pairs (ξ, θ), (ξ', θ') are equivalent to each other if there is an isomorphism $\alpha: \xi \longrightarrow \xi'$ so that $(\alpha + 1)\,\theta \simeq \theta': \varepsilon^{n+k} \longrightarrow \xi' + \nu$, in which '$\simeq$' means 'is homotopic to, through isomorphisms'.

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THEOREM 1.1 (Dupont, Sutherland). *$b_c(\xi, \theta) = b_c(\xi', \theta')$ if and only if $(\xi, \theta), (\xi', \theta')$ are equivalent pairs. Furthermore, for each normal invariant c, there is a pair (ξ, θ) such that $b_c(\xi, \theta) = i$, for each $i \in Z_2$.*

Let the trivialisation $\theta_\iota : \varepsilon^{n+k} \longrightarrow TM + \nu$ be the one coming from an embedding $\iota : M \longrightarrow S^{n+k}$ and the normal invariant $c_\iota : S^{n+k} \longrightarrow T(\nu)$ be the collapse map onto $T(\nu) = N/\partial N$, where N is the normal neighbourhood of ιM in S^n. Recall the semi-characteristic $\chi_{1/2}(M) = \sum_i \operatorname{rank} H_{2i}(M; Z_2)$. Then we have the following theorem.

THEOREM 1.2 (Sutherland). $b_{c_\iota}(TM, \theta_\iota) = \chi_{1/2}(M) \in Z_2$.

As in [**8**], the James–Thomas number of M means the number of isomorphism classes of vector bundles of rank n stably isomorphic to TM. Similarly, the homotopy James–Thomas number of M is the number of fibrewise homotopy equivalence classes of $(n-1)$-sphere fibrations stably fibrewise homotopy equivalent to STM.

Let $c_{\iota'} : S^{n+k} \longrightarrow T(\nu')$ and $\theta_{\iota'} : \varepsilon^{n+k} \longrightarrow TM' + \nu'$ respectively denote the normal invariant and the trivialisation coming from an embedding $\iota' : M' \longrightarrow S^{n+k}$.

Then Theorem 1.3 describes exactly when $\varphi^* TM'$, TM are isomorphic to each other.

THEOREM 1.3. *$\varphi^* TM'$, TM are isomorphic to each other if and only if there is a bundle map $\beta : \nu \longrightarrow \nu'$ covering φ so that $b_{T(\beta) c_\iota}(TM', \theta_{\iota'}) = \chi_{1/2}(M')$.*

THEOREM 1.4. *Assume that M' is a smooth closed manifold of an odd dimension n such that its James–Thomas number is 2 while its homotopy James–Thomas number is 1 and the surgery obstruction group $L_n(\pi_1(M'), w_1(TM'))$ vanishes. Then there is a manifold M and a homotopy equivalence $\varphi : M \longrightarrow M'$ so that $\varphi^* TM'$, TM are stably isomorphic but not isomorphic to each other.*

As asserted by Sutherland [**8**, §7], we have the following lemma.

LEMMA 1.5. *$S^{13} \times P^2$ has the James–Thomas number 2 and the homotopy James–Thomas number 1.*

On the other hand, according to [**10**], $L_{15}(Z_2, w) = 0$. Therefore, we conclude that there is a homotopy equivalence $\varphi : M \longrightarrow M'$ between closed smooth manifolds such that $\varphi^* TM'$, TM are stably isomorphic but not isomorphic to each other.

Let η be a vector bundle over M. Now $\operatorname{Aut}(\eta)$ denotes the group of all equivalence classes of automorphisms of η, where the equivalence relation is 'is homotopic to, through automorphisms'. If $\alpha : TM \longrightarrow TM$ is an automorphism, there is an automorphism of ν, say, $j(\alpha) : \nu \longrightarrow \nu$ so that $\alpha + 1 \simeq 1 + j(\alpha) : TM + \nu \longrightarrow TM + \nu$ (cf. [**2**]). This gives a well defined homomorphism $j : \operatorname{Aut}(TM) \longrightarrow \operatorname{Aut}(\nu)$. If ζ is a sphere fibration, $\operatorname{Aut}^f(\zeta)$ is defined similarly and there is the homomorphism j: $\operatorname{Aut}^f(STM) \longrightarrow \operatorname{Aut}^f(S\nu)$.

Then, in addition, we observe the following.

THEOREM 1.6. *The group $\operatorname{Aut}(\nu)/j\operatorname{Aut}(TM)$ is trivial if the James–Thomas number is 2 and isomorphic to Z_2 otherwise. Similarly, $\operatorname{Aut}^f(S\nu)/j\operatorname{Aut}^f(STM)$ is trivial if the homotopy James–Thomas number is 2 and isomorphic to Z_2 otherwise.*

2. *Proofs*

Proof of Theorem 1.6. Note that there are exactly two equivalence classes of pairs (ξ, θ) according to Theorem 1.1.

Assume that the James–Thomas number is 2. Then (TM, θ) represents the same class for any $\theta : \varepsilon^{n+k} \longrightarrow TM + v$. Let $\beta : v \longrightarrow v$ be any automorphism. Then, since (TM, θ), $(TM, (1+\beta)\,\theta)$ are equivalent to each other, there must be an automorphism $\alpha : TM \longrightarrow TM$ such that $(1+\beta)\,\theta = (\alpha+1)\,\theta$, which means that $j[\alpha] = [\beta]$. This proves that $\mathrm{Aut}(v)/j\,\mathrm{Aut}(TM) = 0$.

The other cases can be dealt with in a similar way to obtain the asserted results. $\square$

Proof of Theorem 1.3. '*If*' *part*: Let $\theta : \varepsilon^{n+k} \longrightarrow \varphi^* TM' + v$ be a trivialisation for which the following diagram commutes up to homotopy through bundle maps:

$$\begin{array}{ccc} \varepsilon^{n+k}_M & \xrightarrow{\theta} & \varphi^* TM' + v \\ {\scriptstyle \bar\varphi}\downarrow & & \downarrow{\scriptstyle \bar\varphi+\beta} \\ \varepsilon^{n+k}_{M'} & \xrightarrow{\theta_{l'}} & TM' + v' \end{array}$$

where the $\bar\varphi$ mean the natural bundle maps covering φ.

Let $g' : Y \longrightarrow \Sigma^l T(TM')$ be the map dual to the unique map $T(a') : T(v'+v') \longrightarrow T(\bar\gamma_\kappa)$ introduced in [3] with respect to any duality between Y and $T(\bar\gamma_\kappa)$ and the duality

$$\begin{aligned} S^{2n+2k} &\cong \Sigma^{n+k} S^{n+k} \xrightarrow{\Sigma^{n+k} c_{l'}} \Sigma^{n+k} T(v') \\ &\cong T(\varepsilon^{n+k}_{M'} + v') \xrightarrow{T(\theta_{l'}+1)} T(TM' + v' + v') \xrightarrow{T(\bar\Delta)} T(TM' \times (v'+v')) \\ &\cong T(TM') \wedge T(v+v). \end{aligned}$$

(For details of the notations above, refer to the beginning paragraphs of [3, §6].)

Likewise, let $g : X \longrightarrow \Sigma^l T(\varphi^* TM)$ be the dual of $T(a) : T(v+v) \longrightarrow T(\bar\gamma_\kappa)$ with respect to the duality determined by $(\varphi^* TM, \theta)$ and the normal invariant c_l.

Let $\overline{\varphi^{-1}} : TM' \longrightarrow \varphi^* TM'$ denote an inverse of $\bar\varphi$ up to homotopy through bundle maps. Then, it is straightforward to see that $\Sigma^l T(\overline{\varphi^{-1}}) : \Sigma^l T(TM') \longrightarrow \Sigma^l T(\varphi^* TM')$ is dual to $T(\beta+\beta) : T(v+v) \longrightarrow T(v'+v')$ with respect to the dualities above. Therefore, we have $g' \simeq \Sigma^l T(\bar\varphi)\, g$.

Let U_ξ denote the Thom class for any vector bundle ξ in Z_2-coefficients. Since $T(\bar\varphi)^* U_{TM'} = U_{\varphi^* TM'}$, it follows that, by definition,

$$b_{c_l}(\varphi^* TM', \theta) = b_{T(\beta)\, c_l}(TM', \theta_{l'}),$$

which is, by assumption,

$$\chi_{1/2}(M') = \chi_{1/2}(M) = b_{c_l}(TM, \theta_l).$$

It follows that $(\varphi^* TM', \theta), (TM, \theta_l)$ are equivalent pairs and, in particular, $\varphi^* TM'$, TM are isomorphic to each other.

'*Only if*' *part*: By assumption, there is a bundle map $\alpha: TM \longrightarrow TM'$ covering φ. There is a bundle map $\beta: \nu \longrightarrow \nu'$ covering φ for which the following diagram commutes up to homotopy through bundle maps (cf. [**2**]):

$$\begin{array}{ccc} \varepsilon_M^{n+k} & \xrightarrow{\theta_\iota} & TM+\nu \\ {\scriptstyle\bar{\varphi}}\big\downarrow & & \big\downarrow{\scriptstyle\alpha+\beta} \\ \varepsilon_{M'}^{n+k} & \xrightarrow{\theta_{\iota'}} & TM'+\nu' \end{array}$$

Let $\alpha^{-1}: TM' \longrightarrow TM$ denote an inverse of α up to homotopy through bundle maps. Then it can be easily seen that the map $g': Y \longrightarrow \Sigma^l T(TM')$, dual to the unique map $T(a'): T(\nu'+\nu') \longrightarrow T(\bar{\gamma}_\kappa)$ with respect to the duality determined by $(TM', \theta_{\iota'})$ and the degree-one map $T(\beta)c_\iota$, factors the map $g: Y \longrightarrow \Sigma^l T(TM)$, dual to the unique map $T(a): T(\nu+\nu) \longrightarrow T(\bar{\gamma}_\kappa)$ with respect to the duality determined by (TM, θ_ι) and the degree-one map c_ι, by the map $\Sigma^l T(\alpha^{-1}): \Sigma^l T(TM') \longrightarrow \Sigma^l T(TM)$. That is, $g \simeq \Sigma^l T(\alpha^{-1}) g'$.

Now we may proceed as in the above to conclude that

$$b_{T(\beta)c_\iota}(TM', \theta_{\iota'}) = b_{c_\iota}(TM, \theta_\iota) = \chi_{1/2}(M) = \chi_{1/2}(M'). \qquad \square$$

Proof of Theorem 1.4. Let $\varrho: S\nu' \longrightarrow S\nu'$ be an automorphism of the sphere fibration such that $b^f_{c_{\iota'}}(STM', (1+\varrho)S\theta_{\iota'}) \neq \chi_{1/2}(M')$. Such a ϱ exists since the homotopy James–Thomas number is 1 while there are exactly two equivalence classes of pairs (ζ, ϑ) consisting of an $(n-1)$-sphere fibration ζ over M' and a trivialisation $\vartheta: \varepsilon \longrightarrow \zeta + S\nu$. (In fact, ϱ represents the non-trivial element in $\mathrm{Aut}^f(S\nu')/j\,\mathrm{Aut}^f(STM')$ of Theorem 1.6.)

Apply the usual transversality argument to $T(\varrho)c_{\iota'}: S^{n+k} \longrightarrow T(\nu')$ to obtain an element $[M, \varphi, F] \in NM(M')$, the normal set over M', where $\varphi: M \longrightarrow M'$ is a degree-one map and $F: \varepsilon_M^{n+k} \longrightarrow TM + \varphi^*\nu'$ is the trivialisation coming from the inclusion $M \hookrightarrow S^{n+k}$ (cf. [**10**]). Since the surgery obstruction group is zero, we may assume that $\varphi: M \longrightarrow M'$ is a homotopy equivalence.

Let $c_\iota: S^{n+k} \longrightarrow T(\nu)$ denote the collapse map onto the normal bundle ν of M coming from the inclusion $\iota: M \hookrightarrow S^{n+k}$. Also note that M comes with a bundle map $\beta: \nu \longrightarrow \nu'$ covering φ. By construction, $T(\beta)c_\iota$ is homotopic to $T(\varrho)c_{\iota'}$. Therefore, we have

$$b_{T(\beta)c_\iota}(TM', \theta_{\iota'}) \neq \chi_{1/2}(M').$$

Furthermore, there cannot be any bundle map $\beta': \nu \longrightarrow \nu'$ covering φ so that $b_{T(\beta')c_\iota}(TM', \theta_{\iota'}) = \chi_{1/2}(M')$; if there were such a β', we choose $\alpha: TM'+\nu' \longrightarrow TM'+\nu'$ so that $(\alpha+1)(\theta_{\iota'}+1)(\bar{\varphi}+\beta): \varepsilon_M^{n+k} \longrightarrow (TM'+\nu')+\nu'$ is homotopic to $(\theta_{\iota'}+1)(\bar{\varphi}+\beta')$ through bundle maps. Then we have $b_{T(\beta)c_\iota}(TM', \alpha\theta_{\iota'}) = b_{T(\beta')c_\iota}(TM', \theta_{\iota'}) = \chi_{1/2}(M')$, which contradicts the fact that the James–Thomas number of M' is 2. $\square$

Proof of Lemma 1.5. As noted by Sutherland in the last paragraph of [**7**], for any vector bundle ξ over $\Sigma^{14}P^2$, the Stiefel–Whitney class $w_i(\xi) = 0$ for $i > 0$. In fact, [**1**, Theorem 2] asserts in general that it is the case for any real vector bundle over a 9-fold suspension of any finite complex. On the other hand, Sutherland also shows that there is a sphere fibration ζ over $\Sigma^{14}P^2$ with $w_{16}(\zeta) \neq 0$.

Let $n \geqslant 3$ be an odd integer. Both [**6**, 1.6] and [**7**, 3.1], respectively in the sphere bundle (with a vector bundle reduction) category and in the sphere fibration category, assert that the number of equivalence classes of $(n-1)$-sphere fibrings over a connected n-complex A in a given stable class α is the order of the quotient of $H^n(A; Z_2)$ by the space spanned by $w_n(\alpha)$ and all the cohomology classes given by (see [**6**, (1.5)])

$$\sigma w_{n+1}(\xi) + \sum_{i=2}^{n} \sigma w_i(\xi)\, w_{n-i+1}(\alpha),$$

for a sphere fibring ξ over ΣA, where σ is the inverse of the suspension $H^i(A; Z_2) \longrightarrow H^{i+1}(\Sigma A; Z_2)$ for $i \geqslant 1$.

To show that the James–Thomas number of $S^{13} \times P^2$ is 2, note that there is a well known homotopy equivalence

$$\Sigma(S^{13} \times P^2) \xrightarrow{\simeq} \Sigma(S^{13} \vee P^2 \vee (S^{13} \wedge P^2)) \cong S^{14} \vee \Sigma P^2 \vee \Sigma^{14} P^2.$$

Now a straightforward calculation shows that there are two isomorphism classes of vector bundles of rank 15 over $S^{13} \times P^2$ in the stable class of the tangent bundle $T(S^{13} \times P^2) \cong TS^{13} \times TP^2$.

To show that the homotopy James–Thomas number is 1, consider the sphere fibration $\zeta' = f^*\zeta$, where $f\colon \Sigma(S^{13} \times P^2) \longrightarrow \Sigma(S^{13} \wedge P^2) \cong \Sigma^{14} P^2$ is the collapse map. Note that $w_{15}(\zeta)$ vanishes. First of all, for any sphere fibration ξ over S^n, $n \neq 2, 4, 8$, we must have $w_n(\xi) = 0$; otherwise there cannot be two fibrewise homotopy equivalence classes of stably trivial $(n-2)$-sphere fibrations over S^{n-1}. Now consider the map $\iota\colon S^{15} \cong \Sigma^{14} P^1 \longrightarrow \Sigma^{14} P^2$, which induces an isomorphism $\iota^*\colon H^{15}(\Sigma^{14} P^2; Z_2) \longrightarrow H^{15}(S^{15}; Z_2)$. But $\iota^* w_{15}(\zeta)$ must be zero. Now a straightforward calculation shows that there is only one fibrewise homotopy equivalence class of 14-sphere fibrations in the stable class of the tangent sphere fibration of $S^{13} \times P^2$. $\square$

References

1. M. F. Atiyah and F. Hirzebruch, 'Bott periodicity and the parallelizability of spheres', *Proc. Cambridge Philos. Soc.* 57 (1961) 223–226.
2. W. Browder, *Surgery on simply-connected manifolds*, Ergebnisse der Mathematik und Ihrer Grenzgebiete 65 (Springer, 1975).
3. Y. Byun, 'Tangent fibration of a Poincaré complex', *J. London Math. Soc.* (2) 59 (1999) 1101–1116.
4. J. L. Dupont, 'On homotopy invariance of the tangent bundle I', *Math. Scand.* 26 (1970) 5–13.
5. J. L. Dupont, 'On homotopy invariance of the tangent bundle II', *Math. Scand.* 26 (1970) 200–220.
6. I. M. James and E. Thomas, 'An approach to the enumeration problem for non-stable vector bundles', *J. Math. Mech.* 14 (1965) 485–506.
7. W. A. Sutherland, 'The fiber homotopy enumeration of non-stable sphere bundles and fibrings over real projective spaces', *J. London Math. Soc.* (2) 15 (1965) 543–556.
8. W. A. Sutherland, 'The Browder–Dupont invariant', *Proc. London Math. Soc.* (3) 33 (1976) 94–112.
9. C. T. C. Wall, 'Poincaré complexes I', *Ann. of Math.* 86 (1967) 213–245.
10. C. T. C. Wall, *Surgery on compact manifolds* (Academic Press, London, 1970).

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