TANGENT FIBRATION OF A POINCARÉ COMPLEX

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Abstract

The unstable tangent fibration of a Poincaré complex is defined so that it is consistent with the manifold case. It exists uniquely for each Poincaré complex X up to fibrewise homotopy equivalence and, furthermore, if a Poincaré embedding structure exists on the diagonal $X \longrightarrow X \times X$, its normal fibration is the tangent fibration.

1. *Introduction*

Let M be a smooth manifold and consider the diagonal embedding Δ: $M \longrightarrow M \times M$. Then there is a tubular neighbourhood N of ΔM, with a retraction $r: N \longrightarrow \Delta M = M$, so that the tangent sphere bundle of M is isomorphic to the one whose projection is $r|_{\partial N}$.

Therefore it seems natural to define the tangent fibration of a Poincaré complex (in the sense of Wall [**14**]) using a notion which corresponds to the normal neighbourhood of the diagonal subspace for smooth manifolds, namely the notion of *Poincaré embedding* (see §4). Unfortunately, there are difficulties in this approach. First of all, it is not clear whether the diagonal $\Delta: X \longrightarrow X \times X$ admits a Poincaré embedding. Even if there are results such as [**9**, **10**] which assert that given any continuous map $f: Y \longrightarrow X$ from a finite Poincaré complex of dimension i to another of dimension n that is simply connected with $n \geqslant 2i+1$, there exists a Poincaré embedding structure on f, they do not apply to our situation for reasons including a dimensional one. Furthermore, even if Poincaré embedding structures on Δ do exist, it is still not clear whether the normal fibrations of different Poincaré embeddings are fibrewise homotopy equivalent, unlike the situation in the stable range.

Even if we have obtained a partial result regarding the existence of a Poincaré embedding structure on the diagonal, the tools involved are remarkably different from those of this paper and it does not look natural for us to include it in the same paper. (To be more specific, we consider the case when the Poincaré complex under concern is formed by gluing two smooth manifolds along their boundaries using a homotopy equivalence. Furthermore, when G denotes the fundamental group of any path component of the boundary, we demand the extra condition that the diagonal subgroup $\Delta G < G \times G$ should satisfy the square-root closed condition (see [**5**]). In particular, this provides a ground for the suspicion that there might be a Poincaré complex for which the diagonal does not admit any Poincaré embedding structure.)

For these reasons, we have adopted Definition 1.1 below as the definition of the tangent fibration. Unfortunately, it exploits the invariants such as $\chi(\xi, U)$ (see Definition 4.5) and $b(\xi, \theta)$ (see §6). The former is just the Euler characteristic in a form

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generalised so that it may encompass the unorientable cases and the latter is the invariant defined by Dupont [**6**, **7**] and Sutherland [**12**] in a slightly revised form so that it may fit our more homotopy theoretic situation. Also $\chi(X)$ is the Euler–Poincaré number (see §3) and $\chi_{1/2}(X) = \sum_i \text{rank } H^{2i}(X; Z_2)$ is the semi-characteristic. Note that we allow a Poincaré complex to be infinite.

Definition 1.1. A *tangent fibration* of a Poincaré complex X, $\dim X = n$, is an $(n-1)$-sphere fibration ξ over X such that (i) ξ is stably inverse to the Spivak normal fibration ν, and (ii) if n is even, there is a Thom class U of ξ so that $\chi(\xi, U) = \chi(X)$, presuming a choice of fundamental class of X and, if n is odd, there is a stable trivialisation $\theta : \varepsilon^{n+k} \longrightarrow \xi + \nu$ so that $b(\xi, \theta) = \chi_{1/2}(X)$, presuming a choice of normal invariant $c : S^{n+k} \longrightarrow T(\nu)$.

Here, a normal invariant is just a degree-one map from the sphere S^{n+k} to the Thom complex $T(\nu)$ when ν is a $(k-1)$-sphere fibration.

The following justifies Definition 1.1.

Theorem 1.2. *Every Poincaré complex admits one and only one tangent fibration up to fibrewise homotopy equivalence.*

A fibrewise homotopy equivalence will exclusively mean one covering the identity map and we shall use the term 'fibre map' to refer in general to a map between fibrations which is fibre preserving and is a homotopy equivalence when restricted to each fibre.

Then, regarding the Poincaré embedding structure on the diagonal, we prove the following theorem.

Theorem 1.3. *Let X be a Poincaré complex. If there is a Poincaré embedding structure on the diagonal $\Delta : X \longrightarrow X \times X$, its normal fibration ν_Δ is the tangent fibration of X.*

Corollary 1.4 immediately follows from Theorems 1.2 and 1.3.

Corollary 1.4. *The tangent fibration of a smooth manifold* (*that is, the reduction of the tangent sphere bundle to a sphere fibration*) *is an invariant of the homotopy type of the manifold up to fibrewise homotopy equivalence.*

The same result is obtained by Benlian and Wagoner [**1**]. In particular, Dupont [**6**, **7**] defined an invariant, which is essentially the same as the one in this paper, to prove the homotopy type invariance of the tangent fibrations for odd-dimensional manifolds.

We note that Theorem 1.3 has the following parallel in smooth category: the normal bundle of any immersion homotopic to the diagonal embedding $\Delta : M \longrightarrow M \times M$ is isomorphic to the tangent bundle of M. This can be proved using a technique such as that in [**1**].

To prove Theorem 1.2 we will first show Proposition 1.5 below. For the precise meanings of such terms as 'equivalence' between (ξ, U) and 'equivalence' between (ξ, θ) below, one must refer respectively to the paragraph preceding Definition 4.5 and to the one preceding Proposition 6.2.

PROPOSITION 1.5. *Let X be a Poincaré complex of formal dimension n. Assume that n is even and fix a fundamental class of X. Then, for each integer κ, there is one and only one equivalence class of pair (ξ, U) such that*

$$\chi(\xi, U) = 2\kappa + \chi(X),$$

where ξ is an $(n-1)$-sphere fibration stably inverse to ν and $U \in H^n(D\xi, S\xi; Z^w)$ is a Thom class of ξ. Likewise, assume that n is odd and fix a normal invariant c: $S^{n+k} \longrightarrow T(\nu)$. Then there is one and only one equivalence class of pair (ξ, θ) such that

$$b(\xi, \theta) = i,$$

for each $i \in Z_2$, where ξ is an $(n-1)$-sphere fibration and $\theta : \varepsilon^{n+k} \longrightarrow \xi + \nu$ is a stable trivialisation.

The author would like to give his special thanks to Professor L. Taylor who suggested the definition of tangent fibration above, which improved the statement of Theorem 1.3, clarifying its proof.

2. *Homology*

We will use the homology theory with $Z\pi_1(X)$-module coefficients. Even if its appearance in literature is not rare, there are special needs to be met for this paper: we have to deal with maps that may not be isomorphisms on the fundamental group level and we should exploit the product operations in a slightly more refined fashion than usual (see §3).

Throughout the paper, a pair will always mean an *NDR*-pair (X, Y) (cf. [**16**, p. 22]) such as a CW-pair, where X is path-connected and pointed. Note that a pair (X, Y) is referred to as an *NDR*-pair if it satisfies the following: (i) X is compactly generated in the sense that X is Hausdorff and any set $C \subset X$ such that $C \cap K$ is closed for any compact $K \subset X$ is itself closed, and (ii) the inclusion map of Y into X is a cofibration. Maps should be understood as basepoint preserving. Also, we will understand X as coming with a universal cover $\tilde{X}$. If $\bar{Y} \subset \tilde{X}$ denotes the inverse image of Y by the covering projection, then the chain complex $C_*(X, Y)$ is the quotient of singular simplicial chain complexes $\Delta_*(\tilde{X})/\Delta_*(\bar{Y})$, which is a chain complex of left $Z\pi_1(X)$-modules.

We will use the involution of $Z\pi_1(X)$ defined by $\overline{\sum n_g g} = \sum n_g g^{-1}$ instead of the other form $\overline{\sum n_g g} = \sum w(g) n_g g^{-1}$, where $w : \pi_1(X) \longrightarrow \{\pm 1\}$ is a given homomorphism. Then, for any left $Z\pi_1(X)$-module B, $C_*(X, Y; B)$ is $C_*(X, Y) \otimes_{Z\pi_1(X)} B$ in which we exploit the right module structure of $C_*(X, Y)$ coming from the involution. Then $H_k(X, Y; B)$ is $H_k(C_*(X, Y; B))$. Similarly, $C^{-*}(X, Y; B)$ is $\mathrm{Hom}_{Z\pi_1(X)}(C_*(X, Y), B)$ and $H^p(X, Y; B)$ is $H_{-p}(C^{-*}(X, Y; B))$.

Any map $f : (X, Y) \longrightarrow (X', Y')$ induces a chain map $f_* : C_*(X, Y) \longrightarrow C_*(X', Y')$ between the chain complexes of abelian groups. To count the actions of fundamental groups, we understand $f_* : C_*(X, Y) \longrightarrow C_*(X', Y')$ as an $(f_\# : Z\pi_1(X) \longrightarrow Z\pi_1(X'))$-homomorphism: in general, if $\rho : \Lambda \longrightarrow \Lambda'$ is a ring homomorphism and B, B' are left Λ, Λ' modules, respectively, then a homomorphism $\alpha : B \longrightarrow B' (\alpha' : B' \longrightarrow B)$ will be referred to as a (co-)ρ-homomorphism if $\alpha(rb) = \rho(r)\alpha(b)$ (if $\alpha'(\rho(r) b') = r\alpha'(b')$) for any $r \in \Lambda$, $b \in B (b' \in B')$. Let $\Lambda = Z\pi_1(X)$, $\Lambda' = Z\pi_1(X')$ and $\rho = f_\# : Z\pi_1(X) \longrightarrow Z\pi_1(X')$. With these notations in mind, we have the following proposition.

PROPOSITION 2.1. *There is a well-defined homomorphism*

$$(f,\alpha)_*: H_*(X, Y; B) \longrightarrow H_*(X', Y'; B').$$

Similarly, there is a well-defined homomorphism

$$(f,\alpha')^*: H^*(X', Y'; B') \longrightarrow H^*(X, Y; B).$$

It is straightforward to define the product operations.

Let B, B' be as in Proposition 2.1. Regard $B \otimes B' = B \otimes_Z B'$ as a $Z(\pi_1(X) \times \pi_1(X')) = Z\pi_1(X \times X')$ module in the obvious way. Then the cross product can be defined with the help of the Eilenberg–Zilber map. Among others, it is a homomorphism

$$\times : H_*(X, Y; B) \otimes H_*(X', Y'; B') \longrightarrow H_*((X, Y) \times (X', Y'); B \otimes B')$$
$$\times : H^*(X, Y; B) \otimes H^*(X', Y'; B') \longrightarrow H^*((X, Y) \times (X', Y'); B \otimes B')$$

where $(X, Y) \times (X', Y')$ denotes the pair $(X \times X', X \times Y' \cup Y \times X')$.

For any continuous map $f: X' \longrightarrow X$ and a left $Z\pi_1(X)$-module B, let f^*B denote the $Z\pi_1(X')$-module such that its underlying abelian group is B itself and the action of $\pi_1(X')$ comes from $f_*: \pi_1(X') \longrightarrow \pi_1(X)$. Note that the identity $1: f^*B \longrightarrow B$ is a co-$f_\#$-homomorphism. Then the cup product is the composite

$$\cup : H^*(X, Y_1; B) \otimes H^*(X, Y_2; B') \xrightarrow{\times} H^*((X, Y_1) \times (X, Y_2); B \otimes B')$$
$$\xrightarrow{(d,1)^*} H^*(X, Y_1 \cup Y_2; d^*(B \otimes B'))$$

where $d: (X, Y_1 \cup Y_2) \longrightarrow (X, Y_1) \times (X, Y_2)$ is the diagonal.

Let $w: \pi_1(X) \longrightarrow \{\pm 1\}$ be a homomorphism. Then B^w means B itself except that the action of $\pi_1(X)$ is slightly changed by the rule $g \cdot b = w(g)gb$ for any $g \in \pi_1(X)$ and $b \in B$. For example, regarding the integers Z as a left $Z\pi_1(X)$-module with the trivial action, we have $d^*(B \otimes Z^w) \cong B^w$. We shall identify $\mathrm{Hom}(\pi_1(X), \{\pm 1\})$ with $H^1(X; Z_2)$ and the additive notation below has been used in this context.

The slant product can be defined in the usual way which again exploits the Eilenberg–Zilber map. It is a homomorphism:

$$/ : H_*((X', Y') \times (X, Y); B \otimes Z^w) \otimes H^p(X, Y; Z^{w'}) \longrightarrow H_{*-p}(X', Y'; B^{w+w'}).$$

Note that $d^*(B \otimes Z) = d^*(B^w \otimes Z^w)$ and $j: B \longrightarrow B^w \otimes Z^w$, $j(b) = b \otimes 1$, is a $d_\#$-homomorphism. The cap product is the composite:

$$\cap : H_k(X, Y_1 \cup Y_2; B) \otimes H^p(X, Y_2; Z^w) \xrightarrow{(d,j)_* \otimes 1}$$
$$H_k((X, Y_1) \times (X, Y_2); B^w \otimes Z^w) \otimes H^p(X, Y_2; Z^w) \xrightarrow{/} H_{k-p}(X, Y_1; B^w).$$

The sign conventions for the products are still at work.

PROPOSITION 2.2. *For* $x \in H_k(X, Y_1 \cup Y_2; B)$, $y \in H_l(X', Y'_1 \cup Y'_2; B')$ *and* $u \in H^p(X, Y_2; Z^w)$, $v \in H^q(X', Y'_2; Z^{w'})$, *we have*

(i) $v \cup u = (-1)^{pq} u \cup v \in H^{p+q}(X, Y_2 \cup Y'_2; Z^{w+w'})$, *if* $X = X'$;

(ii) $(x \times y) \cap (u \times v) = (-1)^{lp}((x \cap u) \times (y \cap v)) \in H_n((X, Y_1) \times (X', Y'_1);\ B^w \otimes B'^{w'})$, $n = k+l-p-q$;

(iii) $x \cap (u \cup v) = (x \cap u) \cap v \in H_{k-p-q}(X, Y_0; B^{w+w'})$, *where* $X = X'$ *and* $Y_1 = Y_0 \cup Y'_2$.

3. *Diagonal cohomology class*

Throughout this section, X will be fixed as a connected Poincaré complex in the sense of Wall [**14**] with an orientation character w and a fundamental class $[X]\in H_n(X;Z^w)$. Note that $X\times X$ is also a Poincaré complex with the fundamental class $[X]\times[X]\in H_{2n}(X\times X;Z^{w\times w})$. Note that, in our convention (see §2), $w_1\times w_2$: $\pi_1(X\times X)\longrightarrow\{\pm 1\}$ means the one identified with $w_1\times 1+1\times w_2\in H^1(X\times X;Z_2)$, for any homomorphism $w_1, w_2:\pi_1(X)\longrightarrow\{\pm 1\}$.

Now let R be either the integer ring Z or the field of rational numbers Q and consider a homomorphism $w':\pi_1(X)\longrightarrow\{\pm 1\}$. We will identify the two coefficients $R^w\otimes R^{w'}$, $R^{w\times w'}$ so that $r\otimes s$ corresponds to rs.

Consider the diagonal map $\Delta: X\longrightarrow X\times X$. Then the identity $1: R^w\longrightarrow R^{(w'+w)\times w'}$ is a $\Delta_\#$-homomorphism and, therefore, we have a homomorphism

$$(\Delta,1)_*: H_*(X,R^w)\longrightarrow H_*(X\times X,R^{(w'+w)\times w'}).$$

From now on, we will write Δ_* to denote $(\Delta,1)_*$. Also we will use the same notation $[X]\in H_n(X;R^w)$ to denote the image of $[X]$ by the homomorphism $H_n(X;Z^w)\longrightarrow H_n(X;R^w)$. The following lemma is clear and we omit the proof.

LEMMA 3.1. *$[X]\cap : H^k(X;R^{w'})\longrightarrow H_{n-k}(X;R^{w'+w})$ is an isomorphism for all $k\in Z$.*

DEFINITION 3.2. The *diagonal cohomology class* $u_{w'}\in H^n(X\times X;R^{w'\times(w'+w)})$ is the unique class satisfying $[X\times X]\cap u_{w'}=\Delta_*[X]\in H_n(X\times X;R^{(w'+w)\times w'})$.

For any pair (X,Y) of finitely dominated spaces, we consider the number

$$\chi_{w',R}(X,Y)=\sum_k(-1)^k\operatorname{rank} H_k(X,Y;R^{w'})\in Z,$$

in which $w':\pi_1(X)\longrightarrow\{\pm 1\}$ is any homomorphism.

In fact, $\chi_{w',R}(X,Y)$ does not depend on the choice of w' or R. Let P_* be a chain complex of finitely generated projective $Z\pi_1(X)$-modules, chain homotopy equivalent to $C_*(X,Y)$ (cf. [**13**]). Then $P_*\otimes_{Z\pi_1(X)}Z_2\cong(P_*\otimes_{Z\pi_1(X)}Z^{w'})\otimes Z_2$, for any w'. Note that $P_i\otimes_{Z\pi_1(X)}Z^{w'}$ is a free abelian group since it is a direct summand of one. Therefore,

$$\chi_{w',R}(X,Y)=\sum_i(-1)^i\operatorname{rank}_Z(P_i\otimes_{Z\pi_1(X)}Z^{w'})=\sum_i(-1)^i\operatorname{rank}_{Z_2}(P_*\otimes_{Z\pi_1(X)}Z_2).$$

The arguments above and the resulting definition below were suggested by Professor Frank Connolly.

DEFINITION 3.3. For any pair (X,Y) of finitely dominated spaces, the *Euler–Poincaré number* of (X,Y), denoted by $\chi(X,Y)$, is the common value $\chi_{w',R}(X,Y)$.

For any $x\in H_p(X;R^{w'})$, $a\in H^p(X;R^{w'})$, we write $\langle x,a\rangle = x\cap a\in H_0(X;R)\cong R$. Note that $\langle x\cap a,b\rangle=\langle x,a\cup b\rangle$, for any $x\in H_{p+q}(X;R^{w'+w''})$, $a\in H^p(X;R^{w'})$ and $b\in H^q(X;R^{w''})$.

The technique of the proof of Proposition 3.4 below can be found in [**11**] without the complications arising from the use of the equivariant homology theory.

PROPOSITION 3.4. *For any homomorphism $w': \pi_1(X) \longrightarrow \{\pm 1\}$, it holds that*

$$\langle \Delta^* u_{w'}, [X] \rangle = \chi(X).$$

Proof. It is enough to show the equality when the coefficient R is Q.

Both $H_*(X; Q^{w'})$ and $H^*(X; Q^{w'})$ are graded vector spaces over Q of finite rank. Note that the map $H^q(X; Q^{w'}) \longrightarrow \mathrm{Hom}(H_q(X; Q^{w'}), Q)$, $a \longrightarrow \langle \cdot, a \rangle$ is an isomorphism.

Let $\{y_1, \ldots, y_\nu\}$ be a basis for $H_*(X; Q^{w'})$ consisting only of homogeneous elements. Then there is a dual basis $\{y^1, \ldots, y^\nu\}$ for $H^*(X; Q^{w'})$, in the sense that $\langle y_i, y^j \rangle = \delta_{ij}$ understanding $\langle y_i, y^j \rangle = 0$ if $\dim y_i \neq \dim y^j$. Here δ_{ij} is the Kronecker delta.

On the other hand, there are unique $x_1, \ldots, x_\nu \in H_*(X; Q^{w'+w})$ such that $\Delta_*[X] = \sum_i x_i \times y_i \in H_n(X \times X; Q^{(w'+w) \times w'})$. Then, by a direct calculation, it holds that $\Delta_*[X]/y^i = x_i$. Furthermore, since the homomorphism

$$\Delta_*[X]/ = [X] \cap : H^*(X; Q^{w'}) \longrightarrow H_{n-*}(X; Q^{w'+w})$$

is an isomorphism, $\{x_1, \ldots, x_\nu\}$ is in fact a basis for $H_*(X; Q^{w'+w})$. Also let $\{x^1, \ldots, x^\nu\}$ be the basis for $H^*(X; Q^{w'+w})$ dual to $\{x_1, \ldots, x_\nu\}$. In particular, we have

$$\langle [X], y^i \cup x^j \rangle = \langle [X] \cap y^i, x^j \rangle = \langle x_i, x^j \rangle = \delta_{ij}.$$

Therefore, we have

$$\langle [X] \cap x^j, y^i \rangle = \langle [X], x^j \cup y^i \rangle = (-1)^{\dim x^j \dim y^i} \delta_{ij}$$

and, by uniqueness of the dual basis, we conclude that

$$[X]_Q \cap x^i = (-1)^{(n - \dim y_i) \dim y_i} y_i.$$

We may write $u_{w'} = \sum y^i \times x'^i$ for some $x'^i \in H^*(X; Q^{w'+w})$, $i = 1, \ldots, \nu$. Then, from the equalities

$$([X] \times [X]) \cap u = \sum (-1)^{n \dim y^i} ([X] \cap y^i) \times ([X] \cap x'^i) = \sum x_i \times y_i$$

we have

$$x'^i = (-1)^{(n - \dim y^i) \dim y^i} (-1)^{n \dim y^i} x^i = (-1)^{\dim y^i} x^i.$$

Therefore, it follows that $u_{w'} = \sum (-1)^{\dim y^i} y^i \times x^i$. Finally, we have

$$\begin{aligned} \langle [X], \Delta^* u_{w'} \rangle &= \sum (-1)^{\dim y_i} \langle [X], y^i \cup x^i \rangle = \sum (-1)^{\dim y_i} \\ &= \sum_p (-1)^p \operatorname{rank} H_p(X; Q^{w'}) = \chi(X). \end{aligned}$$ □

4. *Sphere fibration*

Let ξ be an $(n-1)$-sphere fibration over a space X with projection p. Then we will write $S\xi$ to denote the total space, $D\xi$, the mapping cylinder of p. We will write s for the natural inclusions $X \hookrightarrow D\xi$, $X \longrightarrow (D\xi, S\xi)$. For any subspace A of X, ξ_A will mean the restriction. If η is another $(m-1)$-sphere fibration over another space Y with projection q, the product $\xi \times \eta$ means the $(n+m-1)$-sphere fibration over $X \times Y$ whose total space $S(\xi \times \eta)$ is $S\xi \times D\eta \cup D\xi \times S\eta$, with the obvious projection (replacing it with a fibration in the sense that it satisfies the homotopy lifting property, if necessary). Then, if $Y = X$, the Whitney sum $\xi + \eta$ will mean the pull-back of $\xi \times \eta$ along the diagonal $\Delta: X \longrightarrow X \times X$.

Following Wall [**15**, p. 113], given Poincaré complexes A, X of respective dimension $n, n+q$ and a map $f: A \longrightarrow X$, a *Poincaré embedding structure* on f consists of (i) a $(q-1)$-spherical fibration $\nu_f = \nu$ with projection $p: S\nu \longrightarrow A$, (ii) a Poincaré pair $(Z, S\nu)$ of dimension $n+q$, and (iii) a homotopy equivalence $h: D\nu \cup_{S\nu} Z \longrightarrow X$, so that the following diagram commutes up to homotopy:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "s"'] & X \\ D\nu \arrow[r, hook] & D\nu \cup Z \arrow[u, "h"'] \end{tikzcd}$$

We will call ν_f the *normal fibration* of the Poincaré embedding structure.

For the Poincaré complex X, let ν_X denote the Spivak fibration of X. The following lemma is due to Wall [**15**, p. 115].

LEMMA 4.1 (Wall). *Let A and X be Poincaré complexes. Assume that $f: A \longrightarrow X$ is a continuous map which admits a Poincaré embedding structure with a normal fibration ν_f. Then ν_A is stably fibrewise homotopy equivalent to $\nu_f + f^*\nu_X$.*

Therefore, the normal fibration of a continuous map, if it admits a Poincaré embedding structure, is unique up to stable fibrewise homotopy equivalence. Also, we have the following corollary.

COROLLARY 4.2. *Let X be a Poincaré complex and assume that the diagonal $\Delta: X \longrightarrow X \times X$ admits a Poincaré embedding structure with a normal fibration ν_Δ. Then ν_Δ represents the stable inverse of the Spivak fibration of X.*

The following lemma is well known and can easily be proved using the Thom isomorphism theorem and the fact that $H^{n+k}(D\nu_X, S\nu_X; Z) \cong Z$, where $n = \dim X$ and k is the fibre dimension of ν_X.

LEMMA 4.3. *The orientation character of a Poincaré complex X is the first Stiefel–Whitney class $w_1(\nu_X)$.*

By combining Corollary 4.2 and Lemma 4.3, we have Corollary 4.4.

COROLLARY 4.4. *If there is a Poincaré embedding structure on the diagonal $\Delta: X \longrightarrow X \times X$, $w_1(\nu_\Delta)$ is the orientation character of X.*

In general, if ξ, η are sphere fibrations and $\alpha: \eta \longrightarrow \xi$ is a fibre map, there is the Thom map $T(\alpha): (D\eta, S\eta) \longrightarrow (D\xi, S\xi)$. Moreover, let $U_\xi \in H^k(D\xi, S\xi; Z^{w_\xi})$, $U_\eta \in H^k(D\eta, S\eta; Z^{w_\eta})$ be some presumed Thom classes. Then we shall say α preserves the Thom classes (or the orientations) if $T(\alpha)^* U_\xi = U_\eta$.

From now on, X will be a Poincaré complex with a fixed fundamental class $[X] \in H_n(X; Z^w)$. Then we consider the category of all pairs (ξ, U_ξ) in which ξ is an $(n-1)$-sphere fibration over X whose first Stiefel–Whitney class is w and $U_\xi \in H^n(D\xi, S\xi; Z^w)$ is the Thom class. Two such pairs are *equivalent* to each other if there is a fibrewise homotopy equivalence between the two which preserves the orientations.

DEFINITION 4.5. We will call $\langle [X], s^*U_\xi \rangle \in Z$ the *Euler characteristic* and denote it by $\chi(\xi, U_\xi)$.

It is clear that $\chi(\xi, U_\xi)$ is an invariant of the equivalence class containing (ξ, U_ξ). In particular, if ξ admits an automorphism reversing the orientation, it follows that $\chi(\xi, U_\xi) = 0$.

Let $h\colon D\nu_\Delta \cup_{S\nu_\Delta} W \longrightarrow X \times X$ specify a Poincaré embedding structure on the diagonal $\Delta\colon X \longrightarrow X \times X$. Then the Poincaré embedding structure specifies a choice of Thom class U_τ of $\tau = \nu_\Delta$ as follows. Let $\overline{h^{-1}}\colon X \times X \longrightarrow (D\tau \cup W, W)$ denote the map defined by h^{-1}, a homotopy inverse of h, and let $\imath\colon (D\tau, S\tau) \longrightarrow (D\tau \cup W, W)$ denote the excision map. Consider the maps

$$H_{2n}(X \times X; Z^{w\times w}) \xrightarrow{\overline{h^{-1}}_*} H_{2n}(D\tau \cup W, W; Z^{h^*(w\times w)}) \xleftarrow{\imath_*} H_{2n}(D\tau, S\tau; Z).$$

It is not difficult to see that $\overline{h^{-1}}_*$, $\imath_*$ in the above are isomorphisms. Choose $[D\tau, S\tau] \in H_{2n}(D\tau, S\tau; Z)$ so that

$$\imath_*[D\tau, S\tau] = \overline{h^{-1}}_*([X] \times [X]).$$

Then $U_\tau \in H^n(D\tau, S\tau; Z^w)$ is chosen such that $[D\tau, S\tau] \cap U_\tau = [X]$.

Proposition 4.6. *Let X, τ and U_τ be as above. Then we have*

$$\chi(\tau, U_\tau) = \chi(X).$$

Proof. We will use the same notations as above.

Since $\imath^*\colon H^n(D\tau \cup W, W; Z^{h^*(w\times 1)}) \longrightarrow H^n(D\tau, S\tau; Z^w)$ is an isomorphism, there is a unique class $U' \in H^n(D\tau \cup W, W; Z^{h^*(w\times 1)})$ satisfying $\imath^* U' = U_\tau$ and we write u' to denote $\overline{h^{-1}}^* U'$. Then there is a commutative diagram:

$$\begin{array}{ccccc}
H_{2n}(X\times X; Z^{w\times w}) & \xrightarrow{h_*^{-1}} & H_{2n}(D\tau \cup W, W; Z^{h^*(w\times w)}) & \xleftarrow{\imath_*} & H_{2n}(D\tau, S\tau; Z) \\
\Big\downarrow{\cap u'} & & \Big\downarrow{\cap U'} & & \Big\downarrow{\cap U_\tau} \\
H_{2n}(X\times X; Z^{1\times w}) & \xrightarrow{h_*^{-1}} & H_n(D\tau \cup W; Z^{h^*(1\times w)}) & \xleftarrow{\imath_*} & H_n(D\tau; Z^w).
\end{array}$$

Thus it holds that $h_*^{-1}([X\times X] \cap u') = \imath_*([D\tau, S\tau] \cap U_\tau)$, in which the latter is just $\imath_*[D(\tau)]$. Note that $h^{-1}\Delta \simeq \imath s$ as maps from X into $D\tau \cup W$. We conclude that

$$[X\times X] \cap u' = h_* \imath_*[D(\tau)] = h_* \imath_* s_*[X] = h_* h_*^{-1} \Delta_*[X] = \Delta_*[X].$$

Therefore, u' is none other than the diagonal cohomology class u_w. Since $\overline{h^{-1}}\Delta \simeq \imath s$, it follows that $s^* U_\tau = s^* \imath^* U' = \Delta^* \overline{h^{-1}}^* U' = \Delta^* u_w$, which is enough for the proof considering Proposition 3.4. □

More generally, the following holds, which we include for later use.

Proposition 4.7. *Let ξ denote the stable inverse of ν_X. Then for the nth Stiefel–Whitney class $w_n(\xi)$, we have*

$$w_n(\xi)([X]) \equiv \chi(X) \bmod 2.$$

Note that Proposition 4.7 is clear if the diagonal $\Delta\colon X \longrightarrow X\times X$ admits a Poincaré embedding. If ξ is an $(n-1)$-sphere fibration, $w_n(\xi)$ is the image of the Euler class $e(\xi) = s^* U_\xi$ by the natural homomorphism $H^n(X; Z^w) \longrightarrow H^n(X; Z_2)$ (cf. [**11**, 9.5]). However, $w_n(\xi)$ is an invariant of the stable class of ξ. Since there is a representative $\tau = \nu_\Delta$ for the stable inverse of ν, the assertion follows from Proposition 4.6. The general case can be proved in a straightforward manner using the following facts. If n is odd, $w_n = 0$ and $\chi(X) = 0$. If $n = 2m$, $w_n = v_m \cup v_m$, where $v_m \in H^m(X; Z_2)$ is such that $\langle v_m, x\rangle = \langle x, x\rangle$ for any $x \in H^m(X; Z_2)$ (cf. [**11**]). Here, $\langle\cdot,\cdot\rangle$: $H^m(X; Z_2) \times H^m(X; Z_2) \longrightarrow Z_2$ denotes the intersection form, defined by $\langle x, y\rangle = (x \cup y)([M])$, which is non-degenerate. Also, by duality, $\chi(X) = \operatorname{rank} H^m(X; Z_2)$.

5. *Proof of Proposition* 1.5 *for even dimensions*

Any Poincaré complex admits a Poincaré embedding structure on the inclusion of a one-point space according to [**8**, **14**]. Therefore, given a Poincaré complex X, we may choose a decomposition $X = D^n \cup_{S^{n-1}} L$ where L is homotopic to an $(n-1)$-complex. Then we will follow Dupont [**6**]: given a pair (ξ, U) consisting of an $(n-1)$-sphere fibration ξ over X and its Thom class U, we introduce an operation which alters (ξ, U) systematically to another such pair preserving the stable fibre homotopy equivalence class of ξ. Then we show that the Euler characteristic detects this operation sensitively enough to prove Proposition 1.5 for even dimensions.

Consider the *pinching map*

$$p: X \longrightarrow X/S^{n-1} \cong X \vee S^n.$$

Let BG_n be the classifying space of $(n-1)$-sphere fibrations, for which we fix a basepoint. Let $v: X \longrightarrow BG_n$, $\mu: S^n \longrightarrow BG_n$ be basepoint-preserving maps. Then for any integer k, we consider the composite

$$X \xrightarrow{p} X \vee S^n \xrightarrow{u \vee \mu^k} BG_n,$$

in which $\mu^k: S^n \longrightarrow BG_n$ represents $k\cdot[\mu] \in \pi_n(BG_n)$. We denote the composite $(v \vee \mu^k)p$ by $v \bigstar \mu^k$.

Let $j: BG_k \longrightarrow BG_{k+1}$ be the map stabilising the $(k-1)$-sphere fibrations. We recall the following proposition.

PROPOSITION 5.1 (Dupont). *Assume that v, v' are basepoint-preserving maps from X into BG_n. Then they represent fibrations over X stably fibre homotopy equivalent to each other if and only if v' is homotopic to $v \bigstar \mu$ for some $\mu \in \operatorname{Ker} j_*: \pi_n(BG_n) \longrightarrow \pi_n(BG_{n+1})$ by a basepoint-preserving homotopy.*

In fact, [**6**, 2.1] concerns only the oriented fibrations. However, the proof is still valid without a change. Also the following is well known (cf. [**6**, 2.2]).

PROPOSITION 5.2. *The kernel of $j_*: \pi_n(BG_n) \longrightarrow \pi_n(BG_{n+1})$ is cyclic generated by the map classifying the tangent sphere fibration of S^n. Furthermore,*

$$\operatorname{Ker} j_* = \begin{cases} Z & \text{if } n \text{ is even} \\ 0 & \text{if } n = 1, 3, 7 \\ Z_2 & \text{if } n \text{ is odd and } q \neq 1, 3, 7. \end{cases}$$

Choose a universal $(n-1)$-sphere fibration γ_n over BG_n and orient γ_n by choosing a Thom class $U_{\gamma_n} \in H^n(D\gamma_n, S\gamma_n; Z^w)$, where $w: \pi_1(BG_n) \longrightarrow \{\pm 1\}$ is the isomorphism.

Let Y be any connected complex with a basepoint. For any basepoint-preserving map $v: Y \longrightarrow BG_n$, write $v^*\gamma_n$ for the pull-back fibration and $v^*U_{\gamma_n}$ for the Thom class of $v^*\gamma_n$ which is the pull-back of U_{γ_n} along the natural fibre map. Then the equivalence class of the pair $(v^*\gamma_n, v^*U_{\gamma_n})$ does not depend on the choice of v up to basepoint-preserving homotopy.

Moreover, consider the usual action of $\pi_1(BG_n)$ on the set $[Y, BG_n]_*$ of the basepoint-preserving homotopy class of maps and let $[\rho]\cdot[v] = [\rho\cdot v]$ denote $[v]\in[Y, BG_n]_*$ multiplied by $[\rho]\in\pi_1(BG_n)$.

LEMMA 5.3. *For the non-trivial element $[\rho]\in\pi_1(BG_n)$ and for any $[v]\in[Y, BG_n]_*$, the pair $((\rho\cdot v)^*\gamma_n, (\rho\cdot v)^* U_{\gamma_n})$ is equivalent to $(u^*\gamma_n, -v^*U_{\gamma_n})$.*

Proof. There is a map $G\colon Y\times[0,1]\longrightarrow BG_n$ such that $G(\cdot,0)$ is v and $G(y_0,\cdot)$ is ρ, where $y_0\in Y$ denotes the basepoint. We write $\rho\cdot v$ for $G(\cdot,1)$. Then, $\rho\cdot v$ represents the homotopy class $[\rho]\cdot[v]\in[Y, BG_n]_*$.

Now we have $(G^*\gamma_n)|_{Y\times\{0\}} = v^*\gamma_n$, $(G^*\gamma_n)|_{Y\times\{1\}} = (\rho\cdot v)^*\gamma_n$. Also note that $v^*U_{\gamma_n}$, $(\rho\cdot v)^* U_{\gamma_n}$ are the pull-backs of $G^*U_{\gamma_n}$ by the inclusions $v^*\gamma_n\longrightarrow G^*\gamma_n$, $(\rho\cdot v)^*\gamma_n\longrightarrow G^*\gamma_n$.

Consider $\alpha\colon v^*\gamma_n\longrightarrow(\rho\cdot v)^*\gamma_n$ which is the restriction of a fibrewise homotopy equivalence $\bar\alpha\colon v^*\gamma_n\times I\longrightarrow G^*\gamma_n$ to $Y\times\{1\}$.

Let $*$ denote the basepoint of BG_n and consider the composite, $\bar\alpha'\colon\gamma|_{\{*\}}\times[0,1]\longrightarrow\rho^*\gamma_n\longrightarrow\gamma_n$, which is the restriction of $\bar\alpha$ followed by the natural fibre map.

Then the restrictions of $\bar\alpha$, say, $\alpha'_r\colon\gamma_n|_{\{*\}} = \gamma_n|_{\{*\}}\times\{r\}\longrightarrow\gamma_n|_{\{*\}}$, $r = 0, 1$, would have been homotopic to each other only if ρ represented the trivial element of $\pi_1(BG_n)$. From this it is straightforward to see that $T(\alpha)^*((\rho\cdot v)^* U_{\gamma_n}) = -v^*U_{\gamma_n}$, which completes the proof. □

Proposition 5.4 immediately follows.

PROPOSITION 5.4. *For any oriented $(n-1)$-sphere fibration (ξ, U) over Y, there is a unique class $[v]\in[Y, BG_n]_*$ so that (ξ, U) is equivalent to $(v^*\gamma_n, v^*U_{\gamma_n})$.*

We orient the Poincaré complex X by choosing a fundamental class $[X]\in H_n(X; Z^w)$ and let (ξ, U) denote an oriented $(n-1)$-sphere fibration over X such that $w_1(\xi) = w$.

Let (τ_n, U_n) denote the oriented tangent sphere fibration of S^n, which we consider with the standard orientation $[S^n]$. We choose U_n so that it is consistent with the Poincaré embedding structure on $\Delta\colon S^n\longrightarrow S^n\times S^n$ (see §4).

Let $\mu\colon S^n\longrightarrow BG_n$, $v\colon X\longrightarrow BG_n$ be the classifying maps for (τ_n, U_{τ_n}), (ξ, U) respectively, in the sense of Proposition 5.4. Then we will write $(\xi\star\tau_n^k, U\star U_n^k)$ to denote

$$((v\star\mu^k)^*\gamma_n, (v\star\mu^k)^* U_{\gamma_n}).$$

PROPOSITION 5.5.

$$\chi(\xi\star\tau_n^k, U\star U_n^k) = \begin{cases} \chi(\xi, U)+2k & \text{if } n \text{ is even} \\ \chi(\xi, U) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Note that $\chi(\xi\star\tau_n^k, U\star U_n^k)$ is the evaluation at $[X]$ of the pull-back of $s^*U_{\gamma_n}$ by $v\star\mu^k$ which is the composite

$$X\xrightarrow{p} X\vee S^n\xrightarrow{v\vee\mu^k} BG_n.$$

Note also that $H^n(X\vee S^n; Z^{r^*w})\cong H^n(X; Z^w)\oplus H^n(S^n; Z)$, where $r\colon X\vee S^n\longrightarrow X$ is the map collapsing S^n, and, allowing ourselves a slight abuse of the notations, that $p^*\overline{[X]} = \overline{[X]}$, $p^*\overline{[S^n]} = \overline{[X]}$, where $\overline{[X]}, \overline{[S^n]}$ are the top dimensional cohomology classes dual to the 1s in $H_0(X; Z)$, $H_0(S^n; Z)$, respectively.

Now the proof follows from the fact that $\chi(\tau_n, U_n)$ is 2, if n is even, and 0 if n is odd. □

Proof of Proposition 1.5 *for even dimensions*. Apply [**8**, **15**] to choose a decomposition $X \simeq D^n \cup_{S^{n-1}} L$ so that (L, S^{n-1}) is a Poincaré pair. Then we may define the operation $\bigstar$ of $\pi_n(BG_n)$ on $[X, BG_n]_*$ as above.

Note that the stabilising map, $BG_n \longrightarrow BG_{n+k}, k \geqslant 1$, is an n-equivalence. Therefore, a stable inverse of ν_X can be represented by an $(n-1)$-sphere fibration ξ over the n-complex X (cf. [**16**]).

Now, by Propositions 4.6, 5.1, 5.2, 5.4 and 5.5, we conclude Proposition 1.5 when n is even. □

6. *Proof of Proposition* 1.5 *for odd dimensions*

By Propositions 5.1 and 5.2, given an odd-dimensional Poincaré complex X, there are at most two classes of $(n-1)$-sphere fibrations over X which are stably inverse to the Spivak fibration. To distinguish between the two, we will use an invariant introduced by Dupont [**6**, **7**] and subsequently revised by Sutherland [**12**].

Even if Sutherland confines his construction to vector bundles, most of the strategy also works in our homotopy theoretic situation. In particular, the use of his 'symmetric lifting' is crucial in the process (see below).

Throughout the section, we fix a Poincaré complex X of odd dimension n together with a normal invariant $c: S^{n+k} \longrightarrow T(\nu)$, where $T(\nu)$ denotes the Thom space of the Spivak $(k-1)$-sphere fibration ν of X, $k \geqslant n+2$.

We consider a pair (ξ, θ) which consists of an $(n-1)$-sphere fibration ξ and a *trivialisation* θ which is a fibrewise homotopy equivalence from the trivial $(n+k)$-sphere fibration ε^{n+k} to the Whitney sum $\xi + \nu$. We outline the definition of the so-called b-invariant as follows (for more details, see [**2**, **6**, **7**, **12**]).

In what follows, Z_2-coefficients must be understood for all the (co)homology groups.

Let $\bar{\gamma}_\kappa$, $\kappa \geqslant 2k+2$, denote a universal $(\kappa-1)$-sphere fibration over a space E which classifies $(\kappa-1)$-sphere fibrations whose $(n+1)$th Wu classes v_{n+1} vanish. Here, the total Wu class $v = 1 + v_1 + v_2 + \ldots$ of a sphere fibration ζ over a space Y means $Sq^{-1}w^{-1} \in H^*(Y)$, where w^{-1} is the inverse of the total Stiefel–Whitney class w of ζ and Sq^{-1} is the multiplicative inverse of the total Steenrod square Sq in the Steenrod algebra (cf. [**2**]).

Choose a finite complex Y together with a duality between Y and (a finite skeleton of) $T(\bar{\gamma}_k)$.

On the other hand, the trivialisation θ specifies, together with the normal invariant $c: S^{n+k} \longrightarrow T(\nu)$, a duality between $T(\xi)$ and $T(\nu + \nu)$ as follows:

$$\begin{aligned} S^{2n+2k} &\cong \Sigma^{n+k} S^{n+k} \xrightarrow{\Sigma^{n+k} c} \Sigma^{n+k} T(\nu) \\ &\cong T(\varepsilon^{n+k} + \nu) \xrightarrow{T(\theta+1)} T(\xi + \nu + \nu) \xrightarrow{T(\bar{\Delta})} T(\xi \times (\nu + \nu)) \cong T(\xi) \wedge T(\nu + \nu), \end{aligned}$$

in which $\Sigma^m \cdot$ denotes the mth reduced suspension and $\bar{\Delta}$ denotes the natural fibre map covering the diagonal $\Delta: X \longrightarrow X \times X$.

Furthermore, Sutherland has shown that there is a symmetric lifting which is a fibre map $A: \nu \times \nu + \varepsilon^i \longrightarrow \bar{\gamma}_\kappa$, $i = \kappa - 2k$ such that $A(t+1)$ and A are homotopic to each other through fibre maps, where $t: v\times v \longrightarrow v\times v$ is the fibre map given by exchanging the factors. Also he has shown that $a = A(\bar{\Delta}+1): v+v+\varepsilon^i \longrightarrow \bar{\gamma}_\kappa$ is independent of the choice of A up to homotopy through fibre maps.

Therefore, there is a map $g: Y \longrightarrow \Sigma^l T(\xi)$, for some integer l, well defined up to stable homotopy as the dual of $T(a): T(v+v+\varepsilon^i) \longrightarrow T(\bar{\gamma}_\kappa)$ with respect to the dualities above.

Let the composite

$$Y \xrightarrow{g} \Sigma^l T(\xi) \xrightarrow{\Sigma^l U_\xi} \Sigma^l K_n$$

be denoted by f, where U_ξ is the Thom class and $K_n = K(Z_2, n)$ is the Eilenberg–Maclane space. Then the invariant $b(\xi, \theta)$ is defined as

$$Sq_f^{n+1}(\Sigma^l \iota) \in H^{l+2n}(Y) \cong Z_2,$$

where Sq_f^{n+1} is the functional Steenrod square and $\iota \in H^n(K_n)$ is the fundamental class.

Only a point regarding the well-definedness of $b(\xi,\theta)$ seems worth some comments considering the works by Dupont and Sutherland. We ask whether the functional Steenrod square is well defined. Recall that it is defined via a diagram chase with the commutative diagram

$$\begin{array}{ccccccc}
 & H^{l+n-1}(Y) & \xrightarrow{\delta} & H^{l+n}(f) & \xrightarrow{j^*} & H^{l+n}(\Sigma^l K_n) & \xrightarrow{f^*} \\
 & \downarrow{\scriptstyle Sq^{n+1}} & & \downarrow{\scriptstyle Sq^{n+1}} & & \downarrow{\scriptstyle Sq^{n+1}} & \\
\xrightarrow{f^*} & H^{l+2n}(Y) & \xrightarrow{\delta} & H^{l+2n+1}(f) & \xrightarrow{j^*} & H^{l+2n+1}(\Sigma^l K_n). &
\end{array}$$

To begin the chase, we must have the following lemma (cf. [**7**, 4.2]).

LEMMA 6.1. $f^*(\Sigma^l \iota) = 0$.

Proof. It is enough to show that $g^*\Sigma^l U_\xi = 0$. But $\Sigma^l U_\xi$ is dual to the generator of $H^{n+2k}(T(v+v))$. Therefore, it is also enough to show that $\hat{a}^*: H^n(E) \longrightarrow H^n(X)$ is a zero homomorphism, where $\hat{a}$ is the map covered by the fibre map $a = A(\bar{\Delta}+1)$ (see above).

Note that, writing $\hat{A}$ for the map covered by A, we have $\hat{a} = \hat{A}\Delta$.

Let $u \in H^n(E)$ be any class. Write $\hat{t}: X\times X \longrightarrow X \times X$ for the map transposing the factors. Then, since $\hat{A}\hat{t}$ is homotopic to $\hat{A}$, it follows that $\hat{t}^*\hat{A}^*u = \hat{A}^*u$. Therefore, if we write $\hat{A}^*u = \sum_{\dim a \leqslant n/2} a\times b + \sum_{\dim a \geqslant n/2} a \times b = U_- + U_+$, we must have $\hat{t}^*U_+ = U_-$.

Therefore we conclude that $\hat{a}^*u = \Delta^*\hat{A}^*u = \Delta^*\hat{t}^*U_+ + \Delta^*U_- = \Delta^*(U_- + U_-) = 0$, as desired. □

To see that the indeterminacy is zero, one must refer to [**2**]. In addition, one may easily show that $b(\xi,\theta)$ does not depend on the choice of the duality between Y and $T(\bar{\gamma}_\kappa)$.

We will say that two pairs (ξ_0, θ_0), (ξ_1, θ_1) are *equivalent* if and only if there is a fibrewise homotopy equivalence $\alpha: \xi_0 \longrightarrow \xi_1$ such that $(\alpha+1)\theta_0$, $\theta_1: \varepsilon^{n+k} \longrightarrow \xi_1 + v$ are homotopic to each other through fibrewise homotopy equivalence.

The following is a restatement of [**12**, 2.5] which refines [**7**, 5.2].

PROPOSITION 6.2 (Dupont, Sutherland). *Two pairs* (ξ_0, θ_0) *are equivalent pairs if and only if* $b(\xi_0, \theta_0) = b(\xi_1, \theta_1)$.

In particular, if $b(\xi_0, \theta_0) = b(\xi_1, \theta_1)$, it follows that ξ_0, ξ_1 are fibrewise homotopy equivalent to each other.

On the other hand, note that we have fixed a normal invariant $c: S^{n+k} \longrightarrow T(\nu)$. Lemma 6.3 below clarifies the dependence of the b-invariant on c and the proof will be postponed until the later part of this section.

For any Poincaré complex X of an odd dimension n, let the *homotopy James–Thomas number* of X mean the number of fibrewise homotopy equivalence classes of $(n-1)$-sphere fibrations stably inverse to the Spivak fibration of X.

LEMMA 6.3. *The invariant* $b(\xi, \theta)$ *depends on the choice of the normal invariant* $c: S^{n+k} \longrightarrow T(\nu)$ *if and only if the homotopy James–Thomas number of* X *is* 1.

Now we recall the $\bigstar$ operation introduced in §5. Let η be an $(n-1)$-sphere fibration over S^n with a trivialisation $\alpha: \varepsilon^{n+2} \longrightarrow \varepsilon^2 + \eta$. Then we consider $\xi \bigstar \eta$ with the trivialisation $\theta \bigstar \alpha$ which is the composite

$$\begin{aligned} \varepsilon^{n+k+2} \xrightarrow{1+\theta} \varepsilon^2 + \xi + \nu \simeq ((\varepsilon^2 + \xi) \bigstar \varepsilon^{n+2}) + \nu &\xrightarrow{(1\bigstar\alpha)+1} ((\varepsilon^2 + \xi) \bigstar (\varepsilon^2 + \eta)) + \nu \\ &\simeq \varepsilon^2 + (\xi \bigstar \eta) + \nu. \end{aligned}$$

Let $b(\eta, \theta)$ be given by choosing the collapse map $S^{n+1} \longrightarrow T(\varepsilon^1)$ coming from the standard embedding $S^n \longrightarrow S^{n+1}$ as the normal invariant.

Then the following has been essentially proved by Dupont [**7**, 5.2], in particular, when $n \neq 1, 3, 7$. A careful but straightforward modification of his arguments proves it in general.

PROPOSITION 6.4 (Dupont). $b(\xi \bigstar \eta, \theta \bigstar \alpha) = b(\xi, \theta) + b(\eta, \alpha)$.

Assume that there is a Poincaré embedding structure on $\Delta: X \longrightarrow X \times X$ and let $\tau = \nu_\Delta$ denote the normal fibration and $h: D_\tau \cup_{S_\tau} W \longrightarrow X \times X$ be the homotopy equivalence which specifies the Poincaré embedding structure. Write $\overline{\nu \times \nu} = h^*(\nu \times \nu)$. Then we consider, on the one hand, the degree-one map

$$\Sigma^{n+k} c: S^{2n+2k} \cong \Sigma^{n+k} S^{n+k} \longrightarrow \Sigma^{n+k} T(\nu) \cong T(\varepsilon^{n+k} + \nu),$$

and, on the other hand, the degree-one map $\hat{c}$ which is the composite

$$S^{2n+2k} \xrightarrow{c \wedge c} T(\nu \times \nu) \simeq T(\overline{\nu \times \nu}) \longrightarrow T(\overline{\nu \times \nu}|_{D\tau}) / T(\overline{\nu \times \nu}|_{S\tau}) \simeq T(\tau + \nu + \nu).$$

Then, according to [**14**], there is a unique fibrewise homotopy equivalence $\bar{\theta}: \varepsilon^{n+k} + \nu \longrightarrow \tau + \nu + \nu$ such that $\hat{c}$ is homotopic to $T(\bar{\theta}) \Sigma^{n+k} c$. Furthermore, $\bar{\theta}$ is $\theta_h + 1$ for some unique fibrewise homotopy equivalence $\theta_h: \varepsilon^{n+k} \longrightarrow \tau + \nu$, up to homotopy through fibrewise homotopy equivalence (cf. [**3**]). Once the normal invariant c is fixed, we will refer to θ_h as the trivialisation of $\tau + \nu$ determined by the Poincaré embedding structure on the diagonal.

REMARK 6.5. Note that the equivalence class (τ, θ_h) above depends on the choice of the normal invariant c if and only if the homotopy James–Thomas number of X

is 1, which must be clear from Proposition 6.2 and Lemma 6.3. However, it does not depend on the choice of a Poincaré embedding structure on $\Delta: X \longrightarrow X \times X$ by Corollary 6.8 below.

Once the normal invariant c is fixed, there is a natural duality between $T(\nu+\nu)$ and $T(\tau)$. It is the composite

$$\begin{aligned} S^{2N} \xrightarrow{c \wedge c} T(\nu \times \nu) \longrightarrow T(\nu \times \nu)/T((\nu \times \nu)|_W) \\ \simeq T((\nu+\nu)+\tau) \xrightarrow{T(\bar{\Delta})} T((\nu+\nu)\times\tau) \cong T(\nu+\nu) \wedge T(\tau). \end{aligned}$$

Also there is the duality between $(X \times X)_+$ and $T(\nu \times \nu)$:

$$S^{2N} \xrightarrow{c \wedge c} T(\nu \times \nu) \longrightarrow T(\nu \times \nu) \wedge (X \times X)_+.$$

The following has been proved in [**7**, 3.2] when X is a manifold. The same proof works for any Poincaré embedding structure on $\Delta: X \longrightarrow X \times X$.

Lemma 6.6. *With respect to the dualities above, the dual of* $T(\bar{\Delta}): T(\nu+\nu) \longrightarrow T(\nu \times \nu)$ *is the collapse map* $C_h: (X \times X)_+ \longrightarrow T(\tau) \cong (D\tau \cup_{S\tau} W)/W$.

Let $u \in H^n(X \times X)$ be the diagonal cohomology class in the sense $[X \times X] \cap u = \Delta_*[X]$. Recall the symmetric lifting $A: \nu \times \nu \longrightarrow \bar{\gamma}_k$. Denote by $q: Y \longrightarrow \Sigma^l(X \times X)_+$ the dual of $T(A): T(\nu \times \nu) \longrightarrow T(\bar{\gamma}_k)$ with respect to the same duality between $T(\nu \times \nu)$ and $\Sigma^l(X \times X)_+$ as in Lemma 6.6.

Write K for the kernel of $q^*: H^n(X \times X) \longrightarrow H^n(Y)$. Then there is the quadratic function $\varphi: K \longrightarrow Z_2$ defined by Browder [**2**] with respect to the (**Y**-orientation) q (**q**).

Recall the semi-characteristic $\chi_{1/2}(X) = \sum_i \operatorname{rank} H^{2i}(X; Z_2) \bmod 2$. Even if the following has been proved in [**12**, 2.7], the proof below seems more focused.

Lemma 6.7 (Sutherland). $\varphi(u) = \chi_{1/2(X)}$.

Proof. We invoke the quadratic function $\psi: H^n(X \times X) \longrightarrow Z_4$ defined by Brown [**4**] using the same lifting $A: \nu \times \nu \longrightarrow \bar{\gamma}_k$. In particular, ψ satisfies $\psi(v) = j\varphi(v)$ for any $v \in K$, where $j: Z_2 \longrightarrow Z_4$ is the monomorphism. Furthermore,

$$\psi(v+v') = \psi(v) + \psi(v') + j((v \cup v')([X \times X])).$$

(This explains the terminology 'quadratic function'.)

On the other hand, we have $u = a + t^*a$ for some $a \in H^n(X \times X)$ such that $(a \cup t^*a)([X \times X]) = \chi_{1/2}(X)$ (see the proof of Lemma 6.1). In particular, it follows that $u \in K$.

In fact, $\psi(x) \in jZ_2 \subset Z_4$ for any $x \in H^n(X \times X)$ since $x \cup x = 0$ and ψ is 'quadratic'. Also it is not difficult to see that $\psi(x) = \psi(t^*x)$ as long as we use a symmetric lifting $\nu \times \nu \longrightarrow \bar{\gamma}_k$ to define ψ. Therefore, we have

$$\begin{aligned} j\varphi(u) = \psi(u) = \psi(a+t^*a) &= \psi(a) + \psi(t^*a) + j(a \cup t^*a)([X \times X]) \\ &= j((a \cup t^*a)([X \times X])) = j\chi_{1/2}(X). \end{aligned}$$ □

Corollary 6.8. $b(\tau, \theta_h) = \chi_{1/2}(X)$.

Proof. We consider the duality between $T(\nu \times \nu)$ and $(X \times X)_+$ and the one between $T(\nu+\nu)$ and $T(\tau)$ as in Lemma 6.6. Note that, by construction of θ_h, the latter is the same as the duality between $T(v+v)$ and $T(\tau)$ determined by (τ, θ_h) and the normal invariant c by means of which we define $b(\tau, \theta_h)$ (see the beginning of the section).

Recall $C_h : (X \times X)_+ \longrightarrow T(\tau)$, the collapse map given by the Poincaré embedding. Also fix a duality between $T(\bar{\gamma}_\kappa)$ and Y. Let $g: Y \longrightarrow \Sigma^l T(\tau)$ and $q: Y \longrightarrow \Sigma^l (X \times X)_+$ denote respectively the map dual to $T(a) = T(A)\, T(\bar{\Delta}+1)$ and the map dual to $T(A)$.

Then, the following diagram commutes:

$$\begin{tikzcd} Y \arrow[r, "q"] \arrow[d, "id"'] & \Sigma^l (X\times X)_+ \arrow[r, "\Sigma^l u"] \arrow[d, "\Sigma^l C_h"'] & \Sigma^l K_n \arrow[d, "id"'] \\ Y \arrow[r, "g"] & \Sigma^l T(\tau) \arrow[r, "\Sigma^l U_r"] & \Sigma^l K_n \end{tikzcd}$$

It is straightforward to see that $C_h^* U_\tau$ is the diagonal cohomology class u and, therefore, does not depend on the choice of Poincaré embedding of the diagonal.

By the commutativity of the diagram, we have

$$b(\tau, \theta_h) = Sq^{n+1}_{(\Sigma^l U_\tau)\, g}(\Sigma^l \iota) = Sq^{n+1}_{(\Sigma^l u)\, q}(\Sigma^l \iota) = \varphi(u) = \chi_{1/2}(X). \qquad \square$$

COROLLARY 6.9. *For the tangent sphere fibration τ_n of S^n and the standard trivialisation $\alpha_n : \varepsilon^{n+k} \longrightarrow \varepsilon^k + \tau_n$, we have $b(\tau_n, \alpha_n) = 1 \in Z_2$.*

Here we provide the postponed proofs.

Completion of the proof of Proposition 1.5. When n is even, Proposition 1.5 has been proved in §5. When n is odd, Propositions 5.1, 5.2, 6.2 and 6.4 and Corollary 6.9 together prove Proposition 1.5. $\square$

Proof of Lemma 6.3. Assume that the homotopy James–Thomas number of X is 2. Let ξ represent one of the two classes of $(n-1)$-sphere fibrations stably inverse to v. Then, by Proposition 6.2, it follows that $b(\xi, \theta)$ does not depend on the choice of θ.

Let $c' : S^{n+k} \longrightarrow T(v)$ denote another normal invariant. Then there is a fibrewise homotopy equivalence $\alpha : v \longrightarrow v$ such that $T(\alpha)\, c$ is homotopic to c' [**14**]. For a pair (ξ, θ), let $\theta'' : \varepsilon^{n+k} \longrightarrow \xi + v$ denote the composite

$$\varepsilon^{n+k} \xrightarrow{\theta} \xi + v \xrightarrow{1+\alpha} \xi + v.$$

Then a straightforward calculation shows that $b(\xi, \theta'')$ with respect to the normal invariant c is the same as $b(\xi, \theta)$ with respect to the normal invariant c'. But $b(\xi, \theta'') = b(\xi, \theta)$ when we use the same normal invariant c for both sides of the equality.

Assume that the homotopy James–Thomas number is 1. Let ξ represent the unique $(n-1)$-sphere fibration stably inverse to the Spivak fibration up to fibrewise homotopy equivalence. Then, by Proposition 1.5, there are trivialisations $\theta, \theta' : \varepsilon^{n+k} \longrightarrow \xi + v$ such that $b(\xi, \theta) \neq b(\xi, \theta')$, presuming a fixed normal invariant c.

Let $\alpha : v \longrightarrow v$ be such that θ followed by $1 + a : \xi + v \longrightarrow \xi + v$ is homotopic to θ' through fibrewise homotopy equivalences (cf. [**3**]).

Consider $c' = T(\alpha)\, c : S^{n+k} \longrightarrow T(v)$. Then, it follows that $b(\xi, \theta)$ with respect to c'

is the same as $b(\xi, \theta')$ with respect to c. However, we have $b(\xi, \theta) \neq b(\xi, \theta')$, when we fix the normal invariant c. □

Proof of Theorem 1.2. Once Proposition 1.5 is established, we must show that the tangent fibration does not depend on the choice of the fundamental class or of the normal invariant.

For even dimensions, if the orientation of the Poincaré complex is reversed, the same fibration with the reversed orientation will have the same Euler characteristic.

For odd dimensions, if the homotopy James–Thomas number is 1, there is nothing left to prove and, if it is 2, then the b-invariant does not depend on the choice of the normal invariant by Lemma 6.3. □

Proof of Theorem 1.3. The assertion follows from Corollary 4.2 and Proposition 4.6 if n is even and from Corollaries 4.2 and 6.8 if n is odd. □

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