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CLAIRAUT RELATION FOR GEODESICS OF HOPF TUBES

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Abstract. In this note we use the Hopf map $\pi : \mathbf{S}^3 \to \mathbf{S}^2$ to construct an interesting family of Riemannian metrics h^f on the 3-sphere, which are parametrized on the space of positive smooth functions f on the 2-sphere. All these metrics make the Hopf map a Riemannian submersion. The Hopf tube over an immersed curve γ in $\mathbf{S}^2$ is the complete lift $\pi^{-1}(\gamma)$ of γ, and we prove that any geodesic of this Hopf tube satisfies a Clairaut relation. A Hopf tube plays the role in $\mathbf{S}^3$ of the surfaces of revolution in $\mathbf{R}^3$. Furthermore, we show a geometric integration method of the Frenet equations for curves in those non-standard $\mathbf{S}^3$. Finally, if we consider the sphere $\mathbf{S}^3$ equipped with a family h^f of Lorentzian metrics, then a new Clairaut relation is also obtained for timelike geodesics of the Lorentzian Hopf tube, and a geometric integration method for curves is still possible.

1. Introduction

A surface of revolution in $\mathbf{R}^3$ is generated by the rotation of an arclength parametrized curve $\alpha : I \to \mathbf{R}^2$, $\alpha(s) = \big(x(s), 0, z(s)\big)$ around the Z-axis. It

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can be defined also as the warped product $I \times_x \mathbf{S}^1$ endowed with the (warped) metric $g = ds^2 + x(s)^2 dt^2$, where $x : I \to \mathbf{R}$ works as the warping function.

As early as 1735, A. C. Clairaut obtained the following well known characterization for a geodesic curve of a surface of revolution in $\mathbf{R}^3$:

If θ is the angle between the tangent to the curve and a circle of latitude, and if r is the radius of this circle, then $r \cos\theta = \text{const.}$ *along the curve.*

In this note we prove that a Clairaut relation can be also stated in a very different setting. In fact, the ambient space is now the 3-dimensional sphere $\mathbf{S}^3$ furnished with a family of Riemannian or Lorentzian metrics, and the geodesics live in Hopf tubes of $\mathbf{S}^3$, which play the role in $\mathbf{S}^3$ of the surfaces of revolution in $\mathbf{R}^3$. In fact, a Hopf tube is the inverse image under the Hopf map $\pi : \mathbf{S}^3 \to \mathbf{S}^2$ of a curve γ on $\mathbf{S}^2$, and if γ is a closed curve, $\pi^{-1}(\gamma)$ is a Hopf torus. But a Hopf tube can be seen also as a warped product $I \times_f \mathbf{S}^1$ in $\mathbf{S}^3$ with f a positive smooth function on $\mathbf{S}^2$. Hopf tori appear in the setting of the Riemannian geometry [3, 16] as well as in the Lorentzian geometry [1, 4].

2. Generalized Kaluza–Klein metrics on the 3-sphere

Let $\mathbf{S}^3$ denote the unit 3-sphere in $\mathbf{C}^2$, $\mathbf{S}^3 = \left\{ z = (z_1, z_2) \in \mathbf{C}^2 \mid |z_1|^2 + |z_2|^2 = 1 \right\}$. The unit circle $\mathbf{S}^1$ acts naturally on $\mathbf{S}^3$ by means of

$$(e^{i\theta}, z) \to e^{i\theta} \cdot z = (e^{i\theta} z_1, e^{i\theta} z_2),$$

for any $e^{i\theta} \in \mathbf{S}^1$ and $z \in \mathbf{S}^3$. The Hopf map $\pi : \mathbf{S}^3 \to \mathbf{S}^2$ is a principal fibre bundle with structure group $\mathbf{S}^1$.

For any point $z \in \mathbf{S}^3$ we consider the tangent vector $V(z) = iz \in T_z(\mathbf{S}^3)$. Then V is a global vector field on $\mathbf{S}^3$. Now, if we denote by $\bar{g}$ the standard metric of radius 1 on $\mathbf{S}^3$ and by g the standard metric of radius $1/2$ on $\mathbf{S}^2$, then $\pi : (\mathbf{S}^3, \bar{g}) \to (\mathbf{S}^2, g)$ is a Riemannian submersion with geodesic fibres. The 2-dimensional distribution $\mathcal{H}$ defined by $\mathcal{H}(z) = \left\langle V(z) \right\rangle^{\perp}$ (orthogonal complement with respect to $\bar{g}$), gives a principal connection whose principal 1-form is denoted by ω.

Let $\bar{\nabla}$ and ∇ be the Levi-Civita connections of $\bar{g}$ and g, respectively. From the theory of (semi) Riemannian submersions ([8, 14]) the following equations are well known:

$$\bar{\nabla}_{\widetilde{X}} \widetilde{Y} = \widetilde{\nabla_X Y} - \bar{g}(i\widetilde{X}, \widetilde{Y})V, \quad \bar{\nabla}_{\widetilde{X}} V = \bar{\nabla}_V \widetilde{X} = i\widetilde{X}, \quad \bar{\nabla}_V V = 0,$$

where overtildes stand for horizontal lifts. Notice that the latter equation shows the geodesic nature of the fibres.

An ample and interesting class of semi-Riemannian metrics can be considered on $\mathbf{S}^3$. We start with an arbitrary Riemannian metric h on $\mathbf{S}^2$ and a positive smooth function f on $\mathbf{S}^2$. Now, we define the following metric on $\mathbf{S}^3$:

$$h^f = \pi^*(h) + \varepsilon(u \cdot \pi)^2 \omega^*(dt^2), \tag{1}$$

where $\varepsilon = \pm 1$ and dt^2 is the standard metric on $\mathbf{S}^1$. Then, h^f is Riemannian or Lorentzian according to ε being $+1$ or -1, respectively. The metric h^f is called a *generalized Kaluza–Klein metric*. It should be noticed that these metrics are like local warped product metrics. In particular, if f is chosen to be constant, then it works as a global scaling factor on the fibres. In these cases, the metrics h^f are called *Kaluza–Klein* or *bundle-like* metrics. We will first consider $\varepsilon = +1$, and the Lorentzian case will be studied similarly. For the sake of simplicity, we shall write $f \cdot \pi \equiv f$, etc.

Clearly $\pi : (\mathbf{S}^3, h^f) \to (\mathbf{S}^2, h)$ is a Riemannian submersion with horizontal distribution $\mathcal{H}$. Moreover, the $\mathbf{S}^1$-action is made up of isometries of $(\mathbf{S}^3, h^f)$. Let D^f, D denote the Levi-Civita connections of h^f and h, respectively. Then, a standard computation involving some well known facts from the theory of Riemannian submersions [8], gives

$$D^f_{\widetilde{X}} \widetilde{Y} = \widetilde{D_X Y} - \bar{g}(i\widetilde{X}, \widetilde{Y})V. \tag{2}$$

Note that $h^f(V,V) = f^2$, and hence $\eta = (1/f)V$ is a unitary vertical vector field.

It is clear that $[\widetilde{X}, V] = 0$ for any horizontal lift $\widetilde{X}$, and therefore $D^f_{\widetilde{X}}(f\eta) = D^f_{f\eta}\widetilde{X} = f D^f_\eta \widetilde{X}$. Then we have,

$$D^f_\eta \widetilde{X} = D^f_{\widetilde{X}} \eta + \frac{\widetilde{X}(f)}{f}\eta. \tag{3}$$

Now we compute $D^f_\eta \eta$. It is obvious that it is a horizontal vector field. But from (3), for any horizontal lift $\widetilde{X}$ we have

$$h^f(D^f_\eta \eta, \widetilde{X}) = -h^f(D^f_\eta \widetilde{X}, \eta) = -\frac{\widetilde{X}(f)}{f} = -\widetilde{X}(\log f),$$

and hence we obtain

$$D^f_\eta \eta = -\nabla^f(\log f). \tag{4}$$

Remark 1. (a) Since f is constant along each fibre, then $\nabla^f(\log f)$ is nothing but the horizontal lift of the gradient of $\log f$ in $(\mathbf{S}^2, h)$.

(b) Notice also that (4) implies that all the fibres are geodesics in $(\mathbf{S}^3, h^f)$ if and only if f is a constant positive function (i.e., h^f is a Kaluza–Klein metric).

(c) Generalized Kaluza–Klein metrics can be defined on a principal fibre bundle $P(M, G)$, where G is an m-dimensional compact Lie group equipped with a bi-invariant metric dt^2 [6, 9]. Kaluza–Klein spacetimes have a remarkable interest in physics, including string theory [7, 17], general relativity [13, 20], particle physics [2, 18], quantum field theory [11, 19], etc.

Let γ be an arclength parametrized curve in $(\mathbf{S}^2, h)$, and let $\{T = \gamma', \xi, \kappa\}$ be its Frenet apparatus. The Frenet equations of γ are given by $D_T T = \kappa\xi$, $D_T \xi = -\kappa T$. Let $\widetilde{\gamma}$ be a horizontal lift of γ (which is obviously arclength parametrized), and denote by $\{\widetilde{T} = \widetilde{\gamma}', \xi_2^*, \xi_3^*, \kappa^*, \tau^*\}$ its Frenet apparatus in $(\mathbf{S}^3, h^f)$. Now we shall relate both Frenet frames. To this end, we write down the Frenet equations for $\widetilde{\gamma}$:

$$D^f_{\widetilde{T}}\widetilde{T} = \kappa^*\xi_2^*, \quad D^f_{\widetilde{T}}\xi_2^* = -\kappa^*\widetilde{T} + \tau^*\xi_3^*, \quad D^f\xi_3^* = -\tau^*\xi_2^*.$$

Since $\widetilde{T}$ is horizontal, from (2) we have

$$\kappa^*\xi_2^* = D^f_{\widetilde{T}}\widetilde{T} = \widetilde{D_T T} - \bar{g}(i\widetilde{T}, \widetilde{T})V = \widetilde{D_T T} = \widetilde{\kappa\xi} = \kappa\widetilde{\xi},$$

and then $\kappa^* = \kappa$, $\xi_2^* = \widetilde{\xi}$. This means that κ^* and ξ^* are the horizontal lifts of the curvature and the principal normal of γ, respectively.

The second Frenet equations of γ, $\widetilde{\gamma}$ and (2) give $\tau^*\xi_3^* = -fg(\gamma', \gamma')\eta$, so that $\tau^* = -fg(\gamma', \gamma')$, and the binormal of $\widetilde{\gamma}$ is $\xi_3^* = \eta$. In particular, if the metric h is the standard metric g on $\mathbf{S}^2$ and we take f the constant function $f = 1$, then $\tau^* \equiv -1$.

3. The shape operator of a Hopf tube

Let $\gamma : I \to \mathbf{S}^2$ be an arclength parametrized curve in $(\mathbf{S}^2, h)$. The *Hopf tube over* γ is the surface $M_\gamma = \pi^{-1}(\gamma)$ in $(\mathbf{S}^3, h^f)$ equipped with the induced metric $h^f|_{M_\gamma}$. We can use the nice argument of Pinkall [16] to show that an immersed surface M in $\mathbf{S}^3$ is $\mathbf{S}^1$-invariant if and only if $M = M_\gamma = \pi^{-1}(\gamma)$ for some immersed curve γ in $\mathbf{S}^2$ (i.e., M is a Hopf tube). In particular, if γ is closed, M_γ is a Hopf torus, which is embedded if γ is simple in $\mathbf{S}^2$.

Let $\widetilde{\gamma}$ be a horizontal lift of γ. The universal covering of this Hopf tube is

$$\Psi : \mathbf{R}^2 \to M_\gamma, \quad \Psi(s,t) = e^{it} \cdot \widetilde{\gamma}(s).$$

The metric coefficients of M_γ are given by $g_{11} = 1$, $g_{12} = 0$, $g_{22} = f^2$. The Gaussian curvature of this tube is

$$K^f = -\frac{f_{ss}}{f} = -\frac{1}{f}\frac{d^2}{ds^2}\big(f\big(\gamma(s)\big)\big).$$

In particular, we have obtained:

PROPOSITION 1. *Let γ be a closed curve in $(\mathbf{S}^2, h)$. Then M_γ is a flat torus if and only if $f|_\gamma$ is a constant positive function.*

Since $\{\Psi_s, \eta = (1/f)\Psi_t\}$ is an h^f-orthonormal frame along M_γ in $(\mathbf{S}^3, h^f)$, then $N = \Psi_s \wedge \eta = i\Psi_s$ is the unit normal vector field of this Hopf tube, and its Weingarten endomorphism A^f can be computed with respect to the basis $\{\Psi_s, \eta\}$ as follows:

$$A^f\Psi_s = -D^f_{\Psi_s} i\Psi_s = -\widetilde{D_{\gamma'}\xi} + \bar{g}(i\Psi_s, i\Psi_s)\eta = \kappa\Psi_s + fg(\gamma', \gamma')\eta,$$

$$A^f\eta = -D^f_\eta i\Psi_s = -D^f_{i\Psi_s}\eta + (1/f)i\Psi_s(f)\eta = fg(\gamma', \gamma')\Psi_s - \xi(\log f)\eta,$$

where we have used (2), (3) and the fact that Ψ_s (resp. $i\Psi_s$) is the complete lift of γ' (resp. $\xi = J\gamma'$). Therefore, the matrix of A^f is given by

$$\begin{pmatrix} \kappa & fg(\gamma', \gamma') \\ fg(\gamma', \gamma') & -\xi(\log f) \end{pmatrix}.$$

As a consequence, the mean curvature α^f of M_γ is given by

$$\alpha^f = \frac{1}{2}\big(\kappa - \xi(\log f)\big). \tag{5}$$

REMARK 2. Notice that if h is the standard round metric g on $\mathbf{S}^2$ of radius $1/2$ and we choose f to be the constant function $f = 1$, then the torsion τ^* of a horizontal lift $\widetilde{\gamma}$ of γ is $\tau^* = -1$, the Hopf tube M_γ is flat, and the mean curvature function of this tube is $\alpha^f = (1/2)\kappa$.

4. The Clairaut relation for geodesics of Hopf tubes

A direct computation involving equations (2), (3) and (4) give the following Gauss equations for the surface M_γ:

$$D^f_{\Psi_s}\Psi_s = \kappa\, i\Psi_s, \quad D^f_\eta\Psi_s = \Psi_s(\log f)\eta + fg(\gamma',\gamma')i\Psi_s,$$

$$D^f_{\Psi_s}\eta = fg(\gamma',\gamma')i\Psi_s, \quad D^f_\eta\eta = -\Psi_s(\log f)\Psi_s - i\Psi_s(\log f)i\Psi_s.$$

Now we prove

THEOREM 1. *Let β be a geodesic of the Hopf tube M_γ in $(\mathbf{S}^3, h^f)$ and let θ denote the angle between β and the fibres. Then, the following Clairaut relation is satisfied:* $f\cos\theta = \text{const}$.

PROOF. Let $\Psi(s,t) = e^{it}\cdot\widetilde{\gamma}(s)$ be a parametrization of M_γ and assume that β is an arclength parametrized immersed curve in this tube, i.e., $\beta(z) = \Psi\big(s(z),t(z)\big)$, $\beta'(z) = s'\Psi_s + t'\Psi_t$ and $h^f(\beta',\beta') = (s')^2 + f^2(t')^2 = 1$. Since $[\Psi_s,\Psi_t] = 0$, then $D^f_{\Psi_s}\Psi_t = D^f_{\Psi_t}\Psi_s$, and we have

$$D^f_{\beta'}\beta' = s''\Psi_s + t''\Psi_t + (s')^2 D^f_{\Psi_s}\Psi_s + (t')^2 D^f_{\Psi_t}\Psi_t + 2s't' D^f_{\Psi_s}\Psi_t.$$

But

$$D^f_{\Psi_t}\Psi_s = fD^f_{\frac{1}{f}\Psi_t}\Psi_s = f\Psi_s(\log f)(1/f)\Psi_t + f^2 g(\gamma',\gamma')i\Psi_s,$$

and

$$D^f_{\Psi_t}\Psi_t = f^2 D^f_{(1/f)\Psi_t}(1/f)\Psi_t,$$

because f does not depend on t. Thus,

$$D^f_{\beta'}\beta' = \big(s'' - f^2(t')^2\Psi_s(\log f)\big)\Psi_s + \big(t'' + 2s't'\Psi_s(\log f)f\big)\eta$$
$$+ \big(k(s')^2 - f^2(t')^2 i\Psi_s(\log f) + 2f^2 s't' g(\gamma',\gamma')\big)i\Psi_s, \tag{6}$$

where the first two terms on the right hand side are nothing but $D^{M_\gamma}_{\beta'}\beta'$, where D^{M_γ} is the induced covariant derivative on M_γ. Thus β is a geodesic of M_γ if and only if the following equations are satisfied:

$$s'' - f^2(t')^2\Psi_s(\log f) = 0, \quad t'' + 2s't'\Psi_s(\log f) = 0. \tag{7}$$

Note that M_γ is flat if and only if $\Psi_s(\log f) = 0$, and obviously then its geodesics are the images under Ψ of straight lines. Otherwise $\Psi_s(\log f) \neq 0$ and since $f(s) = f\big(s(z)\big)$ along γ we have $f' = df/dz = \Psi_s(f)s'$. From

$\Psi_s(\log f) = f'/(fs')$, the second equation in (7) gives $t'' + 2t'(f'/f) = 0$, or equivalently,

$$f^2 t' = \text{const.} \tag{8}$$

Let θ denote the angle defined by β and the fibres. Since

$$\cos\theta = \big(1/\|\Psi_t\|\big)\, h^f(\beta', \Psi_t) = f t', \tag{9}$$

from (8) and (9) we have obtained the following Clairaut relation for the geodesic β of the Hopf tube M_γ:

$$f\ \cos\theta = \text{const.} \qquad \square \tag{10}$$

From (7) we see that any horizontal lift $\widetilde{\gamma}$ of a curve γ in $\mathbf{S}^2$ is a geodesic of the Hopf tube M_γ. All these horizontal lifts are as the "meridians" of the Hopf tube, and the fibres are as the "parallels". On the other hand, a fibre $\pi^{-1}(p)$, $p = \gamma(s)$ is a geodesic curve in M_γ, if and only if $s' = s'' = 0$, and it satisfies the equations

$$(t')^2 \Psi_s(\log f) = 0, \quad t'' = 0, \quad f^2 (t')^2 = 1,$$

and therefore $\Psi_s(\log f)\big|_{\pi^{-1}(p)} = 0$, that is, $\frac{df(\gamma(s))}{ds}(p) = 0$. We have,

Corollary 1. *The fibre $\pi^{-1}(p)$ is a geodesic of the Hopf tube M_γ if and only if p is a critical point of $f|_\gamma$.*

The Clairaut relation allows us to draw the following consequence. Let β be a geodesic of the Hopf tube M_γ which meets the fibres under an angle $\theta \neq 0, \pi/2$ (β is neither a fibre nor a lift of γ). Now, assume that $p_1 = \gamma(s_1)$ is a local minimum for the function f along γ. Then, from Corollary 1 the fibre $\pi^{-1}(p_1)$ is a geodesic of the tube. Let $p_0 = \gamma(s_0)$ denote a nearby point of $p_1 = \gamma(s_1)$ and pick any point $m_0 \in \pi^{-1}(p_0)$. Now define β to be the geodesic through m_0 that meets the fibre $\pi^{-1}(p_0)$ under an angle $\theta(p_0) > 0$ such that $\cos\theta(p_0) = \frac{f(p_1)}{f(p_0)}$. Then, the Clairaut relation for this geodesic β gives $f(p_0)$ $\cdot \cos\theta(p_0) = c$ and hence $c = f(p_1) > 0$. Therefore, β meets the fibre $\pi^{-1}(p_1)$ provided $f(p_1) \cdot \cos\theta(p_1) = f(p_1)$, which is impossible. This means that β rolls up to the fibre $\pi^{-1}(p_1)$ asymptotically.

Example. Let $\big(\mathbf{S}^2(1/2), g\big)$ be the standard round sphere of radius $1/2$ and take in $\mathbf{R}^3$ the vector $v = (0,0,1)$. Define $f : (\mathbf{S}^2, g) \to \mathbf{R}$ as the positive smooth function $f(p) = \langle p, v\rangle + 1$, where $\langle , \rangle$ is the Euclidean metric on $\mathbf{R}^3$. The level curves of f are circles in $\mathbf{S}^2$ which yield in planes orthogonal to v.

The Hopf torus over any of these circles is flat and all its fibres are geodesic. Now let γ be the circle in $\mathbf{S}^2$ obtained as the intersection of $\mathbf{S}^2$ with a plane which is not orthogonal to v. If p_1, p_2 are the points of γ where $f|_\gamma$ reaches its critical values, then $\pi^{-1}(\gamma)$ has exactly two fibres $\pi^{-1}(p_1)$ and $\pi^{-1}(p_2)$ which are geodesics.

5. Curvature and torsion of geodesics of M_γ in $(\mathbf{S}^3, h^f)$

Let $\beta(z) = \Psi\big(s(z), t(z)\big)$ be an arclength parametrized geodesic of the Hopf tube M_γ. From (6) we have

$$D^f_{\beta'}\beta' = \big(k(s')^2 - f^2(t')^2 i\Psi_s(\log f) + 2f^2 s't'g(\gamma', \gamma')\big)\, i\Psi_s.$$

Therefore the curvature ρ of β in $(\mathbf{S}^3, h^f)$ is

$$\rho = k(s')^2 - f^2(t')^2 i\Psi_s(\log f) + 2f^2 s't'g(\gamma', \gamma').$$

But if we denote by θ the angle between the geodesic β and the fibres, then

$$\beta'(z) = (\sin\theta)\Psi_s + (\cos\theta)\eta.$$

The curvature ρ of β can be written as

$$\rho = (\sin\theta \quad \cos\theta)\begin{pmatrix} \kappa & fg(\gamma', \gamma') \\ fg(\gamma', \gamma') & -\xi(\log f) \end{pmatrix}\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}. \tag{11}$$

Notice that since β is a geodesic of M_γ, then its unit principal normal in $(\mathbf{S}^3, h^f)$ is nothing but $i\Psi_s$, and its unit binormal in $(\mathbf{S}^3, h^f)$ is $(\cos\theta)\Psi_s - (\sin\theta)\eta$. Consequently, the torsion ν of β in $(\mathbf{S}^3, h^f)$ is given by

$$\nu = -h^f\big(D^f_{\beta'}\big((\cos\theta)\Psi_s - (\sin\theta)\eta\big), i\Psi_s\big) = -(\cos\theta\sin\theta)\kappa$$

$$- (\cos^2\theta)ug(\gamma', \gamma') + (\sin^2\theta)ug(\gamma', \gamma') - (\sin\theta\cos\theta)\xi(\log u),$$

or

$$\nu = (\sin\theta \quad \cos\theta)\begin{pmatrix} \kappa & fg(\gamma', \gamma') \\ fg(\gamma', \gamma') & -\xi(\log f) \end{pmatrix}\begin{pmatrix} -\cos\theta \\ \sin\theta \end{pmatrix} \tag{12}$$

Now a direct computation shows that

$$(-\cos\theta \quad \sin\theta)\begin{pmatrix} \kappa & fg(\gamma', \gamma') \\ fg(\gamma', \gamma') & -\xi(\log f) \end{pmatrix}\begin{pmatrix} -\cos\theta \\ \sin\theta \end{pmatrix} = 2\alpha^f - \rho, \tag{13}$$

where α^f is the mean curvature of the Hopf tube M_γ in $(\mathbf{S}^3, h^f)$. Thus, formulae (11), (12) and (13) yield the following relation:

$$\begin{pmatrix} \kappa & fg(\gamma',\gamma') \\ fg(\gamma',\gamma') & -\xi(\log f) \end{pmatrix} = \begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} \begin{pmatrix} \rho & \nu \\ \nu & 2\alpha^f - \rho \end{pmatrix} \begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix},$$

that is $\begin{pmatrix} \kappa & fg(\gamma',\gamma') \\ fg(\gamma',\gamma') & -\xi(\log f) \end{pmatrix}$ and $\begin{pmatrix} \rho & \nu \\ \nu & 2\alpha^f - \rho \end{pmatrix}$ are congruent matrices. Therefore the following equations are obtained:

$$\begin{cases} \kappa = -\rho\cos 2\theta - \nu\sin 2\theta + 2\alpha^f\cos^2\theta, \\ fg(\gamma',\gamma') = (\rho - \alpha^f)\sin 2\theta - \nu\cos 2\theta, \\ \xi(\log f) = -\rho\cos 2\theta - \nu\sin 2\theta - 2\alpha^f\sin^2\theta. \end{cases} \tag{14}$$

Now we state the following geometric integration method.

THEOREM 2. *Let $\rho, \nu, \alpha^f, \theta : I \to \mathbf{R}$ be smooth functions satisfying $(\rho - \alpha^f)\sin 2\theta - \nu\cos 2\theta > 0$. Then there exist $f \in \mathcal{C}^\infty_+(\mathbf{S}^2)$, an immersed curve γ in $(\mathbf{S}^2, h)$ and a geodesic β of the Hopf tube $M_\gamma = \pi^{-1}(\gamma)$ such that*

1. *ρ, ν are the curvature and torsion, respectively, of β in $(\mathbf{S}^3, h^f)$;*
2. *α^f is the mean curvature of M_γ;*
3. *θ is the h^f-slope of β in M_γ.*

PROOF. Define $\gamma(s)$ to be an arclength parametrized curve in $(\mathbf{S}^2, h)$ with curvature $\kappa(s) = -\rho\cos 2\theta - \nu\sin 2\theta + \alpha^f\cos^2\theta$. Take a positive smooth function $f : \mathbf{S}^2 \to \mathbf{R}$ satisfying

$$f\big(\gamma(s)\big) = \frac{1}{g(\gamma',\gamma')}\big((\rho - \alpha^f)\sin 2\theta - \nu\cos 2\theta\big),$$

$$\xi(\log f) = -\rho\cos 2\theta - \nu\sin 2\theta - 2\alpha^f\sin^2\theta,$$

where g is the standard metric on $\mathbf{S}^2$, and ξ is the normal of γ in $(\mathbf{S}^2, h)$. Then, the geodesic curve β of $M_\gamma = \pi^{-1}(\gamma)$ with h^f-slope θ is a curve in $(\mathbf{S}^3, h^f)$ with curvature ρ and torsion ν. $\square$

6. Lorentzian metrics on the sphere

Now we start with the sphere $\mathbf{S}^3$ endowed with the Lorentzian metric h^f defined by formula (1) when $\varepsilon = -1$:

$$h^f = \pi^*(h) - f^2\omega^*(dt)^2, \tag{15}$$

where π, h, f, ω and dt^2 have the same meaning as in Section 2. It can be proved that (2) and (3) still hold, but (4) changes to

$$D^f_\eta \eta = \nabla^f(\log f). \tag{16}$$

Let $\gamma(s)$ be an arclength parametrized curve immersed in $(\mathbf{S}^2, h)$ with unit tangent vector T, normal ξ and curvature κ. Then, it is easy to prove that any horizontal lift $\widetilde{\gamma}$ of γ has tangent vector $\widetilde{T}$, normal $\widetilde{\xi}$, binormal η, curvature κ and torsion $\tau^* = fg(\gamma', \gamma')$. The metric coefficients of M_γ are $g_{11} = 1$, $g_{12} = 0$, $g_{22} = -f^2$, and its Gauss curvature is still $K^f = -(1/f)(d^2/ds^2)f\bigl(\gamma(s)\bigr)$.

The Hopf tube M_γ can be parametrized as in Section 4, and it is easy to see that the matrix A^f of the Weingarten endomorphism of M_γ in $(\mathbf{S}^3, h^f)$ with respect to the orthonormal basis $\bigl\{\Psi_s, \eta = \frac{1}{f}\Psi_t\bigr\}$ now is

$$\begin{pmatrix} \kappa & -fg(\gamma', \gamma') \\ -fg(\gamma', \gamma') & \xi(\log f) \end{pmatrix},$$

and the mean curvature function α^f of M_γ is given by $2\alpha^f = \kappa - \xi(\log f)$. The Gauss equations now become

$$D^f_{\Psi_s}\Psi_s = \kappa\, i\Psi_s, \quad D^f_\eta \Psi_s = \Psi_s(\log f)\eta - fg(\gamma', \gamma')i\Psi_s,$$

$$D^f_{\Psi_s}\eta = -fg(\gamma', \gamma')i\Psi_s, \quad D^f_\eta \eta = \Psi_s(\log f)\Psi_s + i\Psi_s(\log f)i\Psi_s.$$

Suppose we are given an arclength parametrized geodesic curve $\beta(z) = \Psi\bigl(s(z), t(z)\bigr)$ of the Lorentzian Hopf tube M_γ. Then $\beta'(z) = s'\Psi_s + ft'\eta$ and $h^f(\beta' y, \beta') = (s')^2 - f^2(t')^2 = \varepsilon_1$, where $\varepsilon_1 = \pm 1$ according to the causal character of β.

Some computations similar to those of Section 5 allow us to obtain the differential equations of a geodesic curve in M_γ:

$$\begin{cases} s'' + f^2(t')^2\Psi_s(\log f) = 0, \\ t'' + 2s't'\Psi_s(\log f) = 0, \end{cases} \tag{17}$$

and since the second equation is the same as in the Riemannian case, it can be integrated to obtain $f^2t' = \text{const.}$

Assume that β is a timelike geodesic ($\varepsilon_1 = -1$) such that at the point $q \in M_\gamma$ its unit tangent β'_q satisfies

$$h^f(\beta'_q, \eta_q) < 0 \tag{18}$$

i.e., β'_q belongs to the timecone $\mathcal{C}(\eta_q)$ of the timelike vector η_q which is the unit tangent to the fibre $\pi^{-1}\big(\pi(q)\big)$ at q. As M_γ is time-orientable, this inequality holds along any point of β and we know [14] that there is a unique number $\varphi \geqq 0$, called the *hyperbolic angle* between β' and η, such that

$$\cosh\varphi = -\frac{h^f(\beta', \eta)}{|\beta'|\,|\eta|}.$$

Now the unit tangent to β can be written as

$$\beta' = \sinh\varphi\Psi_s + \cosh\varphi\eta$$

and hence $\cosh\varphi = ft'$. Thus, from $f^2t' = \text{const.}$ we have proved the following Lorentzian version of the Clairaut theorem:

THEOREM 3. *Let β be a timelike geodesic of the Lorentzian Hopf tube $M_\gamma = \pi^{-1}(\gamma)$ in $(\mathbf{S}^3, h^f)$ such that at some point q,* (18) *holds. Then β satisfies the Clairaut relation $f\cosh\varphi = \text{const.}$*

As in the Riemannian case, Corollary 1 also holds.

Now, the curvature ρ and torsion ν in $(\mathbf{S}^3, h^f)$ of a timelike geodesic β of the Lorentzian Hopf tube M_γ can be computed as in the Riemannian case to give the following relations:

$$\begin{cases} \kappa = (\rho + 2\alpha^f)\cosh^2\varphi - \nu\sinh 2\varphi + \rho\sinh^2\varphi, \\ fg(\gamma', \gamma') = (\rho + \alpha^f)\sinh 2\varphi + \nu(1 + 2\sinh^2\varphi), \\ \xi(\log f) = \rho(1 + 2\sinh^2\varphi) - \nu\sinh 2\varphi + 2\alpha^f\sinh^2\varphi. \end{cases} \tag{19}$$

Thus the following geometric integration method can be stated.

THEOREM 4. *Let ρ, ν, α^f, $\varphi, I \to \mathbf{R}$ be smooth functions satisfying $(\rho + \alpha^f)\sinh 2\varphi + \nu(1 + 2\sinh^2\varphi) > 0$. Then there exists $f \in \mathcal{C}^\infty_+(\mathbf{S}^2)$, an immersed curve γ in $(\mathbf{S}^2, h)$, and a timelike geodesic β of the Hopf tube $M_\gamma = \pi^{-1}(\gamma)$ such that*

1. *ρ, ν are the curvature and torsion, respectively, of β in the Lorentzian sphere $(\mathbf{S}^3, h^f)$;*
2. *α^f is the mean curvature function of M_γ;*
3. *φ is the hyperbolic angle defined between β and the fibres of M_γ.*

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