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HOMEOMORPHISMS BETWEEN TOPOLOGICAL MANIFOLDS AND ANALYTIC MANIFOLDS[1]

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1. Existence of the homeomorphisms. By a *topological m-manifold, M*, ($m = 0, 1, \cdots$) we mean a connected topological space which can be covered with a denumerable set of neighborhoods, each of which is an m-cell. We will employ, on M, various coordinate systems, each having an m-cell for domain and each defined by a homeomorphism between its domain and a region in a euclidean m-space, E^m.

Consider a set, X, of coordinate systems whose domains cover M. We will say that M is *analytic* (*in terms of the systems* X) if every transformation

$$v_i = v_i(x) \qquad (i = 1, \cdots, m) \tag{1.1}$$

between two of the systems, $(x) = (x_1, \cdots, x_m)$ and (v), whose domains overlap, is analytic with a non-vanishing jacobian.

Let $(y) = (y_1, \cdots, y_n)$ denote a coordinate system in E^n. A *topological m-manifold in* E^n will mean a set of points

$$M\colon y_i = y_i(p) \qquad (i = 1, \cdots, n) \tag{1.2}$$

where (1) p is a variable point on a topological m-manifold, M', and (2) the correspondence (1.2) between M and M' is a homeomorphism. Let X be a set of coordinate systems whose domains cover M'. As p ranges over the domain of any system (x), the functions $y_i(p)$ can be interpreted as functions of (x). If all such functions are analytic and if every functional matrix $(\partial y_i/\partial x_j)$ is of rank m on its domain, then M will be called an *analytic manifold in* E^n (*in terms of the systems* X *and the correspondence* (1.2)).

Consider a point set, S, in E^n. A k-plane, π^k ($k \geqq 1$), through a point p of S will be called *transversal to* S *at* p if it makes angles bounded away from zero with the secant lines of some neighborhood of p on S. Any plane, π^k, is called *transversal to* S (*in the large*) if it makes angles bounded away from zero with all the secant lines of S.

We will say that a topological m-manifold, M, in E^n is in *normal position* if it is possible to define, through each point p of M, an $(n - m)$-plane, $\pi^{n-m}(p)$, in such a way that (1) $\pi^{n-m}(p)$ varies continuously with p and (2) $\pi^{n-m}(p)$ is transversal to M at p.

Suppose the topological m-manifold, M, can be subdivided into the cells of a

[1] Presented to the American Mathematical Society; March 27, 1937, February 25, 1939, and October 28, 1939.

simplicial complex. It can then be mapped by a homeomorphism into a polyhedral complex, P^m, in an E^n $(n > 2m)$, where the faces of P^m are euclidean simplexes corresponding to the cells into which M is subdivided.

THEOREM I. *Given a topological m-manifold, M, there exists a set of coordinate systems in terms of which M is analytic with an analytic Riemannian metric, if and only if M can be triangulated so as to have a polyhedral representation, P^m, in normal position in some E^n.*

This theorem, in so far as the sufficiency of the condition is concerned, is a consequence of the following.

THEOREM[2] II. *Arbitrarily near any normal position of P^m, there exists an analytic manifold in E^n, homeomorphic to P^m.*

Part of this paper is devoted to an investigation of conditions under which a polyhedral manifold P^m can be put into normal position. By showing this to be always possible when $m = 3$, we obtain the following result. The cases $m < 3$ can easily be dealt with by known methods.

THEOREM III. *If a topological 3-manifold, M, can be triangulated, then there exists a set of coordinate systems in terms of which M is analytic and has an analytic Riemannian metric.*

2. Normal positions and general positions. We now establish the necessity of the conditions in Theorem I. Any analytic m-manifold, M, has a homeomorph, M', which is analytic in some euclidean space E^n (DM, Theorem I). The writer has shown[3] that M' can be so triangulated into cells (σ) that (1) the vertices of each m-cell determine a non-degenerate m-simplex and (2) the totality of the simplexes so determined is a polyhedral manifold, P^m, homeomorphic to M' in such a way that corresponding m-cells have identical vertices and that the tangent m-plane to M' at any point of a cell, σ^m, of (σ) differs arbitrarily little in direction from the m-plane of the corresponding face of P^m. Now suppose (p, q) are corresponding points on (P^m, M') respectively. If $\pi^{n-m}(p)$ is the $(n - m)$-plane through p parallel to the $(n - m)$-plane normal to M' at q, and if P^m is a sufficiently close approximation to M', then $\pi^{n-m}(p)$ is transversal to P^m as required by the definition of normal position. For, since the faces of P^m are approximately tangent to M', the directions of the secant lines of any neighborhood on P^m are approximately the same as in the case of the corresponding neighborhood on M'.

The sufficiency proof for Theorem I will not be complete until §8.

Consider an arbitrary triangulated topological m-manifold, M. We will assume first[4] that there exists an upper bound to the number of cells in a star

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[2] Our proof of Theorem II will involve methods due to Hassler Whitney. See *Differentiable manifolds*, Annals of Mathematics, vol. 37 (1936), pp. 645–680. This paper will be referred to as DM.

[3] *Polyhedral approximations to regular loci*, Annals of Mathematics, vol. 37 (1936), pp. 409–415.

[4] In §9 a method is given which does not involve this hypothesis.

on M. The assumption enables us to imbed P^m in an E^n, for n sufficiently large, so that the vertices on each star of simplexes are linearly independent. P^m is then said to be in *general position.*

LEMMA. *If it is possible to put P^m into normal position, then there exists a general position which is also a normal position.*

PROOF. We commence with an auxiliary result.

(A) If a manifold, M, is in normal position in E^ν, then it is in normal position in any E^n which contains E^ν as a subspace.

For, suppose that $\pi^{\nu-m}(p)$ in E^ν is transversal to M at p and that $\pi^{n-\nu}(p)$ in E^n is transversal to E^ν at p. Then it follows from our definitions that $\pi^{n-m}(p)$, determined by $\pi^{\nu-m}(p)$ and $\pi^{n-\nu}(p)$ is transversal to M at p. It remains only to require that $\pi^{n-\nu}(p)$ be continuous in p on M. We might, for example, use the $(n - \nu)$-plane normal to E^ν at p.

Now let P^m be in normal position in $E^\nu \subset E^n$, n being so large that E^n can contain P^m in general position. Then P^m can be brought into a general position, $*P^m$, by arbitrarily small displacements of its vertices. Let barycentric coordinates be introduced on the simplexes of P^m and, in precisely the same way, on the simplexes of $*P^m$. Two points on $(P^m, *P^m)$, respectively, will correspond if their coordinates are the same. Suppose $\pi^{n-m}(p)$ is transversal to a certain neighborhood, $N(p)$, on P^m. The directions of the secant lines of the corresponding neighborhood, $N(p^*)$, on $*P^m$ can be made arbitrarily close to those of $N(p)$ by suitable restrictions on the displacements carrying P^m into $*P^m$. Hence it can be arranged that $\pi^{n-m}(p^*) \,||\, \pi^{n-m}(p)$ shall be transversal to $*P^m$ at p^*, as required by the definition of normal position.

3. Planes transversal to Brouwer stars.

The triangulated manifold M, or its representation P^m, will be called a *Brouwer manifold*[5] if the star of each vertex on P^m can be mapped into an E^m by a piecewise linear homeomorphism; that is, a homeomorphism which is linear on each simplex of the star.

THEOREM IV. *No P^m can be put into normal position unless it is a Brouwer manifold.*

This follows immediately from the first sentence in the lemma below.

For $m > 3$, it is unknown[6] whether every triangulated m-manifold is a Brouwer manifold. We show, in §8, that this is surely true for $m = 3$. It is obvious for $m < 3$.

Let (π^m, π^{n-m}) be planes, of the indicated dimensions, transversal to each other in the euclidean space E^n. Consider the plane parallel to π^{n-m} through any point p in E^n. This plane meets π^m in a point, p', which will be called the *π^{n-m}-projection of p on π^m.* The locus of p' as p ranges over a point set S will

[5] Brouwer, *Über Abbildungen von Mannigfaltigkeiten,* Mathematische Annalen 71 (1912), pp. 97–115.

[6] Since this was written, examples of non-Brouwer triangulated manifolds have been constructed. See *Triangulated manifolds which are not Brouwer manifolds,* immediately following the present article.

be referred to as the *π^{n-m}-projection of S on π^m*. This same expression will be used for the mapping of S onto π^m in which p and p' correspond. The following result is then obvious.

(A) *The π^{n-m}-projection of S onto π^m is a homeomorphism if π^{n-m} is transversal to S. As a partial converse, if S is bounded and the π^{n-m}-projection of the closure, $\bar{S}$, of S is a homeomorphism, then π^{n-m} is transversal to $\bar{S}$ and hence to S.*

Let $S^k = S^k(s^j)$ be a set of simplexes of dimensions $(j, \cdots, k)$ in E^ν, where (1) S^k is the star of a j-simplex, s^j, and (2) S^k, regarded as a point set, is a k-cell. We will refer to S^k as a *Brouwer k-star* if it has a piecewise linear homeomorph in an E^k.

LEMMA. *If $S^k(s^j)$ can be put into normal position, it is a Brouwer star. Furthermore, every general position of a Brouwer star is also a normal position.*

As a first step in the proof, we make the following easily verified statement.

(B) *If N is an arbitrary neighborhood on $S^k(s^j)$ of a point on s^j, then every secant of $S^k(s^j)$ is parallel to a secant of N.*

It follows at once that, when S^k is in normal position, any $(n - k)$-plane π^{n-k} transversal to S^k at a point of s^j is also transversal to S^k in the large. The π^{n-k}-projection of S^k onto any π^k transversal to π^{n-k} affords a piecewise linear homeomorphism as required by the definition of Brouwer star. This establishes the first statement in the lemma.

Consider, now, a Brouwer star, S^k, in general position in E^n. By definition, there exists a piecewise linear homeomorphism, $\Lambda = \Lambda(S^k)$, mapping S^k onto an E^k. Let $(P_0, \cdots, P_\nu)$ denote the vertices of S^k, the notation being so assigned that $(P_0, \cdots, P_k)$ are the vertices of some k-cell of S^k. Let $(y) = (y_1, \cdots, y_n)$ be a rectilinear coordinate system in E^n, relative to which P_0 is the origin and P_i $(i = 1, \cdots, \nu)$ is unit point on the y_i-axis.

We restrict Λ so that it will map S^k onto the coordinate $(y_1, \cdots, y_k)$-plane with P_i self-corresponding $(i = 0, \cdots, k)$. Let $Q_j : (a_{j1}, \cdots, a_{jk}, 0, \cdots, 0)$ denote the image of P_j $(j = k + 1, \cdots, \nu)$ under Λ, and consider the transformation of coordinates

$$\begin{aligned} x_i &= y_i + \sum_{j=k+1}^{\nu} a_{ji} y_j \qquad && (i = 1, \cdots, k) \\ x_j &= y_j && (j = k + 1, \cdots, n). \end{aligned} \tag{3.1}$$

In terms of the coordinate system (x), P_0 is still the origin, P_i $(i = 1, \cdots, k)$ is the unit point on the x_i-axis, and the following are the coordinates of the remaining P's and of the Q's:

$$\begin{aligned} &P_j\colon(a_{j1}, \cdots, a_{jk}, 0, \cdots, 0, x_j = 1, 0, \cdots, 0) \\ &Q_j\colon(a_{j1}, \cdots, a_{jk}, 0, \cdots, 0) \end{aligned} \qquad (j = k + 1, \cdots, \nu). \tag{3.2}$$

Hence, if π^{n-k} denote the $(x_{k+1}, \cdots, x_n)$-plane, then Λ is the π^{n-k}-projection onto the plane of $(P_0, \cdots, P_k)$. Therefore, by result (A), π^{n-k} is transversal to S^k, and the proof is complete.

4. Spaces of transversal planes. Let S^k be a Brouwer k-star in general position in E^ν. We will denote with $\Pi(S^k, E^\nu)$ the topological space each of whose points is a system of parallel $(\nu - k)$-planes transversal to S^k in E^ν, continuity being defined in terms of direction cosines.[7] The following statements are easy to verify.

(*A*) *If*

$$S^k \subset E^\nu \subset E^n \tag{4.1}$$

then $\Pi(S^k, E^n)$ *is the set of* $(n - k)$*-planes in* E^n *which intersect* E^ν *in planes of the set* $\Pi(S^k, E^\nu)$. [Compare the proof of §2(A)].

(*B*) *If* S_0^k *is a subset of* S^k, *then*

$$\Pi(S^k, E^n) \subset \Pi(S_0^k, E^n). \tag{4.2}$$

We will refer to two topological spaces, Σ_1 and Σ_2, as *β-equivalent*[8] provided the following statement holds: For each j $(j = 0, 1, \cdots)$ every (singular or non-singular) j-sphere in Σ_1 bounds a $(j + 1)$-cell if and only if every j-sphere in Σ_2 also bounds a $(j + 1)$-cell.

LEMMA. *Any space* $\Pi(S^m(s^k), E^n)$ *is β-equivalent to a certain space* $\Pi(S^{m-k}(s^0), E^{n-k})$.

PROOF. We take s^0 as the barycenter of s^k, and E^{n-k} as the $(n - k)$-plane normal to s^k at s^0. The star $S^{m-k} = S^{m-k}(s^0)$ is defined as the projection of $S^m = S^m(s^k)$ onto E^{n-k}. We commence with the following auxiliary result.

(*C*) *Suppose* π^{n-m} *belongs to* $\Pi(S^m, E^n)$, *and let* π^k *denote the k-plane of* s^k. *Then the plane* π^{n-m+k} *determined by* π^{n-m} *and* π^k *belongs to* $\Pi(S^{m-k}, E^n)$.

In the first place, we note that the secants of S^{m-k} are a subset of those of S^m. With the aid of §3(*B*), one can verify that if l is a line in E^n whose π^k-projection is a secant of S^{m-k}, then l is parallel to a secant of S^m. Since every line on π^{n-m+k} is parallel to π^k or else has the same π^k-projection as a line on π^{n-m}, it follows that the only secants of S^m parallel to π^{n-m+k} are also parallel to π^k. Hence no secant of S^{m-k} is parallel to π^{n-m+k}. Since the secants of S^{m-k} are a closed set, result (*C*) follows at once. It also follows that $\Pi(S^{m-k}, E^{n-k})$ consists of the intersections of E^{n-k} with planes such as π^{n-m+k}.

An $(n - m)$-plane π^{n-m} on π^{n-m+k} belongs to $\Pi(S^m, E^n)$ if it contains no line parallel to π^k. Any subset of $\Pi(S^m, E^n)$ consisting of planes whose angles with π^k all exceed $\vartheta > 0$ can therefore be homotopically deformed in $\Pi(S^m, E^n)$ so that each π^{n-m} remains in a single plane such as π^{n-m+k} and is carried into its π^k-projection on E^{n-m}. Now suppose every sphere in $\Pi(S^m, E^n)$ bounds a cell. Since $\Pi(S^{m-k}, E^{n-k})$ is a subset of $\Pi(S^m, E^n)$, any sphere, β^j, in the former space bounds a cell, σ^{j+1}, in the latter. Some deformation of the sort just

[7] S. S. Cairns, *The direction cosines of a p-space in euclidean n-space*, American Mathematical Monthly, vol. 39 (1932), pp. 518–523.

[8] The stronger condition of complete homology equivalence (cf. Alexandroff-Hopf, *Topologie I*, 1935) might be established for the spaces we treat. However, we need only β-equivalence.

described will leave β^j fixed and will carry σ^{j+1} into a $(j+1)$-cell in $\Pi(S^{m-k}, E^{n-k})$. Hence every sphere in $\Pi(S^{m-k}, E^{n-k})$ bounds a cell in $\Pi(S^{m-k}, E^{n-k})$.

Suppose conversely that, in $\Pi(S^{m-k}, E^{n-k})$, every sphere bounds a cell: in other words, can be shrunk to a point. An arbitrary sphere in $\Pi(S^m, E^n)$ can be deformed, as above, into a sphere in $\Pi(S^{m-k}, E^{n-k})$ and then further shrunk to a point. This completes the establishment of the lemma.

5. β-equivalent spaces of maps. Corresponding to a Brouwer k-star, S^k, we define a space of mappings $\Lambda(S^k)$ as follows. Each point of $\Lambda(S^k)$ can be represented by a piecewise linear homeomorphism of S^k into an E^k. Two such homeomorphisms represent the same point of $\Lambda(S^k)$ if and only if one can be carried into the other by a linear transformation of E^k. Let $(P_1, \cdots, P_\nu)$ be the vertices of S^k and $(P'_1, \cdots, P'_\nu)$ their respective images under a piecewise linear homeomorphism representing a point, λ_0, of $\Lambda(S^k)$. A neighborhood, $N(\lambda_0)$, in $\Lambda(S^k)$ will correspond as follows to any set of neighborhoods, $N_i(P'_i)$ $(i = 1, \cdots, \nu)$, in E^k: A point of $\Lambda(S^k)$ belongs to $N(\lambda_0)$ if and only if it can be represented by a piecewise linear homeomorphism carrying P_i into $N_i(P'_i)$ $(i = 1, \cdots, \nu)$. Thus $\Lambda(S^k)$ is defined as a topological space.

LEMMA 5.1. *There exists a homeomorphism between* $\Lambda(S^k)$ *and* $\Pi(S^k, E^\nu)$ *provided* (1) S^k *is in general position in* E^ν *and* (2) *no n-plane with* $n < \nu$ *contains* S^k.

PROOF. Let E^k be determined by the vertices $(P_0, \cdots, P_k)$ of a particular k-simplex of S^k. Since two piecewise linear homeomorphisms of S^k into E^k represent the same point of $\Lambda(S^k)$ if they are related by a linear transformation of E^k, we can obtain unique representations for the points of $\Lambda(S^k)$ by stipulating that $(P_0, \cdots, P_k)$ be self-corresponding. Comparing the proof of §3, Lemma, we see that a homeomorphism between $\Pi(S^k, E^\nu)$ and $\Lambda(S^k)$ is defined if each element $\pi^{\nu-k}$ of the former space be associated with the $\pi^{\nu-k}$-projection of S^k onto E^k.

LEMMA 5.2. *If* $E^n \supset E^\nu$, *then* $\Pi(S^k, E^n)$ *is β-equivalent to* $\Pi(S^k, E^\nu)$ *and hence to* $\Lambda(S^k)$.

This can be proved, on the basis of Lemma 5.1, by reasoning as in the proof of §4, Lemma.

Given $S^k = S^k(s^j)$, let p_0 be the barycenter of s^j. We will denote with $\Lambda^0(S^k)$ the subspace of $\Lambda(S^k)$ consisting of those elements which map all the boundary vertices of S^k onto the unit $(k-1)$-sphere in E^k about the image of p_0.

LEMMA 5.3. *The spaces* $\Lambda(S^k)$ *and* $\Lambda^0(S^k)$ *are β-equivalent.*

PROOF. We represent all the points of $\Lambda(S^k)$ by elements mapping p_0 into the origin, O, in E^k and mapping the vertices of some particular k-cell into prescribed images on the unit $(k-1)$-sphere S^{k-1} about O. Let λ_0 denote any element of $\Lambda(S^k)$, thus restricted, and let q_0 be the image of any vertex, p, of S^k under λ_0. Let q_1 denote the intersection of S^{k-1} with the ray Oq_0. We then denote with λ_t that element of $\Lambda(S^k)$ which carries each vertex p into the point q_t on the segment q_0q_1 such that $q_0q_t = t \cdot q_0q_1$. As t increases from 0 to 1, λ_t

defines a deformation of λ_0 into λ_1. By applying this deformation simultaneously to all the elements λ, we deform the whole space $\Lambda(S^k)$ into $\Lambda^0(S^k)$. Using this deformation, one can complete the proof as in the case of §4, Lemma.

6. The space of triangulations $T(\tau^{k-1})$. Given a Brouwer star $S^k = S^k(s^0)$, let the points of $\Lambda^0(S^k)$ be represented by homeomorphisms carrying the vertices $(s^0, P_1, \cdots, P_k)$ of some k-cell into specified images $(O, Q_1, \cdots, Q_k)$ in E^k. Let Σ^k denote the image of S^k under some such homeomorphism, λ. By definition of $\Lambda^0(S^k)$, the boundary, B^{k-1}, of Σ^k has all its vertices on the unit sphere, S^{k-1}, about O. The central projection from O onto S^{k-1} maps B^{k-1} into a geodesic triangulation, τ^{k-1}, of S^{k-1}; that is, one in which each cell appears as a simplex relative to some local coordinate system in which arcs of great circles are represented as straight lines. This implies that the closure of each cell of τ^{k-1} is on an open hemisphere of S^{k-1}, in other words that each 1-cell is less than 180°. Let $T(\tau^{k-1})$ be the following topological space. Its points are geodesic triangulations of S^{k-1} homeomorphic to τ^{k-1} with Q_i $(i = 1, \cdots, k)$ self-corresponding. If τ_0^{k-1} is any such triangulation and Q_i $(i = 1, \cdots, \nu)$ are its vertices, then a neighborhood, $N(\tau_0^{k-1})$, in $T(\tau^{k-1})$ corresponds as follows to a given set of neighborhoods $N_j(Q_j)$ $(j = k + 1, \cdots, \nu)$ of the points Q_j on S^{k-1}: $N(\tau_0^{k-1})$ consists of all elements of $T(\tau^{k-1})$ for which the vertex corresponding to Q_j lies in $N_j(Q_j)$.

The following lemma is a direct consequence of our definitions.

LEMMA. *The spaces $T(\tau^{k-1})$ and $\Lambda^0(S^k)$ are homeomorphic under the correspondence induced by the central projection from O.*

7. A sufficient condition in the normal position problem. THEOREM V. *A sufficient condition that it be possible to put a Brouwer m-manifold into normal position is that, in every space $\Pi(S^k(s^0), E^\nu)$ [or $\Lambda(S^k)$ or $T(\tau^{k-1})$], every $(m - k - 1)$-sphere bound an $(m - k)$-cell*[9] $(k = 0, \cdots, m - 1)$.

The proof will occupy this section and the next.

(A) Let η be a positive constant less than $1/(m + 1)$. If s^j be any j-simplex, $j \leqq m$, then the *η-core*, γ^j, of s^j will mean the set of points where all the barycentric coordinates for s^j exceed η. In the case $j = 0$, we have $\gamma^0 = s^0$.

We now consider an m-simplex, s^m, and define on it certain neighborhoods, $N(\gamma^j)$, where γ^j is the η-core of a typical bounding simplex, s^j, of s^m. The definition will be recurrent in $j = 0, \cdots, m$. To define $N(\gamma^0)$, we choose barycentric coordinates $(u_0, \cdots, u_m)$ on s^m so that $u_i = 0$, $(i = 1, \cdots, m)$ at γ^0. We then make the definition

$$N(\gamma^0)\colon O \leqq u_i \leqq \eta \qquad (i = 1, \cdots, m). \tag{7.1}$$

To define $N(\gamma^j)$, $O < j < m$, choose the barycentric coordinates so that $u_i = 0$

[9] For $k = 0$, the space $T(\tau^{k-1})$ is vacuous, and the condition becomes trivial.

($i = j + 1, \cdots, m$) on γ^j. Assume the $N(\gamma^k)$ all defined ($k = 0, \cdots, j - 1$). We define a region $N'(\gamma^j)$ as follows:

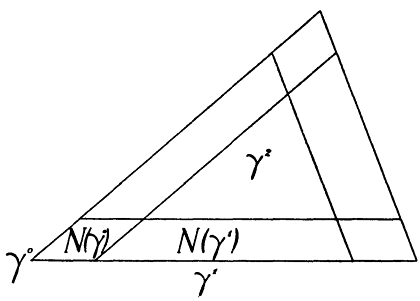
$$N'(\gamma^j)\colon\ 0 \leqq u_i \leqq \eta \qquad (i = j + 1, \cdots, m) \tag{7.2}$$

and then make the definition

$$N(\gamma^j) = \overline{N'(\gamma^j) - \sum_{k<j} N(\gamma^k)}, \tag{7.3}$$

the summation being over the η-cores of all bounding k-simplexes of s^m ($k < j$). Then $N(\gamma^j)$ is a closed, box-like, m-dimensional region with γ^j for one face. The simplex s^m is covered by γ^m plus the regions $N(\gamma^j)$ ($j < m$). These regions are distinct, save for common faces. The accompanying figure illustrates the definitions when $m = 2$.

Any core γ^j, $0 \leqq j < m$, is parallel to a certain bounding simplex, $^*\gamma^j$, of γ^m. In case $j = 0$, $^*\gamma^j$ will denote the vertex of γ^m nearest γ^0. Let σ^{m-j-1} denote the



bounding simplex of γ^m opposite $^*\gamma^j$. Consider any point, q, on γ^j. We will denote with $B^{m-j}(q)$ the intersection of $N(\gamma^j)$ with the $(m - j)$-plane determined by q and σ^{m-j-1}. Then $B^{m-j}(q)$ is a box-like $(m - j)$-dimensional point set. As q ranges over the closure of γ^j, the sets $B^{m-j}(q)$ fill out the region $N(\gamma^j)$ in continuous one-to-one fashion.

(B) If q is on the boundary of γ^j, then $B^{m-j}(q)$ is common to the boundaries of $N(\gamma^j)$ and some $N(\gamma^k)$ where $k < j$.

Now let s^m be any m-simplex of P^m and s^j any bounding simplex of s^m, with γ^j denoting its η-core. We will then use the notation

$$\begin{aligned} \mathfrak{N}(\gamma^j) &= \sum_{s^m \in S(s^j)} N(\gamma^j) \\ \mathfrak{B}(q) &= \sum_{s^m \in S(s^j)} B(q) \qquad q \text{ on } \gamma^j \end{aligned} \tag{7.4}$$

where $S(s^j)$ is the star of s^j on P^m.

Our proof will consist in assuming the condition of the theorem and constructing suitable transversal planes $\pi^{n-m}(p)$. We take P^m in general position in E^n. The method will be recurrent, with the following basic hypothesis.

HYPOTHESIS I. *For some value j of the set* $(1, \cdots, m)$, *a plane* $\pi^{n-m}(p)$ *has been defined so as to vary continuously with p on the sum of the neighborhoods* $\mathfrak{N}(\gamma^k)$ $(k < j)$ *and so as to be transversal to* P^m *at p.*

The initial step of the recurrency falls into two parts. We first select arbitrary transversal planes, $\pi^{n-m}(p)$, at the vertices $(s_1^0, s_2^0, \cdots) = (\gamma_1^0, \gamma_2^0, \cdots)$ [see (A) above] of P^m. This is possible, by §3, Lemma, since P^m is a Brouwer manifold. We then define $\pi^{n-m}(p)$ on the $\mathfrak{N}(\gamma^0)$ by the requirement

$$\pi^{n-m}(p) \parallel \pi^{n-m}(\gamma_i^0), \qquad p \text{ on } \mathfrak{N}(\gamma_i^0), \tag{7.5}$$

together with the requirement that $\pi^{n-m}(p)$ pass through p. The verification of hypothesis I for $j = 1$ depends on §4 (B), to be applied where $k = m$, $S^k = S(s_i^0)$, and S_0^k is the star of any simplex of P^m incident with s_i^0.

8. The general step of the recurrency. Hypothesis I would be a sufficient basis for the proof of Theorem V. The following hypotheses are used to secure regularity restrictions which will enable us to prove Theorem II. The initial step in §7 satisfies all these hypotheses.

HYPOTHESIS II. *If* p_1 *and* p_2 *are common to any* $\mathfrak{B}^k(q)$, *then*

$$\pi^{n-m}(p_1) \parallel \pi^{n-m}(p_2). \tag{8.1}$$

HYPOTHESIS III. *There exists a function* $\xi_{j-1}(p)$ *such that, if* $R(p)$ *denote the set of points on* $\pi^{n-m}(p)$ *within distance* $\xi_{j-1}(p)$ *of p, then the sets* $R(p)$ *fill out, in one-to-one fashion, a closed neighborhood,* R_{j-1}, *in* E^n *of the inner points of the* $\mathfrak{N}(\gamma^k)$ $(k < j)$.

HYPOTHESIS IV. *If* $\pi^{n-m}(p')$ *denote the plane of the set* $\pi^{n-m}(p)$ *through any point* p' *of* R_{j-1}, *then* $\pi^{n-m}(p')$, *regarded as a mapping of* R_{j-1} *into the space of all* $(n - m)$*-planes in* E^n, *is differentiable.*[10]

The general step of the recurrency extends the definition of $\pi^{n-m}(p)$ over the neighborhoods $\mathfrak{N}(\gamma^j)$. We break this step into two parts. In the first part, we extend the definition over a typical η-core, γ^j. In the second part, omitted when[11] $j = m$, we extend it over the rest of $\mathfrak{N}(\gamma^j)$.

The definition of $\pi^{n-m}(p)$ maps $s^j - \gamma^j$ differentiably[10] into $\Pi(S^m(s^j), E^n)$, which is β-equivalent to some space $\Pi(S^{m-j}(s^0), E^{n-j})$ [see §4, Lemma]. Since the boundary of γ^j is a $(j - 1)$-sphere, the condition of Theorem V, read with $k = m - j$, implies that the mapping of $s^j - \gamma^j$ can be extended over γ^j. As a result of Theorem 7 in *DM*, the extension can be made so as to give a differentiable[10] mapping of the whole of s^j into $\Pi(S^m(s^j), E^n)$. By such a mapping, we extend the definition of $\pi^{n-m}(p)$ over γ^j.

If, now, q is any point of γ^j, and p is any point of $\mathfrak{B}^{m-j}(q)$, $\pi^{n-m}(p)$ will mean the $(n - m)$-plane through p parallel to $\pi^{n-m}(q)$. This completes the definition of $\pi^{n-m}(p)$ on the neighborhoods $\mathfrak{N}(\gamma^j)$.

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[10] This term has meaning here, because $\Pi(S^m(s^j), E^n)$ is a subspace of the space of all m-planes in E^n, and this latter space can be interpreted as an analytic manifold in some euclidean space (cf DM, §24).

[11] The proof of the theorem is completed with the first part of the step $j = m$.

The preservation of Hypotheses II and IV is an immediate consequence of the construction. In Hypothesis I, the transversality requirement is easy to verify, for the value j, with the aid of §4 (B), and the continuity requirement follows with the aid of §7 (B) and Hypothesis II. In establishing the preservation of Hypothesis III, it is convenient to consider the two parts of our general step: (1) the extension of $\pi^{n-m}(p)$ over γ^j and (2) the extension over the rest of $\mathfrak{N}(\gamma^j)$. The preservation during part (1) can, since Hypothesis IV is preserved, be proved by the methods of DM, Lemma 21. During part (2), Hypothesis III is preserved by virtue of the parallelism requirement in Hypothesis II.

9. Completion of the proofs of Theorems I and II. (*A*) *If a P^m be in normal position, then transversal planes $\pi^{n-m}(p)$ can be constructed by the recurrency in* §§7 *and* 8.

For, if $\pi'(p)$ denote any set of transversal planes relative to which P^m is in normal position, then one can construct, using the methods of the recurrency with η sufficiently small, an arbitrarily close approximation, $\pi^{n-m}(p)$, to $\pi'(p)$.

We are now ready to establish Theorem II, thus incidentally completing the proof of Theorem I. We assume P^m in normal position, with transversal planes $\pi^{n-m}(p)$ defined as in §§7 and 8. Hypotheses III and IV impose conditions which permit the application of Parts IV and V in DM, read with the following substitutions.

(1) The differentiable manifold, M, in E^n is to be replaced by the polyhedral manifold, P^m, in normal position in E^n.

(2) $\pi^{n-m}(p)$ plays the role of the plane $P(p)$, approximately normal to M at a point p.

(3) For a given point p on P^m, let $s^j(p)$ denote the simplex containing p. Assuming a fixed numbering (s_1^m, s_2^m, $\cdots$) for the m-simplexes of P^m, let s_i^m denote the m-simplex with the smallest subscript belonging to the star of $s^j(p)$. The m-plane, $\tau(p)$, of s_i^m replaces the tangent m-plane, T, to M at a point p.

The relevant parts of Whitney's work can be outlined as follows, in their application to the construction of an analytic manifold M^* homeomorphic to P^m. First, an analytic $(n - 1)$-manifold S, surrounding P^m, is defined. This is done with the aid of a function, $\Phi'(p)$, continuous in $R(P^m) = R_m$ (see Hypothesis III above), zero on P^m, and positive and analytic in $R(P^m) - P^m$. From Φ', there is subtracted a small positive analytic function, $\omega(p)$, such that the equation

$$\Phi'(p) - \omega(p) = 0 \tag{9.1}$$

determines a suitably restricted[12] analytic manifold, S. This manifold is such that if π^{n-m} passes through a point p in $R(P^m)$ and has direction cosines sufficiently close to those of $\pi^{n-m}(p)$ [cf. Hypothesis IV above], then (1) π^{n-m} is transversal to $\tau(p')$, p' being the point where $\pi^{n-m}(p)$ intersects P^m, and (2) π^{n-m} meets S in an analytic $(n - m - 1)$-sphere $S^*(p, \pi^{n-m})$ contained in

[12] The restrictions are obtained by conditions on Φ', ω, and their gradients.

$R(P^m)$. Let $Q^*(p, \pi)$ be the part of π^{n-m} inside $S^*(p, \pi^{n-m})$. The following results are then proved. (1) The vector function $g(p, \pi)$ representing the center of mass of $Q^*(p, \pi)$ is analytic. (2) If $\pi^*(p)$ is a sufficiently close anlytic approximation to $\pi^{n-m}(p)$ through first order derivatives, then the locus of the centers of mass $g(p, \pi^*(p))$ is an analytic manifold, M^*, homeomorphic to P^m. This manifold can be made arbitrarily close to P^m.

If it is desired merely to make P^m, and hence M, differentiable to any given order $r \epsilon (1, 2, \cdots, \infty)$, this can be done as follows. Construct $\pi^{n-m}(p)$ by the recurrency of §§7 and 8 so that it is of class C^r in $R(P^m)$. On each $\tau(p)$, let there be introduced a fixed rectangular cartesian coordinate system, with its domain restricted to the part of $\tau(p)$ inside $R(P^m)$. If p' denote a point on such a domain, then the plane $\pi^{n-m}(p')$ [see Hypothesis IV] meets P^m in a single point near p'. This affords a mapping which carries the coordinate system from $\tau(p)$ onto P^m. In terms of coordinate systems thus defined on P^m, the latter is of class C^r since the transformation between any two such systems agrees with the correspondence established by $\pi^{n-m}(p)$ in $R(P^m)$ between rectangular cartesian systems and two m-planes $\tau(p_0)$ and $\tau(p_1)$. This method enables us to dispense with the hypothesis that there be an upper bound to the number of cells in a star on P^m [see footnote 4]. For we can apply the above argument over a sequence of finite subcomplexes each containing the preceding and, in the limit, covering M. This makes the entire manifold M of class C^r. It then remains only to apply DM, Theorem 1.

The existence of differentiable approximations to polyhedral manifolds was prematurely asserted by the writer[13], who is indebted to Hassler Whitney for calling his attention to the incompleteness of his work. The results of the present paper include the theorem of the abstract only for the case $m = 3$. This is the strongest such theorem which the writer has thus far proved.

10. The Brouwer nature of M ($m \leqq 3$). Lemma. *Let S be any star of an $(m - k)$-cell on a P^m. Then, if $k \leqq 3$, it is possible to map S by a piecewise linear homeomorphism into an E^m.*

Proof. In view of §4, Lemma, and the work in §6, it is sufficient to show that any triangulation (σ) of a $(k - 1)$-sphere, S^{k-1}, can be mapped homeomorphically into a geodesic triangulation of a $(k - 1)$-sphere. The proof is trivial for $k < 3$, so we restrict ourselves to the case $k = 3$. We employ a recurrency with the following basic hypothesis.

Hypothesis. For some value $j > 0$, a subcomplex $(\sigma)_j$ of (σ), consisting of j 2-cells with their boundaries, has been mapped topologically into a geodesic complex (τ_j) on S^2 so that the part of S^2 not covered by $(\tau)_j$ is the sum of a finite number of convex regions, the closure of each of which is a subset of an open hemisphere and none of which has three of its boundary vertices on a

[13] Bulletin of the American Mathematical Society, vol. XL (1934), Abstract 67.

great circle. If β is the boundary of any one of these regions, ρ, then the image, β', of β bounds a subcomplex $(\sigma)^*$ of (σ) containing no 2-cell of $(\sigma)_j$.

For the initial step of the proof, consider any vertex, P, of (σ). It is then a simple matter to map the closure of the star $S(P)$, relative to (σ), into a complex $(\tau)_j$ so that the conditions of the hypothesis are fulfilled for j equal to the number of 2-cells on $S(P)$.

To define the general step, using the notation of the hypothesis, let P be a vertex on β' and let $S(P)$ denote its star relative to the subcomplex $(\sigma)^*$ of (σ). There is no difficulty in mapping $S(P)$ onto $(\beta + \rho)$ so as to secure the conditions of the hypothesis for a larger value of j. After a certain finite number of such steps, the mapping will be complete.

COROLLARY. *Every P^m $(m \leqq 3)$, and hence every triangulated m-manifold $(m \leqq 3)$, is a Brouwer manifold.*

This is the special case of the lemma in which $m \leqq 3$ and $k = m$.

11. Establishment of Theorem III. The case $m = 4$. LEMMA 11.1. *It is possible, for any P^m in general position, to define $\pi^{n-m}(p)$ on the regions $\mathfrak{N}(\gamma^j)$ $(j \geqq m - 3)$ so that $\pi^{n-m}(p)$ will be continuous in p and will be transversal to P^m at p.*

PROOF. We will assume $m \geqq 3$, so that there will exist stars $S(s^{m-3})$. The lower-dimensional cases require only part of the following argument. By §10, Lemma, every $S(s^{m-3})$ is a Brouwer star. Hence, given a point p_0 on any γ^{m-3}, we can define a plane $\pi^{n-m}(p_0)$ transversal to P^m at p_0. We can then extend the definition as p ranges over the γ^{m-3} containing p_0 by the requirement $\pi^{n-m}(p) \parallel \pi^{n-m}(p_0)$. Since the γ^{m-3} are bounded away from one another, this can be done independently for each of them. It is then possible to proceed with the recurrency of §§7 and 8, starting with the second part of the step $j = m - 3$. In order to apply the condition of Theorem V, as in §8, to the remaining steps, we have to note that (1) in any $T(\tau^1)$, every $(m - 3)$-sphere bounds[14] a cell, (2) in $T(\tau^0)$, every $(m - 2)$-sphere bounds a cell, and (3) every $(m - 1)$-sphere bounds a cell in $T(\tau^{-1})$ [see footnote 9].

The case $m \leqq 3$ gives the following result.

COROLLARY. *Every P^m $(m \leqq 3)$ can be put in normal position in some E^n.*

From this corollary and Theorem II, we deduce Theorem III.

LEMMA 11.2. *If every $T(\tau^2)$ is connected, then every Brouwer 4-manifold can be put in normal position.*

PROOF. The connectedness of the $T(\tau^2)$ being assumed, we have only to note that the $T(\tau^1)$ are simply connected[14] and that the higher-dimensional connectivities of $T(\tau^0)$ and of $T(\tau^{-1})$ satisfy the conditions of Theorem V.

Thus the normal position problem for Brouwer 4-manifolds reduces to the following.

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[14] For, $T(\tau^1)$ is a μ-cell, where $\mu + 2$ is the number of vertices of τ^1.

DEFORMATION PROBLEM. Given two geodesic triangulations, (σ) and (τ), of a 2-sphere, which correspond under an orientation-preserving homeomorphism, does there exist a continuously varying geodesic triangulation $(\sigma)_t$ $(0 \leqq t \leqq 1)$, such that $(\sigma)_0 = (\sigma)$, $(\sigma)_1 = (\tau)$, and $(\sigma)_t$ is always homeomorphic to $(\sigma)_0$?

Some of Tietze's work[15] has a bearing on this deformation problem, but the writer has not succeeded in obtaining a solution save in a few special cases.

(*A*) *It follows from Theorem II and Lemma* 11.2 *that every* 4-*dimensional Brouwer manifold can be made into an analytic Riemannian manifold if the deformation problem has an affirmative solution.*

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[15] Renditiconti del circolo matematico di Palermo, vol. 38 (1914), pp. 247–304, especially *Satz IV* on p. 280.