ON STABLE COMMUTATOR LENGTH IN HYPERELLIPTIC MAPPING CLASS GROUPS

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Abstract. We give a new upper bound on the stable commutator length of Dehn twists in hyperelliptic mapping class groups, and determine the stable commutator length of some elements. We also calculate values and the defects of homogeneous quasimorphisms derived from ω-signatures, and show that they are linearly independent in the mapping class groups of pointed 2-spheres when the number of points is small.

1. Introduction

The aim of this paper is to investigate stable commutator length in hyperelliptic mapping class groups and in mapping class groups of pointed 2-spheres. Given a group G and an element $x \in [G, G]$, the commutator length of x, denoted by $\mathrm{cl}_G(x)$, is the smallest number of commutators in G whose product is x, and the stable commutator length of x is defined by the limit $\mathrm{scl}_G(x) := \lim_{n\to\infty} \mathrm{cl}_G(x^n)/n$ (see Definition 2.1 for details).

We investigate stable commutator length in two groups $\mathcal{M}_0^m$ and $\mathcal{H}_g$. Let m be a positive integer greater than 3. Choose m distinct points $\{q_i\}_{i=1}^m$ in a 2-sphere S^2. Let $\mathrm{Diff}_+(S^2, \{q_i\}_{i=1}^m)$ denote the set of all orientation-preserving diffeomorphisms in S^2 which preserve $\{q_i\}_{i=1}^m$ setwise with the C^∞-topology. We define the mapping class group of the m-pointed 2-sphere by $\mathcal{M}_0^m = \pi_0 \mathrm{Diff}_+(S^2, \{q_i\}_{i=1}^m)$. Let Σ_g be a closed connected oriented surface of genus $g \geq 1$. An involution $\iota : \Sigma_g \to \Sigma_g$ defined as in Figure 1 is called the hyperelliptic involution. Let $\mathcal{M}_g$ denote the

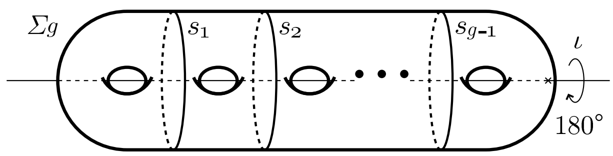

Figure 1. hyperelliptic involution ι and the curves $s_1, \dots s_{g-1}$

mapping class group of Σ_g, that is the group of isotopy classes of orientation-preserving diffeomorphisms of Σ_g, and let $\mathcal{H}_g$ be the centralizer of the isotopy class of a hyperelliptic involution in $\mathcal{M}_g$, which is called the hyperelliptic mapping class group of genus g. Note that $\mathcal{M}_g = \mathcal{H}_g$ when $g = 1,\ 2$. Since there exists a surjective homomorphism $\mathcal{P} : \mathcal{H}_g \to \mathcal{M}_0^{2g+2}$ with finite kernel (see Lemma 3.3 and the paragraph before Remark 3.7), these two groups have the same stable commutator length.

Let s_0 be a nonseparating curve on Σ_g satisfying $\iota(s_0) = s_0$, and let s_h be a separating curve in Figure 1 for $h = 1, \ldots, g-1$. We denote by t_{s_j} the Dehn twist about s_j for $j = 0, 1, \ldots, g-1$. In general, it is difficult to compute stable commutator length, but those of some mapping classes are known. In the mapping class group of a compact oriented surface with connected boundary, Baykur, Korkmaz and the second author [3] determined the commutator length of the Dehn twist about a boundary curve. In the mapping class group of a closed oriented surface, interesting lower bounds on scl of Dehn twists are obtained using Gauge theory. Endo-Kotschick [13], and Korkmaz [17] proved that $1/(18g-6) \leq \mathrm{scl}_{\mathcal{M}_g}(t_{s_j})$ for $j = 0, 1, \ldots, g-1$. For technical reasons, this result is stated in [13] only for separating curves. This technical assumption is removed in [17]. The second author [22] also showed that $1/4(2g+1) \leq \mathrm{scl}_{\mathcal{H}_g}(t_{s_0})$ and $h(g-h)/g(2g+1) \leq \mathrm{scl}_{\mathcal{H}_g}(t_{s_h})$ for $h = 1, \ldots, g-1$.

Stable commutator length on a group is closely related to functions on the group called homogeneous quasimorphisms through Bavard's Duality Theorem. Homogeneous quasimorphisms are homomorphisms up to bounded error called the defect (see Definition 2.2 for details). By Bavard's theorem, if we obtain a homogeneous quasimorphism on the group and calculate its defect, we also obtain a lower bound on stable commutator length. Actually, Bestvina and Fujiwara [4, Theorem 12] proved that the spaces of homogeneous quasimorphisms on $\mathcal{M}_g$ and $\mathcal{M}_0^m$ are infinite dimensional when $g \geq 2$ and $m \geq 5$, respectively. However, it is hard to compute explicit values of these quasimorphisms and their defects. To compute stable commutator length, we consider computable quasimorphisms derived from ω-signature in Gambaudo-Ghys' paper [16] on symmetric mapping class groups.

In Section 3, we review symmetric mapping class groups, which are defined by Birman-Hilden as generalizations of hyperelliptic mapping class groups. We reconsider cobounding functions of ω-signatures as a series of quasimorphisms $\phi_{m,j}$ on a symmetric mapping class group $\pi_0 C_g(t)$. Since there exists a surjective homomorphism $\mathcal{P} : \pi_0 C_g(t) \to \mathcal{M}_0^m$ with finite kernel, the homogenizations $\bar{\phi}_{m,j}$ induce homogeneous quasimorphisms on $\mathcal{M}_0^m$. Let $\sigma_i \in \mathcal{M}_0^m$ be a half twist which permutes the i-th point and the $(i+1)$-th point. We denote by $\tilde{\sigma}_i \in \pi_0 C_g(t)$ a lift of σ_i, which will be defined in Section 3.1.

In Section 6, we calculate $\phi_{m,j}$ and their homogenizations $\bar{\phi}_{m,j}$.

Theorem 1.1. Let r be an integer such that $2 \leq r \leq m$. Then, we have

(i)
$$\phi_{m,j}(\tilde{\sigma}_1 \cdots \tilde{\sigma}_{r-1}) = \frac{2(r-1)j(m-j)}{m(m-1)},$$

(ii)
$$\bar{\phi}_{m,j}(\sigma_1 \cdots \sigma_{r-1}) = -\frac{2}{r}\left\{\frac{jr(m-j)(m-r)}{m^2(m-1)} + \left(\frac{rj}{m} - \left[\frac{rj}{m}\right] - \frac{1}{2}\right)^2 - \frac{1}{4}\right\},$$

where $[x]$ denotes the greatest integer $\leq x$.

Since it requires straightforward and lengthy calculations to prove it, we leave it until the last section. A computer calculation shows that the $([m/2]-1)\times([m/2]-1)$ matrix whose (i,j)-entry is $\bar{\phi}_{m,j+1}(\tilde{\sigma}_1 \cdots \tilde{\sigma}_i)$ is nonsingular when $4 \leq m \leq 30$. Thus, we have:

Corollary 1.2. The set $\{\bar{\phi}_{m,j}\}_{j=2}^{[m/2]}$ is linearly independent when $4 \leq m \leq 30$.

In Section 4, we calculate the defects of the homogenizations of these quasimorphisms. In particular, we determine the defect of $\bar{\phi}_{m,m/2}$ when m is even. Actually, $\bar{\phi}_{m,m/2}$ is the same as the homogenization of Meyer function on the hyperelliptic mapping class group $\mathcal{H}_g$.

Theorem 1.3. Let $D(\phi_{m,j})$ and $D(\bar{\phi}_{m,j})$ be the defects of the quasimorphisms $\phi_{m,j}$ and $\bar{\phi}_{m,j}$, respectively.

(i) For $j = 1, 2, \ldots, [m/2]$,
$$D(\bar{\phi}_{m,j}) \leq D(\phi_{m,j}) \leq m - 2.$$
(ii) When m is even and $j = m/2$,
$$D(\bar{\phi}_{m,m/2}) = m - 2.$$

Remark 1.4. If $\phi : G \to \mathbb{R}$ is a quasimorphism and $\bar{\phi} : G \to \mathbb{R}$ is its homogenization, they satisfy
$$D(\bar{\phi}) \leq 2D(\phi)$$
(see [8] Corollary 2.59). We will claim in Lemma 4.1, when ϕ is antisymmetric and a class function, they satisfy the sharper inequality
$$D(\bar{\phi}) \leq D(\phi).$$

Note that when $g = 2$, the hyperelliptic mapping class group $\mathcal{H}_2$ coincides with $\mathcal{M}_2$. We may think of the lift of $\sigma_i \in \mathcal{M}_0^6$ for $i = 1, 2, 3, 4, 5$ to $\mathcal{M}_2$ as the Dehn twist t_{c_i} along the simple closed curve c_i in Figure 2 (see Section 3.1). Similarly, the Dehn twist $t_{s_1} \in \mathcal{M}_2$ can be considered as a lift of $(\sigma_1\sigma_2)^6 \in \mathcal{M}_0^6$ by the chain relation (see Lemma 2.8). Since Theorem 1.1 (ii) implies $\bar{\phi}_{6,2}((\sigma_1\sigma_2)^6) = -8/5$ and Theorem 1.3 (i) implies $D(\bar{\phi}_{6,2}) \leq 4$, by applying Bavard's duality theorem, we have:

Corollary 1.5.
$$\frac{1}{5} \leq \mathrm{scl}_{\mathcal{M}_2}(t_{s_1}).$$

To the best of our knowledge, for $g \geq 2$, there is not known an element x in $\mathcal{H}_g$ (or $\mathcal{M}_g$) such that $\mathrm{scl}(x)$ is non-zero and can be computed explicitly. By Theorem 1.3 (ii), we can determine the stable commutator length of the following element in $\mathcal{H}_g$.

Theorem 1.6. Let $d_2^+, d_2^-, \ldots, d_{g-1}^+, d_{g-1}^-$ be simple closed curves in Figure 8. Let c be a nonseparating simple closed curve satisfying $\iota(c) = c$ which is disjoint from d_i^+, d_i^- and s_h $(i = 1, \ldots, g,\ h = 1, \ldots, g-1)$. For $g \geq 2$,
$$\mathrm{scl}_{\mathcal{H}_g}(t_c^{2g+8}(t_{d_2^+}t_{d_2^-} \cdots t_{d_{g-1}^+}t_{d_{g-1}^-})^2(t_{s_1} \cdots t_{s_{g-1}})^{-1}) = \frac{1}{2}.$$
In particular, if $g = 2$, then we have $\mathrm{scl}_{\mathcal{H}_2}(t_c^{12}t_{s_1}^{-1}) = 1/2$.

Next, we consider upper bounds on stable commutator length. Korkmaz also gave the upper bound $\mathrm{scl}_{\mathcal{M}_g}(t_{s_0}) \leq 3/20$ for $g \geq 2$ (see [17]). In the case of $g = 2$, the second author showed $\mathrm{scl}_{\mathcal{M}_2}(t_{s_0}) < \mathrm{scl}_{\mathcal{M}_2}(t_{s_1})$ (see [22]). However, these upper bounds do not depend on g. On the other hand, Kotschick proved that there is an estimate $\mathrm{scl}_{\mathcal{M}_g}(t_{s_0}) = O(1/g)$ by using the so-called "Munchhausen trick" (see [18]).

In Section 5 we give the following upper bounds.

Theorem 1.7. Let s_0 be a nonseparating curve on Σ_g, and let G_g be either $\mathcal{M}_g$ or $\mathcal{H}_g$. For all $g \geq 1$, we have

$$\mathrm{scl}_{G_g}(t_{s_0}) \leq \frac{1}{2\{2g+3+(1/g)\}}.$$

2. Preliminaries

2.1. **Stable commutator lengths and quasimorphisms.** Let G denote a group, and let $[G,G]$ denote the commutator subgroup, which is the subgroup of G generated by all commutators $[x,y] = xyx^{-1}y^{-1}$ for $x, y \in G$.

Definition 2.1. For $x \in [G,G]$, the commutator length $\mathrm{cl}_G(x)$ of x is the least number of commutators in G whose product is equal to x. The stable commutator length of x, denoted $\mathrm{scl}(x)$, is the limit

$$\mathrm{scl}_G(x) = \lim_{n\to\infty} \frac{\mathrm{cl}_G(x^n)}{n}.$$

For each fixed x, the function $n \to \mathrm{cl}_G(x^n)$ is non-negative and $\mathrm{cl}_G(x^{m+n}) \leq \mathrm{cl}_G(x^m) + \mathrm{cl}_G(x^n)$. Hence, this limit exists. If x is not in $[G,G]$ but has a power x^m which is in, define $\mathrm{scl}_G(x) = \mathrm{scl}_G(x^m)/m$. We also define $\mathrm{scl}_G(x) = \infty$ if no power of x is contained in $[G,G]$ (we refer the reader to [8] for the details of the theory of the stable commutator length).

Definition 2.2. A quasimorphism is a function $\phi : G \to \mathbb{R}$ for which there is a least constant $D(\phi) \geq 0$ such that

$$|\phi(xy) - \phi(x) - \phi(y)| \leq D(\phi)$$

for all $x, y \in G$. We call $D(\phi)$ the defect of ϕ. A quasimorphism is homogeneous if it satisfies the additional property $\phi(x^n) = n\phi(x)$ for all $x \in G$ and $n \in \mathbb{Z}$.

We recall the following basic facts. Let ϕ be a quasimorphism on G. For each $x \in G$, define

$$\bar{\phi}(a) := \lim_{n\to\infty} \frac{\phi(x^n)}{n}.$$

The limit exists, and defines a homogeneous quasimorphism. Homogeneous quasimorphisms have the following properties. For example, see [8, Section 5.5.2] and [18, Lemma 2.1 (1)].

Lemma 2.3. Let ϕ be a homogeneous quasimorphism on G. For all $x, y \in G$,

(a) $\phi(x) = \phi(yxy^{-1})$,

(b) $xy = yx \Rightarrow \phi(xy) = \phi(x) + \phi(y)$.

Theorem 2.4 (Bavard's Duality Theorem [2])**.** Let Q be the set of homogeneous quasimorphisms on G with positive defects. For any $x \in [G,G]$, we have

$$\mathrm{scl}_G(x) = \sup_{\phi\in Q} \frac{|\phi(x)|}{2D(\phi)}.$$

2.2. **Mapping class groups.**

For $g \geq 1$, the abelianizations of the mapping class group $\mathcal{M}_g$ of the surface Σ_g and its subgroup $\mathcal{H}_g$ are finite (see [24]). Therefore, all elements of $\mathcal{M}_g$ and $\mathcal{H}_g$ have powers that are products of commutators. Dehn showed that the mapping class group $\mathcal{M}_g$ is generated by Dehn twists along non-separating simple closed curves. We review some relations between them. Hereafter, we do not distinguish a simple closed curve in Σ_g and its isotopy class. The following relations are well known. See, for example, Section 3.3, 3.5.1, 5.1.4, and 4.4.1in [15].

Lemma 2.5. Let c be a simple closed curve in Σ_g, and $f \in \mathcal{M}_g$. Then, we have

$$t_{f(c)} = f t_c f^{-1}.$$

From this lemma, the values of scl and homogeneous quasimorphisms on the Dehn twists about nonseparating simple closed curves are constant.

Lemma 2.6. Let c and d be simple closed curves in Σ_g.

(a) If c is disjoint from d, then $t_c t_d = t_d t_c$.
(b) If c intersects d in one point transversely, then $t_c t_d t_c = t_d t_c t_d$.

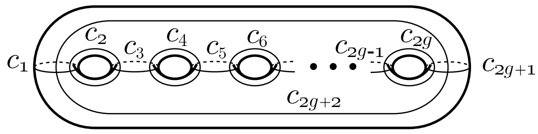

FIGURE 2. The curves $c_1, c_2, \ldots, c_{2g+2}$.

Lemma 2.7 (The hyperelliptic involution)**.** Let $c_1, c_2, \ldots, c_{2g+1}$ be nonseparating curves in Σ_g as in Figure 2. We call the product

$$\iota := \; t_{c_{2g+1}} t_{c_{2g}} \cdots t_{c_2} t_{c_1} t_{c_1} t_{c_2} \cdots t_{c_{2g}} t_{c_{2g+1}}$$

the hyperelliptic involution. For $g = 1$, the hyperelliptic involution ι equals to $t_{c_1} t_{c_2} t_{c_1} t_{c_1} t_{c_2} t_{c_1}$, where c_1 (resp. c_2) is the meridian (resp. the longitude) of Σ_1.

Lemma 2.8 (The chain relation)**.** For a positive integer n, let $a_1, a_2, \ldots, a_n$ be a sequence of simple closed curves in Σ_g such that a_i and a_j are disjoint if $|i-j| \geq 2$, and that a_i and a_{i+1} intersect at one point.

When n is odd, a regular neighborhood of $a_1 \cup a_2 \cup \cdots \cup a_n$ is a subsurface of genus $(n-1)/2$ with two boundary components, denoted by d_1 and d_2. We then have

$$(t_{a_n} \cdots t_{a_2} t_{a_1})^{n+1} = t_{d_1} t_{d_2}.$$

When n is even, a regular neighborhood of $a_1 \cup a_2 \cup \cdots \cup a_n$ is a subsurface of genus $n/2$ with connected boundary, denoted by d. We then have

$$(t_{a_n} \cdots t_{a_2} t_{a_1})^{2(n+1)} = t_d.$$

2.3. **Meyer's signature cocycle.** Let X be a compact oriented $(4n+2)$-manifold for nonnegative integer n, and let Γ be a local system on X such that $\Gamma(x)$ is a finite dimensional real or complex vector space for $x \in X$. If we are given a regular antisymmetric (resp. skew-hermitian) form $\Gamma \otimes \Gamma \to \mathbb{R}$ (resp. $\Gamma \otimes \Gamma \to \mathbb{C}$), we have a symmetric (resp. hermitian) form on $H_{2n+1}(X;\Gamma)$ as in [21, p.12]. For simplicity, we only explain the complex case. It is defined by

$$\begin{aligned} H_{2n+1}(X;\Gamma) \otimes H_{2n+1}(X;\Gamma) &\cong H^{2n+1}(X,\partial X;\Gamma) \otimes H^{2n+1}(X,\partial X;\Gamma) \\ &\xrightarrow{\cup} H^{4n+2}(X,\partial X;\Gamma \otimes \Gamma) \\ &\to H^{4n+2}(X,\partial X;\mathbb{C}) \\ &\xrightarrow{[X,\partial X]} \mathbb{C}, \end{aligned}$$

where the first row is defined by the Poincaré duality, the second row is defined by the cup product of the base space, the third row comes from the skew-hermitian form of Γ as above, and the fourth row is the evaluation by the fundamental class of X. Meyer showed additivity of signatures with respect to this hermitian form (more strongly, he showed Wall's non-additivity formula for G-signatures of homology groups with local coefficients).

Theorem 2.9 ([21, Satz I.3.2])**.** Let X and Γ be as above. Assume that X is obtained by gluing two compact oriented $(4n+2)$-manifold X_- and X_+ along some boundary components.

Then, we have

$$\mathrm{Sign}(H_{2n+1}(X;\Gamma)) = \mathrm{Sign}(H_{2n+1}(X_-;\Gamma|_{X_-})) + \mathrm{Sign}(H_{2n+1}(X_+;\Gamma|_{X_+})).$$

Consider the case when X is a pair of pants, which we denote by P. Let α and β be loops in P as in Figure 3.

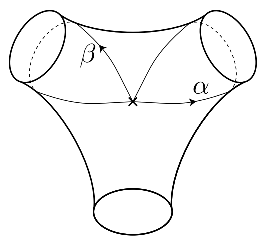

FIGURE 3. loops in a pair of pants

For $\varphi, \psi \in \mathcal{M}_g$, there exists a Σ_g-bundle $E_{\varphi,\psi}$ on P whose monodromies along α and β are φ and ψ, respectively. This is unique up to bundle isomorphism. In this setting, the intersection form on the local system $H_1(\Sigma_g;\mathbb{R})$ induces the symmetric form on $H_1(P;H_1(\Sigma_g;\mathbb{R}))$. Meyer showed that the signature of this symmetric form on $H_1(P;H_1(\Sigma_g;\mathbb{R}))$ coincides with that of $E_{\varphi,\psi}$. Moreover, he explicitly described it in terms of the action of the mapping class group on $H_1(\Sigma_g;\mathbb{R})$ as follows. Fix the symplectic basis $\{A_i, B_i\}_{i=1}^g$ of $H_1(\Sigma_g;\mathbb{Z})$ as in Figure 4, then the action induces a homomorphism $\rho : \mathcal{M}_g \to \mathrm{Sp}(2g;\mathbb{Z})$. Let I denote the identity matrix of rank g, and J denote a matrix defined by

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

For symplectic matrices A and B of rank $2g$, let $V_{A,B}$ denote the vector space defined by

$$V_{A,B} = \{(v,w) \in \mathbb{R}^{2g} \times \mathbb{R}^{2g} \mid (A^{-1} - I)v + (B - I)w = 0\}.$$

Consider the symmetric bilinear form

$$\langle\ ,\ \rangle_{A,B} : V_{A,B} \times V_{A,B} \to \mathbb{R}$$

on $V_{A,B}$ defined by

$$\langle (v_1, w_1), (v_2, w_2) \rangle_{A,B} := (v_1 + w_1)^T J (I - B) w_2.$$

Then, the space $V_{A,B}$ coincides with $H_1(P; H_1(\Sigma_g; \mathbb{R}))$, and the above form $\langle\ ,\ \rangle_{\rho(\varphi),\rho(\psi)}$ corresponds to the symmetric form on $H_1(P; H_1(\Sigma_g; \mathbb{R}))$.

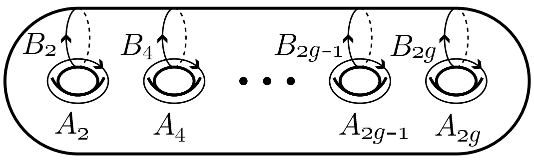

FIGURE 4. A symplectic basis of $H_1(\Sigma_g; \mathbb{Z})$

Meyer's signature cocycle $\tau_g : \mathcal{M}_g \times \mathcal{M}_g \to \mathbb{Z}$ is the map defined by $(\varphi, \psi) \mapsto \mathrm{Sign}(\langle\ ,\ \rangle_{\rho(\varphi),\rho(\psi)})$, which is known to be a bounded 2-cocycle by Theorem 2.9. When we restrict it to the hyperelliptic mapping class group $\mathcal{H}_g$, it represents the trivial cohomology class in $H^2(\mathcal{H}_g; \mathbb{Q})$. Since the first homology $H_1(\mathcal{H}_g; \mathbb{Q})$ is trivial, the cobounding function $\phi_g : \mathcal{H}_g \to \mathbb{Q}$ of τ_g is unique. It is a quasimorphism, called the Meyer function. In [12], Endo computed it to investigate signatures of fibered 4-manifolds called hyperelliptic Lefschetz fibrations. In [23], Morifuji relates it to the eta invariants of mapping tori and the Casson invariants of integral homology 3-spheres.

3. Cobounding functions of the Meyer's signature cocycles on symmetric mapping class groups

As in the introduction, let m be a positive integer greater than 3 and $\{q_i\}_{i=1}^m$ be m distinct points in a 2-sphere S^2. Choose a base point $* \in S^2 - \{q_i\}_{i=1}^m$, and denote by $\alpha_i \in \pi_1(S^2 - \{q_i\}_{i=1}^m, *)$ a loop which rounds the point q_i clockwise as in Figure 5.

For an integer d such that $d \geq 2$ and $d|m$, define a homomorphism $\pi_1(S^2 - \{q_i\}_{i=1}^m) \to \mathbb{Z}/d\mathbb{Z}$ by mapping each generator α_i to $1 \in \mathbb{Z}/d\mathbb{Z}$. This homomorphism induces a d-cyclic branched covering $p_d : \Sigma_h \to S^2$ with m branched points, where Σ_h is a closed oriented surface of genus h. Applying the Riemann-Hurwitz formula, we have $h = (d-1)(m-2)/2$. We denote by $t : \Sigma_h \to \Sigma_h$ the deck transformation which corresponds to the generator $1 \in \mathbb{Z}/d\mathbb{Z}$.

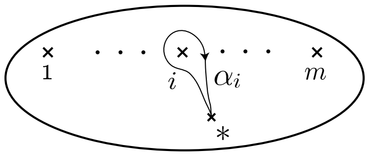

FIGURE 5. a loop α_i

Let η denote the d-th root of unity $\exp(2\pi \boldsymbol{i}/d)$, where $\boldsymbol{i}$ is a square root of -1. The first homology $H_1(\Sigma_h;\mathbb{C})$ decomposes into a direct sum $\bigoplus_{j=1}^{d-1} V^{\eta^j}$, where V^z is an eigenspace whose eigenvalue is $z \in \mathbb{C}$. Note that V^1 is trivial since the quotient space $\Sigma_g/\langle t\rangle$ is a 2-sphere, where $\langle t\rangle$ denotes the cyclic group generated by the deck transformation t. We also denote by $C_h(t)$ the centralizer of t in the diffeomorphism group $\mathrm{Diff}_+\Sigma_h$. We call the path-connected component $\pi_0 C_h(t)$ the symmetric mapping class group of the covering p, which is defined by Birman-Hilden (see, for example, [6]).

In this section, we define 2-cocycles on the symmetric mapping class group $\pi_0 C_h(t)$ using the ω-signature in [16] which derive from the nonadditivity formula.

Let us consider an oriented Σ_h-bundle $E_{\varphi,\psi}$ over P whose structure group is contained in $C_h(t)$, and monodromies along α and β are φ and ψ in $\pi_0 C_h(t)$, respectively. Since coordinate transformations commute with the deck transformation t, we can define a fiberwise $\mathbb{Z}/d\mathbb{Z}$ action on $E_{\varphi,\psi}$. Since the structure group is in $C_h(t)$, not only $H_1(\Sigma_h;\mathbb{C})$ but also each eigenspace V^{η^j} is a local system on P. We can extend the intersection form as a skew-hermitian form $H_1(\Sigma_h;\mathbb{C})\otimes H_1(\Sigma_h;\mathbb{C}) \to \mathbb{C}$ defined by

$$(x_1 + x_2\boldsymbol{i}) \cdot (y_1 + y_2\boldsymbol{i}) = x_1 \cdot y_1 + x_2 \cdot y_2 + (x_1 \cdot y_2 - x_2 \cdot y_1)\boldsymbol{i}.$$

For $v \in V^{\eta^j}$ and $w \in V^{\eta^k}$ $(1 \le j \le d-1,\ 1 \le k \le d-1)$,

$$(tv) \cdot w = (\omega^j v) \cdot w = \omega^{-j}(v \cdot w),$$
$$(tv) \cdot w = v \cdot (t^{-1} w) = v \cdot (\omega^{-k} w) = \omega^{-k}(v \cdot w).$$

Since ω^{-j} is not equal to ω^{-k}, we have $v \cdot w = 0$. Hence, $H_1(\Sigma_h;\mathbb{C})$ decomposes into an orthogonal sum of subspaces $\{V^{\omega^j}\}_{j=1}^{d-1}$. By restricting the intersection form on $H_1(\Sigma_h;\mathbb{C})$ to V^{η^j}, we can define a hermitian form on $H_1(P;V^{\eta^j})$. By Theorem 2.9, we have a 2-cocycle on $\pi_0 C_h(t)$ as follows.

Lemma 3.1. Let j be an integer such that $1 \le j \le m-1$. The map $\tau_{m,d,j}$: $\pi_0 C_h(t) \times \pi_0 C_h(t) \to \mathbb{Z}$ defined by

$$\tau_{m,d,j}(\varphi,\psi) = \mathrm{Sign}(H_1(P;V^{\eta^j}))$$

is a 2-cocycle, where V^{η^j} is the local system on P induced from the oriented Σ_h-bundle $E_{\varphi,\psi} \to P$.

Proof. The proof is the same as for [21, p.43 equation (0)]. Applying additivity of signatures to two oriented Σ_h-bundles on P, we can see that $\tau_{m,d,j}$ satisfies

$$\tau_{m,d,j}(\varphi_1,\varphi_2) + \tau_{m,d,j}(\varphi_1\varphi_2,\varphi_3) = \tau_{m,d,j}(\varphi_1,\varphi_2\varphi_3) + \tau_{m,d,j}(\varphi_2,\varphi_3),$$

for $\varphi_1, \varphi_2, \varphi_3 \in \pi_0 C_h(t)$. □

Since the deck transformation t acts on $H^1(P, \partial P; V^{\eta^j})$ by multiplication of η^j, we can calculate $\mathbb{Z}/d\mathbb{Z}$-signature as

$$\mathrm{Sign}(H_1(P; V^{\eta^j}), t^k) = \eta^{kj}\,\mathrm{Sign}(H_1(P; V^{\eta^j})) = \eta^{kj}\tau_{m,d,j}(\varphi, \psi),$$

for $0 \leq k \leq m-1$. Moreover, in [21, Satz I.2.2], Meyer proved $\mathrm{Sign}(E_{\varphi,\psi}, t^k) = \mathrm{Sign}(H_1(P; H^1(\Sigma_h; \mathbb{C})), t^k)$. Hence, we have:

Lemma 3.2. For $0 \leq k \leq m-1$,

$$\mathrm{Sign}(E_{\varphi,\psi}, t^k) = \sum_{j=1}^{d-1} \eta^{kj}\tau_{m,d,j}(\varphi, \psi).$$

3.1. **The symmetric mapping class groups.** A diffeomorphism $f : \Sigma_h \to \Sigma_h$ in $C_h(t)$ induces a diffeomorphism $\bar{f} : S^2 \to S^2$ which satisfies the commutative diagram

$$\begin{array}{ccc} \Sigma_h & \xrightarrow{f} & \Sigma_h \\ {\scriptstyle p_d}\big\downarrow & & {\scriptstyle p_d}\big\downarrow \\ S^2 & \xrightarrow{\bar{f}} & S^2. \end{array}$$

Moreover, since $\bar{f}$ satisfies $p_d^{-1}(q) = p_d^{-1}(\bar{f}(q))$ for any $q \in S^2$, we have $\bar{f} \in \mathrm{Diff}_+(S^2, \{q_i\}_{i=1}^m)$. Therefore, we have a natural homomorphism $\mathcal{P} : \pi_0 C_h(t) \to \mathcal{M}_0^m$ which maps $[f]$ to $[\bar{f}]$. By a similar way to [5, Theorem 1] (see also [6, Section 5]), we have:

Lemma 3.3. Let $m \geq 4$. The sequence

$$1 \longrightarrow \mathbb{Z}/d\mathbb{Z} \longrightarrow \pi_0 C_h(t) \xrightarrow{\mathcal{P}} \mathcal{M}_0^m \longrightarrow 1$$

is exact.

Let $s_i : S^2 \to S^2$ be a half twist of the disk which exchanges the points q_i and q_{i+1} as in Figure 6. We denote by $\sigma_i \in \mathcal{M}_0^m$ the mapping class represented by

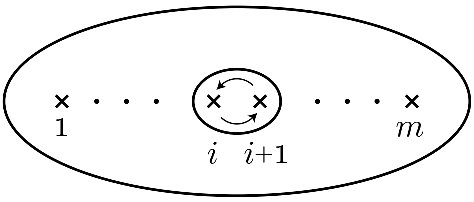

FIGURE 6. the diffeomorphism s_i

s_i. By lifting s_i, we have a unique diffeomorphism $\tilde{s}_i : \Sigma_h \to \Sigma_h$ which satisfies $\mathrm{supp}\,\tilde{s}_i = p_d^{-1}(\mathrm{supp}\, s_i)$. Let us denote the mapping class of $\tilde{s}_i$ by $\tilde{\sigma}_i \in \pi_0 C_h(t)$. Note that when $d = 2$, $\tilde{\sigma}_i$ is the Dehn twist along a nonseparating simple closed curve.

Lemma 3.4. The set $\{\tilde{\sigma}_i\}_{i=1}^{m-1} \subset \pi_0 C_h(t)$ generates the group $\pi_0 C_h(t)$.

Proof. Since $\{\sigma_i\}_{i=1}^{m-1}$ generates the group $\mathcal{M}_0^m$, it suffices to represent $[t] \in \pi_0 C_h(t)$ as a product of $\{\sigma_i\}_{i=1}^{m-1}$. Let $C_h^{(*)}(t)$ denote the subgroup of $C_h(t)$ defined by $C_h^{(*)}(t) = \{f \in C_h(t) \mid f(p_d^{-1}(*)) = p_d^{-1}(*)\}$. In this proof, by abuse of terminology, we use the term "Dehn twist" both for a diffeomorphism and for a mapping class. The diffeomorphism $s_1 \cdots s_{m-2} s_{m-1}^2 s_{m-2} \cdots s_1$ in $\mathrm{Diff}_+(S^2, \{q_i\}_{i=1}^m)$ is isotopic to the product of Dehn twists $t_c^{-1} t_{c'}$ in Figure 7, and it is also isotopic to the Dehn twist t_d^{-1}.

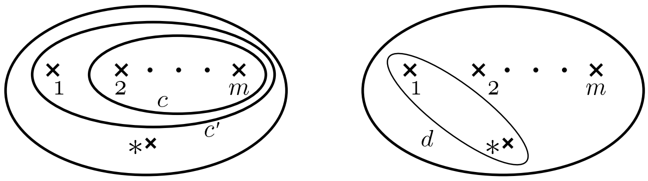

FIGURE 7. the curves c, c', d

Therefore, the lift $\tilde{s}_1 \cdots \tilde{s}_{m-2} \tilde{s}_{m-1}^2 \tilde{s}_{m-2} \cdots \tilde{s}_1$ is isotopic to some lift $\tilde{f}_1 : \Sigma_h \to \Sigma_h$ of t_d^{-1}. Since we can choose the isotopy in $\mathrm{Diff}_+(S^2, \{q_i\}_{i=1}^m)$ so that it does not move $*$, the lift $\tilde{f}_1$ fixes $p^{-1}(*)$ pointwise. Let D be the closed disk which is bounded by d and contains $*$, and $\tilde{f}_2$ denote the lift of t_d which satisfies $\mathrm{supp}\, \tilde{f}_2 \subset p^{-1}(D)$. Since $f_1 f_2$ is a lift of the identity map of S^2, and the action of $\tilde{f}_2$ on $p^{-1}(*)$ coincides with that of t, we have $\tilde{f}_1 \tilde{f}_2 = t \in \mathrm{Diff}_+ \Sigma_h$. Since t_d is isotopic to the identity map in $\mathrm{Diff}_+ \Sigma_h$, we have $[\tilde{f}_2] = 1 \in \pi_0 C_h(t)$. Thus, we obtain

$$\tilde{\sigma}_1 \cdots \tilde{\sigma}_{m-2} \tilde{\sigma}_{m-1}^2 \tilde{\sigma}_{m-2} \cdots \tilde{\sigma}_1 = [\tilde{f}_1] = [\tilde{f}_1 \tilde{f}_2] = [t] \in \pi_0 C_h(t).$$

□

3.2. **The cobounding function of the cocycles $\tau_{m,d,j}$.** Recall that, for an integer d with $d|m$, we have a covering space $p_d : \Sigma_h \to S^2$. Let $g = (m-1)(m-2)/2$. If we consider the case when $d = m$, p_d is the m-cyclic covering on S^2 whose genus of the covering surface is g. thus we identify it with the surface Σ_g, and denote the covering by $p : \Sigma_g \to S^2$.

Since the quotient space $\Sigma_g / \langle t^d \rangle$ is also a d-cyclic covering of S^2 with m branched points, we can identify $\Sigma_h \cong \Sigma_g / \langle t^d \rangle$. Since a diffeomorphism $f \in C_g(t)$ induces a diffeomorphism $\bar{f}$ on $\Sigma_g / \langle t^d \rangle$ which commutes with t, we have a natural homomorphism $\mathcal{P} : \pi_0 C_g(t) \to \pi_0 C_h(t)$ which maps $[f]$ to $[\bar{f}]$. Since $H^*(\pi_0 C_h(t); \mathbb{Q}) \cong H^*(\mathcal{M}_0^m; \mathbb{Q})$, and $H^*(\mathcal{M}_0^m; \mathbb{Q})$ is trivial (see [10] Corollary 2,2), there exists a unique cobounding function of $\tau_{m,d,j}$. Denote it by $\phi_{m,d,j} : \pi_0 C_h(t) \to \mathbb{Q}$. Since $\tau_{m,d,j}$ is bounded, the cobounding function $\phi_{m,d,j}$ is a quasimorphism.

Remark 3.5. Gambaudo and Ghys [16] already constructed almost the same quasimorphisms on the mapping class groups of pointed disks, called ω-signatures. They calculated the value of their quasimorphisms for a similar elements to $\tilde{\sigma}_1 \tilde{\sigma}_2 \cdots \tilde{\sigma}_{r-1} \in \pi_0 C_h(t)$ in [16, Proposition 5.2].

Remark 3.6. This construction is also similar to higher-order signature cocycles in Cochran-Harvey-Horn's paper [9]. They considered von Neumann signatures of surface bundles whose fibers are non-finite regular coverings on a surface with boundary.

Let us recall a natural homomorphism $\pi_0 C_h(t) \to \mathcal{M}_h$ defined by forgetting symmetries of mapping classes. It maps a mapping class $[f] \in \pi_0 C_h(t)$ to $[f] \in \mathcal{M}_h$, and is injective as in Birman-Hilden [6, Theorem 1]. In particular, if we consider the case when m is even and the double covering $p_2 : \Sigma_h \to S^2$, this homomorphism induces isomorphism between $\pi_0 C_h(t)$ and $\mathcal{H}_h$. In this case, the eigenspace V^{-1} coincides with $H_1(\Sigma_h; \mathbb{C})$. Thus, we have:

Remark 3.7. When m is even, $\phi_{m,2,1} : \pi_0 C_h(t) \to \mathbb{Q}$ is equal to the Meyer function $\phi_h : \mathcal{H}_h \to \mathbb{Q}$ on the hyperelliptic mapping class group, under the natural isomorphism $\pi_0 C_h(t) \cong \mathcal{H}_h$.

Lemma 3.8. For $1 \leq j \leq d-1$ and $\varphi \in \pi_0 C_g(t)$,

$$\phi_{m,m,mj/d}(\varphi) = \phi_{m,d,j}(\mathcal{P}(\varphi)).$$

Proof. Since $H_1(\pi_0 C_g(t); \mathbb{Q})$ is trivial, it suffices to show that $\tau_{m,m,mj/d}(\varphi, \psi) = \tau_{m,d,j}(\mathcal{P}(\varphi), \mathcal{P}(\psi))$ for $\varphi, \psi \in \pi_0 C_g(t)$. If $f : E \to P$ is an oriented Σ_g-bundle with structure group $C_g(t)$, the induced map $\bar{f} : E/\langle t^d \rangle \to P$ is an oriented Σ_h-bundle with structure group $C_h(t)$. If we denote the monodromies of f along α and β by φ and ψ, the ones of $\bar{f}$ are $\mathcal{P}(\varphi)$ and $\mathcal{P}(\psi)$.

Let ω be the m-th root of unity $\exp(2\pi \boldsymbol{i}/m)$, and let $q_d : \Sigma_g \to \Sigma_g/\langle t^d \rangle$ denote the projection. To distinguish eigenspaces of $H_1(\Sigma_g; \mathbb{C})$ and $H_1(\Sigma_h; \mathbb{C})$ of the action by t, we denote them by $(V_g)^z$ and $(V_h)^z$ instead of V^z, respectively. The projection q_d induces the isomorphism $H_1(\Sigma_g; \mathbb{C})^{\langle t^d \rangle} \cong H_1(\Sigma_h; \mathbb{C})$. Moreover, we have $(V_g)^{\omega^{mj/d}} \cong (V_h)^{\eta^j}$. Hence, it also induces a natural isomorphism between $H_1(P; (V_g)^{\omega^{mj/d}})$ and $H_1(P; (V_h)^{\eta^j})$, where $(V_g)^{\omega^{mj/d}}$ and $(V_h)^{\eta^j}$ are local systems coming from f and $\bar{f}$.

Let $\tilde{a}$, $\tilde{b}$ be loops in $\Sigma_g - \{q_i\}_{i=1}^m$. We may assume that $q_d(\tilde{a}) \cup q_d(\tilde{b})$ has no triple point. Then, the intersection number $[q_d(\tilde{a})] \cdot [q_d(\tilde{b})]$ in Σ_h coincides with $[q_d^{-1}(q_d(\tilde{a}))] \cdot [\tilde{b}]$ in Σ_g. Hence, we have

$$\begin{aligned}
\sum_{i=0}^{m/d-1} [(t^{di})_* \tilde{a}] \cdot \sum_{j=0}^{m/d-1} [(t^{dj})_* \tilde{b}] &= \sum_{i=0}^{m/d-1} \sum_{j=0}^{m/d-1} [(t^{di-dj})_* \tilde{a}] \cdot [\tilde{b}] \\
&= \frac{m}{d} [q_d^{-1}(q_d(\tilde{a}))] \cdot [\tilde{b}] \\
&= \frac{m}{d} [q_d(\tilde{a})] \cdot [q_d(\tilde{b})].
\end{aligned}$$

Therefore, the isomorphism $H_1(\Sigma_g; \mathbb{C})^{\langle t^d \rangle} \cong H_1(\Sigma_h; \mathbb{C})$ induced by the quotient map $q_d : \Sigma_g \to \Sigma_h$ preserves the intersection form up to constant multiple. Thus, it also preserves the intersection forms on $H_1(P; (V_g)^{\omega^{mj/d}})$ and $H_1(P; (V_h)^{\eta^j})$, and we obtain

$$\begin{aligned}
\tau_{m,m,mj/d}(\varphi, \psi) &= \mathrm{Sign}(H_1(P; (V_g)^{\omega^{mj/d}})) \\
&= \mathrm{Sign}(H_1(P; (V_h)^{\eta^j})) \\
&= \tau_{m,d,j}(\mathcal{P}(\varphi), \mathcal{P}(\psi)).
\end{aligned}$$

□

By Lemma 3.8, it suffices to consider the case when $d = m$, we simply denote $\tau_{m,m,j} = \tau_{m,j}$ and $\phi_{m,m,j} = \phi_{m,j}$.

Lemma 3.9.
$$\phi_{m,j}(\varphi) = \phi_{m,m-j}(\varphi).$$

Proof. By taking complex conjugates, we have an isomorphism $i : V^{\omega^j} \cong V^{\omega^{m-j}}$. Moreover, it induces the isomorphism $i_* : H_1(P; V^{\omega^j}) \cong H_1(P; V^{\omega^{m-j}})$.

Let us denote the hermitian form on $H_1(P; V^{\omega^j})$ by $\langle\ ,\ \rangle_j$. By the definition of the hermitian form, we have $\langle x, y\rangle_j = \overline{\langle i_*x, i_*y\rangle}_{m-j}$ for $x, y \in H_1(P; V^{\omega^j})$, where $\overline{z}$ is a complex conjugate of $z \in \mathbb{C}$. Thus, the signatures of the hermitian forms $\langle\ ,\ \rangle_j$ and $\langle\ ,\ \rangle_{m-j}$ coincide, and the cobounding functions of $\tau_{m,j}$ and $\tau_{m,m-j}$ also coincide. □

4. Defects of homogeneous quasimorphisms

In this section, we will prove Theorem 1.3 and 1.6. In Section 4.1, we give an inequality between the defects of a quasimorphism and its homogenization when the quasimorphism is antisymmetric and a class function (Lemma 4.1), and prove Theorem 1.3 (i). In Section 4.2, we prove Theorem 1.3 (ii) by giving a lower bound on the defect of $\phi_{m,m/2} : \pi_0 C_g(t) \to \mathbb{R}$, which is the cobounding function of the 2-cocycle $\tau_{m,m/2}$. In Section 4.2, we prove Theorem 1.6.

4.1. **Proof of Theorem 1.3 (i).**

In [12] Proposition 3.1, Endo showed that the Meyer function $\phi_g : \mathcal{H}_g \to \mathbb{Q}$ satisfies the conditions in Lemma 4.1. The quasimorphisms $\bar{\phi}_{m,j}$ also satisfy these conditions.

In [25], Turaev defined another 2-cocycle on the symplectic group. In [14] Proposition A.3, Endo and Nagami showed that Turaev's cocycle coincides with the Meyer cocycle up to sign. Since Turaev's cocycle is defined by the signature on a vector space of rank less than or equal to $m-2$, A similar argument shows $D(\phi_{m,j}) \leq m-2$. Thus, Theorem 1.3 (i) follows from Lemma 4.1 below.

Lemma 4.1. Let G be a group, and $\phi : G \to \mathbb{R}$ a quasi-morphism satisfying
$$\phi(xyx^{-1}) = \phi(y),\ \phi(x^{-1}) = -\phi(x).$$
Then, we have
$$D(\bar{\phi}) \leq D(\phi),$$
where $\bar{\phi}$ is the homogenization of ϕ.

Proof of Lemma 4.1. Without loss of generality, we may assume that the quasimorphism $\phi : G \to \mathbb{R}$ is antisymmetric:
$$\phi(x^{-1}) = -\phi(x).$$
Otherwise, pass to the antisymmetrization $\phi' : G \to \mathbb{R}$ defined by
$$\phi'(x) = \frac{\phi(x) - \phi(x^{-1})}{2},$$

which satisfies
$$D(\phi') \leq D(\phi), \text{ and } \bar{\phi}' = \bar{\phi}.$$
For any $x, y \in G$, we have
$$\phi([x,y]) = |\phi([x,y]) - \phi(y) + \phi(y)| = |\phi(xyx^{-1}y^{-1}) - \phi(xyx^{-1}) - \phi(y^{-1})| \leq D(\phi).$$
Thus, for any $g \in [G,G]$,
$$|\phi(g)| \leq (2\,\mathrm{cl}(g) - 1)D(\phi).$$
As observed by Bavard [2, Lemma 3.6],
$$\mathrm{cl}(x^n y^n (xy)^{-n}) \leq \frac{n}{2},$$
for every $n \geq 0$. Therefore, we have $|\phi(x^n y^n (xy)^{-n})| \leq (n-1)D(\phi)$.

Hence, we have
$$|\delta\bar{\phi}(x,y)| = \lim_{n\to\infty} \left| \frac{\phi(x^n) + \phi(y^n) - \phi(x^n y^n)}{n} \right| = \lim_{n\to\infty} \left| \frac{\phi(x^n y^n (xy)^{-n})}{n} \right| \leq D(\phi).$$
□

4.2. **Proof of Theorem 1.3 (ii).** Let m be an even number greater than or equal to 4. By Remark 3.7, we consider the Meyer function ϕ_g on the hyperelliptic mapping class group $\mathcal{H}_g$ instead of $\phi_{m,m/2}$.

Lemma 4.2 ([1] Proposition 3.5)**.** For any $A \in \mathrm{Sp}(2g;\mathbb{Z})$,
$$\mathrm{Sign}(\langle\ ,\ \rangle_{A^k,A}) = \mathrm{Sign}\left(-J \sum_{i=1}^{k} (A^i - A^{-i}) \right).$$

Let c_i, d_i^+, and d_i^- denote the simple closed curves in Figure 8. For simplicity, we also denote by c_i, d_i^+, and d_i^- the Dehn twists along these curves. To prove

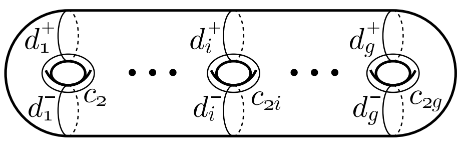

FIGURE 8. curves in Σ_g

Theorem 1.3 (ii), it suffices to show the following.

Lemma 4.3.
$$\delta\bar{\phi}_g(c_2^2 c_4^2 \cdots c_{2g}^2, d_1^+ d_1^- d_2^+ d_2^- \cdots d_g^+ d_g^-) = -2g.$$

Proof of Lemma 4.3. Since the pairs (c_i, c_j), $(d_i^+ d_i^-, d_j^+ d_j^-)$, and $(c_i, d_j^+ d_j^-)$ mutually commute when $i \neq j$, we have
$$\begin{aligned}
&\delta\bar{\phi}_g(c_2^2 c_4^2 \cdots c_{2g}^2, d_1^+ d_1^- d_2^+ d_2^- \cdots d_g^+ d_g^-) \\
&= \bar{\phi}_g(c_2^2 c_4^2 \cdots c_{2g}^2) + \bar{\phi}_g(d_1^+ d_1^- d_2^+ d_2^- \cdots d_g^+ d_g^-) - \bar{\phi}_g(c_2^2 d_1^+ d_1^- c_4^2 d_2^+ d_2^- \cdots c_{2g}^2 d_g^+ d_g^-) \\
&= \sum_{i=1}^{g} (\bar{\phi}_g(c_{2i}^2) + \bar{\phi}_g(d_i^+ d_i^-) - \bar{\phi}_g(c_{2i}^2 d_i^+ d_i^-)).
\end{aligned}$$

Hence, It suffices to prove $\bar{\phi}_g(c_{2i}^2) + \bar{\phi}_g(d_i^+ d_i^-) - \bar{\phi}_g(c_{2i}^2 d_i^+ d_i^-) = -2$ for $1 \le i \le g$. Since $\rho(d_i^+) = \rho(d_i^-)$, we have

$$\begin{aligned}
&\bar{\phi}_g(c_{2i}^2) + \bar{\phi}_g(d_i^+ d_i^-) - \bar{\phi}_g(c_{2i}^2 d_i^+ d_i^-)\\
&= -\lim_{n\to\infty} \frac{1}{n}\left\{\phi_g((c_{2i}^2 d_i^+ d_i^-)^n) - \phi_g((c_{2i}^2)^n) - \phi_g((d_i^+ d_i^-)^n)\right\}\\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n-1}\left\{\tau_g((c_{2i}^2 d_i^+ d_i^-)^k, c_{2i}^2 d_i^+ d_i^-) - \tau_g(c_{2i}^{2i}, c_{2i}^2) - \tau_g((d_i^+ d_i^-)^i, d_i^+ d_i^-)\right\} + \tau_g(c_{2i}^2, d_i^+ d_i^-)\\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n-1}\left\{\tau_g((c_{2i}^2 (d_i^+)^2)^k, c_{2i}^2 (d_i^+)^2) - \tau_g(c_{2i}^{2i}, c_{2i}^2) - \tau_g((d_i^+)^{2i}, (d_i^+)^2)\right\} + \tau_g(c_{2i}^2, (d_i^+)^2).
\end{aligned}$$

There exists a mapping class ψ_i such that $\psi_i c_{2i} \psi_i^{-1} = c_2$ and $\psi_i d_i^+ \psi_i^{-1} = d_i^+$ for $i = 2, \dots, g$. Since the Meyer cocycle satisfies the property

$$\tau_g(xyx^{-1}, xzx^{-1}) = \tau_g(y, z)$$

for $x, y, z \in \mathcal{M}_g$, we have

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{1}{n}\{\tau_g((c_{2i}^2 (d_i^+)^2)^k, c_{2i}^2 (d_i^+)^2) - \tau_g(c_{2i}^{2i}, c_{2i}^2) - \tau_g((d_i^+)^{2i}, (d_i^+)^2)\} + \tau_g(c_{2i}^2, (d_i^+)^2)\\
&= \lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{1}{n}\{\tau_g((c_2^2 (d_1^+)^2)^k, c_2^2 (d_1^+)^2) - \tau_g(c_2^{2i}, c_2^2) - \tau_g((d_1^+)^{2i}, (d_1^+)^2)\} + \tau_g(c_2^2, (d_1^+)^2).
\end{aligned}$$

Let us consider the case when $g = 1$. Since

$$\rho(c_2^2) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},\ \rho((d_1^+)^2) = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix},\ \text{and } \rho(c_2^2 (d_1^+)^2) = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix},$$

we have

$$\begin{aligned}
-J\sum_{k=1}^{n}(\rho(c_2^{2k}) - \rho(c_2^{-2k})) &= \begin{pmatrix} 0 & 0 \\ 0 & 2n(n+1) \end{pmatrix},\\
-J\sum_{k=1}^{n}(\rho(d_1^+)^{2k} - \rho(d_1^+)^{-2k}) &= \begin{pmatrix} 2n(n+1) & 0 \\ 0 & 0 \end{pmatrix},\\
-J\sum_{k=1}^{n}(\rho((c_2^2 (d_1^+)^2)^k) - \rho((c_2^2 (d_1^+)^2)^{-k})) &= \sum_{k=1}^{n} 4k(-1)^k \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\\
&= \{(-1)^n(2n+1) - 1\}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},
\end{aligned}$$

By Lemma 1.1, we obtain

$$\lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{\tau_g(c_2^{2k}, c_2^2)}{n} = \lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{\tau_g((d_1^+)^{2k}, (d_1^+)^2)}{n} = 1, \tag{1}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n-1}\frac{\tau_g((c_2^2 (d_1^+)^2)^k, c_2^2 (d_1^+)^2)}{n} = 0. \tag{2}$$

When $g \ge 2$, the same calculation also shows the equation (1). It is an easy calculation to show that

$$\tau_g(c_2^2, d_1^+ d_1^-) = 0.$$

Therefore, we obtain

$$\bar{\phi}_g(c_{2i}^2) + \bar{\phi}_g(d_i^+ d_i^-) - \bar{\phi}_g(c_{2i}^2 d_i^+ d_i^-) = -2.$$

□

In the same way as the equation (1), we have $\tau_g(s_0^i, s_0) = 1$. Hence, we obtain

$$\begin{aligned}\bar{\phi}_g(s_0) &= -\lim_{n\to\infty} \frac{\sum_{i=1}^{n-1} \tau_g(s_0^i, s_0)}{n} + \phi_g(s_0) = -1 + \phi_g(s_0), \text{ and} \\ \bar{\phi}_g(s_h) &= \phi_g(s_0).\end{aligned}$$

By Lemma 3.3 and 3.5 in Endo [12], we have

$$\bar{\phi}_g(t_{s_0}) = -\frac{g}{2g+1}, \text{ and } \bar{\phi}_g(t_{s_h}) = -\frac{4h(g-h)}{2g+1}.$$

Remark 4.4. By Theorem 1.3 and 2.4, $\bar{\phi}_g$ gives the lower bounds for $\mathrm{scl}_{\mathcal{H}_g}(t_{s_h})$ $(j = 0, \ldots, g-1)$ corresponding to ones given in [22].

Remark 4.5. By Theorem 1.7, Theorem 2.4, and remark 4.4, we have $\mathrm{scl}_{\mathcal{M}_1}(t_c) = 1/12$. Let $\rho : \mathcal{M}_1 \cong SL(2,\mathbb{Z}) \to PSL(2,\mathbb{Z})$ be the natural quotient map. It is easily seen that for all $x \in \mathcal{M}_1$, $\mathrm{scl}_{\mathcal{M}_1}(x) = \mathrm{scl}_{PSL(2,\mathbb{Z})}(\rho(x))$. Louwsma determined $\mathrm{scl}_{PSL(2,\mathbb{Z})}(y) = 1/12$ for $y = \rho(t_c)$ (see [19]).

4.3. **Proof of Theorem 1.6.** If an element $x \in \mathcal{H}_g$ satisfies $|\bar{\phi}_g(x)| = D(\bar{\phi}_g)$ and $|\phi(x)| \leq D(\phi)$ for any homogeneous quasimorphism $\phi : \mathcal{H}_g \to \mathbb{R}$, then we obtain $\mathrm{scl}(x) = 1/2$ by Bavard's duality theorem (Theorem 2.4). We will show that

$$x = c^{2g+8}(d_2^+ d_2^- \cdots d_{g-1}^+ d_{g-1}^-)^2 (s_1 \cdots s_{g-1})^{-1}$$

satisfies this property.

Firstly, we will prove

$$\sum_{j=1}^{g-1} \phi(s_j) = \sum_{i=1}^{g} (\phi(c_{2i}^2 d_i^- d_i^+) - \phi(d_i^- d_i^+)).$$

for any homogeneous quasimorphism $\phi : \mathcal{H}_g \to \mathbb{R}$. By Lemma 2.8, we have

$$(d_1^+ c_2 d_1^-)^4 = s_1, \ (d_i^+ c_{2i} d_i^-)^4 = s_{i-1}s_i \ (i = 2, \ldots, g-1), \text{ and } (d_g^+ c_{2g} d_g^-)^4 = s_{g-1}.$$

Since c_{2i} commutes with s_j, $(c_2 d_1^- d_1^+ c_2 d_1^- d_1^+)^2 = s_1$, $(c_{2i} d_i^- d_i^+ c_{2i} d_i^- d_i^+)^2 = s_{i-1}s_i$, and $(c_{2g} d_g^- d_g^+ c_{2g} d_g^- d_g^+)^2 = s_{g-1}$. By Lemma 2.6, $c_{2i} d_i^- d_i^+ c_{2i}$ commutes with $d_i^- d_i^+$ for $i = 1, \ldots, g$, as is easy to check. It follows that $(c_2 d_1^- d_1^+ c_2)^2 = s_1 (d_1^- d_1^+)^{-2}$, $(c_{2i} d_i^- d_i^+ c_{2i})^2 = s_{i-1}s_i (d_i^- d_i^+)^{-2}$, and $(c_{2g} d_g^- d_g^+ c_{2g})^2 = s_{g-1}(d_g^- d_g^+)^{-2}$. These equations give

$$\begin{aligned} 2\phi(c_2^2 d_1^- d_1^+) &= \phi(s_1) - 2\phi(d_1^- d_1^+), \\ 2\phi(c_{2i}^2 d_i^- d_i^+) &= \phi(s_{i-1}) + \phi(s_i) - 2\phi(d_i^- d_i^+), \quad \textit{and} \\ 2\phi(c_{2g}^2 d_g^- d_g^+) &= \phi(s_{g-1}) - 2\phi(d_g^- d_g^+). \end{aligned}$$

Thus, we obtain $\sum_{j=1}^{g-1} \phi(s_j) = \sum_{i=1}^{g} (\phi(c_{2i}^2 d_i^- d_i^+) - \phi(d_i^- d_i^+))$.

Secondly, we will prove $\bar{\phi}_g(x) = D(\bar{\phi}_g)$. The curves c, s_1, …, s_{g-1}, d_2^+, d_2^-, …, d_{g-1}^+, d_{g-1}^- are mutually disjoint, and c_i is conjugate to c. Hence, by Lemma 2.3 (a) and (b), we have

$$\phi(x) = (g+4)\phi(c^2)+2\sum_{i=2}^{g-1}\phi(d_i^+d_i^-)-\sum_{j=1}^{g-1}\phi(s_i) = \sum_{i=1}^{g}(\phi(c_{2i}^2)+\phi(d_i^+d_i^-)-\phi(c_{2i}^2d_i^-d_i^+)).$$

In the proof of Lemma 4.3, we showed

$$\sum_{i=1}^{g}(\bar{\phi}_g(c_{2i}^2)+\bar{\phi}_g(d_i^+d_i^-)-\bar{\phi}_g(c_{2i}^2d_i^-d_i^+)) = -2g = -D(\bar{\phi}_g).$$

Thus, we obtain $|\bar{\phi}_g(x)| = D(\bar{\phi}_g)$.

Lastly, we will prove $\phi(x) \leq D(\phi)$ for any homogeneous quasimorphism $\phi : \mathcal{H}_g \to \mathbb{R}$.

$$\begin{aligned}
D(\phi) &\geq |\delta(c_2^2\cdots c_{2g}^2, d_1^+d_1^-\cdots d_g^+d_g^-)| \\
&= |\phi(c_2^2\cdots c_{2g}^2)+\phi(d_1^+d_1^-\cdots d_g^+d_g^-)-\phi(c_2^2\cdots c_{2g}^2d_1^+d_1^-\cdots d_g^+d_g^-)| \\
&= |\phi(c_2^2\cdots c_{2g}^2)+\phi(d_1^+d_1^-\cdots d_g^+d_g^-)-\phi((c_2^2d_1^+d_1^-)\cdots(c_{2g}^2d_g^+d_g^-))| \\
&= \left|\sum_{i=1}^{g}(\phi(c_{2i}^2)+\phi(d_i^+d_i^-)-\phi(c_{2i}^2d_i^+d_i^-))\right| \\
&= |\phi(x)|.
\end{aligned}$$

5. Proof of Theorem 1.7

In this section, we prove Theorem 1.7.

Let $c_1, \dots, c_{2g+2}$ be nonseparating simple closed curves on Σ_g as in Figure 2 and let ϕ be a homogeneous quasimorphism on $\mathcal{H}_g$. For simlicity of notation, we write t_i instead of t_{c_i} By $\iota = \iota^{-1}$, we have $t_{2g+1}^2t_{2g}\cdots t_2t_1^2 = (t_{2g}\cdots t_2)^{-1}\iota^{-1}$. Since each of the two boundary components of a regular neighborhood of $c_2 \cup c_3 \cup \cdots \cup c_{2g}$ is c_{2g+2}, by Lemma 2.8 we have $(t_{2g}\cdots t_2)^{2g} = t_{2g+2}^2$. Note that this relation holds in $\mathcal{H}_g$. Therefore, by Definition 2.2 and Lemma 2.3, we have

$$\phi(t_{2g+1}^2t_{2g}\cdots t_2t_1^2) = -\phi(t_{2g}\cdots t_2)+\phi(\iota^{-1}) = -\frac{1}{g}\phi(t_{2g+2}). \tag{3}$$

Applying Lemma 2.3 (a) and 2.6 (a), we can move the factors with single and double underlines alternatively as follows.

$$\begin{aligned}
\phi(t_{2g+1}^2 t_{2g}\cdots t_3 t_2 \underline{t_1^2}) &= \phi(\underline{t_1^2} t_{2g+1}^2 t_{2g}\cdots t_3 t_2) \quad \text{(by Lem. 2.3)}\\
&= \phi(t_{2g+1}^2 t_{2g}\cdots t_3 \underline{t_1^2}\underline{\underline{t_2}}) \quad \text{(by Lem. 2.6)}\\
&= \phi(\underline{\underline{t_2}} t_{2g+1}^2 t_{2g}\cdots t_4 t_3 t_1^2) \quad \text{(by Lem. 2.3)}\\
&= \phi(t_{2g+1}^2 t_{2g}\cdots t_4 \underline{\underline{t_2}}\underline{t_3 t_1^2}) \quad \text{(by Lem. 2.6)}\\
&= \phi(t_{2g+1}^2 t_{2g}\cdots t_6 t_5 \underline{t_3 t_1^2}\underline{\underline{t_4 t_2}}) \quad \text{(by Lem. 2.3 and 2.6)}\\
&= \phi(t_{2g+1}^2 t_{2g}\cdots t_6 \underline{\underline{t_4 t_2}}\underline{t_5 t_3 t_1^2}) \quad \text{(by Lem. 2.3 and 2.6)}\\
&= \phi(t_{2g+1}^2 t_{2g}\cdots t_7 \underline{t_5 t_3 t_1^2}\underline{\underline{t_6 t_4 t_2}}) \quad \text{(by Lem. 2.3 and 2.6)}\\
&= \phi(t_{2g+1}^2 t_{2g}\cdots t_8 \underline{\underline{t_6 t_4 t_2}}\underline{t_7 t_5 t_3 t_1^2}) \quad \text{(by Lem. 2.3 and 2.6)}\\
&= \phi(t_{2g+1}^2 t_{2g}\cdots t_9 \underline{t_7 t_5 t_3 t_1^2}\underline{\underline{t_8 t_6 t_4 t_2}}) \quad \text{(by Lem. 2.3 and 2.6)}\\
&\ \vdots \qquad\qquad\qquad \vdots\\
&= \phi((t_{2g+1}^2 t_{2g-1}\cdots t_5 t_3 t_1^2)(t_{2g} t_{2g-4}\cdots t_4 t_2)).
\end{aligned}$$

From Definition 2.2 and the equation (3)

$$\begin{aligned}
D(\phi) &\geq |\phi((t_{2g+1}^2\cdots t_3 t_1^2)(t_{2g}\cdots t_4 t_2)) - \phi(t_{2g+1}^2\cdots t_3 t_1^2) - \phi(t_{2g}\cdots t_4 t_2)|\\
&= |-\frac{1}{g}\phi(t_{2g+2}) - \phi(t_{2g+1}^2\cdots t_3 t_1^2) - \phi(t_{2g}\cdots t_4 t_2)|,
\end{aligned}$$

where $D(\phi)$ is the defect of ϕ. From Lemma 2.3, Lemma 2.5 and Lemma 2.6 we have

$$D(\phi) \geq \left|\frac{1}{g}\phi(t_1) + (g+3)\phi(t_1) + g\phi(t_1)\right| = (2g+3+1/g)\,|\phi(t_1)|.$$

By Theorem 2.4 we have $\mathrm{scl}_{\mathcal{H}_g}(t_1) \leq \dfrac{1}{2(2g+3+1/g)}$. This completes the proof of Theorem 1.7.

Remark 5.1. By a similar argument to the proof of Theorem 1.7, for all $m \geq 4$, we can show $\mathrm{scl}_{\mathcal{M}_0^m}(\sigma_1) = \dfrac{1}{2\{m+1+2/(m-2)\}}$.

6. Calculation of quasimorphisms

In this section, we prove Theorem 1.1. To prove it, we perform a straightforward and elementary calculation of the Hermitian form $\langle\ ,\ \rangle_{\tilde{\sigma}^k,\tilde{\sigma}}$ on the eigenspace V^{ω^j}.

Let $p : \Sigma_g \to S^2$ be the regular branched m-cyclic covering on S^2 with m branched points as in Section 3.2. Choose a point in $p^{-1}(*)$, and denote it by $\tilde{*} \in \Sigma_g$. We denote by $\tilde{\alpha}_i$ the lift of α_i which starts at $\tilde{*}$. Note that $\tilde{\alpha}_i t(\tilde{\alpha}_{i+1})^{-1}$ is a loop in Σ_g while $\tilde{\alpha}_i$ is an arc. We denote by $e_i(k) \in H_1(\Sigma_g;\mathbb{Z})$ the homology class represented by $t^k(\tilde{\alpha}_i t(\tilde{\alpha}_{i+1})^{-1})$.

Lemma 6.1. The set of the homology classes $\{e_i(k)\}_{\substack{1\leq i\leq m-2\\ 0\leq k\leq m-2}}$ is a basis of $H_1(\Sigma_g;\mathbb{Z})$.

Proof. We use the Schreier method. Let T denote a Schreier transversal $T = \{\alpha_1^k\}_{i=0}^{m-1}$, and S a generating set $S = \{\alpha_i\}_{i=1}^{m-1}$ of $\pi_1(S^2 - \{q_i\}_{i=1}^m)$. The subgroup $\pi_1(\Sigma_g - \{p^{-1}(q_i)\}_{i=1}^m)$ is generated by

$$\{(rs(\overline{rs})^{-1} \mid r \in T, s \in S\} = \{\alpha_1^k \alpha_i \alpha_1^{-k-1}\}_{\substack{2 \le i \le m-1 \\ 0 \le k \le m-2}} \cup \{\alpha_1^{m-1}\alpha_i\}_{1 \le i \le m-1}.$$

By van Kampen's theorem, the group $\pi_1(\Sigma_g)$ is obtained by adding the relation $\alpha_i^m = 1$ to $\pi_1(\Sigma_g - \{p^{-1}(q_i)\}_{i=1}^m)$. Thus, the set $\{\alpha_1^k \alpha_i \alpha_{i+1}^{-1} \alpha_1^{-k}\}_{\substack{1 \le i \le m-2 \\ 0 \le k \le m-2}}$ generates the group $\pi_1(\Sigma_g)$. It implies that $\{e_i(k)\}_{\substack{1 \le i \le m-2 \\ 0 \le k \le m-2}}$ is a generating set of $H_1(\Sigma_g; \mathbb{Z})$.

By the Riemann-Hurwitz formula, $H_1(\Sigma_g; \mathbb{Z})$ is a free module of rank $2g = (m-1)(m-2)$, and it is equal to the order of the set $\{e_i(k)\}_{\substack{1 \le i \le m-2 \\ 0 \le k \le m-2}}$. Therefore, the set $\{e_i(k)\}_{\substack{1 \le i \le m-2 \\ 0 \le k \le m-2}}$ is a basis of the free module $H_1(\Sigma_g; \mathbb{Z})$. □

6.1. **The intersection form and the action of $\tilde{\sigma}_i$.** Let j be an integer with $1 \le j \le m-1$. Firstly, we find a basis of $V^{\omega^j} \subset H_1(\Sigma_g; \mathbb{C})$, and calculate intersection numbers.

Lemma 6.2. The intersection numbers of $\{e_i(k)\}_{\substack{1 \le i \le m-2 \\ 0 \le k \le m-2}}$ are

$$e_i(k) \cdot e_{i'}(k) = \begin{cases} -1 & \text{if } i = i' - 1, \\ 1 & \text{if } i = i' + 1, \\ 0 & \text{otherwise,} \end{cases} \quad e_i(k) \cdot e_{i'}(k+1) = \begin{cases} -1 & \text{if } i = i', \\ 1 & \text{if } i = i' - 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$e_i(k) \cdot e_{i'}(k-1) = \begin{cases} -1 & \text{if } i = i', \\ 1 & \text{if } i = i' + 1, \\ 0 & \text{otherwise,} \end{cases} \qquad e_i(k) \cdot e_{i'}(k') = 0 \quad \text{if } |k - k'| \ge 2.$$

Proof. We only prove the intersection $e_i(k) \cdot e_{i+1}(k+1) = 1$ since the other cases are proved in the same way.

Let l_i be the paths as in Figure 9. Consider the m-copies of the 2-sphere cut

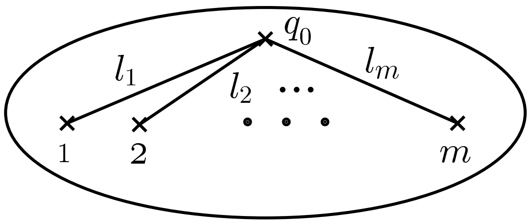

FIGURE 9. paths $l_1, l_2, \ldots, l_m$

along l_i, and number these copies from 1 to m. For convenience, we call the first copy of the 2-sphere the $(m+1)$-th copy. Gluing the left hand side of l_i in the k-th copy to the right hand side of l_i in the $(k+1)$-th copy for $k = 1, 2, \cdots, m$, we obtain a closed connected surface homeomorphic to Σ_g, and it is naturally a covering space on S^2. As in Figure 10 and Figure 11, the loops representing $e_i(k)$ and $e_{i+1}(k+1)$, intersect once positively in the $(k+1)$-th copy.

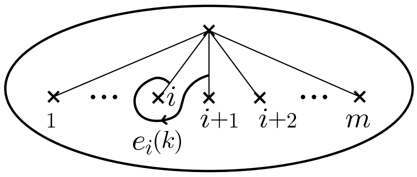

FIGURE 10. the k-th copy

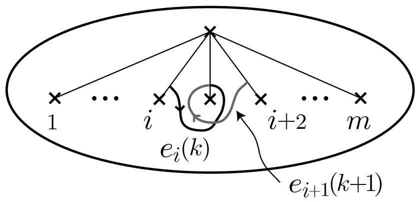

FIGURE 11. the $(k+1)$-th copy

Hence, we have $e_i(k) \cdot e_{i+1}(k+1) = 1$. □

For $1 \leq i \leq m-2$, we denote $w_i = \sum_{k=0}^{m-1} \omega^{-jk} e_i(k)$. Since $te_i(k) = e_i(k+1)$ for $1 \leq k \leq m-2$ and $e_i(m-1) = -\sum_{k=0}^{m-2} e_i(k)$, we have $w_i \in V^{\omega^j}$, and the set $\{w_i\}_{i=1}^{m-2}$ is a basis of V^{ω^j}.

Lemma 6.3. The intersecton numbers of $\{w_i\}_{1 \leq i \leq m-2}$ are

$$w_i \cdot w_{i'} = \begin{cases} d(1-\omega^j), & \text{if } i = i'+1, \\ d(-\omega^{-j}+\omega^j), & \text{if } i = i', \\ d(\omega^{-j}-1), & \text{if } i = i'-1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. By Lemma 6.2, we have

$$w_i \cdot w_i = \sum_{k=0}^{d-1}\sum_{l=0}^{d-1} \omega^{j(k-l)} e_i(k) \cdot e_i(l) = d(-\omega^{-j}+\omega^j),$$
$$w_i \cdot w_{i+1} = \sum_{k=0}^{d-1}\sum_{l=0}^{d-1} \omega^{j(k-l)} e_i(k) \cdot e_{i+1}(l) = d(\omega^{-j}-1),$$

and $w_i \cdot w_k = 0$ when $|i-k| \geq 2$. □

Let $\tilde{\sigma} = \tilde{\sigma}_1 \cdots \tilde{\sigma}_{r-1}$. Secondly, we will find eigenvectors in V^{ω^j} with respect to the action by $\tilde{\sigma}$.

Lemma 6.4. Let i be an integer such that $1 \leq i \leq m-1$. Then, we have

$$(\tilde{\sigma}_i)_* e_l(k) = \begin{cases} e_l(k) + e_{l+1}(k), & \text{if } 2 \leq i \leq m-1, \text{ and } l = i-1, \\ -e_l(k-1), & \text{if } l = i, \\ e_{l-1}(k-1) + e_l(k), & \text{if } l = i+1, \\ e_l(k), & \text{if } l \neq i-1, i, i+1. \end{cases}$$

Proof. Recall that $e_i(k)$ is the homology class represented by the loop $\tilde{\alpha}_1^k\tilde{\alpha}_i\tilde{\alpha}_{i+1}^{-1}\tilde{\alpha}_1^{-k}$. In the fundamental group $\pi_1(S^2-\{q_i\}_{i=1}^m)$, we have

$$\begin{aligned}(\sigma_i)_*(\alpha_{i-1}\alpha_i^{-1}) &= \alpha_{i-1}\alpha_{i+1}^{-1} = (\alpha_{i-1}\alpha_i^{-1})(\alpha_i\alpha_{i+1}^{-1}),\\ (\sigma_i)_*(\alpha_i\alpha_{i+1}^{-1}) &= \alpha_{i+1}(\alpha_{i+1}^{-1}\alpha_i\alpha_{i+1})^{-1} = \alpha_{i+1}^{-1}(\alpha_i\alpha_{i+1}^{-1})^{-1}\alpha_{i+1},\\ (\sigma_i)_*(\alpha_{i+1}\alpha_{i+2}^{-1}) &= (\alpha_{i+1}^{-1}\alpha_i\alpha_{i+1})\alpha_{i+2}^{-1} = \alpha_{i+1}^{-1}(\alpha_i\alpha_{i+1}^{-1})\alpha_{i+1}(\alpha_{i+1}\alpha_{i+2}^{-1}).\end{aligned}$$

By lifting these loops to the covering space Σ_g, we obtain what we want. $\square$

By Lemma 6.4, the matrix representations of the actions of $\{\tilde{\sigma}_i\}_{i=1}^{m-1}$ on V^{ω^j} with respect to the basis $\{w_i\}_{1\leq i\leq m-2}$ are calculated as

$$(\tilde{\sigma}_1)_* = \begin{pmatrix} -\omega^{-j} & \omega^{-j} & O\\ 0 & 1 & O\\ O & O & I_{m-4}\end{pmatrix}, \qquad (\tilde{\sigma}_i)_* = \begin{pmatrix} I_{i-1} & O & O\\ O & L & O\\ O & O & I_{m-i-4}\end{pmatrix},$$

$$(\tilde{\sigma}_{m-2})_* = \begin{pmatrix} I_{m-4} & O & O\\ O & 1 & O\\ O & 1 & -\omega^{-j}\end{pmatrix},\ (\tilde{\sigma}_{m-1})_* = \begin{pmatrix} I_{m-3} & v\\ O & -1+\sum_{k=1}^{m-2}\omega^{-jk}\end{pmatrix},$$

where

$$L = \begin{pmatrix} 1 & 0 & 0\\ 1 & -\omega^{-j} & \omega^{-j}\\ 0 & 0 & 1\end{pmatrix}, v = (1, 1+\omega^{-j}, \ldots, \sum_{k=0}^{m-3}\omega^{-jk})^T.$$

Let r be an integer with $2\leq r\leq m$, and put

$$e_r'(k) = [\tilde{a}_1^k\tilde{a}_r(\tilde{a}_1\tilde{a}_2\cdots\tilde{a}_r)^{-1}\tilde{a}_1^{-1}(\tilde{a}_1\tilde{a}_2\cdots\tilde{a}_r)\tilde{a}_1^{-k}].$$

By Lemma 6.4, we have

$$\begin{aligned}\tilde{\sigma}_*e_i(k) &= e_{i+1}(k),\ \text{when } 1\leq i\leq r-2,\\ \tilde{\sigma}_*e_r(k) &= -e_r'(k)+e_r(k),\\ \tilde{\sigma}_*e_{r-1}(k) &= e_r'(k),\\ \tilde{\sigma}_*e_r'(k) &= e_1(k-r+1).\end{aligned}$$

If we put $w_r' = \sum_{k=0}^{m-1}\omega^{-jk}e_r'(k)$, w_r' is contained in V^{ω^j}. For $i=1,2,\ldots,r-2$, we have

$$\begin{aligned}\tilde{\sigma}_*w_i &= \tilde{\sigma}_*\sum_{k=0}^{m-1}\omega^{-jk}e_i(k) = \sum_{k=0}^{m-1}\omega^{-jk}e_{i+1}(k) = w_{i+1},\\ \tilde{\sigma}_*w_{r-1} &= \sum_{k=0}^{m-1}\omega^{-jk}(e_{r-1}(k)+e_r'(k)) = w_{r-1}+w_r',\\ \tilde{\sigma}_*w_r' &= \sum_{k=0}^{m-1}\omega^{-jk}e_1(k-r+1) = \sum_{k=0}^{m-1}\omega^{-j(k+r-1)}e_1(k) = \omega^{-(r-1)j}w_1.\end{aligned}$$

Let $\zeta = \exp(2\pi\boldsymbol{i}/r)$ and $v_i = \sum_{k=1}^{r-1} \omega^{(k-1)j}\zeta^{-(k-1)i}w_k + \omega^{(r-1)j}\zeta^{-(r-1)i}w'_r$. Then, we have

$$\begin{aligned}
\tilde{\sigma}_* v_i &= \sum_{k=1}^{r-1} \omega^{(k-1)j}\zeta^{-(k-1)i}(\tilde{\sigma})_* w_k + \omega^{(r-1)j}\zeta^{-(r-1)i}(\tilde{\sigma})_* w'_r \\
&= \sum_{k=1}^{r-2} \omega^{(k-1)j}\zeta^{-(k-1)i}w_{k+1} + \omega^{(r-2)j}\zeta^{-(r-2)i}w'_r + \omega^{(r-1)j}\zeta^{-(r-1)i}\omega^{-pj}w_1 \\
&= \omega^{-j}\zeta^{i}\left(\sum_{k=1}^{r-1} \omega^{(k-1)j}\zeta^{-(k-1)i}w_k + \omega^{(r-1)j}\zeta^{-(r-1)i}w'_r\right) \\
&= (\omega^{-j}\zeta^{i})v_i.
\end{aligned}$$

Hence, v_i is an eigenvector with eigenvalue $\omega^{-j}\zeta^{i}$ with respect to the action by $\tilde{\sigma}$. Note that the subspace generated by $\{w_i\}_{i=1}^{r-1}$ coincides with one generated by $\{v_i\}_{i=1}^{r-1}$. Since $\tilde{\sigma}$ acts trivially on $\{w_i\}_{i=r+1}^{m-1}$, they are also eigenvectors with eigenvalue 0. Moreover, the set $\{v_i\}_{i=1}^{r-1} \cup \{w_i\}_{i=r+1}^{m-2}$ is linearly independent.

Lemma 6.5. Let i, i' be integers such that $1 \leq i \leq r-1$ and $1 \leq i' \leq r-1$. Then, we have

$$v_i \cdot v_{i'} = \begin{cases} 8rd\boldsymbol{i}\sin\dfrac{\pi i}{r}\sin\dfrac{\pi j}{m}\sin\pi\left(\dfrac{i}{r} - \dfrac{j}{m}\right), & \text{if } i = i', \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Since the action of the mapping class group $\pi_0 C_g(t)$ preserves the intersection form,

$$\begin{aligned}
v_i \cdot v_i &= \sum_{k=0}^{r-1}\sum_{l=0}^{r-1} \omega^{(l-k)j}\zeta^{-(l-k)i}(\tilde{\sigma}_*^k w_1 \cdot \tilde{\sigma}_*^l w_1) \\
&= \sum_{k=0}^{r-1}\sum_{l=0}^{r-1} \omega^{(l-k)j}\zeta^{-(l-k)i}(w_2 \cdot \tilde{\sigma}_*^{l-k+1} w_1).
\end{aligned}$$

Thus, Lemma 6.3 implies

$$\begin{aligned}
v_i \cdot v_i &= \omega^{(r-1)j}\zeta^{-(r-1)i}(w_2 \cdot \tilde{\sigma}_*^r w_1) + \omega^{j}\zeta^{-i}(r-1)(w_2 \cdot \tilde{\sigma}_*^2 w_1) + r(w_2 \cdot \tilde{\sigma}_* w_1) \\
&\quad + \omega^{-j}\zeta^{i}(r-1)(w_2 \cdot w_1) + \omega^{-(r-1)j}\zeta^{(r-1)i}(w_2 \cdot \tilde{\sigma}_*^{-r+2} w_1) \\
&= r\{(\omega^{-j}\zeta^{i})w_2 \cdot w_1 + (\omega^{j}\zeta^{-i})w_2 \cdot w_3 + w_2 \cdot w_2\} \\
&= 8rd\boldsymbol{i}\sin\frac{\pi i}{r}\sin\frac{\pi j}{m}\sin\pi\left(\frac{i}{r} - \frac{j}{m}\right).
\end{aligned}$$

□

6.2. **Calculation of ω-signatures and the cobounding functions $\phi_{m,j}$.** Lastly, we will calculate the Hermitian form $\langle\ ,\ \rangle_{\tilde{\sigma}^k,\tilde{\sigma}}$ and the ω-signature. We have already found the set of eigenvectors $\{v_i\}_{i=1}^{r-1} \cup \{w_i\}_{i=r+1}^{m-2}$ with respect to the action by $\tilde{\sigma}$ which is linearly independent. Since $\dim V^{\omega^j} = m-2$, we need to find another eigenvector.

Lemma 6.6.
$$\sum_{k=1}^{rm} \tau(\tilde{\sigma}^k, \tilde{\sigma}) = rm - 2|mi - rj|.$$

Proof. We first consider the case when rj/m is not an integer. Put
$$\beta = \sum_{i=1}^{r} w_i - \frac{1}{r}\sum_{k=1}^{r} \frac{1}{1-\omega^j\zeta^{-k}} v_k.$$
The subspace generated by $\{v_i\}_{i=1}^{r-1}$ and that generated by $\{w_i\}_{i=1}^{r-1}$ coincides. Thus, the set $\{v_i\}_{i=1}^{r-1}$, β, $\{w_i\}_{i=r+1}^{m-2}$ forms a basis of V^{ω^j} when $1 \le r \le m-2$, and the set $\{v_i\}_{i=1}^{m-2}$ forms a basis of V^{ω^j} when $r = m-1$. We have
$$\begin{aligned}
\tilde{\sigma}_*\beta &= \sum_{i=2}^{r} w_i - \frac{1}{r}\sum_{k=1}^{r} \frac{\omega^j\zeta^{-k}}{1-\omega^j\zeta^{-k}} v_k \\
&= \sum_{i=2}^{r} w_i + \frac{1}{r}\sum_{k=1}^{r} v_k - \frac{1}{r}\sum_{k=1}^{r} \frac{1}{1-\omega^j\zeta^{-k}} v_k \\
&= \sum_{i=1}^{r} w_i - \frac{1}{r}\sum_{k=1}^{r} \frac{1}{1-\omega^j\zeta^{-k}} v_k \\
&= \beta.
\end{aligned}$$
Note that β and $\{w_i\}_{i=r+1}^{m-2}$ are in the annihilator of the Hermitian form $\langle\ ,\ \rangle_{\tilde{\sigma}^k,\tilde{\sigma}}$ since they have eigenvalue 1 with respect to the action by $\tilde{\sigma}$.

By Lemma 4.2, we have
$$\begin{aligned}
\tau(\tilde{\sigma}^k, \tilde{\sigma}) &= \sum_{i=1}^{r} \operatorname{sign} \langle v_i, v_i\rangle_{\tilde{\sigma}^k,\tilde{\sigma}} \\
&= -\sum_{i=1}^{r} \operatorname{sign}\left((v_i \cdot v_i) \sum_{l=1}^{k} ((\omega^{-j}\zeta^i)^l - (\omega^j\zeta^{-i})^l) \right) \\
&= -\sum_{i=1}^{r} \operatorname{sign}\left((v_i \cdot v_i)(1-\omega^j\zeta^{-i})(1-\omega^{-j}\zeta^i) \sum_{l=1}^{k} ((\omega^{-j}\zeta^i)^l - (\omega^j\zeta^{-i})^l) \right).
\end{aligned}$$

By the equations
$$\begin{aligned}
&(1-\omega^j\zeta^{-i})(1-\omega^{-j}\zeta^i)\sum_{l=1}^{k}((\omega^{-j}\zeta^i)^l - (\omega^j\zeta^{-i})^l) \\
&= 8\boldsymbol{i}\sin\left(-\frac{\pi(k+1)j}{m} + \frac{\pi(k+1)i}{r}\right)\sin\left(-\frac{\pi kj}{m} + \frac{\pi ki}{r}\right)\sin\left(-\frac{\pi j}{m} + \frac{\pi i}{r}\right)
\end{aligned}$$
and Lemma 6.5, we have
$$\tau((\tilde{\sigma})^k, \tilde{\sigma}) = \sum_{i=1}^{r-1} \operatorname{sign}\left(\sin k\pi\left(\frac{i}{r} - \frac{j}{m}\right)\sin(k+1)\pi\left(\frac{i}{r} - \frac{j}{m}\right)\right).$$

Since rj/m is not an integer, $i/r - j/m$ is not zero. Thus, we obtain

$$\begin{aligned}\sum_{k=1}^{rm} \tau((\tilde{\sigma})^k, \tilde{\sigma}) &= \sum_{i=1}^{r-1} \sum_{k=1}^{rm} \operatorname{sign}\left(\sin k\pi\left(\frac{i}{r} - \frac{j}{m}\right) \sin(k+1)\pi\left(\frac{i}{r} - \frac{j}{m}\right)\right) \\ &= \sum_{i=1}^{r-1} (rm - 2|mi - rj|).\end{aligned}$$

Next, consider the case when rj/m is an integer and $1 \leq r \leq m-1$. Denote this integer rj/m by i_0. Then, the eigenvalue of v_{i_0} is 1, and v_{i_0} and $\{w_i\}_{i=r+1}^{m-2}$ are in the annihilator of $\langle\ ,\ \rangle_{\tilde{\sigma}^k, \tilde{\sigma}}$. If we put

$$\beta' = \sum_{i=1}^{r} w_i - \frac{1}{r} \sum_{\substack{1 \leq k \leq r \\ k \neq i_0}} \frac{1}{1 - \omega^j \zeta^{-k}} v_k,$$

the set of the homology classes $\{v_i\}_{i=1}^{r-1}$, β', $\{w_i\}_{i=r+1}^{m-2}$ forms a basis of V^{ω^j}. We have

$$\begin{aligned}\tilde{\sigma}\beta' &= \sum_{i=2}^{r} w_i - \frac{1}{r} \sum_{\substack{1 \leq k \leq r \\ k \neq i_0}} \frac{\omega^j \zeta^{-k}}{1 - \omega^j \zeta^{-k}} v_k \\ &= \sum_{i=2}^{r} w_i + \frac{1}{r} \sum_{\substack{1 \leq k \leq r \\ k \neq i_0}} v_k - \frac{1}{r} \sum_{\substack{1 \leq k \leq r \\ k \neq i_0}} \frac{1}{1 - \omega^j \zeta^{-k}} v_k \\ &= \sum_{i=1}^{r} w_i - \frac{1}{r} v_{i_0} - \frac{1}{r} \sum_{\substack{1 \leq k \leq r \\ k \neq i_0}} \frac{1}{1 - \omega^j \zeta^{-k}} v_k \\ &= \beta' - \frac{1}{r} v_{i_0}.\end{aligned}$$

By Lemma 4.2,

$$\langle \beta', \beta' \rangle_{\tilde{\sigma}^k, \tilde{\sigma}} = \beta' \cdot \frac{1}{r} \sum_{i=1}^{k} 2i v_{i_0} = \frac{k(k+1)}{r} \sum_{i=1}^{r} w_i \cdot v_{i_0}.$$

Since the eigenvalues of $\{v_i\}_{i=1}^{r-1}$ are different from 1, the intersection $v_i \cdot v_{i_0} = 0$ for $1 \leq i \leq r-1$. Since the subspace generated by $\{w_i\}_{i=1}^{r-1}$ and that generated by $\{v_i\}_{i=1}^{r-1}$ coincides, we also have $w_i \cdot v_{i_0} = 0$. Thus, we have

$$\begin{aligned}\frac{r}{k(k+1)} \langle \beta', \beta' \rangle_{\tilde{\sigma}^k, \tilde{\sigma}} &= w_r \cdot v_{i_0} \\ &= w_r \cdot (w_{r-1} + w_r') \\ &= w_r \cdot \left(w_{r-1} - \sum_{k=0}^{r-1} \omega^{(k-r)j} w_k\right) \\ &= (1 - \omega^{-j}) w_r \cdot w_{r-1} \\ &= (1 - \omega^{-j})(1 - \omega^j) > 0.\end{aligned}$$

Moreover, since $v_i \cdot v_{i_0} = 0$, Lemma 4.2 implies $\langle v_i, \beta' \rangle_{\tilde{\sigma}^k, \tilde{\sigma}} = 0$ for $1 \le i \le r-1$. Therefore, we have

$$\begin{aligned}
\sum_{k=1}^{rm} \tau(\tilde{\sigma}^k, \tilde{\sigma}) &= \sum_{k=1}^{rm} \left(\sum_{i=1}^{k} \operatorname{sign}(\langle v_i, v_i \rangle_{\tilde{\sigma}^k, \tilde{\sigma}}) + \operatorname{sign}(\langle \beta', \beta' \rangle_{\tilde{\sigma}^k, \tilde{\sigma}}) \right) \\
&= \sum_{k=1}^{rm} \left(\sum_{\substack{1 \le i \le r-1 \\ i \ne i_0}} \operatorname{sign} \left(\sin k\pi \left(\frac{i}{r} - \frac{j}{m} \right) \sin(k+1)\pi \left(\frac{i}{r} - \frac{j}{m} \right) \right) + 1 \right) \\
&= \sum_{\substack{1 \le i \le r-1 \\ i \ne i_0}} (rm - 2|mi - rj|) + rm \\
&= \sum_{i=1}^{r-1} (rm - 2|mi - rj|).
\end{aligned}$$

In the case when $r = m$, the set $\{v_i\}_{i=1}^{r-2}$ forms a basis of V^{ω^j}. By a similar calculation, we can also prove what we want. $\square$

Lemma 6.7. For $r = 2, 3, \ldots, m$,

$$\phi_{m,j}(\tilde{\sigma}) - \bar{\phi}_{m,j}(\tilde{\sigma}) = \frac{2}{r} \left\{ \left(\frac{rj}{m} - \left[\frac{rj}{m} \right] - \frac{1}{2} \right)^2 - \frac{r^2 j(m-j)}{m^2} - \frac{1}{4} \right\}.$$

Proof.

$$\tau(\tilde{\sigma}^k, \tilde{\sigma}) = \sum_{i=1}^{r-1} \operatorname{sign} \left(\sin k\pi \left(\frac{i}{r} - \frac{j}{m} \right) \sin(k+1)\pi \left(\frac{i}{r} - \frac{j}{m} \right) \right),$$

Since we have $\tau(\tilde{\sigma}^{k+rm}, \tilde{\sigma}) = \tau(\tilde{\sigma}^k, \tilde{\sigma})$,

$$\begin{aligned}
\phi_{m,j}(\tilde{\sigma}) - \bar{\phi}_{m,j}(\tilde{\sigma}) &= \frac{1}{rm} \sum_{k=1}^{rm} \tau(\tilde{\sigma}^k, \tilde{\sigma}) \\
&= \frac{1}{rm} \sum_{i=1}^{r-1} (rm - 2|mi - rj|) \\
&= r - 1 - \frac{2}{rm} \left(\sum_{i=1}^{[\frac{rj}{m}]} (rj - mi) + \sum_{[\frac{rj}{m}]+1}^{r-1} (mi - rj) \right) \\
&= \frac{2}{r} \left\{ \left(\frac{rj}{m} - \left[\frac{rj}{m} \right] - \frac{1}{2} \right)^2 + \frac{r^2 j(m-j)}{m^2} - \frac{1}{4} \right\}.
\end{aligned}$$

$\square$

Proof of Theorem 1.1. Applying Lemma 6.7 to the case when $r = m$, we have

$$\phi_{m,j}(\tilde{\sigma}_1 \cdots \tilde{\sigma}_{m-1}) - \bar{\phi}_{m,j}(\tilde{\sigma}_1 \cdots \tilde{\sigma}_{m-1}) = \frac{2j(m-j)}{m}.$$

Since

$$\bar{\phi}_{m,j}(\tilde{\sigma}_1 \cdots \tilde{\sigma}_{m-1}) = \frac{1}{m} \bar{\phi}_{m,j}((\tilde{\sigma}_1 \cdots \tilde{\sigma}_{m-1})^m) = 0,$$

we have

$$\phi_{m,j}(\tilde{\sigma}_1 \cdots \tilde{\sigma}_{m-1}) = \frac{2j(m-j)}{m}.$$

Put $\varphi = \tilde{\sigma}_1\tilde{\sigma}_3 \cdots \tilde{\sigma}_{m-1}, \psi = \tilde{\sigma}_2\tilde{\sigma}_4 \cdots \tilde{\sigma}_{m-2}$ when m is even, and $\varphi = \tilde{\sigma}_1\tilde{\sigma}_3 \cdots \tilde{\sigma}_{m-2}, \psi = \tilde{\sigma}_2\tilde{\sigma}_4 \cdots \tilde{\sigma}_{m-1}$, when m is odd. As we saw in Section 5, $\tilde{\sigma}_1 \cdots \tilde{\sigma}_{m-1}$ is conjugate to $\varphi\psi$. By direct computation, if $(\varphi_*^{-1} - I_{2g})x + (\psi_* - I_{2g})y = 0$ for $x, y \in V^{\omega^j}$, we have $(\varphi_*^{-1} - I_{2g})x = (\psi_* - I_{2g})y = 0$. Hence, we have $\tau_g(\varphi, \psi) = 0$.

In the same way, for $i = 1, 2, \ldots, [(m-1)/2]$, we have

$$\tau_g(\tilde{\sigma}_1\tilde{\sigma}_3 \cdots \tilde{\sigma}_{2i+1}, \tilde{\sigma}_2\tilde{\sigma}_4 \cdots \tilde{\sigma}_{2i}) = \tau_g(\tilde{\sigma}_1\tilde{\sigma}_3 \cdots \tilde{\sigma}_{2i+1}, \tilde{\sigma}_2\tilde{\sigma}_4 \cdots \tilde{\sigma}_{2i+2}) = 0,$$
$$\tau_g(\tilde{\sigma}_1\tilde{\sigma}_3 \cdots \tilde{\sigma}_{2i-1}, \tilde{\sigma}_{2i+1}) = \tau_g(\tilde{\sigma}_2\tilde{\sigma}_4 \cdots \tilde{\sigma}_{2i}, \tilde{\sigma}_{2i+2}) = 0.$$

Thus,

$$\phi_{m,j}(\tilde{\sigma}) = (r-1)\phi_{m,j}(\tilde{\sigma}_1) = \frac{r-1}{m-1}\phi_{m,j}(\tilde{\sigma}_1 \cdots \tilde{\sigma}_{m-1}) = \frac{2(r-1)j(m-j)}{m(m-1)}.$$

Hence, we obtain

$$\begin{aligned}
\bar{\phi}_{m,j}(\sigma_1 \cdots \sigma_{r-1}) &= \bar{\phi}_{m,j}(\tilde{\sigma}) \\
&= \phi_{m,j}(\tilde{\sigma}) - (\phi_{m,j}(\tilde{\sigma}) - \bar{\phi}_{m,j}(\tilde{\sigma})) \\
&= -\frac{2}{r}\left\{\frac{jr(m-j)(m-r)}{m^2(m-1)} + \left(\frac{rj}{m} - \left[\frac{rj}{m}\right] - \frac{1}{2}\right)^2 - \frac{1}{4}\right\}.
\end{aligned}$$

□

By the values of $\bar{\phi}_{m,1}$, we see:

Remark 6.8. Let r be an integer such that $2 \leq r \leq m$.

$$\bar{\phi}_{m,1}(\tilde{\sigma}_1 \cdots \tilde{\sigma}_{r-1}) = 0.$$

However, we do not know whether the quasimorphism $\bar{\phi}_{m,1}$ is trivial or not.

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