On the Maslov Index

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Dedicated to Henry McKean.

Abstract

In this paper we give four definitions of Maslov index and show that they all satisfy the same system of axioms and hence are equivalent to each other. Moreover, relationships of several symplectic and differential geometric, analytic, and topological invariants (including triple Maslov indices, eta invariants, spectral flow and signatures of quadratic forms) to the Maslov index are developed and formulae relating them are given. The broad presentation is designed with a view to applications both in geometry and in analysis. © 1994 John Wiley & Sons, Inc.

0. Introduction

The object of this paper is to give a systematic and unified treatment of the Maslov index and some related invariants. In the literature, the Maslov index has often been described as an integer invariant associated to any one of the following situations:

(0.1)

(i) *A smooth one-parameter family* $\{(L_1(t), L_2(t)) \colon a \leqq t \leqq b\}$ *of pairs of Lagrangian subspaces* $L_1(t)$, $L_2(t)$ *in a symplectic vector space* V.

(ii) *A triple* (L_1, L_2, L_3) *of Lagrangian subspaces.*

(iii) *A pair* $(\widetilde{L}_1, \widetilde{L}_2)$ *of elements in the universal covering* $\widetilde{\mathrm{Lag}}(V)$ *of the space* $\mathrm{Lag}(V)$ *of Lagrangian subspaces in a symplectic vector space* V.

Here all three will be considered and compared with each other in Sections 1 to 9. Following [5] and [12], we regard the setting (i) as the main theme, whereas the others are variations.

Let $(V, \{\,,\,\})$ be a fixed symplectic vector space with symplectic pairing $\{\,,\}$, and let $\mathrm{Lag}(V)$ be the space of Lagrangian subspaces in V. To a continuous and

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piecewise smooth path $f(t) = (L_1(t), L_2(t))$, $a \leqq t \leqq b$, in $\mathrm{Lag}(V) \times \mathrm{Lag}(V)$, there is an associated integer invariant

$$\text{(0.2)} \qquad \textit{the Maslov index} \quad \mu_V(f)\ .$$

In the literature, this has been defined only under the assumption that $f\colon [a,b] \to \mathrm{Lag}(V) \times \mathrm{Lag}(V)$ is a *proper* path (i.e., $L_1(t) \cap L_2(t) = 0$ for $t = a$ and $t = b$). In the present treatment, we will see that *this assumption is unnecessary*.

In Section 1, the Maslov index $\mu_V(f)$ will be *characterized axiomatically* by six simple properties. After some preliminaries in Sections 2 and 3, we will give in Sections 4 through 7, four equivalent and elementary definitions of $\mu_V(f)$. Precisely, it is this wealth of different perspectives and the work of unifying them which forms the main object of this paper.

The first two definitions of $\mu_V(f)$ are geometrical in nature, denoted respectively by $\mu_{\mathrm{geo},1}$, $\mu_{\mathrm{geo},2}$. The first $\mu_{\mathrm{geo},1}$ is given as the intersection number of an oriented line segment with a codimension one submanifold of $[a,b] \times \mathrm{Lag}(V)$. As for the second, $\mu_{\mathrm{geo},2}$ is defined in terms of a determinant line bundle on $\mathrm{Lag}(V)$. Underlying both constructions is the basic lemma (2.1), which allows us to eliminate the assumption of transversality $L_1(t) \cap L_2(t) = 0$ at the end points $t = a$, $t = b$.

The third and fourth definitions of $\mu_V(f)$ are analytical: $\mu_{\mathrm{anal},1}$ is defined in Section 6 via an eta invariant

$$\text{(0.3)} \qquad \eta(D(L_1, L_2))$$

of a self-adjoint operator $D(L_1, L_2)$ associated to two Lagrangians L_1, L_2; $\mu_{\mathrm{anal},2}$ is defined in Section 7 via the spectral flow of $\{D(L_1(t), L_2(t))\colon a \leqq t \leqq b\}$. The operator can be taken to be either $D(L_1, L_2)\colon L_1^2([0,1]; L_1, L_2) \to L^2([0,1])$ defined on the interval $[0,1]$ or $D^{\#}(L_1, L_2)\colon L_1^2(E) \to L^2(E)$ over the circle S^1 with coefficients in a flat, Hermitian bundle E; all these operators utilize a choice of complex structure J and Hermitian inner product $\langle\ ,\ \rangle$ on V compatible with $\{\ ,\ \}$. The definitions and basic analytical properties of these self-adjoint operators $D(L_1, L_2)$ and $D^{\#}(L_1, L_2)$ will be discussed in Section 3.

In Section 8, the Maslov triple index $\tau(L_1, L_2, L_3)$ is also characterized by a set of axioms. It is shown that the definition of the triple index by Kashiwara (see [24]) satisfies these axioms as does (-1) times the index defined by Wall (see [31]) in his study of Novikov additivity of signatures. Both of these generalize the usual treatment (see [18]) where $L_1 \cap L_3 = 0$. A geometrical interpretation of the Maslov triple index, following Wall, and of higher indices, e.g., Hörmander's fourfold index, is given in Section 12.

In his study of the Morse index in variational calculus, Duistermaat (see [11]) introduced an integer invariant associated to a single smooth path of Lagrangians. In Section 10, we express this invariant in terms of a Maslov index.

There are useful relations between the triple index $\tau(L_1, L_2, L_3)$, the eta invariant $\eta(D(L_1, L_2))$, and the Maslov index $\mu_V(f)$, as one may expect. For example, by checking the axioms for $\tau(L_1, L_2, L_3)$, the following formula is proved in Section 8.

THEOREM 0.1.

$$\tau(L_1, L_2, L_3) = \eta(D(L_1, L_2)) + \eta(D(L_2, L_3)) + \eta(D(L_3, L_1)) \,. \tag{0.4}$$

In view of the Atiyah-Patodi-Singer index theorem and the interpretation of $\eta(D(L_1, L_2))$ as the eta invariant of $D^{\#}(L_1, L_2)$: $\Gamma(E) \to \Gamma(E)$, the above formula allows us to further interpret $\tau(L_1, L_2, L_3)$ as the index of a first-order elliptic operator $\bar\partial$ coupled to a flat unitary vector bundle over the twice punctured disk. (For details, see Proposition 8.6.)

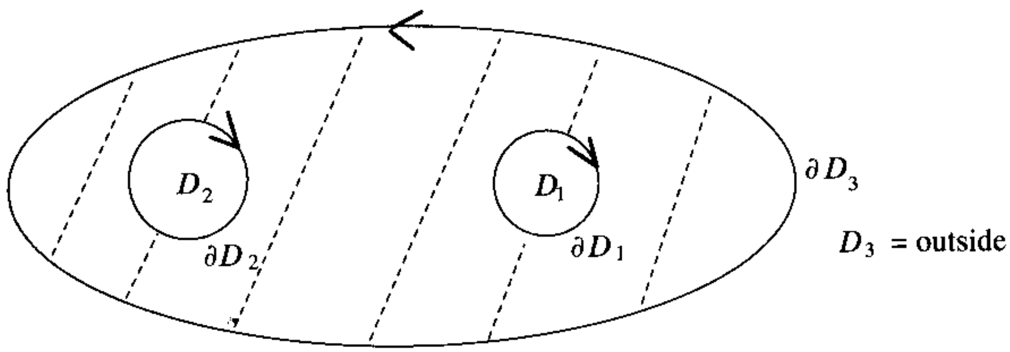

Diagram 1. Twice punctured disk in sphere S^2.

Similarly, $\mu_V(f)$ and $\tau(L_1, L_2, L_3)$ are related. Let $\{L_j(t); a \leqq t \leqq b\}$, for $j = 1, 2, 3$, be three continuous and piecewise smooth paths of Lagrangians in $\mathrm{Lag}(V)$. Set $e(t) = (L_1(t), L_2(t))$, $f(t) = (L_2(t), L_3(t))$, $g(t) = (L_3(t), L_1(t))$ and $h_{jk}(t) = \dim L_j(t) \cap L_k(t)$, then in Section 8 the following formula is proved:

THEOREM 0.2.

$$\begin{aligned} &\frac{1}{2}[\tau(L_1(b), L_2(b), L_3(b)) - \tau(L_1(a), L_2(a), L_3(a))] \\ &= \mu_V(e) + \mu_V(f) + \mu_V(g) + \frac{1}{2}\left\{\left[\sum_{j<k} h_{jk}(b)\right] - \left[\sum_{j<k} h_{jk}(a)\right]\right\} . \end{aligned} \tag{0.5}$$

The analytical definition $\mu_{\mathrm{anal},1}(f)$ in Section 6 of the Maslov index gives the following:

THEOREM 0.3.

$$\mu_V(f) = \int_a^b [L_2^*(\omega) - L_1^*(\omega)] + \frac{1}{2}[\eta(D(L_1(b), L_2(b)) - \eta(D(L_1(a), L_2(a))] + \frac{1}{2}[-h_{12}(b) + h_{12}(a)] \tag{0.6}$$

where ω is a 1-form on Lag(V), defined in Section 6. Comparing the three terms on the right-hand side of (0.6) with the corresponding terms in the Atiyah-Patodi-Singer index formula (see [3], p. 65), it is not surprising that $\mu_V(f)$ has an interpretation as a spectral flow of the real self-adjoint operators

$$\{D(L_1(t), L_2(t)) \colon a \leqq t \leqq b\} .$$

Indeed, in Section 7, we prove:

THEOREM 0.4.

$$\mu_V(f) = \varepsilon\text{-}\textit{spectral flow of } D(L_1(t), L_2(t)),\ a \leqq t \leqq b . \tag{0.7}$$

Here the phrase ε-spectral flow is due to the following convention. Since in our setting the solution space ker $D(L_1(t), L_2(t))$, isomorphic to $L_1(t) \cap L_2(t)$ (see Section 2), may be nonzero for $t = a$, $t = b$, the usual definition of spectral flow does not necessarily apply. Instead, to define a notion of spectral flow quite generally, we choose an $\varepsilon > 0$ such that at the end points $t = a$, $t = b$, no eigenvalues λ of $D(L_1(t), L_2(t))$ lie in the range $0 < \lambda \leqq \varepsilon$, and count the number of eigenvalues crossing $\lambda = +\varepsilon$ with signs (+1 for increasing and –1 for decreasing). This definition of ε-spectral flow agrees with the usual one when the latter is defined.

There are similar definitions of generalized spectral flows: for example, adapted specially to applications in gauge theory, one such definition can be found in the works of Floer and of Fintushel and Stern; see [13] and [14]. In Section 7, we will explain their approach and compare it with our definition of ε-spectral flow.

As indicated in (0.1) (iii), another common usage of the term "Maslov index" is the index $m(\alpha, \beta)$ for a pair of elements α, β in the universal covering space $\pi\colon \widetilde{\text{Lag}}(V) \to \text{Lag}(V)$ of Lag(V). This index $m(\alpha, \beta)$ is defined in Section 9. It can be expressed in terms of $\mu_V(f)$: Choose a smooth path in $\widetilde{\text{Lag}}(V)$ from α to β, and denote by $L_1(t)$, $a \leqq t \leqq b$, its projection in Lag(V). Then $f(t) = (L_1(b), L(t))$, $a \leqq t \leqq b$, gives a path of pairs of Lagrangians and from this a well-defined invariant $\mu_V(f)$ of (α, β). In Section 9, we show that

THEOREM 0.5.

$$m(\alpha, \beta) = 2\mu_V(f) + (n - h) \tag{0.7}$$

where $n = \dim V/2$ *and* $h = \dim \pi(\alpha) \cap \pi(\beta)$.

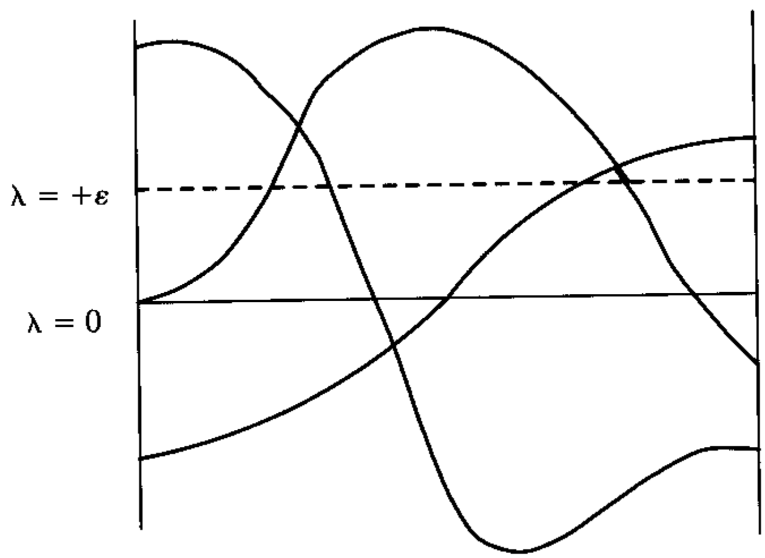

Diagram 2. The spectral lines of $D(L_1(t), L_2(t))$.

An important feature in our treatment is the choice of a complex structure J together with a Hermitian inner product $\langle\ ,\ \rangle$ on (V, J) such that

$$\{v, w\} = (-1)\ \mathrm{Im}\langle v, w\rangle \text{ for } v, w \text{ in } V\ . \tag{0.8}$$

From [18] it is known that such a complex structure J and Hermitian pairing $\langle\ ,\ \rangle$ always exists and in fact the space of all these structures $(J, \langle\ ,\ \rangle)$ forms a cell.

Important to our applications is the *geometrical observation* that, once $(J, \langle,\rangle)$ is chosen, any two Lagrangians L_1 and L_2 can be deformed to a transverse position in an almost canonical manner. (For details, see Section 2.) Another important observation from the *analytical* side is the assignment to every pair (L_1, L_2) of a real self-adjoint operator $D(L_1, L_2)$ whose kernel is $L_1 \cap L_2$ (see Section 3).

Among other things, this last step provides us with a link between the spaces of Lagrangians and self-adjoint operators. A map expressing such a relation between these spaces can be seen by combining Bott periodicity (see [4]) with the theory of skew-adjoint operators in [2]. Consider the symplectic space $\mathbb{R}^{2n}$ as embedded in $\mathbb{R}^{2n+2} = \mathbb{R}^{2n} \oplus \mathbb{R}\, e_{n+1} \oplus \mathbb{R}\, f_{n+1}$ by taking the orthogonal sum with the hyperbolic plane $\mathbb{R}e_{n+1} \oplus \mathbb{R}f_{n+1}$. This induces an inclusion $\mathrm{Lag}(\mathbb{R}^n) \hookrightarrow \mathrm{Lag}(\mathbb{R}^{n+2})$ by sending a Lagrangian subspace L in $\mathrm{Lag}(\mathbb{R}^n)$ to the direct sum $L \oplus \mathbb{R}e_{n+1}$. As $n \to \infty$, we obtain $\mathrm{Lag}(\mathbb{R}^\infty) = \cup_n \mathrm{Lag}(\mathbb{R}^{2n})$ which has the same homotopy type as U/O because $\mathrm{Lag}(\mathbb{R}^{2n}) = U_{2n}/O_{2n}$. On the other hand, let $\mathbb{H}$ denote a real Hilbert space and SelfAdj($\mathbb{H}$) denote the topological space of real self-adjoint Fredholm operators $T\colon \mathbb{H} \to \mathbb{H}$, $T = T^*$. Then, as shown in [2], this last space SelfAdj($\mathbb{H}$) has the same homotopy type as $\Omega^7(BO)$. By Bott periodicity, $\Omega^7(BO) \sim U/O$, and therefore the two spaces Lag($\mathbb{R}^\infty$) and SelfAdj($\mathbb{H}$) are of the same homotopy type.

One further perspective on the Maslov index $\mu_V(f)$ which is not treated here is its interpretation as the index of the Cauchy-Riemann operator $\bar{\partial} \otimes V$ with coefficients in V and over a lens-shaped disk. On the two sides of the disk, the two families of Lagrangians $\{L_1(t);\, a \leqq t \leqq b\}$ and $\{L_2(t);\, a \leqq t \leqq b\}$ determine the boundary conditions. Of course, this is in agreement with (0.7) where $\mu_V(f)$ is viewed as a spectral flow. The reader may refer to Floer's paper, [14], for this perspective and its generalization as the Viterbo index; see also the recent paper of Otsuki and Furutani, [26].

Following Floer, and Salamon and Zehnder (see [27]), in Section 11 we consider a real, first-order elliptic operator on $\mathbb{R}^{2n}$ valued functions on a cylinder defined using a path of symplectic matrices. There, we recapitulate (and slightly extend) a recent result of Salamon and Zehnder expressing the index of such operators in terms of Maslov indices. Section 12 relates the Maslov indices to Wall's work on Novikov additivity and Hörmander's fourfold index. Section 13 interprets the Maslov index of a path, following Floer and Walker, as that of an associated loop. Further, there are natural generalizations of the Maslov index to certain infinite dimensional settings, e.g., in works of Mrowka, Nicolaesco, [34], and of the authors, [8].

Of course, it is not possible to list here all the varied contexts in which the various Maslov indices have arisen. There are certainly early precursers, e.g., in the context of phase change in crossing caustics, as notably in the work of J. B. Keller; see [20] and [21].

During our research, we have benefitted from a wealth of excellent books and research articles of various authors on symplectic geometry and the Maslov index: Arnold, [1], Bott, [5], Duistermaat, [11], Edwards, [12], Gossen, [17], Guillemin-Sternberg, [18], Hörmander, [19], Leray, [22] and [23], Lions-Vergne, [24], Maslov, [25], Salamon-Zehnder, [27], Turaev, [28], Viterbo, [29], Wall, [31], and Weinstein, [32]. Throughout this paper, we refer quite frequently to their works. (In particular, [18] will be used as a standard reference.) It was, however, Wall's paper, [31], on the nonadditivity of signature, extending the Novikov additivity principle, which first introduced us to this beautiful subject. Cheeger's work in [10] illuminated the role of Lagrangian boundary conditions in index theory. The works of Floer, [14], [15], and [16], led to the development of connections between Maslov index and spectral flow. Such connections are made in the paper of Yoshida, [33], and in the papers of the present authors, [6], [7], [8], and [9]. The work of Atiyah, Patodi, and Singer, [3], has influenced us throughout.

The following section titles outline the topics covered in the paper: Section 1, The Maslov Index $\mu_V(f)$; Section 2, Preliminary and Geometrical Observations; Section 3, Analytical Observations; Section 4, First Geometrical Definition of the Maslov Index: $\mu_{\text{geo},1}(f)$; Section 5, Second Geometrical Definition of the Maslov Index: $\mu_{\text{geo},2}(f)$; Section 6, First Analytical Definition of the Maslov Index:

$\mu_{\mathrm{anal},1}(f)$; Section 7, Second Analytical Definition of the Maslov Index: $\mu_{\mathrm{anal},2}(f)$; Section 8, The Maslov Triple Index: $\tau(L_1, L_2, L_3)$; Section 9, The Maslov Index $m(x, y)$ of a Pair in $\widetilde{\mathrm{Lag}}(V)$; Section 10, The Duistermaat Index of a Path of Lagrangians; Section 11, The Maslov Index of a Path of Symplectic Matrices and a Theorem of Salamon and Zehnder; Section 12, A Geometric Interpretation of the Maslov Triple Index (Following Wall) and Hörmander's Fourfold Index; Section 13, $\mu_V(f)$ as the Maslov Index of Closed Loops à la Floer and Walker.

1. The Maslov Index $\mu_V(f)$

Let $(V, \{\ ,\})$ be a fixed symplectic vector space. For $a < b$, we denote by $P([a, b]; V)$ the space of continuous and piecewise smooth maps

$$f\colon [a, b] \to \{\text{pairs of Lagrangian subspaces in } V\}$$

That is, for each t, $a \leqq t \leqq b$, we have an ordered pair of Lagrangians $f(t) = (L_1(t), L_2(t))$, and as t varies the Lagrangians $L_1(t), L_2(t)$ varies continuously and piecewise smoothly. As for the topology on $P([a, b]; V)$, it is given by the usual compact open topology.

The Maslov index can be characterized as an integer-valued map

$$\mu_V\colon P([a, b]; V) \to \mathbb{Z} \tag{1.1}$$

of $P([a, b]; V)$ which satisfies the following:

Property I (Affine Scale Invariance). For $k > 0$, $\ell \geqq 0$, we have the affine map $\psi\colon [a, b] \to [ka + \ell, kb + \ell]$ defined by $\psi(t) = kt + \ell$. Given a path f in $P([ka + \ell, kb + \ell]; V)$ we denote by $f \circ \psi$ the composite path in $P([a, b]; V)$. Then

$$\mu_V(f) = \mu_V(f \circ \psi)\ . \tag{1.2}$$

Property II (Deformation Invariance rel. End Points). If $f(s)(t) = (L_1(s, t), L_2(s, t))$, $0 \leqq s \leqq 1$, $a \leqq t \leqq b$, defines a continuous map $f(s)\colon [0, 1] \to P([a, b]; V)$ with the endpoints $(L_1(s, a), L_2(s, a))$ and $(L_1(s, b), L_2(s, b))$ fixed (independent of s), then

$$\mu_V(f(0)) = \mu_V(f(1))\ . \tag{1.3}$$

Property III (Path Additivity). If $a < b < c$ and if $f(t) = (L_1(t), L_2(t))$, $a \leqq t \leqq c$, is a continuous and piecewise smooth path in $P([a, c]; V)$, then

$$\mu_V(f) = \mu_V(f\,|\,[a, b]) + \mu_V(f\,|\,[b, c])\ . \tag{1.4}$$

Property IV (Symplectic Additivity). Let V and W be symplectic vector spaces, and let f and g denote respectively elements in $P([a,b];V)$ and $P([a,b];W)$. Define the path $f \oplus g$ in $P([a,b];V \oplus W)$ by

$$(f \oplus g)(t) = (L_1(t) \oplus \hat{L}_1(t)\ ,\ L_2(t) \oplus \hat{L}_2(t))\ , \qquad a \leqq t \leqq b$$

where $L_1(t)$, $L_2(t)$ are the components of $f(t)$ and $\hat{L}_1(t)$, $\hat{L}_2(t)$ are those of $g(t)$. Then

$$\text{(1.5)} \qquad \mu_{V\oplus W}(f \oplus g) = \mu_V(f) + \mu_W(g)\ .$$

Property V (Symplectic Invariance). Let $f(t) = (L_1(t), L_2(t))$, $a \leqq t \leqq b$, be an element in $P([a,b],V)$ and let $\phi_t \colon V \to V$, $a \leqq t \leqq b$, be a continuous and piecewise smooth family of symplectic automorphisms in $\mathrm{Sp}(V)$. Define $\phi_* f$ in $P([a,b];V)$ by the formula

$$(\phi_* f)(t) = (\phi_t(L_1(t))\ ,\ \phi_t(L_2(t)))\ , \qquad a \leqq t \leqq b\ .$$

Then

$$\text{(1.6)} \qquad \mu_V(\phi_* f) = \mu_V(f)\ .$$

Property VI (Normalization). Let $\mathbb{C} = \mathbb{R}^2$ be the symplectic vector space with the inner product

$$\begin{aligned}\{(x_1,y_1),(x_2,y_2)\} &= x_1y_2 - y_1x_2 = -\mathrm{Im}(x_1 + iy_1)\overline{(x_2+iy_2)} \\ &= \mathrm{Re}\left(i(x_1+iy_1)\cdot\overline{(x_2+iy_2)}\right)\ .\end{aligned}$$

Define the path of Lagrangians $f(t)$ in $P([-\pi/4,\pi/4];\mathbb{R}^2)$ by the formula

$$f(t) = (\mathbb{R}\{1\}, \mathbb{R}\{e^{it}\})\ , \qquad -\pi/4 \leqq t \leqq \pi/4\ .$$

Then

$$\text{(1.7)} \qquad \begin{aligned}&\text{(i)} && \mu_{\mathbb{R}^2}(f\,|\,[-\pi/4,\pi/4]) = 1\ ;\\ &\text{(ii)} && \mu_{\mathbb{R}^2}(f\,|\,[-\pi/4,0]) = 0\ ;\\ &\text{(iii)} && \mu_{\mathbb{R}^2}(f\,|\,[0,\pi/4]) = 1\ .\end{aligned}$$

The situation in (1.7) can be interpreted diagramatically as in the following figure. First, the space $\mathrm{Lag}(\mathbb{R}^2)$ can be viewed as a circle $\{e^{2i\theta}\,|\,0 \leqq \theta \leqq \pi\}$ via the identification, $e^{i\theta} \leftrightarrow \mathbb{R}\{e^{i\theta}\}$, and likewise $\mathrm{Lag}(\mathbb{R}^2)\times[-\pi/4,\pi/4]$ as a cylinder. Inside this cylinder, the graphs of $\mathbb{R}\{1\}$ and $\mathbb{R}\{e^{i\theta}\}$ can be represented respectively by a straight line and a spiral. As these curves intersect transversely at $t = 0$, formula (1.7)(i) agrees with the interpretation of counting the intersection number of the two Lagrangian paths.

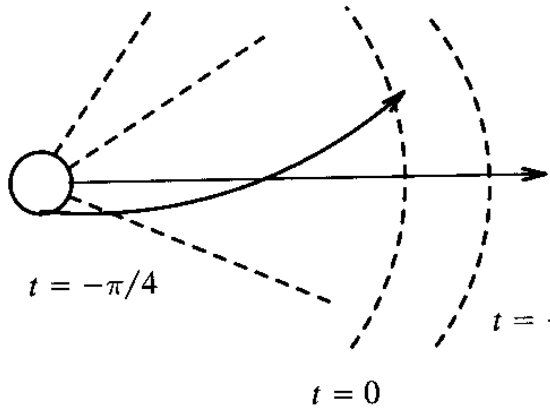

Diagram 3. The cylinder $\mathrm{Lag}(\mathbb{R}^2) \times [-\pi/4, \pi/4]$.

THEOREM 1.1.

(i) *There exists a collection of functions* μ_V: $P([a,b];V) \to \mathbb{Z}$ *satisfying properties* I–VI *above.*

(ii) *Maslov index* $\mu_V(f)$ *is uniquely characterized by Properties* I–VI *above.*

(iii) *If a collection of functions* $\hat{\mu}_V$: $P([a,b];V) \to \mathbb{Z}$ *satisfies properties* I–V, *then*

$$\hat{\mu}_V(f) = (x+y)\,\mu_V(f) + y[h_{12}(b) - h_{12}(a)]$$

where x *and* y *are some fixed integers and* $h_{12}(t) = \dim L_1(t) \cap L_2(t)$ *with* $f(t) = (L_1(t), L_2(t))$.

(iv) *The Maslov index* μ_V *satisfying* I–VI *also enjoys the additional properties* VII–XII *listed below.*

Theorem 1.1 will be proved at the end of Section 4. Four different ways of defining Maslov index: $\mu_{\mathrm{geo},1}(f)$, $\mu_{\mathrm{geo},2}(f)$, $\mu_{\mathrm{anal},1}(f)$, $\mu_{\mathrm{anal},2}(f)$ satisfying I–VI will be given in Sections 4–7. From the uniqueness statement (1.1)(ii), all these four are the same. By (1.1)(iv), each of them satisfies the additional properties VII–XII below. Some of these properties cannot be deduced easily from a particular definition. For instance, the Nullity Property can be easily seen from $\mu_{\mathrm{geo},1}(f)$; however it is not apparent from $\mu_{\mathrm{geo},2}(f)$.

Here are the additional properties of Maslov index:

Property VII (Normalization). Let $g(t)$ be the path of Lagrangian pairs in $P([-\pi/4, \pi/4]; \mathbb{R}^2)$ defined by

$$g(t) = (\mathbb{R}\{1\}, \mathbb{R}\{e^{-it}\}), \qquad -\pi/4 \leqq t \leqq \pi/4 .$$

Then

$$(1.8) \qquad \begin{aligned} &\text{(i)} \quad \mu_V(g) = -1 \; ; \\ &\text{(ii)} \quad \mu_V(g|[-\pi/4, 0]) = -1 \; ; \\ &\text{(iii)} \quad \mu_V(g|[0, \pi/4]) = 0 \; . \end{aligned}$$

Property VIII (Nullity). For a path of Lagrangian pairs

$$f(t) = (L_1(t), L_2(t)) \; , \quad a \leqq t \leqq b \; , \quad \text{in } P([a,b]; V)$$

with

$$h_{12}(t) = \dim L_1(t) \cap L_2(t)$$

independent of t, and $L_1(t) \cap L_2(t)$ varying continuously, then

$$(1.9) \qquad \mu_V(f) = 0 \; .$$

Property IX (Reparametrization Invariance). Let ψ: $[c,d] \to [a,b]$ be a continuous and piecewise smooth function with $\psi(c) = a$, $\psi(d) = b$, $a < b$, $c < d$. For $f \in P([a,b]; V)$, we denote by $f \circ \psi$ the composite Lagrangian path in $P([c,d]; V)$. Then

$$(1.10) \qquad \mu_V(f \circ \psi) = \mu_V(f) \; .$$

Property X (Reversal). Let $f(t) = (L_1(t), L_2(t))$, $a \leqq t \leqq b$ be a path of Lagrangian pairs in $P([a,b]; V)$. Denote the same path traveled in the reverse direction in $P([-b,-a]; V)$ by

$$\hat{f}(s) = (L_1(-s), L_2(-s)) \; .$$

Then

$$(1.11) \qquad \mu_V(\hat{f}) = -\mu_V(f) \; .$$

Property XI (Symmetry). Let $f(t) = (L_1(t), L_2(t))$, $a \leqq t \leqq b$ be a path of Lagrangian pairs in $P([a,b]; V)$. Denote by $f_{\text{flip}}(t) = (L_2(t), L_1(t))$, $a \leqq t \leqq b$ the path obtained by flipping the two components. Then

$$(1.12) \qquad \mu_V(f_{\text{flip}}) = -\mu_V(f) + [h_{12}(a) - h_{12}(b)]$$

where $h_{12}(t) = \dim L_1(t) \cap L_2(t)$.

Property XII (Proper Path). Let $f(t) = (L_1(t), L_2(t))$, $a \leqq t \leqq b$, be a proper path in $P([a,b]; V)$. That is, it satisfies the transversality condition at the endpoints: $L_1(a) \cap L_2(a) = L_1(b) \cap L_2(b) = 0$. Then $\mu_V(f)$ equals the Maslov index of the proper path f as defined by Guillemin and Sternberg in [18].

The symmetry property XI may seem strange, especially in view of the correction terms in (1.12). As an illustration, we consider the multiplication of the path $f(t) = (\mathbb{R}\{1\}, \mathbb{R}\{e^{it}\})$ by e^{-it}, $-\pi/4 \leqq t \leqq \pi/4$ to get $\hat{g}(t) = (\mathbb{R}\{e^{-it}\}, \mathbb{R}\{1\})$. Flipping the two components of $\hat{g}$, it becomes the path $g(t) = (\mathbb{R}\{1\}, \mathbb{R}\{e^{-it}\})$ of VIII. By V and XI, we have

$$\begin{aligned} \mu_{\mathbb{R}^2}(g|[-\pi/4,0]) &= -\mu_{\mathbb{R}^2}(\hat{g}|[-\pi/4,0]) + (0-1) \\ &= -\mu_{\mathbb{R}^2}(f|[-\pi/4,0]) - 1 \\ &= -1 \end{aligned} \tag{1.13}$$

which is in agreement with VII. Similarly we have

$$\begin{aligned} \mu_{\mathbb{R}^2}(g|[0,\pi/4]) &= -\mu_{\mathbb{R}^2}(\hat{g}|[0,\pi/4]) + (1-0) \\ &= -\mu_{\mathbb{R}^2}(f|[0,\pi/4]) + 1 \\ &= 0 \end{aligned} \tag{1.14}$$

which is in agreement with VII. These formulae will be obvious once the definition of $\mu_{\text{geo},1}(f)$ is given in Section 4.

2. Preliminary and Geometrical Observations

Throughout this section, we fix a complex structure J on V, $J^2 = -\text{id}$, which preserves the symplectic inner product $\{\ ,\ \}$. Associated to this situation, there is a Hermitian inner product $\langle\ ,\ \rangle$ on V with

$$\begin{aligned} \{v,w\} &= -\text{Im}\langle v,w\rangle\ , \text{ or equivalently} \\ \langle v,w\rangle &= -\{Jv,w\} - i\{v,w\}\ . \end{aligned} \tag{2.1}$$

By definition, a Lagrangian subspace L in V is a real vector subspace of dimension $n = (\dim V)/2$ with the additional property

$$\{v,w\} = 0 \quad \text{for all} \quad v,w \text{ in } L\ .$$

In particular, for an orthonormal basis $\{e_j\}_{1 \leqq j \leqq n}$ for L under the real inner product

$$(v,w) = \text{Re}\langle v,w\rangle = -\{Jv,w\} \tag{2.2}$$

then $\{e_j\}_{1\leqq j \leqq n}$ is also an orthonormal basis of the complex vector space $\{V,J\}$ with respect to the Hermitian inner product $\langle\ ,\ \rangle$. Thus, in general, any Lagrangian subspace L' in V is the real span

$$L' = \bigoplus_{j=1}^{n} \mathbb{R}\{e'_j\} \tag{2.3}$$

of some orthonormal basis $\{e'_j\}_{1\leqq j\leqq n}$ of the complex vector space $\{V,J\}$.

There is a natural mapping

$$\phi\colon U(n) \to \operatorname{Lag} V \tag{2.4}$$

sending A in $U(n)$ to the real span $\oplus_{j=1}^{n}\mathbb{R}\{A\cdot e_j\}$. From the discussion in the previous paragraph, it is clear that ϕ is surjective and in fact gives rise to a bijection

$$\phi(L)\colon U(n)/O(n) \xrightarrow{\cong} \operatorname{Lag}(V) \tag{2.5}$$

which depends only on L (not on the choice of basis $\{e_j\}_{1\leqq j\leqq n}$) and the choice of complex structure J. In addition, there is an imbedding $j\colon U(n)/O(n) \to U(n)$ defined by $j(A \bmod O(n)) = A\cdot {}^tA$, and combining with (2.5) we have an identification

$$\operatorname{Lag}(V) \overset{\phi}{\leftarrow} U(n)/O(n) \overset{j}{\hookrightarrow} U(n) \tag{2.6}$$

of $\operatorname{Lag}(V)$ with a subspace in $U(n)$.

Having chosen the complex structure J, the basic geometrical observation is that any two Lagrangians L_1 and L_2 may be perturbed (almost canonically) into a transverse position.

LEMMA 2.1. (BASIC GEOMETRIC OBSERVATION) *Let L_1 and L_2 be two Lagrangian subspaces in V. Then*

(i) *$e^{J\theta}\cdot L_2$ is a Lagrangian subspace of V for all θ;*

(ii) *There exists an ε, $0<\varepsilon<\pi$, such that $L_1\cap e^{J\theta}\cdot L_2=\{0\}$ for all θ with $0<|\theta|<\varepsilon$. A similar lemma with restriction on L_1 and L_2 can be found in [1].*

Proof of Lemma 2.1: First of all, in view of the discussion in (2.3), the first statement (i) is clear. In order to prove (ii), let $a=\dim_{\mathbb{R}}(L_1\cap L_2)$ and let $\{e_j\}_{1\leqq j\leqq n}$ be an orthonormal basis of L_1 such that the first a of the $\{e_j\}_{1\leqq j\leqq n}$ forms a basis of $L_1\cap L_2$.

Let U and W be the complex subspaces in V generated respectively by $\{e_j\}_{1\leqq j\leqq a}$ and $\{e_j\}_{a+1\leqq j\leqq n}$, i.e.

$$U=\bigoplus_{j=1}^{a}\mathbb{C}\{e_j\}\,,\qquad W=\bigoplus_{j=a+1}^{n}\mathbb{C}\{e_j\}\,. \tag{2.7}$$

Then we have

$$V=U\oplus W \quad \text{with} \quad L_1\cap L_2\subset U\,. \tag{2.8}$$

By dimension count, the real span of $\{e_j\}_{1\leqq j\leqq a}$ is $L_1\cap L_2$ and the real span of $\{e_j\}_{a+1\leqq j\leqq n}$ is $L_1\cap W$. Thus we have $L_1=(L_1\cap L_2)\oplus(L_1\cap W)$.

In a similar manner, we choose a basis $\{e'_j\}_{1 \leqq j \leqq n}$ for L_2 orthonormal with respect to the product $(\, , \,) = \mathrm{Re}\langle \, , \, \rangle$ such that $e'_j = e_j$, $1 \leqq j \leqq a$. Let W' be the complex subspace in V generated by $\{e'_j\}_{a+1 \leqq j \leqq n}$, i.e., $W' = \oplus_{j=a+1}^{n} \mathbb{C}\{e'_j\}$. Then we have the following orthogonal sum decomposition:

$$V = U \oplus W' \,, \qquad L_2 = (L_1 \cap L_2) \oplus (L_2 \cap W') \,.$$

Since the inner product $(\, , \,) = \mathrm{Re}\langle \, , \, \rangle$ is nondegenerate on U, it has a unique orthogonal complement and in particular $W = W'$. Thus we have two orthogonal sum decompositions:

$$\begin{aligned} L_1 &= (L_1 \cap L_2) \oplus (L_1 \cap W) \quad \text{in} \quad U \oplus W = V \\ L_2 &= (L_1 \cap L_2) \oplus (L_2 \cap W) \quad \text{in} \quad U \oplus W = V \,. \end{aligned} \tag{2.9}$$

We now turn to the proof of (2.1)(ii). Since $\{e_j\}_{1 \leqq j \leqq a}$ is a real basis for $L_1 \cap L_2$ and a complex basis for U, the multiplication by $e^{J\theta}$, $0 < |\theta| < \pi$, on $\{e_j\}_{1 \leqq j \leqq a}$ gives us $\mathbb{R}$-linearly independent elements $\{e^{J\theta} \cdot e_j\}_{1 \leqq j \leqq a}$. In other words, we have

$$(L_1 \cap L_2) \cap e^{J\theta} \cdot (L_1 \cap L_2) = 0 \tag{2.10}$$

for $0 < |\theta| < \pi$. On the other hand, we have

$$(L_1 \cap W) \cap (L_2 \cap W) = (L_1 \cap L_2) \cap W \subset U \cap W = (0) \,,$$

and so the subspaces $L_1 \cap W$ and $L_2 \cap W$ are transverse to each other. Since transversality is an open condition, there exists $\varepsilon > 0$ such that

$$(L_1 \cap W) \cap e^{J\theta} \cdot (L_2 \cap W) = 0 \quad \text{for all} \quad |\theta| < \varepsilon \,. \tag{2.11}$$

Combining (2.9), (2.10), (2.11), we have for this ε, $0 < \varepsilon < \pi$, the transversality condition

$$L_1 \cap e^{J\theta} \cdot L_2 = \left[(L_1 \cap L_2) \cap e^{J\theta}(L_1 \cap L_2) \oplus (L_1 \cap W) \cap e^{J\theta}(L_2 \cap W) \right] = 0$$

whenever $0 < |\theta| < \varepsilon$. This completes the proof.

3. Analytical Observations

As in (2.1), we fix a complex structure J and Hermitian inner product $\langle \, , \, \rangle$ throughout. The basic analytical observation is that, associated to a pair of Lagrangians L_1 and L_2, there are two natural, self-adjoint operators $D(L_1, L_2)$ and $D^\sharp(L_1, L_2)$. Here the first is a real operator and the second complex; however

$$\dim_{\mathbb{R}} \ker D(L_1, L_2) = \dim_{\mathbb{C}} \ker D^\sharp(L_1, L_2) = \dim_{\mathbb{R}}(L_1 \cap L_2) \,. \tag{3.1}$$

The first operator $D(L_1, L_2)$ is implicit in the work of Cheeger (see [10]) and Floer (see [14], [15], and [16]) and is discussed explicitly by Yoshida in [33]. Defined as a real operator on the space of differentiable, vector valued functions ϕ: $[0,1] \to V$ with boundary condition

$$\phi(0) \in L_1 \text{ and } \phi(1) \in L_2\,, \tag{3.2}$$

it is given by the formula

$$D(L_1, L_2)\phi = -J\frac{d\phi}{dt}\,. \tag{3.3}$$

To define the operator $D^\#(L_1, L_2)$, we observe that, as in Section 2, there exists a unitary matrix A in $U(n)$ with $L_2 = A \cdot L_1$. Using $A \cdot A^t$, we have the space of functions ψ: $[0,1] \to \mathbb{C}^n$ satisfying the boundary condition

$$\psi(1) = A \cdot A^t \cdot \psi(0) \tag{3.4}$$

The operator $D^\#(L_1, L_2)$ is an operator on this complex vector space defined by

$$D^\#(L_1, L_2)\psi = -i\frac{d\psi}{dt}$$

Note that, in the discussion of (2.6), the matrix $A \cdot A^t$ depends only on the pair (L_1, L_2), not the choice of A. Another explanation of $A \cdot A^t$ is that it is the clutching function of the flat, Hermitian vector bundle $\xi \to S^1$,

$$\xi = [0,1] \times \mathbb{C}^n /\ \text{modulo the relation } (0,x) \sim (1, A \cdot A^t \cdot x) \tag{3.5}$$

defined over the circle $S^1 = [0,1]/\text{modulo } 0 \sim 1$. The operator $D^\#(L_1, L_2)$ may be regarded as $-i\,d/dt$ acting on the space of smooth sections of ξ.

LEMMA 3.1. (BASIC ANALYTICAL OBSERVATION)

(i) *Let $L^2_1([0,1]; L_1, L_2)$ denote the Sobolev completion of the smooth functions ϕ: $[0,1] \to V$ satisfying the boundary condition* (3.2). *Here the Sobolev norm is defined by $\int_0^1 \sqrt{\mathrm{Re}\langle \phi, \phi\rangle + \mathrm{Re}\langle d\phi/dt, d\phi/dt\rangle}\,dt$. Let $L^2([0,1]; V)$ denote the L^2-completion of the smooth functions ϕ: $[0,1] \to V$. Then $-J\,d/dt$ defines a real, self-adjoint operator*

$$D(L_1, L_2)\colon L^2_1([0,1]; L_1, L_2) \to L^2_1([0,1]; V) \tag{3.6}$$

which is Fredholm of unbounded, pure spectrum without limit points.

(ii) *The kernel of $D(L_1, L_2)$ coincides with the space of constant functions $f\colon [0,1] \to L_1 \cap L_2$ and in particular is isomorphic to $L_1 \cap L_2$.*

(iii) *Let* $L_1^2([0,1]; A\cdot A^t)$ *denote the completion of the complex vector space of functions* ψ: $[0,1] \to \mathbb{C}^n$ *satisfying* (3.4). *Define on* $L_1^2([0,1]; A\cdot A^t)$ *and* $L^2([0,1]; V)$, *the Hermitian inner products:*

$$\langle \psi_1, \psi_2 \rangle_{L_1^2} = \int_0^1 \langle \psi_1, \psi_2 \rangle + \left\langle \frac{d\psi_1}{dt}, \frac{d\psi_2}{dt} \right\rangle d\tau$$

$$\langle \phi_1, \phi_2 \rangle_{L^2} = \int_0^1 \langle \phi_1, \phi_2 \rangle \, dt$$

Then $-i\, d/dt$ *defines a complex, self-adjoint operator*

(3.7) $$D^\#(L_1, L_2) : L_1^2([0,1]; A\cdot A^t) \to L^2([0,1]; \mathbb{C}^n)$$

which is Fredholm of unbounded, pure, point spectrum without limit points.

(iv) *The kernel of* $D^\#(L_1, L_2)$ *coincides with the space of constant functions* ψ: $[0,1] \to (\mathbb{R}^n \cap A\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$ *and in particular is isomorphic to* $(L_1 \cap L_2) \otimes \mathbb{C}$.

The above two operators $D(L_1, L_2)$, $D^\#(L_1, L_2)$ are closely related; one aspect is the dimension formula (3.1) and the other, as will be explained in (6.3), is that the eta invariant of these two operators are the same

(3.8) $$\eta(D(L_1, L_2)) = \eta(D^\#(L_1, L_2)) \, .$$

Thus, it is immaterial whether we use $\eta(D(L_1, L_2))$ or $\eta(D^\#(L_1, L_2))$ to provide the analytical definition of Maslov index $\mu_{\text{anal},j}(t)$, $j = 1, 2$. For simplicity $\eta(D(L_1, L_2))$ is used throughout; only in Section 8 do we shift to $\eta(D^\#(L_1, L_2))$.

Proof of Lemma 3.1: Given two functions f, g: $[0,1] \to V$ satisfying (3.2), we have

$$\begin{aligned}(D(L_1, L_2)f, g) - (f, D(L_1, L_2)g) &= \int_0^1 \operatorname{Re}\left(\left\langle -J\frac{df}{dt}, g \right\rangle - \left\langle f, -J\frac{dg}{dt} \right\rangle \right) dt \\ &= -\operatorname{Re} \int_0^1 i\frac{d}{dt}\langle f, g\rangle \, dt = \operatorname{Im} \langle f(t), g(t)\rangle \Big|_0^1 \\ &= -\{f(1), g(1)\} + \{f(0), g(0)\} = 0 \, .\end{aligned}$$

By the standard theory of elliptic partial differential equations, the remaining proof that $D(L_1, L_2)$ is Fredholm, self-adjoint, etc., is now apparent. From the definition of $D(L_1, L_2)$, $\ker(D(L_1, L_2))$ consists of precisely the constant functions $f(t) = k$. From the boundary condition, it follows that this constant k lies in $L_1 \cap L_2$. This proves (i) and (ii).

In a similar manner, because $A \cdot A^t$ is unitary, we have

$$\begin{aligned}&\langle D^\#(L_1, L_2)f, g\rangle - \langle f, D^\#(L_1, L_2)g\rangle \\ &= \int_0^1 \left\langle -i\frac{df}{dt}, g \right\rangle - \left\langle f, -i\frac{dg}{dt} \right\rangle dt \\ &= -i\int_0^1 \frac{d}{dt}\langle f, g\rangle \, dt = -i\langle f(t), g(t)\rangle \Big|_{t=a}^{t=b} \\ &= -i\left[\langle A\cdot A^t f(0), A\cdot A^t g(0)\rangle - \langle f(0), g(0)\rangle\right] = 0\end{aligned}$$

for $f, g\colon [0,1] \to V$ with $f(1) = A \cdot A^t \cdot f(0)$, $g(1) = A \cdot A^t \cdot g(0)$. Again standard elliptic theory yields (iii).

The kernel of $D^\#(L_1, L_2)$ consists of constant functions $f(t) = \nu$ for all $0 \leqq t \leqq 1$, with $A \cdot A^t \nu = \nu$ for $\nu \in \mathbb{C}^n$,

$$\ker D^\#(L_1, L_2) = \{\nu \in \mathbb{C}^n | A \cdot A^t \cdot \nu = \nu\} .$$

We write $\nu = x + iy$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$. Then a straightforward computation shows that the equation $A \cdot A^t \cdot \nu = \nu$ is equivalent to

$$A \cdot A^t \cdot x = x \quad \text{and} \quad A \cdot A^t \cdot y = y$$

(since taking conjugates (by A unitary) gives $(A^t)^{-1} A^{-1} \bar{\nu} = \bar{\nu}$ or $\bar{\nu} = A \cdot A^t \bar{\nu}$). In addition, we have the following sequence of implications

$$\begin{aligned} &A \cdot A^t \cdot x = x \text{ for } x \in \mathbb{R}^n \Leftrightarrow A^t \cdot x = A^{-1} \cdot x \text{ for } x \in \mathbb{R}^n \\ &\Leftrightarrow \overline{A^{-1} x} = A^{-1} x \text{ for } x \in \mathbb{R}^n \Leftrightarrow A^{-1} x \in \mathbb{R}^n \text{ for } x \in \mathbb{R}^n \\ &\Leftrightarrow x \in \mathbb{R}^n \cap A \cdot \mathbb{R}^n . \end{aligned}$$

Thus

$$\begin{aligned} \ker D^\#(L_1, L_2) &= \{\nu \in \mathbb{C}^n | A \cdot A^t \nu = \nu\} \\ &\cong \{x \in \mathbb{R}^n \mid A \cdot A^t x = x\} \otimes \mathbb{C} \\ &\cong (\mathbb{R}^n \cap A \cdot \mathbb{R}^n) \otimes \mathbb{C} \\ &\cong (L_1 \cap L_2) \otimes \mathbb{C} . \end{aligned}$$

This proves Lemma 3.1.

4. First Geometrical Definition of the Maslov Index: $\mu_{\mathrm{geo},1}(f)$

Given a continuous and piecewise smooth path of Lagrangians

$$f(t) = (L_1(t), L_2(t)) , \qquad a \leqq t \leqq b \tag{4.1}$$

we have by the basic Lemma 2.1 an ε, $0 < \varepsilon < \pi$, such that for all θ with $0 < |\theta| < \varepsilon$

$$L_1(a) \cap e^{J\theta} \cdot L_2(a) = 0 , \qquad L_1(b) \cap e^{J\theta} \cdot L_2(b) = 0 . \tag{4.2}$$

In particular, the perturbed path with $0 < \theta' < \varepsilon$

$$g_{\theta'}(t) = (L_1(t), e^{-J\theta'} \cdot L_2(t)) , \qquad a \leqq t \leqq b \tag{4.3}$$

is proper; that is, at $t = a$ and $t = b$, the intersection of the two Lagrangians equals $\{0\}$. For a proper path, say $h(t) = (\hat{L}_1(t), \hat{L}_2(t))$, $a \leqq t \leqq b$, one already has a definition of the Maslov index; see [18].

$$\mu_{\mathrm{proper}}(h) \quad \text{in} \quad \mathbb{Z} . \tag{4.4}$$

Hence for a general path f, we set

$$\mu_{\mathrm{geo},1}(f) = \mu_{\mathrm{proper}}(g_{\theta'}) \tag{4.5}$$

with $g_{\theta'}$ as in (4.3). In this section, we will show that this is well defined and satisfies Properties I–VI of Section 1.

Before proceeding further, let us *review the definition of* $\mu_{\mathrm{proper}}(h)$. The idea is to count with signs and multiplicities the number of times that $\hat{L}_1(t) \cap \hat{L}_2(t) \neq (0)$ as t ranges from $t = a$ to $t = b$. First we consider the situation when $h(t) = (\hat{L}_1(t), \hat{L}_2(t))$ is a smooth path for $a \leqq t \leqq b$.

Let $\mathcal{Z}$ be the subspace in $[a,b] \times \mathrm{Lag}(V)$ consisting of all pairs (t, L) which has the property

$$\hat{L}_1(t) \cap L \neq 0\,. \tag{4.6}$$

As shown in [18], Proposition 3.5, the intersection $\mathcal{Z} \cap (\{t\} \times \mathrm{Lag}\ V)$ is a codimension one subvariety of $\{t\} \times \mathrm{Lag}\ V$ and has singularities of codimension 3 in $\{t\} \times \mathrm{Lag}\ V$. Moreover, by a fundamental lemma of Arnold (see [1]). the top stratum of $\mathcal{Z} \cap (\{t\} \times \mathrm{Lag}\ V)$ has a canonical transverse orientation. Indeed, Arnold proved that if $\{t\} \times L$ is a point on this top stratum, then the path of Lagrangian $\{t\} \times e^{J\theta} \cdot L$ crosses the stratum transversely as θ increases. It defines the desired transverse orientation.

Hence given a proper path $h(t) = (\hat{L}_1(t), \hat{L}_2(t))$, $a \leqq t \leqq b$, we may, by a slight perturbation keeping the endpoints fixed, modify the oriented path

$$\gamma = \{(t, \hat{L}_2(t))\,, \qquad a \leqq t \leqq b\} \tag{4.7}$$

to a new path γ' intersecting $\mathcal{Z}$ only at points of the top smooth stratum and crossing them transversely. Define $\mu_{\mathrm{proper}}(h)$ to be the geometric intersection number, counted with signs, of the oriented path γ' with the top stratum of z, i.e.,

$$\mu_{\mathrm{proper}}(h) = \#(\mathcal{Z} \cap \gamma' \text{ in } [a,b] \times \mathrm{Lag}\ V)\,. \tag{4.8}$$

Since $h(t)$, and so $L_1(t)$, is assumed to be smooth, the union of strata

$$\coprod_{a<t<b} \{\text{top strata of } \mathcal{Z} \cap (\{t\} \times \mathrm{Lag}\ (V))\}$$

forms a smooth open manifold in Z and the singularities in $\mathcal{Z}$ are of codimension at least 3. Because of this codimension 3 property, two different choices of γ' may be deformed from one to another avoiding the singularities. Thus the intersection number of (4.8) is well defined and independent of the choice of γ'.

Similarly if $h(t) = (L_1(t), L_2(t))$ is continuous and piecewise smooth, then we can approximate $h(t)$ by a smooth path and define $\mu_{\mathrm{proper}}(h)$ by (4.8) using this smooth approximation. Again the codimension 3 property of the singularities guarantees the independence from the choice of approximation.

From the properties of geometric intersection number, the function

$$\mu_{\text{proper}}\colon \{\text{proper continuous and piecewise smooth maps } h\colon [a,b] \to \operatorname{Lag} V \times \operatorname{Lag} V\} \to \mathbb{Z} \tag{4.9}$$

satisfies the properties of Section 1: I. Affine Scale Invariance; II. Deformation Invariance (Fixed End Points); III. Path Additivity; IV. Symplectic Additivity; V. Symplectic Invariance. The normalization condition VI is of the form:

$$\mu_{\mathbb{R}^2}(f\,|\,[-\pi/4, +\pi/4]) = 1$$

where $f(t) = (\mathbb{R}\{1\}, \mathbb{R}\{e^{it}\})$, $-\pi/4 \leqq t \leqq \pi/4$. This computation can be easily seen by referring to Diagram 3. In addition we have:

Property XIII (Proper Deformation Invariance). If $(L_1(s,t), L_2(s,t))$, $a \leqq t \leqq b$, $0 \leqq s \leqq 1$, defines a continuous mapping

$$[a,b] \times [0,1] \to \operatorname{Lag} V \times \operatorname{Lag} V$$

such that for each s, $0 \leqq s \leqq 1$, the path

$$h_s(t) = (L_1(s,t), L_2(s,t)), \qquad a \leqq t \leqq b$$

is a proper, continuous and piecewise smooth path in $\operatorname{Lag} V \times \operatorname{Lag} V$, then

$$\mu_{\text{proper}}(h_0) = \mu_{\text{proper}}(h_1). \tag{4.10}$$

Note that in XIII, the endpoints of $h_s(t)$ may move so long as the properness condition is satisfied. This last condition ensures that all intersections are confined to a compact subset of the open interval (a,b) away from the end points. By a small perturbation, the above mapping $[a,b] \times [0,1] \to \operatorname{Lag} V \times \operatorname{Lag} V$ may be moved so that the new mapping image is disjoint from the codimension 3 singular strata and is transverse to the top smooth stratum. From this, it follows that $\mu_{\text{proper}}(h_0) = \mu_{\text{proper}}(h_1)$.

Property XIV (Proper Nullity). If $h(t) = (L_1(t), L_2(t))$, $a \leqq t \leqq b$, satisfies the condition $L_1(t) \cap L_2(t) = 0$ for all t, then $\mu_{\text{proper}}(h) = 0$.

Again this is obvious because h gives rise to a path in $\operatorname{Lag} V \times \operatorname{Lag} V$ which does not intersect $\mathcal{Z}$.

Proof of well definedness of $\mu_{\text{geo},1}(f)$ in (4.5): First, as θ' varies with $0 < \theta' < \theta$, the corresponding path $g_{\theta'} = \{(L_1(t), e^{J\theta'} \cdot L_2(t)),\ a \leqq t \leqq b\}$ is a continuously varying, piecewise smooth path which is *proper*. Hence, by Property XIII of proper deformation, $\mu_{\text{proper}}(g_{\theta'})$ is independent of θ' with $0 < \theta' < \theta$. Similarly by choosing $0 < \theta' < \varepsilon$ with θ' small, we ensure that (4.2) holds as we smoothly interpolate between any two choices of J, $\langle\ \rangle$. As in [18], these choices form a cell. Again by Property XIII, this proves that $\mu_{\text{proper}}(g_{\theta'})$ is independent of the choice of complex structure J and the Hermitian inner product $\langle\ ,\ \rangle$. Thus $\mu_{\text{geo},1}(f)$ is well defined and depends only on the symplectic structure of V.

Proof of Properties I–VI: By the above discussion of Properties I, II, IV of $\mu_{\text{proper}}(h)$, the analogous properties of $\mu_{geo,1}(f)$ follow immediately from the definition. In II, the endpoint conditions ensure that once θ is chosen $0 < \theta < \varepsilon$ satisfying (4.2) for the path $f(0)$, it also satisfies the same condition (4.2) for all $f(s)$, $0 \leqq s \leqq 1$. Hence,

$$\mu_{\text{geo},1}(f(s)) = \mu_{\text{proper}}(g(s)_{\theta'})\ , \qquad 0 \leqq s \leqq 1\ ,$$

and applying Property XIV to

$$g(s)_{\theta'} = \{(L_1(s,t)\,,\, e^{-J\theta'}L_2(s,t))\ ,\quad a \leqq t \leqq b\}\ ,$$

we have $\mu_{\text{geo},1}(f(0)) = \mu_{\text{geo},1}(f(1))$.

To prove the Path Additivity Property III for $f(t) = (L_1(t), L_2(t))$, $a \leqq t \leqq c$, and $a < b < c$, we choose ε, $0 < \varepsilon < \pi$ such that $L_1(t) \cap e^{J\theta} \cdot L_2(t) = 0$ for all θ with $0 < |\theta| < \varepsilon$ and *for* $t = a$ *and* $t = b$ *and* $t = c$. Hence, for $0 < \theta' < \varepsilon$

$$\begin{aligned} \ell &= \{(L_1(t), e^{-J\theta'} \cdot L_2(t))\ ,\quad a \leqq t \leqq b\} \\ m &= \{(L_1(t), e^{-J\theta'} \cdot L_2(t))\ ,\quad b \leqq t \leqq c\} \\ n &= \{(L_1(t), e^{-J\theta'} \cdot L_2(t))\ ,\quad a \leqq t \leqq c\} \end{aligned}$$

are proper paths, and by Proper Path Additivity,

$$\mu_{\text{proper}}(n) = \mu_{\text{proper}}(\ell) + \mu_{\text{proper}}(m)\ . \tag{4.11}$$

Thus Property III holds for $\mu_{\text{geo},1}(f)$.

In a similar fashion, Property V (Symplectic Invariance) is proved. Given ϕ: $[a,b] \to \mathrm{Sp}(V)$, we choose a piecewise smooth map ψ: $[a,b] \times [0,1] \to \mathrm{Sp}(V)$ such that $\psi(t,1) = \phi(t)$ and $\psi(t,0) = \mathrm{id}$ for $a \leqq t \leqq b$. For $t = a$ or b, we consider the largest ε_s, $0 < \varepsilon_s \leqq \pi/2$, such that

$$\psi(t,s) \cdot L_1(t) \cap e^{-J\theta} \cdot \psi(t,s)L_2(t) = (0) \tag{4.12}$$

for all θ with $0 < |\theta| < \varepsilon_s$. Since ε_s varies in an upper-semicontinuous manner on s, we may choose ε such that (4.12) holds for all s, all θ with $0 < |\theta| \leqq \varepsilon$ and for $t = a$ or b. With this choice ε and any θ, $0 < \theta < \varepsilon$, the path

$$\{(\psi(t,s) \cdot L_1(t), e^{-J\theta} \cdot \psi(t,s) \cdot L_2(t)\,,\ a \leqq t \leqq b\}$$

is a proper path for all s. By Property XIII, we have $\mu_{\mathrm{geo},1}(\phi_* f) = \mu_{\mathrm{geo},1}(f)$ because $\phi(t) = \psi(t,1)$, $\psi(t,0) = \mathrm{id}$.

The Normalization Property VI is easily verified for $\mu_{\mathrm{geo},1}(f)$. The following diagram indicates that, after a perturbation, the graphs of $f(t) = (\mathbb{R}\{1\}, \mathbb{R}\{e^{it}\})$, $-\pi/4 \leqq t \leqq \pi/4$, become $g_{\theta'}(t) = (\mathbb{R}\{1\}, \mathbb{R}\{e^{i(t-\theta')}\})$, $-\pi/4 \leqq t \leqq \pi/4$. In this new position, the graphs $(t, \mathbb{R}\{1\})$ and $(t, \mathbb{R}e^{i(t-\theta')})$ have no intersection between $-\pi/4 \leqq t \leqq 0$, and have intersection $+1$ in $0 \leqq t \leqq \pi/4$. This proves formula (1.7)(i)–(iii).

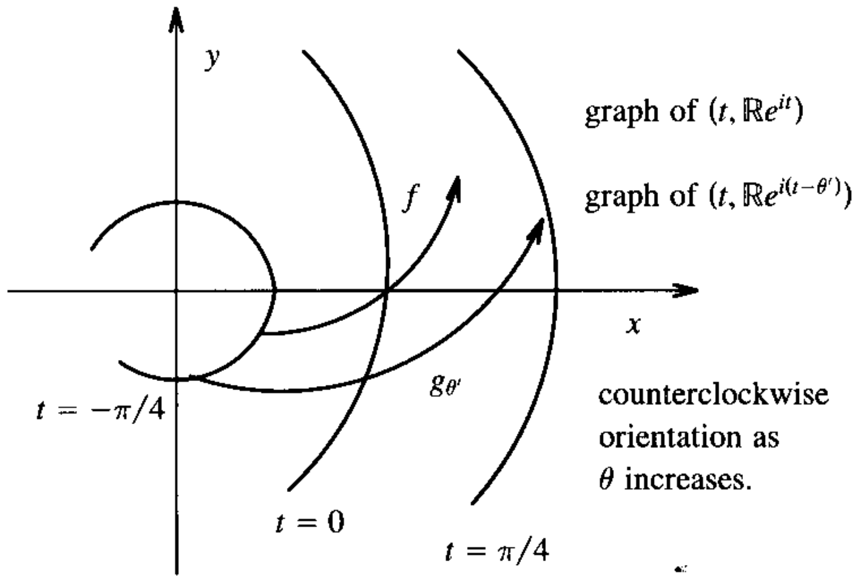

Diagram 4.

The proof of Property VII is identical and we omit the details.

We now come to the proof of Properties VIII (Nullity) and XII (Proper Path). If $L_1(t) \cap L_2(t)$ has constant dimension and varies continously with t, $a \leqq t \leqq b$, we can choose ε, $0 < \varepsilon < \pi$, such that $L_1(t) \cap e^{J\theta} \cdot L_2(t) = 0$ for all $0 < |\theta| < \varepsilon$ and $a \leqq t \leqq b$. After choosing θ', $0 < \theta' < \varepsilon$, we have

$$\mu_{\mathrm{geo},1}(f) = \mu_{\mathrm{proper}}\{L_1(t), e^{-J\theta'} \cdot L_2(t))\} = 0$$

by the Nullity Property of a proper path.

Suppose we are given a proper path $g(t) = (L_1(t), L_2(t))$, $a \leqq t \leqq b$. Then there exists an ε, $0 < \varepsilon < \pi$ with

$$L_1(t) \cap e^{i\theta} \cdot L_2(t) = 0\,, \quad \text{for} \quad t = a \text{ or } b\,, \text{ and } |\theta| < \varepsilon\,.$$

Here θ can approach zero because g is proper. By the deformative invariance XIII of $\mu_{\text{proper}}(\ )$, we have, for $0 < \theta' < \varepsilon$,

$$\begin{aligned}\mu_{\text{geo},1}(g) &= \mu_{\text{proper}}(\{L_1(t), e^{-J\theta'} \cdot L_2(t)\}) \\ &= \mu_{\text{proper}}(\{L_1(t), L_2(t)\}) \\ &= \mu_{\text{proper}}(g)\ .\end{aligned}$$

Since the above definition (4.8) of $\mu_{\text{proper}}(g)$ agrees with that of Guillemin and Sternberg (see [18]), Property XII is proved.

For future reference, we summarize the above results in the following:

PROPOSITION 4.1. (i) *The mapping given by* (4.5)

$$\mu_{\text{geo},1}\colon P([a,b];V) \to \mathbb{Z}$$

is well defined, independent of the choice of $(J, \langle\ ,\ \rangle)$.

(ii) $\mu_{\text{geo},1}$ *satisfies properties* I, II, III, IV, V, VI, VII, VIII, XII *of Section* 1.

As far as the existence of Maslov index is concerned, the above proposition proves Part (i) of Theorem 1.1. In order to prove Parts (ii) and (iii), we assume that $\mu_V(f)$ satisfies Properties I to V.

First note that for the *constant* path $\{(L_1, L_2),\ a \leqq t \leqq b\}$ we have by the Scale Invariance Property I and Path Additivity III:

$$\begin{aligned}\mu_V[\{(L_1,L_2);\ 0 \leqq t \leqq 1\}] &= \mu_V[\{(L_1,L_2);\ 0 \leqq t \leqq 2\}] \\ &= \mu_V[\{(L_1,L_2);\ 0 \leqq t \leqq 1\}] \\ &\quad + \mu_V[\{(L_1,L_2);\ 1 \leqq t \leqq 2\}] \\ &= 2\mu_V[\{(L_1,L_2);\ 0 \leqq t \leqq 1\}]\ .\end{aligned}$$

Thus, $\mu_V[\{(L_1,L_2);\ a \leqq t \leqq b\}] = \mu_V[\{(L_1,L_2);\ 0 \leqq t \leqq 1\}] = 0$.

Let $\{(L_1(t), L_2(t));\ a \leqq t \leqq b\}$ be any path of continuous and piecewise smooth Lagrangians such that the dimension of the intersection $L_1(t) \cap L_2(t)$ is constant dimensional and varies smoothly for $a \leqq t \leqq b$. It is not difficult to show that $\{(L_1(t), L_2(t));\ a \leqq t \leqq b\}$ is symplectic equivalent to a pair $\{(\widehat{L}_1, \widehat{L}_2);\ a \leqq t \leqq b\}$ where $\widehat{L}_1, \widehat{L}_2$ are fixed Lagrangian and $\dim(\widehat{L}_1 \cap \widehat{L}_2) = k$. Hence, by the Symplectic Invariance Property V, it follows that

$$\mu_V[\{(L_1(t), L_2(t));\ a \leqq t \leqq b\}] = 0$$

for any such path. In particular, $\mu_V(f) = 0$ when the two Lagrangians have empty intersection, $L_1(t) \cap L_2(t) = 0$, for $a \leqq t \leqq b$.

Next, we apply the Deformation Invariance Property to a path which first traces along a path γ and then back in the opposite direction $-\gamma$ to the original position.

Such a path $\gamma * -\gamma$ can be deformed, relative to fixed end points, to a constant path. Since, by the above discussion, a constant path has $\mu_V = 0$, we have by Path Additivity:

$$\mu_V(\text{path } \gamma) = -\mu_V(\text{path } -\gamma \text{ traced in the opposite direction}) .$$

As a special case,

$$\begin{aligned}\mu_V[\{(\mathbb{R}, \mathbb{R}e^{it}); a \leqq t \leqq b\}] &= -\mu[\{(\mathbb{R}, \mathbb{R}e^{-it}); -b \leqq t \leqq -a\}] \\ &= \mu[\{(\mathbb{R}e^{it}, \mathbb{R}); -b \leqq t \leqq -a\}] .\end{aligned}$$

To prove the uniqueness property in Theorem 1.1 (ii), we develop a formula to calculate $\mu_V(f)$ for a general path $f(t) = (L_1(t), L_2(t))$, $a \leqq t \leqq b$, in terms of two numbers $x = \mu_V[\{(\mathbb{R}, \mathbb{R}e^{it}); 0 \leqq t \leqq \pi/4)]$ and $y = \mu_V[\{(\mathbb{R}, \mathbb{R}e^{it}); -\pi/4 \leqq t \leqq 0\}]$. To simplify the problem, we choose a family of symplectic automorphism $\phi(t)$ such that $L_1(t) = \phi(t) \cdot \widehat{L}$ for a fixed $\widehat{L}$. Then, by Symplectic Invariance,

$$\mu_V(f) = \mu_V[\{(\widehat{L}, \widehat{L}_2(t)); a \leqq t \leqq b\}]$$

where $\widehat{L}_2(t) = \phi(t)^{-1} \cdot L_2(t)$. In other words, the problem can be reduced to one $(\widehat{L}, \widehat{L}_2(t))$ where the first Lagrangian is fixed.

Next we reduce the problem to one for a proper path. At the end point $t = a$ and up to a symplectic automorphic, we can present the data as: $V = \mathbb{C}^n$, and $L_1(a) = \mathbb{R}^n$, and $L_2(a) = \mathbb{R}^\alpha \oplus i \cdot \mathbb{R}^{n-\alpha}$. Here $L_1(a) \cap L_2(a) = \mathbb{R}^\alpha$ and $\alpha = \dim L_1(a) \cap L_2(a)$. The idea of the reduction is to add a "tail" to f. For this we introduce the path

$$\gamma_1(\alpha)(t) = (\mathbb{R}^n, \mathbb{R}^\alpha e^{it} \oplus \mathbb{R}^{n-\alpha}); 0 \leqq t \leqq \pi/4 .$$

Then the "tail" consists of first traveling along $\gamma_1(\alpha)(t - (a - \pi/2))$ for $a - \pi/2 \leqq t \leqq a - \pi/4$ and then along the reverse path for $a - \pi/4 \leqq t \leqq a$. Denote by Δ the composite of the "tail" with the original path $\{(\widehat{L}, \widehat{L}_2(t)); a \leqq t \leqq b\}$, and $\widehat{\Delta}$ the part of Δ from $a - \pi/4$ to b. Then, by the above discussion,

$$\mu_V(f) = \mu_V(\Delta) = \mu_V[\{\gamma_1(\alpha)(t); 0 \leqq t \leqq \pi/4\}] + \mu_V(\widehat{\Delta}) .$$

Note that at the beginning point $t = a - \pi/4$ the path $\widehat{\Delta}$ is proper.

In a similar manner, at $t = b$, we may arrange that $V = \mathbb{C}^n$, $L_1(b) = \mathbb{R}^n$, $L_2(b) = \mathbb{R}^\beta \oplus i\mathbb{R}^{n-\beta}$, and consider the path

$$\gamma_2(\beta)(t) = (\mathbb{R}^n, e^{it} \cdot \mathbb{R}^\beta \oplus i\mathbb{R}^{n-\beta}), 0 \leqq t \leqq \pi/4$$

and the composite path $\hat{\hat{\Delta}}$ of $\hat{\Delta}$ and $\gamma_2(\beta)(\pi/4 - t - b)$, $b \leqq t \leqq b + \pi/4$. Then, after repeating the same argument, we have

$$\mu_V(f) = \mu_V[\{\gamma_1(\alpha)(t); 0 \leqq t \leqq \pi/4\}] + \mu_V[\{\gamma_2(\beta)(t); 0 \leqq t \leqq \pi/4\}] + \mu_V(\hat{\hat{\Delta}})$$

where $\hat{\hat{\Delta}}$ is proper.

Next we modify $\hat{\hat{\Delta}} = \{(\hat{L}_1, \hat{\hat{L}}_2(t)); a - \pi/4 \leqq t \leqq b + \pi/4\}$ by continuously deforming $\hat{\hat{L}}_2(t)$ to a smooth path which intersects $\hat{L}_1$ transversely. At these finite intersection points $\{t_i\}$, we may assume $\hat{\hat{\Delta}}$ is locally isomorphic to one of the following two types:

$$\begin{aligned}&(1)\quad \{(\mathbb{R}^n, e^{i(t-t_i)}\mathbb{R}^1 \oplus \mathbb{R}^{n-1}),\ |t - t_i| < \delta\}\ ,\\ &(2)\quad \{(\mathbb{R}^n, e^{-i(t-t_i)} \cdot \mathbb{R}^1 \oplus \mathbb{R}^{n-1}),\ |t - t_i| < \delta\}\ .\end{aligned}$$

Outside these intervals $|t - t_i| < \delta$, the two Lagrangians have trivial intersection.

From the definition of x and y, it follows that

$$\begin{aligned}&\mu_V(\text{type (1) intersection}) = x + y\ ,\\ &\mu_V(\text{type (2) intersection}) = -(x + y)\ ,\\ &\mu_V[\{\gamma_1(\alpha)(t); 0 \leqq t \leqq \frac{\pi}{4}\}] = \alpha x\ ,\\ &\mu_V[\{\gamma_2(\beta)(t); 0 \leqq t \leqq \frac{\pi}{4}\}] = \beta y\ .\end{aligned}$$

Thus if after making $\hat{\hat{\Delta}}$ transverse to $\hat{L}$ there are p intersection points of type (1) and q intersection points of type (2), then the above yields the computation:

$$\mu_V(f) = \alpha x + (p - q)(x + y) + \beta y\ .$$

For the invariant $\mu_{\text{geo},1}(f)$, we have $x = 1$ and $y = 0$ by Property VI. Applying the above formula,

$$\mu_{\text{geo},1}(f) = \alpha + (p - q)\ .$$

The same formula holds for $\mu_V(f)$ whenever it satisfies properties I to VI, or, in other words, these properties determine $\mu_V(f)$. This proves Part (ii) of (1.1).

Note that if a collection of functions $\hat{\mu}_V\colon \mathscr{P}([a,b]; V) \to \mathbb{Z}$ satisfies properties I–V, then

$$\begin{aligned}\hat{\mu}_V(f) &= [\alpha + (p - q)](x + y) + (\beta - \alpha)y\\ &= \mu_{\text{geo},1}(f) \cdot [x + y] + [h_{12}(b) - h_{12}(a)]y\ .\end{aligned}$$

This proves Part (iii) of (1.1).

In the above discussion we proved the following: Property VIII and Property XII for $\mu_{\text{geo},1}(f)$; Property VIII (Nullity) and Property IX (Reparametrization Invariance) and Property X (Reversal) for general $\mu_V(f)$. There remains only Property XI (Symmetry). Note that $\mu_{\text{geo},1}(f_{\text{flip}})$ can be regarded as a system of Maslov functions satisfying I–VI. Hence $\mu_{\text{geo},1}(f_{\text{flip}}) = [\mu_{\text{geo},1}(f)](x+y) + [h_{12}(b) - h_{12}(a)]y$ where the constants x and y are given by

$$\begin{aligned}x &= \mu_{\text{geo},1}[\{(e^{it}\mathbb{R}, \mathbb{R}); 0 \leqq t \leqq \pi/4\}] = 0\ ,\\ y &= \mu_{\text{geo},1}[\{(e^{it}\mathbb{R}, \mathbb{R}); -\pi/4 \leqq t \leqq 0\}] = -1\ .\end{aligned}$$

Hence $\mu_{\text{geo},1}(f_{\text{flip}}) = -\mu_{\text{geo},1}(f) - [h_{12}(b) - h_{12}(a)]$ and completes the proof of (1.1).

5. Second Geometrical Definition of the Maslov Index: $\mu_{\text{geo},2}(f)$

Recall from Section 2 that a Lagrangian subspace L of V is the real span

$$L = \bigoplus_{j=1}^{n} \mathbb{R}\{e_j\} \tag{5.1}$$

of an orthonormal basis of the complex vector space (V, J), with respect to the Hermitian inner product $\langle\ ,\ \rangle$. This gives rise to a natural identification

$$\phi(L)\colon U(n)/O(n) \xrightarrow{\cong} \text{Lag}\ (V) \tag{5.2}$$

sending A mod $O(n)$ to $A \cdot L$.

Since the determinant of a real orthogonal matrix equals ± 1, the square of the determinant map $(\det)^2\colon U(n) \to S^1$ factors through $U(n)/O(n)$ and hence defines a natural mapping

$$(\det)^2\colon \text{Lag}\ V \to S^1 \tag{5.3}$$

by $(\det)^2(A \cdot L) = (\det A)^2$. By a theorem in [18], page 122, the fundamental group $\pi_1(\text{Lag}\ V)$ of Lag V equals $\mathbb{Z}$ and the above map $(\det)^2$ induces an isomorphism

$$\pi_1(\text{Lag}\ V) \xrightarrow{\cong} \pi_1(S^1) = \mathbb{Z}\,. \tag{5.4}$$

For a closed loop in Lag V, the above isomorphism (5.4) gives an integer which is again referred to as the Maslov index; see [18]. It counts the number of times, with signs and multiplicities, that $L(t)$ intersects a fixed Lagrangian. Our $\mu_{\text{geo},2}(f)$ will be so defined that it becomes identical to the above procedure when f is a closed loop (i.e., $f(t) = (\hat{L}_{\text{fixed}}, L_2(t))$; $a \leqq t \leqq b$ with $L_2(a) = L_2(b)$).

It will be convenient to have a functorial description of $(\det)^2$ which does not invoke a reference Lagrangian L as in (5.2). If $\{e_j\}_{1 \leqq j \leqq n}$ is an orthonormal basis for (V, J) with respect to $\langle\ ,\ \rangle$, giving a real basis for L', then

$$\left(\bigwedge_{j=1}^{n} e_j\right) \text{ is an element of } \Lambda^n_{\mathbb{C}} V\,, \text{ the top exterior complex power of } \Lambda^n_{\mathbb{C}} V\,. \tag{5.5}$$

The above element is of norm 1, with respect to the induced metric $\langle\ ,\ \rangle$ on $\Lambda^n_{\mathbb{C}} V$, and different choices of orthonormal basis for L' change $\Lambda^n_{j=1} e_j$ by a factor $\pm$ 1. Thus we obtain, after squaring, a well-defined mapping

$$\text{“}\det^2\text{”}\colon \text{Lag}\ (V) \to S^1[(\Lambda^n_{\mathbb{C}} V)^2] \tag{5.6}$$

by sending L' to the tensor square $(\bigwedge_{j=1}^n e_j)^{\otimes 2}$ in $(\Lambda_{\mathbb{C}}^n V)$. Here the notation $S^1[(\Lambda_{\mathbb{C}}^n V)^{\otimes 2}]$ stands for the unit circle in the complex line $(\Lambda_{\mathbb{C}}^n V)^{\otimes 2}$. Given another Lagrangian $L^\sharp = \oplus_{j=1}^n \mathbb{R} A \cdot e_j$ with $A \in U(n)$, we have

$$\text{“det}^2\text{”}(L') = \left(\bigwedge_{j=1}^n A \cdot e_j\right)^2 = (\det A)^2 \cdot \left(\bigwedge_{j=1}^n e_j\right)^{\otimes 2} = (\det A)^2 \cdot \text{“det}^2\text{”}(L) .$$

Hence, under the identification

$$S^1[(\Lambda_{\mathbb{C}}^n V)^{\otimes 2}] \cong S^1$$

which sends “det^2”(L) to $+1$, the mapping “det^2” of (5.6) coincides with (det^2) of (5.3).

Before we define $\mu_{\text{geo},2}(f)$ as an intersection number in the cylinder

$$[a,b] \times S^1[(\Lambda_{\mathbb{C}}^n V)^{\otimes 2}] , \tag{5.7}$$

we need to clarify our orientation conventions:

(i) The space $(\Lambda_{\mathbb{C}}^n V)^{\otimes 2}$, being a complex line, has a natural orientation. As the oriented boundary of the unit disk, $S^1[(\Lambda_{\mathbb{C}}^n V)^{\otimes 2}]$ inherits an oriention (counterclockwise).

(ii) The cylinder $[a,b] \times S^1[(\Lambda_{\mathbb{C}}^n V)^{\otimes 2}]$ can be identified with the annulus $\{\omega\colon 1 < |\omega| < 1 + (b-a)\}$ by sending $(p, \vec{\nu}) \mapsto [1 + (p-a)]\vec{\nu}$. The orientation on $[a,b] \times S^1[(\Lambda_{\mathbb{C}}^n V)^{\otimes 2}]$ is induced from the complex orientation on $(\Lambda_{\mathbb{C}}^n V)^{\otimes 2}$.

To define $\mu_{\text{geo},2}(f)$ for $f(t) = (L_1(t), L_2(t))$, $a \leqq t \leqq b$, in $P([a,b]; V)$, we proceed as before by choosing ε, $0 < \varepsilon < \pi$, such that

$$L_1(t) \cap e^{J\theta} \cdot L_2(t) = 0 \quad \text{for all} \quad 0 < |\theta| < \varepsilon \text{ and } t = a \text{ and } t = b . \tag{5.8}$$

Choose ε with $0 < n\varepsilon < \pi$ where $\dim V = 2n$ and choose θ' with $0 < \theta' < \varepsilon$. By [18], page 118, the space of Lagrangians transverse to a fixed Lagrangian L forms a cell. Hence there exist two paths of Lagrangians in V (unique up to homotopy)

$$\gamma_\ell(t) \, , \; a-1 \leqq t \leqq a \, , \quad \text{and} \quad \gamma_r(t) \, , \; b \leqq t \leqq b+1 \tag{5.9}$$

such that

$$\begin{cases} \gamma_\ell(t) \cap L_1(a) = \{0\} & \text{for all } a-1 \leqq t \leqq a \\ \gamma_\ell(a-1) = e^{-J\theta'} \cdot L_1(a) \, , \; \gamma_\ell(a) = e^{-J\theta'} \cdot L_2(a) \, . \end{cases}$$
$$\begin{cases} \gamma_2(t) \cap L_1(b) = \{0\} & \text{for all } b \leqq t \leqq b+1 \\ \gamma_r(b) = e^{-J\theta'} \cdot L_2(b) \, , \; \gamma_r(b+1) = e^{-J\theta'} \cdot L_1(b) \, . \end{cases} \tag{5.10}$$

Let Γ' be the composite path

$$\Gamma' = \gamma_\ell * \{e^{-J\theta'} L_2(t) \; , \; a \leqq t \leqq b\} * \gamma_r \tag{5.11}$$

of γ_ℓ and $e^{-J\theta'} \cdot L_2(t)$ and γ_r. This Γ' is to be compared to the composite path

$$\Gamma = [L_1(a)] * \{L_1(t),\ a \leqq t \leqq b\} * [L_1(b)] \tag{5.12}$$

where $[L_1(a)]$ and $[L_1(b)]$ stand respectively for the constant paths with $a - 1 \leqq t \leqq a$ and $b \leqq t \leqq b + 1$. These two paths Γ and Γ' have end points $e^{-J\theta'} \cdot L_1(a)$, $e^{-J\theta'} \cdot L_1(b)$ and respectively $L_1(a)$, $L_1(b)$. Diagramatically we have:

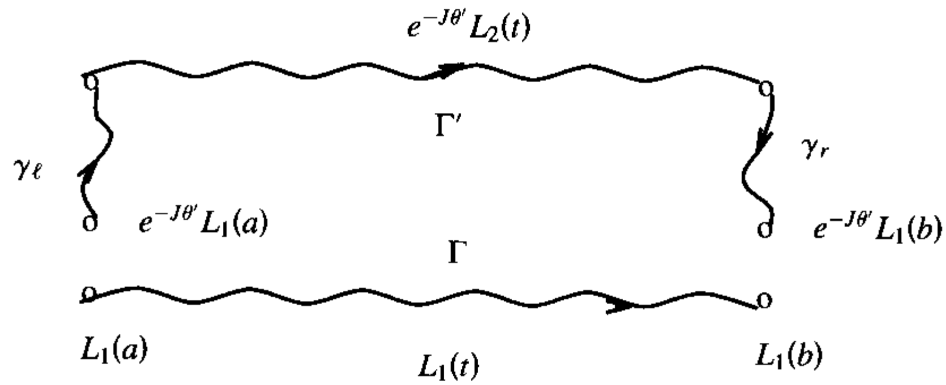

Diagram 5.

Note that for $0 < \theta' < \varepsilon < \pi/n$ we have

$$\text{“det}^2\text{”}(e^{-J\theta'} \cdot L_1(t)) = e^{-2nJ\theta'} \cdot \text{“det}^2\text{”}(L_1(t)) \neq \text{“det}^2\text{”}(L_1(t))$$

in the circle $S^1[(\Lambda^n_{\mathbb{C}} V)^{\otimes 2}]$. Thus, in the cylinder $[a - 1, b + 1] \times S^1$, there are two paths $\overline{\Gamma}$ and $\overline{\Gamma}'$ given by the formula:

$$\begin{cases} \overline{\Gamma} = \{(t, \text{“det}^2\text{”}(\Gamma(t))),\ a - 1 \leqq t \leqq b + 1\} \\ \overline{\Gamma}' = \{(t, \text{“det}^2\text{”}(\Gamma'(t))),\ a - 1 \leqq t \leqq b + 1\} \end{cases} \tag{5.13}$$

These two paths are oriented according to the increasing t-direction, and they have *disjoint* endpoints lying on the boundary of the cylinder. Consequently, the intersection number of $\overline{\Gamma}$, $\overline{\Gamma}'$ is well-defined, and $\mu_{\text{geo},2}(f)$ is defined to be this intersection number

$$\mu_{\text{geo},2}(f) = \sharp(\overline{\Gamma} \cap \overline{\Gamma}') \tag{5.14}$$

The numbers $\sharp(\overline{\Gamma} \cap \overline{\Gamma}')$ are independent of the choice of θ',$0 < \theta' < \varepsilon$, because changing θ' smoothly varies the paths $\overline{\Gamma}$ and $\overline{\Gamma}'$ smoothly with end points disjoint throughout. Similarly the structure $(J, \langle\ ,\ \rangle)$ may vary without changing $\sharp(\overline{\Gamma} \cap \overline{\Gamma}')$. Thus (5.14) gives a well-defined map

$$\mu_{\text{geo},2} = P([a, b]; V) \to \mathbb{Z}$$

which depends only on the symplectic structure $\{\ ,\ \}$ of V.

PROPOSITION 5.1. *The functions* $\mu_{\text{geo},2}$: $P([a,b];V) \to \mathbb{Z}$ *satisfy Properties* I–XII *of Section* 1. *In particular,* $\mu_{\text{geo},2}(f) = \mu_{\text{geo},1}(f)$ *for all* f.

Proof of Proposition 5.1: In view of (1.1), it suffices to show that $\mu_{\text{geo},2}$ satisfies Properties I–VI. Since the intersection number of two oriented arcs in $[a-1, b+1] \times S^1[(\Lambda^n_{\mathbb{C}}V)^{\otimes 2}]$ remains the same under perturbation, the proofs of Properties I and II are clear.

Similarly, to prove the Symplectic Invariance Property IV, it suffices to choose θ' with $0 < \theta' < \varepsilon < \pi/n$ so that

$$\phi(s,t)\cdot L_1(t) \cap [e^{J\theta}\cdot\phi(s,t)\cdot L_2(t)] = 0 \tag{5.15}$$

for $0 < |\theta| < \varepsilon$, $0 \leqq s \leqq 1$, and $t = a$ and $t = b$. Here $\phi(s,t)$ is a continuous mapping ϕ: $[0,1]\times[a,b] \to \text{Sp}(V)$ such that $\phi(1,t)$ is the given symplectic automorphism $\phi(t)$ in Property IV, $\phi(0,t) = \text{id}$, and for each fixed s, $\phi(s)(t) = \phi(s,t)$ are piecewise smooth. Thus, for each s the above procedure with the data $(\phi(t,s)\cdot L(t),$ $\phi(s,t)\cdot L_2(t))$, $a \leqq t \leqq b$, constructs a pair of oriented arcs $\overline{\Gamma}(s)$ and $\overline{\Gamma}'(s)$. The deformation invariance of $\sharp(\overline{\Gamma}(s), \overline{\Gamma}'(s))$ yields the desired identity:

$$\mu_{\text{geo},2}(f) = \mu_{\text{geo},2}(\phi(0,t)_* f) = \mu_{\text{geo},2}(\phi(1,t)_* f) = \mu_{\text{geo},2}(\phi_* f)\ .$$

After choosing ε with $\varepsilon < \pi/(m+n)$, $m = \dim W$, $n = \dim V$, we have pairs of arcs $\overline{\Gamma}(V)$, $\overline{\Gamma}'(V)$ on $[a,b]\times S^1[(\Lambda^n_{\mathbb{C}}V)^{\otimes 2}]$ and $\overline{\Gamma}(W)$, $\overline{\Gamma}'(W)$ on $[a,b]\times S^1[(\Lambda^n_{\mathbb{C}}W)^{\otimes 2}]$. The Symplectic Additivity (IV) is a consequence of the following:

LEMMA 5.2. *Let* $\gamma_1, \gamma_2, \gamma_1^\sharp, \gamma_2^\sharp$: $[0,1] \to S^1$ *be continuous mappings such that*

$$\begin{aligned} \gamma_2(0) &= e^{-i\theta'}\gamma_1(0)\ , & \gamma_2(1) &= e^{-i\theta'}\gamma_1(1) \\ \gamma_2^\sharp(0) &= e^{-i\theta'}\gamma_1^\sharp(0)\ , & \gamma_2^\sharp(1) &= e^{-i\theta'}\gamma_1^\sharp(1) \end{aligned} \tag{5.16}$$

with $0 < \theta' < \pi/2$. *Form the paths*

$$\begin{aligned} (\gamma_1\cdot\gamma_1^\sharp)(t) &= \gamma_1(t)\cdot\gamma_1^\sharp(t)\ , & 0 \leqq t \leqq 1 \\ (\gamma_2\cdot\gamma_2^\sharp)(t) &= \gamma_2(t)\cdot\gamma_2^\sharp(t)\ , & 0 \leqq t \leqq 1 \end{aligned} \tag{5.17}$$

Then the intersection numbers in $[0,1]\times S^1$ *of the graphs* Γ_{γ_1}, Γ_{γ_2}, $\Gamma_{\gamma_1^\sharp}$, $\Gamma_{\gamma_2^\sharp}$, $\Gamma_{\gamma_1\cdot\gamma_1^\sharp}$, $\Gamma_{\gamma_2\cdot\gamma_2^\sharp}$ *of the functions* γ_1, γ_2, *etc., are related by:*

$$\sharp(\Gamma_{\gamma_1\cdot\gamma_1^\sharp} \cap \Gamma_{\gamma_2\cdot\gamma_2^\sharp}) = \sharp(\Gamma_{\gamma_1}\cap\Gamma_{\gamma_2}) + \sharp(\Gamma_{\gamma_1^\sharp}\cap\Gamma_{\gamma_2^\sharp})\ . \tag{5.18}$$

Proof of Lemma 5.2: The mappings $\gamma_1, \gamma_2, \gamma_1^\sharp, \gamma_2^\sharp$, can easily be deformed continuously so that throughout the deformation condition (5.16) holds and at the end, the mappings enjoy the additional properties:

$$\gamma_1(1) = \gamma_1(0)\ , \qquad \gamma_1^\sharp(1) = \gamma_1^\sharp(0)\ . \tag{5.19}$$

Since this process leaves the terms in (5.18) unchanged, it suffices to treat only those cases satisfying (5.19).

Note (5.19) implies that $(\gamma_1 \cdot \gamma_1^\#)(1) = (\gamma_1 \cdot \gamma_1^\#)(0)$ and $(\gamma_2 \cdot \gamma_2^\#)(1) = (\gamma_2 \cdot \gamma_2^\#)(0)$. Hence, by forming the torus $T^2 = S^1 \times S^1 = ([0,1]/0 \sim 1) \times S^1$, the arcs Γ_{γ_1}, Γ_{γ_2}, $\Gamma_{\gamma_1^\#}$, $\Gamma_{\gamma_2^\#}$, $\Gamma_{\gamma_1 \cdot \gamma_1^\#}$, $\Gamma_{\gamma_2 \cdot \gamma_2^\#}$ can be closed up to produce loops on T^2 which are denoted respectively by $\overline{\Gamma}_{\gamma_1}$, $\overline{\Gamma}_{\gamma_2}$, $\overline{\Gamma}_{\gamma_1^\#}$, $\overline{\Gamma}_{\gamma_2^\#}$, $\overline{\Gamma}_{\gamma_1 \cdot \gamma_1^\#}$, $\overline{\Gamma}_{\gamma_2 \cdot \gamma_2^\#}$. Since the endpoints of the pair of arcs $(\Gamma_{\gamma_1}, \Gamma_{\gamma_2})$, $(\Gamma_{\gamma_1^\#}, \Gamma_{\gamma_2^\#})$, $(\Gamma_{\gamma_1 \cdot \gamma_1^\#}, \Gamma_{\gamma_2 \cdot \gamma_2^\#})$ are disjoint, the close-up operations do not produce new intersection points and so

$$\begin{aligned}
\#(\Gamma_{\gamma_1} \cap \Gamma_{\gamma_2}) &= \#(\overline{\Gamma}_{\gamma_1} \cap \overline{\Gamma}_{\gamma_2}) \text{ in } S^1 \times S^1 , \\
\#(\Gamma_{\gamma_1^\#} \cap \Gamma_{\gamma_2^\#}) &= \#(\overline{\Gamma}_{\gamma_1^\#} \cap \overline{\Gamma}_{\gamma_2^\#}) \text{ in } S^1 \times S^1 , \\
\#(\Gamma_{\gamma_1 \cdot \gamma_1^\#} \cap \Gamma_{\gamma_2 \cdot \gamma_2^\#}) &= \#(\overline{\Gamma}_{\gamma_1 \cdot \gamma_1^\#} \cap \overline{\Gamma}_{\gamma_2^\# \cdot \gamma_2^\#}) \text{ in } S^1 \times S^1 .
\end{aligned}$$

Let $H_1(T^2, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ with generators $x = [S^1 \times 1]$ and $y = [1 \times S^1]$; then the above closed loops with the obvious orientation give rise to cohomology classes

$$\begin{aligned}
[\overline{\Gamma}_{\gamma_1}] &= x + py , \qquad [\overline{\Gamma}_{\gamma_2}] = x + qy , \\
[\overline{\Gamma}_{\gamma_1^\#}] &= x + p^\# y , \qquad [\overline{\Gamma}_{\gamma_2^\#}] = x + q^\# y .
\end{aligned}$$

As for the cycles $\overline{\Gamma}_{\gamma_j \cdot \gamma_j^\#}$, $j = 1, 2$, they can be constructed from the graphs $\overline{\Gamma}_{\gamma_j}, \overline{\Gamma}_{\gamma_j^\#}$ by multiplication on the second factor $\phi((t, e^{i\theta}), (t, e^{i\phi})) = (t, e^{i(\theta+\phi)})$. Since the multiplication $S^1 \times S^1 \to S^1$, $(e^{i\theta}, e^{i\phi}) \mapsto (e^{i(\theta+\phi)})$ induces addition on homology, it follows that

$$[\overline{\Gamma}_{\gamma_1 \cdot \gamma_1^\#}] = x + (p + p^\#) y , \qquad [\overline{\Gamma}_{\gamma_1 \cdot \gamma_1^\#}] = x + (q + q^\#) y .$$

Appealing to the relation between cup product and intersection number, we have

$$\begin{aligned}
[\overline{\Gamma}_{\gamma_1 \cdot \gamma_1^\#}] \cap [\Gamma_{\gamma_2 \cdot \gamma_2^\#}] &= (q + q^\#) - (p + p^\#) = (q - p) + (q^\# - p^\#) \\
&= (\overline{\Gamma}_{\gamma_1}] \cap [\Gamma_{\gamma_1^\#}]) + ([\overline{\Gamma}_{\gamma_2}] \cap [\overline{\Gamma}_{\gamma_2^\#}])
\end{aligned}$$

and hence Lemma 5.2.

To prove *Path Additivity Property III* for $f(t) = (L_1(t), L_2(t))$, $a \leqq t \leqq c$ with $a < b < c$, we choose ε, $0 < \varepsilon < \pi/n$, so that (5.8) holds for $t = a, b, c$. Construct γ_ℓ, γ_r as above for $f|[a,b]$. Let $\gamma_\ell^\#(s)$ be $\gamma_r(t)$, but travelled in the opposite direction as s ranges from $b - 1 \leqq s \leqq b$, i.e.,

$$\gamma_\ell^\#(s) = \gamma_r(2b - s) , \qquad b - 1 \leqq s \leqq b .$$

Using the transversality of $\bar{e}^{J\theta'} \cdot L_2(c)$ and $\bar{e}^{J\theta'} \cdot L_1(c)$ to $L_1(c)$, we choose a path $\gamma_r^\#(s)$, $c \leqq s \leqq c + 1$ with

$$\begin{aligned}
\gamma_r^\#(s) \cap L_1(c) &= 0 , \qquad c \leqq s \leqq c + 1 , \\
\gamma_r^\#(c) &= \bar{e}^{J\theta'} \cdot L_2(c) , \qquad \gamma_r^\#(c+1) = e^{-J\theta'} \cdot L_1(c) .
\end{aligned}$$

With these choices, $\mu_{\mathrm{geo},2}(f\,|\,[a,b])$, respectively $\mu_{\mathrm{geo},2}(f\,|\,[b,c])$, is computed from the paths

$$\overline{\Gamma}' = \text{“}\det^2\text{”}[\gamma_\ell * \{e^{-J\theta'}\cdot L_2(t);\ a \leqq t \leqq b\} * \gamma_r]\,,$$
$$\overline{\Gamma} = \text{“}\det^2\text{”}[\{L_1(a)\} * \{L_1(t);\, a \leqq t \leqq b\} * \{L_2(a)\}]\,,$$

respectively,

$$\overline{\Gamma}^{\#'} = \text{“}\det^2\text{”}[\gamma_\ell^\# * \{e^{-J\theta'}\cdot L_2(t);\, b \leqq t \leqq c\} * \gamma_r^\#]$$
$$\overline{\Gamma}^{\#} = \text{“}\det^2\text{”}[\{L_1(b)\} * \{L_1(t);\, b \leqq t \leqq c\} * \{L_2(c)\}]$$

where the parameter t lies in $[a-1, b+1]$ for $\overline{\Gamma}'$, $\overline{\Gamma}$ and in $[b-1, c+1]$ for $\overline{\Gamma}^{\#'}$, $\overline{\Gamma}^{\#}$. Let $\overline{\Gamma}, \overline{\Gamma}', \overline{\Gamma}^{\#}, \overline{\Gamma}^{\#'}$ denote the corresponding paths in the cylinder.

Reparametrize $\overline{\Gamma}'$, $\overline{\Gamma}$ by shifting s to $s-2$ and replacing the interval of definition $[a-1, b+1]$ by $[a-3, b-1]$. Then the end points of $\overline{\Gamma}'$ and $\overline{\Gamma}$ coincide with the starting points of $\overline{\Gamma}^{\#'}$ and, respectively $\overline{\Gamma}^{\#}$ with the same parameter $t = b-1$. This allows us to form the composite paths $\overline{\Gamma}' * \overline{\Gamma}^{\#'}$ and $\overline{\Gamma} * \overline{\Gamma}^{\#}$.

From the definition, we have

$$\mu_{\mathrm{geo},2}(f\,|\,[a,b]) + \mu_{\mathrm{geo},2}(f\,|\,[b,c]) = \#(\overline{\Gamma} * \overline{\Gamma}^{\#}, \overline{\Gamma}' * \overline{\Gamma}^{\#'}) \tag{5.20}$$

where the intersection number is evaluated in the ambient space $[a-3, c+1] \times S^1[(\Lambda^n_{\mathbb{C}}V)^{\otimes 2})]$. Since the path $\gamma_\ell * \gamma_r^\#$ can be continously deformed to the constant path $\{e^{-J\theta'}\cdot L_2(b)\}$, the term on the right-hand side of (5.20) equals the intersection of $\{L_1(a)\} * \{L_1(t);\, a \leqq t \leqq b\} * \{L_1(b)\} * \{L_1(t);\, a \leqq t \leqq c\} * L_1(c)$ and $\{\gamma_\ell(t)\} * \{e^{-J\theta'}\cdot L_2(t);\, a \leqq t \leqq b\} * \{e^{-J\theta'}\cdot L_2(b)\} * \{e^{-J\theta'}\cdot L_2(t);\, b \leqq t \leqq c\} * \{\gamma_r^\#(t)\}$. By rescaling and shrinking the middle constant path, this last intersection number is precisely that from $\{L_1(a)\} * \{L_1(t);\, a \leqq t \leqq c\} * \{L_1(c)\}$ and $\{\gamma_\ell(t)\} * \{e^{-J\theta'}\cdot L_2(t);\, a \leqq t \leqq c\} * \{\gamma_r^\#(t)\}$. Clearly this is $\mu_{\mathrm{geo},2}(f\,|\,[a,b])$ and hence

$$\mu_{\mathrm{geo},2}(f\,|\,[a,b]) + \mu_{\mathrm{geo},2}(f\,|\,[b,c]) = \mu_{\mathrm{geo},2}(f\,|\,[a,c])\,,$$

as claimed.

The *Normalization Property VI* for $\mu_{\mathrm{geo},2}(f)$ can be verified directly from the definition. The following diagrams illustrate the procedure of constructing Γ, Γ' in each of the cases (1.7)(i)–(iii).

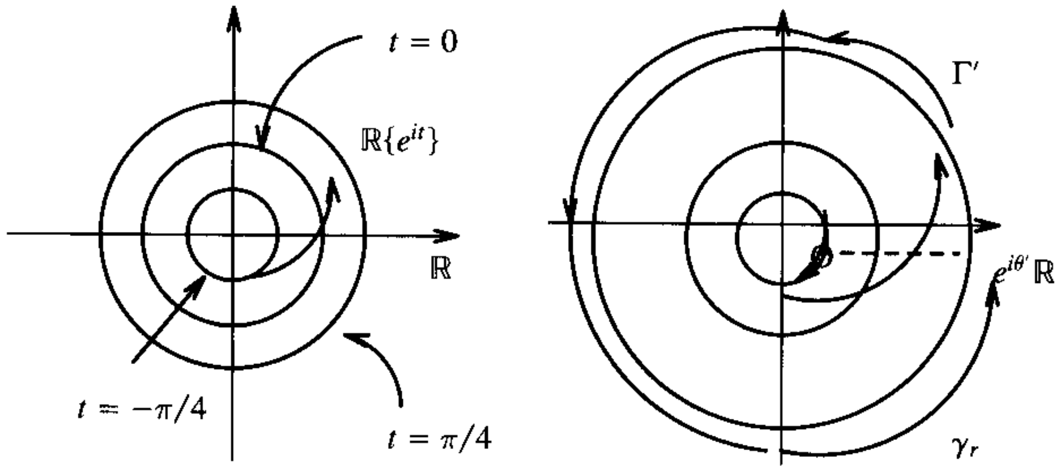

Diagram 6. For (1.7)(i), we obtain (+1) intersection on the right-hand side.

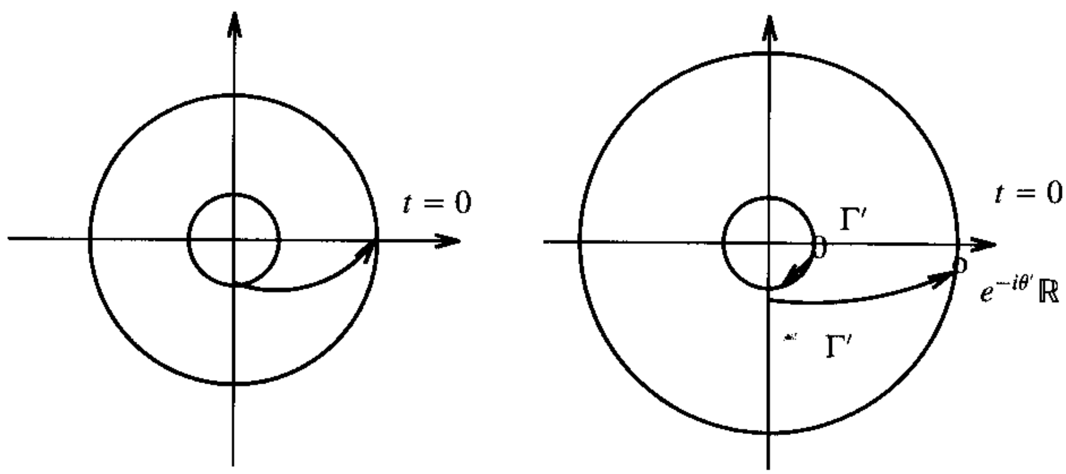

Diagram 7. For (1.7)(ii), we have no intersection on the right.

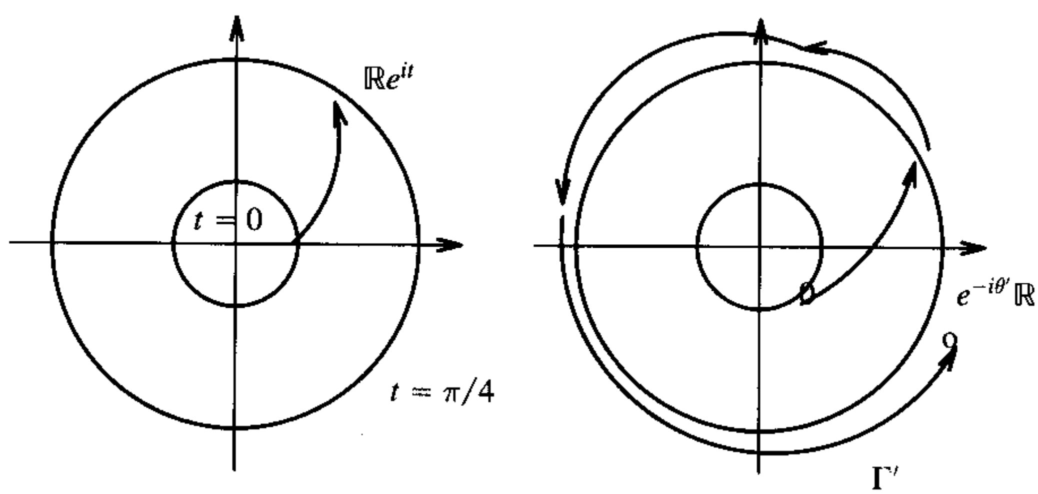

Diagram 8. For (1.7)(iii), we have (+1)-intersection.

6. First Analytical Definition of the Maslov Index: $\mu_{\text{anal},1}(f)$

Given a pair of Lagrangian subspaces L_1, L_2 of V, we consider the first order elliptic operator $-J\frac{d}{dt}$ on the functions ϕ: $[0,1] \to V$ satisfying the boundary condition

$$\phi(0) \in L_1 \quad \text{and} \quad \phi(1) \in L_2 \tag{6.1}$$

As explained in (3.6), this operator

$$D(L_1, L_2)\colon L_1^2([0,1]; L_1, L_2) \to L^2([0,1]; V) \tag{6.2}$$

is self-adjoint and has as its kernel the space of constant function $\phi(t) = k$, with $k \in L_1 \cap L_2$.

The eta invariant $\eta(D(L_1, L_2))$ of $D(L_1, L_2)$ is defined in [3] by first forming the meromorphic continuation of the function $\eta(s) = \sum_{\lambda_j \neq 0}(\text{sign } \lambda_j)|\lambda_j|^{-s}$, where λ_j are eigenvalues of $D(L_1, L_2)$, and then setting $s = 0$, i.e., $\eta(D(L_1, L_2)) = \eta(0)$. In the present situation, this eta invariant can be computed explicitly.

Let $L_1 = \oplus_{j=1}^n \mathbb{R}\hat{e}_j$, where $\{\hat{e}_j\}_{1 \leqq j \leqq n}$ is an orthogonal basis of (V, J) under $\langle\ \rangle$. Then for some unitary matrix A, we can write L_2 as $\oplus_{j=1}^n \mathbb{R}\{A \cdot \hat{e}_j\}$. In fact, we diagonalize A:

$$A = B^{-1} \cdot D(e^{i\theta_1}, \ldots, e^{i\theta_n}) \cdot B \tag{6.3}$$

where B is unitary and $D(e^{i\theta_1}, \ldots, e^{i\theta_n})$ is the diagonal matrix with (j, j)-entry $e^{i\theta_j}$. Under the unitary transformation

$$B\colon V \to V\,, \qquad \nu \mapsto B \cdot \nu\,,$$

the pair of Lagrangians (L_1, L_2) is sent to $(BL_1, D(e^{i\theta_1}, \ldots, e^{-i\theta_n}) \cdot BL_2)$. Thus the transformation induces an isometry

$$L_1^2([0,1]; L_1, L_2) \to L_2^1([0,1]; BL_1, D(e^{i\theta_1}, \ldots, e^{i\theta_n})\, BL_1)\,,$$

which in turn commutes with $-J\frac{d}{dt}$. Or in other words, the operator $D(L_1, L_2)$ is transformed into $D[BL_1, D(e^{i\theta_1}, \ldots, e^{i\theta_n}) \cdot BL_1]$. Since the spectrum is a unitary invariant, the spectrum of $D(L_1, L_2)$ is identical to that of $D[BL_1, D(e^{i\theta_1}, \ldots, e^{i\theta_n})BL_1]$ and so

$$\eta(D(L_1, L_2)) = \eta[D(BL_1, D(e^{i\theta_1}, \ldots, e^{i\theta_n}) \cdot BL_2]\,.$$

Choosing Be_j, $1 \leqq j \leqq n$, as the orthonormal basis $\{f_j\}$ of V, we can identify BL_1 as $\mathbb{R}^n$ in $V = \mathbb{C}^n$ and $D(e^{i\theta_1}, \ldots, e^{i\theta_n})B \cdot L$ as the subspace $\oplus_{j=1}^n \mathbb{R}\{e^{i\theta_j} \cdot f_j\}$. In this way, the operator $D[BL_1, D(e^{i\theta_1}, \ldots, e^{i\theta_n})\, BL_1]$ becomes

$$D\left(\mathbb{R}^n, \bigoplus_{j=1}^n \mathbb{R}\{e^{i\theta_j} f_j\}\right). \tag{6.4}$$

Since $e^{i\theta} = -e^{i(\theta+\pi)}$, we can choose θ_j with the property that

$$0 \leqq \theta_j < \pi \,, \qquad j = 1, \ldots, n \,. \tag{6.5}$$

PROPOSITION 6.1. *Given a pair of Lagrangians (L_1, L_2) we choose θ_j as in (6.5). Then*

$$\begin{aligned} \eta(D(L_1, L_2)) &= \eta\left(D\left(\mathbb{R}^n, \bigoplus_{j=1}^{n} \mathbb{R}\{e^{j\theta_j} f_j\}\right)\right) \\ &= \sum_{0<\theta_j<\pi} \{1 - 2(\theta_j/\pi)\} \,. \end{aligned} \tag{6.6}$$

Note that the angles $\theta_j = 0$ or $\pi/2$ make no contribution to $\eta(D(L_1, L_2))$. The discontinuity of $\eta(D(L_1, L_2))$ occurs whenever θ_j becomes 0 or equivalently π. For θ_j small and positive, the contribution of $e^{i\theta_j}$ to (6.6) is approximately $+1$; while for θ_j small and negative we must use $\pi + \theta_j$ in (6.6) and so the contribution is -1. Hence $\eta(D(L_1, L_2))$ jumps by $+2$ as θ_j crosses 0, or in other words, the spectral flow is $+1$. Despite this discontinuity, for smoothly varying $\theta_j(t)$, the left and right derivatives $-\frac{2}{\pi}\frac{d\theta_j}{dt}$ remain the same. That is, $\frac{d}{dt}\eta(D(L_1(t), L_2(t))$ is a smooth function if $L_1(t)$, $L_2(t)$ are smooth.

Proof of Proposition 6.1: The eigenvectors for $D(\mathbb{R}^n, \oplus_{j=1}^n \mathbb{R}\{e^{i\theta_j} f_j\})$ are

$$f_{j,m_j} = \exp(i(\pi m_j + \theta_j)t) \cdot f_{j,m_j} \,, \qquad 1 \leqq j \leqq n$$

where $m_j \in \mathbb{Z}$. Since $-i\frac{d}{dt} f_{j,m_j} = (\pi m_j + \theta_j) f_{j,m}$, the eigenvalue of f_{j,m_j} is $(\pi m_j + \theta_j)$.

Recall that the generalized zeta function $\zeta(s, \omega)$ is defined for $0 < \omega \leqq 1$ by the formula

$$\zeta(s, \omega) = \sum_{\nu=0}^{\infty} (\nu + \omega)^{-s}$$

and has a unique meromorphic extension to $s = 0$ with

$$\zeta(0, \omega) = (1/2) - \omega \,, \qquad 0 < \omega \leqq 1 \,.$$

In terms of $\zeta(s, \omega)$, the eta invariant $\eta(D(\mathbb{R}^n, \oplus_{j=1}^n \mathbb{R}\{e^{i\theta_j} f_j\})$ for $0 \leqq \theta_j < \pi$ is given by

$$\eta(s) = \frac{1}{(\pi)^s} \sum_{0<\theta_j<\pi} [\zeta(s, \theta_j/\pi) - \zeta(s, 1 - (\theta_j/\pi)] \,.$$

Thus we obtain

$$\eta(0) = \sum_{0<\theta_j<\pi} 1 - 2(\theta_j/\pi)$$

as claimed.

COROLLARY 6.2. *For $L_2 = A \cdot L_1$ with $A \in U(n)$ we have*

$$\exp\left[-2\pi i\left(\frac{\eta(D(L_1,L_2)) + h}{2}\right)\right] = (-1)^n\,(det\ A)^2$$

where $h = \dim L_1 \cap L_2$ and $n = \dim V/2$.

Since the value $(\det A)^2$ is independent of the choice of B, we take A to be of the above form with $0 \leqq \theta_j < \pi$. Hence by Proposition 6.2,

$$\exp[-\pi i\eta(D(L_1,L_2))] = \exp\left[-\pi i \sum_{0<\theta_j<\pi}\left(1 - 2\left(\frac{\theta_j}{\pi}\right)\right)\right]$$
$$= \exp\left[-\pi i(n-h) + 2i\sum_{j=1}^{n}\theta_j\right] = (-1)^n\cdot(-1)^h\cdot(\det A)^2\ ,$$

as claimed.

In Section 3, the flat unitary bundle

$$\xi_A = [0,1]\times\mathbb{C}^n/(0,\nu) \sim (1, A\cdot A^t\nu) \tag{6.7}$$

over S^1 was introduced and the operator $D^\#(L_1,L_2)$ was defined as $-i\frac{d}{dt}$ acting on the sections of ξ_A. As shown in (3.1), $D^\#(L_1,L_2)$ is complex self-adjoint with $\dim_{\mathbb{C}} \ker D^\#(L_1,L_2) = \dim_{\mathbb{R}}(L_1\cap L_2)$. Equivalently $D^\#(L_1,L_2)$ may be viewed as $-i\frac{d}{dt}$ acting on functions ψ: $[0,1]\to\mathbb{C}^n$ with boundary conditions

$$\psi(1) = A\cdot A^t\psi(0)\ . \tag{6.8}$$

If we diagonalize $A\cdot A^t$ as

$$A\cdot A^t = C^{-1}D(e^{i\phi_1},\ldots,e^{i\phi_n})C \tag{6.9}$$

by a unitary matrix C and choose ϕ_j to lie in $[0,2\pi)$, then, as in Proposition 6.1, the eigenvalues of $D^\#(L_1,L_2)$ may be easily computed. They are $\{2\pi m_j + \phi_j | m_j \in \mathbb{Z}\}_{1\leqq j\leqq n}$ and the associated eta function $\eta(s)$ is

$$\frac{1}{(2\pi)^s}\sum_{0<\phi_j<2\pi}\left\{\sum \text{sign}(m_j + \phi_j/2\pi)|m_j + (\phi_j/2\pi)|^{-s}\right\}$$
$$= \frac{1}{(2\pi)^s}\sum_{0<\phi_j<2\pi}\{\zeta(s,(\phi_j/2\pi) - \zeta(s, 1-\phi_j/2\pi)\}\ .$$

Consequently, we arrive at the formula for the eta invariant

$$\eta(D^\sharp(L_1, L_2)) = \sum_{0<\phi_j<2\pi} (1 - 2(\phi_j/2\pi)) . \tag{6.10}$$

The relation between the eigenvalues $\{e^{i\theta_j}\}$ of A and $\{e^{i\phi_j}\}$ of $A \cdot A^t$ is not obvious; however, their determinants satisfy

$$\det(A \cdot A^t) = (\det A)^2$$

and so $e^{i(\Sigma\phi_j)} = e^{i(\Sigma 2\theta_j)}$.

In particular, we have

$$\frac{1}{2\pi}\left(\sum_{0<\phi_j<2\pi} \phi_j\right) = \frac{1}{2\pi}\left(\sum_{0<\theta_j<\pi} 2\theta_j\right) \mod \mathbb{Z} .$$

As a consequence (by Corollary 6.2 for $\eta(D)$ and $\eta(D^\sharp)$), the eta invariants $\eta(D(L_1, L_2))$, $\eta(D^\sharp(L_1, L_2))$ are related by

$$\frac{1}{2}[\eta(D(L_1, L_2)) + h] = \frac{1}{2}[\eta(D^\sharp(L_1, L_2)) + h] \mod \mathbb{Z} . \tag{6.11}$$

Here, $h = \dim L_1 \cap L_2 = \dim_{\mathbb{R}} \ker D(L_1, L_2) = \dim_{\mathbb{C}} \ker D^\sharp(L_1, L_2)$ by Lemma 3.1. In fact, we have a stronger relation as in the following:

PROPOSITION 6.3. *The two eta invariants $\eta(D(L_1, L_2))$ and $\eta(D^\sharp(L_1, L_2))$ are the same.*

Because of (6.3), we may interchange $\eta(D(L_1, L_2))$ and $\eta(D^\sharp(L_1, L_2))$, and similarly the spectral the flow of the two families of operators $D(L_1(t), L_2(t))$ and $D^\sharp(L_1(t), L_2(t))$.

Proof of Proposition 6.3: Let $B(s)$ be a smooth family of unitary matrices such that $B(0) = \text{id}$, $B(1) = B$, and $L_2 = B^{-1}D(e^{i\theta_1}, \dots, e^{i\theta_n})B \cdot L_1$, $0 \leqq \theta_j < \pi$, as discussed above. Set $A(s) = B^{-1}(s)D(e^{i\theta_1}, \dots, e^{i\theta_n})B$ and define the function

$$h(s) = \eta(D(L_1, A(s)L_1)) - \eta(D^\sharp(L_1, A(s)L_1)) .$$

Now $\dim L_1 \cap A(s).L_1 = \dim[B(s)L_1] \cap [D(e^{i\theta_1}, \dots, e^{i\theta_n}) \cdot B(s)L_1] = \{\sharp\, \theta_j$ with $\theta_j = 0\}$ is independent of s and is equal to $\dim \ker D(L_1, A(s)L_2)$ and also $\dim_{\mathbb{C}} \ker D^\sharp(L_1, A(s)L_1)$ by Lemma 3.1. Moreover, the kernel of $D^\sharp(L_1, A(s)L_2)$, by Lemma 3.1, coincides with the complexification of $\mathbb{R}^n \cap A(s) \cdot \mathbb{R}^n = \mathbb{R}^n \cap B(s)^{-1}D(e^{I\theta_1}, \dots, e^{i\theta_n})B(s)\mathbb{R}^n = B^{-1}(s)[B(s)\mathbb{R}^n \cap D(e^{i\theta_1}, \dots, e^{i\theta_n})B(s)\mathbb{R}^n]$, which has constant dimension ($= \dim L_1 \cap L_2$) and varies continuously with respect to s. In particular there is zero spectral flow, and by the continuity of these kernels we conclude that $h(s)$ is a continuous function of s. As observed in (6.11), however, $h(s)$ is integer valued and hence $h(1) = h(0)$.

In the specal case when $A(0) = D(e^{i\theta_1}, \dots, e^{i\theta_n})$, we have $A(0) \cdot A(0)^t = D(e^{i2\theta_1}, \dots, e^{i2\theta_n})$ and so

$$\eta(D^\sharp(L_1, A(0)L_1)) = \sum_{0 \leqq \theta_j < \pi} 1 - 2\left(\frac{2\theta_j}{2\pi}\right) = \eta(D(L_1, A(0)L_1)) ,$$

by (6.10). Hence, $h(1) = h(0) = 0$, proving Proposition 6.2.

In order to define $\mu_{\text{anal},1}(f)$, one more ingredient is needed: a canonical one form ω on Lag V, depending only on the choice $\{J, \langle\, , \rangle\}$. Fix a reference Lagrangian L_0, we have the map

$$(\det)^2\colon \text{Lag } V \to S^1 , \qquad \det{}^2(A \cdot L_0) = (\det A)^2$$

defined as before. Pulling back the standard 1-form $(1/2\pi)\,d\theta$ on the circle $S^1 = \{e^{i\theta}\}$ via this map $(\det)^2$, we have the canonical 1-form (sometimes called the Keller-Arnold-Maslov form)

$$\omega = (\det{}^2)^* \left(\frac{1}{2\pi}\, d\theta\right) \qquad \text{on Lag } V . \tag{6.12}$$

Since $(1/2\pi)\,d\theta$ represents the generator of $H^1(S^1, \mathbb{Z}) \cong \mathbb{Z}$ and $(\det)^2$ induces an isomorphism on cohomology (see [18], p. 116),

$$H^1(\text{Lag } V; \mathbb{Z}) \xrightarrow{\cong} H^1(S^1; \mathbb{Z}) \cong \mathbb{Z} ,$$

the above 1-form ω represents in $H^1(\text{Lag } V; \mathbb{R})$ an integral generator of

$$H^1(\text{Lag } V; \mathbb{Z}) \cong \mathbb{Z} .$$

Note that another choice of the reference Lagrangian $L_0' = B \cdot L_0$ would change $(\det^2)$ by the multiplication of a constant $(\det B)^{-2}$. Since such a multiplication leaves $(1/2\pi)\,d\theta$ invariant, the 1-form ω in (6.12) is independent of the choice of L_0.

Definition of $\mu_{\text{anal},1}(f)$. Given a pair of continuous and piecewise smooth Lagrangians $f(t) = (L_1(t), L_2(t))$, $a \leqq t \leqq b$, we define (see Theorem 0.3 in the Introduction),

$$\begin{aligned} \mu_{\text{anal},1}(f) = {} & \frac{1}{2}\left[\eta(D(L_1(b), L_2(b))) - \eta(D(L_1(a), L_2(a)))\right] \\ & + \int_a^b \left[L_2^*(\omega) - L_1^*(\omega)\right] + \frac{1}{2}(-h_{12}(b) + h_{12}(a)) . \end{aligned} \tag{6.13}$$

where $L_j^*(\omega)$, $j = 1, 2$; is the pullback of ω by the map $L_j\colon [a, b] \to \text{Lag } V$, $t \mapsto L_j(t)$.

In Section 7, we will compare the above formula (6.13) with the Atiyah-Patodi-Singer Index Theorem. Note that, by Proposition 6.2, we may replace $\eta(D(L_1, L_2))$ by $\eta(D^\sharp(L_1, L_2))$.

PROPOSITION 6.4. *The formula* (6.13) *defines* $\mu_{\text{anal},1}(f)$ *as an integer invariant depending only on the symplectic structure and* $\mu_{\text{anal},1} : P([a,b], V) \to \mathbb{Z}$ *satisfies Properties* I *to* XII *of Section* 2. *That is,* $\mu_{\text{anal},1}(f)$ *is a Maslov index.*

Proof of Proposition 6.4: From our previous discussion, it suffices to check that $\mu_{\text{anal},1}(f)$ is an integer depending on $\{\,,\,\}$ and satisfies Properties I to VI.

Note that all the terms except the integrals $\int_a^b L_j^*(\omega)$ depend only on the end points. Since $\int_a^b L_j^*(\omega)$, $j = 1, 2$, are invariant under the deformation of f keeping the end points fixed, the Deformation Property II is apparent. Similarly, Affine Scale Invariance holds because these integrals are unchanged and Path Additive Property III follows because

$$\int_a^b L_j^*(\omega) + \int_b^c L_j^*(\omega) = \int_a^c L_j^*(\omega)\ .$$

As for Symplectic Additivity Property IV, we need only to observe that under the inclusion

$$i\colon\ \text{Lag}\, V \times \text{Lag}\, W \hookrightarrow \ \text{Lag}\,(V \oplus W)$$

the pullback 1-form $i^*(\omega)$ equals $\pi_1^*\omega + \pi_2^*\omega$ where π_j, $j = 1, 2$, denote respectively the projection on the first and second factor. This computation for $i^*(\omega)$ follows from a similar additive property for the map $\det^2$.

Note that the dependence of $\mu_{\text{anal},1}(f)$ only on the symplectic structure $\{\ \}$ and the Symplectic Invariance Property V follows once $\mu_{\text{anal},1}(f)$ is shown to be an integer. Indeed, under such deformations the dimension $h_{12}(t) = \dim L_1(t) \cap L_2(t) = \ker D(L_1(t), L_2(t))$ is unchanged and so are the eta invariant terms; hence, $\mu_{\text{anal},1}(f)$ varies continuously. As an integer valued invariant, $\mu_{\text{anal},1}(f)$ is constant for such deformations.

To show $\mu_{\text{anal},1}(f)$ is an integer, we choose a reference Lagrangian L_0, and two continuous and piecewise smooth families of unitary matrices $A_1(t)$, $A_2(t)$ with

$$L_1(t) = A_1(t) \cdot L_0\ , \qquad L_2(t) = A_2(t) \cdot L_0$$

for $a \leqq t \leqq b$. Diagonalize $A_j(t)$, $j = 1, 2$ in a piecewise smooth manner

$$A_j(t) = B_j^{-1}(t) \cdot D(e^{i\theta_{1j}(t)}, \ldots, e^{i\theta_{nj}(t)}) \cdot B_j(t)$$

where $\theta_{\alpha j}(t)$, $1 \leqq \alpha \leqq n$, $j = 1, 2$, are continuous and likewise $B_j(t) \in U(n)$. Thus

$$(\det^2)(L_j(t)) = e^{2i(\sum_{\alpha=1}^n \theta_{\alpha j}(t))}$$

and so

$$\int_a^b L_j^*(\omega) = \frac{1}{\pi}\int_a^b \sum_{\alpha=1}^n \frac{d\theta_{\alpha j}}{dt}\,dt\ .$$

Hence,

$$\begin{aligned}&\exp 2\pi i\left(\int_a^b (L_2^*(\omega) - L_1^*(\omega))\right)\\ &= \exp 2i\left\{\sum_{\alpha=1}^n [\theta_{\alpha_2}(b) - \theta_{\alpha_1}(b)] - [\theta_{\alpha_2}(a) - \theta_{\alpha_1}(a)]\right\}\\ &= [\det A_2(b)A_1(b)^{-1}]^2 / [\det A_2(a)A_1(a)^{-1}]^2\ .\end{aligned}$$

On the other hand, by Corollary 6.2,

$$\begin{aligned}&\exp\left(-2\pi i\left\{\frac{1}{2}[n(D(b)) - \eta(D(a)] + \frac{1}{2}[-h_{12}(b) + h_{12}(a)]\right\}\right)\\ &= \exp\left(-\pi i\{[\eta(D(b)) + h_{12}(b)] - [\eta(D(a)) + h_{12}(a)]\}\right)\\ &= [\det A_2(b)A_1(b)^{-1}]^2 / [\det A_2(a)A_1(a)^{-1}]^2\ .\end{aligned}$$

Thus $\exp 2\pi i \mu_{\mathrm{anal},1}(f) = 0$. That is, $\mu_{\mathrm{anal},1}(f)$ is an integer as claimed. (Here $\eta(D(t))$ is short for $\eta(D(L_1(t), L_2(t)))$.)

To complete the proof of Proposition 6.3 it remans to check the Normalization Property VI. Applying (6.1) to the setting $\mathbb{R}^2 \cong \mathbb{C}$, $L_1(\theta) = \mathbb{R}$, $L_2(\theta) = \mathbb{R}e^{i\theta}$, we have

$$\eta(D(L_1, L_2)) = \begin{cases} 1 - \frac{2\theta}{\pi} & \text{for } 0 < \theta \leqq \frac{\pi}{4}\\ 0 & \text{for } \theta = 0\\ 1 - \frac{2(\theta+\pi)}{\pi} & \text{for } -\frac{\pi}{4} \leqq \theta < 0\end{cases}$$

and also

$$L_1^*(\omega) = 0\ ,\quad L_2^*(\omega) = \frac{d\theta}{\pi}\ ,\qquad -\frac{\pi}{4} \leqq \theta \leqq \frac{\pi}{4}\ .$$

For $f(t) = (\mathbb{R}, e^{it}\mathbb{R})$, $-\frac{\pi}{4} \leqq t \leqq \frac{\pi}{4}$, the term $h_{12}(-\frac{\pi}{4}) = h_{12}(\frac{\pi}{4}) = 0$, $h_{12}(0) = 1$. Putting these data into (6.13), it is straightforward to see that

$$\begin{aligned}\mu_{\mathrm{anal},1}\left(f\left|\left[-\frac{\pi}{4}, 0\right]\right.\right) &= \frac{1}{2}\left[0 - \left\{1 - 2\left(\frac{3\pi/4}{\pi}\right)\right\}\right] + \int_{-\pi/4}^0 \frac{d\theta}{\pi} + \frac{1}{2}[-1+0]\\ &= 0\end{aligned}$$

while $\mu_{\mathrm{anal},1}(f|[0,\pi/4]) = 1$ and $\mu_{\mathrm{anal},1}(f|[-\pi/4,\pi/4]) = 1$. This completes the proof of Proposition 6.4.

Note that if $L_2 = AL_1$ with $A = B^{-1}D(e^{i\theta_1}, \ldots, e^{i\theta_n})B$ and $0 \leqq \theta_j < \pi$, then $L_1 = A^{-1}L_2$ with A as above. By $e^{-i\theta_j} = -e^{i(\pi-\theta_j)}$ and Proposition 6.1, then:

PROPOSITION 6.5.

$$\begin{aligned}\eta(D(L_2, L_1)) &= \sum_{0 \leqq \theta_j < \pi} \left[1 - 2\left(\frac{\pi - \theta_j}{\pi}\right)\right] \\ &= \sum_{0 \leqq \theta_j < \pi} \left\{-1 + 2\left(\frac{\theta_j}{\pi}\right)\right\} \\ &= -\eta(D(L_1, L_2))\,.\end{aligned}$$

In many applications the interpretation of the Maslov index as $\mu_{\text{anal},1}(f)$ in (6.13) is very useful and flexible. We will see examples of this in proving Theorems 0.1 and 0.2 of Section 0.

7. Second Analytical Definition of the Maslov Index: $\mu_{\text{anal},2}(f)$

Let $f(t) = (L_1(t), L_2(t))$, $a \leqq t \leqq b$, be a continuous and piecewise smooth family of pairs of Lagrangians. As shown in Lemma 3.1, there is a corresponding family of self-adjoint, elliptic operators $D(L_1(t), L_2(t))$, $a \leqq t \leqq b$. At the endpoints $t = a$, $t = b$ there exists an $\varepsilon > 0$ which is smaller than the absolute value of any nonzero eigenvalue of $D(L_1(a), L_2(a))$ and $D(L_1(b), L_2(b))$. Then the self-adjoint operator $D(L_1(t), L_2(t)) - \varepsilon \cdot (\text{id})$ has no 0-eigensolutions for $t = a, b$, and allows us to define

(7.1) $$\mu_{\text{anal},2}(f) = \text{ spectral flow of } [D(L_1(t), L_2(t)) - \varepsilon \text{ id }]\,, \qquad a \leqq t \leqq b$$

This is the ε-spectral flow of Section 0; it counts with signs (+1 for increasing value, −1 for decreasing) and multiplicities the number of eigenvalues of $D(L_1(t), L_2(t))$ crossing the line $\lambda = \varepsilon$. The situation is represented schematically in Diagram 2.

Clearly $\mu_{\text{anal},2}(f)$ is independent of the choice of $\varepsilon > 0$. It is also easy to verify that it satisfies Properties (I)–(V). For example, in verifying III we only have to choose $\varepsilon > 0$ small enough so that for $t = a, b, c$, the operator $D(L_1(t), L_2(t))$ has no eigenvalues in the interval $(0, \varepsilon]$. Then the additivity of $\mu_{\text{anal},2}(f)$ follows from the additivity of spectral flow. For example, the deformation property of (II) holds because these deformations leave $h_{12}(a)$ and $h_{12}(b)$ unchanged and so we can choose $\varepsilon > 0$ so that no eigenvalues for $D(L_1(b), L_2(b))$ and $D(L_1(a), L_2(a))$ are in the range $(0, \varepsilon]$ throughout the deformation. Similarly, for the symplectic invariance of V.

For the Normalization Property VI, we recall the discussion preceding Proposition 6.1 that for $-\pi/4 \leqq \theta \leqq \pi/4$ the operator $D(\mathbb{R}, \mathbb{R}e^{i\theta})$ has eigenvalues $\pi m + \theta$. Thus in the situation of (1.7)(i)–(iii), the spectral curves of $f(t) = (\mathbb{R}, \mathbb{R}e^{it})$ are straight lines.

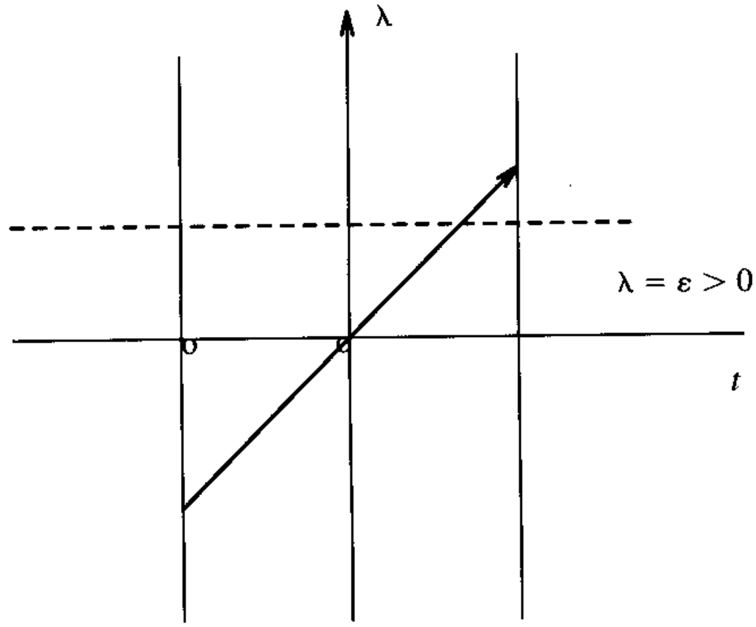

Diagram 9. For eigenvalues in the range $(-\pi/2, \pi/2)$, a single spectral line of $D(\mathbb{R}, \mathbb{R}e^{i\theta})$.

From Diagram 9 above, it is apparent that the ε-spectral flow is 0 on $-\pi/4 \leqq t \leqq 0$ and 1 on $0 \leqq t \leqq \pi/4$, as desired. Thus we obtain the proof of the following (by using Theorem 1.1).

PROPOSITION 7.1. *The invariant $\mu_{\text{anal},2}(f)$ depends only on the symplectic structure $\{\,,\,\}$ not on the choice of complex structure $J, \langle\ \rangle$. Moreover,*

$$\mu_{\text{anal},2}\colon \mathscr{P}([a,b],V) \to \mathbb{Z}$$

satisfies properties I *to* XII *of Section* 2 *and* $\mu_{\text{geo},1}(f) = \mu_{\text{geo},2}(f) = \mu_{\text{anal},1}(f) = \mu_{\text{anal},2}(f)$.

To clarify the two definitions $\mu_{\text{anal},1}(f)$ in (6.13) and $\mu_{\text{anal},2}(f)$ in (7.1), we compare the quantity $\mu_{\text{anal},1}(f) - h_{12}(a) = \mu_{\text{anal},2}(f) - h_{12}(a)$ with the Index Theorem in [3]. Note that $\mu_{\text{anal},1}(f) - h_{12}(a)$ equals by definition

$$\frac{1}{2}[\eta(D(L_1(b), L_2(b)) - \eta(D(L_1(a), L_2(a))]$$
$$+ \int_a^b [L_2^*(\omega) - L_1^*(\omega)] - \frac{1}{2}(h_{12}(b) + h_{12}(a))\,.$$

On the other hand, $\mu_{\text{anal},2}(f) - h_{12}(a)$ is the spectral flow (or $(-\varepsilon/+\varepsilon)$ spectral flow) counting the number of eigenvalues of $D(L_1(t), L_2(t))$ crossing the tilted line from $(a, -\varepsilon)$ to (b, ε). This is seen by deforming this line to a line passing from $(a, -\varepsilon)$ to $(a, +\varepsilon)$ thence straight to $(b, +\varepsilon)$.

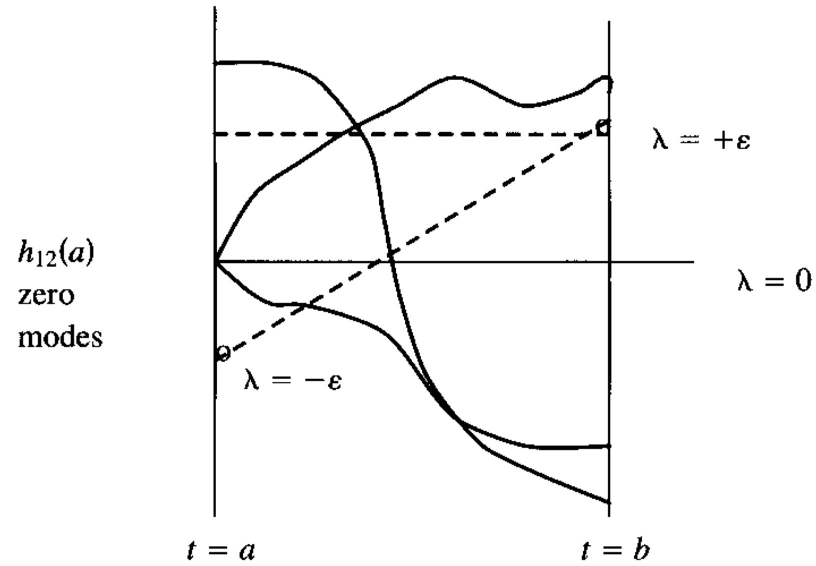

Diagram 10. The difference between the two crossing numbers with $\lambda = -\varepsilon$ and the tilted line is $h_{12}(a)$.

To relate these expressions to the Atiyah-Patodi-Singer Theorem, we recall that

$$\text{Index } D = \int_X \alpha_0(x)\,dx - \frac{1}{2}\left[\eta\left(\widehat{D}\right) + h\right],$$

where D is a first-order, elliptic operator on an oriented manifold Y with boundary $X = \partial Y$. On a product neighborhood $X \times [0,1]$ in Y, the operator D is asumed to be of the form $\pi^*\sigma \circ \left(\frac{\partial}{\partial s} + \pi^*\widehat{D}\right)$ where s the normal coordinate, $\partial/\partial s$ points inward, $\pi\colon [0,1]\times X \to X$ is the projection, and $\widehat{D}$ is a self-adjoint elliptic operator on X. Let $\mathscr{H}$ be the kernel of $\widehat{D}$, and let P_+ (respectively P_-) be the span of the positive (negative) eigenvectors for $\widehat{D}$. The operator D becomes Fredholm when the boundary condition $\phi|X \in P_-$ is imposed on sections ϕ over Y. Index D refers to the index of this Fredholm operator; $\eta(\widehat{D})$ to the eta invariant of $\widehat{D}$, and $h = \dim \ker \widehat{D} = \dim \mathscr{H}$.

Now consider the situation when $Y = \Sigma \times [a,b]$ and $D = \frac{\partial}{\partial t} + \widehat{D}(t)$ where $\widehat{D}(t)$, $a \leqq t \leqq b$, is a smooth family of self-adjoint operators on Σ. Since $+\partial/\partial t$ points inward into $\Sigma \times [a,b]$ at a and outward at b, the Atiyah-Patodi-Singer formula reads:

$$\text{Index } D = \int \alpha_0(x)\,dx - \frac{1}{2}\left[-\eta\left(\widehat{D}(b)\right) + \eta\left(\widehat{D}(a)\right)\right] - \frac{1}{2}\left[h(b) + h(a)\right].$$

Here the sign $-\eta(\widehat{D}(b))$ is due to the change from $\widehat{D}(b)$ to $-\widehat{D}(b)$ in our formula $\frac{\partial}{\partial t}+\widehat{D}=(-1)\cdot(-\frac{\partial}{\partial t}+(-\widehat{D}))$ at $t=b$. As for $h(t)=$ dim ker $\widehat{D}$, it is independent of this sign change. Thus if we consider $\widehat{D}(t)$ as a smooth family of operators, the spectral flow of this variation is by definition the change in Index D. In particular, if $h(a)=h(b)=0$, then

$$\text{Index}(D|\Sigma\times[a,b]) = \text{spectral flow of } \{\widehat{D}(t);\ a \leqq t \leqq b\}\ .$$

In all cases,

$$\begin{aligned}\text{Index}(D|\Sigma\times[a,b]) + h(a) = \int \alpha_0(x)\,dx &+ \frac{1}{2}[\eta(\widehat{D}(b)) - \eta(\widehat{D}(a))]\\ &+ \frac{1}{2}(-h(b)+h(a))\end{aligned}$$

which is the analog of $\mu_{\text{anal},1}(f)$ for $f(t)=(L_1(t),L_2(t))$, $a \leqq t \leqq b$, as defined in (6.13).

We claim as in Theorem 0.4 that, analogous to $\mu_{\text{anal},2}(f)$, the quantity Index $(D|\Sigma\times[a,b]) + h(a)$ is the ε-spectral flow of $\widehat{D}(t)$, $a \leqq t \leqq b$. Here $\varepsilon>0$ is chosen so that no nonzero eigenvalues $\lambda \neq 0$ of $\widehat{D}(a)$ or $\widehat{D}(b)$ lies in $[-\varepsilon,\varepsilon]$, and the ε-spectral flow is computed by counting the crossings of spectral curves with respect to the line $\lambda=\varepsilon$. As noted earlier, for $D(L_1(t),L_2(t))$, this means that Index$(D|\Sigma\times[a,b])$ is the spectral flow of $\widehat{D}(t)$ with respect to a tilted line. For operators in gauge theory, a discussion of such spectral flow can be found in [13].

To prove the above claim, we may assume, by a smooth perturbation, that the 0-eigenvalues for $\widehat{D}(a)$, $\widehat{D}(b)$ bifurcate transversally into nonzero eigenmodes in the interval $[a,a+\delta]$, $[b-\delta,b]$. Let $h_+(a), h_+(b)$ denote the numbers of positive eigenmodes after the bifurcation and $h_-(a), h_-(b)$, the numbers of negative eigenmodes. Then

$$\begin{aligned}&\lim_{\delta\to 0}\eta(\widehat{D}(b-\delta)) - [h(b)_+ - h(b)_-] = \eta(\widehat{D}(b))\\ &\lim_{\delta\to 0}\eta(\widehat{D}(a+\delta)) - [h(a)_+ - h(a)_-] = \eta(\widehat{D}(a))\ .\end{aligned}$$

By the Atiyah-Patodi-Singer formula, we have (dropping $\widehat{D}$ for convenience)

$$\begin{aligned}&\text{Index } D|\Sigma\times[a,b]\\ &= \int_a^b \alpha_0(x)\,dx + \frac{1}{2}(\eta(b)-\eta(a)) - \frac{1}{2}(h(b)+h(a))\\ &= \int_a^b \alpha_0(x)\,dx + \frac{1}{2}\lim_{\delta\to 0}[\eta(b-\delta)-\eta(a+\delta)] + \frac{1}{2}(-h(b)_+ + h(a)_+)\\ &\quad - \frac{1}{2}(-h(b)_- - h(a)_-) - \frac{1}{2}(h(b)+h(a))\\ &= \lim_{\delta\to 0}\{\text{spectral flow for } \widehat{D}(t);\ a+\delta \leqq t \leqq b-\delta\} - h(b)_+ - h(a)_-)\ .\end{aligned}$$

From the following Diagram 11, it is easy to see that the last expression is the $(-\varepsilon/\varepsilon)$-spectral flow of $\hat{D}(t)$, $a \leqq t \leqq b$, and so

$$\text{Index } D|\Sigma \times [a,b] + h(a) = \{\varepsilon\text{-spectral flow of } \hat{D}(t)\ ,\ a \leqq t \leqq b\}\ .$$

This proves Theorem 0.4.

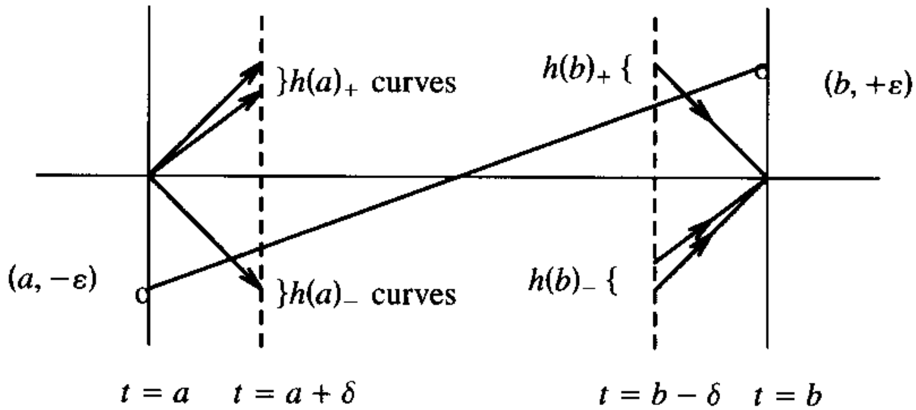

Diagram 11.

8. The Maslov Triple Index: $\tau(L_1, L_2, L_3)$

Given three Lagrangians L_1, L_2, L_3 in a symplectic vector space $(V, \{\ ,\ \})$, Kashiwara (see [24]) defines an integer $\tau_V(L_1, L_2, L_3)$ as the signature of the symmetric quadratic form

$$Q: L_1 \oplus L_2 \oplus L_3 \to \mathbb{R}$$
$$Q(x_1, x_2, x_3) = \{x_1, x_2\} + \{x_2, x_3\} + \{x_3, x_1\}\ .$$

This invariant τ_V can be shown to have the following properties:

Property I (Skew Symmetry). For a permutation σ of the three letters $\{1,2,3\}$,

$$\tau_V(L_{\sigma(1)}, L_{\sigma(2)}, L_{\sigma(3)}) = \text{Sign}(\sigma) \cdot \tau(L_1, L_2, L_3)\ .$$

Property II (Symplectic Additivity). For three Lagrangians $L_j \subset V$, $j = 1,2,3$, and three Lagrangians $L_j^\# \subset W$, $j = 1,2,3$,

$$\tau_{V\oplus W}(L_1 \oplus L_1^\#, L_2 \oplus L_2^\#, L_3 \oplus L_3^\#) = \tau_V(L_1, L_2, L_3) + \tau_W(L_1^\#, L_2^\#, L_3^\#)\ .$$

Property III (Symplectic Invariance). For a symplectic automorphism $g \in \text{Sp}(V)$,

$$\tau_V(L_1, L_2, L_3) = \tau_V(g(L_1), g(L_2), g(L_3))\ .$$

Property IV (Normalization). For the Lagrangians $\mathbb{R}$, $\mathbb{R}(1+i)$, $\mathbb{R}(i)$, in $\mathbb{R}^2 = \mathbb{C}$ with the standard skew symmetric pairing $\{\ ,\ \}$ in $\mathbb{C}$,

$$\tau_{\mathbb{C}}(\mathbb{R}, \mathbb{R}(1+i), \mathbb{R}(i)) = 1 .$$

Two more properties, V and VI, are proved in [24].

Property V (L_1 Transverse to L_3). If $L_1 \cap L_3 = 0$, then $\tau_V(L_1, L_2, L_3)$ is the signature of the quadratic form

$$q: L_2 \to \mathbb{R}, \qquad q(x) = \{x', x''\}$$

where $x = x' + x''$ is the decomposition of x in $L_1 + L_3 = V$ (cf. [18], p. 40).

Property VI (Cocycle Condition). For four Lagrangians L_1, L_2, L_3, L_4 in V,

$$\tau_V(L_1, L_2, L_3) = \tau(L_1, L_2, L_4) + \tau_V(L_2, L_3, L_4) + \tau_V(L_3, L_1, L_4)$$

(cf. [24], p. 42).

Property V can be taken to be the definition of triple index $\tau(L_1, L_2, L_3)$ when $L_1 \cap L_3 = 0$. For example, Guillemin and Steinberg defined

$$i(L_1, L_2, L_3) = \frac{1}{2} \text{ signature of quadratic form}$$
$$x \mapsto \{x'', x\} = -\{x', x''\}, \quad x \in L_2$$

which is $-\frac{1}{2}$ of Kashiwara's index (cf. [24], p. 126).

Analogous to the characterization of Maslov index $\mu_V(f)$ in Theorem 1.1, we will prove the following:

THEOREM 8.1.
(i) *There is a unique system of functions* $\tau_V(L_1, L_2, L_3)$ *which satisfies the above Properties* I, II, III, *and* IV.
(ii) *Any such system which satisfies Properties* I *through* IV *equals Kashiwara's* $\tau_V(L_1, L_2, L_3)$ *and so satisfies Properties* V *and* VI.
(iii) *If a system* $\tilde{\tau}_V(L_1, L_2, L_3)$ *satisfies only Properties* I, II, *and* III, *then*

$$\tilde{\tau}(L_1, L_2, L_3) = k\tau_V(L_1, L_2, L_3)$$

where $k = \tilde{\tau}_{\mathbb{C}}(\mathbb{R}, \mathbb{R}(1+i), \mathbb{R}(i))$.

We consider C. T. C. Wall's definition $\tau_V(L_1, L_2, L_3)_{\text{Wall}}$ of the triple in [31] as an application of (8.1) (iii). Let $\mathscr{V}(L_1, L_2, L_3)$ denote the subspace in $L_1 \oplus L_2 \oplus L_3$

consisting of (x_1, x_2, x_3) with $x_1 + x_2 + x_3 = 0$. Defined on $\mathscr{V}(L_1, L_2, L_3)$ there is the quadratic form

$$\mathscr{Q}\colon \mathscr{V}(L_1, L_2, L_3) \to \mathbb{R}$$

given by $\mathscr{Q}'(x_1, x_2, x_3) = 3\{x_1, x_2\}$. Then $\tau_V(L_1, L_2, L_3)_{Wall}$ is the signature of this quadratic form $\mathscr{Q}(x_1, x_2, x_3)$. From the equality

$$\begin{aligned} Q(x_1, x_2, x_3) &= \{x_1, x_2\} + \{x_2, x_3\} + \{x_3, x_1\} \\ &= \{x_1, x_2\} + \{x_2, -x_1 - x_2\} + \{-x_1 - x_2, x_1\} = \mathscr{Q}'(x_1, x_2, x_3) \end{aligned}$$

on $\mathscr{V}(L_1, L_2, L_3)$, the properties (I), (II), (III) are immediate. On $\mathscr{V}(\mathbb{R}, \mathbb{R}(1+i), \mathbb{R}(i))$, this quadratic form equals

$$t \mapsto (t; -t(1+i), ti) \mapsto 3\{t, -(1+i)t\} = -3t^2$$

and hence $\tau_V(\mathbb{R}, \mathbb{R}(1+i), \mathbb{R}(i)) = -1$.

PROPOSITION 8.2. *Let $\tau_V(L_1, L_2, L_3)_{\text{Wall}}$ be defined as above. Then*

$$\tau_V(L_1, L_2, L_3)_{\text{Wall}} = -\tau_V(L_1, L_2, L_3)\ .$$

Another application of Theorem 8.1 is in proving Theorem 0.1 which is restated here:

PROPOSITION 8.3. *For a triple of Lagrangians L_1, L_2, L_3 in V,*

$$\tau(L_1, L_2, L_3) = \{\eta(D(L_1, L_2)) + \eta(D(L_2, L_3)) + \eta(D(L_3, L_1))\}\ .$$

Proof of Proposition 8.3: By Theorem 8.1, we only have to show: first, the right-hand side is independent of the choice of complex structure J and $\langle\ \rangle$; second, the right-hand side satisfies properties I, II, III, IV.

Now if $\{J_t, \langle\ ,\ \rangle_t\}_{0\leqq t\leqq 1}$ is a smooth, 1-parameter family of complex structures and Hermitian inner products, then there exist smoothly varying unitary matrices $A_1(t), A_2(t)$ with

$$L_2 = A_1(t)\cdot L_1\ , \quad L_3 = A_2(t)\cdot L_2\ , \qquad 0 \leqq t \leqq 1\ .$$

By Proposition 6.1,

$$\eta(D(L_1, L_2) \text{ for } J_t, \langle\ \rangle_t) = \sum_{0<\theta_j<\pi}\left[1 - 2\left(\frac{\theta_j(t)}{\pi}\right)\right]$$

where $\pm e^{i\theta_j(t)}$ are the eigenvalues of $A_1(t)$, chosen so that $0 < \theta_j(t) \leqq \pi$ and hence

$$\exp\left[2\pi i\eta(D(L_1, L_2) \text{ for } J_t, \langle\ \rangle_t)\right] = \exp\left[-2\pi i\left(\sum_{j=1}^{n} \frac{2\theta_j(t)}{\pi}\right)\right] = [\det A_1(t)]^{-4} .$$

Similarly, we have

$$\begin{aligned}\exp\left[2\pi i\eta(D(L_2, L_3) \text{ for } J_t, \langle\ \rangle_t)\right] &= [\det A_2(t)]^{-4}\\ \exp\left[2\pi i\eta(D(L_3, L_1) \text{ for } J_t, \langle\ \rangle_t)\right] &= [\det(A_1(t))^{-1}\cdot(A_2(t))^{-1}]^{-4}\end{aligned}$$

and hence

$$\begin{aligned}&\exp\left\{2\pi i\left[\eta(D(L_1, L_2)) + \eta(D(L_2, L_3)) + \eta(D(L_3, L_1)) \text{ for } J_t, \langle\ \rangle_t\right]\right\}\\ &= [\det A_1(t)]^4[\det A_2(t)]^{-4}\cdot[\det A_1(t)^{-1}\cdot\ \det A_2(t)^{-1}]^{-4} = 1 .\end{aligned}$$

This last equation means that $\eta(D(L_1, L_2)) + \eta(D(L_2, L_3)) + \eta(D(L_3, L_1))$ is an integer for all t. On the other hand, the kernel of the operator $D(L_j, L_k)$ equals $L_j \cap L_k$ and is independent of t. It follows that $\eta(D(L_j, L_k))$ is continuous and hence the triple sum $\eta(D(L_1, L_2)) + \eta(D(L_2, L_3)) + \eta(D(L_3, L_1))$ is a fixed integer. Since two complex structures can be connected up by a path $J_t, \langle\ \rangle_t$, we have shown the independence of complex structure.

By Proposition 6.5, the right-hand side is skew-symmetric with respect to interchanging of a pair L_j, L_k, $j \neq k$. Since these are the generators of a symmetric group, Property I follows immediately. Property II is clear from the definition of $\eta(D(L_j, L_k))$. Property III is proved as above by deforming g to the identity. As for Property IV, it follows from a straightforward calculation using Proposition 6.1. Since the subspaces $\mathbb{R}(1+i)$, $\mathbb{R}(i)$, $\mathbb{R}$ can be written respectively as $\exp(i\frac{\pi}{4})\cdot\mathbb{R}$, $\exp(i\frac{\pi}{4})\cdot\mathbb{R}(1+i)$, $\exp(i\frac{\pi}{2})\cdot\mathbb{R}(i)$, we have

$$\begin{aligned}&\left\{\eta(D(\mathbb{R}, \mathbb{R}(1+i))) + \eta(D(\mathbb{R}(1+i), \mathbb{R}(i))) + \eta(D(\mathbb{R}(i), \mathbb{R}))\right\}\\ &= \left[1 - 2\left(\frac{\pi/4}{\pi}\right)\right] + \left[1 - 2\left(\frac{\pi/4}{\pi}\right)\right] + \left[1 - 2\left(\frac{\pi/2}{\pi}\right)\right]\\ &= 1\end{aligned}$$

This proves Proposition 8.3.

The above proposition may be combined with the definition of $\mu_{\mathrm{anal},1}(f)$ to prove Theorem 0.2. This is reformulated as:

PROPOSITION 8.4. *Let $L_j(t)$, $a \leqq t \leqq b$, $j = 1, 2, 3$, be three families of smoothly varying Lagrangians. Let $e(t) = (L_1(t), L_2(t))$, $f(t) = (L_2(t), L_3(t))$, $g(t) = (L_3(t), L_1(t))$, $a \leqq t \leqq b$, and let $h_{jk}(t)$ denote the dimension of $L_j(t) \cap L_k(t)$. Then* *the Maslov indices $\mu_V(e)$, $\mu_V(f)$, $\mu_V(g)$ are related to the triple indices $\tau(L_1(t), L_2(t), L_3(t))$ at $t = a, b$, by the formula:*

$$\frac{1}{2}\left[\tau(L_1(b), L_2(b), L_3(b)) - \tau(L_1(a), L_2(a), L_3(a))\right]$$
$$= \mu_V(e) + \mu_V(f) + \mu_V(g) + \frac{1}{2}\left\{\left[\sum_{j<k} h_{jk}(b)\right] - \left[\sum_{j<k} h_{jk}(a)\right]\right\}.$$

Proof of Proposition 8.4: Using the definition of $\mu_{\text{anal},1}(f)$, the sum $\mu_V(e) + \mu_V(f) + \mu_V(g)$ equals

$$\frac{1}{2}\left[\eta(D(L_1(b), L_2(b))) + \eta(D(L_2(b), L_3(b)) + \eta(D(L_3(b), L_1(b)))\right]$$
$$- \frac{1}{2}\left[\eta(D(L_1(a), L_2(a))) + \eta(D(L_2(a), L_3(a)) + \eta(D(L_3(a), L_1(a)))\right]$$
$$+ \frac{1}{2}\left[-\left(\sum_{j<k} h_{jk}(b)\right) + \left(\sum_{j<k} h_{jk}(a)\right)\right].$$

By Proposition 8.3, the first and second sum in the above can be replaced by Maslov triple indices, and hence $\mu_V(e) + \mu_V(f) + \mu_V(g)$ equals $\frac{1}{2}[\tau_V(L_1(b), L_2(b), L_3(b)) - \tau_V(L_1(a), L_2(a), L_3(a)] + \frac{1}{2}(-[\sum_{j<k} h_{jk}(b)] + [\sum_{j<k} h_{jk}(a)])$. This proves Proposition 8.4.

Another application of Proposition 8.3 is another proof of Property VI (Cocycle Condition). By Proposition 8.3, the right-hand side of the cocycle formula can be rewritten as $[\eta(D(L_1, L_2)) + \eta(D(L_2, L_4)) + \eta(D(L_4, L_1))] + [\eta(D(L_2, L_3)) + \eta(D(L_3, L_4)) + \eta(D(L_4, L_2)] + [\eta(D(L_3, L_1)) + \eta(D(L_1, L_4)) + \eta(D(L_4, L_3))]$. Since $\eta(D(L_j, L_k)) = -\eta(D(L_k, L_j))$, four of the terms in the last expression cancel leaving $\eta(D(L_1, L_2)) + \eta(D(L_2, L_3)) + \eta(D(KL_3, L_1))$, which is $\tau_V(L_1, L_2, L_3)$, by Proposition 8.3.

The combination of cocycle condition and symplectic invariance implies that the function

$$z\colon \mathrm{Sp}(2n, \mathbb{R}) \times \mathrm{Sp}(2n, \mathbb{R}) \times \mathrm{Sp}(2n, R) \to \mathbb{Z}$$
$$(g_1, g_2, g_3) \mapsto \tau(g_1 \cdot \mathbb{R}^n, g_2 \cdot \mathbb{R}^n, g_3 \cdot \mathbb{R}^n)$$

is a homogeneous 2-cocycle and so defines a cohomology class

$$[z] \in H^2(\mathrm{Sp}(2n, \mathbb{R})^\delta; \mathbb{Z})$$

where $\mathrm{Sp}(2n; \mathbb{R})^\delta$ stands for the real symplectic group endowed with discrete topology. Proposition 8.3 implies that regarding $[z]$ as a 2-cocycle in $\mathbb{R}$, the restriction $[z]|U(n)^\delta$ to the discrete unitary group $U(n)^\delta \subset \mathrm{Sp}(2n; \mathbb{R})^\delta$ is a coboundary,

$$z = \delta\omega \quad \text{with} \quad \omega(g_1, g_2) = \eta(D(g_1 \cdot \mathbb{R}^n, g_2 \cdot \mathbb{R}^n)) .$$

It is well known that the restriction of $[z]$ to $\mathrm{Sp}(2n;\mathbb{Z})$ in $\mathrm{Sp}(2n;\mathbb{R})^\delta$ gives a generator $[z|\mathrm{Sp}(2n;\mathbb{Z})]$ of $H^2(\mathrm{Sp}(2n;\mathbb{Z});\mathbb{Z}) \cong \mathbb{Z}$ for $n \geqq 2$ (cf. [24]).

As still another application of Proposition 8.3, we relate $\tau(L_1, L_2, L_3)$ to the index of a Cauchy-Riemann operator $\bar\partial \otimes \xi$ with coefficients on a flat Hermitian vector bundle $\xi(L_1, L_2, L_3)$ over a twice punctured disk X of Diagram 1. The holonomy is defined as $A \cdot A^t$, $B \cdot B^t$, over ∂D_1,∂D_2 with $L_2 = AL_1$, $L_3 = BL_2$.

As in [3], page 62, the tangential component of the Cauchy-Riemann operator $\bar\partial$ near ∂D_j can be identified with $-i\frac{\partial}{\partial\theta_j}$. Hence, forming the $\bar\partial$ operator coupled to $\xi(L_1, L_2, L_3)$, we obtain an elliptic operator $\bar\partial \otimes \xi(L_1, L_2, L_3)$ whose tangential component near ∂D_j coincides with $-i\frac{\partial}{\partial\theta_j}$ operating on sections of the flat, Hermitian vector bundles $\xi_{A\cdot A^t}$, $\xi_{B\cdot B^t}$. Here the notation is the same as in (3.5), and the tangential components of $\bar\partial \otimes \xi(L_1, L_2, L_3)$ are nothing but $D^\sharp(L_1, L_2), D^\sharp(L_2, L_3)$ on ∂D_1, respectively ∂D_2.

We regard X as embedded in the extended complex plane $\mathbb{CP}^1 \cong S^2$ with local coordinates z_1, z_2, z_3 on neighborhoods of the three disks D_1, D_2, D_3. The space $\mathbb{CP}^1$ is endowed with a metric so that on these disks $|z_i| \leqq 1$ and near the boundary ∂D_i the metric is $|dz_j/z_j|^2 = [d\log r_j]^2 + (d\theta_j)^2$ where $z_j = r_j e^{i\theta_j}$. With this metric, a neighborhood of $\partial X = \partial D_1 \cup \partial D_2 \cup \partial D_3$ is metrically a product $\partial X \times [0, \varepsilon)$.

In order to describe the operator $\bar\partial \otimes \bar\partial\xi(L_1, L_2, L_3)$ we need the following:

LEMMA 8.5. *Let L_1, L_2, L_3 be three Lagrangian subspaces in V. Then there exists a complex structure $(J, \langle\ \rangle)$ on V together with two unitary, diagonal matrices $A = D(e^{i\alpha_1}, \dots, e^{i\alpha_n})$, $B = D(e^{i\beta_1}, \dots, e^{i\beta_n})$ such that $L_2 = A \cdot L_1$ and $L_3 = B \cdot L_2$.*

Choosing A, B as above, we have a flat, Hermitian vector bundle $\xi(L_1, L_2, L_3)$ with fiber $\mathbb{C}^n$ such that, around the two boundary curves ∂D_1, ∂D_2 in the counterclockwise direction, the holonomies are respectively

$$A \cdot A^t = D(e^{2i\alpha_1}, \dots, e^{2i\alpha_n}) \quad \text{and} \quad B \cdot B^t = D(e^{2i\beta_1}, \dots, e^{2i\beta}) .$$

Hence the holonomy around the outside circle is

$$[(B \cdot B^t) \cdot (A \cdot A)^t]^{-1} = D(e^{-2i(\alpha_1+\beta_1)}, \dots, e^{-2i(\alpha_n+\beta_n)}) .$$

Thus, over ∂D_3 we get the operator $D^\sharp(L_3, L_1)$.

Note, by Proposition 6.3,

$$\eta(D^\sharp(L_1, L_2)) = \eta(D(L_1; L_2)) \text{ and } \dim \ker_{\mathbb{C}} D^\sharp(L_1, L_2) = \dim_{\mathbb{R}} \ker D(L_1, L_2) .$$

Hence, applying the Atiyah-Patodi-Singer theorem to $\bar\partial \otimes \xi(L_1, L_2, L_3)$ on X, it yields:

$$\begin{aligned}\text{Index } \bar\partial \otimes \xi(L_1, L_2, L_3) &= \int \alpha_0(x)\, dx \\ &\quad - \frac{1}{2}[\eta(D(L_1, L_2)) + \eta(D(L_2, L_3)) + \eta(D(L_3, L_1)] \\ &\quad - \frac{1}{2}[h_{12} + h_{23} + h_{31}] .\end{aligned}$$

Since $\xi(L_1, L_2, L_3)$ is flat, the integral term is n times the corresponding values for $\bar{\partial}$. As in [3], page 62, this last term is given by

$$\begin{aligned} \frac{1}{2}\int_X c_1 &= \frac{1}{2}(2 - 2\,\text{genus}(X) - \sharp\text{ of comp.}) \\ &= \frac{1}{2}(2-3) = -\frac{1}{2}\,. \end{aligned}$$

This proves the following:

PROPOSITION 8.6. *Let L_1, L_2, L_3, A, B, be defined as above. Then*

$$\begin{aligned} \text{Index } \bar{\partial} \otimes \xi(L_1, L_2, L_3) = &-\frac{n}{2} - \frac{1}{2}(\tau(L_1, L_2, L_3)) \\ &-\frac{1}{2}\left[\sum_{j<k} \dim\,(L_j \cap L_k)\right]. \end{aligned}$$

The proof of Lemma 8.5 is an immediate consequence of the following:

PROPOSITION 8.7. *Let L_1, L_2, L_3 be three Lagrangian subspaces in a symplectic vector space V. Then there exists a symplectic basis e_j, f_j, $1 \leqq j \leqq n$ such that*

(i) $$L_1 = \bigoplus_{j=1}^{n} \mathbb{R}e_j\,,$$

(ii) $$L_2 = \bigoplus_{j=1}^{n} \mathbb{R}(\alpha_j e_j + \beta_j f_j) \text{ where } (\alpha_j, \beta_j) \neq (0,0)$$

are pairs of real numbers ,

(iii) $$L_3 = \bigoplus_{j=1}^{n} \mathbb{R}(\gamma_j e_j + \delta_j f_j) \text{ where } (\gamma_j, \delta_j) \neq (0,0)$$

are pairs of real numbers .

Note that, any triple of Lagrangians (L_1, L_2, L_3) in $\mathbb{R}^2 \cong \mathbb{C}$ is symplectically equivalent to one of the following:

Type (1) $(\mathbb{R}, \mathbb{R}(1+i), \mathbb{R}(i))$
Type (2) $(\mathbb{R}, \mathbb{R}(i), \mathbb{R}(1+i))$
Type (3) $(\mathbb{R}, \mathbb{R}, \mathbb{R}(i))$, $(\mathbb{R}, \mathbb{R}(i), \mathbb{R})$, $(\mathbb{R}(i), \mathbb{R}, \mathbb{R})$

Type (4) ($\mathbb{R}$, $\mathbb{R}$, $\mathbb{R}$)

More generally, by Proposition 8.7, any triple (L_1, L_2, L_3) is symplectically equivalent to a direct sum of, say α of Type (1), β of Type (2), γ of Type (3), δ of Type (4). If $\tilde{\tau}_V(L_1, L_2, L_3)$ is a function satisfying Property I (Skew Symmetry), then $\tilde{\tau}_V$ vanishes on Type (3) and (4) and takes on opposite signs for Type (1) and Type (2). If $\tilde{\tau}_V$ also enjoys Property II (Symplectic Additivity) and Property III (Symplectic Invariance), then

$$\tilde{\tau}_V(L_1, L_2, L_3) = (\alpha - \beta)k$$

with $k = \tilde{\tau}_{\mathbb{C}}(\mathbb{R}, \mathbb{R}(1+i), \mathbb{R}(i))$. This proves Theorem 8.1, Part (i). Since Kashiwara's τ_V function satisfies Properties I to VI, the proof of Theorem 8.1 is complete after proving Proposition 8.7.

Proof of Proposition 8.7: If the three Lagrangians are transverse to each other, $L_1 \cap L_2 = L_2 \cap L_3 = L_3 \cap L_1 = 0$, then Proposition 8.7 is proved by Lions and Vergne; see [24], page 41. The general case can be deduced from this transverse situation by the following device.

Let L be an isotropic subspace in V and $L^\perp$ be its perpendicular subspace $\{x \in V | \{x, L\} = 0\}$. Then the restriction of $\{\ \}$ to $L^\perp$ has $L \cap L^\perp$ as its null space and hence gives an induced symplectic structure on $L^\perp/L$. Moreover, there is an induced pairing $L \times (V/L^\perp) \to \mathbb{R}$, which in turn gives rise to an isomorphism $V/L^\perp \cong L^*$. By choosing a splitting of the sequence $0 \to L^\perp \to V \to V/L^\perp \to 0$ by an isotropic subspace in V, we obtain an orthogonal decomposition

$$V = L^\perp/L \oplus L \oplus L^* \ .$$

Now consider L to be the intersection $L_1 \cap L_2 \cap L_3$. Choose a basis $\{e_\alpha(123)\}$ of $L_1 \cap L_2 \cap L_3$ and a partial basis $\{f_\alpha(123)\}$ in V, with $\{e_\alpha(123), f_\beta(123)\} = \delta_{\alpha\beta}$ and $\{f_\alpha(123), f_\beta(123)\} = 0$. By the above discussion, we have a decomposition:

$$V = (L_1 \cap L_2 \cap L_3)^\perp / (L_1 \cap L_2 \cap L_3) \oplus \mathrm{Span}\{f_\alpha(123)\} \\ \oplus \mathrm{Span}\{e_\alpha(123)\} \ .$$

Note that $L_j \subset (L_1 \cap L_2 \cap L_3)^\perp$ for $j = 1, 2, 3$, and, in fact, they induce Lagrangian subspaces $L_1/(L_1 \cap L_2 \cap L_3)$, $L_2/(L_1 \cap L_2 \cap L_3$, $L_3/(L_1 \cap L_2 \cap L_3)$ in the symplectic subspace $(L_1 \cap L_2 \cap L_3)^\perp / (L_1 \cap L_2 \cap L_3)$. In other words, we can reduce the proof of Proposition 8.7 to the situation $L_1 \cap L_2 \cap L_3 = 0$.

Next, consider the isotropic subspace $L_{12} = L_1 \cap L_2$ in the situation when $L_1 \cap L_2 \cap L_3 = 0$. Note that L_1 and L_2 are maximal isotropic subspaces in $L_{12}^\perp$, and their quotients L_1/L_{12}, L_2/L_{12} form a pair of transverse Lagrangians in $L_{12}^\perp/L_{12}$. As for L_3, it can be written as a sum

$$L_3 = (L_3 \cap L_{12}^\perp) \oplus \mathrm{Span}\{f_\beta(12)\}$$

where $f_\beta(12)$ is part of a dual basis $\{f_\gamma(12)\}$ to $L_{12} = \oplus_\gamma \mathbb{R}e_\gamma(12)$.

Applying the work of Lions and Vergne to the situation L_1/L_{12}, L_2/L_{12}, and $(L_3 \cap L_{12})/L_{12}$ in $L_{12}^\perp/L_{12}$, we obtain symplectic basis $\{e'_\alpha, f'_\alpha \mid 1 \leqq \alpha \leqq \dim L_{12}^\perp/L_{12}\}$ so that L_1/L_{12} is spanned by $\{e'_\alpha\}$, L_2/L by $\{f'_\alpha\}$, and $(L_3 \cap L_{12}/L_{12}$ by $\{a'_\alpha e'_\alpha + b'_\alpha f'_\alpha \mid 1 \leqq \alpha < \dim L_3 \cap L_{12}^\perp/L_{12}\}$. Clearly, we can pull back e'_α to elements e_α in L_1 and f'_α to f_α in L_2. Combining $\{e_\alpha, f_\alpha\}$ with $e_\gamma(12)$ in L_{12} and $f_\gamma(12)$ in L_{12}^*, we obtain a symplectic basis for V. It is easy to verify that the Lagrangian subspaces L_1, L_2, L_3 are in the desired position with respect to $\{e_\alpha, f_\alpha, e_\gamma(12), f_\gamma(12)\}$. This completes the proof.

9. The Maslov Index $m(x, y)$ of a Pair in $\widetilde{\text{Lag}}(V)$

Let π: $\widetilde{\text{Lag}}(V) \to \text{Lag}(V)$ denote the universal covering space. Given a pair (x, y) in $\widetilde{\text{Lag}}(V)$, then by the unique path lifting property, there exists a path $\tilde{\gamma}$: $[0, 1] \to \widetilde{\text{Lag}}(V)$, unique up to homotopy relative boundary, such that

$$\gamma(0) = x\,, \quad \gamma(1) = y\,. \tag{9.1}$$

In particular, for any pair (x, y) in $\widetilde{\text{Lag}}(V)$ and any Lagrangian L_0 in V, we have a well-defined invariant given by

$$\overline{\mathscr{M}}(L_0; x, y) = \mu_V([L_0], \gamma) \tag{9.2}$$

where $[L_0]$ is the constant path and $\gamma = \pi\tilde{\gamma}$.

A natural choice to use for L_0 is one of the end points. For instance, setting $L_0 = \pi(y) = \gamma(1)$, we may define

$$\overline{\mathscr{M}}(x, y) = \mu_V([\gamma(1)], \gamma)\,.$$

If x, y, z are in $\widetilde{\text{Lag}}(V)$, the quantity $\overline{\mathscr{M}}(x, y) + \overline{\mathscr{M}}(y, z) + \overline{\mathscr{M}}(z, x)$ can be easily computed via the definiton of $\mu_{\text{anal},1}(\ )$. Let γ_1 be a path from $\pi(x)$ to $\pi(y)$ and γ_2 from $\pi(y)$ to $\pi(z)$ which lift to paths $\tilde{\gamma}_1$, $\tilde{\gamma}_2$ from x to z, respectively from y to z in $\widetilde{\text{Lag}}(V)$. Then the composite $\gamma_1 \circ \gamma_2$ lifts to $\tilde{\gamma}_1 \circ \tilde{\gamma}_2$ from x to z, and so the reversed path $(\gamma_2 \circ \gamma_1)^{-1}$ serves to compute $\overline{\mathscr{M}}(z, x)$. The integral terms in (6.13) for $\mu_V([\gamma_i(1)], \gamma_i)$ are $\int \gamma_i^*(\omega)$ while $\mu_V([(\gamma_2 \circ \gamma_1)^{-1}(1)], (\gamma_2 \circ \gamma_1)^{-1})$ gives $\int [(\gamma_2 \circ \gamma_1)^{-1}] * (\omega) = -\int \gamma_1^*(\omega) - \int \gamma_2^*(\omega)$. Thus, the last integral cancels the other two.

The eta invariant terms in (6.13) are

$$\frac{1}{2}[\eta(D(\pi(y), \pi(y))) - \eta(D(\pi(y), \pi(x)))]\,,$$
$$\frac{1}{2}[\eta(D(\pi(z), \pi(z))) - \eta(D(\pi(z), \pi(y)))]\,,$$
$$\frac{1}{2}[\eta(D(\pi(x), \pi(x))) - \eta(D(\pi(x), \pi(z))]\,.$$

Since $\eta(D(L_0, L_0)) = 0$ and $\eta(D(L_1, L_2)) = -\eta(D(L_2, L_1))$, these three terms add up to

$$\begin{aligned}&\frac{1}{2}[\eta(D(\pi(x),\pi(y))) + \eta(D(\pi(y),\pi(z))) + \eta(D(\pi(z),\pi(x)))]\\ &= \frac{1}{2}\tau(\pi(x),\pi(y),\pi(z)) .\end{aligned}$$

Finally, the zero-mode terms are

$$\frac{1}{2}(-n + h_{12}) , \quad \frac{1}{2}(-n + h_{23}) , \quad \frac{1}{2}(-n + h_{31})$$

where $h_{jk} = \dim L_j \cap L_k$, and $(L_1, L_2, L_3) = (\pi(x), \pi(y), \pi(z))$. In toto, this proves the cocycle equation:

$$\begin{aligned}&\overline{\mathcal{M}}(x,y) + \overline{\mathcal{M}}(y,z) + \overline{\mathcal{M}}(z,x)\\ &= \frac{1}{2}\tau(\pi(x),\pi(y),\pi(z)) - \frac{3n}{2} + \frac{1}{2}(h_{12} + h_{23} + h_{31}) .\end{aligned} \tag{9.3}$$

To compare $\overline{\mathcal{M}}(x,y)$ with $\overline{\mathcal{M}}(y,x)$, we consider as before the path γ from $\pi(x)$ to $\pi(y)$ and use the reversed path γ^{-1} from $\pi(y)$ to $\pi(x)$. Utilizing the formula of $\mu_{\text{anal},1}(\ )$, we have

$$\begin{aligned}\overline{\mathcal{M}}(y,x) &= \mu_{\text{anal},1}([\pi(x)], \gamma^{-1})\\ &= \frac{1}{2}[\eta(D(\pi(x),\pi(x))) - \eta(D(\pi(x),\pi(y)))]\\ &\quad + \int(\gamma^{-1})^*\omega + \frac{1}{2}(-n + h_{12})\\ &= -\frac{1}{2}[\eta(D(\pi(x),\pi(y))] - \int\gamma^*\omega + \frac{1}{2}(-n + h_{12})\\ &= -\mu_{\text{anal},1}([\pi(y)],\gamma) + (-n + h_{12})\\ &= -\overline{\mathcal{M}}(x,y) - n + h_{12} .\end{aligned}$$

In particular, this proves the following (Theorem 0.5):

PROPOSITION 9.1. *Let x, y be elements in $\widetilde{\text{Lag}}\,(V)$ and γ be a path from $\pi(x)$ to $\pi(y)$ lifting to a path $\tilde{\gamma}$ from x to y as before. Set*

$$\begin{aligned}m(x,y) &= 2\overline{\mathcal{M}}(x,y) + (n - \dim \pi(x) \cap \pi(y))\\ &= 2\mu_V([\pi(y)],\gamma) + (n - \dim \pi(x)\cap\pi(y)) .\end{aligned}$$

Then $m(x,y)$ satisfies

(i) *it is independent of the choice in $\tilde{\gamma}$,*
(ii) $m(x,y) + m(y,z) + m(z,x) = \tau_V(\pi(x),\pi(y),\pi(z))$,

(iii) $m(x, y) = -m(y, x)$.

In [17] Gossen defined a Maslov index $\mu(x, y)$ and showed that $\mu(x, y)$ is characterized uniquely by two properties:

$$(9.4)\qquad \begin{aligned} &\text{(i)} \quad \mu(x, y) + \mu(x, z) + \mu(y, z) = \tau_V(\pi(x), \pi(y), \pi(z)) \\ &\text{(ii)} \quad \mu(x, y) - \tau(\pi(x), \pi(y), L_0) \text{ is locally} \\ &\qquad \text{constant for } x, y \text{ transversal to } L_0 \text{. That is,} \\ &\qquad \pi(x) \cap L_0 = \pi(y) \cap L_0 = 0 . \end{aligned}$$

Comparing (9.4) (i)–(ii) with Proposition 9.1 (i)–(iii), we have

PROPOSITION 9.2. *For all x, y in $\widetilde{\mathrm{Lag}}(V)$, the index $\mu(x, y)$ of Gossen is the same as $m(x, y)$.*

Proof: It suffices to prove properties (9.4) (i), (ii) for $m(x, y)$. The condition (9.4) (i) is clear from (9.1) (ii). As for (9.4) (ii), by letting $L_0 = \pi(z)$ in (9.4) (i), it can be reformulated as

$$(9.5)\qquad \begin{aligned} &- m(x, z) + m(y, z) \text{ is locally constant} \\ &\text{on the subspace } \{(x, y, z) | \pi(x) \cap \pi(z) = \pi(y) \cap \pi(z) = 0\} . \end{aligned}$$

To comprehend (9.5) analytically, we take a path γ_1 from $L_1 = \pi(x)$ to $L_3 = \pi(z)$, and a path γ_2 from $L_2 = \pi(y)$ to $L_3 = \pi(z)$ in the same manner as before lifting to paths from x to z, y to z respectively. Then by the transversality of L_1, L_3, we have

$$\begin{aligned} m(x, z) &= 2\overline{\mathcal{M}}(x, z) + n \\ &= 2\mu_V([L_3], \gamma_1) + n \\ &= 2\left\{\frac{1}{2}[\eta(D(L_3, L_3)) - \eta(D(L_3, L_1))] + \int \gamma_1^*(\omega) + \frac{1}{2}(-n)\right\} + n \\ &= -\eta(D(L_3, L_1)) + 2\int \gamma_1^*(\omega) \end{aligned}$$

and similarly,

$$m(y, z) = -\eta(D(L_3, L_2)) + 2\int \gamma_2^*(\omega) .$$

Combining the two, we have

$$\begin{aligned} &- m(x, z) + m(y, z) \\ &\quad = [\eta(D(L_3, L_1)) - \eta(D(L_3, L_2))] + 2\int -\gamma_1^*(\omega) + \gamma_2^*(\omega) . \end{aligned}$$

On the other hand, by the transversality of L_1, L_3 and L_2, L_3, the eta invariants $\eta(D(L_3, L_1))$ and $\eta(D(L_3, L_2))$ vary smoothly on the subspace $\{(x, y, z) | \pi(x) \cap \pi(z) =$

$\pi(x) \cap \pi(y) = 0\}$, and so do $\gamma_1^*(\omega)$, $\gamma_2^*(\omega)$. Hence, the integer $-m(x,z) + m(y,z)$ is continuous and therefore locally constant on this subspace. This proves (9.5) and so Proposition 9.2.

In [17], page 28a, Gossen further identifies $\frac{1}{2}(\mu(x,y) + n)$ with the definition of Maslov index $\hat{\mu}(x,y)_{\text{Arnold}}$ by Arnold for $\pi(x) \pitchfork \pi(y) = 0$; see [1]. In view of the above proposition, we have for $\pi(x) \pitchfork \pi(y)$

$$\begin{aligned} \hat{\mu}(x,y)_{\text{Arnold}} &= \frac{1}{2}(m(x,y) + n) \\ &= \overline{\mathcal{M}}(x,y) + n \\ &= \mu_V([\pi(y)], \gamma) + n \,. \end{aligned}$$

10. The Duistermaat Index of a Path of Lagrangians

Duistermaat has introduced an integer invariant $\mathcal{D}(\{L_1(t): a \leqq t \leqq b\})$ associated to a *single* smooth path of Langrangians $L(t)$: $a \leqq t \leqq b$. This symplectic invariant is a homotopy invariant relative to fixed end points and plays a pivotal role in his analysis of the Morse index in variational calculus; see [11]. In this section we show that

$$\begin{aligned} \mathcal{D}(\{L_1(t): a \leqq t \leqq b\}) &= \left[\int_a^b L_1^*(\omega)\right] + \frac{1}{2}\eta(L_1(a), L_1(b)) \\ &= \mu_V(\{(L_1(a), L_1(t)): 0 \leqq t \leqq 1\}) \\ &\quad + \frac{1}{2}[\dim(L_1(a) \cap L_1(b)) - n] \,. \end{aligned} \tag{10.1}$$

First we recall Duistermaat's definition. Choose a fixed Lagrangian α with $\alpha \pitchfork L_1(a)$, $\alpha \pitchfork L_1(b)$. Choose a path $\hat{\gamma}$ from $L_1(b)$ to $L_1(a)$ which is always transverse to α. Thus the composite path

$$\gamma * \hat{\gamma} \quad \text{with} \quad \gamma = \{L_1(t): a \leqq t \leqq b\}$$

is a loop. Then the number of times α is not transverse to the loop $\gamma * \hat{\gamma}$ is counted via

$$\mu_V([\alpha], \gamma * \hat{\gamma}) = \mu_V(\{([\alpha], L_1(t)): a \leqq t \leqq b\}) \,.$$

This quantity depends on the choice of α; however, Duistermaat normalizes it by defining

$$\mathcal{D}(\{L_1(t): a \leqq t \leqq b\} = \mu_V([\alpha], \gamma * \hat{\gamma}) + \tfrac{1}{2}s$$

with s = signature of the quadratic form in $L_1(a)$ given by $x \mapsto \{x', x''\}$ with $x \in L_1(a)$, $x = x' + x''$ and $x' \in \alpha$, $x'' \in L_1(b)$.

Equivalently, utilizing Property IV of Section 8

$$\mathscr{D}(\{L_1(t)\colon a \leqq t \leqq b\}) = \{\mu_V((\alpha, L_1(t))\colon a \leqq t \leqq b) + \tfrac{1}{2}\tau(\alpha, L_1(a), L_1(b))\} .$$

The close relation between $\mathscr{D}(\ )$ and the $m(\ ,\ )$ of Section 9 is clear from this definition.

For α *arbitrary* we may use definition $\mu_{\mathrm{anal},1}(\ )$ and Proposition 8.3 to compute

$$\begin{aligned}
&\mu_V((\alpha, L_1(t))\colon a \leqq t \leqq b) + \tfrac{1}{2}\tau(\alpha, L_1(a), L_1(b)) \\
&= (\tfrac{1}{2}[\eta(\alpha, L_1(b)) - \eta(\alpha, L_2(a))] \\
&\quad + \int_a^b (L_1^*(\omega) - 0) + \tfrac{1}{2}(-\dim \alpha \cap L_1(b) + \dim \alpha \cap L_1(a)) \\
&\quad + (\tfrac{1}{2}[\eta(\alpha, L_1(a)) + \eta(L_1(a), L_1(b)) + \eta(L_1(b), \alpha)]) \\
&= \int_a^b L_1^*(\omega) + \tfrac{1}{2}\eta(L_1(a), L_1(b)) \\
&\quad + (\tfrac{1}{2})(-\ \dim(\alpha \cap L_1(b)) + \dim(\alpha \cap L_1(a))) .
\end{aligned}$$

In particular for $\alpha \pitchfork L_1(a)$, $\alpha \pitchfork L_2(b)$ we have the first equality of (10.1)

$$\mathscr{D}(\{L_1(t)\colon a \leqq t \leqq b\}) = \int_a^b L_1^*(\omega) + \tfrac{1}{2}\eta(L_1(a), L_2(b))$$

and for *any* α, we also have

$$\begin{aligned}
&\int_{[\alpha,b]} L_1^*(\omega) + \tfrac{1}{2}\eta(L_1(a), L_2(b)) \\
&= \mu_V((\alpha, L_1(t))\colon a \leqq t \leqq b) + \tfrac{1}{2}\tau(\alpha, L_1(a), L_1(b)) \\
&\quad + (\tfrac{1}{2}) \cdot [\dim(\alpha \cap L_1(b)) - \dim(\alpha \cap L_1(a))] .
\end{aligned}$$

Setting $\alpha = L_1(a)$ gives the second equality in (10.1). This last formula makes the independence of α and independence of choice of J and $\langle\ ,\ \rangle$ (used in computing $\eta(\ )$ and $\int L_1^*(\omega)$) clear. Moreover, the left-hand side has the desired homotopy invariance relative to endpoints. The right-hand side has the desired symplectic invariance property.

For a composite path $\gamma_1 * \gamma_2$ where γ_1 traces from x to y and γ_2 from y to z in $\mathrm{Lag}(V)$ as t traces from a to b and b to c, then (using (6.13) and Proposition 8.3)

$$\begin{aligned}
&\int_a^c (\gamma_1 * \gamma_2)^* \omega + \tfrac{1}{2}\eta(x, z) \\
&= \int_a^b \gamma_1^*(\omega) + \int_b^c \gamma_2^*(\omega) + \tfrac{1}{2}[-\tau(x, y, z) + \eta(x, y) + \eta(y, z)] .
\end{aligned}$$

Thus,

$$\mathscr{D}(\gamma_1 * \gamma_2) = \mathscr{D}(\gamma_1) + \mathscr{D}(\gamma_2) - (\tfrac{1}{2})\tau(x, y, z) .$$

This is the additivity formula for $\mathscr{D}(\cdot)$.

11. The Maslov Index of a Path of Symplectic Matrices and a Theorem of Salamon and Zehnder

Let $\psi(t)$: $0 \leqq t \leqq \tau$ be a smooth path in $\mathrm{Sp}(2n, \mathbb{R})$, the $2n \times 2n$ symplectic matrices. Regarding $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$ as a symplectic vector space via the new symplectic inner product

$$\{(v_1, v_2), (w_1, w_2)\}_0 = +\{v_1, w_1\} - \{v_2, w_2\}\ , \tag{11.1}$$

then the path of subspaces (graphs)

$$\Gamma_{\psi(t)} = \{(v, \psi(t)v) \mid v \in \mathbb{R}^{2n}\} \tag{11.2}$$

is a path of Lagrangian subspaces. This path of Lagrangians may be compared with the constant path

$$\Gamma_{\psi(0)}\colon 0 \leqq t \leqq \tau\ .$$

We will call the associated Maslov index,

$$\hat{\mu}(\{\psi(t)\}) \overset{def}{=} \tau_{V\oplus V}(\{(\Gamma_{\psi(0)}, \Gamma_{\psi(t)})\colon 0 \leqq t \leqq \tau\})\ , \tag{11.3}$$

the Maslov index of the path $\{\psi(t)\}$ of symplectic matrices. Note that

$$\Gamma_{\psi(0)} \cap \Gamma_{\psi(t)} \cong \{v \in \mathbb{R}^{2n} \mid \psi(0)v = \psi(t)v\} \tag{11.4}$$

and note the similarity to $m(x, y)$ of Section 9 and $\mathscr{D}(\cdot)$ of Section 10.

Now let $\psi(s, t)$: $0 \leqq s \leqq 1$, $0 \leqq t \leqq \tau$ be a smooth parameter family of symplectic automorphisms. This defines a smooth map

$$\psi\colon [0, 1] \times [0, \tau] \to \mathrm{Sp}(2n, \mathbb{R})\ . \tag{11.5}$$

For technical convenience we assume that

$$\psi(s, t) = \psi(0, t) \text{ for } 0 \leqq s \leqq \varepsilon\ ;\ \psi(s, t) = \psi(1, t) \text{ for } 1 - \varepsilon \leqq s \leqq 1 \tag{11.6}$$

for some fixed $\varepsilon > 0$.

In this section we will derive formulas for the difference

$$\hat{\mu}(\{\psi(1, t)\colon 0 \leqq t \leqq \tau\}) - \hat{\mu}(\{\psi(0, t)\colon 0 \leqq t \leqq \tau)\ . \tag{11.7}$$

The main formula expresses this difference in terms of the index of an operator F_ψ on the cylinder $[0, 1] \times (\mathbb{R}/\tau\mathbb{Z})$. This will recapitulate (and slightly extend) a recent result of Salamon and Zehnder; see [27]. We refer to their paper for a beautiful application of this general result. Let

$$J_0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

By $\psi(s,t)$ symplectic, the matrix

$$S(s,t) = -J_0(\partial\psi/\partial t)\psi^{-1} \tag{11.8}$$

is a real $2n\times 2n$ symmetric matrix. Conversely, $\psi(s,t)$ may be recovered by solving

$$\frac{\partial\psi(s,t)}{\partial t} = J_0 S(s,t)\,\psi(s,t) \tag{11.9}$$

with initial conditions $\psi(s,0)$ given.

In particular, the real operator

$$B_s = J_0\frac{\partial}{\partial t} + S(s,t)\colon L_1^2(\mathbb{R}/\tau\mathbb{Z}\colon \mathbb{R}^{2n}) \to L^2(\mathbb{R}/\tau\mathbb{Z}, \mathbb{R}^{2n}) \tag{11.10}$$

is real self-adjoint. (Note $(J_0)^T = -J_0$, $(\partial/\partial t)^* = -\partial/\partial t$.) Here

$$\begin{aligned} \ker B_s &= \{f(t) \mid f(t) = \psi(s,t)v \text{ with } f(0) = f(\tau)\} \\ &\cong \{v \mid \psi(s,0)v = \psi(s,\tau)v\} \\ &\cong \Gamma_{\psi(s,0)} \cap \Gamma_{\psi(s,1)} \end{aligned} \tag{11.11}$$

depends *only* on the symplectic matrices $\psi(s,0)$, $\psi(s,1)$. Of course B_s is a mild generalization of $-J\frac{\partial}{\partial t}$ considered above.

Following Salamon and Zehnder, [27], and Floer, [15], introduce the real, first-order, elliptic operator

$$F_\psi = \frac{\partial}{\partial s} + J_0\frac{\partial}{\partial t} + S(s,t) \tag{11.12}$$

acting on functions $[0,1]\times(\mathbb{R}/\tau\mathbb{Z}) \to \mathbb{R}^{2n}$. If $S \equiv 0$ this is the $\bar\partial$ operator regarded as a real operator.

The methods of Atiyah-Patodi-Singer apply to F_ψ since F_ψ is product-like near $\{0,1\}\times(\mathbb{R}/\tau\mathbb{Z})$ by assumption (11.6).

The associated tangential operators at $\{0\}\times(\mathbb{R}/\tau\mathbb{Z})$ and $\{1\}\times(\mathbb{R}/\tau\mathbb{Z})$ are B_0, respectively $-B_1$. Introducing the Sobolev space $L_1^2([0,1]\times(\mathbb{R}/\tau\mathbb{Z}); \mathbb{R}^{2n}, \hat\partial)$, the L_1^2 closure of the smooth functions f which satisfy the additional nonlocal boundary conditions:

$$\begin{aligned} &f\,|\{0\}\times(\mathbb{R}/\tau\mathbb{Z}) \in \{\text{closure of eigensolutions of } B_0\phi_\lambda = \lambda\phi_\lambda \text{ with } \lambda < 0\} \\ &f\,|\{1\}\times(\mathbb{R}/\tau\mathbb{Z}) \in \{\text{closure of eigensolutions of } B_1\phi_\lambda = \lambda\phi_\lambda \text{ with } \lambda > 0\}\,, \end{aligned} \tag{11.13}$$

the operator F_ψ yields a Fredholm map

$$F_\psi\colon L_1^2([0,1]\times(\mathbb{R}/\tau\mathbb{Z}); \mathbb{R}^{2n}, \hat\partial) \to L^2([0,1)\times(\mathbb{R}/\tau\mathbb{Z}); \mathbb{R}^{2n})\,. \tag{11.14}$$

Moreover, from the work of Atiyah-Patodi-Singer the index of F_ψ is given by the formula

(11.15) $$\text{Index } F_\psi = \{\text{normalized spectral flow of } B_s\colon 0 \leqq s \leqq \tau\}\ .$$

See the discussion in Section 7.

Additional formulas for the index of F_ψ are given by the following result, in which the equality of the first (Index F_ψ) and the last, is a theorem of Salamon and Zehnder in [27] (stated there under the assumptions that $\psi(s,0) \equiv id$, $0 \leqq s \leqq \tau$, $\Gamma_{\psi(0,0)} \cap \Gamma_{\psi(0,\tau)} = \{0\}$, and $\Gamma_{\psi(1,0)} \cap \Gamma_{\psi(1,\tau)} = \{0\}$). Moreover, they give an alternative (but equivalent) definition of the Maslov index of $\{\psi(s,t)\colon 0 \leqq t \leqq \tau\}$.

THEOREM 11.1.

$$\begin{aligned}
&\text{Index } F_\psi \\
&\quad = \{\textit{normalized spectral flow of } B_s\colon 0 \leqq s \leqq \tau\} && (1)\\
&\quad = \mu_{V\oplus V}(\{\Gamma_{\psi(s,0)}, \Gamma_{\psi(s,\tau)}\}) + \dim \Gamma_{\psi(0,0)} \cap \Gamma_{\psi(0,\tau)}\} && (2)\\
&\quad = \{\hat\mu\{\psi(1,t)\colon 0 \leqq t \leqq \tau\} - \hat\mu\{\psi(0,t)\colon 0 \leqq t \leqq \tau\} \\
&\qquad +\dim\, (\Gamma_{\psi(0,0)} \cap \Gamma_{\psi(0,\tau)})\} && (3)\\
&\quad = \{\textit{Maslov index of } \{\psi(1,t)\}\textit{-Maslov index of } \{\psi(0,t)\} && (4)\\
&\qquad +\dim\, (\Gamma_{\psi(0,0)} \cap \Gamma_{\psi(0,\tau)})\}.
\end{aligned}$$

Proof of Theorem 11.15: Equality (1) is (11.15) above.
Equality (2) is quite reasonable since

$$\mu_{V\oplus V}(\{\Gamma_{\psi(s,0)}, \Gamma_{\psi(s,\tau)}\}) + \dim(\Gamma_{\psi(0,s)} \cap \Gamma_{\psi(0,\tau)})$$

(by Section 8) equals the normalized spectral flow of the operators

$$D(\Gamma_{\psi(s,0)}, \Gamma_{\psi(s,\tau)}) \quad \text{or} \quad D^\#(\Gamma_{\psi(s,0)}, \Gamma_{\psi(s,\tau)})\ .$$

The proof that

(11.16) $$\left\{\begin{matrix}\text{normalized SF}\\ \text{of } B_s\colon 0 \leqq s \leqq \tau\end{matrix}\right\} = \left\{\begin{matrix}\text{normalized SF of}\\ D^\#(\Gamma_{\psi(s,0)}, \Gamma_{\psi(s,\tau)})\end{matrix}\right\}$$

is apparent since by (11.11)

$$\ker B_s \cong \Gamma_{\psi(s,0)} \cap \Gamma_{\psi(s,\tau)} \cong \ker D(\Gamma_{\psi(s,0)}, \Gamma_{\psi(s,\tau)})\ .$$

A simple computation shows that these zero modes (which can be assumed to cross 0 transversely) cross with the same sign. This shows equality (2). (The $-$ sign in $-i\frac{d}{dt}$ of $D(\ ,\ )$ is cancelled by the choice $(+,-)$ in (11.1).)

As for (3), we deform the path $\Gamma_{\psi(s,\tau)}\colon 0 \leqq s \leqq t$ (via ψ) to the composite path $\gamma_1 * \gamma_2 * \gamma_3$ with

$$\gamma_2 \ \Big\uparrow \quad \Leftarrow \quad \uparrow \ \Gamma_{\psi(s,\tau)}$$

γ_1 tracing along $\{\Gamma_{\psi(0,t)}: 0 \leqq t \leqq \tau\}$ backwards, γ_2 tracing $\{\Gamma_{\psi(0,s)}: 0 \leqq s \leqq 1\}$, and γ_3 tracing $\{\Gamma_{\psi(1,t)}: 0 \leqq t \leqq \tau\}$. We may replace $\{\Gamma_{\psi(s,0)}: 0 \leqq s \leqq 1\} = \gamma_2$ by the path that stays at $\Gamma_{\psi(0,0)}$ a while, then traces γ_2, then stays at $\Gamma_{\psi(1,0)}$. By the path additivity property and reversal property, we conclude

$$\begin{aligned}\mu_{V\oplus V}(\{(\Gamma_{\psi(s,0)}, \Gamma_{\psi(s,\tau)}): 0 \leqq s \leqq 1) = &-\mu_{V\oplus V}(\{(\Gamma_{\psi(0,0)}, \Gamma_{\psi(0,t)}): 0 \leqq s \leqq \tau\}) \\ &+ \mu_{V\oplus V}(\{(\Gamma_{\psi(s,0)}, \Gamma_{\psi(s,0)}): 0 \leqq s \leqq 1\}) \\ &+ \mu_{V\oplus V}(\{(\Gamma_{\psi(1,0)}, \Gamma_{\psi(1,t)}: 0 \leqq t \leqq \tau\}) .\end{aligned}$$

Since the middle term equals 0 by nullity, we have shown (3) and hence (4).

12. A Geometric Interpretation of the Maslov Triple Index (Following Wall) and Hörmander's Fourfold Index

For an oriented $4k$-dimensional manifold L which is cut into two pieces L' and L'' by a closed $(4k-1)$-dimensional submanifold, Novikov showed that the signature is additive,

$$\sigma(L) = \sigma(L') + \sigma(L'') .$$

When, more generally, the $(4k-1)$-dimensional submanifold M_2 along which L is cut is not necessarily closed, but is properly embedded (so that $\partial M_2 \subset \partial L$) and separating, Wall found a nonadditivity signature correction. Here we redo Wall's argument by identifying the signature of a "three prong manifold" with a Maslov triple index and then apply standard Novikov additivity to get his general result. We then consider the index of n-prong manifolds, and for $n = 4$ compare that with Hörmander's index.

Let C^{4k+2} be a $(4k+2)$-dimensional oriented closed manifold. Then the intersection pairing in the middle dimensional homology $V = H_{2k+1}(C;\mathbb{R})$ induces a nondegenerate pairing

$$\{\,,\,\}: V \times V \to \mathbb{R}$$

which is skew symmetric (by $(2k+1)$ odd). If C is identified with the boundary ∂B_j of a closed oriented manifold B_j, then a standard observation (due to Thom) is that the kernel

$$L_j = \ker\{H_{2k+1}(\partial B_j;\mathbb{R}) \to H_{2k+1}(C;\mathbb{R})\} \tag{12.1}$$

is a Lagrangian subspace of $V \cong H_{2k+1}(\partial B_j, \mathbb{R})$.

In particular, if $C = \partial B_1 = \partial B_2 = \partial B_3$ for three such $(4k+3)$-manifolds we get three Lagrangians in V. We may identify $\tau_V(L_1, L_2, L_3)$ with the signature $\sigma(A_3)$ of an oriented $(4k+4)$-manifold A,

$$\tau_V(L_1, L_2, L_3) = \sigma(A) . \tag{12.2}$$

The construction of A is made by forming the disjoint pieces

$$C \times D^2, B_1 \times [-1, +1], B_2 \times [-1, +1], B_3 \times [-1, +1]$$

($D^2 = 2-$ disk) and gluing them together via the prescription suggested by Diagram 12. That is, $\partial B_1 \times [-1, +1]$, $\partial B_2 \times [-1, +1]$; $\partial B_3 \times [-1, +1]$ with a part of $C \times S^1$ (by using $C = \partial B_1 = \partial B_2 = \partial B_3$ and an identification of the three segments $[-1, +1]$)

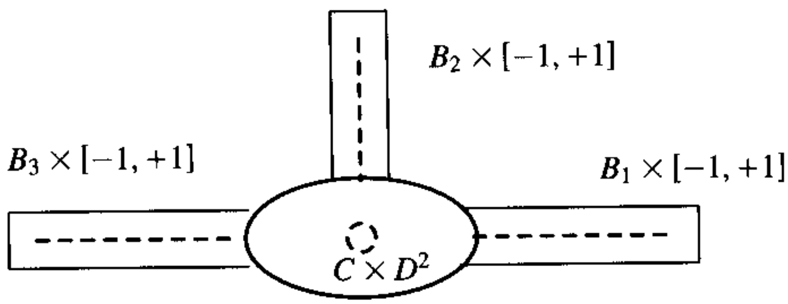

Diagram 12.

(with $[-\pi/4, \pi/4]$, $[3\pi/4, 5\pi/4]$, $[7\pi/4, 9\pi/4]$ in the circle). A is to be given the orientation which agrees with that of $C \times D^2$ with standard orientation on D^2.

Since the cycles of A which are in the image of ∂A are in the radical of the intersection pairing (we may just push them off of each other), we immediately see that the signature of C is computed via considering the cycles z of C which are a sum $z_1 + z_2 + z_3 + w$ when z_j is a chain on $B_j \times [-1, +1]$ with ∂z_j in $\partial B_j \times [-1, +1]$, $j = 1, 2, 3$, and w is a chain on $C \times D^2$ with $\partial w = -(\partial z_1 + \partial z_2 + \partial z_3)$. That is, the relevant cycles of A determine elements $x_j \in L_j$, $j = 1, 2, 3$ with $x_1 + x_2 + x_3 = 0$ in V. Setting

$$V(L_1, L_2, L_3) = \{(x_1, x_2, x_3) \in L_1 \times L_2 \times L_3 \mid x_1 + x_2 + x_3 = 0\}$$

we see that the bilinear mapping of cycles (via intersection)

$$Z_{4k+2}(A) \times Z_{4k+2}(A) \to \mathbb{R}$$

factors through $V(L_1, L_2, L_3)$ and induces

$$V(L_1, L_2, L_3) \times V(L_1, L_2, L_3) \to \mathbb{R} .$$

A simple check shows that the associated quadratic pairing is given by (up to a positive multiple)

$$-3\{x_1, x_2\} = -Q'(x_1, x_2, x_3) \text{ of Section 8 .}$$

By Section 8 which identifies the signature of $-Q'(x_1, x_2, x_3)$ on $V(L_1, L_2, L_3)$ with the Kashiwara index $\tau_V(L_1, L_2, L_3)$, we hence obtain (12.2).

Note that the special case of (12.2) combined with the Novikov additivity of the signature (cutting along closed submanifolds) yields Wall's general nonadditivity theorem as follows. Consider a compact oriented $4k+4$ manifold L^{4k+4} which is split into two pieces L', L'' by a codimension one submanifold M_2^{4k+3} with boundary $\partial M_2^{4k+3} = N^{4k+2}$ in ∂L. Then ∂L is the union of two pieces, $M_1 = (\partial L) \cap L'$ and $M_3 = (\partial L) \cap L''$, so

$$\partial M_1 = \partial M_2 = \partial M_3 = N^{4k+2}$$

as in Diagram 13.

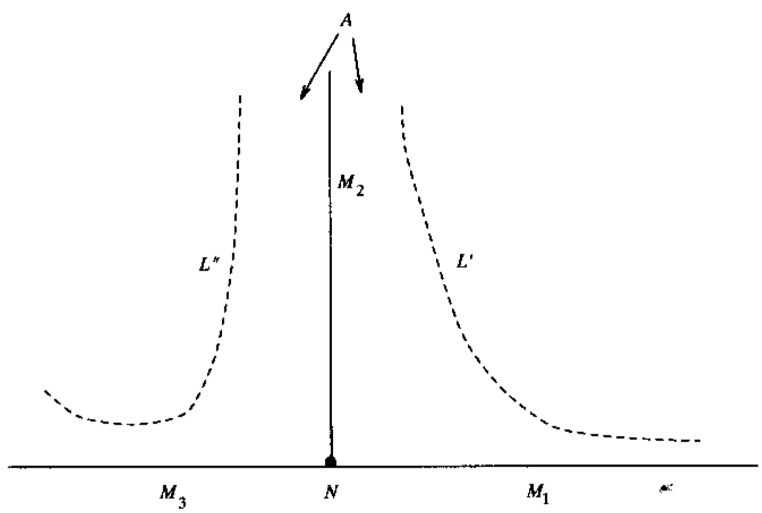

Diagram 13. $L=L''\cup A\cup L'$.

(Here N is oriented by $[N] = \partial[M_1]$ and $[M_1]$ oriented as in $[M_1 \cup M_3] = [\partial L] = \partial[L]$.) If we form the closed manifolds $M_1 \cup M_2$, $M_2 \cup M_3$ and push them into the interior of L as indicated in Diagram 13 (two dashed lines), we may cut along these closed submanifolds yielding three pieces. Two of these are homeomorphic to L', L'', the remaining central one A is obtained by the above construction applied to $N = \partial M_1 = \partial M_2 = \partial M_3$. Thus by Novikov additivity and (12.2), we get

$$\begin{aligned}\sigma(L) &= \sigma(L') + \sigma(L'') + \sigma(A) \\ &= \sigma(L') + \sigma(L'') + \tau_V(L_1, L_2, L_3)\ .\end{aligned}$$

This is Wall's formula for the nonadditivity of the signature.

Of course, we may apply the same discussion to the case of four orientable manifolds B_1, B_2, B_3, B_4 with $\partial B_j = C$, $j = 1,2,3,4$, $\dim C = 4k+2$ and oriented. In this case, we get four Lagrangians L_j, $j = 1,2,3,4$ and using Novikov additivity the resulting "4-prong" may be decomposed in either of two ways yielding two "3-prongs." See Diagram 14.

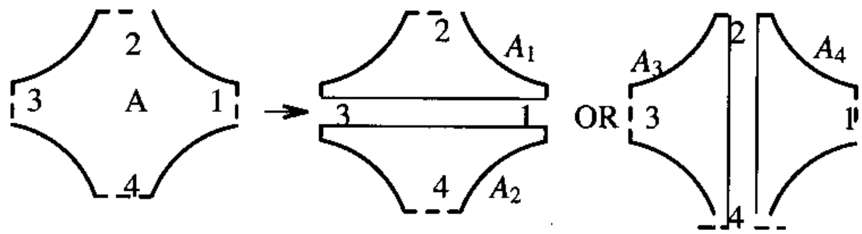

Diagram 14. Cutting along closed manifolds.

The resulting equality of signatures, from Novikov additivity,

$$\sigma(A) = \sigma(A_1) + \sigma(A_2) = \sigma(A_3) + \sigma(A_4)$$

is familiar to us as the basic cocycle condition on the Maslov triple index

$$\begin{aligned}\sigma(A) &= \tau_V(L_1, L_2, L_3) + \tau_V(L_3, L_4, L_1)\\ &= \tau_V(L_2, L_3, L_4) + \tau_V(L_4, L_1, L_2)\ .\end{aligned}$$

This combination of two triple Maslov indices equals the Hörmander four-fold index $\tau(L_1, L_2, L_3, L_4)$ (see [19]) and so provides a geometrical interpretation of it.

Similarly, we may consider "n-prongs." If $C = \partial B_j$, $j = 1, \ldots, n$, set $L_j = \ker H_{2j+1}(C, \mathbb{R}) \to H_{2k+1}(B_j; \mathbb{R})$.

The resulting $4k + 4$–manifold A_n may be decomposed into $(n - 2)$ "three prongs" in many ways. One such diagram is $(n = 6)$:

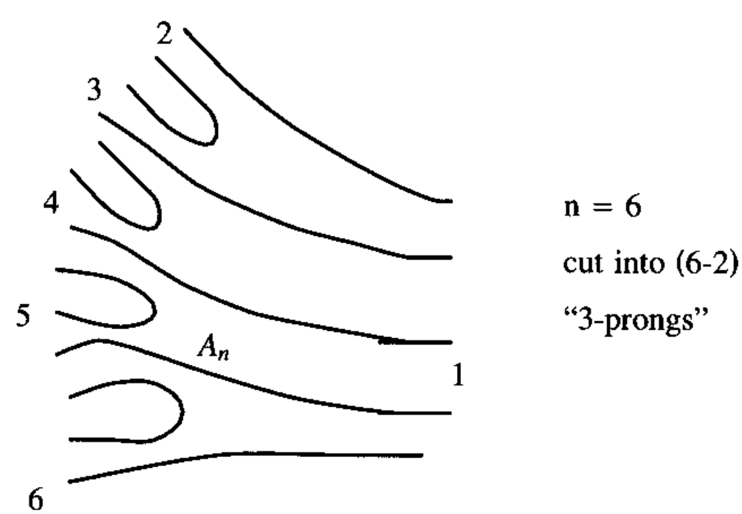

Diagram 15.

The resulting $4k + 4$–manifold has signature

$$\begin{aligned}\sigma(A_n) = \{&\tau(L_1, L_2, L_3) + \tau(L_1, L_3, L_4) + \tau(L_1, L_4, L_5)\\ &+ \cdots + \tau(L_1, L_{n-1}, L_n)\}\ .\end{aligned}$$

All of the standard formulas about $\tau(L_1, L_2, L_3)$ (see [24]) may be illustrated this way. They are most easily proved, however, by using Proposition 8.3 above, i.e.,

$$\tau(L_1, L_2, L_3) = \{\eta(D(L_1, L_2)) + \eta(D(L_2, L_3)) + \eta(D(L_3, L_1))\} .$$

As in the above analysis of the case $n = 3$, the signature of A_n may be directly computed by considering cycles in A_n. In this way, we may easily prove

$$\sigma(A_n) = -\sigma(V, L_1, L_2, \dots, L_n) .$$

Here $\sigma(V; L_1, \dots, L_n)$ is the signature of the quadratic form

$$Q(x_1, \dots, x_n) = \sum_{j=1}^{n-1} \{x_j, x_{j+1}\} + \{x_n x_1\}$$

on the subspace $V(L_1, \dots, L_n)$ of $L_1 \oplus L_2 \oplus \cdots \oplus L_n$ consisting of n-tuples $(x_1, \dots, x_n)$ with $\sum x_j = 0$.

With some additional effort we can prove that for any n-Lagrangians $L_1, \dots,$ L_n in a symplectic space V there is an equality

$$\sum_{j=2}^{n-1} \tau_V(L_1, L_j, L_{j+1}) = -\sigma(V; L_1, \dots, L_n) .$$

(When the L_j's arise geometrically as above, this formula is proved by observing that both sides equal $\sigma(A_n)$.)

Of course, in the context of manifolds equipped with finite group actions, the discussion in this section of signature invariants and signature additivity apply equally well to equivariant signatures in all even dimensions.

13. $\mu_V(f)$ as the Maslov Index of Closed Loops à la Floer and Walker

For $f(t) = (L_1(t), L_2(t))$ $a \leqq t \leqq b$ a smooth path of pairs of Lagrangians, let $(L_1(t)), (L_2(t))$ be the paths of individual Lagrangians $a \leqq t \leqq b$. Let $-(L_2(t))$ be the path $(L_2(-t) : -b \leqq t \leqq -a)$, that is, $L_2(t)$ traversed in the opposite direction. Once we have chosen smooth paths of Lagranigan $\nu_a(t), \nu_b(t), 0 \leqq t \leqq 1$, such that

$$\text{(13.1)} \qquad \begin{cases} \nu_b(0) = L_1(1), \nu_b(1) = L_2(1) \\ \nu_a(0) = L_2(0), \nu_a(1) = L_1(0), \end{cases}$$

then the composition of paths

$$\text{(13.2)} \qquad \hat{f}(\nu_b, \nu_a) = [(L_1(t))] * \nu_b * [-(L_2(t))] * \nu_a$$

is a closed loop of Lagrangians. Here we traverse $(L_1(t))$ first, then ν_b, then $-(L_2(t))$ (or $L_2(*)$ backwards), then ν_a.

Now following Floer, [16], or Walker, [30], there are two natural choices for each of the paths ν_1, ν_2 up to homotopy rel. end points. These are described as follows:

Let A, B be arbitrary Lagrangians in V. Choose a complex structure J and Hermitian inner product $\langle\,,\rangle$ on V compatible with the symplectic product $\{\,,\}$. Such choices form a contractible cell; see [18]. By Lemma 2, the path

$$A(\theta) = (A \cap B) \oplus e^{J\theta}[(A \cap B)^{\perp} \cap A] \tag{13.3}$$

is a family of Lagrangians such that $A(\theta) \cap A = A \cap B$ for $0 < \theta < \pi/4$. Since $A(\pi/4) \cap A = A \cap B, A(-\pi/4) \cap A = A \cap B$ and the space of Lagrangians $\mathscr{E}$ with $\mathscr{E} \cap A = A \cap B$ is a cell and so contractible (see [18]), there are unique paths $C(t), D(t) : \pi/4 \leqq t \leqq 1$ up to homotopy rel. end points with $C(0) = A(\pi/4), e(1) = B, e(t) \cap A = A \cap E$ and $D(0) = A(-\pi/4), D(1) = B, D(t) \cap A = A \cap E$.

We set

$$\nu_+(A,B)(t) = \begin{cases} A(t) : 0 \leqq t \leqq (\pi/4) \\ C(t) : (\pi/4) \leqq t \leqq 1 \end{cases},$$
$$\nu_-(A,B)(t) = \begin{cases} A(-t) : 0 \leqq t \leqq \pi/4 \\ D(t) : \pi/4 \leqq t \leqq 1 \end{cases}. \tag{13.4}$$

Here $\nu_+(A,B), \nu_-(A,B)$ are paths of Lagrangians starting at A and ending at B. Up to homotopy rel. end points, ν_+, ν_- are uniquely specified by A, B independent of the choices of $J, \langle\,,\rangle, e(t)\cdot, D(t)$. Thus ν_+ rotates "positively" and then connects to B, while ν_- rotates "negatively" and then connects to B.

For $f(t) = (L_1(t), L_2(t)) : a \leqq t \leqq b$, let $\nu_{b\pm} = \nu_\pm(L_1(b), L_2(b))$ which interpolates from $L_1(b)$ to $L_2(b)$ and $\bar{\nu}_{a\pm} = \nu_\pm(L_2(a), L_1(a))$, which interpolates from $L_2(a)$ to $L_1(a)$. Defining with $\alpha, \beta = \pm$

$$\hat{f}_{\alpha\beta} = [(L_1(t))] * \nu_{\beta\alpha} * [-(L_2(t))] * \bar{\nu}_{a\beta} \tag{13.5}$$

yields four closed paths $\hat{f}_{\pm\pm}$.

By definition the *closed* paths $\hat{f}_{\pm\pm}$ depend up to homotopy only on the path $\delta(t) = (L_1(t), L_2(t)) : a \leqq t \leqq b$. Indeed, the homotopy class $[\hat{f}_{\pm\pm}]$ in $\pi_1(\mathrm{Lag}(V))$ depends only on the homotopy class of $\delta(t)$ rel. end points.

Recall from [18], and as discussed above, that a Maslov index

$$\hat{\mu} : \pi_1(\mathrm{Lag}) \overset{\approx}{\to} Z$$

can be defined using

$$[\ell(t) : a \leqq t \leqq b, \ell(a) = \ell(b)] \to \mu_V((\gamma, \ell(t)),\ a \leqq t \leqq b$$

for any smooth loop of Lagrangians. Here γ is any fixed Lagrangian. It is not surprising that the Maslov index $\hat{\mu}[\hat{f}_{\pm\pm}]$ is intimately related to the Maslov index of $(L_1(t), L_2(t))$.

PROPOSITION 13.1. *Let $\delta(t) = (L_1(t), L_2(t)) : a \leqq t \leqq b$ be a smooth path of pairs of Lagrangians, then with the closed loop $\hat{\delta}_{\pm\pm}$ defined by* (13.5), *the Maslov indices $\hat{\mu}[\hat{f}_{\pm\pm}]$ are given by:*

$$\hat{\mu}[\hat{f}_{+-}] = \mu_V((L_2(t), L_1(t)) : a \leqq t \leqq b) \tag{13.6}$$

$$\hat{\mu}[\hat{f}_{+,+}] = \hat{\mu}[\hat{f}_{+-}] + (n - \dim L_1(a) \cap L_2(a)) \tag{13.7}$$

$$\hat{\mu}[\hat{f}_{--}] = \hat{\mu}[\hat{f}_{+-}] - (n - \dim L_1(b) \cap L_2(b)) \tag{13.8}$$

$$\hat{\mu}[\hat{f}_{-+}] = \{\hat{\mu}[f_{+-}] + (n - \dim L_1(a) \cap L_2(a)) - (n - \dim L_1(b) \cap L_2(b)\} \tag{13.9}$$

Note that the right-hand side of (13.6) is $\mu_V((L_2(t), L_1(t))$ and this is

$$\{+\mu_V((L_1(t), L_2(t))) + \dim L_1(a) \cap L_2(a) - \dim L_1(b) \cap L_2(b)\}$$

by Property XI (Symmetry).

Proof of Proposition 13.1: The path $\nu_+(A, B) * \nu_-(B, A)$ from A to A is homotopic rel. end points to the constant path, as is $\nu_-(B, A) * \nu_+(A, B)$. Also via the Normalization Property IV, the closed paths $\nu_+(A, B) * \nu_+(B, A), \nu_-(A, B) * \nu_-(B, A)$ have Maslov indices

$$\hat{\mu}(\nu_+(A, B) * \nu_+(A, B)) = [n - \dim(A \cap B)] ,$$

$$\hat{\mu}(\nu_-(A, B) * \nu_-(A, B)) = -[n - \dim(A \cap B)] .$$

Consequently, $\hat{\mu}[f_{++}] = \hat{\mu}[f_{+-}] + \hat{\mu}[\nu_+(A, B) * \nu_+(B, A)] = \hat{\mu}[f_{+-}] + [n - \dim(A \cap B)]$ with $A = L_2(a), B = L_1(a)$. This reduces the proof of (13.7) to (13.6). Similarly, (13.8), (13.9) follow from (13.6).

In order to prove (13.6) we need but prove that $\hat{\mu}[\hat{f}_{+-}]$ satisfies the axioms I–VI characterizing the Maslov index $\mu_V((L_2(t), L_1(t))$.

In view of the fact that the homotopy class of $[\hat{f}_{+-}]$ in $\pi_1(\text{Lag})$ depends only on the path $f_{\text{flip}} = (L_2(t), L_1(t)) : a \leqq t \leqq b$ up to homotopy rel. end points, Properties I, II, and V are apparent. Symplectic Additivity IV is true by construction and the commutative diagram

$$\begin{array}{ccc} \pi_1(\text{Lag}V) \times \pi_1(\text{Lag}W) & \rightarrow & \pi_1(\text{Lag}V \oplus W) \\ \downarrow \hat{\mu} \times \hat{\mu} & & \downarrow \hat{\mu} \\ \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{+} & \mathbb{Z} . \end{array}$$

That this diagram commutes is easily checked by direct computation on the generators of $\pi_1(\mathrm{Lag}V) \equiv \mathbb{Z}, \pi_1(\mathrm{Lag}W) \equiv \mathbb{Z}$.

If $a \leqq c \leqq b$ with $f(t) = (L_1(t), L_2(t)) : a \leqq t \leqq b$, then we have

$$(\hat{f}[[a,b])_{+-} = (L_1(t)[[a,b]) * \nu_{b+} * (L_2(-t)|[-b, l-a]) * \nu_{a-}$$

Now $L_1(t))[a,b]$ is homotopic rel. end points to

$$(L_1(t)|[a,c]) * \nu_+(L_1(c), L_2(c)) * \nu_1(L_2(c), L_1(c)) * (L_1(t)|[c,b])$$

and $L_2(-t)|[-b,-a]$ is homotopic rel. end points to

$$(L_2(-t)|[-b,-c]) * \nu_-(L_2(c), L_1(c)) * \nu_+(L_1(c), L_2 * (L_2(-t)|[-c,-a] \; .$$

Inserting these expressions into $(\hat{f}([a,b])_{++}$ above and using the path additivity of $\mu_V([\gamma], -)$ with (γ) the constant path, shows that $\hat{\mu}[(\hat{f}|[a,b])_{+-}]$ is the sum of eight terms of the form $\mu_V([\gamma], -)$. The sum of four of these terms gives $\hat{\mu}((\hat{f})[a,c])_+ = \mu_V([\gamma], (f)[a,c])_{+-})$, the other four give $\hat{\mu}((\hat{f}|[b,c])_{+-})$. This proves the Path Additivity Property III.

The Normalization Property IV is simply checked. For $g = (L_1(t), L_2(t)) = (\mathbb{R}\{e^{it}\}, \mathbb{R})$ in $\mathbb{R}^2$, so $g_{\text{flip}} = (\mathbb{R}, \mathbb{R}\{e^{it}\})$, we have by VI

$$\mu_{\mathbb{R}^2}(g_{\text{flip}}|[0, \pi/4)) = 1, \mu_{\mathbb{R}^2}(g_{\text{flip}}|[-\pi/4, 0]) = 0 \; .$$

The corresponding loops $(\hat{g}|[0,\pi/4])_{+-}, (\hat{g}/[-\pi/4,0])_{+-}$ are easily drawn. This proves Proposition 13.1.

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