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Imbeddings and homology cobordisms of lens spaces

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In this paper we consider the existence of smooth or PL imbeddings of manifolds in Euclidean space with codimension one. The manifolds we treat are made from lens spaces (or homotopy lens spaces) by removing a disc or by taking a connected sum. (It is easy to see [R2] that a homotopy lens space must be punctured in order to imbed in Euclidean space of one higher dimension.) The results of [GL, R2] show that this problem reduces to the problem of finding a homology cobordism (i.e. one with the homology of a product) between two homotopy lens spaces. It is shown in [R2] that for (linear) lens spaces L with $\pi_1(L)$ of prime power order, the existence of such a homology cobordism implies the existence of an s-cobordism, and hence that a lens space L imbeds punctured if and only if L admits an automorphism satisfying certain conditions. It is straightforward to explicitly describe all such lens spaces. Further, the connected sum of two such lens spaces imbeds if and only if they are diffeomorphic. Hence in both problems, the homology cobordism may be taken to be a product.

The present paper will demonstrate that the situation changes when the order of $\pi_1(L)$ is divisible by more than one prime and when L is allowed to be a homotopy lens space. The invariants used in [R2] as obstructions to imbedding were equivariant signatures associated to coverings of prime-power degree; in the general case considered here they do not characterise a homotopy lens space, even up to h-cobordism. Nevertheless, we show that in dimensions greater than three, the signature invariants used in [R2] do determine a homotopy lens space up to homology cobordism within its normal cobordism class. Hence only a small portion of the invariants used in [W1] to classify homotopy lens spaces comes into play; in particular Reidemeister torsion plays no role. This classification up to homology cobordism leads to necessary and sufficient conditions for punctured imbeddings and imbeddings of connected sums.

The fact that only the invariants associated to prime-power coverings come into play has an analog in other parts of topology, most notably in the theory of transformation groups. In that context, Smith theory [B1] provides restrictions on

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the homology of fixed-point sets of actions of finite groups. These homological restrictions often turn out to be (with some additional conditions) sufficient to construct actions with specified fixed-point set [J, W2]. So, as in Jones' converse to Smith theory [J], only prime-power restrictions arise.

1. Definitions and notation

The quotient of S^{2k-1} by a [PL] free action of a cyclic group will be called a *homotopy lens space*; if the action is the restriction of a representation L is simply a lens space. For each L, fix a generator g of $\pi_1(L)$ and an orientation of L. Let $\psi:\pi_1(L)\to\mathbf{Z}_d$ be a homomorphism taking g to 1; this gives an action of $\mathbf{Z}_d$ on $\tilde{L}$. Some multiple (say s) of this action bounds a free action of $\mathbf{Z}_d$ on some manifold $\tilde{W}^{2k}$; examining the $\mathbf{Z}_d$ action on $H_k(\tilde{W})$ gives the multisignature ρ_d [W1]. We follow [W1] in regarding $\rho_d(L)$ as an element of the ring $\mathbf{Q}[\chi]/\Sigma$ where $\Sigma=\sum_{i=0}^{d-1}\chi^i$, and χ is a generator of $\operatorname{Hom}(\mathbf{Z}_d, S^1)$. We can thus view $\rho_d(L)$ as a function from $\pi_1(L)-\{e\}$, or as a polynomial $\sum_{i=0}^{d-1}\sigma_i\chi^i$ well-defined up to addition of multiples of Σ. The numbers σ_i are $1/s$ times the eigenspace signatures of the action of $\mathbf{Z}_d$ on $H_k(\tilde{W})$. If $\pi_1(L)=\mathbf{Z}_d$ then $\rho_d(L)$ is denoted $\rho(L)$ in [W1].

If $d\mid n$ and the homomorphism ψ factors through the obvious surjection $\mathbf{Z}_n\to\mathbf{Z}_d$, then the invariants ρ_d and ρ_n are related by a formula due to Hirzebruch: If $n=md$ and ρ_n is written as $\sum_{k=0}^{n-1}\sigma_k\chi_n^k$, then according to [H], ρ_d will be given as $m\sum_{k=0}^{d-1}\sigma_{km}\chi_d^k$. For M a closed manifold, M_0 will denote the punctured manifold M-(open ball).

1.1. DEFINITION. Suppose M and M' are oriented manifolds. A homology cobordism from M to M' is an oriented cobordism $(W;M,M')$ with $H_*(W;M)=H_*(W,M')=0$.

The obstructions to homology cobordism and imbeddings which we discuss are equally valid in the three-dimensional case and in high dimensions. However our positive results are valid (so far as we know) only in dimensions greater than three, so that all lens spaces considered from now on will have dimension ≥ 5. We have phrased our results in terms of PL manifolds and imbeddings; we will indicate at appropriate places the modifications necessary for the smooth case.

2. Homology cobordisms and imbeddings

2.1. LEMMA. *If (W^{2k}, L, L') is a homology cobordism between the homotopy lens space L and L', there is a retraction $r:W\to L$ whose restriction to L' is a homotopy equivalence.*

Proof. View L as the $(2k-1)$-skeleton of $K(\mathbf{Z}_n, 1)$. Using the fact that $H_1(W, L) = 0$, it is easy to extend the inclusion of L in $K(\mathbf{Z}_n, 1)$ to a map of W to $K(\mathbf{Z}_n, 1)$. But $W = L \cup$ cells of dimension $\leq 2k-1$ so this map compresses (rel L) into the $(2k-1)$-skeleton of $K(\mathbf{Z}_n, 1)$, i.e. into L. Since r is a retraction, it induces a surjection on homology, so that the restriction of r to L' is a surjection on homology and hence on π_1 as well. Since the lens spaces have the same homology groups, r_* is an isomorphism, so r must be a homotopy equivalence.

2.2. PROPOSITION. *Suppose L, L' are $(2k-1)$-dimensional oriented homotopy lens spaces.*

1. *If $L \# -L'$ imbeds in S^{2k}, then there is a homology cobordism $(W; L, L')$.*
2. *If L_0 imbeds in S^{2k} then there is a homology cobordism from L to itself such that the induced homotopy equivalence $r: L \to L$ satisfies $r_*(g) = g^a$, where a is a unit in $\pi_1(L) = \mathbf{Z}_n$ satisfying the conditions:*

$$a^k \equiv 1 \pmod n, \qquad (a^j - 1, n) = 1 \quad \text{for} \quad j < k. \tag{$*$}$$

3. *If there is a homology cobordism as in (2) with fundamental group $\mathbf{Z}_n$, then L_0 imbeds in S^{2k}.*
4. *If L_0 imbeds in S^{2k} and there is a homology cobordism from L to L' with fundamental group $\mathbf{Z}_n$, then $L \# -L'$ (and hence L_0') imbeds in S^{2k}.*

Proof. (1) and (2) are shown in [R2, theorem 6]; W is essentially a component of $S^{2k} - (L \# -L')$ or of $S^{2k} - L_0 \times I$. Suppose we have $(W; L, L)$ as in (2). Glue L to itself via the identity map, resulting in a homology $S^1 \times S^{2k-1}$ by a Mayer–Vietoris calculation. (The point is that the conditions (*) describe the induced map on the homology of L.) Surgery on an imbedded circle hitting L transversally in one point produces a homotopy S^{2k}, hence a PL sphere with L_0 imbedded in it. Finally let $(W; L, L')$ be a homology cobordism with $\pi_1(W)$ cyclic. Remove an arc from L to L', and glue two copies along L_0' to get a homology cobordism from L_0 to itself. If now L_0 is imbedded in S^{2k}, split open S^{2k} along L_0 and insert this new homology cobordism. The result is again S^{2k} now with $L \# -L'$ imbedded.

From (2) we get an easy restriction on what lens spaces could conceivably imbed in S^{2k}.

2.3. COROLLARY. *If L_0 imbeds in S^{2k} and $\pi_1(L) = \mathbf{Z}_n$ then $p > k$ for all prime factors p of n. In particular if $2 \mid n$ then L_0 does not imbed in S^{2k}.*

Proof. The above conditions are clearly necessary for there to be an element of order exactly k in $\mathbf{Z}_n^*$.

Proposition 2.2 reduces the imbedding problem to the question of finding a homology cobordism whose induced retraction acts in a given way on $\pi_1(L)$. It is not hard to find obstructions to homology cobordism of homotopy lens spaces; because we are primarily interested in the application to the imbedding problem we restrict to the case when π_1 is of odd order. By Corollary 2.3 this does not lose any generality.

2.4. PROPOSITION. *Suppose* $(W; L, L')$ *is a homology cobordism with* $r: L' \to L$ *the induced homotopy equivalence.*

1. *r is normally cobordant* [B2] *to* id_L.
2. *For all prime powers d dividing the order of* $\pi_1(L)$, $\rho_d(L')(\chi) = \rho_d(L)(\chi^r)$.

Proof. The first part is shown in ([CS], p. 307); the point is that a homology equivalence between two spaces induces a bijection between their sets of stable bundles, hence the stable normal bundle of $L \times I$ comes from a bundle on W. So W itself provides the normal cobordism. Part (2) is shown in [R2] and depends on a Smith-theory argument of Gilmer [G1].

Our main theorem is the converse of this proposition.

2.5. THEOREM. *Suppose* $k > 2$, $r: L \to L'$ *is a homotopy equivalence, and that*

1. *r is (PL) normally cobordant to* id_L.
2. *For all prime powers d dividing n,* $\rho_d(L')(\chi) = \rho_d(L)(\chi^r)$.

Then there is a PL homology cobordism W from L to L' whose induced homotopy equivalence is r, and with $\pi_1(W) = \mathbf{Z}_n$.

As an immediate consequence of Theorem 2.5 and Proposition 2.4 we get necessary and sufficient conditions for $L_0 \subset S^{2k}$ and for $L \# -L' \subset S^{2k}$. For the rest of this section we assume that $k > 2$.

2.6. THEOREM. *Let L, L' be homotopy lens spaces with* $\pi_1 = \mathbf{Z}_n$.

1. $L_0 \subset S^{2k}$ *if and only if there is an* $a \in \mathbf{Z}_n$ *such that* $a^k \equiv 1 \pmod n$, $(a^j - 1, n) = 1$ $(j < k)$ *for which the following hold:*
   (a) $\rho(L)(\chi^a) \equiv \rho(L)(\chi) \pmod{\mathbf{Z}}$.
   (b) $\rho_d(L)(\chi^a) = \rho_d(L)(\chi)$ *for all prime-powers d dividing n.*
2. *If* $L_0 \subset S^{2k}$, *then* $L \# -L' \subset S^{2k}$ *if and only if there is an orientation preserving homotopy equivalence* $r: L \to L'$ *with:*
   (a) $\rho(L)(\chi^a) \equiv \rho(L')(\chi) \pmod{\mathbf{Z}}$.
   (b) $\rho_d(L)(\chi^a) = \rho_d(L')(\chi)$ *for all prime-powers d dividing n.*

Proof. This follows directly from parts 3 and 4 of Proposition 2.2 and the above theorem.

To prove Theorem 2.5, we use the homology surgery of Cappell–Shaneson [CS]. Suppose $f: W^{2k} \to L \times I$ is a normal map such that $\partial W = L' \cup L$, $f \mid L' = r$, and $f \mid L = id_L$. Then an element $\sigma(f) \in \Gamma^h_{2k}(\mathbf{Z}[\mathbf{Z}_n] \to \mathbf{Z})$ is defined which is the obstruction to doing surgery on W to make f into a homology equivalence. The obstruction group is not even finitely generated; what makes $\sigma(f)$ computable is the fact that $f \mid \partial W$ is a homotopy equivalence. This implies that $\sigma(f)$ is in the image of the natural map from $L^h_{2k}(\mathbf{Z}[\mathbf{Z}_n])$. So to prove Theorem 2.5 we need to compute (enough of) $\Gamma^h_{2k}(\mathbf{Z}[\mathbf{Z}_n] \to \mathbf{Z})$ to detect the image of $L^h_{2k}(\mathbf{Z}[\mathbf{Z}_n])$. For the computation of Γ^h_{2k} we use the work of J. Smith [S2] which interprets Γ^h_{2k} as the L-group of a certain localization of $\mathbf{Z}[\mathbf{Z}_n]$. (Smith's work holds in more generality; in the case stated below, the result is due to Capell and Shaneson (unpublished). See [V] for related results.)

2.7. DEFINITION. Let $\epsilon: \mathbf{Z}[\mathbf{Z}_n] \to \mathbf{Z}$ be the augmentation, and set $S = \{\psi \mid \epsilon(\psi) = 1\} = 1 + \ker(\epsilon)$. Define the localized ring $\Lambda = S^{-1}\mathbf{Z}[\mathbf{Z}_n]$.

2.8. THEOREM [S2]. $\Gamma^h_{2k}(\mathbf{Z}[\mathbf{Z}_n] \to \mathbf{Z}) \cong L^h_{2k}(\Lambda)$. *The map* $L^h_{2k}(\mathbf{Z}[\mathbf{Z}_n]) \to \Gamma^h_{2k}(\mathbf{Z}[\mathbf{Z}_n] \to \mathbf{Z}) \to L^h_{2k}(\Lambda)$ *is induced by the localization map* $s: \mathbf{Z}[\mathbf{Z}_n] \to \Lambda$.

The computation of $L^h_{2k}(\Lambda)$ reduces, via the Ranicki–Rothenberg sequence [R1]

$$\to H^1(\mathbf{Z}_2; \tilde{K}_0(\Lambda)) \to L^h_{2k}(\Lambda) \to L^p_{2k}(\Lambda) \to$$

to understanding $L^p_{2k}(\Lambda)$, $\tilde{K}_0(\Lambda)$, and the maps from $L^p_{2k}(\mathbf{Z}[\mathbf{Z}_n])$ and $\tilde{K}_0(\mathbf{Z}[\mathbf{Z}_n])$. We summarise the computations in:

2.9. THEOREM. 1. *For all* $n, \tilde{K}_0(\Lambda) = 0$. *In particular,* $L^p_*(\Lambda) \cong L^h_*(\Lambda)$.

2. *If* $x \in L^p_{2k}(\mathbf{Z}[\mathbf{Z}_n])$ *has* $\rho_d(x) = 0$ *for all prime-powers d dividing n, and has* $\mathrm{Arf}(x) = 0$ *if k is odd, or signature* $(x) = 0$ *if k is even, then* $s_*(x) = 0$ *in* $L^p_{2k}(\Lambda)$.

We will prove this in the next section, but first we deduce Theorem 2.5 from it.

Proof of Theorem 2.5. Let $f: W \to L \times I$ be a normal cobordism of the given homotopy equivalence r to id_L. Since r (and id_L) is a homotopy equivalence, there is an obstruction $\sigma(f) \in L^h_{2k}(\mathbf{Z}[\mathbf{Z}_n])$ to doing surgery on W (rel ∂) to make it into an h-cobordism; we would like to know that $s_*(\sigma(f)) \in L^h_{2k}(\Lambda)$ is trivial. Note that in the PL case we can kill the simply connected surgery obstruction (the signature of Arf invariant, depending on the dimension) by taking the connected sum with a standard surgery problem. This is the only part of the argument where the PL case differs from the smooth one. The localization map $\mathbf{Z}[\mathbf{Z}_n] \to \Lambda$ induces a map between Ranicki–Rothenberg sequences:

$$\begin{array}{ccccccc} \longrightarrow H^1(\mathbf{Z}_2; \tilde{K}_0(\mathbf{Z}[\mathbf{Z}_n])). & \longrightarrow & L^h_{2k}(\mathbf{Z}[\mathbf{Z}_n]) & \longrightarrow & L^p_{2k}(\mathbf{Z}[\mathbf{Z}_n]) & \longrightarrow \\ \downarrow & & & & \downarrow & \\ H^1(\mathbf{Z}_2; \tilde{K}_0(\Lambda)) & \longrightarrow & L^h_{2k}(\Lambda) & \longrightarrow & L^p_{2k}(\Lambda) & \end{array}$$

By assumption, $\rho_d(\sigma(f)) = 0$ for all prime-powers d, so by 2.9(2) $s_*(\sigma(f)) = 0$ in $L^p_{2k}(\Lambda)$. Since $\tilde{K}_0(\Lambda)$ is trivial, $L^h_*(\Lambda)$ is isomorphic to $L^p_*(\Lambda)$. Hence $s_*(\sigma(f)) = 0$ and the theory of [CS] provides a homology cobordism from L' to L.

2.10. COROLLARY. *If L' is homology-cobordant to L^{2k-1} there is a $(k-1)$-connected homology cobordism from L' to L.*

Proof. If L' is homology cobordant to L the prime-power multisignatures ρ_d are all equal. The homology cobordism provided by Theorem 2.5 can be taken to be $(k-1)$-connected, by performing preliminary low-dimensional surgeries.

3. Algebraic computations

The idea behind our computation is that (roughly speaking) the ring $\mathbf{Z}[\mathbf{Z}_n]$ splits up as a product of rings according to the various factors of n. Upon passing to the localized ring Λ, the rings associated to composite factors of n become trivial, while those associated to prime-powers remain. The tools which are used in carrying out this idea are the Mayer–Vietoris sequences in K- and L-theory due to Milnor [M2] and Ranicki [Ri] respectively. We remind the reader that a diagram of rings

$$\begin{array}{ccc} R_1 & \longrightarrow & R_2 \\ \downarrow & & \downarrow \\ R_3 & \longrightarrow & R_4 \end{array}$$

is *cartesian* if the associated sequence of the additive groups

$$0 \to R_1 \to R_2 \oplus R_3 \to R_4 \to 0$$

is exact.

By definition the multiplicative set which we invert to obtain Λ is $S = 1 + \ker(\epsilon) = 1 + (T-1)\mathbf{Z}[\mathbf{Z}_n]$. Any other ring we will localize will be a quotient of $\mathbf{Z}[\mathbf{Z}_n]$, and the multiplicative set will be simply the image of S. It is a standard exercise [A] to show that the localizations of the rings in a cartesian square still form a cartesian square.

NOTATION. We will denote the d^{th} cyclotomic polynomial by Φ_d, so that $\prod_{d|n} \Phi_d(T) = T^n - 1$. If δ_d is a primitive d^{th}-root of unity, then $\mathbf{Z}(\zeta_d) = \mathbf{Z}[T, T^{-1}]/\Phi_d(T)$. We call d composite if d is divisible by more than one prime. Finally, we will write n as a product $n = \prod d$, where the d's are powers of distinct primes.

The key algebraic facts which distinguish prime-powers from composite numbers is the following well-known lemma (cf. [L]) and its corollary, which shows how (for n composite) a large portion of $\mathbf{Z}[\mathbf{Z}_n]$ gets killed upon localization.

3.1. LEMMA. $\Phi_d(1) = \begin{cases} 1 & (d \textit{ composite}) \\ p & (d = p^r, p \text{ a prime}). \end{cases}$

3.2. COROLLARY. *Let $R = \mathbf{Z}[T, T^{-1}]/I$ where I is an ideal containing an element of the form $f(T) = \prod \Phi_d(T)$ where all of the d's are composite. Let $S = 1 + (T-1)R$; then $S^{-1}R$ is trivial.*

Proof. $f(1) = \prod \Phi_d(1) = 1$ by the lemma, hence $(T-1) \mid (f(T) - 1)$, i.e. $f(T) \in S$. But $f(T) \in I$, so $0 \in S$. This forces $S^{-1}R$ to be trivial [A].

The first step is to split up the ring $\mathbf{Z}[\mathbf{Z}_n]$ into pieces corresponding to the factorization of $T^n - 1$ into a product of cyclotomic factors, where we group separately the polynomials corresponding to composite and prime-power factors of n. The result is summarized in the following lemma, whose proof we omit.

3.3. LEMMA. *Let Φ_{comp} be the polynomial*

$$\prod_{\substack{r|n \\ r \text{ composite}}} \Phi_r.$$

If d denotes the power of a prime d in n, then there is a cartesian square:

$$\begin{array}{ccc} \mathbf{Z}[\mathbf{Z}_n] & \longrightarrow & \mathbf{Z}[T, T^{-1}]/\Phi_{comp} \\ \downarrow & & \downarrow \\ \prod_p \mathbf{Z}[\mathbf{Z}_d] & \longrightarrow & \prod_p \mathbf{Z} \end{array}$$

The map from $\mathbf{Z}[\mathbf{Z}_n]$ to $\prod \mathbf{Z}[\mathbf{Z}_d]$ is given by the obvious projections, and the map from $\prod \mathbf{Z}[\mathbf{Z}_d]$ to $\prod \mathbf{Z}$ is given by the product of the augmentations.

Hence to localize $\mathbf{Z}[\mathbf{Z}_n]$, we must determine the localization of each piece in the above cartesian square. According to Lemma 3.2, the 'composite piece' becomes zero when we invert the elements in S, so it suffices to understand what happens to the ring $\mathbf{Z}[\mathbf{Z}_d]$.

3.4. LEMMA. *For d a power of the prime p, there is a cartesian square*

$$\begin{array}{ccc} S^{-1}\mathbf{Z}[\mathbf{Z}_d] & \longrightarrow & \mathbf{Z} \\ \downarrow & & \downarrow \\ \mathbf{Z}_{(p)}[\mathbf{Z}_d] & \longrightarrow & \mathbf{Z}_{(p)} \end{array}$$

Proof. First we need to construct the left-hand vertical map; the horizontal maps are given by augmentations and the map $\mathbf{Z} \to \mathbf{Z}_{(p)}$ is the obvious inclusion. To construct the map $S^{-1}\mathbf{Z}[\mathbf{Z}_d] \to \mathbf{Z}_{(p)}[\mathbf{Z}_d]$, we need to show that if $g \in \mathbf{Z}[\mathbf{Z}_d]$ has $\epsilon(g) = 0$, then $1 + g$ is invertible in $\mathbf{Z}_{(p)}[\mathbf{Z}_d]$. (Here ϵ denotes the augmentation.) To see this, note first that a polynomial in $\mathbf{Z}[\mathbf{Z}_d]$ is invertible in $\mathbf{Z}_{(p)}[\mathbf{Z}_d]$ if and only if it is invertible in $\mathbf{Z}_p[\mathbf{Z}_d]$. But it is easy to verify that if $\epsilon(g) = 0$, then $g^d \equiv 0 \pmod p$, so that $1 + g$ is invertible $(\text{mod}\, p)$. Hence we can define the desired map as $f/g \to f \cdot g^{-1}$.

To prove that the square is cartesian, we must verify that any $h \in \mathbf{Z}_{(p)}[\mathbf{Z}_d]$ with integral augmentation $\epsilon(h)$ may be written as a quotient f/g for $f, g \in \mathbf{Z}[\mathbf{Z}_d]$ with $\epsilon(g) = 1$. We may write such an h in the form $\sum (a_i/m)x^i$, where $\sum a_i \equiv 0 \pmod m$, and m and p are relatively prime. Choose an integer u with $d \cdot u \equiv 1 \pmod m$, then working modulo integral terms:

$$\left(\frac{1}{m} \sum_{i=0}^{d-1} a_i x^i\right)\left(u \sum_{i=0}^{d-1} x^i + 1 - d \cdot u\right) \equiv u \cdot \epsilon(h) \in \mathbf{Z}.$$

Since $\epsilon(\sum_{i=0}^{d-1} x^i) = d$, the second term has $\epsilon = 1$, and so the proof of the lemma is completed.

Putting together Lemmas 3.2 and 3.4, we obtain the desired splitting of the localized ring Λ:

3.5. LEMMA. *There is a cartesian square:*

$$\begin{array}{ccc} \Lambda & \longrightarrow & \mathbf{Z} \\ \downarrow & & \downarrow \\ \prod_p \mathbf{Z}_{(p)}[\mathbf{Z}_d] & \longrightarrow & \prod_p \mathbf{Z}_{(p)} \end{array}$$

We are now able to verify the first part of Theorem 2.9.

Proof of 2.9(1). According to Milnor [M2], the square in Lemma 3.5 yields an exact sequence in (reduced) K-theory:

$$\to \underset{K_1(\prod_p \mathbf{Z}_{(p)}[\mathbf{Z}_d])}{\overset{K_1(\mathbf{Z})}{\bigoplus}} \to K_1\Big(\prod_p \mathbf{Z}_{(p)}\Big) \to \tilde{K}_0(\Lambda) \to \underset{\tilde{K}_0(\prod_p \mathbf{Z}_{(p)}[\mathbf{Z}_d])}{\overset{\tilde{K}_0(\mathbf{Z})}{\bigoplus}} \to \tilde{K}_0\Big(\prod_p \mathbf{Z}_{(p)}\Big).$$

Since $\mathbf{Z}_{(p)}$ is a local ring, $K_1(\mathbf{Z}_{(p)}) = \text{units of } \mathbf{Z}_{(p)}$, and so the map $K_1(\prod_p \mathbf{Z}_{(p)}[Z_d]) \to K_1(\prod_p \mathbf{Z}_{(p)})$ is a surjection. Therefore it suffices to show that $\tilde{K}_0(\mathbf{Z}_{(p)}[\mathbf{Z}_d]) = 0$.

But $\mathbf{Z}_{(p)}[\mathbf{Z}_d]$ fits into its Rim diagram [M2]

$$\begin{array}{ccc} \mathbf{Z}_{(p)}[\mathbf{Z}_d] & \longrightarrow & \mathbf{Z}_{(p)}(\zeta_d) \\ \downarrow & & \downarrow \\ \mathbf{Z}_{(p)} & \longrightarrow & \mathbf{Z}_p \end{array}$$

Both $\mathbf{Z}_{(p)}$ and $\mathbf{Z}_{(p)}(\zeta_d)$ are local rings (for the latter, see e.g. [S1]), and so have vanishing $\tilde{K}_0$-groups. $K_1(\mathbf{Z}_{(p)})$ evidently surjects onto $K_1(\mathbf{Z}_p)$, so the Mayer–Vietoris sequence shows the vanishing of $\tilde{K}_0(\mathbf{Z}_{(p)}[\mathbf{Z}_d])$ as well.

The second part of 2.9 follows in a similar manner.

Proof of 2.9(2). Since all the $\tilde{K}_0$-groups vanish, the square in Lemma 3.5 yields a Mayer–Vietoris sequence [R1, §6.3.1] in L^p-theory (remember that $0 = \bigoplus_p L^p_{2k+1}(\mathbf{Z}_{(p)})$):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^p_{2k}(\Lambda) & \longrightarrow & L^p_{2k}(\mathbf{Z}) \oplus \bigoplus_p L^p_{2k}(\mathbf{Z}_{(p)}[\mathbf{Z}_d]) & \longrightarrow & \bigoplus_p L^p_{2k}(\mathbf{Z}_{(p)}) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & & & \\ & & L^p_{2k}(\mathbf{Z}[\mathbf{Z}_n]) & \xrightarrow{\sigma \oplus \bigoplus \rho_d} & L^p_{2k}(\mathbf{Z}) \oplus \bigoplus_p L^p_{2k}(\mathbf{Z}[\mathbf{Z}_d]). & & & & \end{array}$$

The bottom square commutes, where σ represents the simply-connected surgery obstruction, either signature or Arf-invariant. Hence if an element $x \in L^p_{2k}(\mathbf{Z}[\mathbf{Z}_n])$ has all the signatures $\rho_d(x) = 0$ and $\sigma(x) = 0$, it will go to zero in $L^p_{2k}(\Lambda)$.

4. Computations and applications

The criteria of Theorem 2.6 for existence of homology cobordisms lead to new examples of imbeddings of punctured lens spaces.

EXAMPLE. Let L^{2k-1} be a lens space with an imbedding of L_0 in S^{2k}, for example one of the fibered imbeddings constructed in [R2]. Let $x \in L^h_{2k}(\mathbf{Z}[\mathbf{Z}_n])$ with multisignature $\rho(x) = 4(\chi + (-1)^k\chi^{-1})$, and let L' be the homotopy lens space obtained as the boundary of a normal cobordism from L whose surgery obstruction is the element x [W1]. $\rho(L') = \rho(L) + \rho(x)$, and it follows that L' cannot be h-cobordant to L. Likewise, $\rho_d(L') = \rho_d(L) + \rho_d(x)$, and we compute:

4.1. LEMMA. $\rho_d(x) = 0$ *for all* $d \mid n$, $d \neq n$.

Proof. In general, from [H], we have that if $\rho(x) = \sum_{r=0}^{n-1} a_r\chi^r$, then $\rho_d(x) = m\sum_{k=0}^{d-1} a_{km}\chi_d^k$ $(m = n/d)$. In our case, then, $\rho_d(x)$ is evidently 0 for all $d \neq n$.

4.2. THEOREM. *There is a homotopy lens space* L' *for which* L_0' *imbeds in* S^{2k}, *but does not imbed as the fiber of a fibered knot.*

Proof. Take L in the example above to be $L(n; 1, c, \dots, c^{k-1})$, where c satisfies the condition (*) of [R2], and where n is composite. Perform the construction indicated to get the homotopy lens space L'. By construction, L' is normally cobordant to L, so by the calculation above and Theorem 2.5, L' is homology cobordant to L and hence it too imbeds in S^{2k}. However it cannot imbed punctured in a fibered manner. For let $f: L' \to L'$ be the monodromy of the fibration; it induces a homotopy equivalence g from L to itself whose mapping torus is a homology $S^1 \times S^{2k-1}$. But it is easy to see that this implies that g_* must be multiplication by c^j for some $j < k$. Since f is a homeomorphism,

$$\begin{aligned} \rho(L')(\chi) &= \rho(L')(f_*(\chi)) \\ \rho(L)(\chi) + \rho(x) &= \rho(L)(g_*(\chi)) + g_*(x) \\ &= \rho(L)(\chi) + f_*(x) \end{aligned}$$

since g is in fact realized by a homeomorphism. Therefore, $x = f_*(x)$, which is clearly not so.

Similar examples presumably arise from 2-torsion elements of $L^h_{2k}(\mathbf{Z}[\mathbf{Z}_n])$. Such elements abound, e.g. the torsion subgroup of $L^h_{2k}(\mathbf{Z}[\mathbf{Z}_{15}])$ has an extra $\mathbf{Z}_2$ coming from $\tilde{K}_0$ [KM]. To get examples of lens spaces which do not imbed punctured in this way we need such elements which are not invariant under appropriate automorphisms of π_1.

Our criteria for homology cobordism and imbeddings, while complete in principle, have two unfortunate aspects. One concerns our original motivation for this work – the imbedding question for linear lens spaces. The homotopy lens space constructed in Theorem 4.2 is not a linear lens space, and it is not clear how to carry out such a construction to get a linear lens space. In fact, extensive computer calculations done on the CYBER computer at Courant have found that for lens linear spaces of dimension 5 or 7, and n = product of ≤ 4 primes from the list 7, 13, 19, 31 (respectively 5, 13, 17, 37), $L_0 \subset S^6$ ($\subset S^8$, respectively) if and only if L_0 imbeds fibered, and that the connected sum of two such lens spaces imbeds if and only if the two are diffeomorphic. On the other hand there are examples [GL] of non-diffeomorphic 3-dimensional lens spaces which satisfy the criteria for $L \# L'$ to imbed in S^4. However, recent work of Fintushel–Stern [FS] on Yang–Mills theory indicates that L is smoothly homology cobordant to L' if and only if $L = L'$. (This has been extended to more general 3-manifolds [M1, R3].) It is not clear whether or not our theorem extends to give topological imbeddings, because homology surgery does not work in general in dimension 4 [CG], even topologically.

The other aspect is that the criteria for homology cobordism are not completely independent. It is known [W1] that the class of $\rho(L) \bmod \mathbf{Z}$ is a normal cobordism invariant, and it is easy to verify that the same is true for all the $\rho_d(L)$. So the condition that $\rho_d(L) = \rho_d(L')$ for $d = p^r$ already places some restriction on the normal cobordism class of L'. In fact in low dimensions, the condition about normal cobordism in Theorem 2.5 is superfluous.

4.3. THEOREM. *If $r : L' \to L$ is homotopy equivalence of 5-dimensional homotopy lens spaces, and $\rho_d(L)(\chi^r) = \rho_d(L')(\chi)$ for all prime-powers d dividing n, then r is normally cobordant to id_L.*

Proof. We follow the determination of normal cobordism classes of maps into L as given in [W1]. By the computation on p. 208, there are n normal cobordism classes in $[L^5, G/PL]$ for each homotopy type. Hence it suffices to find, for each lens space L^3, n homotopy lens spaces L^5_i, with $L_i \simeq L_j$, but with $\rho_d(L_i) - \rho_d(L_j)$ not integral for some prime-power d dividing n.

CLAIM. (See below for proof.) Let L^3 *be a 3-dimensional lens space, and* $0 \leq j \leq n$. Then there is a 5-dimensional homotopy lens space L_j with $\rho(L_j) = \rho(\Sigma L) + 4j(\chi + \chi^{-1})(1 + \chi)/(1/\chi)$.

(L_j is constructed as a sort of suspension of L.)

Using the formula of Hirzebruch [H], we compute that for $n = md$,

$$\rho_d(L_j) - \rho_d(L_i) = 16(m/n)(j - i) \sum_{l=1}^{d-1} l\chi_d^l.$$

If $j \neq i \pmod n$, then we can choose a prime p with $(j - i, p) = 1$, and let $d =$ the largest power of p dividing n. It then follows from the above formula that $\rho_d(L_j) - \rho_d(L_i)$ is not integral. Therefore, $\{\rho_d\}$ determines the normal invariant.

Proof of claim. Let x_j be a hermitian form with multisignature $\rho(x_j) = 4j(\chi + \chi^{-1})$, and let (W^4, L_j, L) be a normal cobordism which realizes x_j. (L_j will be $\mathbf{Z}[\mathbf{Z}_n]$-homology equivalent to L.) If L_j were $S^3/\mathbf{Z}_n$, we could suspend the $\mathbf{Z}_n$ action on S^3 to get a $\mathbf{Z}_n$ action on S^5 with ρ as desired. It is unlikely that $\tilde{L}_j$ is S^3, (in fact it can't be for $n = 3^k$ by [R4]), but we can still 'suspend' the action as follows:

Let $E \to L_j$ be the D^2-bundle with Euler class Poincare dual to the generator of $\pi_1(L_j)$ corresponding to a fixed generator of $\pi_1(L)$. Note that $\partial E = S^1 \times \tilde{L}_j$ is $\mathbf{Z}[\mathbf{Z}_n]$-homology equivalent to $S^1 \times S^3$. We would like to make ∂E the boundary of a homotopy circle; the only obstruction to doing this is the μ-invariant of $\tilde{L}_j$. But since n is odd, we can arrange that $\mu(\tilde{L}_j)$ be zero by connected summing L_j with a homology sphere; this evidently doesn't affect the ρ-invariant. Hence $\partial E = \partial V^5$, where $V \simeq S^1$.

Let $L_j^5 = E \cup V$; it follows that L_j^5 is a homotopy lens space. By crossing the whole construction with CP^2, one can show that the ρ-invariant of L_j^5 is exactly that of a suspension, or in other words

$$\rho(L_j^5) = \rho(L_j^3)(1 + \chi)/(1 + \chi^{-1})$$

which is equivalent to the claimed formula.

Remark. The proof just given can be used instead of the argument given on pp. 213–214 of [W1] to construct all normal cobordism classes of 5-dimensional homotopy lens spaces.

A final question raised by these investigations is whether a knot constructed as the boundary of an imbedded punctured homotopy lens space is determined by its complement. Recall that there are at most two knots with a given complement, and that these differ by a 'Gluck twist' around the knot [G2, K]. All linear lens spaces admit S^1-actions with codimension-two fixed point sets. This implies that a knot which has a punctured lens space for a Seifert surface is determined by its complement. For one can concentrate the Gluck twist to be non-trivial on $K \times I \subset K \times S^1$ where $K = \partial L_0$ and $K \times I \subset L_0 \times I = \nu(L_0)$, and use the circle action on L_0 to extend the twist.

In fact the same is true if the Seifert surface is just a punctured homotopy lens space. To see this, note that if L is a homotopy lens space, there is a linear lens space L' and a homotopy equivalence $f: L \to L'$. Conjugating the self-diffeomorphism of $L_0' \times I$ just described by the homotopy equivalence f, we obtain a self-homotopy inverse F on $L_0 \times I$ which extends the Gluck twist on $\partial L_0 \times I$. It is easy to see that F will be in fact a simple homotopy equivalence, and that we can arrange that F be the identity on $L_0 \times \partial I$. If now L_0 is a Seifert surface for a knot, F extends by the identity to give a simple homotopy equivalence of the knot complement to itself which extends the Gluck twist on the boundary of the tubular neighborhood of the knot. The surgery argument in [C] now shows that this simple homotopy equivalence may be replaced by a PL homeomorphism, so that the knot is determined by its complement.

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