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The Annals of Mathematics, 2nd Ser., Vol. 134, No. 2 (Sep., 1991), 325-374.

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Annals of Mathematics, 134 (1991), 325–374

Singular spaces, characteristic classes, and intersection homology

By Sylvain E. Cappell[1] and Julius L. Shaneson [1]

Introduction

This paper begins a series in which we try to understand the relation between the local and global topological structure of stratified spaces. Here the global topological invariant to be studied will be the L-classes

$$L_i(X) \in H_i(X; \mathbf{Q}),$$

X a stratified pseudomanifold with even-codimension strata. For manifolds, these characteristic classes are the Poincaré duals of the Hirzebruch polynomials in the Pontrjagin classes; Goresky-MacPherson extended the definition to stratified spaces with even-codimension strata using intersection homology [GM1]. Cheeger [Ch] defined L-classes analytically using his development of L^2-cohomology theory of singular Riemannian spaces. See also [S], [CSW]. The importance of these classes is illustrated by the famous theorem of Browder and Novikov: The homeomorphism type of a compact simply connected manifold of dimension at least five is determined by its homotopy type and its L-classes, up to a finite number of possibilities. (For 4-manifolds, the homotopy type alone suffices [F].) By recent work of Cappell-Weinberger (see [We]), similar results hold for stratified spaces with simply connected even-codimension strata and simply connected links, with respect to isovariant homotopy type and L-classes of the space and its strata.

The study of L-classes will take place in the context of a stratified pseudomanifold X^n of real dimension n, embedded, say piecewise linearly, in a manifold M^m of dimension $m = n + 2$. For example X might be a hypersurface in a projective complex algebraic variety. For X a smoothly or PL locally flatly embedded submanifold, the L-classes are given by the classical formula

$$L(X) = [X] \cap i^*\mathscr{L}\Big(P(M) \cup \big(1 + \chi^2\big)^{-1}\Big),$$

$\mathscr{L}$ the total Hirzebruch L-polynomial, $P(M) \in H^*(M)$ the total Pontrjagin class of M, i the inclusion, and χ the Poincaré dual of $i_*[X]$.

[1]Both authors partially supported by NSF Grants, and S.E.C. by a J. S. Guggenheim Fellowship.

In general, we assume given a stratification of the pair (M, X) (see §1), with only even-codimension strata. Let $\mathscr{V}$ be the collection of components of singular strata of the induced stratification of X.

THEOREM (see (6.2)). *Assume each $V \in \mathscr{V}$ is simply connected. Then*

$$L(X) = [X] \cap i^*\mathscr{L}\left(P(M) \cup (1+\chi^2)^{-1}\right) - \sum_{\mathscr{V}} \sigma(\mathfrak{B}_V)\left(i_{V_*} L(\overline{V})\right).$$

In this theorem i_V is the inclusion of the closure $\overline{V}$ of the stratum $V \in \mathscr{V}$, and $\sigma(\mathfrak{B}_V)$ is a signature invariant associated to the link pair of V. Specifically $\mathfrak{B}_V$ is a $(-1)^{c-1}$-symmetric Hermitian torsion linking form, with values in $\mathbf{Q}(t)/\mathbf{Q}[t, t^{-1}]$ (the rational functions modulo the Laurent polynomials), on the intersection homology group with local coefficients $IH^{\overline{m}}_{c-1}(G; \mathbf{Q}[t, t^{-1}])$. Here (G, F) is the link pair of a top simplex of V, $\dim G = 2c - 1$, and the local system on $G - F$ is determined by $\alpha \mapsto t^{l(F,\alpha)}$, $\alpha \in \pi_1(G - F)$ and $l(F, \alpha)$ the linking number. The integer $\sigma(\mathfrak{B}_V)$ is the signature of the form $T \circ \mathfrak{B}_V$,

$$T: \mathbf{Q}(t)/\mathbf{Q}\left[t, t^{-1}\right] \to \mathbf{Q}$$

an appropriate (Trotter) trace (see §6). A mild condition, guaranteeing finite dimensionality and always satisfied for varieties, is needed for this signature to be defined and hence for the formula of the theorem to make sense; see §2. If (G, F) happens to be a smooth knot pair, this signature is just the usual signature of the intersection pairing on a Seifert surface.

When the strata are not simply connected, the terms in the sum must be replaced by the twisted L-class associated to a local system of $\pm$symmetric forms, as in (6.1) below. In some cases this can be decomposed as a product of a characteristic class measuring twisting (cf. [At]), and the L-class of the stratum (see §6).

In the final section, we illustrate this theorem by calculating the L-class and in particular the signature ($= L_0(X)$) for some simple singular hypersurfaces in projective space.

Non-locally smoothable embeddings in real codimension two also arise in the following topological context: Let ξ^k be a vector bundle over the manifold N^n, with total space $E(\xi)$ and fiber $\mathbf{R}^k$.

THEOREM [CS2, 4]. *Suppose that ξ is fiber-homotopy equivalent to the Whitney sum of an oriented 2-plane bundle and a trivial bundle. Then there exists a piecewise linear regular neighborhood W^{n+2} of N and a* PL *homeomorphism of pairs*

$$(E(\xi), \mathfrak{Z}(N)) \cong \left(W \times \mathbf{R}^{k-2}, N \times \{0\}\right),$$

$\mathfrak{Z}(N)$ the zero-section.

In this situation $i^*P(W) = P(N) \cup P(\xi)$; hence the above formula (or (6.1) below) can be rewritten in the form

$$P(\xi) = (1 + i^*\chi^2)\left[1 + P^{-1}(N) \cup \hat{\mathscr{L}}\left\{D \sum_{\mathscr{V}} \sigma(\mathfrak{B}_V)\left(i_{V_*} L(\bar{V})\right)\right\}\right],$$

(or with twisted L-classes for non-simply connected strata) where D is Poincaré duality and $\hat{\mathscr{L}}$ satisfies

$$x_n = \hat{\mathscr{L}}_n(\mathscr{L}_1(x_1), \mathscr{L}_2(x_1, x_2), \ldots, \mathscr{L}_n(x_1, \ldots, x_n)),$$

where we assume the stratification of (W, N) satisfies the hypothesis. Note that for some integer q, the q-fold Whitney of sum ξ with itself will be fiber homotopy trivial. Note also that in the special case ξ = stable normal bundle of N, the formula of (6.2) becomes the formula

$$L(N) = -\sum_{\mathscr{V}} \sigma(\mathfrak{B}_V)\left(i_{V_*} L(\bar{V})\right)$$

for the L-classes of the manifold N, or equivalently

$$P(N) = -\hat{\mathscr{L}}\left\{D \sum_{\mathscr{V}} \sigma(\mathfrak{B}_V)\left(i_{V_*} L(\bar{V})\right)\right\}.$$

These results raise (at least) the following questions:

1. Can W in the previous theorem be chosen so that the stratification of (W, N) has only even-codimension strata?

2. If (or when) this is the case, is there a canonical choice for W and/or a procedure to construct it explicitly from the geometry of N and ξ?

There are many other situations in which codimension-two embeddings or immersions of PL *manifolds* arise with necessarily large singular (= non-locally smooth) sets. These include embeddings of homotopy projective spaces and homotopy lens spaces, immersions of homotopy products of spheres in Euclidean space, etc. (see [CS2, 4, 5] [Ho]), and similar questions apply.

Although the main results on L-classes can be stated in terms of the PL chain version of intersection homology of [GM1], our overall approach to stratified spaces makes use of sheaf theory and the derived category [GM2], [Bo], and we rely heavily on the basic material in these references. We also use some foundational material on the category of perverse sheaves [BBD].

Note. We will always use the indexing conventions of [GM2].

We obtain our results as a consequence of the study of certain complexes of sheaves that are naturally related to characteristic classes.

One of the basic features of intersection homology is that it satisfies Poincaré duality over a field, e.g., **Q** or the function field $\mathbf{Q}(t)$, but not in general over rings, e.g., **Z** or $\mathbf{Q}[t, t^{-1}]$. Recall from [GS] that integral Poincaré duality for intersection homology theory does hold under the hypothesis that the links are torsion free, at least in appropriate dimensions. We exhibit a duality ("superduality") that holds at the opposite extreme—the intersection homology of the links is all torsion. Perversities $\bar{p}$ and $\bar{q}$ will be said to be superdual

$$p(k) + q(k) = k - 1, \qquad k \geqq 2.$$

(See (1.3) below concerning perversities with $p(2) = 1$.) Let R be a Dedekind ring.

(3.2) THEOREM. *Let Y^m be a stratified pseudomanifold, let $\mathfrak{L}$ and $\mathfrak{M}$ be local systems over $Y - \Sigma$ with coefficients in finitely generated R-modules, and let $\bar{p}$ and $\bar{q}$ be superdual perversities. Suppose that if $y \in \Sigma$, the stalks $\mathbf{H}^i(\mathbf{IC}^{\cdot}_{\bar{p}}(Y; \mathfrak{M})_y)$ are torsion modules over R. Then a perfect pairing*

$$\mathfrak{L} \otimes_R \mathfrak{M} \to R_{Y-\Sigma}$$

and an orientation of Y over R induce a canonical isomorphism

$$\mathbf{IC}^{\cdot}_{\bar{q}}(Y; \mathfrak{L}) \cong R\operatorname{Hom}\big(\mathbf{IC}^{\cdot}_{\bar{p}}(Y; \mathfrak{M}), \mathbf{D}^{\cdot}_Y\big)[m]$$

in the derived category $D^b(Y)$.

Of course, for trivial coefficients the hypothesis is never satisfied, but for non-trivial local systems this often happens. For example, in the case of Y a sphere, $\Sigma \subseteq Y$ a smooth knot, and $R = \mathbf{Q}[t, t^{-1}] = \mathfrak{L}_y$, the induced pairing on (co-)homology is the familiar Blanchfield pairing.

The (lower) middle and (upper) logarithmic perversities, $\bar{m} = (0, 0, 1, 1, 2, \ldots)$ and $\bar{l} = (1, 2, 2, 3, 3, 4, \ldots)$ are superdual. (Note that this differs from the logarithmic perversity of [GM2].) Assume R has an involution α and, in (3.2), an identification $\mathfrak{M} = \mathfrak{L}^{\mathrm{op}}$, is given. The peripheral complex $\mathbf{R}^{\cdot}_Y(\mathfrak{L})$ is defined by the distinguished triangle

$$\begin{array}{ccc} \mathbf{IC}^{\cdot}_{\bar{m}}(Y; \mathfrak{L}) & \longrightarrow & \mathbf{IC}^{\cdot}_{\bar{l}}(Y; \mathfrak{L}) \\ & {\scriptstyle [1]}\nwarrow \quad \swarrow & \\ & \mathbf{R}^{\cdot}_Y(\mathfrak{L}) & \end{array} .$$

Then (3.2) implies a duality isomorphism

$$\mathbf{R}^{\cdot}_Y(\mathfrak{L}) \cong \mathfrak{D}(\mathbf{R}^{\cdot}_Y(\mathfrak{L}))[m+1]^{\mathrm{op}}, \tag{4.1}$$

$\mathfrak{D}\mathbf{A}^{\cdot} = R\operatorname{Hom}(\mathbf{A}^{\cdot}; \mathbf{D}^{\cdot}_Y)$ the Verdier dual of a complex of sheaves $\mathbf{A}^{\cdot}$. For $m = 2n$, this induces a perfect $(-1)^n$-Hermitian torsion pairing on hypercohomology,

$$\mathscr{H}^{-n}(Y; \mathbf{R}^{\cdot}_Y(\mathfrak{L})) \otimes \mathscr{H}^{-n}(Y; \mathbf{R}^{\cdot}_Y(\mathfrak{L}))^{\mathrm{op}} \xrightarrow{\mathfrak{B}_Y} F/R,$$

F the field of quotients of R.

Now suppose that Y has only even-codimension strata, and let V be a component of the open stratum of codimension $2c$. Let $\mathfrak{G}^V$ be the local system over the open stratum V with stalk at y the image of

$$IH^{\bar{m}}_{c-1}(L_y;\mathfrak{L}) \to IH^{\bar{l}}_{c-1}(L_y;\mathfrak{L}),$$

L_y the link of this stratum near y. Let

$$\mathfrak{B}_V : \mathfrak{G}^V \otimes_R (\mathfrak{G}^V)^{\mathrm{op}} \to F/R$$

be the perfect pairing induced by superduality, as applied to L_y. Then $\mathfrak{B}_V$ induces

$$\mathfrak{B}_{V_*} : IH^{\bar{m}}_{n-c}(\bar{V};\mathfrak{G}^V) \otimes_R IH^{\bar{m}}_{n-c}(\bar{V};\mathfrak{G}^V)^{\mathrm{op}} \to F/R.$$

Given morphisms

$$\mathbf{X}^{\cdot} \xrightarrow{u} \mathbf{Y}^{\cdot} \xrightarrow{v} \mathbf{Z}^{\cdot}$$

in $D^b(Y)$ with $v \circ u$ trivial, let $\mathbf{C}^{\cdot}_{u,v} = \mathbf{C}^{\cdot}_{u'}$ be the algebraic mapping cylinder of a lift u' of u to $\mathbf{C}^{\cdot}_v[-1]$. This will be well-defined when we assume $\mathrm{Hom}_{D^b}(\mathbf{X}^{\cdot}, \mathbf{Z}^{\cdot})[-1] = 0$.

A complex $\mathbf{R}^{\cdot}$ in the derived category $D^b(Y)$ supported on the singular set of Y, with torsion stalks, and with a given duality isomorphism

$$\mathbf{R}^{\cdot} \cong \mathfrak{D}(\mathbf{R}^{\cdot})[m+1]$$

will be called a self-dual complex of torsion sheaves over Y. If $\mathbf{Y}^{\cdot} = \mathbf{R}^{\cdot}$, $\mathbf{Z}^{\cdot} \cong \mathfrak{D}(\mathbf{X}^{\cdot})[m+1]^{\mathrm{op}}$ (with respect to these duality isomorphisms), $u = \mathfrak{D}(v)[m+1]^{\mathrm{op}}$, $\mathbf{X}^{\cdot}$ has torsion stalks and is supported on the singular set, then $\mathbf{R}^{\cdot}_1 = \mathbf{C}^{\cdot}_{u,v}$ admits an obvious structure as a self-dual complex of torsion sheaves as well. We then say that $\mathbf{R}^{\cdot}$ is cobordant to $\mathbf{R}^{\cdot}_0$ if the latter is obtained from $\mathbf{R}^{\cdot}$ by a series of such operations.

THEOREM (see (4.2) and (5.6)). *$\mathbf{R}^{\cdot}_Y(\mathfrak{L})$ is cobordant to the orthogonal sum*

$$\sum_{\mathscr{V}} j_* \mathbf{IC}^{\cdot}_{\bar{m}}(\bar{V};\mathfrak{G}^V)[c(V)],$$

$\mathscr{V}$ the set of components of singular strata of Y. As a consequence, the pairings $\mathfrak{B}_Y$ and

$$\sum_{\mathscr{V}} \mathfrak{B}_{V_*}$$

represent the same element in the Witt group $\mathscr{W}(F/R)$ of torsion pairings.

These results are consequences of a general splitting theorem, up to cobordism, for an arbitrary self-dual perverse torsion sheaf; see (4.4) and (5.5).

In applying these results to characteristic classes, we proceed as follows (§§6, 7): Let $X \subset M$ be a stratified pseudomanifold, embedded in the smooth manifold M. Let W be a closed regular neighborhood of X in M, and consider the space

$$Y = W \cup_{\partial W} c(\partial W)$$

obtained by attaching to W the cone on its boundary. Then Y has a stratification with singular set

$$\Sigma_Y = X \cup N \cup c(\partial \overline{N}),$$

N a proper submanifold dual to $\chi(W, X)$ (§1), with a local system with stalks $\mathbf{Q}[t, t^{-1}]$ on $Y - \Sigma_Y$ given by linking number with $X \cup -N$. The element represented by $\mathfrak{B}_Y$ in a Witt group of torsion pairings can be calculated from the intersection pairing on $IH_*^{\overline{m}}(Y; \mathbf{Q}(t))$ and vanishes in this case. Hence the sum (m even)

$$\sum_{\mathscr{V}} \mathfrak{B}_{V_*}$$

represents zero in this Witt group. By applying a generalized (Trotter) trace (§6), taking signatures, and interpreting the results (§7), we obtain a signature formula (7.0) that is the zero-dimensional case of our main result. Our main result, (6.1), then follows essentially by the definition of L-classes (§6) in terms of signatures.

In the final section we use our results to study the signature of non-locally smooth knots and we give some calculations for a few hypersurfaces in projective space. In future papers, we will consider knot-theory-valued characteristic classes, equivariant (Atiyah-Singer) characteristic classes for group actions on stratified spaces, characteristic classes in generalized homology theories (e.g., G/TOP), etc., and special features of the algebraic case.

Some less general and precise forms of the present results were announced in [CS6].

1. Stratifications and local systems

A stratification of a pair of paracompact Hausdorff (Y, X) spaces is a filtration

$$\phi = Y_{-1} \subset Y_0 \subset \cdots Y_{m-2} \subset Y_{m-1} \subset Y_m = Y$$

such that for each point $y \in Y_i - Y_{i-1}$ there exist a distinguished neighborhood N, a compact Hausdorff pair (G, F), a filtration

$$\phi = G_{-1} \subset G_0 \subset \cdots \subset G_{m-i-1} = G,$$

and a homeomorphism

$$\varphi\colon D^i \times c(G, F) \to (N, N \cap X)$$

that carries $D^i \times c(G_{j-1}, G_{j-1} \cap F)$ onto $(Y_{i+j}, Y_{i+j} \cap X)$. Here c denotes the cone, and by convention $c(\phi) = \{y\}$. If (Y, X) is a piecewise linear (PL) pair, then such a stratification always exists, with Y_i, φ, etc. piecewise linear, and refines the intrinsic stratification [A], [St]. Further, the pair (G, F) depends up to PL homeomorphism only upon the component V of y in $Y_i - Y_{i-1}$. Any loop in V determines a PL homeomorphism of (G, F), and if $\mathscr{H}$ is any functor on the category of PL pairs and PL homeomorphisms to an algebraic category (groups, modules, rings, bilinear forms, etc.), then $\mathscr{H}(G, F)$ will be a local system of objects in the category, over V. The pair (G, F) will be called the link pair of V, written

$$(G, F) = \mathrm{lk}(V).$$

In this paper it will generally be assumed that Y is an oriented connected PL manifold of dimension m and that X is a compact connected oriented PL pseudomanifold of dimension n; i.e., each $(n - 1)$ simplex in a triangulation is a face of exactly two n-simplices. It can always be assumed that

$$Y_{m-1} = \cdots = Y_{n+1} = Y_n = X,$$

and, since $S^0 \subset S^{m-1}$ is always unknotted, that

$$Y_{n-1} = Y_{n-2}$$

also. We will call X a PL *sub-pseudomanifold of* Y, with the understanding that any stratification of (Y, X) to be considered will satisfy these properties.

Suppose that $m = n + 2$ and that the fundamental class

$$[X] \in H_n(X; \mathbf{Z})$$

maps trivially to $H_n(Y)$. Then the map

$$\alpha \mapsto t^{l(X, \alpha)},$$

where l denotes linking number, defines a homomorphism

$$\pi_1(Y - X) \to \{t^i | i \in \mathbf{Z}\}.$$

Hence the ring ($\mathbf{Q}$ = rational numbers)

$$\Lambda = \mathbf{Q}[t, t^{-1}]$$

of Laurent polynomials becomes the coefficients of a local system over $Y - X$, and hence, for any perversity $\bar{p}$, the intersection chain complex

$$\mathbf{IC}^{\cdot}_{\bar{p}}(Y; \Lambda)$$

is defined, as in [GM2]. (We have varied the notation slightly; in [GM2] this would be denoted, suppressing the perversity, as $\mathbf{IC}^{\cdot}_{Y}(\mathfrak{L})$, $\mathfrak{L}$ the local system with coefficients in Λ just given.)

More generally, let N be a proper, locally flat, oriented submanifold of Y, meeting X transversally [St], such that $d[N]$, d an integer, represents the image of $[X]$ in the locally finite homology of $H_n^{\infty}(Y, \varepsilon)$ of Y relative its ends. One constructs such an N as the transverse inverse image of $CP^{k-1} \subset CP^k$, k large, under a map $Y \to CP^k$ that pulls back the Chern class of the canonical complex line bundle to a class ξ_N with

$$d\xi_N = \chi(Y, X) \in H^2(Y),$$

$\chi(Y, X)$ the Poincaré dual of the image of $[X]$ in $H_n^{\infty}(Y, \varepsilon)$; every such N arises in this way. (The image of $\chi(Y, X)$ in $H^2(X)$ is by definition the Euler class of a regular neighborhood of X in Y [CS2].) Hence any such submanifold will be called a *dual submanifold* (of multiplicity d) to $\chi(Y, X)$. The homomorphism

$$\alpha \mapsto t^{l(X \cup -dN, \alpha)}$$

will then be defined for $\alpha \in \pi_1(Y - X \cup N)$; let $\mathfrak{L}_N$ denote the resulting local system over $Y - X \cup N$ with coefficients in Λ. Then $X \cup N$ is also a PL sub-pseudomanifold of Y and $\mathbf{IC}^{\cdot}_{\bar{p}}(Y; \mathfrak{L}_N)$ is also defined.

Note. In general the definition of linking number will depend upon the choice of a locally finite chain bounding $X \cup -dN$. It can be shown that different choices will vary the linking number with α precisely by evaluation of an element of $H^1(Y)$ on the image of α in $H_1(Y)$. Thus for $H^1(Y) = 0$, there is no ambiguity. In general the ambiguity in the definition of $\mathfrak{L}_N$ will be suppressed (but see (1.2)).

Obviously, we may take $N = \phi$ if and only if $[X]$ maps trivially into $H_n(Y)$. Then the following is clear:

(1.1) When $N = \phi$, $\mathbf{IC}^{\cdot}_{\bar{p}}(Y; \mathfrak{L}_N) = \mathbf{IC}^{\cdot}_{\bar{p}}(Y; \Lambda)$.

For k sufficiently large, a map $Y \to CP^k$ is determined up to homotopy by the pull-back of the Chern class of the canonical line bundle. Standard transversality arguments and linking number considerations then yield:

(1.2) Let N_i, $i = 0, 1$, be dual submanifolds to $\chi(Y, X)$, with $\xi_{N_0} = \xi_{N_1}$. Then there exists a proper submanifold P of $Y \times \mathbf{R}$, meeting $X \times \mathbf{R}$ transversally, with

$$P \cap (Y \times \{i\}) = N_i \times \{i\}, \qquad i = 0, 1.$$

Hence, after a possible change of $\mathfrak{L}_{N_1}$ by evaluation of an element of $H^1(Y)$, there is a (unique) local system $\mathfrak{L}_P$ over $Y \times \mathbf{R} - \{(X \times \mathbf{R}) \cup P\}$ with

$$\mathfrak{L}_P \big| (Y - X \cup N_i) \times \{i\} = \mathfrak{L}_{N_i}$$

and

$$\mathbf{IC}^{\cdot}_{\bar{p}}(Y \times \mathbf{R}; \mathfrak{L}_P) \big| Y \times \{i\} = \mathbf{IC}^{\cdot}_{\bar{p}}(Y; \mathfrak{L}_{N_i})$$

for $i = 0, 1$.

(1.3) *Note.* In [GM1, 2] perversities always satisfy $p(2) = 0$. Here we will also need to allow perversities $\bar{p}$ with $p(2) = 1$ (and $p(k+1) = p(k)$ or $p(k) + 1$). We may take as the definition of $\mathbf{IC}^{\cdot}_{\bar{p}}$ the Deligne construction: Let Y be a stratified pseudomanifold, let $U_k = Y - Y_{n-k}$, and let $i_k \colon U_k \to U_{k+1}$ be the inclusion. Suppose $\mathfrak{L}$ is a local system of R-modules over U_2, R a ring. Set

$$\mathbf{IC}^{\cdot}_{\bar{p}}(U_2; \mathfrak{L}) = \mathfrak{L},$$

and inductively define

$$\mathbf{IC}^{\cdot}_{\bar{p}}(U_{k+1}; \mathfrak{L}) = \tau^{\leqq p(k)-n} R(i_k)_* \mathbf{IC}^{\cdot}_{\bar{p}}(U_k; \mathfrak{L}).$$

This complex of sheaves satisfies the axioms [AX1] of [GM2] and is characterized by these axioms in the derived category (compare Prop. 4.3 below). Alternatively, $\mathbf{IC}^{\cdot}_{\bar{p}}(Y; \mathfrak{L})$ can be defined as the complex of locally finite PL chains ξ satisfying the admissibility conditions

$$\dim(|\xi| \cap Y_{n-k}) \leqq \dim \xi - k + p(k),$$
$$\dim(|\partial\xi| \cap Y_{n-k}) \leqq \dim \partial\xi - k + p(k).$$

The key point in defining the boundary operator is that, in view of the conditions, a principal face of a simplex of ξ that lies in the singular set cannot be contained in the support of $\partial\xi$. Clearly (1.1) and (1.2) hold for this wider notion of a perversity.

2. Knots and regular neighborhoods

Let X^n be a PL sub-pseudomanifold of the PL manifold Y^{n+2}, and suppose that Y is an open PL regular neighborhood of X. By uniqueness of regular neighborhoods,

$$Y = \operatorname{Int} W,$$

where W is a compact PL manifold with boundary and a regular neighborhood of X. Also W is unique up to PL homeomorphism relative X. By definition the *Euler class* of Y,

$$\chi(Y) = \chi(W) \in H^2(X),$$

is the image of $\chi(Y, X)$ under the map induced by inclusion. A *dual submanifold* to $\chi(Y)$ will be defined as any dual submanifold N to $\chi(Y, X)$ whose closure $\overline{N}$ in W is a proper locally flat submanifold (with boundary) of $(W, \partial W)$ and a regular neighborhood of the PL pseudomanifold $X \cap N$.

(2.1) A dual submanifold N to $\chi(Y)$ always exists. Further, given any two, N_1 and N_2, with $\xi_{N_1} = \xi_{N_2}$ the submanifold $P \subset Y \times \mathbf{R}$ of (1.2) relating them can also be chosen to have as closure in $W \times \mathbf{R}$ a proper submanifold with boundary that is also a regular neighborhood of $(X \times \mathbf{R}) \cap P$.

The proof is essentially the same as the discussion of [CS2, p. 186 and p. 189].

Definitions. A knot is a sub-pseudomanifold $X^n \subset S^{n+2}$ of a sphere; it is said to be of finite (homological) type if the homology groups $H_i(S^{n+2} - X; \Lambda)$ with local coefficients in Λ are finite-dimensional over $\mathbf{Q}$. A sub-pseudomanifold $X \subset Y$ is said to be of finite local type if the link of each component of any stratification is of finite type. It is an easy observation that a knot is of finite type if and only if its k-fold suspension is; we leave it to the reader to use this to show that a sub-pseudomanifold is of finite local type if and only if it has one stratification with links of finite type. It is also not hard to see that the link pairs of components of strata of a sub-pseudomanifold of finite local type also have finite local type.

(2.2) Proposition. *If the knot $X \subset S^{n+2}$ is algebraic or if X has the rational homology of S^n, then the knot $X \subset S^{n+2}$ is of finite type. Hence if Y is a neighborhood of a hypersurface, X, in a smooth complex algebraic variety or if X is a rational homology manifold, then $X \subset Y$ is of finite local type.*

The assertion for algebraic knots follows from [M1, §4]. Recall that an algebraic knot is just the intersection of a (possibly singular) hypersurface $f^{-1}(0)$ with small sphere about the origin, $f: \mathbf{C}^n \to \mathbf{C}$ a complex polynomial. The result for a rational homology sphere is proven in the same way as for smooth knots, as in [L2], for example.

Example. The standard smooth embedding $S^p \times S^q \subset S^{p+q+2}$ has complement homotopy equivalent to $S^1 \vee S^{p+1} \vee S^{q+1}$ and hence does not have finite type. The suspension of this embedding yields a sub-pseudomanifold of S^{p+q+3} that is not of finite local type.

(2.3) Proposition. *Let Y^{n+2} be a* PL *manifold regular neighborhood of the sub-pseudomanifold $X^n \subset Y$, of finite local type. Let N be a dual submanifold to*

$\chi(Y)$. Then the intersection homology groups

$$IH_i^{\bar{p}}(Y;\mathcal{L}_N) = \mathcal{H}^{-i}(Y;\mathbf{IC}^{\cdot}_{\bar{p}}(Y;\mathcal{L}_N))$$

are finite-dimensional over $\mathbf{Q}$.

This result will be proven in conjunction with a similar statement for knots.

(2.4) PROPOSITION. *Let $X^n \subset S^{n+2}$ be a knot that is of finite local type and of finite type. Then the groups $IH_i^{\bar{p}}(S^{n+2};\Lambda)$ are finite-dimensional over* $\mathbf{Q}$.

Proof of (2.3) *and* (2.4). First observe that if $n = -1$, i.e., $X = \phi$,

$$IH_i^{\bar{p}}(S^1;\Lambda) = H_i(S^1;\Lambda) = \begin{cases} \mathbf{Q}, & i = 0 \\ 0, & i \neq 0 \end{cases}.$$

Thus (2.4) holds for $n = -1$.

Now we prove (2.3) for $n \geqq 0$, under the inductive hypothesis that (2.4) holds for knots of dimensions -1 through $n-1$.

(2.3.1) LEMMA. *$X \cup N \subset Y$ has finite local type.*

Assuming the lemma, let $\mathbf{IC}^{\cdot} = \mathbf{IC}^{\cdot}_{\bar{p}}(Y;\mathcal{L}_N)$. Let $\{Y_i\}$ be a stratification of $(Y, X \cup N)$, let $i \leqq n$, let $y \in Y_i - Y_{i-1}$, and let (G, F) be the link pair of the component of y in $Y_i - Y_{i-1}$. Then $\mathcal{L}_N|(G-F) = \mathcal{L}_\phi$; and, as in [GM2, 2.4], the stalk homology is given by

$$\mathbf{H}^{-i}(\mathbf{IC}^{\cdot})_y = \begin{cases} IH_{n-i+1}^{\bar{p}}(G;\Lambda) & \text{if } i \geqq (n+2) - p(n+2-i) \\ 0 & \text{if } i < (n+2) - p(n+2-i) \end{cases}.$$

But $F \subset G \cong S^{n+1-i}$ has finite type and finite local type. Hence $\mathbf{H}^{-i}(\mathbf{IC}^{\cdot})_y$ is finite-dimensional for $y \in X \cup N$, whereas

$$\mathbf{H}(\mathbf{IC}^{\cdot})|(Y - X \cup N) = \mathcal{L}_N[n+2].$$

It then follows from the spectral sequence for hypercohomology $\mathcal{H}^*$ that, modulo the class of finite-dimensional vector spaces,

$$IH_i^{\bar{p}}(Y;\mathcal{L}_N) \cong H^{n+2-i}(Y;\alpha_!\mathcal{L}_N),$$

α the inclusion of $Y - X \cup N$ in Y. This may also be seen from the distinguished triangle (since $\alpha^*\mathbf{IC}^{\cdot} \cong \mathcal{L}_N[n+2]$)

$$\begin{array}{ccc} R\alpha_!\alpha^*\mathbf{IC}^{\cdot} & \longrightarrow & \mathbf{IC}^{\cdot} \\ & {\scriptstyle [1]}\nwarrow \quad \swarrow & \\ & R\beta_*\beta^*\mathbf{IC}^{\cdot} & \end{array},$$

β the inclusion of $X \cup N$. However, standard arguments in regular neighborhood theory imply that $Y - X \cup N$ has the structure of a product with $\mathbf{R}$, and it follows readily that

$$H^{n+2-i}(Y; \alpha_! \mathfrak{L}_N) = 0.$$

Finally, we prove (2.4) for a knot $X^n \subset S^{n+2}$. Let

$$\mathbf{IC}^{\cdot}_{\bar{p}} = \mathbf{IC}^{\cdot}_{\bar{p}}(S^{n+2}; \Lambda),$$

and consider the distinguished triangle

$$\begin{array}{ccc} R\gamma_! \gamma^* \mathbf{IC}^{\cdot}_{\bar{p}} & \longrightarrow & \mathbf{IC}^{\cdot}_{\bar{p}} \\ & {}_{[1]}\nwarrow \quad \swarrow & \\ & R\delta_* \delta^* \mathbf{IC}^{\cdot}_{\bar{p}} & \end{array}.$$

Here δ is the inclusion of X and γ of $S^{n+2} - X$. Again, by the inductive hypothesis the stalk homology of $R\delta_* \delta^* \mathbf{IC}^{\cdot}_{\bar{p}}$ is finite-dimensional for $y \in X$; hence so is its hypercohomology. Since $X \subset S^{n+2}$ has finite type,

$$\mathscr{H}^{-i}\left(S^{n+2}; R\gamma_! \gamma^* \mathbf{IC}^{\cdot}_{\bar{p}}\right) = \mathscr{H}^{-i}\left(S^{n+2}; R\gamma_! \mathfrak{L}[n+2]\right) \cong H_i(S^{n+2} - X; \Lambda)$$

is also finite-dimensional. Hence, by the long exact sequence for hypercohomology associated to a distinguished triangle,

$$IH_i^{\bar{p}}(S^{n+2}; \Lambda) = \mathscr{H}^{-i}\left(S^{n+2}; \mathbf{IC}^{\cdot}_{\bar{p}}\right)$$

is also finite-dimensional over $\mathbf{Q}$.

Proof of (2.3.1). Because N meets X transversely, there are stratifications $\{Z_i\}$ of (Y, X) and $\{Y_i\}$ of $(Y, X \cup N)$ with

$$Y_i = Z_i \cup (Z_{i+2} \cap N).$$

Further, if

$$y \in (Y_i - Y_{i-1}) \cap N = (Z_{i+2} - Z_{i+1}) \cap N,$$

let (G_1, F_1) be the link pair in (Y, X) of the component of y in $Z_{i+2} - Z_{i+1}$ (so that $G_1 \cong S^{n-i-1}$). Then the link pair (G, F) in $(Y, X \cup N)$ of the component of y in $Y_i - Y_{i-1}$ is given as

$$S^{n-i+1} \cong G = S^1 * G_1 \supset \left(S^1 * F_1\right) \cup G_1 = F,$$

where $A * B$ denotes the join of A and B. It follows that

$$G - F = S^1 \times (c^\circ G_1 - c^\circ F_1) = S^1 \times (G_1 - F_1) \times \mathbf{R},$$

where c° = open cone. Moreover, $\mathfrak{L}_N | (G - F) = \mathfrak{L}_\phi$ is given by linking numbers with F_1 on $(G_1 - F_1)$ (i.e., $\mathfrak{L}_\phi$ for the pair (G_1, F_1)) and on S^1 by sending a generator of $\pi_1(S^1)$ to $t^d \in \Lambda$, d the multiplicity of the dual submanifold N.

Hence

$$H_j(G - F; \Lambda) \cong H_j(C; \mathbf{Q}),$$

C the d-fold cyclic covering of $G_1 - F_1$ corresponding to linking numbers with F_1 reduced modulo d. But $G_1 - F_1$ has the homotopy type of the finite complex $G_1 - \operatorname{Int} N(F_1)$; hence these groups are finite-dimensional.

Points in $X - X \cap N$ have the same link pairs in (Y, X) and in $(Y, X \cup N)$; hence these are also of finite type. Finally, the link pair at any point in $N - X \cap N$ will be (S^1, ϕ), also of finite type.

(2.5) PROPOSITION. *Let $X \subset Y$ have finite local type, and let P be, as in* (2.1), *a submanifold of $Y \times \mathbf{R}$ relating two dual submanifolds to $\chi(Y)$, and let $\mathfrak{L}_P$ be as in* (1.2). *Then*

$$IH_i^{\bar{p}}(Y \times \mathbf{R}; \mathfrak{L}_P)$$

is finite-dimensional over $\mathbf{Q}$, *for all* i.

The proof of (2.5) is left to the reader.

3. Superduality

Let R be a Dedekind ring. Let Y be a stratified pseudomanifold of dimension m, with singular set $\Sigma = Y_{m-2}$. Let $\mathfrak{L}$ and $\mathfrak{M}$ be systems over $Y - \Sigma$ of local coefficients in finitely generated modules over R. A pairing

$$\mathfrak{L} \otimes_R \mathfrak{M} \to R_{Y-\Sigma}$$

is called perfect if the induced map

$$\mathfrak{L} \to \operatorname{Hom}_R(\mathfrak{M}, R_{Y-\Sigma})$$

is an isomorphism.

(3.1) *Example.* Let Y be a PL manifold, let $X^n \subset Y$, $m = n + 2$, be a PL sub-pseudomanifold, and let N be a dual submanifold to $\chi(Y)$. For any complex of sheaves $\mathbf{A}^\cdot$ over Λ, let $(\mathbf{A}^\cdot)^{\mathrm{op}}$ be obtained by composing all module structures with the involution $f(t) \mapsto f(t^{-1})$. Then a perfect (Hermitian) pairing

$$\mathfrak{L}_N^{\mathrm{op}} \otimes_\Lambda \mathfrak{L}_N \to \Lambda_{Y - X \cup N}$$

is determined by the pairing

$$(f(t), g(t)) \mapsto f(t^{-1})g(t)$$

on the stalk over a basepoint.

Two perversities $\bar{p}$ and $\bar{q}$ will be called superdual if

$$p(k) + q(k) = k - 1, \qquad k \geqq 2.$$

See (1.3) above concerning perversities with $p(2) = 1$. $\mathbf{D}^{\cdot}_Y \cong \mathfrak{D}(R_Y)$ will denote the Verdier dualizing complex over R, so that, in the derived category,

$$\mathfrak{D}(\mathbf{A}^{\cdot}) \cong R\,\mathrm{Hom}(\mathbf{A}^{\cdot}, \mathbf{D}^{\cdot}_Y).$$

(3.2) THEOREM. *Let Y^m be a stratified pseudomanifold, let $\mathfrak{L}$ and $\mathfrak{M}$ be local systems over $Y - \Sigma$ with coefficients in finitely generated R-modules, and let $\bar{p}$ and $\bar{q}$ be superdual perversities. Suppose that if $y \in \Sigma$, the stalks $\mathbf{H}^i(\mathbf{IC}^{\cdot}_{\bar{p}}(Y; \mathfrak{M})_y)$ are torsion modules over R. Then a perfect pairing*

$$\mathfrak{L} \otimes_R \mathfrak{M} \to R_{Y-\Sigma}$$

and an orientation of Y over R induce a canonical isomorphism

$$\mathbf{IC}^{\cdot}_{\bar{q}}(Y; \mathfrak{L}) \cong R\,\mathrm{Hom}\big(\mathbf{IC}^{\cdot}_{\bar{p}}(Y; \mathfrak{M}), \mathbf{D}^{\cdot}_Y\big)[m]$$

in the derived category $D^b(Y)$.

Proof. Assume that $p(2) = 1$ and $q(2) = 0$. Let $\mathbf{IC}^{\cdot} = \mathbf{IC}^{\cdot}_{\bar{p}}(Y; \mathfrak{M})$ and let $\mathbf{S}^{\cdot} = \mathfrak{D}(\mathbf{IC}^{\cdot})[m]$. Then it will be shown that $\mathbf{S}^{\cdot}$ satisfies the axioms [AX 1] of [GM2, 3.3], for the perversity $\bar{q}$. The pairing and the orientation give the isomorphism $\mathbf{S}^{\cdot}|(Y - \Sigma) \cong \mathfrak{L}[m]$ of [AX 1](a). Axiom (b), $\mathbf{H}^i(\mathbf{S}^{\cdot}) = 0$ for $i < -m$, is left to the reader.

Let $y \in Y_{n-k} - Y_{n-k-1}$, $k \geqq 2$, and let $j_y \colon \{y\} \to Y$ be the inclusion. Then

$$\begin{aligned} H^i\big(j_y^! \mathbf{S}^{\cdot}\big) &\cong H^{i+m}\big(j_y^! \mathfrak{D}(\mathbf{IC}^{\cdot})\big) \\ &\cong H^{i+m}\big(\mathfrak{D}\big(j_y^* \mathbf{IC}^{\cdot}\big)\big) \cong \mathrm{Ext}\big(H^{-i-m+1}\big(j_y^* \mathbf{IC}^*\big), R\big). \end{aligned}$$

Hence $H^i(j_y^! \mathbf{S}^{\cdot}) = 0$ for $-i - m + 1 > p(k) - m$, i.e. for $i \leqq -p(k)$, by [AX 1](c) for $\mathbf{IC}^{\cdot}$. Since

$$-p(k) = q(k) - k + 1,$$

this proves [AX 1](d″).

Similarly,

$$\begin{aligned} H^i\big(j_y^* \mathbf{S}^{\cdot}\big) &\cong H^{i+m}\big(j_y^* \mathfrak{D}(\mathbf{IC}^{\cdot})\big) \\ &\cong H^{i+m}\big(\mathfrak{D}\big(j_y^! \mathbf{IC}^{\cdot}\big)\big) \\ &\cong \mathrm{Hom}\big(H^{-i-m}\big(j_y^! \mathbf{IC}^{\cdot}\big), R\big) \oplus \mathrm{Ext}\big(H^{-i-m+1}\big(j_y^! \mathbf{IC}^{\cdot}\big), R\big). \end{aligned}$$

Hence, by [AX 1](d″) for $\mathbf{IC}^{\cdot}$ (see (1.3)), $H^i(j_y^* \mathbf{S}^{\cdot}) = 0$ for $-i - m + 1 \leqq p(k) - k + 1$, i.e., for $i \geqq -p(k) + k - m$. Since $k - p(k) = q(k) + 1$ for $k \geqq 2$, this proves [AX 1](c).

Now we return to the Dedekind ring

$$\Lambda = \mathbf{Q}\big[t, t^{-1}\big].$$

(3.3) THEOREM. *Let X be a sub-pseudomanifold of the manifold Y, of finite local type, with dual submanifold N. Let* $\bar{p}$ *and* $\bar{q}$ *be superdual perversities. Then*

$$\mathbf{IC}^{\cdot}_{\bar{q}}(Y;\mathfrak{L}_N)^{\mathrm{op}} \cong \mathfrak{D}(\mathbf{IC}^{\cdot}_{\bar{p}}(Y;\mathfrak{L}_N))[m],$$

$m = \dim Y$. *Hence*

$$IH^{\bar{q}}_i(Y;\mathfrak{L}_N)^{\mathrm{op}} \cong \mathrm{Hom}(IH^{\bar{p}}_{m-i}(Y;\mathfrak{L}_N),\Lambda) \oplus \mathrm{Ext}(IH^{\bar{p}}_{m-i-1}(Y;\mathfrak{L}_N),\Lambda).$$

Proof. By (2.4) and [GM2, 2.4], the stalks $\mathbf{H}^i(\mathbf{IC}^{\cdot}_{\bar{p}}(Y;\hat{\mathfrak{L}}_N)_y)$ are torsion modules for $y \in X = \Sigma$.

(3.4) COROLLARY. *Let* $X \subset S^{n+2}$ *be a knot of finite type and finite local type. Let* $\bar{p}$ *and* $\bar{q}$ *be superdual perversities. Then there is a canonical isomorphism*

$$IH^{\bar{q}}_i(S^{n+2};\Lambda)^{\mathrm{op}} \cong \mathrm{Hom}(IH^{\bar{p}}_{n-i+1}(S^{n+2};\Lambda),\mathbf{F}/\Lambda),$$

$\mathbf{F} = \mathbf{Q}(t)$ *the field of rational functions.*

The isomorphism of (3.4) can be interpreted as a perfect Hermitian linking pairing

$$IH^{\bar{q}}_i(S^{n+2},\Lambda) \times IH^{\bar{p}}_{n-i+1}(S^{n+2},\Lambda) \to \mathbf{F}/\Lambda.$$

For locally flat spherical knots, with $n = 2k-1$ and $i = k$, this is just the usual Blanchfield pairing of knot theory. If $K \subset S^{n+2}$ is any locally flat submanifold, then

$$IH^{\bar{p}}_i(S^{n+2};\Lambda) = \begin{cases} H_i(C;\Lambda), & p(2) = 0 \\ H_i(C,\partial C;\Lambda), & p(2) = 1 \end{cases},$$

where C is the complement of the interior of a tubular neighborhood of K.

4. The peripheral complex

Let $\bar{m}$ and $\bar{l}$ be the middle and logarithmic perversities, respectively. Thus,

$$m(s) = [(s-1)/2]$$

and

$$l(s) = [(s+1)/2];$$

i.e., $\bar{m} = (0,0,1,1,2,\dots)$ and $\bar{l} = (1,2,2,3,3,4,\dots)$. Let Y^m be a stratified pseudomanifold. Let $\mathfrak{L}$ be a local system over $Y - \Sigma$ with coefficients in a finitely generated R-module, R a Dedekind ring with quotient field F. Then the peripheral complex $\mathbf{R}^{\cdot}_{Y}(\mathfrak{L})$ is defined by the distinguished triangle:

$$\begin{array}{ccc} \mathbf{IC}^{\cdot}_{\overline{m}}(Y;\mathfrak{L}) & \longrightarrow & \mathbf{IC}^{\cdot}_{\overline{l}}(Y;\mathfrak{L}) \\ {\scriptstyle [1]}\nwarrow & & \swarrow \\ & \mathbf{R}^{\cdot}_{Y}(\mathfrak{L}) & \end{array}.$$

Let $\mathfrak{L}^*$ be defined by setting

$$\mathfrak{L}^*_y = \operatorname{Hom}(\mathfrak{L}_y; R), \qquad y \in Y - \Sigma_Y,$$

and by letting $g \in \pi_1(Y - \Sigma_Y, y)$ act on $\varphi \in \mathfrak{L}^*_y$ by

$$(g \cdot \varphi)(\alpha) = \varphi(g^{-1} \cdot \alpha).$$

We will also assume that $\mathfrak{L}$ is bidual (the natural map $\mathfrak{L} \to \mathfrak{L}^{**}$ is an isomorphism), so that the pairing

$$\mathfrak{L} \otimes_R \mathfrak{L}^* \to R_{Y-\Sigma}$$

given by evaluation is perfect. Further, we will suppose that the ring R has an involution and that we are given an identification $\mathfrak{L}^* = \mathfrak{L}^{\mathrm{op}}$, as in the example $R = \Lambda$, $\mathfrak{L}_y = R$ of (3.1). Then, when we assume the stalks of $\mathbf{IC}^{\cdot}_{\overline{m}}(Y;\mathfrak{L})$ over the singular set are torsion modules, as in the hypotheses of (3.2), superduality induces a canonical isomorphism in $D^b(Y^m)$:

$$\mathbf{R}^{\cdot}_{Y}(\mathfrak{L}) \cong \mathfrak{D}(\mathbf{R}^{\cdot}_{Y}(\mathfrak{L}))[m+1]^{\mathrm{op}}. \tag{4.1}$$

We will also assume that R is an algebra over a field over which torsion R-modules are finite-dimensional.

By a self-dual complex of torsion sheaves over Y we will mean a pair consisting of an object $\mathbf{R}^{\cdot}$ in $D^b(Y)$ whose stalks are finitely generated torsion R-modules and whose support is contained in the singular set, and an isomorphism

$$\mathbf{R}^{\cdot} \cong \mathfrak{D}(\mathbf{R}^{\cdot})[m+1]^{\mathrm{op}}.$$

Thus the peripheral complex is a self-dual complex of torsion sheaves. Another type of example can be constructed as follows: Let V be a singular stratum of Y, let $\mathfrak{G}$ be a local system of finitely generated torsion modules over R and let

$$\mathfrak{B}: \mathfrak{G} \otimes_R \mathfrak{G}^{\mathrm{op}} \to (F/R)_V$$

be a perfect pairing, F the field of quotients of R. Then $\mathfrak{B}$ induces an isomorphism ($\dim V = m - 2c$),

$$\mathbf{IC}^{\cdot}_{\overline{m}}(\overline{V};\mathfrak{G}) \cong \mathfrak{D}\big(\mathbf{IC}^{\cdot}_{\overline{m}}(\overline{V};\mathfrak{G})\big)[m-2c+1]^{\mathrm{op}}.$$

(This is proved in a way similar to the proof of the basic duality result of [GM2], when we use the fact that everything is torsion as in the proof of superduality and take into account the dimension shift in the adjoint of φ, viewed as an isomorphism $\mathfrak{G} \cong \mathfrak{D}(\mathfrak{G})[1]^{\mathrm{op}}$. We leave the details to the reader.) Hence

$$j_*\mathbf{IC}^\cdot_{\bar{m}}(\bar{V}; \mathfrak{G})[c] \cong \mathfrak{D}\big(j_*\mathbf{IC}^\cdot_{\bar{m}}(\bar{V}; \mathfrak{G})[c]\big)[m+1]^{\mathrm{op}},$$

j the inclusion of the closure $\bar{V}$ of V in Y. Thus $j_*\mathbf{IC}^\cdot_{\bar{m}}(\bar{V}; \mathfrak{G})[c]$, with this isomorphism, is also a self-dual complex of torsion sheaves.

Given morphisms

$$\mathbf{X}^\cdot \xrightarrow{u} \mathbf{Y}^\cdot \xrightarrow{v} \mathbf{Z}^\cdot$$

in $D^b(Y)$ with $v \circ u$ trivial, let $\mathbf{C}^\cdot_{u,v} = \mathbf{C}^\cdot_{u'}$ be the algebraic mapping cylinder of a lift of u to $\mathbf{C}^\cdot_v[-1]$. This will be well-defined when $\mathrm{Hom}_{D^b}(\mathbf{X}^\cdot, \mathbf{Y}^\cdot)[-1] = 0$. The reader can check that $\mathbf{C}^\cdot_{u,v}$ is isomorphic to $\mathbf{C}^\cdot_{v'}[-1]$, where v' is a factorization of v through $\mathbf{C}^\cdot_u$. In particular, from the isomorphisms $\mathfrak{D}\mathbf{C}^\cdot_w \cong \mathbf{C}^\cdot_{\mathfrak{D}w}[-1]$, $w = v$ or u', it follows that $\mathfrak{D}\mathbf{C}^\cdot_{v,u} \cong \mathbf{C}^\cdot_{\mathfrak{D}u, \mathfrak{D}v}$.

Now we suppose that $\mathbf{Y}^\cdot$ is a self-dual torsion complex, that $\mathbf{X}^\cdot$ is also supported on the singular set and has torsion R-modules as stalks, and that we are given an isomorphism $\mathbf{Z}^\cdot \cong \mathfrak{D}(\mathbf{X}^\cdot)[m+1]^{\mathrm{op}}$ such that the following commutes:

$$\begin{array}{ccc} \mathbf{Y}^\cdot & \xrightarrow{\quad v \quad} & \mathbf{Z}^\cdot \\ \cong\downarrow & & \downarrow\cong \\ \mathfrak{D}(\mathbf{Y}^\cdot)[m+1]^{\mathrm{op}} & \xrightarrow{\mathfrak{D}u[m+1]^{\mathrm{op}}} & \mathfrak{D}(\mathbf{X}^\cdot)[m+1]^{\mathrm{op}}. \end{array}$$

This will usually be abbreviated by writing $v = \mathfrak{D}u[m+1]^{\mathrm{op}}$; with this notation we obtain an isomorphism

$$\mathbf{C}^\cdot_{v,u} \cong \mathfrak{D}\mathbf{C}^\cdot_{\mathfrak{D}u,\mathfrak{D}v} = \mathfrak{D}\big(\mathbf{C}^\cdot_{v,u}[-m-1]^{\mathrm{op}}\big) = \mathfrak{D}(\mathbf{C}^\cdot_{v,u})[m+1]^{\mathrm{op}}.$$

Thus $\mathbf{Y}^\cdot_1$ is also a self-dual complex of torsion sheaves. We will say in this circumstance that $\mathbf{Y}^\cdot_1$ is obtained from $\mathbf{Y}^\cdot$ by an elementary cobordism. We will say that $\mathbf{Y}^\cdot$ is cobordant to $\hat{\mathbf{Y}}^\cdot$ if there is a sequence $\mathbf{Y}^\cdot = \mathbf{Y}^\cdot_0, \mathbf{Y}^\cdot_1, \ldots, \mathbf{Y}^\cdot_t = \hat{\mathbf{Y}}^\cdot$ such that $\mathbf{Y}^\cdot_i$ is obtained from $\mathbf{Y}^\cdot_{i-1}$ by an elementary cobordism. (It can be shown that cobordism is an equivalence relation.)

(4.2) Theorem. *Let Y^m have only even-codimension strata. Assume that for each component V of a singular stratum with link L_y at a point $y \in V$, the groups $IH_i^{\bar{m}}(L_y; \mathfrak{L})$ are torsion R-modules. Let $\mathfrak{G}^V$ be the local system over V with stalk at y the image of the natural map*

$$IH^{\bar{m}}_{c(V)-1}(L_y; \mathfrak{L}) \to IH^{\bar{l}}_{c(V)-1}(L_y; \mathfrak{L}),$$

where $2c(V)$ = *codimension of* V. *Let*

$$\mathfrak{B}_V\colon \mathfrak{G}^V \otimes (\mathfrak{G}^V)^{\mathrm{op}} \to (F/R)_V$$

be the perfect pairing induced on each stalk by the superduality isomorphism

$$IH^{\bar{l}}_{c-1}(L_y;\mathfrak{L}) \cong \mathrm{Hom}\big(IH^{\bar{m}}_{c-1}(L_y;\mathfrak{L}), F/R\big)^{\mathrm{op}}.$$

Then $\mathbf{R}^{\cdot}_Y(\mathfrak{L})$ *is cobordant to the orthogonal sum*

$$\sum_{\mathscr{V}} j_* \mathbf{IC}^{\cdot}_{\bar{m}}(\bar{V};\mathfrak{G}^V)[c(V)],$$

$\mathscr{V}$ *the set of components of (open) singular strata of* Y.

The following result will be used in the proof of (4.2).

(4.3) PROPOSITION. *Let* $\mathbf{A}^{\cdot}$ *and* $\mathbf{B}^{\cdot}$ *be bounded complexes of sheaves over* R *on the stratified pseudomanifold* Z^m, *constructible with respect to the stratification* $\{Z_k\}$. *Let* $\bar{p}$ *be a perversity and assume that for* $z \in Z_{m-k} - Z_{m-k-1}$, $k \geqq 2$.

(i) $$H^i(j_z^*\mathbf{A}^{\cdot}) = 0 \quad \textit{for} \quad i > p(k) - m.$$

Then restriction to $Z - Z_{m-2} = Z - \Sigma$ *induces an injection*

$$\mathrm{Hom}_{D^b(Z)}(\mathbf{A}^{\cdot},\mathbf{B}^{\cdot}) \to \mathrm{Hom}_{D^b(Z-\Sigma)}(\mathbf{A}^{\cdot}|(Z-\Sigma),\mathbf{B}^{\cdot}|(Z-\Sigma))$$

if

(ii) $$H^i\big(j_z^!\mathbf{B}^{\cdot}\big) = 0 \quad \textit{for} \quad i \leqq p(k) - k,$$

and an isomorphism if

(iii) $$H^i\big(j_z^!\mathbf{B}^{\cdot}\big) = 0 \quad \textit{for} \quad i \leqq p(k) - k + 1.$$

Further, if $\mathbf{H}^i(\mathbf{A}^{\cdot}|(Z-\Sigma)) = 0$ *for* $i > s$ *and* $\mathbf{H}^i(\mathbf{B}^{\cdot}|(Z-\Sigma)) = 0$ *for* $i < s$, *there is an injection*

$$\mathrm{Hom}_{D^b(Z)}(\mathbf{A}^{\cdot},\mathbf{B}^{\cdot}) \subset \mathrm{Hom}(\mathbf{H}^s(\mathbf{A}^{\cdot}|(Z-\Sigma)),\mathbf{H}^s(\mathbf{B}^{\cdot}|(Z-\Sigma))),$$

if (i) *and* (ii) *hold which is an isomorphism if* (i) *and* (iii) *hold.*

The conditions (i) and (iii) are precisely the stalk and costalk conditions [AX1](c) and [AX1](d″) of [GM2, 3.3], and the proof is readily extracted from the arguments of [GM2, 3.5]. In fact, let $U_k = Z - Z_{n-k}$, and let $i_k\colon U_k \to U_{k+1}$ be the inclusion. Write $\mathbf{A}^{\cdot}_k = \mathbf{A}^{\cdot}|U_k$, $\mathbf{B}^{\cdot}_k = \mathbf{B}^{\cdot}|U_k$. By induction, assume the results hold over U_k. As in [GM2, 3.3], the weaker (stronger) costalk condition implies that the canonical adjunction map

$$\mathbf{B}^{\cdot}_{k+1} \to R(i_k)_*(\mathbf{B}^{\cdot}_k),$$

obviously a quasi-isomorphism over U_k, also induces isomorphisms of cohomology stalks over $U_{k+1} - U_k$ in dimensions at most $p(k) - m - 1$ and a surjective map (isomorphism) in dimension $p(k) - m$. Also $\mathbf{H}^i(\mathbf{A}^\cdot_{k+1}) = 0$ for $i > p(k) - m$ (since $p(k)$ is non-decreasing). From the argument used in the lifting result given in [GM2, 1.15], we then have that

$$\operatorname{Hom}_{D^b(U_{k+1})}(\mathbf{A}^\cdot_{k+1}, \mathbf{B}^\cdot_{k+1})$$

injects into (is isomorphic to)

$$\operatorname{Hom}_{D^b(U_{k+1})}(\mathbf{A}^\cdot_{k+1}, R(i_k)_*(\mathbf{B}^\cdot_k)) \cong \operatorname{Hom}_{D^b(U_{k+1})}(R(i_k)_*(\mathbf{A}^\cdot_k), R(i_k)_*(\mathbf{B}^\cdot_k))$$
$$\cong \operatorname{Hom}_{D^b(U_k)}(\mathbf{A}^\cdot_k, \mathbf{B}^\cdot_k),$$

which by induction injects (is isomorphic) to $\operatorname{Hom}_{D^b(U_2)}(\mathbf{A}^\cdot_2, \mathbf{B}^\cdot_2)$. The final statement of (4.3) follows by applying the first proposition in [GM2, 1.15].

We will prove (4.2) as a consequence of a general result on perverse torsion sheaves with duality pairings. More precisely, a perverse self-dual torsion sheaf on Y^m will be a pair consisting of an object $\mathbf{R}^\cdot$ in $D^b(Y)$, and a duality isomorphism

$$d = d_{\mathbf{R}}: \mathbf{R}^\cdot \xrightarrow{\cong} \mathfrak{D}(\mathbf{R}^\cdot)[m+1]^{\mathrm{op}};$$

the object $\mathbf{R}^\cdot$ is required to be supported on Y_{m-2}, to have torsion R-modules as stalks (i.e. $(\mathbf{R}^\cdot, d)$ is a self-dual complex of torsion sheaves), and to satisfy

$$H^i(j_y^*\mathbf{R}^\cdot) = 0$$

for $y \in Y_{m-2c} - Y_{m-2c-2}$ and $i > c - m$. It follows that $\mathbf{R}^\cdot$ is a "perverse sheaf" (with respect to the perversity $\overline{m}$) in the usual sense that it and its dual satisfy the appropriate vanishing condition on stalks. (Recall we use the indexing conventions of [GM2], leading to the definition as in [M, §9].) By duality the above vanishing condition is equivalent to the condition

$$\mathbf{H}^i(j_V^!\mathbf{R}^\cdot) = 0$$

for V a component of a stratum of codimension c, j_V its inclusion into Y, and $i < c - m$.

Let $\mathbf{R}^\cdot$ be a perverse self-dual torsion sheaf over Y. For each component V of a stratum of Y, let $\mathfrak{F}^V(\mathbf{R}^\cdot)$ be the local system

$$\mathfrak{F}^V(\mathbf{R}^\cdot) = \mathbf{H}^{c(V)-m}(j_V^!\mathbf{R}^\cdot).$$

The morphism

$$j_V^!\mathbf{R}^\cdot \to j_V^*\mathbf{R}^\cdot \xrightarrow{d} j_V^*\mathfrak{D}(\mathbf{R}^\cdot)[m+1]^{\mathrm{op}} = \mathfrak{D}(j_V^!\mathbf{R}^\cdot)[m+1]^{\mathrm{op}}$$

induces upon applying $\mathbf{H}^{c(V)-m}$ a map

$$\mathfrak{F}^V(\mathbf{R}^\cdot) \to (\mathfrak{F}^V(\mathbf{R}^\cdot))^*$$

and hence a perfect pairing

$$\mathfrak{B}_V(\mathbf{R}^\cdot)\colon \mathfrak{G}^V(\mathbf{R}^\cdot) \otimes_R (\mathfrak{G}^V(\mathbf{R}^\cdot))^{\mathrm{op}} \to (F/R)_V$$

on the image $\mathfrak{G}^V(\mathbf{R}^\cdot)$ of this map.

(4.4) THEOREM. *Let* $\mathbf{R}^\cdot$ *be a perverse self-dual torsion sheaf over* Y. *Then* $\mathbf{R}^\cdot$ *is cobordant to the orthogonal sum*

$$\sum_{\mathscr{V}} j_* \mathbf{IC}^\cdot_{\overline{m}}(\overline{V}; \mathfrak{G}^V(\mathbf{R}^\cdot))[c(V)].$$

Proof. Let V be a component of a stratum of maximal codimension such that the local system

$$\mathfrak{F}^V = \mathfrak{F}^V(\mathbf{R}^\cdot) = \mathbf{H}^{c(V)-m}(j_V^! \mathbf{R}^\cdot)$$

is not trivial.

Let $j = j_{\overline{V}}$ be the inclusion of $\overline{V}$ and let

$${}^{\overline{m}}j^!\mathbf{R}^\cdot = \tau_{\leq \overline{m}} j^!\mathbf{R}^\cdot = \mathbf{H}^{\overline{m}}(j^!\mathbf{R}^\cdot)$$

be the application to $\mathbf{R}^\cdot$ of the functor on the category of perverse sheaves obtained from $j^!$ on passage to this category [BBD, 2.1.7]. Thus, $j_*{}^{\overline{m}}j^!\mathbf{R}^\cdot$ will be a perverse sheaf over Y, or equivalently, ${}^{\overline{m}}j^!\mathbf{R}^\cdot[-c]$ will be a perverse sheaf over $\overline{V}$. (Recall that we use the indexing conventions of [GM2].) Further, from the definition in terms of the truncation functor (see in particular 1.4.10 of [BBD], "recollement"), there is a natural morphism

$${}^{\overline{m}}j^!\mathbf{R}^\cdot \to j^!\mathbf{R}^\cdot$$

which, for W a singular stratum of V, induces isomorphisms

$$\mathbf{H}^i(j_W^!\,{}^{\overline{m}}j^!\mathbf{R}^\cdot) \to \mathbf{H}^i(j_W^! j^!\mathbf{R}^\cdot)$$

and isomorphisms

$$\mathbf{H}^i(j_W^*\,{}^{\overline{m}}j^!\mathbf{R}^\cdot) \to \mathbf{H}^i(j_W^* j^!\mathbf{R}^\cdot),$$

for $i \leq k - m$, W of codimension $2k$ in Y. In particular,

$$\mathfrak{F}^V = \mathbf{H}^{c(V)-m}({}^{\overline{m}}j_V^!\mathbf{R}^\cdot).$$

From the above maximality, it follows that $j^!\mathbf{R}^\cdot[-c]$ satisfies the costalk axiom [AX1](d or d″) of [GM2] for $\overline{m}$; hence ${}^{\overline{m}}j^!\mathbf{R}^\cdot[-c]$ also satisfies this axiom. Hence, by (4.3) there is a unique morphism

$$\lambda\colon \mathbf{IC}^\cdot_{\overline{m}}(\overline{V}; \mathfrak{F}^V)[c] \to {}^{\overline{m}}j^!\mathbf{R}^\cdot$$

inducing the identity on stalks over y in dimension $c - m$.

Let $(\mathfrak{F}^V)^*$ be the local system on V defined by

$$(\mathfrak{F}^V)^*_y = \operatorname{Hom}_R\left(\mathfrak{F}^V_y; F/R\right)^{\mathrm{op}}$$

for $y \in V$. Then

$$\mathbf{H}^{c-m}(j_V^*\mathbf{R}^\cdot)_y \cong \mathbf{H}^{c-m}\left(j_V^*\mathfrak{D}(\mathbf{R}^\cdot)[m+1]^{\mathrm{op}}\right)_y \cong \mathbf{H}^{c+1}\left(\mathfrak{D}\left(j_V^!\mathbf{R}^\cdot\right)^{\mathrm{op}}\right)_y$$

$$\cong \mathbf{H}^{c+1}\left(\mathfrak{D}\left(j_y^!\mathbf{R}^\cdot\right)^{\mathrm{op}}\right) \cong H^{c+1}\left(\mathfrak{D}\left(\mathfrak{F}^V_y[-c]^{\mathrm{op}}\right)\right) = (\mathfrak{F}^V)^*_y.$$

We will write

$$\mathbf{H}^{c-m}(j_V^*\mathbf{R}^\cdot) = (\mathfrak{F}^V)^*.$$

It follows from the duality

$$^{\overline{m}}j^*\mathbf{R}^\cdot \cong \mathfrak{D}\left(^{\overline{m}}j^!\mathbf{R}^\cdot\right)[m+1]^{\mathrm{op}}$$

induced by d that $^{\overline{m}}j^*\mathbf{R}^\cdot[-c]$ satisfies the stalk axioms for the middle perversity. Hence there exists a unique morphism

$$\mu\colon {}^{\overline{m}}j^*\mathbf{R}^\cdot \to \mathbf{IC}^\cdot_{\overline{m}}\left(\overline{V}; (\mathfrak{F}^V)^*\right)[c]$$

inducing the identity on $\mathbf{H}^{c-m}$, restricted to V. By uniqueness, λ and μ correspond under the duality isomorphism induced by d and the duality of the intersection complex; we write

$$\mu = \mathfrak{D}(\lambda)[m+1]^{\mathrm{op}}.$$

Let $\mathfrak{A}^V$ be the kernel of the map $\mathfrak{F}^V \to (\mathfrak{F}^V)^*$ induced on $\mathbf{H}^{c-m}$ restricted to V by the canonical morphism

$$j^!\mathbf{R}^\cdot \to j^*\mathbf{R}^\cdot;$$

then there is an exact sequence

$$0 \to \mathfrak{A}^V \to \mathfrak{F}^V \to (\mathfrak{F}^V)^* \to (\mathfrak{A}^V)^* \to 0.$$

Let

$$\mathfrak{G}^V = \mathfrak{F}^V/\mathfrak{A}^V.$$

Then the map $\mathfrak{F}^V \to (\mathfrak{F}^V)^*$ induces an isomorphism

$$\varphi_V\colon \mathfrak{G}^V \to (\mathfrak{G}^V)^* = \ker\left((\mathfrak{F}^V)^* \to (\mathfrak{A}^V)^*\right).$$

By (4.3), the composite morphism

$$^{\overline{m}}j^!\mathbf{R}^\cdot[-c] \xrightarrow{\lambda[-c]} \mathbf{IC}^\cdot_{\overline{m}}\left(\overline{V}; (\mathfrak{F}^V)^*\right) \to \mathbf{IC}^\cdot_{\overline{m}}\left(\overline{V}; (\mathfrak{A}^V)^*\right)$$

is trivial. Since the exact sequence

$$0 \to \mathfrak{G}^V \to (\mathfrak{F}^V)^* \to (\mathfrak{A}^V)^* \to 0$$

splits (not naturally) over the ground field of R, we also have the distinguished triangle

$$\begin{array}{ccc} \mathbf{IC}^{\cdot}_{\bar{m}}(\bar{V};\mathfrak{G}^V) & \longrightarrow & \mathbf{IC}^{\cdot}_{\bar{m}}(\bar{V};(\mathfrak{F}^V)^*) \\ {}_{[1]}\nwarrow & & \swarrow \\ & \mathbf{IC}^{\cdot}_{\bar{m}}(\bar{V};(\mathfrak{A}^V)^*) & \end{array},$$

in which the map raising degrees induces zero on stalks and costalks and the other maps are the obvious ones. Hence there is a morphism

$$\lambda_1 \colon {}^{\bar{m}}j^!\mathbf{R}^{\cdot} \to \mathbf{IC}^{\cdot}_{\bar{m}}(\bar{V};\mathfrak{G}^V)[c],$$

unique by (4.3), that induces the quotient projection $\mathfrak{F}^V \to \mathfrak{G}^V$ on $\mathbf{H}^{c-m}$ restricted to V.

Similarly, there exists a unique morphism

$$\mu_1 \colon \mathbf{IC}^{\cdot}_{\bar{m}}(\bar{V};(\mathfrak{G}^V)^*)[c] \to {}^{\bar{m}}j^*\mathbf{R}^{\cdot},$$

inducing the inclusion of $(\mathfrak{G}^V)^*$ in $(\mathfrak{F}^V)^*$. By uniqueness, these correspond under duality, as for λ and μ; i.e.,

$$\mu_1 = \mathfrak{D}(\lambda_1)[m+1]^{\mathrm{op}}.$$

Hence if $\mathbf{A}^{\cdot}_1$ and $\mathbf{B}^{\cdot}_1$ are defined by distinguished triangles

$$\begin{array}{ccc} \mathbf{A}^{\cdot}_1 & \xrightarrow{u_1} & {}^{\bar{m}}j^!\mathbf{R} \\ {}_{[1]}\nwarrow & & \swarrow{}_{\lambda_1} \\ & \mathbf{IC}^{\cdot}_{\bar{m}}(\bar{V};\mathfrak{G}^V)[c] & \end{array}$$

and

$$\begin{array}{ccc} \mathbf{IC}^{\cdot}_{\bar{m}}(\bar{V};(\mathfrak{G}^V)^*)[c] & \xrightarrow{\mu_1} & {}^{\bar{m}}j^*\mathbf{R}^{\cdot} \\ {}_{[1]}\nwarrow & & \swarrow{}_{v_1} \\ & \mathbf{B}^{\cdot}_1 & \end{array},$$

then there is an induced isomorphism

$$\mathbf{B}^{\cdot}_1 \cong \mathfrak{D}(\mathbf{A}^{\cdot}_1)[m+1]^{\mathrm{op}},$$

and

$$v_1 = \mathfrak{D}(u_1)[m+1]^{\mathrm{op}}.$$

Note that $\lambda_1 \circ \lambda[-c]$ is the unique morphism of intersection complexes induced by the surjection $\mathfrak{F}^V \to \mathfrak{G}^V$, which splits over the ground field. Hence this composite, and therefore λ_1 as well, induces surjections on stalks and costalks. Hence u_1 induces injections on stalks and costalks. Similarly, or by duality, v_1 induces surjections. In particular, $\mathbf{A}_1^{\cdot}[-c]$ and $\mathbf{B}_1^{\cdot}[-c]$ are perverse sheaves over $\bar{V}$ (equivalently, $j_*\mathbf{A}_1^{\cdot}$ and $j_*\mathbf{B}_1^{\cdot}$ are perverse sheaves on Y), $\mathbf{A}_1^{\cdot}[-c]$ satisfies the costalk axiom of [AX1] (for $\bar{m}$) and $\mathbf{B}^{\cdot}[-c]$ the stalk axiom. Note also that $\mathbf{A}_1^{\cdot}|V \cong \mathfrak{A}^V[m-c]$ and $\mathbf{B}_1^{\cdot}|V = (\mathfrak{A}^V)^*[m-c]$.

By the injectivity statement in (4.3), the square in the diagram

$$\begin{array}{ccccc} \mathbf{A}_1^{\cdot} & \xrightarrow{u_1} & {}^{\bar{m}}j^!\mathbf{R}^{\cdot} & \xrightarrow{\lambda_1} & \mathbf{IC}_{\bar{m}}^{\cdot}(\bar{V}; \mathfrak{G}^V)[c] \\ & & \downarrow & & \downarrow \cong \\ \mathbf{B}_1^{\cdot} & \xleftarrow{v_1} & {}^{\bar{m}}j^*\mathbf{R}^{\cdot} & \xleftarrow{\mu_1} & \mathbf{IC}_{\bar{m}}^{\cdot}(V; (\mathfrak{G}^V)^*)[c] \end{array}$$

commutes. Hence the composite

$$\mathbf{A}_1^{\cdot} \xrightarrow{u_1} {}^{\bar{m}}j^!\mathbf{R}^{\cdot} \to {}^{\bar{m}}j^*\mathbf{R}^{\cdot} \xrightarrow{v_1} \mathbf{B}_1^{\cdot}$$

is trivial.

Let $\mathbf{A}^{\cdot} = j_*\mathbf{A}_1$, $\mathbf{B}^{\cdot} = j_*\mathbf{B}_1^{\cdot}$, and let u and v be the composites

$$\mathbf{A}^{\cdot} \xrightarrow{j_*u_1} j_*{}^{\bar{m}}j^!\mathbf{R}^{\cdot} \longrightarrow \mathbf{R}^{\cdot}$$

and

$$\mathbf{R}^{\cdot} \longrightarrow j_*{}^{\bar{m}}j^*\mathbf{R}^{\cdot} \xrightarrow{j_*v_1} j_*\mathbf{B}^{\cdot}.$$

Then from the preceding, we obtain an isomorphism

$$\mathbf{B}^{\cdot} \cong \mathfrak{D}(\mathbf{A}^{\cdot})[m+1]^{\text{op}},$$

and we have that

$$v = \mathfrak{D}(u)[m+1]^{\text{op}},$$

and that $v \circ u = 0$. Hence we can construct an elementary cobordism from $\mathbf{R}^{\cdot}$ to the self-dual complex $\mathbf{R}_0^{\cdot} = \mathbf{C}_{u,v}^{\cdot}$.

Clearly $\mathbf{R}_0^{\cdot}|(Y - \bar{V}) \cong \mathbf{R}^{\cdot}|(Y - \bar{V})$. It also follows readily that

$${}^{\bar{m}}j^!\mathbf{R}_0^{\cdot} = \mathbf{IC}_{\bar{m}}^{\cdot}(\bar{V}; \mathfrak{G}^V) \oplus \mathbf{B}_1^{\cdot}[-1],$$

that

$${}^{\bar{m}}j^*\mathbf{R}_0^{\cdot} = \mathbf{IC}_{\bar{m}}^{\cdot}(\bar{V}; (\mathfrak{G}^V)^*) \oplus \mathbf{A}_1^{\cdot}[1],$$

and, with the aid of (4.3), that the canonical morphism between these is the sum of the isomorphism induced by the isomorphism

$$\varphi_V\colon \mathfrak{G}^V \to (\mathfrak{G}^V)^*$$

and the zero map on $\mathbf{B}_1^{\cdot}[-1]$. This implies that $\mathbf{R}_0^{\cdot}$ has an orthogonal decomposition

$$\mathbf{R}_0^{\cdot} = j_* \mathbf{IC}_{\bar{m}}^{\cdot}(\bar{V}; \mathfrak{G}^V)[c\,] \oplus \hat{\mathbf{R}}^{\cdot},$$

where $\hat{\mathbf{R}}^{\cdot}$ is a self-dual perverse torsion sheaf on Y satisfying

$$\mathbf{H}^{c(W)-m}\left(j_W^! \hat{\mathbf{R}}^{\cdot}\right) = 0$$

for W a stratum of Y of codimension $2c(W) > 2c(V)$ *and* for $W = V$.

Hence by induction we see that $\mathbf{R}^{\cdot}$ is cobordant to a sum of intersection complexes, as in the statement of the result, and a self-dual perverse torsion sheaf $\mathbf{R}_1^{\cdot}$ with $\mathbf{H}^{c(V)-m}(j_V^! \mathbf{R}_1^{\cdot})$ trivial for all singular strata V of Y. Thus $\mathbf{R}_1^{\cdot}$ satisfies the costalk axiom [AX1](d) for the intersection complex with the middle perversity. By duality, it also satisfies the stalk axiom. Since $\mathbf{R}_1^{\cdot} | (Y - Y_{m-2})$ is trivial, it follows by [GM2, §3] that $\mathbf{R}_1^{\cdot} \cong 0$. □

Proof of (4.2). It follows easily from the triangle defining $\mathbf{R}_Y^{\cdot}(\mathfrak{L})$ and the stalk (or costalk) axioms for $\mathbf{IC}_{\bar{m}}^{\cdot}$ and $\mathbf{IC}_{\bar{l}}^{\cdot}$ that the self-dual torsion complex $\mathbf{R}_Y^{\cdot}(\mathfrak{L})$ is also a perverse sheaf, so that (4.4) applies.

Let V be a component of a singular stratum of Y, of codimension $2c$. Let $\bar{r}$ and $\bar{s}$ be the perversities

$$r(2k) = \begin{cases} k-1 & \text{for } k \neq c \\ k & \text{for } k = c \end{cases},$$

$$s(2k) = \begin{cases} k-1 & \text{for } k = c \\ k & \text{for } k \neq c \end{cases}.$$

Then $\bar{r}$ and $\bar{s}$ are superdual. Let $\mathbf{S}^{\cdot}$ and $\mathbf{I}^{\cdot}$ be defined by

$$\begin{array}{ccc} \mathbf{IC}_{\bar{m}}^{\cdot}(Y;\mathfrak{L}) & \longrightarrow & \mathbf{IC}_{\bar{r}}^{\cdot}(Y;\mathfrak{L}) \\ {}_{[1]}\nwarrow & & \swarrow \\ & \mathbf{S}^{\cdot} & \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{IC}_{\bar{s}}^{\cdot}(Y;\mathfrak{L}) & \longrightarrow & \mathbf{IC}_{\bar{l}}^{\cdot}(Y;\mathfrak{L}) \\ {}_{[1]}\nwarrow & & \swarrow \\ & \mathbf{T}^{\cdot} & \end{array}.$$

Then there are obvious morphisms

$$\mathbf{S}^{\cdot} \to \mathbf{R}_Y^{\cdot}(\mathfrak{L}) \to \mathbf{T}^{\cdot},$$

and hence a commutative diagram

$$\begin{array}{ccccc} j_V^! \mathbf{S}^{\cdot} & \longrightarrow & j_V^! \mathbf{R}_Y^{\cdot}(\mathfrak{L}) & \longrightarrow & j_V^! \mathbf{T}^{\cdot} \\ \downarrow & & \downarrow & & \downarrow \\ j_V^* \mathbf{S}^{\cdot} & \longrightarrow & j_V^* \mathbf{R}_Y^{\cdot}(\mathfrak{L}) & \longrightarrow & j_V^* \mathbf{T}^{\cdot} \end{array},$$

and superduality induces an isomorphism

$$\mathbf{T}^{\cdot} \cong \mathfrak{D}(\mathbf{S}^{\cdot})[m+1]. \tag{4.5}$$

From [GM2, 5.5], it follows that $\mathbf{S}^{\cdot}$ and $\mathbf{T}^{\cdot}$ are supported on the closure of the strata of codimension $2c$; hence the extreme vertical maps are isomorphisms in $D^b(Y)$. A straightforward argument using the defining triangles and the computation of stalks of intersection complexes [GM2, §2] shows that the lower right horizontal morphism induces an isomorphism

$$\mathbf{H}^{c-m}(j_V^* \mathbf{R}_Y^{\cdot}(\mathfrak{L})) \to \mathbf{H}^{c-m}(j_V^* \mathbf{T}^{\cdot}).$$

Similarly, or by superduality, the upper left horizontal morphism also induces an isomorphism on $\mathbf{H}^{c-m}$. Hence the local system $\mathfrak{G}^V(\mathbf{R}_Y^{\cdot}(\mathfrak{L}))$ of (4.4) is isomorphic to the image of

$$\mathbf{H}^{c-m}(j_V^* \mathbf{S}^{\cdot}) \to \mathbf{H}^{c-m}(j_V^* \mathbf{T}^{\cdot})$$

and $\mathfrak{B}_V(\mathbf{R}_Y^{\cdot}(\mathfrak{L}))$ is carried by this isomorphism to the non-singular pairing on this image induced at $y \in V$ by the isomorphism

$$\mathbf{H}^{c-m}(j_V^* \mathbf{T}^{\cdot})_y = \mathbf{H}^{c-m}(j_V^! \mathbf{T}^{\cdot})_y \cong \operatorname{Hom}\big(\mathbf{H}^{c-m}(j_V^* \mathbf{S}^{\cdot})_y;\, F/R\big)^{\mathrm{op}}$$

obtained from (4.5) upon passage to cohomology. From the defining triangles, the stalk conditions, [GM2, §2], and [GS, §6] (the analogue of [GM2, 2.4] for costalks), there are isomorphisms

$$\mathbf{H}^{c-m}(j_V^* \mathbf{S}^{\cdot})_y \cong IH^{\bar{m}}_{c-1}(L_y; \mathfrak{L}),$$
$$\mathbf{H}^{c-m}(j_V^* \mathbf{T}^{\cdot})_y \cong IH^{\bar{l}}_{c-1}(L_y; \mathfrak{L}),$$

that carry the above map to the natural map

$$IH^{\bar{m}}_{c-1}(L_y; \mathfrak{L}) \to IH^{\bar{l}}_{c-1}(L_y; \mathfrak{L})$$

and the above pairing to the superduality isomorphism

$$IH^{\bar{l}}_{c-1}(L_y; \mathfrak{L}) \cong \operatorname{Hom}\big(IH^{\bar{m}}_{c-1}(L_y; \mathfrak{L}),\, F/R\big)^{\mathrm{op}}. \qquad \square$$

Remark. It is possible to work with an alternative notion of cobordism, by taking advantage of the abelian category structure on perverse sheaves to use kernels and cokernels in place of distinguished triangles. Instead of the functors ${}^{\bar{m}}j^!$, ${}^{\bar{m}}j^*$, etc., one would rely on the short exact sequence

$$0 \to \mathbf{H}^{c-m}(j_V^* \ker \alpha) \to \mathbf{H}^{c-m}(j_V^* \mathbf{A}^{\cdot}) \to \mathbf{H}^{c-m}(j_V^* \mathbf{B}^{\cdot})$$
$$\to \mathbf{H}^{c-m}(j_V^* \operatorname{coker} \alpha) \to 0$$

and the corresponding (dual) one for $j_V^!$, for $\alpha\colon \mathbf{A}^{\cdot} \to \mathbf{B}^{\cdot}$ a morphism of perverse sheaves.

5. Blanchfield torsion pairings

We continue with the notation of the preceding section. Let $(\mathbf{R}^{\cdot}, d_{\mathbf{R}^{\cdot}})$ be a self-dual complex of torsion sheaves over Y, which we now assume to be compact and of even dimension $m = 2n$. By the hypercohomology spectral sequence, $\mathscr{H}^i(Y; \mathbf{R}^{\cdot})$ will be a torsion module, i.e., a finite-dimensional vector space over the ground field $\mathfrak{R}$ of R. Hence $d_{\mathbf{R}^{\cdot}}$ induces an isomorphism

$$\mathscr{H}^{-n}(Y; \mathbf{R}^{\cdot}) \cong \mathscr{H}^{-n}\big(Y; \mathfrak{D}(\mathbf{R}^{\cdot})[m+1]^{\mathrm{op}}\big) = \mathscr{H}^{n+1}(Y; \mathfrak{D}\mathbf{R}^{\cdot})^{\mathrm{op}}$$
$$= \mathrm{Ext}_R(\mathscr{H}^{-n}(Y; \mathbf{R}^{\cdot}), R)^{\mathrm{op}} = \mathrm{Hom}_R(\mathscr{H}^{-n}(Y; \mathbf{R}^{\cdot}), R)^{\mathrm{op}},$$

and hence a non-singular torsion pairing

$$\mathfrak{P}_{\mathbf{R}^{\cdot}} : \mathscr{H}^{-n}(Y; \mathbf{R}^{\cdot}) \otimes_R \mathscr{H}^{-n}(Y; \mathbf{R}^{\cdot})^{\mathrm{op}} \to F/R.$$

Let $\mathscr{W}(F/R)$ denote the Witt group of such pairings. Thus $\mathscr{W}(F/R)$ is the Grothendieck group associated to the orthogonal sum operation on isomorphism classes of perfect Hermitian torsion pairings on finitely generated torsion modules, reduced by requiring that

$$\varphi : H \otimes_R H^{\mathrm{op}} \to F/R$$

represent zero if there is a submodule $K \subset H$ with

$$\dim_{\mathfrak{R}} H = \tfrac{1}{2} \dim_{\mathfrak{R}} H$$

and with $\varphi(x, y) = 0$ for x and y in K. Let

$$[\mathfrak{P}_{\mathbf{R}^{\cdot}}] \in \mathscr{W}(F/R)$$

be the element represented by $\mathfrak{P}_{\mathbf{R}^{\cdot}}$.

A morphism

$$\alpha : \mathbf{A}^{\cdot} \to \mathfrak{D}(\mathbf{A}^{\cdot})[m]^{\mathrm{op}}$$

in $D^b(Y)$ satisfying (recall $\mathfrak{D}^2 = \mathrm{id}$)

$$\alpha = \mathfrak{D}(\alpha)[m]^{\mathrm{op}}$$

determines a self-dual complex of sheaves $(\mathbf{R}^{\cdot}_{\alpha}, d_{\alpha})$, where $\mathbf{R}^{\cdot}_{\alpha}$ is the third term in the triangle containing α. The morphism α will be said to be a resolution of the self-dual complex of torsion sheaves $(\mathbf{R}^{\cdot}, d^{\cdot}_{\mathbf{R}})$ if there is an isomorphism in the derived category $(\mathbf{R}^{\cdot}, d) \cong (\mathbf{R}^{\cdot}_{\alpha}, d_{\alpha})$. Note that this implies that

$$d = \mathfrak{D}(d)[m+1]^{\mathrm{op}},$$

and hence $\mathfrak{P}_{\mathbf{R}^{\cdot}}$ will be Hermitian symmetric (n even) or skew symmetric (n odd).

Let α be a resolution of $\mathbf{R}^{\cdot}$. Then (recall the quotient field F is flat over R)

$$\alpha \otimes F : \mathbf{A}^{\cdot} \otimes F \to \mathfrak{D}(\mathbf{A}^{\cdot} \otimes F)[m]^{\mathrm{op}}$$

is a quasi-isomorphism. Hence it induces a non-singular Hermitian pairing (symmetric or skew-symmetric)

$$\mathfrak{Q}_\alpha : \mathscr{H}^{-n}(Y; \mathbf{A}^{\cdot}) \otimes_F \mathscr{H}^{-n}(Y; \mathbf{A}^{\cdot})^{\mathrm{op}} \to F.$$

Let

$$[\mathfrak{Q}_\alpha] \in \mathscr{W}(F)$$

be the element in the Witt group of such pairings represented by $\mathfrak{Q}_\alpha$. Let

$$\delta : \mathscr{W}(F) \to \mathscr{W}(F/R)$$

be as, for example, in [AHV] (for $\mathbf{Q}$ and $\mathbf{Z}$). Then by the same type of argument as in [GS], one obtains:

(5.1) PROPOSITION. *Let α be a resolution of $\mathbf{R}^{\cdot}$. Then*

$$\delta([\mathfrak{Q}_\alpha]) = [\mathfrak{P}_{\mathbf{R}^{\cdot}}].$$

We will actually only need this proposition in the case when

$$\mathscr{H}^{-i}(Y; \mathbf{A}^{\cdot} \otimes F) = \mathscr{H}^{-i}(Y; \mathbf{A}^{\cdot}) \otimes F = 0$$

for $i = n$ or $n - 1$. In this case, the exact sequence of torsion modules (with $\mathbf{B}^{\cdot} = \mathfrak{D}(\mathbf{A}^{\cdot})[m]^{\mathrm{op}}$)

$$\mathscr{H}^{-n}(Y; \mathbf{A}^{\cdot}) \to \mathscr{H}^{-n}(Y; \mathbf{B}^{\cdot}) \to \mathscr{H}^{-n}(Y; \mathbf{R}^{\cdot}) \to \mathscr{H}^{-n+1}(Y; \mathbf{A}^{\cdot}) \to \mathscr{H}^{-n+1}(Y; \mathbf{B}^{\cdot})$$

is self-dual with respect to $\mathrm{Hom}_R(-, F/R)$, and $\mathfrak{P}_{\mathbf{R}^{\cdot}}$ vanishes on the image of the second map from the left. This is exactly the same situation as in the usual argument that the signature of a boundary is trivial.

The proof of the next result is left as an exercise.

(5.2) PROPOSITION. *Let α: $\mathbf{A}^{\cdot} \to \mathfrak{D}(\mathbf{A}^{\cdot})[m + 1]$ be a resolution of $\mathbf{R}^{\cdot}$ and suppose that $\mathbf{R}^{\cdot}$ is cobordant to $\mathbf{R}_1^{\cdot}$. Then $\mathbf{R}_1^{\cdot}$ has a resolution α_1 for which there is a commutative diagram*

$$\begin{array}{ccc} \mathbf{A}^{\cdot} \otimes F & \xrightarrow{\alpha \otimes F} & \mathfrak{D}(\mathbf{A}^{\cdot} \otimes F)[m]^{\mathrm{op}} \\ \cong \downarrow & & \downarrow \cong \\ \mathbf{A}_1^{\cdot} \otimes F & \xrightarrow{\alpha_1 \otimes F} & \mathfrak{D}(\mathbf{A}^{\cdot} \otimes F)[m]^{\mathrm{op}}. \end{array}$$

(5.3) COROLLARY. *Suppose $\mathbf{R}^{\cdot}$ has a resolution and $\mathbf{R}^{\cdot}$ is cobordant to $\mathbf{R}_1^{\cdot}$. Then*

$$[\mathfrak{P}_{\mathbf{R}^{\cdot}}] = [\mathfrak{P}_{\mathbf{R}_1^{\cdot}}].$$

In particular, if $\mathbf{R}^\cdot$ *has a resolution*

$$\alpha: \mathbf{A}^\cdot \to \mathfrak{D}(\mathbf{A}^\cdot)[m]$$

with $\mathscr{H}^{-i}(Y; \mathbf{A}^\cdot \otimes F) = 0$, $i = n, n-1$, *and* $\mathbf{R}^\cdot$ *is cobordant to* $\mathbf{R}_1^\cdot$ *then*

$$[\mathfrak{P}_{\mathbf{R}_1^\cdot}] = 0.$$

It is only the final statement of this corollary that will figure in the applications (§7). For completeness we briefly discuss the general invariance of $[\mathfrak{P}_{\mathbf{R}^\cdot}]$ under cobordism. If $\mathbf{R}^\cdot$ is cobordant to $\mathbf{R}_1^\cdot$, then $\mathbf{R}^\cdot \oplus -\mathbf{R}^\cdot$ is cobordant to $\mathbf{R}_1^\cdot \oplus -\mathbf{R}^\cdot$, where $-\mathbf{R}^\cdot$ is obtained from $\mathbf{R}^\cdot$ by using the same complex, but setting $d_{-\mathbf{R}^\cdot} = -d_{\mathbf{R}^\cdot}$. On the other hand, it can be shown, in analogy with a familiar result for forms, that $\mathbf{R}^\cdot \oplus -\mathbf{R}^\cdot$ is isomorphic to $\mathbf{R}^\cdot \oplus \mathfrak{D}(\mathbf{R}^\cdot)[m+1]$, with the obvious duality map; i.e., $\mathbf{R}^\cdot \oplus -\mathbf{R}^\cdot$ has the resolution

$$\mathbf{R}^\cdot \xrightarrow{0} \mathfrak{D}(\mathbf{R}^\cdot)[m].$$

Since $\mathbf{R}^\cdot \otimes F = 0$, it follows from (5.3) that

$$[\mathfrak{P}_{\mathbf{R}^\cdot \oplus -\mathbf{R}_1^\cdot}] = [\mathfrak{P}_{\mathbf{R}^\cdot}] - [\mathfrak{P}_{\mathbf{R}_1^\cdot}] = 0.$$

Thus we have the following general result:

(5.4) PROPOSITION. *Let the self-dual complexes of torsion sheaves* $\mathbf{R}^\cdot$ *and* $\mathbf{R}_1^\cdot$ *be cobordant. Then*

$$[\mathfrak{P}_{\mathbf{R}^\cdot}] = [\mathfrak{P}_{\mathbf{R}_1^\cdot}]$$

in $\mathscr{W}(F/R)$.

Let V be a stratum of Y, $\mathfrak{G}$ a local system of torsion R-modules on V, and

$$\mathfrak{B}: \mathfrak{G} \otimes_R \mathfrak{G}^{\mathrm{op}} \to (F/R)_V$$

a perfect pairing. Recall from Section 4 that $j_*\mathbf{IC}^\cdot_{\bar{m}}(V; \mathfrak{G})[c]$ is a self-dual complex of torsion sheaves, with duality induced by $\mathfrak{B}$ as described above. Clearly

$$\mathscr{H}^{-n}(Y; j_*\mathbf{IC}^\cdot_{\bar{m}}(V; \mathfrak{G})[c]) = IH^{\bar{m}}_{n-c}(V; \mathfrak{G})$$

and $\mathfrak{P}_{j_*\mathbf{IC}^\cdot_{\bar{m}}(V;\mathfrak{G})[c]}$ is just the usual pairing [GM1 or 2]

$$\mathfrak{B}_*: IH^{\bar{m}}_{n-c}(V; \mathfrak{G}) \otimes_R IH^{\bar{m}}_{n-c}(V: \mathfrak{G}) \to F/R$$

induced by $\mathfrak{B}$. Hence (5.4) and (4.4) imply:

(5.5) THEOREM. *Let* $\mathbf{R}^\cdot$ *be a perverse self-dual torsion sheaf over* Y. *Then, in* $\mathscr{W}(F/R)$,

$$[\mathfrak{P}_{\mathbf{R}^\cdot}] = \sum_{\mathscr{V}} [\mathfrak{B}_V(\mathbf{R}^\cdot)_*].$$

The peripheral complex $\mathbf{R}^{\cdot}_{Y}(\mathfrak{L})$ has the resolution given by the natural map

$$\alpha_{Y}(\mathfrak{L}) \colon \mathbf{IC}^{\cdot}_{\bar{m}}(Y;\mathfrak{L}) \to \mathbf{IC}^{\cdot}_{\bar{l}}(Y;\mathfrak{L}) \cong \mathfrak{D}(\mathbf{IC}^{\cdot}_{\bar{m}}(Y;\mathfrak{L}))[m]^{\mathrm{op}}.$$

(5.6) COROLLARY. *Let the hypotheses and notation be as in* (4.2). *Then*

$$\delta[\mathfrak{Q}_{\alpha_Y(\mathfrak{L})}] = \sum_{\mathscr{Y}} [\mathfrak{B}_{V_*}].$$

In particular, if

$$IH^{\bar{m}}_{i}(Y;\mathfrak{L} \otimes_R F) = IH^{\bar{m}}_{i}(Y;\mathfrak{L}) \otimes_R F = 0,$$

for $i = n,\ n-1$, *then*

$$\sum_{\mathscr{Y}} [\mathfrak{B}_{V_*}] = 0.$$

The final sentence will be used in Section 7.

6. Characteristic classes of sub-pseudomanifolds

Let Λ be the Dedekind domain $\mathbf{Q}[t, t^{-1}]$ with quotient field the rational functions $\mathbf{Q}(t)$. Let

$$T \colon \mathbf{Q}(t)/\Lambda \to \mathbf{Q}$$

be defined as follows (compare [T] and [Ra, p. 833]): By partial fractions, $\mathbf{Q}(t)$ splits over $\mathbf{Q}$ as a sum of a copy of Λ consisting of sums of polynomials and proper fractions with (in lowest terms) denominator a power of t, and the subspace A consisting of 0 and all proper fractions with denominator prime to t. Define T to be the $\mathbf{Q}$-linear map which vanishes on the copy of Λ and with

$$T(f) = f(0)$$

for $f \in A$. Note that this is defined because $f \in A$ if and only if its denominator does not vanish at 0. Since T vanishes on polynomials, we may view it as defined on $\mathbf{Q}(t)/\Lambda$.

Given a non-singular Hermitian pairing (with respect to the involution $\alpha(p(t)/q(t)) = p(t^{-1})/q(t^{-1})$,

$$B \colon G \otimes_{\Lambda} G^{\mathrm{op}} \to \mathbf{Q}(t)/\Lambda$$

on a finitely generated torsion Λ-module,

$$T \circ B \colon G \times G \to \mathbf{Q},$$

defined by

$$(T \circ B)(x, y) = T(B(x \otimes y)),$$

will be a non-singular symmetric bilinear form on a finite-dimensional vector space over $\mathbf{Q}$, and t will act as an isometry. Similarly, a skew-Hermitian torsion pairing over Λ yields a skew-symmetric form over $\mathbf{Q}$.

Given such a torsion pairing B, the signature σ_B is defined as the usual signature of the form $T \circ B$, i.e., zero in the skew case and the number of positive entries less the number of negative ones in a diagonalization over the reals in the symmetric case. More generally, let X be a stratified, oriented pseudomanifold with even-codimension strata, and let $\mathfrak{B}$ be a local system on $X - \Sigma$ of non-singular Hermitian or skew-Hermitian torsion pairings on a local system $\mathfrak{G}$ of finitely generated torsion modules over Λ. Suppose that X has dimension $2k$, k odd in the skew case and even otherwise. Then, by [GM2], $T \circ \mathfrak{B}$ defined by

$$(T \circ \mathfrak{B})_x = T \circ B_x$$

induces a symmetric, bilinear, non-singular intersection pairing

$$IH_k^{\bar{m}}(X; \mathfrak{G}) \times IH_k^{\bar{m}}(X; \mathfrak{G}) \to \mathbf{Q}.$$

Let $\sigma_{\mathfrak{B}}(X)$ be the signature of this pairing, and set $\sigma_{\mathfrak{B}}(X) = 0$ for $\dim X \not\equiv 0 \pmod 4$ and $\mathfrak{B}$ Hermitian or $\dim X \not\equiv 2 \pmod 4$ and $\mathfrak{B}$ skew-Hermitian.

Given X and $\mathfrak{B}$ as above, we define twisted L-classes

$$L_k^{\mathfrak{B}}(X) \in H_k(X; \mathbf{Q})$$

by the procedure used in [GM1] for the L-classes associated to the usual signature. Thus, stably and up to non-zero multiples, an element ξ of $H^k(X; \mathbf{Q})$ can be represented by the map

$$f\colon X \to S^k$$

that is transverse to $y \in S^k$. Then $L_k^{\mathfrak{B}}(X)$ is defined to satisfy

$$\langle \xi, L_k^{\mathfrak{B}}(X) \rangle = \sigma_{\mathfrak{B}|\{f^{-1}(y) - \Sigma \cap f^{-1}(y)\}}(f^{-1}(y))$$

and is uniquely determined by this formula. In particular, for $X - \Sigma$ connected and $\mathfrak{G}$ a trivial local system of Λ-modules,

$$L_k^{\mathfrak{B}}(X) = \sigma(\mathfrak{B}_x) L_k(X).$$

Note also that $L_{\dim X}^{\mathfrak{B}}(X) = \sigma(\mathfrak{B}_X)[X]$, $[X]$ the orientation cycle, $L_0^{\mathfrak{B}}(X) = \sigma_{\mathfrak{B}}(X)$, and $L_k^{\mathfrak{B}}(X) = 0$ for $k \not\equiv \dim X - 2 \pmod 4$ in the case $\mathfrak{B}$ is a skew-Hermitian and $k \not\equiv \dim X \pmod 4$ for $\mathfrak{B}$ Hermitian.

If M is a manifold, let $\mathscr{L}_k(M) \in H^{4k}(M; \mathbf{Q})$ be the Thom-Hirzebruch L-class. Thus, the Poincaré dual of $\mathscr{L}_k(M)$ is $L_{n-4k}(M)$. Recall that the total $\mathscr{L}$-class

$$\mathscr{L}(M) = 1 + \mathscr{L}_1(M) + \mathscr{L}_2(M) + \cdots$$

can be viewed, for M smooth, as a sum of the Hirzebruch polynomials

$$\mathscr{L} = 1 + \mathscr{L}_1 + \mathscr{L}_2 + \cdots,$$

evaluated on the rational Pontrjagin classes $p_i(M) \in H^{4i}(M; \mathbf{Q})$. Thus, $\mathscr{L}_1(p_1) = \frac{1}{3}p_1$, $\mathscr{L}_2(p_1, p_2) = \frac{1}{45}(7p_2 - p_1^2)$, etc.,

$$\mathscr{L}(P) = 1 + \mathscr{L}_1(p_1) + \mathscr{L}_2(p_1, p_2) + \cdots,$$

where

$$P = 1 + p_1 + p_2 + \cdots$$

and

$$\mathscr{L}(M) = \mathscr{L}(P(M)).$$

(See, e.g., [MSt] for more details.) Let

$$L(X) = [X] + L_{n-4}(X) + L_{n-8}(X) + \cdots$$

and let the total twisted L-class $L^{\mathfrak{B}}(X)$ be defined similarly.

(6.1) THEOREM. *Let X be an oriented sub-pseudomanifold of the* (PL) *oriented manifold M of* (*real*) *codimension two and of finite local type. Let $\mathscr{V}$ be the set of components of the* (*open*) *singular strata of X in a stratification of the pair* (M, X). *Assume that all elements of $\mathscr{V}$ have even codimension. Let N be a dual submanifold to $\chi(M, X)$ of multiplicity one. For $V \in \mathscr{V}$, let $\mathfrak{B}_V$ be the system of torsion pairings,*

$$\mathfrak{B}_V\colon \mathfrak{G}^V \otimes_\Lambda (\mathfrak{G}^V)^{\mathrm{op}} \to (\mathbf{Q}(t)/\Lambda)_V$$

defined by superduality on the local system $\mathfrak{G}^V$ over $V - V \cap N$, as in (4.2), *with stalks*

$$\mathfrak{G}_y^V = \mathrm{Image}\left\{IH_{c-1}^{\bar{m}}(L_y; \mathfrak{L}_N) \to IH_{c-1}^{\bar{l}}(L_y; \mathfrak{L}_N)\right\},$$

L_y the link in M of a top simplex of V with y in its interior, and $2c = \dim M - \dim V$. *Let i: $X \subset M$ and i_V: $\bar{V} \subset X$ be inclusions. Then, with $\chi = \chi(M, X)$,*

$$[X] \cap i^*\mathscr{L}\left(P(M) \cup (1 + \chi^2)^{-1}\right) = L(X) + \sum_{\mathscr{V}} (i_V)_* L^{\mathfrak{B}_V}(\bar{V}).$$

For X a smooth submanifold of M, this is just a consequence of the usual equality

$$i^*P(M) = P(X) \cup P(v),$$

v the normal bundle, obtained from the bundle equation

$$i^*\tau(M) = \tau(X) \oplus v.$$

Thus the above formula describes the deviation from the equality of the smooth case in terms of twisted L-classes of the singular strata.

There are several situations in which the terms in the summation can be further decomposed. The simplest is the following:

(6.2) COROLLARY. *Assume, in addition to the hypotheses of* (6.1), *that for* $V \in \mathscr{V}$, *the local system* $\mathfrak{G}^V$ *is trivial. Then*

$$[X] \cap i^* \mathscr{L}\left(P(M) \cup (1+\chi^2)^{-1}\right) = L(X) + \sum_{\mathscr{V}} \sigma(\mathfrak{B}_V)\left(i_{V_*} L(\overline{V})\right),$$

where $\sigma(\mathfrak{B}_V) = \sigma((\mathfrak{B}_V)_y)$.

Remark. It follows from the PL homogeneity of the pair (M, X) along a stratum V that $\mathfrak{G}^V$ always has an extension to the open stratum V. Hence, (6.2) applies in particular when the components of the singular strata, i.e., the elements of $\mathscr{V}$, are simply connected.

The next simplest case involves the assumption that the local system $\mathfrak{G}^V$ and the pairing $\mathfrak{B}^V$ extend to a local system $\mathfrak{G}^{\overline{V}}$ and a pairing $\mathfrak{B}^{\overline{V}}$ on $\overline{V}$. In this case the characteristic class

$$\mathrm{ch}(\overline{V}) = \mathrm{ch}(\mathrm{sign})(T \circ \mathfrak{B}^{\overline{V}}) \in H^*(\overline{V}; \mathbf{Q})$$

is defined as in [At]. (Recall that it actually comes from a canonical class in $H^*(B\pi_1(\overline{V}))$.) One readily derives:

(6.3) COROLLARY. *Assume in addition to the hypotheses of* (6.1) *that* G^V *and* $\mathfrak{B}^V$ *extend as above. Then*

$$[X] \cap i^* \mathscr{L}\left(P(M) \cup (1+\chi^2)^{-1}\right) = L(X) + \sum_{\mathscr{V}} (i_V)_* \{\mathrm{ch}(\overline{V}) \cap L(\overline{V})\}.$$

Next we consider the case in which $\overline{V}$ is a manifold, but without assuming the above extensions exist. Then given any mapping

$$f \colon P^{4k} \to \overline{V},$$

P an oriented closed smooth manifold, it may be assumed by transversality that, after a small homotopy, $f^{-1}(\Sigma_{\overline{V}})$ has codimension two in P. Then $f^*\mathfrak{G}^V$ and $f^*\mathfrak{B}^V$ will be defined in the complement of a codimension-two subpolyhedron of P, and $T \circ f^*\mathfrak{B}^V$ will induce a nonsingular pairing

$$IH_{2k}^{\overline{m}}(P; f^*\mathfrak{G}^V) \otimes_{\mathbf{Q}} IH_{2k}^{\overline{m}}(P; f^*\mathfrak{G}^V) \to \mathbf{Q}.$$

Further, it is not hard to see that the signature of this pairing depends only upon the oriented bordism class of f, and that the resulting homomorphism

$$s \colon \Omega_{4k}(\overline{V}) \to \mathbf{Z}$$

is multiplicative with respect to the signature. It is a well-known part of the characteristic variety theorem (see [MS]) that there exist canonical cohomology classes

$$\mathfrak{C}(\overline{V}) = 1 + \mathfrak{C}_1(\overline{V}) + \mathfrak{C}_2(\overline{V}) + \cdots \in H^{4*}(\overline{V}; \mathbf{Q})$$

such that

$$s[P, f] = \left(\mathfrak{C}(\overline{V}) \cap f_* L(P)\right)_0.$$

(This is immediate if we instead use Witt bordism, apply the result of Siegel [S] that, rationally, Witt bordism is the homology theory associated to real K-theory, and take the Pontrjagin character.) Hence we obtain:

(6.4) COROLLARY. *Under the hypotheses of* (6.1) *and the assumption that for* $V \in \mathscr{V}$, $\overline{V}$ *is a manifold,*

$$[X] \cap i^*\mathscr{L}\left(P(M) \cup (1 + \chi^2)^{-1}\right) = L(X) + \sum_{\mathscr{V}} (i_V)_* \{\mathfrak{C}(\overline{V}) \cap L(\overline{V})\}.$$

There is a formula with a similar appearance to (6.4) for algebraic varieties, but with a much more subtle interpretation. First of all, one can show [CS7] that $L(\overline{V})$ has a lift to a sum of classes coming (probably canonically) from the intersection homology $IH_*^{\overline{m}}$ of the closed strata of $\overline{V}$. Further, the class $\mathfrak{C}(\overline{V})$ above can always be defined at least as a homology class, for general stratified spaces, but it may not lift to a cohomology class. For varieties, it has a lift similar to that just described for the L-class. The product $\mathfrak{C}(\overline{V}) \cap L(\overline{V})$ is then interpreted to mean the sum of the images of the homology classes of the closed strata of $\overline{V}$ obtained as the intersection products of the terms in the decompositions of $\mathfrak{C}$ and L corresponding to each stratum. This will be the subject of another paper.

7. Proof of Theorem 6.1

We will first show that the desired formula holds in dimension zero, i.e., that the formula

$$\text{(7.0)} \qquad \left([X] \cap i^*\mathscr{L}\left(P(M) \cup (1 + \chi^2)^{-1}\right)\right)_0 = \sigma(X) + \sum_{\mathscr{V}} \sigma_{\mathfrak{B}_V}(\overline{V})$$

is valid. It clearly suffices to consider the case where M is the interior of a manifold regular neighborhood W of X, and N is the interior of a proper locally flat submanifold $\overline{N}$ of W with $\overline{N}$ a regular neighborhood of the transverse intersection $X \cap N$. We will also start with the assumption that X is even-dimensional (actually no restriction by crossing with S^1).

Let

$$Y = W \cup_{\partial W} c(\partial W)$$

be the stratified space obtained by attaching to W the cone on its boundary. Since $Y - \{c\}$, c the cone point, is just the union of W and an open boundary collar, $\mathfrak{L} = \mathfrak{L}_N$ has an obvious extension, also denoted $\mathfrak{L}$, to a local system over $Y - (X \cup N \cup c(\partial \overline{N}))$, with $\mathfrak{L}_x \cong \Lambda$. Then Y has a stratification with singular set

$$\Sigma_Y = X \cup N \cup c(\partial \overline{N}),$$

and with (open) strata

$$X_\circ - X_\circ \cap N,\ V_N = N \cup c(\partial \overline{N}) - N \cap X - \{c\},\ V - V \cap N,\ V \cap N,$$

for $V \in \mathscr{V}$, and $\{c\}$. All these strata have even codimension. Here $X_\circ = X - \Sigma_X$. We wish to apply (4.2) and (5.6) to this situation.

We assert that for $y \in \Sigma$, and $\bar{p}$ a perversity, the stalk $\mathbf{H}^i(\mathbf{IC}^{\cdot}_{\bar{p}}(Y; \mathfrak{L}))_y$ is a torsion module over Λ, or equivalently, that it is finite-dimensional over $\mathbf{Q}$. By (2.3.1), $X \cup N \subset M$ has finite local type. The assertion then follows, by (2.4), applied to the link of the stratum containing y, and [GM2, 2.4], for $y \in X \cup N$. Since $(Y - \{c\}, \Sigma_Y - \{c\})$ is PL homeomorphic to $(M, X \cup N)$, it remains only to check the assertion for $y = c$. As the link of c is ∂W and

$$(M - X, N - X \cap N) \cong (\partial W \times \mathbf{R}, \partial \overline{N} \times \mathbf{R}),$$

it suffices to check that the groups $IH_i^{\bar{p}}(M - X; \mathfrak{L})$ are finite dimensional over $\mathbf{Q}$.

Let $\alpha\colon M - X \to M$ be the inclusion and consider the distinguished triangle:

$$\begin{array}{ccc} Ri_* i^! \mathbf{IC}^{\cdot}_{\bar{p}} & \longrightarrow & \mathbf{IC}^{\cdot}_{\bar{p}} = \mathbf{IC}^{\cdot}_{\bar{p}}(M; \mathfrak{L}) \\ & {\scriptstyle [1]}\nwarrow \quad \swarrow & \\ & R\alpha_* \alpha^* \mathbf{IC}^{\cdot}_{\bar{p}} & \end{array} .$$

By (2.3), $\mathscr{H}^{-i}(M; \mathbf{IC}^{\cdot}_{\bar{p}})$ is finite dimensional over $\mathbf{Q}$. By superduality,

$$\mathfrak{D}\left(i^! \mathbf{IC}^{\cdot}_{\bar{p}}\right) \cong i^* \mathfrak{D}\left(\mathbf{IC}^{\cdot}_{\bar{p}}\right) \cong i^* \mathbf{IC}^{\cdot}_{\bar{q}}[-\dim M],$$

$\bar{p}$ and $\bar{q}$ superdual. Since $X \subset M$ has finite local type, the stalks of $\mathbf{IC}^{\cdot}_{\bar{q}}$, and hence of $i^* \mathbf{IC}^{\cdot}_{\bar{q}}$, will be finite-dimensional, by (2.4). By the spectral sequence for hypercohomology, $\mathscr{H}^{-i}(X; \mathfrak{D}(i^! \mathbf{IC}^{\cdot}_{\bar{p}}))$ will therefore be finite-dimensional; hence

$$\mathscr{H}^{-i}\left(M; Ri_* i^* \mathbf{IC}^{\cdot}_{\bar{p}}\right) = \mathscr{H}^{-i}\left(X; i^* \mathbf{IC}^{\cdot}_{\bar{p}}\right) = \operatorname{Ext}\left(\mathscr{H}^{i+1}\left(X; i^* \mathbf{IC}^{\cdot}_{\bar{p}}\right), \Lambda\right)$$

will also be finite-dimensional. Thus, from the long exact sequence for hypercohomology associated to the above distinguished triangle,

$$IH_i^{\bar{p}}(M - X; \mathfrak{L}) = \mathscr{H}^{-i}\big(M; R\alpha_* \alpha^* \mathbf{IC}^{\boldsymbol{\cdot}}_{\bar{p}}\big)$$

is also finitely generated over $\mathbf{Q}$. Hence the results of Sections 4 and 5 apply to Y.

It also follows readily from the finite dimensionality of the stalks and of $IH_i^{\bar{p}}(M - X; \mathfrak{L})$, by either a Mayer-Vietoris type argument or the hypercohomology spectral sequence, that $IH_i^{\bar{p}}(Y; \mathfrak{L})$ is also finite-dimensional. Therefore,

$$IH_i^{\bar{p}}\big(Y; \mathfrak{L} \otimes_\Lambda \mathbf{Q}(t)\big) = 0.$$

Hence by the last sentence of (5.6), together with the fact that the signature of torsion pairings defines a homomorphism

$$\sigma \colon \mathscr{W}(\mathbf{Q}(t)/\Gamma) \to \mathbf{Z},$$

(which is zero on a skew form), we obtain

$$(7.1) \qquad \sum_{\mathscr{V}} \Big\{\sigma_{\mathfrak{B}_V}(\bar{V}) + \sigma_{\mathfrak{B}_{V \cap N}}(V \cap N)^- \Big\}$$
$$+ \sigma_{\mathfrak{B}_{X_\circ - X_\circ \cap N}}(X) + \sigma_{\mathfrak{B}_{V_N}}\big(N \cup c(\partial \bar{N})\big) + \sigma_{\mathfrak{B}_{\{c\}}}(\{c\}) = 0.$$

Let (G, F) be the link pair of a component of $V \cap N$ in $(M, X \cup N)$, as in the proof of (2.3.1), $V \in \mathscr{V}$. Then we claim that

$$IH_i^{\bar{l}}(G; \mathfrak{L}) = 0, \qquad i \neq \dim G - 1,$$

and

$$IH_i^{\bar{l}}(G; \mathfrak{L}) = \mathbf{Q}, \qquad i = \dim G - 1.$$

In particular, $\mathfrak{G}_y^{V \cap N} \subset IH_{c-1}^{\bar{l}}(G; \mathfrak{L}) = 0$, $\dim G = 2c - 1$, and hence the corresponding signature will vanish.

To prove this claim, recall that, as in the proof of (2.3.1),

$$(7.2) \qquad G = S^1 * G_1 \supset \big(S^1 * F_1\big) \cup G_1 = F,$$

where (G_1, F_1) is the link pair of V in (Y, X) or, equivalently, the link pair of $V \cap N$ in $(N, X \cap N)$. The restriction of $\mathfrak{L}$ to $G \cap (M - X \cup N) = G - F$ is given by

$$(7.3) \qquad \alpha \mapsto t^{l(S^1 * F_1 \cup -G, \alpha)} \in \Lambda,$$

for $\alpha \in \pi_1(G - F)$. Let $\mathbf{IC}^{\boldsymbol{\cdot}} = \mathbf{IC}^{\boldsymbol{\cdot}}_{\bar{l}}(G; \mathfrak{L})$. The link of the codimension-two stratum $G_1 - G_1 \cap (S^1 * F_1)$ of G is a circle that maps to $t^{-1} \in \Lambda$ under $\mathfrak{L}$;

hence by the analogue of (2.4) of [GM2],

$$\mathbf{H}^i(\mathbf{IC}^\cdot)|\{G_1 - G_1 \cap (S^1 * F_1)\} = 0$$

for $i > l(2) - \dim G = 1 - \dim G$, whereas

$$\mathbf{H}^{\varepsilon - \dim G}(\mathbf{IC}^\cdot)_y = H_{1-\varepsilon}(S^1; \Lambda) = \begin{cases} \mathbf{Q}, & \varepsilon = 1 \\ 0, & \varepsilon = 0 \end{cases},$$

for $y \in G_1 - G_1 \cap (S^1 * F_1)$. Further, G_1 is a locally flat submanifold of G and intersects $S^1 * F_1$ transversely. Hence the link pair in (G, F) of a stratum of $G_1 \cap (S^1 * F_1)$ and the restriction of $\mathfrak{L}$ will have the same form as in (7.2) and (7.3). Thus, by induction on dimension and [GM2, (2.4)],

$$\mathbf{H}^i(\mathbf{IC}^\cdot)|(G_1 \cap S^1 * F_1) = 0,$$

for $i \neq 1 - \dim G$, whereas

$$\mathbf{H}^{1 - \dim G}(\mathbf{IC}^\cdot)_y = \mathbf{Q}$$

for y in $G_1 \cap (S^1 * F_1)$. Now, for $y \in G_1$,

$$\mathbf{H}^i(\mathbf{IC}^\cdot)_y = \begin{cases} \mathbf{Q}, & i = 1 - \dim G \\ 0, & i \neq 1 - \dim G \end{cases}.$$

Since G_1 is simply connected (in fact a sphere of dimension at least three), it follows that

$$\mathbf{IC}^\cdot|G_1 \cong \mathbf{Q}_{G_1}[\dim G - 1].$$

Thus the distinguished triangle

$$\begin{array}{ccc} R\gamma_!\gamma^*\mathbf{IC}^\cdot & \longrightarrow & \mathbf{IC}^\cdot \\ & {}_{[1]}\nwarrow \quad \swarrow & \\ & R\delta_*(\mathbf{IC}^\cdot|G_1), & \end{array}$$

γ and δ the inclusion of $G - G_1$ and G_1, respectively, yields the long exact sequence

$$IH_i^{\bar{l}, c}(G - G_1; \mathfrak{L}) \to IH_i^{\bar{l}}(G; \mathfrak{L}) \to H_{i-1}(G_1; \mathbf{Q}) \xrightarrow{d} IH_{i-1}^{\bar{l}, c}(G - G_1; \mathfrak{L}).$$

(Recall that in [GM2], [Bo], $IH_*^{\bar{p}}$ refers to intersection homology with locally finite supports. Thus, $IH_*^{\bar{p}, c}$, intersection homology with compact supports, is just what was originally called $IH_*^{\bar{p}}$ in [GM1].)

However,

$$(G - G_1, F - F \cap G_1) \cong (c^\circ G_1 \times S^1, c^\circ F_1 \times S^1),$$

and $\mathfrak{L}$ is given on

$$(c^\circ G_1 - c^\circ F_1) \times S^1 = (G_1 - F_1) \times \mathbf{R} \times S^1$$

by sending $\alpha \in \pi_1(G_1 - F_1)$ to $t^{l(F_1, \alpha)}$ and a generator of $\pi_1 S^1$ to t. (Compare the proof of (2.3.1).) Then a standard argument (on the chain level) shows that

$$IH_i^{\bar{m}, c}(c^\circ G_1 \times S^1; \mathfrak{L}) = IH_i^{\bar{m}, c}(c^\circ G_1; \mathbf{Q}) = H_i(c^\circ G_1; \mathbf{Q}),$$

as G_1 is a sphere; clearly, d is an isomorphism for $i = 0$. Thus $IH_i^{\bar{l}}(G; \mathfrak{L})$ vanishes for $i - 1 \neq \dim G_1 = \dim G - 2$, i.e., for $i = \dim G - 1$, and $IH^{\bar{l}}_{\dim G - 1}(G, \mathfrak{L}) = \mathbf{Q}$, as claimed.

Hence (7.1) reduces to

$$(7.4) \quad \sum_{\mathscr{V}} \{\sigma_{\mathfrak{B}_V}(\bar{V})\} + \sigma_{\mathfrak{B}_{X_\circ - X_\circ \cap N}}(X) + \sigma_{\mathfrak{B}_{V_N}}(N \cup c(\partial \bar{N})) + \sigma_{\mathfrak{B}_{\{c\}}}(\{c\}) = 0.$$

The link of the stratum $X_\circ - X_\circ \cap N$ in W (or Y) is a circle S^1, and $\mathfrak{L}$ carries a generator of $\pi_1 S^1$ to t. Hence $\mathfrak{G}^{X_\circ - X_\circ \cap N} = \mathbf{Q}_{X_\circ - X_\circ \cap N}$, as $IH_0^{\bar{m}}(S^1; \mathfrak{L}) = IH_0^{\bar{l}}(S^1; \mathfrak{L}) = H_0(S^1; \mathfrak{L}) = \mathbf{Q}$. Since $X_\circ X_\circ \cap N$ is the non-singular set of a stratification of X (refining the stratification of the pair (W, X)), it follows that

$$(7.5) \qquad \sigma_{\mathfrak{B}_{X_\circ - X_\circ \cap N}} = \sigma(X),$$

the Goresky-MacPherson signature of X. Similarly, $\sigma_{\mathfrak{B}_{V_N}}(N \cup c(\partial \bar{N}))$ is the signature $\sigma(-N \cup c(\partial \bar{N}))$. (Recall that $\mathfrak{L}$ was defined using linking with $X \cup -N$, and the orientations must be compatible with this.) The (Novikov) signature $\sigma(\bar{N})$ of a manifold with boundary is defined as the signature of the intersection pairing (write $\dim N = 2s$) on the image of the natural map

$$H_s(\bar{N}; \mathbf{Q}) \to H_s(\bar{N}, \partial \bar{N}; \mathbf{Q}).$$

It is well-known and easy to see (see [GM1]) that the Novikov signature is the Goresky-MacPherson signature of the stratified space obtained by adding the cone on the boundary. Thus

$$(7.6) \qquad \sigma_{\mathfrak{B}_{V_N}}(N \cup c(\partial \bar{N})) = -\sigma(\bar{N}).$$

Finally we look at the stratum $\{c\}$. The link of this stratum is the manifold ∂W, and $\mathfrak{L}$ is defined on $\partial W - \partial \bar{N}$ via linking number with the locally flat submanifold $\partial \bar{N}$. In particular, on a meridional S^1 about $\partial \bar{N}$ (i.e., the link of this stratum), $\mathfrak{L}$ is given by sending a generator of π_1 to t. Let γ and δ be the inclusions of $\partial W - \partial \bar{N}$ and $\partial \bar{N}$ in ∂W, respectively. Then, from the stalk conditions and the vanishing of $H_1(S^1; \Lambda)$, $\delta^* \mathbf{IC}^{\cdot}_{\bar{m}}(\partial W; \mathfrak{L}) = 0$. Similarly, but from the costalk conditions (or by superduality), $\delta^! \mathbf{IC}^{\cdot}_{\bar{l}}(\partial W; \mathfrak{L}) = 0$. Hence, from the usual triangles, $\mathbf{IC}^{\cdot}_{\bar{m}}(\partial W; \mathfrak{L}) \cong R\gamma_! \mathfrak{L}[m]$, $\mathbf{IC}^{\cdot}_{\bar{l}}(\partial W; \mathfrak{L}) = R\gamma^{\circ}\mathfrak{L}[m]$, $m = \dim W$, and by uniqueness, the morphism $\mathbf{IC}^{\cdot}_{\bar{m}} \to \mathbf{IC}^{\cdot}_{\bar{l}}$ is the obvious one.

Let v be the normal bundle of $\partial\overline{N}$ in ∂W, with disk bundle $D(v) \subset \partial W$. Linking numbers with $X \cup -\overline{N}$ defines a homomorphism

$$\pi_1(S(v)) \to \{\pm t^i\}$$

on the sphere bundle that is an isomorphism on each fiber. This homomorphism determines a trivialization $D(v) \cong \partial\overline{N} \times S^1$ of the SO(2)-bundle v, and a map

$$p\colon \partial W - \operatorname{Int} D(v) \to S^1$$

with $p|S(v)$ the projection on the second factor with respect to the trivialization of v. Let $W_0 = \partial W - \operatorname{Int} D(v)$. Then

$$\begin{aligned} H_i(W_0; \Lambda) &= H^{m-i}(W_0, \partial W_0; \Lambda) = H^{-i}(W_0, \partial W_0; \mathfrak{L}[m\,]) \\ &= \mathscr{H}^{-i}(\partial W; R\gamma_! \mathfrak{L}[m\,]) = IH_i^{\bar{m}}(\partial W; \mathfrak{L}), \end{aligned}$$

and

$$H_i(W_0, \partial W_0; \Lambda) = IH_i^{\bar{l}}(\partial W; \mathfrak{L}),$$

similarly. The natural map $(m = 2n)$

$$IH_n^{\bar{m}}(\partial W; \mathfrak{L}) \to IH_n^{\bar{l}}(\partial W; \mathfrak{L})$$

is the inclusion induced map

$$H_n(W_0; \Lambda) \to H_n(W_0, \partial W_0; \mathfrak{L}),$$

and $\mathfrak{B}_{\{c\}}$ is the torsion pairing B_0 on the image of this map given by the usual linking numbers in the manifold W_0. We take B_0 as defined using the orientation of W_0 as part of the boundary of W, whereas for $\mathfrak{B}_{\{c\}}$ the natural orientation comes from viewing ∂W as the boundary of the cone. Thus

$$\sigma_{\mathfrak{B}_{\{c\}}} = -\sigma_{B_0}. \tag{7.7}$$

It may be assumed that p is transverse to a given point $z \in S^1$; let $P_0 = p^{-1}(z)$. Then it is well-known (by an argument going back at least to Novikov) that

$$\sigma_{B_0} = \sigma(P_0). \tag{7.8}$$

(The argument is based on the decomposition of the infinite cyclic cover into blocks, each obtained by cutting W_0 along P_0; compare [M2], [R]. We leave the details to the reader, but mention the following variation: By codimension-one surgery [C] it may be assumed that these blocks are rational homology products. The result is then obtained from arguments implicit in [T] or [L2], and the invariance of the signature under surgery.)

Let J be a radial line from $z \in S^1$ to the origin, so that under the above trivialization of v,

$$\partial\overline{N} \times J \subset D(v) \cong \partial\overline{N} \times D^2.$$

Let

$$P = P_0 \cup (\partial\overline{N} \times J) \cup \overline{N} \subset W.$$

By Novikov additivity (see [AS, III, 7.1]),

$$\sigma(P) = \sigma(P_0) + \sigma(\overline{N}). \tag{7.9}$$

By pushing W into $M = \operatorname{Int} W$ using a boundary collar, we may view P as a locally flat oriented submanifold of M. An argument using the Mayer-Vietoris sequence shows that $[P] = [X]$ in $H_{2k}(M)$. It follows from the definition that the restriction to P of a homotopy inverse g to the inclusion $i\colon X \subset M$ pulls back the Euler class $i^*\chi \in H^2(X)$ to the Euler class

$$(g|P)^* i^*\chi = \chi(\eta) = \chi(\operatorname{Int} D(\eta), P)|P$$

of the normal bundle η of P in M. It follows from the (block) bundle equation

$$\tau M|P = \tau P \oplus \eta$$

that

$$\sigma(P) = L_0(P) = \left\{[P] \cap i_P^* \mathscr{L}\left(P(M) \cup (1 + \chi^2)^{-1}\right)\right\}_0,$$

i_P the inclusion of P in M. Since $i_{P*}[P] = i*[X]$, it follows with the help of the identity $f*(x \cap f*y) = f*x \cap y$ that

$$\sigma(P) = \left([X] \cap i^* \mathscr{L}\left(P(M) \cup (1 + \chi^2)^{-1}\right)\right)_0. \tag{7.10}$$

Clearly (7.4)–(7.10) imply the signature formula (7.0).

Theorem (6.1) now follows by a familiar argument: Let

$$\xi \in H^k(X; \mathbf{Q}) = H^k(W; \mathbf{Q}).$$

It suffices to show that

$$\left\langle \xi, [X] \cap i^* \mathscr{L}\left(P(M) \cup (1 + \chi^2)^{-1}\right)\right\rangle = \langle \xi, L(X)\rangle + \sum_{\mathscr{V}} \left\langle i_V^* \xi, L^{\mathfrak{B}_V}(\overline{V})\right\rangle.$$

After crossing with a sphere, as in [MSt] and multiplying by a large enough positive integer, we may assume that

$$\xi = f^*\iota,$$

$\iota \in H^k(S^k)$ is a generator and

$$f\colon W \to S^k$$

is a continuous function. Let $z \in S^k$. By standard techniques, it may be assumed that f is transverse to y, and that the manifold $Q = f^{-1}(y)$ meets X transversely in the sub-pseudomanifold $Z = (f|X)^{-1}(y)$. Using the equations $\xi \cap [W, \partial W] = j_*[Q, \partial Q]$ and $\xi \cap [X] = j_*[Z]$ (with j the appropriate inclusion in each formula) relating fundamental cycles, one readily checks that $\chi(Q, Z)$ is the restriction to Q of $\chi(W, X)$. It follows by standard arguments that, after a small ambient isotopy, it may also be assumed that the dual submanifold N to $\chi(W, X)$ meets Q transversely in a dual submanifold to $\chi(Q, Z)$.

By definition,

$$\langle \xi, L(X) \rangle = \sigma(Z).$$

The strata of Z are precisely $\{V \cap Z | V \in \mathscr{V}\}$, the link pair of $V \cap Z$ in (Q, Z) is the link pair of V in (W, X) (if $V \cap Z \neq \emptyset$), and

$$\mathfrak{B}_{V \cap Z} = \mathfrak{B}_V | \{V \cap Z - N \cap V \cap Z\}.$$

Moreover, $f|\overline{V}$ is transverse to y and

$$(f'|\overline{V})^{-1}(y) = \overline{V} \cap Z.$$

Hence, by definition,

$$\left\langle i_V^* \xi, L^{\mathfrak{B}_V}(\overline{V}) \right\rangle = \sigma_{\mathfrak{B}_{V \cap Z}}.$$

Finally, since the normal bundle of Q in W is trivial,

$$\begin{aligned}
&\left\langle \xi, [X] \cap i^* \mathscr{L}\left(P(M) \cup (1 + \chi^2)^{-1}\right)\right\rangle \\
&= \left\langle 1, \xi \cap [X] \cap i^* \mathscr{L}\left(P(M) \cup (1 + \chi^2)^{-1}\right)\right\rangle \\
&= \left\langle 1, j_*[Z] \cap i^* \mathscr{L}\left(P(M) \cup (1 + \chi^2)^{-1}\right)\right\rangle \\
&= \left\langle 1, [Z] \cap j^* i^* \mathscr{L}\left(P(M) \cup (1 + \chi^2)^{-1}\right)\right\rangle \\
&= \left\langle 1, [Z] \cap i_Z^* j^* \mathscr{L}\left(P(M) \cup (1 + \chi^2)^{-1}\right)\right\rangle \\
&= \left\langle 1, [Z] \cap i_Z^* \mathscr{L}\left(P(Q) \cup \left(1 + \chi(Q, Z)^2\right)^{-1}\right)\right\rangle \\
&= \left\langle [Z] \cap i_Z^* \mathscr{L}\left(P(Q) \cup \left(1 + \chi(Q, Z)^2\right)^{-1}\right)\right\rangle_0,
\end{aligned}$$

i_Z is the inclusion of Z in Q. Thus the desired formula is just (7.0), applied to the pair (Q, Z). □

8. Examples

(8.1) *Signatures of non-locally flat knots*. Let κ: $K^{2k-1} \subset S^{2k+1}$ be a knot, i.e., a PL sub-pseudomanifold of the sphere, with even-codimension strata, of finite type and of finite local type. Define the signature of κ, $\sigma(\kappa)$ as

$$\sigma(\kappa) = \sigma\big((\mathfrak{B}^{k+1})_y\big).$$

Here, $\mathfrak{B}^{k+1}$ is as in Section 5, where y is the cone point in $Y = c^\circ(S^{2k+1})$ and $\mathfrak{L}$ the local system with coefficients in $\Lambda = \mathbf{Q}[t, t^{-1}]$ given on $c^\circ S^{2k+1} - c^\circ K$ by linking numbers with $c^\circ K$. Thus, $\mathfrak{B}_y^{k+1}$ is the non-singular $(-1)^k$-Hermitian pairing on the image of the natural map

$$IH_k^{\bar{m}}(S^{2k+1}; \Lambda) \to IH_k^{\bar{l}}(S^{2k+1}; \Lambda)$$

induced by the superduality linking pairing

$$IH_k^{\bar{m}}(S^{2k+1}; \Lambda) \times IH_k^{\bar{l}}(S^{2k+1}; \Lambda) \to \mathbf{Q}(t)/\Lambda$$

as in (3.4). By definition, $\sigma(\kappa)$ is the signature of the $(-1)^k$-symmetric bilinear form $T \circ \mathfrak{B}_y^{k+1}$,

$$T: \mathbf{Q}(t)/\Lambda \to \mathbf{Q},$$

the generalized (Trotter) trace, as defined in Section 6.

If the knot κ: $K \subset S^{2k+1}$ is smooth or even PL locally flat, then K bounds a codimension-one submanifold

$$K = \partial U \subset U^{2k} \subset S^{2k+1},$$

the Seifert surface for K. In this case, it can be shown that $\sigma(\kappa)$ as defined above agrees with the (Novikov) signature $\sigma(U)$ or, equivalently, the usual (Goresky-MacPherson if K is not a sphere) signature of the space $U \cup_K (cK)$ obtained by adding to U the cone on K. Note that K may be a manifold, or even a sphere, but the knot κ may fail to be locally flat. It would be interesting to have a construction of a Seifert surface for an arbitrary knot, giving the correct signature.

The following result is a generalization of the cobordism invariance of signatures of smooth or locally flat knots:

(8.1.1) THEOREM. *Let $X \subset S^{2k+2}$ be a sub-pseudomanifold of codimension two of an even-dimensional sphere. Let $\mathscr{V}$ be the set of singular strata of X in a stratification of (S^{2k+2}, X), and assume that each $V \in \mathscr{V}$ is simply connected and has even dimension. Let κ_V be the link pair of V. Then*

$$\sum_{\mathscr{V}} \sigma(\bar{V})\sigma(\kappa_V) + \sigma(X) = 0.$$

The case $X = S^{2k}$, where, obviously, $\sigma(X) = 0$ and $\mathscr{V}$ is a collection of points, is the invariance of signatures under smooth cobordism. The present result is immediate from (6.2) and the facts that $\mathscr{L}(S^{2k+2}) = 1$ and $\chi(S^{2k+1}, X) \in H^2(S^{2k+2}) = 0$.

This result can be promoted to a statement in the smooth knot cobordism group of Kervaire and Levine, generalizing the result of [FM] on embeddings of even-dimensional spheres with isolated singular point. This will be the subject of a future paper.

As a corollary, we obtain a substantial strengthening of the cobordism invariance property of signatures of smooth knots.

(8.1.2) COROLLARY. *Let κ_i: $K_i \subset S^{2k+1}$ be smooth knots. Let $J \subset S^{2k+1} \times [0,1]$ be a* PL *embedded h-cobordism that meets the boundary in $K_0 \cup K_1$ transversely and is smoothly embedded near the boundary. Assume that the singular strata of a stratification of* Int J *in* $S^{2k+1} \times (0,1)$ *have codimension congruent to* 2 mod 4 *and are simply connected. Then*

$$\sigma(\kappa_0) = \sigma(\kappa_1).$$

Recall that smooth or PL locally flat knot cobordism is not quite periodic, and that this failure occurs precisely in the behavior of the signature; see [L1], [CS3]. Thus, for k > 1, there exist locally flat spherical (i.e., K PL homeomorphic to a sphere) knots in S^{4k+1} with signature 8, whereas for $k = 1$, the signature must be divisible by 16. The next result illustrates how this discrepancy is reflected in higher dimensions in terms of the size of the singularities of a PL cobordism.

(8.1.3) COROLLARY. *Let κ_i, $i = 0, 1$, be locally flat* PL (*or smooth*) *knots of a* (*homotopy*) S^7 *in* S^9, *with*

$$\sigma(\kappa_0) - \sigma(\kappa_1) \not\equiv 0 \pmod 8.$$

Let $J \subset S^9 \times [0,1]$ be a PL *embedded h-cobordism, locally flat near the boundary, meeting $S^9 \times \{i\}$ transversely in the knot κ_i. Let $\mathscr{V}$ be the singular stratum of* Int J *in a stratification of* $(S^9 \times (0,1), \operatorname{Int} J)$, *and assume that* $\dim V > 0$, $\dim V$ *is even, and V is simply connected, for all* $V \in \mathscr{V}$. *Then $\mathscr{V}$ has an element of dimension six.*

Proof. If there is no 6-dimensional element of $\mathscr{V}$, then the link pair of any 4-dimensional component will be a locally flat spherical knot in S^5, and hence will have signature divisible by 16. Then, by application of (8.1) to

$$c(K_0) \cup J \cup c(K_1) \subset c(S^9 \times \{0\}) \cup S^9 \times I \cup c(S^9 \times \{1\}) = S^{10},$$

we have

$$\sigma(\kappa_1) \equiv \sigma(\kappa_0) \pmod{16},$$

a contradiction to the assumptions.

Remarks. 1. This result probably remains true with only the weaker hypothesis that $\mathscr{V}$ has no zero-dimensional elements.

2. It seems unlikely that a knot κ of S^3 in S^5 need have signature divisible by 16. For example, if the singular set has a single component consisting of knotted curve k in S^3, with link pair ω, then one might conjecture that

$$\sigma(\kappa) \equiv 8a(\omega)a(k) \pmod{16},$$

where $a(\omega) \in \{0, 1\}$ denotes the Arf invariant of the knot ω, and cobordisms J with 6-dimensional singularities should actually exist. In higher dimensions one can use the results of [CS1, §13] to construct knots with a given embedded lower-dimensional sphere as singular stratum and with an arbitrary spherical knot of the correct dimension as link pair. In the present situation, an explicit low-dimensional construction is needed.

(8.2) *Hypersurfaces in* $\mathbf{P}^n$. Let $\mathbf{P}^n$ denote complex projective n-space, with homogeneous coordinates $[X_0;\dots;X_n]$. Let $c \in H^2(\mathbf{P}^n)$ be the canonical generator, and let $\mu_j \in H_{2j}(\mathbf{P}^n)$ be the Poincaré dual of c^{n-j}. We will consider a few singular hypersurfaces in $\mathbf{P}^n$, beginning with quadrics. Given any set $\mathscr{F}$ of homogeneous polynomials in n variables, let

$$V(\mathscr{F}) = \{[X_0;\dots;X_n] | f(X_0,\dots,X_n) = 0 \text{ for } f \in \mathscr{F}\}$$

be the variety they define. We will discuss only the images of the L-classes of a hypersurface X in the homology of $\mathbf{P}^n$, thereby avoiding the computation of the homology of X. These will be denoted $iL_*(X)$; in particular, the signature of X, $i_* L_0(X) = \sigma(X)$ will be calculated.

We begin with some notation. Let $L_d(n-1)$ be the image in $H_*(\mathbf{P}^n; \mathbf{Q}) \subset H_*(\mathbf{P}^\infty; \mathbf{Q})$ of the total L-class of the nonsingular hypersurface of $\mathbf{P}^n$ of degree d. Thus ([H]),

$$L_d(n-1) = d\mu_{n-1} \cap \left\{\mathscr{L}\left((1+c^2)^{n+1}(1+d^2c^2)^{-1}\right)\right\}.$$

For example, $L_1(n-1) = i_* \mathbf{P}^{n-1}$; hence,

$L_1(0) = 1, \quad L_1(1) = \mu_1, \quad L_1(2) = \mu_2 + 1,$

$L_1(3) = \mu_3 + \frac{4}{3}\mu_1, \quad L_1(4) = \mu_4 + \frac{5}{3}\mu_2 + 1,$

$L_1(5) = \mu_5 + 2\mu_3 + \frac{23}{15}\mu_1, \quad L_1(6) = \mu_6 + \frac{7}{3}\mu_4 + \frac{98}{45}\mu_2 + 1,$ etc.,

$L_2(1) = 2\mu_1, \quad L_2(2) = 2\mu_2, \quad L_2(3) = 2\mu_3 - \frac{2}{3}\mu_1, \quad L_2(4) = 2\mu_4 + \frac{4}{3}\mu_2 + 2,$

$L_2(5) = 2\mu_5 + \mu_3 + \frac{32}{15}\mu_1, \quad L_2(6) = 2\mu_6 + \frac{8}{3}\mu_4 + \frac{136}{45}\mu_2,$

$L_2(7) = 2\mu_7 + \frac{10}{3}\mu_5 + \frac{58}{15}\mu_3 + \frac{128}{567}\mu_1, \quad L_2(8) = 2\mu_8 + 4\mu_6 + \frac{74}{15}\mu_4 + \frac{76}{35}\mu_2 + 2,$

$L_3(1) = 3\mu_1, \quad L_3(2) = 3\mu_2 - 5, \quad L_3(3) = 3\mu_3 - 4\mu_1,$

$L_3(4) = 3\mu_4 - 3\mu_2 + 19,$ etc.

We will also write $\sigma_d(n)$ for the signature of this hypersurface, i.e., the constant term of $L_d(n)$. Hence, by [H], [KW],

$$\sigma_d(n-1) = \left\langle \mathscr{L}\left((1+c^2)^{n+1}(1+d^2c^2)^{-1}\right), d\mu_{n-1} \right\rangle$$
$$= \left\langle (\tanh dc)(c \coth c)^{n+1}, \mu_n \right\rangle.$$

Let

$$Q(n,r) = V\left(X_0^2 + X_1^2 + \cdots + X_r^2\right) \subset \mathbf{P}^n.$$

For $r = n$, this is a smooth hypersurface, and hence

$$i_* LQ(n,n) = L_2(n-1).$$

By [KW], the signature is given by

$$\sigma_2(n-1) = \sigma(Q(n,n)) = \begin{cases} 2 \text{ for } n \equiv 1 \pmod 4 \\ 0 \text{ for } n \not\equiv 1 \pmod 4 \end{cases}.$$

For $r < n$, $Q(n,r)$ is a sub-pseudomanifold with singular stratum the projective space $V(X_0, \ldots, X_r) \cong \mathbf{P}^{n-r-1}$. By considering an affine neighborhood of the point $[0; 0; \ldots; 0; 1]$ in the projective space $V(X_{r+1}, \ldots, X_{n-1})$, one sees that the link of the stratum is the algebraic knot in a $2r+1$ sphere given by the intersection of the affine variety

$$X_0^2 + X_1^2 + \cdots + X_r^2 = 0$$

in $\mathbf{C}^{r+1} = \mathbf{R}^{2r+2}$ with a small sphere about 0. It is well-known that for r even, the intersection form for the Milnor fiber of this knot is (-2), and hence its signature is -1 for r even and zero for r odd. Hence, by (6.2):

(8.2.1) For r even

$$i_* L(Q(n,r)) = L_2(n-1) - L_1(n-r-1),$$

and for r odd,

$$i_* L(Q(n,r)) = L_2(n-1).$$

In particular, for $n \equiv 1 \pmod 4$,

$$\sigma(Q(n,r)) = \begin{cases} 2 \text{ for } r \text{ odd} \\ 1 \text{ for } r \text{ even} \end{cases},$$

and for $n \not\equiv 1 \pmod 4$,

$$\sigma(Q(n,r)) = \begin{cases} 0 & \text{for } r(n-r) \equiv 0 \pmod 4 \\ -1 & \text{otherwise.} \end{cases}$$

For example,

$$i_* L(Q(5,2)) = 2\mu_4 + \tfrac{1}{3}\mu_2 + 1,$$
$$i_* L(Q(9,2)) = 2\mu_8 + 3\mu_6 + \tfrac{39}{15}\mu_4 - \tfrac{2}{315}\mu_2 + 1, \text{ etc.}$$

Next we consider the degree-three hypersurfaces

$$T(n,r) = V\big(X_1^3 + X_0\big(X_2^2 + X_3^2 + \cdots + X_r^2\big)\big) \subset \mathbf{P}^n,$$

$n > 2$ and $2 \leqq r \leqq n$. The open strata of the singular set of a stratification of the pair $(\mathbf{P}^n, T(n,r))$ are

$$V\big(X_0, X_1, X_2^2 + \cdots + X_r^2\big) - V(X_0, \ldots, X_r),$$
$$V(X_1, \ldots, X_r) - V(X_0, \ldots, X_r), \quad \text{and}$$
$$V(X_0, X_1, \ldots, X_r).$$

The closures are homeomorphic to $Q(n-2, r-2)$, $\mathbf{P}^{n-r}$, and $\mathbf{P}^{n-r-1}$, respectively, $n > 2$. The components of the open strata are simply connected; note that the non-singular part of $Q(n-2, r-2)$ is a vector bundle over the non-singular $Q(r-2, r-2) \subset \mathbf{P}^{r-2}$, which is simply connected, $r > 3$, see e.g., [KW], a point if $r = 2$ and two points for $r = 3$.

To describe the link pair κ_Q of the stratum $Q(n-2, r-2)$, it suffices to study an affine neighborhood of, e.g., the isolated point singularity $[0;0;i;1;0;\ldots;0]$ in the pair $(V(X_4, \ldots, X_r), V(X_4, \ldots, X_r) \cap T(n,r))$. This is just the knot obtained by intersecting a small sphere about $(0,0,i)$ in affine space $\mathbf{C}^3$ with the affine variety

$$X_1^3 + X_0\big(X_2^2 + 1\big) = 0.$$

Setting $Y = 1 + iX_2$, so that $1 - iX_2 = 2 - Y$, $X_1 = X$, $X_0 = Z$, we obtain the isolated singularity at 0 given by

$$X^3 - ZY^2 + 2ZY = 0.$$

An easy calculation with the partial derivatives shows that throughout the family

$$X^3 - aZY^2 + 2ZY, \quad 0 \leqq a \leqq 1$$

0 is an isolated singularity with multiplicity ("Milnor number") two. Hence the signature is constant throughout this family ([LR], [Sz1, 2]). The bilinear form $2ZY$ is obviously non-degenerate at 0, and hence is equivalent over the complex

numbers to $X^2 + Y^2$. Hence κ_Q is homeomorphic to the generalized trefoil, i.e., the algebraic knot associated to $X^3 + Y^2 + Z^2$ (i.e., type A_2). Hence ([B], [A]), $\sigma(\kappa_Q) = -2$.

Similarly, the link pair $\kappa_{\mathbf{P}^{n-r}}$ of the stratum $\mathbf{P}^{n-r}$ will be the algebraic knot given by the isolated singularity at 0 of the affine variety

$$X_1^3 + X_2^2 + \cdots + X_r^2 = 0$$

in $\mathbf{C}^r$. Hence

$$\sigma(\kappa_{\mathbf{P}^{n-r}}) = \begin{cases} -2 & \text{for } r \text{ odd} \\ 0 & \text{for } r \text{ even} \end{cases}.$$

Finally, for $r < n$, the link pair $\kappa_{\mathbf{P}^{n-r-1}}$ of $\mathbf{P}^{n-r-1}$ will be the (non-locally flat) algebraic knot k_r given by the non-isolated singularity at 0 of the affine variety in $\mathbf{C}^{r+1}$

$$X_1^3 + X_0(X_2^2 + \cdots + X_r^2) = 0,$$

i.e., the intersection of this variety with a small sphere.

LEMMA. *For r even,*

$$\sigma(k_r) = \sigma_3(r) + 5.$$

Thus, $\sigma_3(k_2) = 0$, $\sigma_3(k_4) = 24$, $\sigma_3(k_6) = 58$, *etc.*

Applying (6.2) now yields:

(8.2.2) For r odd

$$i_* L(T(n,r)) = L_3(n-1) - 2L_2(n-3) - 2L_1(n-r).$$

For r even,

$$i_* L(T(n,r)) = L_3(n-1) - 2L_2(n-3) + (\sigma_3(r) + 7)L_1(n-r-1).$$

In particular, for r odd and n odd,

$$\sigma(T(n,r)) = \begin{cases} \sigma_3(n-1) - 2 & \text{for } n \equiv 3 \pmod 4 \\ \sigma_3(n-1) + 2 & \text{for } n \equiv 1 \pmod 4 \end{cases},$$

and for r even and n odd,

$$\sigma(T(n,r)) = \begin{cases} \sigma_3(n-1) + \sigma_3(r) + 3 & \text{for } n \equiv 3 \pmod 4 \\ \sigma_3(n-1) + \sigma_3(r) + 7 & \text{for } n \equiv 1 \pmod 4. \end{cases}$$

For example,

$$i_* L(T(7,4)) = 3\mu_6 - 5\mu_4 + (31\tfrac{1}{15})\mu_2 + 75,$$
$$i_* L(T(7,6)) = 3\mu_6 - 5\mu_4 + (\tfrac{76}{5})\mu_2 + 109.$$

Proof of lemma (outline). Let $k_r = (S^{2r+1}, K_r)$. Recall that $\sigma(k_r)$ is the signature of $T \circ B$, where T is the generalized trace (see §6) and B is the pairing induced by superduality on the image of the natural map

$$IH_r^{\bar{m}}(S^{2r+1}; \Lambda) \to IH_r^{\bar{l}}(S^{2r+1}; \Lambda).$$

The quotient of (S^{2r+1}, K_r) by the free circle action

$$\xi \cdot (X_0, \dots, X_r) = (\xi X_1, \dots, \xi X_r)$$

is $(\mathbf{P}^r, T(r,r))$. Since the singular set of $T(r,r)$ in a stratification of the pair is the disjoint union of $\mathbf{P}^0$ and the manifold $Q(r-2, r-2)$, the singular set of K_r in a stratification of (S^{2r+1}, K_r) will be the disjoint union of a circle and the total space V^{2r-5} of an S^1-bundle over $Q(r-2, r-2)$ (actually a Stiefel manifold of 2-frames in real $r-2$ space).

Let $y \in V^{2r-5}$, and let $(S^5, J^3) = \kappa_Q$ be the link pair of the stratum; this is a smooth algebraic knot. Then, with $\mathbf{IC}_{\bar{m}}^{\cdot} = \mathbf{IC}_{\bar{m}}^{\cdot}(S^{2r+1}; \Lambda)$,

$$\mathbf{H}^i(\mathbf{IC}_{\bar{m}}^{\cdot})_y = \begin{cases} IH_{4-2r-i}^{\bar{m}}(S^5; \Lambda) = H_{4-2r-i}(S^5 - j; \Lambda) & \text{for } i \leqq 1 - 2r \\ 0 & \text{for } i > 1 - 2r \end{cases}.$$

By [M1], $H_j(S^5 - J; \Lambda) = 0$ for $j > 2$. Hence $\mathbf{H}^i\mathbf{IC}_{\bar{m}}^{\cdot}|V^{2r-5} = 0$ for all i. A similar argument shows that $\mathbf{IC}_{\bar{m}}^{\cdot}$ is also trivial on the singular circle. Hence, if $N(K_r)$ is a regular neighborhood of K_r and $C = S^{2r+1} - \operatorname{Int} N(K_r)$, it follows from the appropriate distinguished triangle that $IH_r^{\bar{m}}(S^{2r+1}; \Lambda) = H_r(C; \Lambda)$. By the same type of argument, or by superduality,

$$IH_r^{\bar{l}}(S^{2r+1}; \Lambda) = H_r(C, \partial C; \Lambda).$$

Linking number with K_r is induced on π_1 by a homotopy class of maps $C \to S^1$. Let $(U, \partial U) \subset (C, \partial C)$ be the transverse inverse image of a point; for a smooth knot this would be the Seifert surface, but in general $\partial U \neq K_r$. From the preceding paragraph and an argument of knot theory (compare (7.8)),

$$\sigma(k_r) = \sigma(U).$$

(By [M1], the map $C \to S^1$ can actually be taken to be a fibration, with U as a fiber.)

Let $N(T(r,r))$ be a regular neighborhood of $T(r,r)$ in $\mathbf{P}^r$, and let π: $S^{2r+1} \to \mathbf{P}^r$ be the projection. Without loss of generality, it may be assumed that $N(K_r) = \pi^{-1}N(T(r,r))$. Hence C will be the total space of an S^1-bundle over $D = \mathbf{P}^r - \operatorname{Int} N(T(r,r))$. The Chern class of this bundle will be the image of the canonical class under the surjective map

$$H^2(\mathbf{P}^r; \mathbf{Z}) \to H^2(D; \mathbf{Z}) = \mathbf{Z}/3\mathbf{Z}$$

induced by the inclusion. From this it can be shown that the above homotopy class contains a fibration with fiber the 3-fold cyclic cover $\hat{D}$ of D (and monodromy the covering translation). Hence,

$$\sigma(k_r) = \sigma(\hat{D}).$$

By Novikov additivity, with appropriate orientations,

$$\sigma(\hat{D}) = \sigma(\hat{\mathbf{P}}^r) - \sigma(\hat{N}),$$

where $\hat{\mathbf{P}}^r$ is the 3-fold branched cyclic cover of $\mathbf{P}^r$, branched along $T(r, r)$ and $\hat{N}$ is the part of this cover lying over $N(T(r, r))$. The map

$$H_r(\hat{N}; \mathbf{Q}) \to H_r(\hat{N}, \partial\hat{N}; \mathbf{Q})$$

can be identified with the map

$$H_r(T(r, r); \mathbf{Q}) \to H_{r-2}(T(r, r); \mathbf{Q})$$

given by intersection with $3\mu_{r-1}$. It follows from the hard Lefschetz theorem (note that $T(r, r)$ is a rational homology manifold, so that rational homology and intersection homology agree) that

$$\sigma(\hat{N}) = -1,$$

with respect to the appropriate orientation. Thus,

$$\sigma(k_r) = \sigma(\hat{\mathbf{P}}^r) + 1.$$

However, $\hat{\mathbf{P}}^r$ is homeomorphic to the hypersurface

$$V\left(X_1^3 + X_0\left(X_2^2 + \cdots + X_r^2\right) + X_{r+1}^3\right) \subset \mathbf{P}^{r+1}.$$

The singular set consists of the smooth simply connected variety

$$V\left(X_0, X_1, X_2^2 + \cdots + X_r^2, X_{r+1}\right) \cong Q(r-2, r-2)$$

of real codimension six in $\hat{\mathbf{P}}^r$, and the isolated point $[1; 0; \ldots; 0]$. The link pair of the isolated point is the algebraic knot associated to the isolated singularity at 0 of the affine variety

$$X_1^3 + X_{r+1}^3 + X_2^2 + \cdots + X_2^2 = 0$$

in affine space $\mathbf{C}^{r+1}$. By [B] or [Ph], this knot has signature -4. Hence, by (6.2), in dimension zero,

$$\sigma(\hat{\mathbf{P}}^r) = \sigma_3(r) + 4,$$

and hence

$$\sigma(k_r) = \sigma_3(r) + 5. \qquad \square$$

Clearly, similar methods apply to many other hypersurfaces, e.g.,

$$V\left(X_1^d + \cdots + X_s^d + X_0\left(X_{s+1}^{d-1} + \cdots + X_r^{d-1}\right)\right) \subset \mathbf{P}^n.$$

A more systematic study of hypersurfaces, non-locally flat knots, and branched covers will appear elsewhere.

Added in proof. Note that in 6.1 and 6.2, the left-hand side of the formulae can be rewritten, as in the case of smooth hypersurfaces [H], as $((\tanh i^*\chi)/i^*\chi) \cup i^*\mathscr{L}(P(M)) \cap [X]$. Consequently, under the hypothesis of 6.2, letting $j_V = i \cdot i_V \colon V \to M$, we get:

THEOREM 6.2.1. $(\tanh \ \chi \cup \mathscr{L}(P(M))) \cap [M] = i_* L(X) + \Sigma\sigma(B_V) j_{V*} (L(\overline{V}))$, *in* $H_*(M)$.

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(Received February 5, 1990)