On Four Dimensional Surgery and Applications

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in: Commentarii mathematici Helvetici | Commentarii Mathematici Helvetici |
Article
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Introduction

Let

$$\begin{array}{ccc} \nu^k(M) & \xrightarrow{b} & \eta^k \\ \downarrow & & \downarrow \\ (M^n, \partial M) & \xrightarrow{f} & (Y, X) \end{array} \qquad (k \gg n)$$

be a normal map of degree one [B1]; i.e. M^n is a smooth n-manifold, Y is a connected smooth n-manifold, or even just a finite Poincaré complex, b is a linear bundle map over f, and f has degree one with respect to given ("local" in the nonorientable case) orientations. Assume also that Y is connected and $f \mid \partial M : \partial M \to X$ is a homotopy equivalence; we do not exclude the case $\partial M = \emptyset$. Let $\pi = \pi_1 Y$ and $w : \pi \to \{\pm 1\}$ be the orientation homomorphism; i.e. the first Stiefel-Whitney class. Then there is an invariant $\sigma(f, b)$, in an abelian group $L_n^h(\pi, w) = L_n(\pi, w)$, of the normal cobordism class *relative the boundary* of (f, b) (see [B1], [B2, p. 7]), which vanishes when f is itself a homotopy equivalence. For $\pi = \{e\}$, this invariant was defined by Browder and Novikov (see [B1], [N]), and earlier by Kervaire and Milnor [M2], [K4] in a special case. The general case is due to Wall [W2], [W3]. (Our notation is slightly at variance with [W3].) The functors L_n are periodic of period four; i.e. $L_n = L_{n+4}$, and $\sigma(f, b) = \sigma((f, b) \times CP^2)$, where CP^2 is the complex projective plane.

Perhaps the two most important widely used properties of this invariant are the following:

(i) If $n \geqslant 5$, then $\sigma(f, b) = 0$ *if and only if* (f, b) is normally cobordant relative the boundary to (g, c), where g is a homotopy equivalence.

(ii) Let $n \geqslant 6$, and suppose $Y = Z \times I$ and $h : (M, \partial M) \to (Z, \partial Z)$ is a homotopy equivalence. Let $c : \nu^k(M) \to \eta^k$ be a linear bundle map over h. Let $\gamma \in L_n(\pi, w)$. Then $\exists$ a normal cobordism *relative* the boundary,

$$\begin{array}{ccc} \nu(W) & \xrightarrow{b} & \eta \times I \\ \downarrow & & \downarrow \\ W & \xrightarrow{f} & Z \times I \end{array}$$

of $(h, c) = ((f \mid \partial_- W, b \mid \partial_- W)$ to the homotopy equivalence $f \mid \partial_+ W : \partial_+ W \to Z \times 1$ with $\sigma(f, b) = \gamma$.

This property is a consequence of the Plumbing theorem of Kervaire-Milnor for $\pi = \{e\}$. The general case is due to Wall [W3]. The absence of such results for $n = 4$, 5

[1]) Both authors were partially supported by an NSF Grant.

is a serious obstacle to the study of 4- and 5-manifolds. In this paper we prove the following (S^2 =2-sphere, # denotes connected sum, and $t(S^2 \times S^2)$ denotes t copies of $S^2 \times S^2$).

THEOREM 2.1. *Let*

$$\begin{array}{ccc} \nu(M) & \xrightarrow{b} & \eta \\ \downarrow & & \downarrow \\ (M, \partial M) & \xrightarrow{f} & (Y, X) \end{array}$$

be a normal map as above. Let M have dimension four. Then $\sigma(f, b)=0$ *if and only if for some* $t \geqslant 0$, $(f, b) \# t(S^2 \times S^2) = (f \# \mathrm{id}_{t(S^2 \times S^2)}, b \# \mathrm{id}_{t(S^2 \times S^2)})$ *is normally cobordant relative the boundary to a homotopy equivalence.*

In order to obtain this result, we solve a special case of the embedding problem for two-spheres in four-manifolds. Say P is a connected smooth four-manifold, and $M=P \# (S^2 \times S^2)$. Let $\xi \in \pi_2(M)$ be a class on whose Hurewicz image in $H_2(M)$ the 2nd Stiefel-Whitney class of M vanishes. Suppose $\exists \tau$ satisfying this condition also, with the intersection number [K1] [W2] $\xi \cdot \tau$ in $\mathbf{Z}[\pi_1 M]$ equal one.

THEOREM 2.2. (Compare [W1]). *The class* ξ *is represented by an embedded sphere in* $M \# (S^2 \times S^2)$ *if and only if the self intersection invariant* $\mu(\xi)$ *(see* §1 *for the definition) vanishes, in which case there is an embedding such that the inclusion of the complement of the image into* $M \# (S^2 \times S^2)$ *induces an isomorphism of fundamental groups.*

As an application of 2.1, we find some new four-manifolds:

THEOREM 2.4. *Let* $\mathbf{RP}^4$ *denote real projective four-space. Then, for some* $t \geqslant 0$, $\exists$ *a smooth manifold K that has the simple homotopy type of* $(\mathbf{RP}^4) \# t(S^2 \times S^2)$ *but is not diffeomorphic or even smooth or* P.L. *h-cobordant or normally cobordant to* $\mathbf{RP}^4 \# t(S^2 \times S^2)$.

So far as we know, this is the first such example known in dimension four [2]). The manifold K is *stable* in the sense that $\forall r \geqslant 0$, $K \# r(S^2 \times S^2)$ is never diffeomorphic to $\mathbf{RP}^4 \# (t+r)(S^2 \times S^2)$. K is topologically h-cobordant to $\mathbf{RP}^4 \# t(S^2 \times S^2)$.

Property (ii) for n odd is related to (i) for n even in that, loosely speaking, a nontrivial obstruction in $L_n(\pi, w)$ is obtained by constructing a normal cobordism (= "doing surgery") in two different ways to get a homotopy equivalence from a suitable normal map in dimension $(n-1)$ with vanishing surgery invariant, and gluing the results together along the normal map of dimension $(n-1)$. Hence we can use our methods to obtain the following type of result:

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[2]) The products of suitable 3-dimensional lens spaces with a circle yield examples of normally cobordant homotopy equivalent 4-manifolds that are not diffeomorphic.

Special case of THEOREM 3.1. *Let* $\gamma \in L_5(\pi, w)$, *and let* Y, Z, h, M, *and* c *be as in* (ii), *but with* $n=5$; *i.e.* $\dim M=4$. *Then if* $Z=Q \# r(S^2 \times S^2)$ *for a sufficiently large* r, $\exists$ *a normal cobordism* (f, b) *as in* (ii), *with* $\sigma(f, b)=\gamma$.

The same method of proof also leads to new examples of non-trivial 5-dimensional h-cobordisms; see Prop. 3.3. The minimum value we need for r depends upon γ; see §3 for more details.

The splitting theorem of [C], valid in dimensions at least six and generalizing [F1], led to algebraic splitting theorems for the Wall groups, just as [F1] led to the formula [S1]

$$L_n^s(\mathbf{Z} \times \pi) \cong L_{n-1}^h(\pi) \oplus L_n^s(\pi),$$

omitting the orientation map from the notation. In general, this algebraic splitting involves the groups $L_n^B(\pi, w)$, where $B \subset \mathrm{Wh}(\pi)$ is a subgroup of the Whitehead group of π invariant under the conjugation in $\mathbf{Z}[\pi]$ determined by $g \to w(g)\, g^{-1}$ for $g \in \pi$; these groups are the target of an invariant representing, in high enough dimensions, the obstruction to performing surgery to get a homotopy equivalence with torsion in B. For $B \subset B'$, there is an exact sequence analogous to that of [S1, Prop. 4.1] relating L_n^B and $L_n^{B'}$. Theorem 3.1 below is just the generalization of the special case $B=\mathrm{Wh}(\pi)$ already mentioned above to the general case $L_5^B(\pi, w)$. Theorem 2.1 can also be generalized in this way; this generalization is not of any use to us here.

In [S1] and [S2] an algebraic splitting theorem for the groups $L_n(\pi)$ was used, in a very special case, to obtain information about 5-manifolds with fundamental group Z; for example, a splitting theorem for homotopy equivalences into closed, fibered 5-manifolds over S^1 [S2, Theorem 1.2] was proven. Let $(Y^5, X) \supset (V, U) \times (-1, 1) \supset \supset (V, U)$ be a manifold pair, $\dim Y=5$, Y connected, V connected, with codimension one submanifold pair with trivial normal bundle, or even a suitable Poincaré pair (i.e., a B-Poincaré pair for suitable B – see §4) and subpair, with $\pi_1 V \to \pi_1 Y$ an inclusion that is (algebraically) 2-sided (§4). We consider a homotopy equivalence

$$h:(W^5, \partial W) \to (Y, X),$$

$(W, \partial W)$ a smooth manifold pair, h transverse to (V, U), and

$$h \mid \partial W:(\partial W; \partial W - h^{-1}U, h^{-1}U) \to (X; X-U, U)$$

a homotopy equivalence of triples. We say that h is *S-split* if, for some $V' \subset X$ obtained by exchanging *trivial* 2-handles in the right (in particular $V'=V \# t(S^2 \times S^2)$ – see §4), h is transverse to V' and

$$h:(W; h^{-1}V', W-h^{-1}V') \to (Y; V', Y-V')$$

is a homotopy equivalence. We say h is *S-splittable* if it is homotopic relative the boundary to an S-split homotopy equivalence.

If $Y-V$ is connected, let $B \subset \mathrm{Wh}(\pi)$ be the image $\mathrm{Wh}(\pi_1(Y-V))$ in $\mathrm{Wh}(\pi_1 Y)$ under the inclusion induced map. If $Y-V=Y_1 \cup Y_2$ has two components, let B be the subgroup in $\mathrm{Wh}(\pi)$ generated by the images of $\mathrm{Wh}(\pi_1 Y_i)$, $i=1, 2$, under the natural maps. We say h is *algebraically splittable* if it has torsion $\tau(h) \in B$. Using the algebraic splitting [C] of the Wall groups and the basic results 2.1 and 3.1, we prove the following:

THEOREM 5.1. *Under the above hypothesis, h is S-splittable if and only if it is algebraically splittable.*

In the case $\pi = \pi_1 Y$ is isomorphic to $\mathbf{Z} \times_\alpha \pi_1(V)$ via an isomorphism consistent with the inclusion induced map, the algebraic 2-sidedness condition is always satisfied. For this case, a somewhat weaker form of 5.1 was proven first by W.-C. Hsiang. Our proof, even for this case, is independent and quite different from this.

The proof of 2.1 and 3.1 depends on the existence of certain diffeomorphisms of four-manifolds constructed in §1. In an appendix, we give some further results on diffeomorphisms of four-manifolds that may have independent interest.

Using the methods of this paper, one can prove weak forms for dimension five of the other splitting theorems of [C1]. For example, one can derive a theorem for splitting up to h-cobordism provided a suitable torsion type invariant vanishes.

§1. Diffeomorphisms of Four-Manifolds

This section contains a generalization of some of the results of [W1] to non-simply connected manifolds. The most comprehensive results possible are not given; we confine ourselves to what we will need to use in this paper.

Let M be a compact, smooth, connected (not necessarily closed) four-manifold. We fix a closed 4-disk $U \subset M$ and when appropriate we choose a suitable basepoint in U, often without explicit mention. Let $\Lambda = \mathbf{Z}[\pi_1 M]$ be the integral group ring of $\pi_1 M$, the fundamental group of M. Let $H_2(M; \Lambda) = H_2(\tilde{M}; \mathbf{Z})$, the 2nd integral homology group of $\tilde{M}$, $\tilde{M}$ the universal cover of M; this group acquires a (left) Λ-module structure via the induced maps of covering translations. Let $\tilde{U} \subset \tilde{M}$ be a disk lying over U; whenever we choose a basepoint $U \subset M$, we choose the corresponding basepoint in $\tilde{U}$ and use it to determine liftings of maps to $\tilde{M}$. Let

$$\varrho : H_2(\tilde{M}; \mathbf{Z}) \times H_2(\tilde{M}; \mathbf{Z}) \to \mathbf{Z}$$

be the usual intersection pairing of homology classes; ϱ is $\pi_1 M$-equivariant. For α, β in $H_2(M; \Lambda)$, we define their intersection number $\alpha \cdot \beta$ in Λ by

$$\alpha \cdot \beta = \sum_{g \in \pi_1 M} \varrho(\alpha, g\beta)\, g .$$

Let $w : \pi_1 M \to \{\pm 1\}$ be the orientation map; i.e., $w(x)=1$ iff x preserves orientation in $\tilde{M}$. Let $^-$ be the antiautomorphism of Λ defined by $(\sum_g \lambda_g g)^- = \sum_g w(g)\lambda_g g^{-1}$. Then this pairing is **Z**-bilinear, and for $\tau \in \Lambda$,

(i) $(\tau\alpha)\cdot\beta = \tau(\alpha\cdot\beta)$ and (ii) $\beta\cdot\alpha = (\alpha\cdot\beta)^-$.

Let $w_2(M) = w_2 : H_2(M; \mathbf{Z}) \to \mathbf{Z}_2$ be the second Stiefel-Whitney class, and let $H: \pi_2(M) \to H_2(M; \mathbf{Z})$ be the Hurewicz homomorphism. Let $\alpha \in \ker(w_2 \circ H)$, and let $f: S^2 \to M$ represent α. Then $f^*\tau_M$ is trivial, where τ_M is the tangent bundle of M. Hence there is a bundle equivalence of $\tau_{S^2} \oplus \varepsilon^2$ with $f^*\tau_M$, ε^2 a trivial bundle; there are two such equivalences up to isotopy because $\pi_2(O(4)) = 0$. It is easy to see that they both determine isotopic monomorphisms $\tau S^2 \to f^*\tau_M$. So, by the immersion theory of [H3], we have an immersion $\hat{f}: S^2 \to M$, representing α, with trivial normal bundle; and this immersion is unique up to regular homotopy. Given such an immersion, we may take its self-intersection invariant as in [K1], [W2], or [W3]; this determines a well-defined map

$$\mu: \ker(w_2 \circ H) \to \Lambda/I,$$

where $I = \{\lambda - \bar{\lambda} \mid \lambda \in \Lambda\}$. The following properties are satisfied:

(iii) $\alpha\cdot\alpha = \mu(\alpha) + \overline{\mu(\alpha)}$; (note that the right is really a well-defined element of Λ)

(iv) $\mu(\alpha+\beta) = \mu(\alpha) + \mu(\beta) + \alpha\cdot\beta \pmod{I}$; and

(v) $\mu(\lambda\alpha) = \bar{\lambda}\mu(\alpha)\lambda$.

See [K1], [W2], or [W3] for more details.

Now we begin constructing diffeomorphisms. Let $S^1 \times D^3 \subset U \subset M$ be a tubular (disk-bundle) neighborhood of an embedded circle so that there is an embedded disk $C \subset U$ which meets $S^1 \times \partial D^3$ transversely with

$$\partial C = C \cap (S^1 \times D^3) = S^1 \times z,$$

some $z \in S^2 = \partial D^3$. Choose a basepoint on the circle $S^1 \times O$, $O \in D^3$ the origin.

Suppose given an ambient isotopy φ_t of M, $0 \leqslant t \leqslant 1$, so that φ_t preserves basepoint and $\varphi_1 \mid S^1 \times O$ is the identity map. Then φ_t induces a map of the 2-torus $T^2 = S^1 \times I/\{(x,0) \sim (x,1), x \in S^1\}$ into M, which preserves basepoints. Choosing an orientation of T^2 to be specified momentarily, the image of the orientation class (under a basepoint preserving lift to $\tilde{M}$) determines an element $\omega \in H_2(M; \Lambda)$. Given ω, we can use a theorem of Whitney [W6] as in [W1] to find an isotopy that determines it in this way. We may suppose that $\varphi_1 \mid S^1 \times D^3$ is an orthogonal bundle map that fixes $S^1 \times z$ pointwise, $z \in S^2 = \partial D^3$. We may also assume (by transversality) that $\varphi_1(C) \cap (S^1 \times D^3) = S^1 \times z$.

Let $j_*: H_2(M; \Lambda) \to H_2(M, S^1 \times D^3; \Lambda)$ be the natural map. (For $A \subset M$, $H_2(M, A; \Lambda) = H_2(\tilde{M}, p^{-1}(A); \mathbf{Z})$, p the covering projection.) Let $[C]$ and $[\varphi_1 C]$ denote classes in the target of j_* represented by these disks. A homotopy

$$f_t: (S^1 \times [0,1] \cup C, S^1 \times 0) \to (M, S^1 \times D^3), \quad 0 \leqslant t \leqslant 1,$$

(identifying $S^1 \times 1 = S^1 \times z = \partial C$) is defined as follows:

$$f_t(x, s) = \varphi_{st}(x, z) \quad x \in S^1, \quad 0 \leqslant s \leqslant 1; \quad \text{and}$$
$$f_t \mid C = \varphi_t \mid C.$$

Then f_0 represents $[C]$ and f_1 represents $[\varphi_1 C] \pm j_* \omega$.

So, choosing the appropriate orientation of T^2, we have

LEMMA 1.1. $[\varphi_1 C] = [C] + j_* \omega$.

Let M_0 and M_1 be obtained from M by *surgery* [M2] using $S^1 \times D^3$ and $\varphi_1 \mid S^1 \times D^3$, respectively; e.g., M_1 is obtained from the disjoint union of $\mathrm{cl}(M - \varphi_1(S^1 \times D^3))$ and $D^2 \times S^2$ by identifying u with $\varphi_1(u)$ for $u \in S^1 \times S^2$ and smoothing corners. Let $h: M_0 \to M_1$ be the diffeomorphism

$$(\varphi_1 \mid \mathrm{cl}(M - S^1 \times D^3)) \cup (\mathrm{id}_{D^2 \times S^2}).$$

Let $x_0 \in H_2(M_0; \Lambda)$ be the class represented by $C \cup (D^2 \times z)$; let y_0 be represented by $O \times S^2$. Let x_1 and y_1 be similarly defined. (To do all this, move the basepoint to (O, z).) By standard arguments, we have the orthogonal decompositions

$$H_2(M_i; \Lambda) = H_2(M; \Lambda) \oplus \{x_i, y_i\}_\Lambda, \qquad i = 0, 1;$$

the 2nd summand denotes the submodule generated by x_i and y_i. Clearly $\pi_1 M_i = \pi_1 M$, $i = 1, 2$, and, with respect to the natural identification of these fundamental groups, h induces the identity on π_1.

From Lemma 1.1 it follows easily, using the usual exact sequences and the excision isomorphisms $H_2(M_i, D^2 \times S^2; \Lambda) \cong H_2(M, S^1 \times D^3; \Lambda)$, that

$$h_*(x_0) = x_1 + \omega + \gamma y_1, \qquad \gamma \in \Lambda.$$

Clearly $h_*(y_0) = y_1$. Now φ_1 is homotopic to the identity; it follows from this and another simple argument with exact sequences, that for $\xi \in H_2(M; \Lambda)$, $h_*(\xi) = \xi + \lambda y_1$.

But $0 = x_0 \cdot \xi = (x_1 + \omega + \gamma y_1) \cdot (\xi + \lambda y_1) = \bar{\lambda} + \omega \cdot \xi$, as $x_1 \cdot y_1 = 1$ and $y_1 \cdot y_1 = 0$. So $\lambda = -\xi \cdot \omega$.

It remains to consider the coefficient γ. By considering intersection numbers, we have

$$x_0 \cdot x_0 - x_1 \cdot x_1 = \omega \cdot \omega + \gamma + \bar{\gamma}.$$

This information is not precise enough for our purposes. Hence we assume $w_2 H(\omega) = 0$. We also assume that the tubular neighborhood of $S^1 \times D$ was chosen so that $w_2 H(x_0) = 0$. Then $w_2 H(x_1) = 0$, and we have

$$\mu(x_0) = \mu(x_1) + \mu(\omega) + \gamma, (\mathrm{mod}\, I).$$

Now, we can identify M_i with the connected sum (in the interior of M),

$M \# (S^2 \times S^2)$, $i=0, 1$, exactly as in [W1]. If e and f in $H_2(M \# (S^2 \times S^2), \Lambda)$ are the standard generators of the summand $H_2(S^2 \times S^2; \Lambda)$, they correspond to $x_i - (x_i \cdot x_i) y_i$ and y_i, respectively, under the respective identifications. (Note that $\mu(x_0)$ and $\mu(x_1)$ are integers.) So we have, so far:

PROPOSITION 1.2. *Let $\omega \in H_2(M; \Lambda)$ with $w_2 H(\omega)=0$. Then there is a (base-point preserving) diffeomorphism*

$$h: M \# (S^2 \times S^2) \to M \# (S^2 \times S^2),$$

inducing the identity on the fundamental group and preserving local orientations with

$$h_*(e) = e + \omega + \gamma f, \gamma \in \Lambda \quad \text{with} \quad \gamma = -\mu(\omega) (\text{mod} I);$$
$$h_*(f) = f;$$

and

$$h_*(\xi) = \xi - (\xi \cdot \omega) f, \quad \text{for} \quad \xi \in H_2(M; \Lambda).$$

Note. Connected sum is well-defined in terms of the "local-orientation" of a non-orientable manifold. We assume M has been given a "local-orientation". That h preserves local orientations means that it induces the identity on $H_4(M \# (S^2 \times S^2); \mathbf{Z}^t) = \mathbf{Z}$, the top homology group with twisted integer coefficients. See [W2].

LEMMA 1.3. *Let M be as above. Let $\delta \in \pi_1 M$. Then there is a (base-point preserving) diffeomorphism*

$$g: M \# (S^2 \times S^2) \to M \# (S^2 \times S^2)$$

which preserves local orientations and induces the identity on the fundamental group, so that

$$g_*(\xi) = \xi \quad \text{for} \quad \xi \in H_2(M; \Lambda),$$
$$g_*(e) = \delta e$$
$$g_*(f) = w(\delta) \delta f.$$

Proof. Let $f: [0, 1] \to M$ be a closed loop representing δ. Assume f is differentiable. Then let φ_t be an isotopy of M so that $\varphi_t(*) = f(t)$, $*$ the basepoint of M; this exists by the isotopy extension theorem. Let χ_t be a similar isotopy for $M \# (S^2 \times S^2)$. If δ is orientation preserving (i.e., $w(\delta)=1$), put $g = (\varphi_1^{-1} \# \text{id}_{S^2 \times S^2}) \circ \chi_1$. If $w(\delta) = -1$, let $g = (\varphi_1^{-1} \# (\text{id} \times a)) \circ \chi_1$, where a is a diffeomorphism of S^2 of degree -1.

LEMMA 1.4. *Let M be as above. Suppose in addition that $M = P \# (S^2 \times S^2)$. Then there is a (basepoint-preserving) diffeomorphism k of $M \# (S^2 \times S^2)$ that pre-*

serves local orientation and induces the identity on π_1, so that

$$k_*(\xi)=\xi \quad \text{for} \quad \xi\in H_2(M;\Lambda);$$
$$k_*(e)=e+(\bar{\lambda}-\lambda)f; \quad \text{and} \quad k_*(f)=f;$$

for any given element λ of Λ.

Proof. It suffices to assume $\lambda=\delta\in\pi_1 M$. Let e' and f' denote the classes in $H_2(P \# (S^2\times S^2);\Lambda)$ carried by the first and second spheres of $S^2\times S^2$, respectively. Let $\bar{h}_1$ be a diffeomorphism of $(S^2\times S^2) \# (S^2\times S^2)$ with $\bar{h}_1(e)=e+e'$, $\bar{h}_1(f)=f$, $\bar{h}_1(e')=e'$, $\bar{h}_1(f')=f'-f$; this exists by [W1]. Let $g:M \# (S^2\times S^2)\to \to M \# (S^2\times S^2)$ be as in the conclusion of Lemma 1.3 for this δ. Let

$$h_1=g^{-1}\circ(\mathrm{id}_P \# \bar{h}_1)\circ g.$$

Then let $\bar{h}_2$ be a diffeomorphism of $(S^2\times S^2) \# (S^2\times S^2)$ with itself with

$$\bar{h}_2(e)=e+f', \quad \bar{h}_2(f)=f, \quad \bar{h}_2(e')=e'-f, \quad \bar{h}_2(f')=f';$$

and let $h_2=(\mathrm{id}_P \# \bar{h}_2)\circ h_1$.

Now let $h_3=\mathrm{id}_P \# (a\times a) \# \mathrm{id}_{S^2\times S^2}$, where $a:S^2\to S^2$ is a diffeomorphism of degree -1. Then $k=(h_3h_2)^2$ is the desired diffeomorphism.

(*Note.* In taking connected sum of diffeomorphisms, we must assume they are the identity on appropriate disks. Clearly this can always be arranged.)

THEOREM 1.5. *Let M be a compact, connected, smooth manifold of dimension four, and suppose $M=P \# (S^2\times S^2)$ for some smooth manifold P. Let $\omega\in H_2(M;\Lambda)$, $w_2H(\omega)=0$. Let $\lambda\in\Lambda=\mathbf{Z}[\pi_1M]$ be any element such that $\mu(\omega)\equiv\lambda(\mathrm{mod}\, I)$. Then there is a (basepoint-preserving) diffeomorphism φ of $M \# (S^2\times S^2)$ with itself which preserves local orientations and induces the identity on $\pi_1(M \# (S^2\times S^2))$, so that*

$$\varphi_*(e)=e+\omega-\lambda f; \quad \varphi_*(f)=f; \quad \text{and}$$
$$\varphi_*(\xi)=\xi-(\xi\cdot\omega)f \quad \text{for} \quad \xi\in H_2(M;\Lambda).$$

(*Note.* e and f are defined in the paragraph preceding 1.2.)

Proof. Compose a diffeomorphism of Prop. 1.2 with the appropriate diffeomorphism provided by Lemma 1.4.

One can use Theorem 1.5 and some algebra to build up results on which automorphisms of $H_2(M \# (S^2\times S^2);\Lambda)$ that preserve intersections and self-intersections can be realized by diffeomorphisms as in 1.5. We do this in an appendix. Here we offer the following example to show that, unlike the simply-connected case [W1], not every such automorphism can be realized by a diffeomorphism regardless how many copies of $S^2\times S^2$ are added by connected sum.

EXAMPLE 1.6. Let π be infinite cyclic, let $\Lambda = \mathbf{Z}[\pi]$ be the group ring with involution $(\sum \alpha_g g)^{-1} = \sum \alpha_g g^{-1}$. Let $SU_r(\Lambda)$ be as in [W3, §6]; see also §3 below. Let $\alpha \in SU_r(\Lambda)$ be an element representing a generator of $L_5(\pi) = \mathbf{Z}$. Then there is no orientation preserving diffeomorphism of

$$K = (S^3 \times S^1) \# (S^2 \times S^2) \# \cdots \# (S^2 \times S^2)$$

(connected sum r times) inducing $(\mathrm{id}_{S^3 \times S^1})_* \oplus \alpha$ on $H_2(K; \Lambda)$.

According to an unpublished result of R. Lee, one can take $r = 1$.

We outline the proof. Suppose such a diffeomorphism exists, φ say. Let $S_i^2 \times D^2$, $1 \leqslant i \leqslant r$, be disjoint tubular neighborhoods of the spheres $S_i^2 \times pt$ in the ith copy of $S^2 \times S^2$. Using the images of these tubes under φ to perform surgery on the normal map (π, b) $\pi: K \to S^3 \times S^1$ the natural projection and b an appropriate bundle map, we obtain a homotopy equivalence $h: Q \to S^3 \times S^1$ that is *normally cobordant* [B1] to the identity, via a normal cobordism whose surgery obstruction is a generator of $L_5(\mathbf{Z})$. From the construction, Q is diffeomorphic to $S^3 \times S^1$. But by [S1, Thm 5.1] and periodicity of surgery obstructions [W3], $Q \times S^1$ must be the exotic manifold of [S1, Thm 7.2]. Alternatively, one can apply the arguments of [S3, §2] directly to the normal cobordism to get a contradiction to Rohlin's theorem [R].

§2. Stable Surgery in Dimension Four and Some New 4-Manifolds

Let (Y, X) be a Poincaré pair, of (formal) dimension four, as defined for example on page 224 of [W2]. Assume also that Y is a connected finite complex. We may have $X = \emptyset$. Let ζ^k, $k \geqslant 4$, be a vector bundle of fibre dimension k over Y, and let

$$\begin{array}{ccc} \nu^k(W) & \xrightarrow{b} & \zeta^k \\ \downarrow & & \downarrow \\ (W, \partial W) & \xrightarrow{h} & (Y, X) \end{array}$$

be a *normal map* of degree one which induces a homotopy equivalence of the boundaries; i.e., $h \mid \partial W: \partial W \to \partial X$ is a homotopy equivalence, h has degree one with respect to the (given) local orientations, and b is a linear bundle map covering f from the k-dimensional normal bundle of the compact smooth manifold W, to ζ. Given such a normal map *of degree one* we define the connected sum $(h, b) \# (S^2 \times S^2)$ in the obvious way. (For the notion of connected sum of Poincaré complexes, see [W3, §2].)

Let $\pi = \pi_1 X$; let $w: \pi \to \{\pm 1\}$ be the first Stiefel-Whitney class. Then according to Wall [W2] there is an abelian group $L_4(\pi, w)$ (often written $L_4^h(\pi, w)$, as in [S1]), functorial in (π, w), and an invariant $\sigma(h, b) \in L_4(\pi, w)$ that depends only upon the normal cobordism class relative boundary [B1], [B2] of (h, b) and vanishes if h is a homotopy equivalence. This invariant is defined as follows: After a normal cobordism relative the boundary we may assume, as in [W2] for example, that f is 2-connected. By [W2, Lemma 2.4],

$$K = K_2(W) = \ker(h_*: H_2(W; \Lambda) \to H_2(Y; \Lambda)), \quad \Lambda = \mathbf{Z}[\pi_1 Y],$$

is a stably free Λ-module. Furthermore, the restriction of intersection numbers to K defines a non-singular sesquilinear form $\lambda: K \times K \to \Lambda$ which is associated with $\mu \mid K: K \to \mathbf{R}/I$; i.e., (K, λ, μ) satisfies (P2)–(P6) of [W2, p. 236] with $\eta = 1$, provided K is given a right module structure by $x\lambda = \bar{\lambda}x$.

By definition $\sigma(h, b)$ is the class in $L_4(\pi, w)$ of (K, λ, μ); $L_4(\pi, w)$ is defined as the *reduced* Grothendieck group of non-singular sesquilinear forms over Λ on stably free Λ-modules; the reduction is carried out by requiring *kernels*, as defined in [W2, p. 237] to represent zero. The details are as in [W2]. In particular, that σ is an invariant under normal cobordisms relative the boundary is essentially a consequence of Lemma 7.3 of [W3].

THEOREM 2.1. *Let (Y, X) be a Poincaré pair of formal dimension four, with Y a connected finite complex. Let*

$$\begin{array}{ccc} \nu^k(W) & \xrightarrow{b} & \eta^k \\ \downarrow & & \downarrow \\ (W, \partial W) & \xrightarrow{h} & (Y, X) \end{array}$$

be a degree one normal map inducing a homotopy equivalence of boundaries. Then $\sigma(h, b) = 0$ if and only if $\exists t \geqslant 0$ such that $(h, b) \# t(S^2 \times S^2)$ is normally cobordant relative the boundary to (h', b') with $h': (W', \partial W') \to (Y \# t(S^2 \times S^2), X)$ a homotopy equivalence.

To prove 2.1, we note first that

$$\sigma(h, b) = \sigma[(h, b) \# t(S^2 \times S^2)];$$

this follows easily from the definition. So the vanishing of $\sigma(h, b)$ is clearly necessary for t to exist so that the last statement of the theorem is satisfied.

We will say an element $\xi \in H_2(M; \Lambda)$ with $w_2 H(\xi) = 0$ (see § 1) is *strongly primitive* if $\exists \tau \in H_2(M; \Lambda)$ with $\xi \cdot \tau = 1$ and $w_2 H(\tau) = 0$. The main part of the proof of 2.1 is also the proof of the next result:

THEOREM 2.2. *Let M be a compact, connected, smooth four-manifold of the form $P \# (S^2 \times S^2)$. Let $\xi \in \pi_2(M) = H_2(M; \Lambda)$, $\Lambda = \mathbf{Z}[\pi_1 M]$, with $w_2 H(\xi) = 0$, be strongly primitive. Assume $\mu(\xi) = 0$. Then ξ is represented by a smooth embedding $S^2 \subset M \# (S^2 \times S^2)$, with trivial normal bundle, so that the inclusion $M \# (S^2 \times S^2) - S^2 \subset M \# (S^2 \times S^2)$ induces an isomorphism of fundamental groups.*

Remarks. 1) If $\xi \in H_2(P; \Lambda)$, one need not assume $w_2 H(\tau)=0$. For then if $w_2 H(\tau) \neq 0$, one can show $P \# (S^2 \times S^2) \cong P \# T$, T the non-trivial 2-sphere bundle over S^2, and use this to find τ' with $\tau' \cdot \xi = 1$ and $w_2 H(\tau')=0$. (Compare [W1].)

2) If $\mu(\xi) \in \mathbf{Z} \subset \Lambda$, one can show that ξ is represented by an embedding satisfying all the conclusions of 2.2 except that the normal bundle will not be trivial.

Proof of 2.2. Let e and f denote, as usual, the classes in $H_2(M \# (S^2 \times S^2); \Lambda)$ determined by the first and second spheres of the second summand, respectively. Let $\tau \in H_2(M; \Lambda)$ be such that $\xi \cdot \tau = 1$, and $w_2 H(\tau)=0$. Let φ and ψ be diffeomorphisms of $M \# (S^2 \times S^2)$ with itself so that

$$\varphi_*(e) = e + \xi, \quad \varphi_*(f) = f, \quad \varphi_*(\xi) = \xi,$$
$$\varphi_*(\tau) = \tau - f, \quad \psi_*(e) = e, \quad \psi_*(f) = f + \tau - \gamma e,$$
$$\psi_*(\xi) = \xi - e, \quad \psi_*(\tau) = \tau - (\gamma + \bar{\gamma}) e.$$

Here $\gamma \in \Lambda$ is some element with $\gamma \equiv \mu(\tau) \pmod{I}$. The diffeomorphism φ exists by Theorem 1.5; ψ exists by composing the appropriate diffeomorphism provided by Prop. 1.2 with a diffeomorphism that interchanges e with f and leaves everything else fixed. Then $\psi\varphi(e)=\xi$. This proves the theorem, since e is represented by an embedding, with trivial normal bundle, such that the inclusion of the complement in $M \# (S^2 \times S^2)$ induces an isomorphism of fundamental groups.

Proof of 2.1. We may assume that h is 2-connected, after a normal cobordism. Since $\sigma(h, b)=0$, we may also assume, after replacing W by $W \# q(S^2 \times S^2)$ (and changing notation) if necessary, that $K_2(W)=\ker(h_*: H_2(W; \Lambda) \to H_2(Y; \Lambda))$ is a free Λ-module with basis $\xi_1, \ldots, \xi_t, \tau_1, \ldots, \tau_t$, such that

$$\xi_i \cdot \xi_j = \tau_i \cdot \tau_j = \mu(\xi_i) = \mu(\tau_i) = 0, \quad 1 \leqslant i, j \leqslant t, \quad \text{and}$$
$$\xi_i \cdot \tau_j = \delta_{ij}.$$

By considering $(h, b) \# (S^2 \times S^2)$ (and then changing notation) if necessary, we may assume that $W=P \# (S^2 \times S^2)$, some P. Hence, by Theorem 2.1, ξ_1 is represented by an embedded sphere $S^2 \subset W \# (S^2 \times S^2)$, as in 2.1. This has a framing $g: S^2 \times D^2 \subset W \# (S^2 \times S^2)$ and we may use this embedding to perform framed surgery ([W2], [B1], [B3]) on the normal map $(h, b) \# (S^2 \times S^2)$. Note that because $\pi_2(O(k))=0$, there is no problem extending the bundle map b to the elementary cobordism $W \times I \cup_{(g,1)} D^3 \times D^2$.

So let (h', b'), $h': W' \to Y$, be the normal map obtained by performing this elementary normal cobordism.

$$W' = \text{cl}(W \# (S^2 \times S^2) - g(S^2 \times D^2)) \cup_g D^3 \times S^1 .$$

Hence, by Van Kampen's theorem and the last assertion of 2.2 about our embedding, g induces an isomorphism of fundamental groups. In fact, g is 2-connected, and essentially the same argument as in the proof of Theorem 3.3 of [W2] now shows that $K_2(W')$ is a free module over Λ with basis $\xi_2, \ldots, \xi_t, \tau_2, \ldots, \tau_t$ with $\mu(\xi_i) = \mu(\tau_i) = = \xi_i \cdot \xi_j = \tau_i \cdot \tau_j = 0$ for $2 \leqslant i, j \leqslant t$, and $\xi_i \cdot \tau_j = \delta \cdot j$. Continuing inductively, we obtain the result.

In the preceding proof we can actually have that W' is diffeomorphic to W. For the 2-sphere which we used in surgery was the image of $S^2 \times pt \subset W \# (S^2 \times S^2)$ under a diffeomorphism, and we can obtain the framing from the standard framing of $S^2 \times pt$ via this diffeomorphism. So we have

COROLLARY 2.3. *Let*

$$\begin{array}{ccc} \nu^k(P) & \xrightarrow{c} & \zeta^k \\ \downarrow & & \downarrow \\ (P^4, \partial P) & \xrightarrow{k} & (Y^4, X) \end{array}$$

be a normal map of degree one that is a homotopy equivalence on the boundary. Assume that k is 2-connected and dim $(Y, X) = 4$. *Suppose that $\sigma(k, c) = 0$. Then $(P \# r(S^2 \times S^2), \partial P)$ has the same homotopy type as $(Y \# t(S^2 \times S^2), X)$ for some $t \geqslant 0$ and $r \geqslant 0$.*

We conclude this section with examples of four-manifolds that have the same (simple) homotopy type, but are not diffeomorphic or even smoothly or P.L. *h*-cobordant.

For any closed four-manifold M, we say that two degree-one normal maps

$$\begin{array}{ccc} \nu(M_i) & \xrightarrow{b_i} & \eta_i \\ \downarrow & & \downarrow \\ M_i & \xrightarrow{f_i} & M \end{array} \qquad i = 1, 2,$$

are *equivalent* if $\exists$ a bundle equivalence $b: \eta_1 \to \eta_2$ such that (f_1, bb_1) and (f_2, b_2) are normally cobordant. A theorem due to Sullivan, at least in the simply-connected case, asserts that the equivalence classes of normal maps into M are in one-one correspondence with $[M; G/O]$, the set of homotopy classes of maps of M into the classifying space G/O for stable fibre-homotopy trivializations of vector bundles. (See [S4], [B1], [W3].) Let $\eta(f, b)$ denote the class in $[M; G/O]$ represented by (f, b); if f is a homotopy equivalence, b is irrelevant and so we write $\eta(f)$, and call this the *normal invariant* of f. We write $\sigma: [M; G/O] \to L_4(\pi_1 M; wM)$ for $\sigma(f, b) = \sigma(\eta(f, b))$.

Now let $M = \mathbf{RP}^4$ be real projective four-space; i.e., the quotient of S^4 by the antipodal map. Then, as in [M4], [W4], $[M; G/O] = \mathbf{Z}_4$ and

$$\sigma: [M; G/O] \to L_4(\mathbf{Z}_2, -) = \mathbf{Z}_2$$

is the natural non-trivial map. (Recall that in these low dimensions there is no difference between G/PL and G/O.) So if $\theta \in [M; G/O]$ is the generator, there is a normal map

$$\begin{array}{ccc} \nu(K_0) & \to & \eta \\ \downarrow & & \downarrow \\ K_0 & \to & M \end{array}$$

with $\sigma(f, b)=0$ and $\eta(f, b)=2\theta \neq 0$. (In fact, η is equivalent to $\nu(M)$.)

We may assume, as usual, that f is 2-connected. Let $K=K_0 \# r(S^2 \times S^2)$. Then by Cor. 2.3, K, for suitable r, has the homotopy type of $M \# t(S^2 \times S^2)$, some $t \geqslant 0$; actually it has the *simple* homotopy type of this manifold, as $\mathrm{Wh}(\mathbf{Z}_2)=0$.

Suppose that K were h-cobordant to $M \# t(S^2 \times S^2)$. Using the h-cobordism, it is easy to find a normal map (f', b'), with $f': M \# t(S^2 \times S^2) \to M$, equivalent to (f, b). Since $\pi_2 M=0$, we may perform surgery using the t copies of $S^2 \times pt$ contained in $M \# t(S^2 \times S^2)$. The result is a homotopy equivalence $h: M \to M$ with $\eta(h)= =2\theta \neq 0$. But every homotopy equivalence of $\mathbf{RP}^4$ with itself is homotopic to the identity. So we have proven most of the following:

THEOREM 2.4. *Let $M=\mathbf{RP}^4$ be real projective four-space. Then $\exists$, for some $t \geqslant 0$, a smooth manifold K that has the (simple) homotopy type of $M \# t(S^2 \times S^2)$ but is not diffeomorphic or even smooth or* P.L. *normally cobordant to $M \# t(S^2 \times S^2)$; in fact for all $r \geqslant 0$, $K \# r(S^2 \times S^2)$ is not h-cobordant to $M \# (t+r)(S^2 \times S^2)$. K is unique up to h-cobordism, and is topologically h-cobordant to $M \# t(S^2 \times S^2)$.*

The assertion about $K \# r(S^2 \times S^2)$ follows easily. The uniqueness up to h-cobordism is a consequence of the fact that $L_5(\mathbf{Z}_2, -)=0$ [W4]. Since $[M; G/\mathrm{Top}]= =\mathbf{Z}_2 \oplus \mathbf{Z}_2$, as follows from results of [K5], the image of 2θ in $[M; G/\mathrm{Top}]$ is trivial; the last statement follows from this and results of [K5]. One can also show that $K-pt$ is not diffeomorphic to $M \# t(S^2 \times S^2)-pt$.

Using the results of this section, one can actually show that K is smoothly h-cobordant to $M \# N$, N a suitable simply-connected almost parallelizable manifold of index sixteen. Of course, this indicates that 2.3 is really not needed for this example. The present method applies to give non-trivial elements of $\mathcal{S}(M)$ for other M^4, but to deduce the existence of new manifolds, one also needs to solve the technical problem of what normal invariants are represented by homotopy equivalences of M with itself. (See [B1], [S4], [W3] or §§3, 5 below for the definition of $\mathcal{S}(M)$.)

§3. Realizing Surgery Obstructions

In order to prove our splitting theorem, we have to generalize slightly the surgery obstructions in [W3] in dimensions $4k+1$; this is done in [S1] and in [C] more general ly. Let π be a finitely presented goup and let $w:\pi\to\{\pm 1\}$ be a homomorphism. Let $\Lambda=\mathbf{Z}[\pi]$ be the integral group ring, with involution $(\sum \alpha_g g)^- = \sum w(g)\,\alpha_g g^{-1}$. Let $B\subset \mathrm{Wh}(\pi)$ be a subgroup that is invariant under the involution * induced on $\mathrm{Wh}(\pi)$; i.e., $B=B^*$.

Let $\kappa_r=(K_r,\varphi_r,\mu_r)$ be the *kernel* ([W1, §4], [W3]) of dimension $2r$. That is K_r is a free Λ-module with basis $e_1,\dots,e_r,f_1,\dots,f_r$, φ_r is a non-singular sesquilinear form associated with $\mu_r\colon K_r\to\Lambda/I$ (see §2 above and [W1], [W3]) and $\varphi_r(e_i,e_j)=\varphi_r(f_i,f_j)=\mu_r(e_i)=\mu_r(f_j)=0$ and $\varphi_r(e_i,f_j)=\delta_{ij}$. Let $U_r^B(\Lambda)$ be the group of automorphism of κ_r whose torsion, with respect to the basis $\{e_1,\dots,e_r,f_1,\dots,f_r\}$ (hereafter called the standard basis) lies in B. Let $TU_r^B(\Lambda)\subset U_r^B(\Lambda)$ be those automorphisms that leave $\{e_1,\dots,e_r\}_\Lambda$ invariant and induce an automorphism of this module with torsion B, with respect to this basis. Let σ_0 be defined by $\sigma_0(e_1)=f_1$, $\sigma_0(f_1)=e_1$, and σ_0 = identity on $\{e_2,\dots,e_r,f_2,\dots,f_r\}_\Lambda$, and let $RU_r^B(\Lambda)$ be the subgroup generated by $TU_r^B(\Lambda)$ and σ_0.

There is a natural inclusion of $U_r^B(\Lambda)$ in $U_{r+1}^B(\Lambda)$; we extend $\alpha\in U_r^B(\Lambda)$ by setting $\alpha(e_{r+1})=e_{r+1}$ and $\alpha(f_{r+1})=f_{r+1}$. Let $U^B(\Lambda)=\lim_{r\to\infty}U_r^B(\Lambda)$, and let $RU^B(\Lambda)=\lim_{r\to\infty}RU_r^B(\Lambda)$. Then as in [W3, Theorem 6.3] for the case $B=\{0\}$, $RU^B(\Lambda)$ is a normal subgroup containing $[U^B(\Lambda),U^B(\Lambda)]$. So we define an abelian group

$$L_{4k+1}^B(\pi,w)=U^B(\Lambda)/RU^B(\Lambda).$$

When $B=\mathrm{Wh}(\pi)$, we omit it from the notation.

Let V be a Poincaré complex and a finite connected CW complex. Let $C\subset\mathrm{Wh}(\pi_1 V)$ be a subgroup. We say V is a *C-Poincaré complex* if the fundamental class (in $H_m(V,\mathbf{Z}^t)$) has a representative on the chain level (possibly infinite), cap product with which induces a homotopy equivalence of the (finitely supported) co-chains $C^*(V;\mathbf{Z}[\pi_1 V])$ with the chains $C_*(V;\mathbf{Z}[\pi_1 V])$ with torsion in C. Let (Y,X) be a Poincaré pair, with Y and X finite CW complexes and Y connected, and with $(\pi_1 Y, wY)=(\pi,w)$. Let $B\subset\mathrm{Wh}(\pi)$ be as above. Let $V_1,\dots,V_s$ be the components of X, and let $C_i\subset\mathrm{Wh}(\pi_1 V_i)$ be the inverse image of B under the natural map. Let $\partial[Y]=[V_1]+\dots+[V_s]$. Then we say (Y,X) is a *B-Poincaré pair* if $[Y]\in H_{m+1}(Y,X;\mathbf{Z}^t)$ has a chain representative cap products with which induces a chain homotopy equivalence from $C^*(Y;\Lambda)$ to $C_*(Y,X;\Lambda)$ with torsion in B; and if V_i is a C_i-Poincaré complex with fundamental class $[V_i]$. For $B=\{0\}$, we have a slight generalization of a *finite Poincaré* pair as defined in [W3, §2]. In particular, a manifold pair is a B-Poincaré pair.

Let (Y,X) be a B-Poincaré pair as in the preceding paragraph. Let

$$\begin{array}{ccc} \nu(M) & \xrightarrow{b} & \eta \\ \downarrow & & \downarrow \\ (M,\partial M) & \xrightarrow{h} & (Y,X) \end{array}$$

be a degree one normal map. Assume $h \mid \partial M: \partial M \to X$ is a homotopy equivalence. Assume the images in $\mathrm{Wh}(\pi_1 Y)$ of the torsions of the restrictions of h to components of ∂M are all in B. (This is slightly more restrictive than necessary.) We then say that $h \mid \partial M$ is a *B-homotopy equivalence*. Assume $\dim Y = 4k+1$, $k \geq 1$. Then, there is an invariant $\sigma(h, b) \in L_{4k+1}(\pi, w)$ of the normal cobordism class relative the boundary of (h, b). It vanishes when h is also a homotopy equivalence with torsion in B, i.e., for h a *B-homotopy equivalence of pairs*. The details are as in [W3, §6] for $B = \{0\}$. In particular, $\sigma(h, b) = \sigma((h, b) \times CP^2)$, where CP^2 denotes the complex projective plane.

For $k \geq 2$, Theorem 6.5 of [W3] asserts that every element of $L_{4k+1}(\pi, w)$ can be realized as the surgery obstruction $\sigma(h, b)$ for a suitable normal map in dimension $4k+1$.

THEOREM 3.1. *Let (Y, X) be a B-Poincaré pair, where $(\pi, w) = (\pi_1 Y, wY)$ and $B \subset \mathrm{Wh}(\pi)$ satisfies $B = B^*$. Let $\gamma \in L_5^B(\pi, w)$. Let $\alpha \in U_r^B(\Lambda)$ be a representative of γ. Let*

$$g: (M \# r(S^2 \times S^2), \partial M) \to (Y, X)$$

be a B-homotopy equivalence of pairs. Then there is a normal cobordism

$$\begin{array}{ccc} \nu(W) & \xrightarrow{b} & \eta \times I \\ \downarrow & & \downarrow \\ W & \xrightarrow{h} & Y \times I \end{array}$$

relative the boundary, from $(g, b \mid M \# r(S^2 \times S^2))$, to $(h \mid \partial_+ W, b \mid \partial_+ W)$, so that

(i) *$h \mid \partial_+ W: \partial_+ W \to Y \times 1$ is a B-homotopy equivalence; and*

(ii) *The obstruction $\sigma(h, b) \in L_5^B(\pi, w)$, defined in view of* (i), *is precisely γ.*

Proof. It suffices to assume $Y = M \# r(S^2 \times S^2)$ and g is the identity. To get the general case, one simply realizes $(g^{-1})_* \gamma$ and composes with $(g \times 1, c \times 1)$, $c: \nu \times (M \# r(S^2 \times S^2)) \to \eta$ a bundle map.

Let $K = M \# r(S^2 \times S^2)$. Let e_i' and f_i', $1 \leq i \leq r$, be the classes in $H_2(K; \Lambda)$ represented by the ith copies of $S^2 \times pt$ and $pt \times S^2$, respectively. Let e_i and f_i, $1 \leq i \leq r$, denote dimilar classes in $H_2(K \# r(S^2 \times S^2); \Lambda)$. This identifies $\{e_1, \ldots, e_r, f_1, \ldots, f_r\}_\Lambda$ with κ_r by identifying these classes with the standard basis of K_r; note that this identification is consistent with intersection and μ-forms.

Let $T_j \subset H_2(K \# r(S^2 \times S^2); \Lambda)$ be spanned over Λ by $\{e_i', f_i' \mid 1 \leq i \leq r, j \neq i\} \cup \cup \{e_i, f_i \mid 1 \leq i \leq r\} \cup H_2(M; \Lambda)$. Let φ_j, $1 \leq j \leq r$, be a diffeomorphis of $K \# r(S^2 \times S^2)$ with itself so that

$$(\varphi_j)_*(e_j') = e_j' + \alpha(e_j), \quad (\varphi_j)_*(f_j') = f_j', \quad \text{and}$$
$$(\varphi_j)_*(\xi) = \xi - (\xi \cdot \alpha(e_j)) f_j' \quad \text{for} \quad \xi \text{ in } T_j.$$

Let ψ_j, $1 \leqslant j \leqslant r$, be a diffeomorphism with

$$(\psi_j)_*(e_j') = e_j', \quad (\psi_j)_*(f_j') = f_j' + \alpha(f_j), \quad \text{and}$$
$$(\psi_j)_*(\xi) = \xi - (\xi \cdot \alpha(f_j))\, e_j' \quad \text{for} \quad \xi \text{ in } T_j.$$

These exist by Theorem 1.5; note that $\mu(e_i) = \mu(f_i) = 0$, $1 \leqslant i \leqslant r$. Let

$$\varphi = (\psi_1 \varphi_1)(\psi_2 \varphi_2) \ldots (\psi_r \varphi_r).$$

Then $\varphi_*(e_i') = \alpha(e_i)$ for $i = 1, \ldots, r$; this follows easily from the fact that $\alpha(e_i) \cdot \alpha(e_j) = \alpha(f_i) \cdot \alpha(f_j) = 0$ and $\alpha(e_i) \cdot \alpha(f_j) = \delta_{ij}$, $1 \leqslant i, j \leqslant r$.

Now we construct a normal cobordism, as follows. First perform surgery on trivial circles to get a normal cobordism relative the boundary, (h_1, b_1), $h_1 : W_1 \to K \times [0, \frac{1}{2}]$, with $\partial_- W_1 = K$ and $h_1 \mid \partial_- W_1 = (\mathrm{id}_K 2 0)$; and with $\partial_+ W_1 = K \# r(S^2 \times S^2)$ and $h_1 \mid \partial_+ W_1$ the natural quotient map (i.e., $\mathrm{id}_K \# rp$, $p : S^2 \times S^2 \to S^2$ of degree one.) Now let

$$S_i^2 \times D^2 \subset M \# r(S^2 \times S^2) \# r(S^2 \times S^2)$$

be a standard embedding of $S^2 \times D^2$ in the ith summand of the first group of $S^2 \times S^2$'s; in particular, $S_i^2 \times D^2$ represents e_i'. Using the embeddings $\varphi \mid S_i^2 \times D^2$, we perform framed surgery on $h_1 \mid \partial_+ W$; note that these classes represent elements in the kernel of the map $h_1 \mid \partial_+ W$ induces on $H_2(\partial_+ W; \Lambda)$ and that, as in §2, there is no obstruction to extending the framing. Let (h_2, b_2) be the resulting normal cobordism, relative the boundary,

$$h_2 : W_2 \to K \times [\tfrac{1}{2}, 1].$$

Let $W = W_1 \cup W_2$ (recall $\partial_+ W_1 = \partial_- W_2$), let $h = h_1 \cup h_2$ and $b = b_1 \cup b_2$.

Clearly the inclusion $\partial_+ W_1 - \bigcup_{i=1}^r (S_i^2 \times D^2) \subset \partial_+ W_1$ induces an isomorphism of fundamental groups. It follows easily from this and the Van Kampen theorem that $h_2 \mid \partial_+ W$ induces an isomorphism of fundamental groups. The same argument as in [W3, §6] now goes through to show that $h \mid \partial_+ W$ is a B-homotopy equivalence and that $\sigma(h, b) = \gamma$. (In fact, that argument doesn't go through in general in this dimension precisely because of the possible lack of suitably embedded two-spheres.) This completes the proof.

Note that $\partial_+ W$ is diffeomorphic to $\partial_- W = K$. However, let $\mathscr{S}(M)$, M^4 a closed smooth 4-manifold, denote the smooth (or P.L.) s-cobordism classes of homotopy equivalences $f : Q \to M$, Q a closed smooth 4-manifold. Then, using Theorem 3.1, for γ the generator of $L_5(\mathbf{Z})$, and an argument similar to that of [S3], for example, one can show the following:

PROPOSITION 3.2. *For r sufficiently large, the set of elements in* $\mathscr{S}((S^3 \times S^1) \# r(S^2 \times S^2))$ *with vanishing normal invariant has precisely two elements.*

There are similar theorems for T^4 and $S^2 \times T^2$, i.e., theorems asserting the existence of an appropriate number of non-trivial elements with vanishing normal invariant. By the unpublished result of R. Lee mentioned in example 1.6, we can actually take $r=1$. The existence of a homeomorphism representing the non-trivial element of $\mathscr{S}((S^3 \times S^1) \# r(S^2 \times S^2))$ is equivalent (using [L2]) with the existence of an almost parallelizable closed topological four-manifold of index eight.

Let $A=((a_{ij}))$ be a non-singular $r \times r$ matrix of Λ. Let $A^*=((\overline{a_{ji}}))$. Then the matrix

$$\begin{pmatrix} A & O \\ O & (A^*)^{-1} \end{pmatrix}$$

with respect to the standard basis, determines an element $U^{\mathrm{Wh}(\pi)}(\Lambda)$. Of course, this represents the trivial element in $L_5(\pi, w)$. In fact, if we carry out the instructions in the proof of 3.1 we get an h-cobordism; computing its torsion we have the following (compare [M1, Thm. 11.1] and [S5]):

PROPOSITION 3.3. *Let A be as above. Let $K=M \# r(S^2 \times S^2)$, with $(\pi_1 M, wM) = (\pi, w)$. Then $\exists$ an h-cobordism $(W; \partial_+ W, \partial_- W)$, with $\partial_- W=K$ and with torsion $\tau(W, K)$ equal to the element of* $\mathrm{Wh}(\pi)$ *represented by A.*

We leave the details to the reader. A realization theorem for h-cobordisms with torsion represented by a unit in $\mathbf{Z}[\pi]$ was proven by Stallings in [S5].

§4. S-Splitting up to Normal Cobordism

Let (Y, X) be a Poincaré pair of finite complexes, of formal dimension five, with X possibly empty. Let

$$(V, U) \subset (V \times [-1, 1], U \times [-1, 1]) \subset (Y, X)$$

be a Poincaré sub-pair of subcomplexes, V connected, of codimension one and with a product regular neighborhood. In this situation we will often refer to V as the *fibre*. Using a circle smoothly (or P.L.) embedded in a suitable 4-simplex Δ of V and a smooth (or P.L.) disk it bounds in $\Delta \times [0, 1)$ that does not meet $\Delta \times 0$ except at the boundary, we may exchange a *trivial* 2-handle as is done in [B3] for manifolds. Alternatively, we may take a copy of $S^2 \times S^2$ embedded trivially in $\Delta \times (0, 1)$ and join it up to $V-\Delta'$, $\Delta' \subset \Delta$ a suitable open disk, by a tube. In either case, we replace V by $V \# (S^2 \times S^2)$. We call this process a *trivial exchange of a 2-handle (on the right)*.

If $Y-V$ is connected, we let B be the image in $\mathrm{Wh}(\pi_1(Y-V))$ under the map induced by inclusion. If $Y-V=Y_1 \cup Y_2$ has 2-components, we let B be the image of the map

$$Wh(\pi_1 Y_1) \oplus Wh(\pi_1 Y_2) \to Wh(\pi_1 Y)$$

which is the sum of the two inclusion induced maps. We will always assume the following in the rest of this paper:

(i) (Y, X) is a B-Poincaré pair; and

(ii) The map $\pi_1 V \to \pi_1 Y$ induced by inclusion is a monomorphism.

We consider a homotopy equivalence

$$h:(W, \partial W) \to (Y, X)$$

which is transverse[3]) to (V, U) and *split* along U on the boundary; i.e.,

$$h \mid \partial W:(\partial W; h^{-1}(U), \partial W - h^{-1}(U)) \to (X; U, X - U)$$

is a homotopy equivalence of triples. We say that h is *split along* V if

$$h:(W; h^{-1}(V), W - h^{-1}(V)) \to (Y; V, Y - V)$$

is also a homotopy equivalence. (Because of (ii), this is equivalent to requiring merely that $h \mid h^{-1} V : h^{-1} V \to V$ is a homotopy equivalence.) We say h is *splittable along* V if it is homotopic relative the boundary to a map that is split along V. (This is the same as saying that it is homotopic as a map from

$$(W; h^{-1}(U), \partial W - h^{-1}(U)) \to (Y; U, X - U)$$

to a split map; to see this, use a boundary collar.)

We say that h is *S-splittable* along V if it is splittable along some V' obtained from V by a finite number of exchanges of trivial 2-handles on the right.

Note that given h as above, if V' is obtained from V by exchanging a trivial 2-handle, we can find

$$h':(W; h^{-1}(V), W - h^{-1}(V)) \to (Y; V, Y - V),$$

homotopic to h as a map of these triples and relative the boundary, so that the inverse image of a suitable cell under $h' \mid V$ is a cell, $(h')^{-1}(V')$ is obtained from $h^{-1}(V)$ by exchanging a trivial 2-handle, and

$$h' \mid (h')^{-1}(V') = (h' \mid V) \# \mathrm{id}_{S^2 \times S^2}.$$

From now on, we assume without explicit mention that, where appropriate, this has always been arranged.

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[3]) Transversality depends only upon the linear structure in the normal directions, and so makes sense here; i.e., $h^{-1}(V)$ is a manifold. From now on, we always assume the appropriate transversalities without explicit mention.

We say a homotopy equivalence h as above is *algebraically* splittable if its torsion $\tau(h)\in \mathrm{Wh}(\pi_1 Y)$ lies in the subgroup B. It is not hard to show, as in [C] and [F] for the case $\pi_1 Y = \mathbf{Z}\times_\alpha G$ that if h is S-splittable along V, then it is algebraically splittable.

Finally, we need the following definition, introduced in [C]: Let H be a subgroup of the group G. Then we say H is (algebraically) two-sided in G if $\forall x\in G$, $(HxH = Hx^{-1}H)$ implies $x\in H$. See [C] for discussion of this condition and, in particular, for the following: if H is normal in G and G/H has no 2-torsion, then H is two-sided in G. If $\nu_+, \nu_- : H\to G$ are one-one group homomorphisms and if K is obtained from $Z*G$ by dividing by the least normal subgroup containing all the elements $t\nu_+(x)\, t^{-1}\nu_-(x^{-1})$, for $t\in\mathbf{Z}$ a generator, $x\in H$ (write $K=\mathbf{Z}*G/\{t\nu_+(x)\,t^{-1}=\nu_-(x)\}$), then $H\subset K$ is 2-sided if and only if $\nu_+(H)\subset G$ and $\nu_-(H)\subset G$ are two sided. In particular, $G\subset G\times_\alpha \mathbf{Z}$ is 2-sided. If $H\subset G_1$ and $H\subset G_2$, let $K=G_1 *_H G_2$, the amalgamated free product. Then $H\subset K$ is 2-sided if and only if $H\subset G_i$, $i=1, 2$, are two-sided.

Throughout the rest of this paper we also assume

(iii) $\pi_1 V\subset \pi_1(Y)$ is two-sided.

THEOREM 4.1. *Let $(V, U)\subset(Y, X)$ be as above (in particular satisfying* (i)–(iii)*). Let $h:(W^5, \partial W)\to(Y, X)$ be an algebraically splittable map that is also split along U on the boundary. Let ξ be a (high-dimensional) vector bundle over Y so that $\exists$ a bundle map $b:\nu(W)\to\xi$ covering h. Then the normal map (h, b) is normally cobordant, relative the boundary, to an S-splittable homotopy equivalence.*

Remark. For the conclusion of 4.1 to hold, one really needs to assume only that $\tau(h)$ is in the kernel of a homomorphism $\mathrm{Wh}(\pi_1 Y)\to\tilde{K}_0(\pi_1 Y)$ defined by Waldhausen. Also, one apparently can eliminate (iii). In the interest of simplicity, we do not carry out these improvements.

Proof of 4.1. We have to consider separately the separating and non-separating cases.

Case I. $Y-V$ connected. Let $K=\pi_1 Y$, $H=\pi_1 V$, $G=\pi_1(Y-V)$. Let $\nu_+, \nu_- : V\to Y-V$ be obtained by pushing right and left, respectively, in the bicollar neighborhood of V; e.g., $\nu_+(x)=(x, \frac{1}{2})\in V\times[-1, 1]\subset Y$. We will also write ν_+ and ν_- for the respective induced homomorphisms of these maps on fundamental groups and Wall groups. Then (by Van Kampen's theorem)

$$K = \mathbf{Z} * G/\{t\nu_+(x)\, t^{-1} = \nu_-(x) \mid x\in H\},$$

t a generator of $\mathbf{Z}$, and B is the image of the inclusion induced map $\mathrm{Wh}(G)\to\mathrm{Wh}(K)$; by naturality $B=B^*$. Using the fact that $h \mid \partial W$ is split, it follows (see [F] [C]) that $h\mid\partial W:\partial W\to X$ is a B-homotopy equivalence. (See §3 for the definition.)

Consider the normal cobordism

$$\Xi:\quad \begin{array}{ccc} \nu(h^{-1}V) & \xrightarrow{b\,|\,h^{-1}V} & \xi\,|\,V \\ \downarrow & & \downarrow \\ (h^{-1}V, h^{-1}U) & \xrightarrow{h\,|\,h^{-1}V} & (V, U) \end{array}$$

Then, by Theorem $\alpha\cdot\beta$ of [C], the surgery obstruction $\sigma(\Xi\times CP^2)$ vanishes in $L_8(H)$. (We omit the orientation homomorphism from the notation.) This also follows from the existence of a well-defined map $L_9^B(K)\to L_8^h(H)$ as in [C] or [S1] in the case $K=\mathbf{Z}\times H$. So, by periodicity of Wall obstructions,

$$\sigma(\Xi)=0.$$

So by Theorem 2.1, $(\Xi)\ \#\ t(S^2\times S^2)$ is normally cobordant, relative the boundary, to a homotopy equivalence. So by the "cobordism extension theorem", (h, b) is normally cobordant relative the boundary to a normal map

$$\begin{array}{ccc} \nu(Q) & \xrightarrow{c} & \xi \\ \downarrow & & \downarrow \\ Q & \xrightarrow{g} & Y \end{array}$$

so that $g\,|\,g^{-1}P: g^{-1}P\to P$ is a homotopy equivalence, where P is obtained from V by exchanging trivial 2-handles on the right.

Let Y_P and $Q_{g^{-1}P}$ be obtained by splitting Y and Q along P and $g^{-1}P$, respectively. (So, for example, $Y_P\cong Y-V\times(-\varepsilon,\varepsilon)$.) Let (g_P, c_P), $g_P: Q_{g^{-1}P}\to Y_P$, be the normal map induced by (g, c). If we had

$$\sigma(g_P, c_P)=0$$

in $L_5(G)$, we would be finished; we could perform surgery *relative the boundary* to get a homotopy equivalence and glue the result back up along the codimension one submanifold (i.e., $g^{-1}P$) or subcomplex (i.e., P) to get the required homotopy equivalence. (Compare [S1, Lemma 5.3].)

Let $\gamma\in L_5(H)$. We exchange further trivial handles on the right if necessary, but keep the same notation. After doing this, we may find a normal cobordism, relative boundary, $(\bar e, \bar a)$, $\bar e: T\to P\times I$, with $\partial_- T=g^{-1}P$ and $(\bar e,\bar a)\,|\,\partial_- T=(g, c)\,|\,g^{-1}P$, and with $e\,|\,\partial_+ T:\partial_+ T\to P\times 1$ a homotopy equivalence, so that

$$\sigma(\bar e, \bar a)=\gamma;$$

this is just Theorem 3.1. By the "cobordism extension theorem", we can find (e, a), a normal cobordism relative the boundary, $e: R\to Y\times I$, with

$$\partial R = Q,\quad (e, a)\,|\,\partial_- R=(g, c),\quad e^{-1}(P\times I)=T,$$

and

$$(e, a) \mid e^{-1}(P \times I) = (\bar{e}, \bar{a}).$$

Now split along $P \times I$ and apply a result of Wall (§3 of [W3]) as stated, for example, in 1.2 of [S1]. This result says roughly that if the boundary of a normal map with connected target has several components, the sum of the images of the surgery obstructions of each under the natural maps into the Wall group of π_1 of the target of the normal map vanishes. The result in [S1] is stated for dimensions greater than six; we can handle the present situation either by taking products with CP^2 and using periodicity or by appealing directly to [W3, Thms. 3.1 and 3.2]. So we have, (up to a sign at least, depending on orientation conventions):

$$(4.2) \quad + \sigma(g_P, c_P) - \sigma(e_P, a_P) = \nu_+(\gamma) - \nu_-(\gamma).$$

Thus, to complete the proof, it suffices to show that

$$\sigma(g_P, c_P) = \nu_+(\gamma) - \nu_-(\gamma)$$

for some $\gamma \in L_5(H)$; for then we would have $\sigma(e_P, a_P) = 0$, and, as above, we would be done.

Let $j_*: L_5^h(G) \to L_5^B(K)$ be the map induced by inclusion. Algebraically, this is induced by the extension of coefficients homomorphism of $U_r^{\mathrm{Wh}(G)}(\mathbf{Z}[G])$ to $U_r(\mathbf{Z}[K])$. It is not hard to prove from the definitions that

$$(4.3) \quad j_* \sigma(g_P, c_P) = \sigma(g, c).$$

But by [C], the following is exact:

$$L_5^h(H) \xrightarrow{\nu_+ - \nu_-} L_5^h(G) \xrightarrow{j_*} L_5^B(K)$$

Since $\sigma(g, c) = \sigma(h, b) = 0$ because h is an algebraically splittable homotopy equivalence, this completes the proof.

Case II. $Y - V = Y_1 \cup Y_2$ has two components. The argument in this case is analogous. If $H = \pi_1 M$, $G_i = \pi_1 Y_i$, $i = 1, 2$, $K = \pi_1 W$, then $K = G_1 *_H G_2$. Again we first do surgery on the codimension one normal map, suitably altered, to get a normal map that is at least a homotopy equivalence along the (modified) codimension one submanifold. When we split, we get two surgery problems. We use the sequence [C]

$$L_5(H) \xrightarrow{\nu_+ \oplus (-\nu_-)} L_5(G_1) \oplus L_5(G_2) \xrightarrow{(j_1)_* + (j_2)_*} L_5^B(K)$$

to show that the direct sum of the two surgery obstructions of these problems is in the image of $L_5(H)$ under $\nu_+ \oplus (-\nu_-)$. (Here $j_i: Y_i \subset Y$ are inclusions.) Hence these obstructions can be cancelled by extension of a suitable normal cobordism of the codimension one normal map. We leave the details for this case to the reader.

§5. S-Splitting in Dimension Five

Let $(Y, X) \supset (V, U)$ be as in §4, and let $g: P \to X$ be a homotopy equivalence that is split along U. Let $\mathscr{S}^B(Y, g)$ denote the equivalence classes of homotopy equivalences $h: (W, \partial W) \to (Y, X)$ that are algebraically splittable and satisfy the following: $\exists$ a diffeomorphism $\varphi: \partial W_1 \to P$ with $g\varphi = h \mid \partial W$. The equivalence relation is the following: h and $h_1: (W_1, \partial W_1) \to (Y, X)$ are equivalent if $\exists$ an h-cobordism, $(Z; W, W_1)$, *relative the boundary*, of W with W_1, and a map $F: Z \to Y \times I$ satisfying

(a) the torsion of F lies in B,

(b) $F \mid W = (h, 0)$ and $F \mid W_1 = (h_1, 1)$

(c) $F \mid (\partial Z - W \cup W_1)^- = h \times \mathrm{id}_I$.

Note that (a) is equivalent to the assertion that $F_*(\tau(Z, W)) \in B$, by [M1, Lemma 7.8].

We define $NM(Y, g)$ to be the equivalence classes of normal maps

$$\begin{array}{ccc} \nu(W) & \xrightarrow{b} & \xi \\ \downarrow & & \downarrow \\ W & \xrightarrow{f} & y \end{array}$$

such that $\exists$ a diffeomorphism $\varphi: \partial W \to P$ with $g\varphi = f \mid \partial W$. Two such, (f, b) and $(f_1, b_1): \nu(W_1) \to \xi_1$, are equivalent if $\exists$ a (stable) bundle equivalence $\bar{b}: \xi \to \xi_1$ so that $(f, \bar{b}b)$ and (f_1, b_1) are normally cobordant relative the boundary.

As usual ([B1], [S4], [W1]) there is a natural map

$$\eta: \mathscr{S}^B(Y, g) \to NM(Y, g).$$

Namely, given a homotopy equivalence

$$h: (W, \partial W) \to (Y, X)$$

representing an element x of $\mathscr{S}^B(Y, g)$, we choose a (stable) bundle ξ so that there is a bundle map $b: \nu(W) \to \xi$ and let $\eta(x)$ be represented by (h, b). Theorem 4.1 asserts that $\eta(x)$ always has an S-splittable representative.

We also need to introduce the Wall group $L^B_{4k+2}(\pi, \omega)$. This is the *reduced* Grothendieck group of triples (K, φ, μ); K is a stably free and stably B-based left module over Λ (i.e., a stable class of basis is assumed given up to equivalence with torsion in B); φ and μ are as in [§4, W2] and satisfy [P2]–[P6] of [W3, p. 236] for the right module structure on K given by $x\lambda = \bar{\lambda}x$ (and with $\eta = -1$); and $A\varphi: K \to \mathrm{Hom}_\Lambda(K; \Lambda)$ given by $A\varphi(x)(y) = \varphi(y, x)$ has vanishing torsion modulo B with respect to the given class of stable bases and its dual (note that the torsion of $A\varphi$ is only defined modulo B). The *reduction* is accomplished by requiring *kernels* to represent zero; a kernel is defined as in [W2, 4.5], with the additional proviso that the basis exhibiting a triple as a kernel must be in the given class of stable bases.

For $B=\{0\}$ this is just the definition of [W3] for $L^S_{4k+2}(\pi, \omega)$; for $B=\mathrm{Wh}(\pi)$ it is essentially the definition of $L_6(\pi, \omega)$ in [W2]. In particular, the analogue of [W3, Thm. 5.8] holds, as well as periodicity under products with CP^2 [W3, §9]. Hence we may define the action of $L_6(\pi_1 Y)$ on $\mathscr{S}^B(W, g)$ as follows:

Let $h:(W, \partial W)\to(Y, X)$ representing $x\in\mathscr{S}^B(W, g)$ and $\gamma\in L^B_6(\pi_1 Y)$ be given. Let $b:\nu(W)\to\xi$ be a bundle map, as above, covering h. Then $\exists$ a normal cobordism (H, B), relative the boundary, of (h, b) to (h', b') with h' a B-homotopy equivalence and $\sigma(H, B)=\gamma$. We define

$$\gamma\cdot x=[h', b'].$$

It is not hard to check that this is a well-defined action (in particular $(\gamma+\gamma')\cdot x=\gamma\cdot(\gamma'\cdot x)$) and that $\eta(x)=\eta(y)$ if and only if x and y are in the same orbit.

THEOREM 5.1. *Let $(Y, X)\supset(V, U)$ be as in Theorem 4.1. Let $h:(W, \partial W)\to(Y, X)$ be an algebraically splittable homotopy equivalence (i.e., a B-homotopy equivalence) with $h \mid \partial W$ already split along U. Then h is S-splittable along V.*

Note. The hypothesis (iii) from §4, algebraic two-sidedness, appears here only because it appears in the splitting theorems of [C]. If it can be eliminated there, it can be eliminated here.

Proof of 5.1. Let $g=h \mid \partial W$, and let B be as in §4. Then, using the existence and uniqueness theorems for 6-dimensional h-cobordisms, relative the boundary, in terms of their Whitehead torsion (see [M1, 11.1 and 11.3], [S5], and [K2]), it is not hard to show that every representative of an element of $\mathscr{S}^B(Y, g)$ is S-splittable if and only if some representative is S-splittable.

Thus, in view of Theorem 4.1 and the preceding discussion, it suffices to study the action of $L^B_6(\pi_1 Y)$ on an element of $\mathscr{S}^B(Y, g)$ represented by an S-splittable homotopy equivalence

$$k:(W_1, \partial W_1)\to(Y, X)$$

with ∂W_1 diffeomorphic to ∂W via a diffeomorphism φ with $(h \mid \partial W)\circ\varphi=k \mid \partial W_1$. So we may as well assume (after a change of notation for the fibre) that k is actually split along V, as well as along the various fibres we shall obtain from V by exchanging trivial 2-handles as necessary.

Case I. Suppose $Y-V$ is connected. Let $K=\pi_1 Y$, $H=\pi_1 V$, $G=\pi_1(Y-V)$, and let $\nu_+, \nu_-:H\to G$ be induced as in §4, by pushing V in the positive and negative directions. We also denote by ν_+ and ν_- the respective maps induced on Wall groups. By [C] (or essentially, by [S1], in case $K=\mathbf{Z}\times H$), and by periodicity of the Wall groups, there is an exact sequence

$$(5.2)\quad L^h_6(G)\xrightarrow{j_*}L^B_6(K)\xrightarrow{\alpha}L^h_5(H)\xrightarrow{\nu_+-\nu_-}L^h_5(G).$$

Here j_* is induced by inclusion. Again we omit mention of the orientation homomorphism.

The homomorphism α can be defined as follows:

Let $\gamma \in L_6^B(K) = L_{10}^B(K)$. Let

$$\begin{array}{ccc} \nu(Q) & \xrightarrow{B} & \xi \\ \downarrow & & \downarrow \\ (Q^{10}, \partial Q) & \xrightarrow{F} & (Y, X) \times CP^2 \times I \end{array}$$

be a normal map so that

$$\partial Q = \partial_- Q \cup \partial(\partial_- Q) \times I \cup \partial_+ Q$$

(with $\partial(\partial_+ Q) = \partial(\partial_- Q) \times 1$), with

$$F \mid \partial_- Q : (\partial_- Q, \partial(\partial_- Q)) \to (Y, X) \times CP^2 \times 0 \quad \text{and}$$
$$F \mid \partial_+ Q : (\partial_+ Q, \partial(\partial_+ Q)) \to (Y, X) \times CP^2 \times 1$$

split along the pair $(V, U) \times CP^2$, i.e., split along $V \times CP^2$ and on the boundary along $U \times CP^2$; with

$$F \mid (\partial(\partial_- Q) \times I)) = (F \mid \partial(\partial_- Q)) \times \mathrm{id}_I ;$$

and with $\sigma(F, B) = \gamma$. (CP^2 = complex projective 2-space.) Such a normal map exists by the realization theorem ([W3, 5.8] for $B = \{0\}$) for surgery obstructions in dimensions at least six, by the splitting theorem of [C], and because we have assumed the existence of some split homotopy equivalence of a manifold pair with (Y, X), namely k. We may also suppose F is transverse to $V \times I$. Let Ξ be the following normal map:

$$\begin{array}{ccc} \nu(F^{-1}(V \times CP^2 \times I)) & \xrightarrow{\bar{B}} & \xi \mid V \times CP^2 \times I \\ \downarrow & & \downarrow \\ F^{-1}(V \times CP^2 \times I) & \xrightarrow{\bar{F}} & V \times CP^2 \times I, \end{array}$$

where $\bar{F}$ and $\bar{B}$ are restrictions of F and B, respectively. Then

$$\alpha(\gamma) = \sigma(\Xi) \in L_9(H) = L_5(H).$$

(The proof that α is well-defined seems to use [W3, §9] as well as the geometric splitting theorem of [C]. It seems that in the present case, by some extra effort, we might avoid any appeal to [W3, §9].)

Returning to the situation at hand, let us study the action of $\gamma \in L_6^B(K)$ on the element of $\mathscr{S}^B(Y, g)$ represented by $k:(W_1, \partial W_1) \to (Y, X)$. Choose a bundle map $c: \nu(W_1) \to \xi$ covering k. Let $(\bar{f}, \bar{b})$ be a normal cobordism, relative the boundary, of $(k \mid k^{-1}V, c \mid k^{-1}V) \# r(S^2 \times S^2)$ to another homotopy equivalence, with $\sigma(\bar{f}, \bar{b}) = \alpha(\gamma)$. For r sufficiently large, this exists by Theorem 3.1.

Let $\bar{T}$ be the domain of $\bar{f}$, i.e.,

$$\bar{f}:\bar{T}\to(V \# r(S^2\times S^2))\times I.$$

Let $P\subset Y$ be obtained from V by exchanging r trivial 2-handles on the right. By the "cobordism extension theorem", $\exists$ a normal cobordism (f_1, b_1), relative the boundary, of (k, c) to (f_+, b_+), so that $(f_1)^{-1}(P\times I)=\bar{T}$, $f_1 \mid \bar{T}=\bar{f}$, and $b_1 \mid \bar{T}=\bar{b}$. So $f_1: T_1\to \to Y\times I$, with $\partial_- T_1 = W_1$, $f_1 \mid \partial_- T_1 = k$, $f_1 \mid \partial_+ T_1 = f_+$, and $b_1 \mid \partial_+ T_1 = b_+$.

By the same argument as was used to derive formula (4.2) in the proof of Theorem 4.1, and using the same notation, we have (up to sign) the following in $L_5(G)$:

$$\sigma((f_+)_P, (b_+)_P) = \nu_+(\alpha(\gamma)) - \nu_-(\alpha(\gamma)).$$

(In (4.2) there was another term on the right, corresponding here to $\sigma(k_P, c_P)=0$.) By exactness of (5.2), the right side vanishes. Hence we may perform surgery *relative the boundary* on $((f_+)_P, (b_+)_P)$ to get a homotopy equivalence and glue up the resulting normal cobordism along the appropriate parts of the boundaries to get a normal cobordism (f_2, b_2) of (f_+, b_+) to (k_2, c_2) with k_2 a split homotopy equivalence along P. (Compare [S1, Lemma 5.3].) Let T_2 be the domain of f_2; we view f_2 as a map from T_2 to $Y\times[1, 2]$ with

$$(k_2, 2) = f_2 \mid \partial_+ T_2 : \partial_+ T_2 \to Y\times 2.$$

Let $T=T_1\cup_{\partial_+ T_1} T_2$, and let $(f, b)=(f_1, b_1)\cup(f_2, b_2)$;

$$f: T\to Y\times[0, 2],$$

and (f, b) is a normal cobordism relative the boundary from (k, c) to (k_2, c_2). We have

$$\alpha(\sigma(f, b)) = \alpha(\sigma(f, b)\times CP^2).$$

By definition of α, the right side is just $\sigma((\bar{f}, \bar{b})\times CP^2)$, which by construction and periodicity is just $\alpha(\gamma)$. So $\alpha(\sigma(f, b))=\alpha(\gamma)$. Let $\tau=\gamma-\sigma(f, b)$. Then

$$\gamma\cdot[k] = (\tau+\sigma(f, b))\cdot[k] = \tau\cdot(\sigma(f, b)\cdot[k]) = \tau\cdot[k_2];$$

here $[k_2]$, for example, denotes the class of k_2 in $\mathscr{S}^B(Y, g)$. So it suffices to show that $\tau[k_2]$ has a split representative along P.

Now, $\alpha(\tau)=0$. So, by the exact sequence (5.2), $\tau=j_*\omega$, $\omega\in L_6(G)$. Let

$$(k_2)_P: (\partial_+ T)_{(k_2)^{-1}P}\to Y_P$$

be obtained from k_2 by splitting along P and $(k_2)^{-1}P$. A representative of $\tau[k_2]$ can be obtained by first constructing a (six-dimensional) normal cobordism Ξ_1, *relative the boundary*, of $(k_2)_P$ to another homotopy equivalence, with $\sigma(\Xi_1)=\omega$, and then gluing up along the portions of the boundary corresponding to $P\times I$ and $(k_2^{-1}P)\times I$. By

(4.3), if Ξ is the resulting normal cobordism, $\sigma(\Xi)=j_*\omega=\tau$. Clearly Ξ is a normal cobordism from (k_2, c_2) to (k_3, c_3), where k_3 is split along P. This completes the proof of Case I.

(One could also carry out the last part of the argument by appealing to the "local character" of surgery obstructions; this says that since τ comes from $L_6(G)$, we can construct a normal cobordism to obtain this element, *relative the complement* of a regular neighborhood of a subcomplex whose fundamental group maps isomorphically to G under the inclusion induced map. In particular, we can construct τ without disturbing our codimension one submanifolds.)

Case II. $Y-V=Y_1\cup Y_2$ has two components. Let $K=\pi_1 V$, $C_i=\pi_1 Y_i$, $i=1,2$. Then the proof proceeds in a fashion analogous to the proof for Case I. We use the sequence [C]

$$L_6^h(G_1)\oplus L_6^h(G_2)\xrightarrow{\varepsilon} L_6^B(K)\xrightarrow{\alpha} L_5^h(H)\xrightarrow{\nu} L_5^h(G_1)\oplus L_5^h(G_2),$$

where $\varepsilon(x,y)=(j_1)_*\,x+(j_2)_*\,(y)$, j_i inclusion maps, $\nu(x)=(\nu_+(x), -\nu_-(x))$, and α is defined geometrically using periodicity and codimension one splitting theorems of [C]. We leave the details to the reader.

Appendix: More Diffeomorphisms of Four-Manifolds

We return to the notation of §1. In particular, M is a smooth, compact, connected four manifold, and $\Lambda=\mathbf{Z}[\pi_1 M]$. We assume $M=P\ \#\ (S^2\times S^2)$. Suppose

$$H_2(M;\Lambda)=L\oplus K_0$$

is a direct sum decomposition of Λ-modules, orthogonal with respect to the intersection paring, so that K_0 has a basis $e_2,\dots,e_r, f_2,\dots,f_r$ with $e_i\cdot e_j=f_i\cdot f_j=0$ and $e_i\cdot f_j=\delta_{ij}$, $2\leqslant i, j\leqslant r$. This implies that the 2nd Stiefel-Whitney class vanishes on K_0 because the Euler classes of the normal bundles of immersions representing the above basis will varnish. Hence μ is defined on K_0; we assume that $\mu(e_i)=\mu(f_i)=0$, $2\leqslant i\leqslant r$.

Let $N=M\ \#\ (S^2\times S^2)$. Let e_1 and f_1 be the classes represented by the first second spheres in the second summand. Then

$$H_2(N;\Lambda)=L\oplus K,\quad \text{where}\quad K=K_0\oplus\{e_1,f_1\}_\Lambda;$$

these are orthogonal direct sums, $\mu(e_1)=\mu(f_1)=0$, and $e_1\cdot f_1=1$.

LEMMA A.1. $\exists$ *diffeomorphisms* $\sigma^i:N\to N$, $2\leqslant i\leqslant r$, *preserving basepoint and inducing the identity on* $\pi_1 N$, *so that* $(\sigma^i)_*(e_1)=e_i$, $(\sigma^i)_*(f_1)=f_i$, $(\sigma^i)_*(e_i)=e_1$, $(\sigma^i)_*(f_i)=f_1$; $(\sigma^i)_*(e_j)=e_j$ *and* $(\sigma^i)_*(f_j)=f_j$ *for* $j\neq 1$, i, *and* $(\sigma^i)_*\mid L=$identity.

Proof. Let φ and ψ be basepoint preserving diffeomorphisms of N which induce the identity on $\pi_1 N$ and satisfy the following: $\varphi_*(e_1)=e_1+e_i$, $\varphi_*(f_1)=f_1$, and $\varphi_*(\xi)=\xi-(\xi\cdot e_i)f_1$ for $\xi\in L\oplus K_0$; and $\psi_*(e_1)=e_1$, $\psi_*(f_1)=f_1+f_i$, $\psi_*(\xi)=\xi-$ $-(\xi\cdot f_i)\, e_1$ for $\xi\in L\oplus K_0$. These diffeomorphisms exist by Theorem 1.5. Let $\delta=\mathrm{id}_M\ \#$ $\#\ (a\times a)$, where a is a diffeomorphism of S^2 of degree -1. Then $\sigma^i=\delta\varphi\psi\varphi$ is the desired diffeomorphism.

Let $U_r(\Lambda)$ denote the automorphisms of K which preserve the intersection and μ-forms. (This is consistent with the notation of §3.) Let E_{ij} denote the $r\times r$ matrix with the single entry 1 in the (i,j)th position. Let $I_r=I$ be the identity $r\times r$ matrix. Let $SL_r(\Lambda)$ denote the subgroup of the automorphism group of the free Λ-module $\{e_1,\ldots,e_r\}_\Lambda$ generated by automorphisms with the following matrices with respect to the basis $e_1,\ldots,e_r$:

$$I+\lambda E_{ij},\quad \lambda\in\Lambda,\quad 1\leqslant i,\quad j\leqslant r,\quad i\neq j\,; \tag{1}$$

and

$$\begin{pmatrix} \pm g & & & \\ & 1 & & 0 \\ & & \ddots & \\ & 0 & & 1 \end{pmatrix}. \tag{2}$$

Let $SLU_r(\Lambda)\subset U_r(\Lambda)$ denote the subgroup of automorphisms that preserve $\{e_1,\ldots,e_r\}_\Lambda$ and induce on it an element of $SL_r(\Lambda)$. Let $RLU_r(\Lambda)$ be the group generated by $SLU_r(\Lambda)$ and σ, where $\sigma(e_1)=f_1$, $\sigma(f_1)=e_1$, and $\sigma\mid K_0=$identity.

THEOREM A.2 (Compare example 1.6). *Let $\alpha\in RLU_r(\Lambda)$. Then $\exists$ a diffeomorphism $\varphi: N\to N$, preserving the basepoint and inducing the identity on $\pi_1 N$, so that on $H_2(N;\Lambda)$,*

$$\varphi_*=(\mathrm{id}_L)\oplus\alpha\,.$$

Proof. It is obvious how to realize $(\mathrm{id}_L)\oplus\sigma$.

Let $UU_r(\Lambda)\subset U_r(\Lambda)$ be those elements which restrict to the identity on $\{e_1,\ldots,e_r\}_\Lambda$. Then there is a split exact sequence:

$$1\to UU_r(\Lambda)\to SLU_r(\Lambda)\overset{R}{\to} SL_r(\Lambda)\to 1\,;$$

R denotes restriction and is split by the following homomorphism H: if $\beta\in SL_r(\Lambda)$ is represented by the matrix $A=((a_{ij}))$ over Λ, then $H(A)$ is represented by

$$\begin{pmatrix} A & O \\ O & (A^*)^{-1}\end{pmatrix},$$

with respect to the basis $\{e_1,\ldots,f_r\}$. Here $A^*=(\overline{(a_{ji})})$, where $^-:\Lambda\to\Lambda$ is the involution in §1.

So it suffices to consider separately the cases $\alpha = H(\beta)$ and $\alpha \in UU_r(\Lambda)$. For the case $\alpha = H(\beta)$, we must consider types (1) and (2), above. For (2), *Theorem* A.2 is just Lemma 1.3. So suppose

$$A = I + \lambda E_{ij},$$

$i \neq j$, $\lambda \in \Lambda$, is the matrix for α. We can also assume $i = 1$, since if $i \neq 1$ we have

$$\alpha = (\sigma^i)_* \, \alpha_1 \, (\sigma^i)_*$$

where α_1 has matrix $I + \lambda E_{1j}$, and σ^i is as in Lemma A.1. So we have $\alpha(e_1) = e_1 + \lambda e_j$, $\alpha(e_i) = e_i$, $2 \leqslant i \leqslant r$, $\alpha(f_i) = f_i$ for $i \neq j$, $\alpha(f_j) = f_j - \bar{\lambda} f_1$. But for this α, the existence of the required diffeomorphism clearly follows from Theorem 1.5.

Now, with respect to the basis $\{e_1, \ldots, f_r\}$, elements of $UU_r(\Lambda)$ have the form

$$\begin{pmatrix} I & O \\ C & I \end{pmatrix}$$

where C is an $(r \times r)$ matrix $((c_{ij}))$ with $c_{ij} = -\overline{c_{ji}}$ for $i \neq j$ and $c_{ii} = d_i - \bar{d}_i$ for some $d \in \Lambda$. Furthermore

$$\begin{pmatrix} I & O \\ C & I \end{pmatrix} \begin{pmatrix} I & O \\ C' & I \end{pmatrix} = \begin{pmatrix} I & O \\ C + C' & I \end{pmatrix};$$

i.e., composition of automorphisms in $U_r(\Lambda)$ corresponds to addition of matrices. So it suffices to prove our result for the case where C has either two off-diagonal non-zero entries, all other entries being zero, or only one non-zero entry, on the diagonal. In the latter case, the result follows by conjugating a diffeomorphism given by Lemma 1.4 with a suitable σ^i. If $c_{kl} = -\bar{c}_{lk}$, $l \neq k$, are the only non-zero entries, by conjugation with σ^k we may assume $k = 1$, in which case it is an easy consequence of *Theorem* 1.5, interchanging the roles of e_1 and f_1, that the required diffeomorphism exists.

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Received January 7, 1971