Contemporary Mathematics
Volume 35, 1984

LINKING NUMBERS IN BRANCHED COVERS

Sylvain E. Cappell* and Julius L. Shaneson*

INTRODUCTION

Let $\alpha: S^1 \to N^3$ be a knot in a 3-dimensional manifold and let $f: \hat{N} \to N$ denote a branched covering space of N branched along α. This note sketches a method based on a 4-dimensional construction for studying invariants of $\hat{N}$ and of the branch set $f^{-1}(\alpha) \subset N$. Our method gives a way of relating a noncyclic branched cover of α to a branched cyclic cover of a different associated knot β, which we call a characteristic knot for α. Here our results will be discussed only for $N = S^3$ and f an (irregular) dihedral branched covering set; the invariant studied in the present note will be the linking numbers of the components of the branch set $f^{-1}(\alpha)$. The method can be used to study other invariants, or other branched covers, as well. The 4-dimensional construction itself was announced and described some 10 years ago in [CS2].

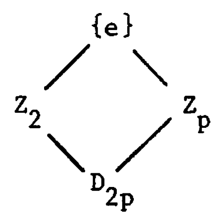
The particular interest of dihedral covering space lies in their extraordinary simplicity and generality. Classically, it was studied as the simplest "non-abelian" cover of a knot and thus gave rise to the simplest "non-abelian" (i.e. not obtained from the cyclic covers) invariants of knots [Re]. More recently, M. Hilden and J. Montesinos showed that every oriented 3-manifold is such a 3-fold dihedral branched covering space of S^3 branched along a knot [Hi], [Mo]. In [CS2] we announced a formula for the Rohlin μ-invariant of any mod 2 3-dimensional homology sphere presented as a 3-fold dihedral covering space. That formula, in terms of various linking numbers, could be extended to all dihedral covers, provided that a certain conjecture on the linking numbers of the components of branch sets, a conjecture apparently long familiar to students of this subject, were verified. That conjecture is the theorem of the present note.

Our study of Rohlin μ-invariants of dihedral branched covers will be presented elsewhere. For certain special classes of knots, e.g. ribbon knots,

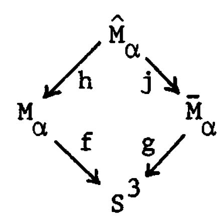
*Work supported by NSF Grants

this formula simplifies. As we noted in [CS2] this can be used to show that various (algebraically slice) knots are not ribbon. As noted in [CS2] these methods can also be used to compute Atiyah-Singer invariants used by Casson and Gordon [CG] in their study of ribbon and slice knots. An extensive study of that has been made by Litherland [Li].

Precisely, let $\alpha: S^1 \to S^3$ be a knot and $\rho: G \to D_{2p}$ a homomorphism of the knot group $G = \pi_1(S^3 - \alpha(S^1))$ onto the dihedral group of order $2p$, p odd. The p-fold irregular (respectively: regular) dihedral cover of α is the branched cover of S^3, branched along α, associated to the subgroup $\rho^{-1}(Z_2)$ (resp., $\rho^{-1}(e)$) of G, for $e \in Z_2 \subset D_{2p}$. Let $f: M_\alpha \to S^3$ (resp., $\hat{f}: \hat{M}_\alpha \to S^3$) denote this covering space of degree p (resp. $2p$). Consideration of the diamond of subgroups of D_{2p}



gives a corresponding diamond of covering spaces,



degree $h = 2$
degree $f = p$
degree $j = p$
degree $g = 2$
$\bar{f} = f \ h$

where $\bar{M}_\alpha \to S^3$ is the 2-fold cyclic cover of S^3 branched along α, $\hat{M}_\alpha \to \bar{M}_\alpha$ is a p-fold cyclic unbranched covering space, M_α is the quotient of a lift to $\hat{M}_\alpha$ of the covering translation of period 2 of $\bar{M}_\alpha$.[1]

[1] Classically, one sees from this that dihedral covers of S^3 branched along α correspond to elements of order p in $H^1(\bar{M}_\alpha;\mathbb{Z}_p)$ producing $\pi_1(\bar{M}_\alpha) \to \mathbb{Z}_p$ and the associated cover $\hat{M}_\alpha \to \bar{M}$. Recall that the order of $H_1(\bar{M}_\alpha;\mathbb{Z})$ is just $\Delta_\alpha(-1)$, for $\Delta_\alpha(t)$ the Alexander polynomial of α. (A conceptual explanation of $|H_1(\bar{M}_\alpha;\mathbb{Z})| = |\Delta(-1)|$ was provided using 4-manifolds in [CS1].) Thus one concludes classically that for p odd and square-free, α has a p-fold dihedral covering space if and only if $\Delta_\alpha(-1) \equiv 0$ (mod p); for p prime there is a unique such cover if $\Delta_\alpha(-1) \not\equiv 0$ (mod p^2) (cf. [F1]).

From this, we read off easily a description of the branch set, the inverse image of α, in each of these covers. Clearly $g^{-1}(\alpha)$ is a single circle of branching index 2. Hence, $\bar{f}^{-1}(\alpha) = j^{-1}g^{-1}(\alpha)$ consists of p circles $\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_{p-1}$ each of branching index 2; here these circles are indexed by the convention $T^i\alpha_0 = \alpha_i$ for T a fixed choice of a generator of the covering translation group Z_p of the map $j\colon \hat{M}_\alpha \to \bar{M}_\alpha$. The covering translation of period 2, $\phi\colon \hat{M}_\alpha \to \hat{M}_\alpha$, is associated to the 2-fold covering space $h\colon \hat{M}_\alpha \to M_\alpha$. Notice that ϕ and T are just generators for the dihedral group D_{2p} acting as covering translation on $\hat{M}_\alpha$; the action of D_{2p} on the components of $\bar{f}^{-1}(\alpha)$ is equivalent to that of D_{2p} on the p verticies of a polygon with p sides. As $M_\alpha = \hat{M}_\alpha$/action of ϕ, in M_α, $f^{-1}(\alpha)$ consists of one circle α_0, of branching index 1, and $(p-1)/2$ circles of branching index 2, $\alpha_1, \ldots, \alpha_{(p-1)/2}$; thus, $\hat{M}_\alpha$ can be viewed as a 2-fold covering space of M_α branched along α_0 with $h^{-1}(\alpha_i) = \hat{\alpha}_i \cup \hat{\alpha}_{p-1}$, $1 \leq i \leq (p-1)/2$.

Fixing an orientation for S^3 and α, the covers M_α and $\hat{M}_\alpha$ are correspondingly oriented, as are the branch curves α_i and $\hat{\alpha}_i$. When M_α is a rational homology sphere, let $v_{i,j}$ denote the linking number of α_i with α_j, $i \neq j$, $0 \leq i,j \leq (p-1)/2$; when M_α is a mod 2 homology sphere these $v_{i,j}$ are rational numbers with odd denominator.

The study of the behavior of these numbers is one of the oldest topics in topology. This is partially because these are the simplest "non-abelian invariants" that can be used to distinguish knots. Calculations of them for this purpose were used by Reidemeister [Re]. An early paper of Bankwitz and Schumann [BS] stated that if α is a 2-bridge knot, then $v_{0,i} = \pm 2$; their proof is difficult to reconstruct; clear and more precise modern proofs of this were given by Perko [Pe1] and by Burde [B1]. Note that if α is a 2-bridge knot, by considering its Heegard genus it is easy to show that then M_α is actually S^3 [B2]. While it is not hard to develop methods for calculating the $v_{i,j}$ (cf. [Re], [F3]) really efficient general algorithms were developed by K. Perko [Pe2] and further studied by Hartley and Murasugi [HM].

The following was perhaps conjectured by everyone who has thought about linking numbers in branched covers; it generalizes to all knots the classical result for 2-bridge knots and is suggested by calculating examples. It is, moreover, needed in understanding other invariants (e.g. μ-invariants) of branched covers.

THEOREM I. <u>If the</u> p-<u>fold dihedral branched covering space</u> M_α <u>is a</u> mod 2 <u>homology sphere, then the linking numbers of the branch curves satisfy</u>
$v_{i,0} \equiv 2 \pmod 4$, $1 \leq i \leq (p-1)/2$ and $v_{i,j} \equiv 0 \pmod 2$, $1 \leq i,j \leq (p-1)/2$, $i \neq j$.

Counterexamples to the converse of this theorem are provided, according to calculations of Ken Perko, by some 10-crossing knots with $p=3$ [Pe1].

Actually, as noted by Perko, the numbers $v_{i,0}$, $1 \le i \le (p-1)/2$ determine all the $v_{i,j}$. This follows readily from the following transfer argument. First of all, note that as $\hat{M}_\alpha$ is a 2-fold branched cyclic cover of M_α, $\hat{M}_\alpha$ is a mod 2 homology sphere if and only if M_α is. (The homology of a 2-fold cyclic branched cover is given by the Alexander polynomial at (-1); cf. [CS1].) Let u_j denote the linking number of $\hat{\alpha}_i$ with $\hat{\alpha}_{i+j}$, $1 \le j \le p-1$; this is independent of i, $0 \le i \le p-1$, as the $\hat{\alpha}_i$ are permuted by the covering translations. For the same reason, $u_i = u_{p-i}$.

Standard transfer considerations show, as noted by Perko [Pe1] that:

$$v_{i,j} = u_{i+j} + u_{|i-j|} \ , \ 0 \le i,j \le \frac{p-1}{2} \ , \ i \ne j$$

and, in particular, $v_{i,0} = 2u_i$, so that

$$v_{i,j} = \frac{1}{2}\left(v_{\min(i+j,p-i-j),0} + v_{|i-j|,0}\right) \ ; \ 1 \le i,j \le \frac{p-1}{2} \ .$$

Hence the numbers $v_{i,0}$ determine all the $v_{i,j}$ and the main theorem of this note will follow from:

THEOREM II. _If the 2p-fold (regular) dihedral branched covering space $\hat{M}_\alpha$ is a mod 2 homology sphere, then the linking numbers of the branch curves satisfy_

$$u_i \equiv 1 \ (\text{mod} 2) \ , \quad 1 \le i \le (p-1) \ .$$

Outline of Method

Here is a summary of our approach to this and related problems on branch covers.

Step 1. An effective method for studying 3-manifolds M described as branched covers of S^3 along a knot α is to utilize a 4-manifold W^4, with $\partial W = M$ obtained by letting W be a branched cover of D^4 along K^2, where $K \cap S^3 = \alpha(S^1)$. It is easy to do this for cyclic covers; just let K be a Seifert surface of α pushed into D^4; this method of studying cyclic covers was introduced by us in [CS2] and independently by L. Kauffman. However, it will not work for more general branched covers as the fundamental group of D^4-{pushed in Seifert surface} is Z and thus has no nonabelian covers.

For noncylcic covers, we employ instead for K a certain (non-manifold) 2-complex. This works at least for all metacyclic covers; in particular, for dihedral covers the resulting W^4 is a manifold even though K^2 is not. (In other settings, the singularity which arises is readily understood and can be resolved.)

Step 2. We relate questions about linking numbers of branch curves in $M^3 = \partial W^4$ to intersection numbers of parts of 2-dimensional surfaces in the branch set of W^4.

Step 3. We get information on these intersection numbers by relating these 2-dimensional surfaces to a kind of equivariant second Stiefel-Whitney class of W^4 and then get our result from an equivariant version of the standard fact that in an oriented 4-manifold, w_2^2 is just the Euler characteristic mod2.

Of course, this method can be used to study many other invariants of such branched covers. An interesting way to view the geometrical procedure outlined in Step 1, and carried out in Section 1 below for dihedral covers, is that it reveals a close relationship between a dihedral (or metacyclic) cover of S^3 branched along α and a cyclic cover of a characteristic knot β associated below to α.

1. Characteristic knots and a cobordism construction.

Fix an orientation of S^3 and adopt the unique conventions so that the circles in Figure 1 have linking number +1.

Fig. 1

If α is a (smooth or P.L. locally flat) knot in S^3, let $\Delta_\alpha(t)$ denote its Alexander polynomial.

Definition. Let α and β be (oriented) knots in S^3. Then β is called a mod p characteristic knot for α if there exists an oriented Seifert surface of α, V, $\partial V = \alpha$, so that $\beta \subset \mathring{V}$ represents a nonzero (primitive) class $[\beta]$ of $H_1(V)$ and so that

$$(L_V + L_V')\beta \equiv 0 \pmod p .$$

L the linking pairing of V in S^3. More precisely, $L_V(x,y) = \ell(f_+x,y)$, where f_+ is induced by pushing V off itself using a positive normal, and ℓ denotes linking numbers and $L_V(x,\beta) + L_V(\beta,x) \equiv 0 \pmod p$, all $x \in H_1(V)$.

Note: If α is a nontrivial knot with Seifert surface V with p square-free, and if $p|\Delta_\alpha(-1)$, then α has a mod p characteristic knot embedded in V. (Proof: Note that $\Delta_\alpha(-1) = \pm\det(L_V + L_V')$ and use the well-known fact that a primitive class in $H_1(V)$ is represented by an embedded circle.)

Suppose α is a knot with Seifert surface V and $\beta \subset \mathring{V}$ is a mod p characteristic knot of α. We proceed to construct a cobordism relating the dihedral covering spaces of S^3 with branch sets α to the cylcic cover of S^3 with branch set β.

Let $\pi: \Sigma(\beta,p) \to S^3$ be the p-fold cyclic branched cover of S^3, branched along β. If $x \in H_1(V-\beta)$, then the intersection number on V, $x\cdot\beta = 0$; hence

$$L_V(x,\beta) - L_V(\beta,x) = x(L_V - L_V')\beta = 0 .$$

Since $(L+L')\beta \equiv 0 \pmod p$, it follows that $2L_V(\beta,x) \equiv 0 \pmod p$. Since $\det(L+L;) \equiv \det(L-L') \pmod 2$, and since $\det(L-L') = \pm 1$ by Poincare duality, p is odd. Hence $L_V(\beta,x) \equiv 0 \pmod p$. Therefore

$$\pi^{-1}(V) = V_0 \cup V_1 \cup \ldots \cup V_{p-1} ,$$

$\pi|V_i : V_i \to V$ a P.L. homeomorphism and

$$V_i \cap V_j = \pi^{-1}(\beta) \quad , \quad i \neq j .$$

Let $T: \Sigma \to \Sigma$ be a generator of the group of covering translations corresponding to a positively oriented meridional circle of β in S^3 (i.e. $T|$fiber of a neighborhood of $\pi^{-1}\beta$ is rotation by $2\pi/p$). Assume the indices have been chosen so that $TV_i = V_{i+1}$, $0 \leq i \leq p-2$, and $TV_{p-1} = V_0$.

Let $V \times [-1,1] \subset S^3$ be a neighborhood of $V = V \times 0$, and let $h(x,t) = (x,-t)$ for $x \in V$ and $t \in [-1,1]$. Then

$$\pi^{-1}(V \times [-1,1]) = J_0 \cup \ldots \cup J_{p-1} = J ,$$

with $\pi|J_i: J_i \to V \times [-1,1]$ a P.L. homeomorphism and with $V_i \subset J_i$. Clearly, $\pi^{-1}(V \times [-1,1]) = J$ is the p-fold branched cyclic cover of $V \times [-1,1]$ along β. Let

$$\bar{h}: J \to J$$

be a lift of h, i.e. $\pi\bar{h} = h \ (\pi|J)$, with $\bar{h}(V_0) \subset V_0$. Then $\bar{h}(J_i) \subset J_{p-i}$, $1 \leq i \leq p-1$, $\bar{h}(J_0) = J_0$, and $\bar{h}$ fixes precisely V_0.

Let $\Sigma = \Sigma(\beta,p)$ and let

$$Y = \Sigma \times [0,1] \,/\, \{(x,1) = (\bar{h}(x),1) \text{ for } x \in J\}$$

the space obtained by identifying $(x,1)$ and $(\bar{h}(x),1)$ in $\Sigma \times I$. Let $\pi': Y \to S^3 \times I/\{(x,t) = (x,t) = (x,-t),\ x \in V\} \cong S^3 \times I$ be induced by $\pi \times 1_{[0,1]}$. Y is evidently an orientable (smooth) cobordism of Σ to a closed manifold, $M_{\alpha,\beta}$, say, and π' is a branched covering projection with branching set B the image of $V \times 0 \cup \beta \times I$ in $S^3 \times I/\{(x,t) = (x,-t),\ x \in V\}$, which is canonically P.L. homeomorphic to V. Orient Y so that π' has positive degree. (Usual convention: $\partial[S^3 \times I] = [S^3 \times 1] - [S^3 \times 0]$, and thus $\partial Y = [M] - [\Sigma]$.)

Clearly the tuple

$$(S^3 - \mathrm{Int}(V \times [-1,1]), \partial(V \times [-1,1]), \alpha)/\{(x,t) = (x,t) \mid (x,t \in \partial(V \times [-1,1])\}$$

is canonically P.L. homeomorphic to (S^3,V,α). Hence the restriction ω of π' to $M_{\alpha,\beta}$ is a branched covering of S^3 along α. Note that $\omega^{-1}(\alpha)$ has $(p+1)/2$ components, with branching index 2 on $(p-1)/2$ of them. In fact, if V_i' denotes the image of $V_i \times 1$ in Y (so $V_i' = V_{p-1}'$, $1 \le i \le p-1$), then $\omega^{-1}(\alpha) = \partial V_0' \cup \ldots \cup V_{(p-1)/2}'$, and $\partial V_0'$ is the component with branching index 1. Write $\alpha_i = \partial V_i'$, $0 \le i \le \frac{p-1}{2}$.

<u>Proposition 1.1</u>. $M_{\alpha,\beta} \xrightarrow{\omega} S^3$ <u>is a dihedral metacylcic, branched covering space of</u> S^3 <u>along</u> α.

<u>Proof</u>: Let $D_p = \{u,\tau \mid \tau^2 = 1;\ u^p = 1,\ \tau u = u^{-1}\tau\}$. The group $\pi_1(S^3-\alpha)$ has the form (Higman-Neumann-Neumann construction)

$$Z * G/\{ti_+(x)t^{-1} = i_-(x),\ x \in H\}$$

where G is the fundamental group of $S^3 - V$, H that of V, t is a generator of the infinite cyclic group, represented by a meridian m of α, and i_+ and i_- are induced by pushing V into its complement along positive and negative normal vectors, respectively.

Define $\rho: G \to D_p$ by

$$\rho(\xi) = u^{\ell(\xi,\beta)},$$

and let $\rho(t) = \tau$. Since $L_V(x,\beta) \equiv -L_V(\beta,x)$ (modp), these definitions determine a homeomorphism

$$\rho: \pi_1(S^3 - \alpha) \to D_p .$$

Assuming $M_{\alpha,\beta}$ is connected, the fundamental group of the unbranched covering $M_{\alpha,\beta} - \omega^{-1}(\alpha)$ also has an (HNN)-representation. In particular, using Van Kampen's theorem (and a base point near α_0), $\pi_1(M_{\alpha,\beta}-\omega^{-1}(\alpha))$ is generated by a meridian m_0 of α_0 with $\omega(m_0) = m$ and elements in the image of $\pi_1(M_{\alpha,\beta} - \omega^{-1}(V))$. Let $\omega' = \omega | M_{\alpha,\beta}-\omega^{-1}(\alpha)$. Since, by construction, $\omega | M_{\alpha,\beta}-\omega^{-1}V$ is the cylcic cover $\pi|\Sigma-\pi^{-1}(\mathrm{Int}\,V \times [-1,1])$, of $S^3 - \mathrm{Int}(V\times[-1,1]) = S^3 - V$, it follows that $M_{\alpha,\beta}-\omega^{-1}(\alpha)$ is connected and that $\ell(\omega_*'\eta,\beta) \equiv 0$ (mod p) for $\eta \in \pi_1(M_{\alpha,\beta}-\omega^{-1}(V))$, and so $\rho(\omega_*'\eta)$ is the trivial element. Clearly $\rho(\omega_*'[m_0]) = \rho([m]) = \rho(t) = \tau$. Thus the image of $\rho \circ \omega_*'$ is $\{\tau,1\}$, which proves the result; in particular we have the following:

Proposition 1.2. Dihedral p-fold branched covers of α are in 1 to 1 correspondence to equivalence classes of characteristic knots viewed as representing elements of order p in the kernel of the mod p reduction of $(L_V + L_V^t)$, modulo the action of $\mathbb{Z}_p^*$.

Let F_0 be a stable framing of the tangent bundle of S^3, compatible with the orientation. Let $N_1 = N(V_0' \cup \ldots \cup V'_{(p-1)/2} \cup \pi^{-1}(\beta) \times I)$ be a regular neighborhood of $(\pi')^{-1}(B)$, meeting the boundary regularly. Clearly N_1 may be chosen so that the restriction of π' to a neighborhood $V_0' - V_0' \cap N_1$ is a homeomorphism. Therefore the stable framing of $Y - \mathring{N}_1$ induced from $F_0 \times I$ via the unbranched covering $\pi'|Y-\mathring{N}_1$ extends to a framing F' of $Y-\mathring{N}_2$, N_2 a regular neighborhood of $V_1' \cup \ldots \cup V'_{(p-1)/2} \cup \pi^{-1}(\beta) \times I$.

Recall that given a q-fold covering map $S^1 \to S^1$ and a stable framing of S^1 that extends over D^2, the induced framing extends over D^2 iff q is odd. Therefore $F'|(\Sigma - \mathring{N}_1 \cap \Sigma)$ extends to a fiber of the tubular neighborhood $N_1 \cap \Sigma$ of β in Σ; i.e. to the complement of a cell in Σ. Hence, as $\pi_2(SO) = 0$, it extends to all of Σ. It follows easily (recall $\pi' = \pi \times id_{[0,\varepsilon)}$ near Σ) that F' extends to a stable framing F of $Y - V_1' \cup \ldots \cup V'_{(p-1)/2}$. The sole obstruction to extending $F|\Sigma$ to all of Y is an element

$$\theta(F|\Sigma) \in H^2(Y;\Sigma;\mathbb{Z}_2)$$

Proposition 1.3. Let $D\colon H^2(Y;\Sigma;\mathbb{Z}_2) \to H_2(Y,M;\mathbb{Z}_2)$ be the Poincare duality isomorphism. Then

$$D(\theta(F|\Sigma)) = [V_1']_2 + \cdots + [V'_{(p-1)/2}]_2 \;,$$

where $[V_i']_2$ is the element of $H_2(Y,M;\mathbb{Z}_2)$ represented by $(V_i', \partial V_i')$.

Proof: $\theta(F|\Sigma)$ is the restriction of $\theta(F) \in H^2(Y;Y-V_1' \cup \ldots \cup V'_{(p-1)/2};\mathbb{Z}_2)$. Hence, by Poincare duality, $D(\theta(F|\Sigma))$ is the image of

$$D(\theta(F)) \in H_2(V_1' \cup \ldots \cup V'_{(p-1)/2} \cup M,M;\mathbb{Z}_2)$$

$$\cong H_2(V_1' \cup \quad \cup V'_{(p-1)/2};\alpha_1 \cup \quad \cup \alpha_{(p-1)/2};\mathbb{Z}_2)\ .$$

Using Meyer-Vietoris, the right side is of course just $\bigoplus_{i=1}^{(p-1)/2} H_2(V_i';\alpha_i)$. Hence $D(\theta(F|\Sigma))$ is a linear combination of the classes $[V_1'],\ldots,[V'_{(p-1)/2}]_2$.

Now let (D_i^2,S_i^1), $i=1,\ldots,\frac{p-1}{2}$ be the disjoint fibers of the normal tubes of $V_1',\ldots,V'_{(p-1)/2}$, respectively. Clearly $\pi'|S_i^1$ is a two-fold covering map; hence, as noted above, $F|S_i^1$ does not extend to D_i^2. It follows (e.g. represent any element of $H_2(Y;\mathbb{Z}_2)$ by a 2-manifold transverse to all V_i' and consider the obstruction to framing a neighborhood of this 2-manifold that $D(\theta(F))$ is as stated.

Remark. This argument could be reformulated as an instance of the general principle that if $P^4 \xrightarrow{f} Q^4$ is a branched covering space of orientable 4-manifolds, then $D(w_2(P)) = D(f^* w_2(Q)) + [S^2]$, where S^2 is the subset of the branching set in Y consisting of points of even branching degree. (This follows from the familiar simplicial formula for Stiefel-Whitney classes.)

Corollary 1.4. The image of $[V_1']_2 + \cdots + [V'_{(p-1)/2}]_2$ in $H_2(Y,\partial Y;\mathbb{Z}_2)$ is precisely $Dw_2(Y)$, $w_2(Y) \in H^2(Y;\mathbb{Z}_2)$ the second Stiefel Whitney class of Y.

Now having constructed a cobordism Y^4 of M to Σ it is easy to further produce a compact manifold with Σ on the boundary. In fact, we just observe that $\Sigma = \partial P^4$ where $\phi\colon P^4 \to D^4$ is obtained as in [CS1] as the branched cyclic cover of D^4 along E^2, a Seifert surface, of the characteristic knot β, whose interior has been pushed into the interior of D^4. See [CS1] for details. Then set $W = Y \cup_\Sigma W$; clearly $\partial W = (\partial Y) - M$.

This 4-manifold W^4 can be described directly as a branched covering space as follows. The maps constructed above $\pi'\colon Y \to S^3 \times I$ and $\phi\colon P^4 \to D^4$ can be glued together to get a map $\Phi\colon W = Y \cup_\Sigma P^4 \to S^3 \times I \cup_{S^3} D^4 = D^4$. This map Φ is then seen to be a branched dihedral covering space. The total branching set in D^4 is a 2-complex K^2 obtained by attaching to V^2 a Seifert surface E^2 of β glued to V along $\beta \times \frac{1}{2}$.

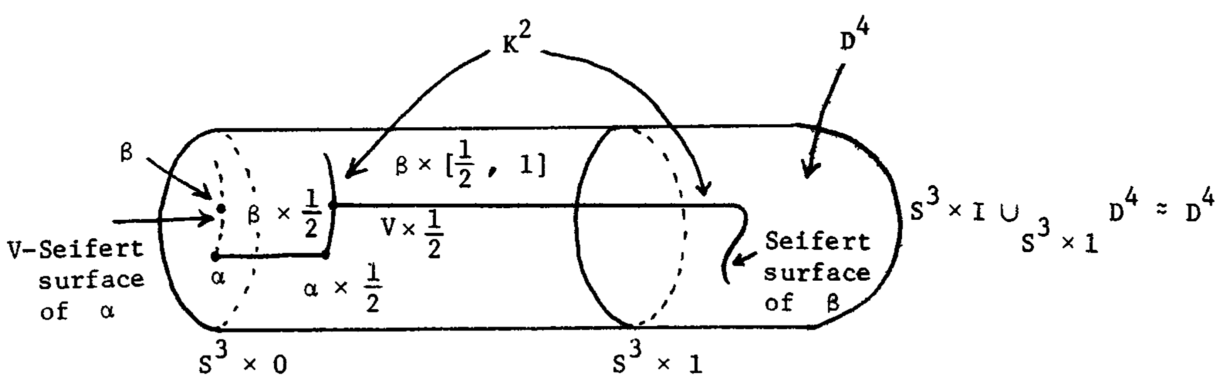


Fig. 2

This branching set in $D^4 = S^3 \times I \cup_{S^3 \times 1} D^4$ fails to be a manifold around the circle $\beta \times \frac{1}{2}$. Nevertheless, as we have seen W^4, the corresponding branched dihedral covering space is a manifold.

Remark. As the branching set in D^4 is not a manifold, it may seem surprising that the branched cover W^4 if a manifold; we explain this directly from another perspective. Consider again the branched dihedral cover W of D^4 along K^2. This is clearly a manifold except in a neighborhood of the inverse of the singularity circle $\beta \times \frac{1}{2}$ lying on K^2. See Figure 2. Now in a neighborhood of $\beta \times \frac{1}{2}$ the pair (D^4, K^2) looks like $S^1 \times (D^3, Q)$ where Q denotes a "figure Y", as can be seen near $\beta \times \frac{1}{2}$ in Figure 2. The dihedral cover of this neighborhood of the circle $\beta \times \frac{1}{2}$ would then be just $S^1 \times$ {a dihedral cover of D^3 branched along $Q\}$. As D^3 = cone on S^2, this cover will be just $S^1 \times \{$cone on the branched cover of S^2 along $S^2 \cap Q\}$. Now $S^2 \cap Q$ = 3 points.

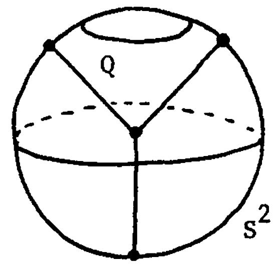


Fig. 3

Last, note that the branched dihedral cover of S^2 along these three points is again S^2. This follows by calculating its Euler characteristic using the fact that there the meridian about each point represents an element of order 2 in the dihedral group. Hence W has no singularity and is a P.L. manifold.

The same geometrical methods used above can be used to extend Proposition 1.4 to the following:

Proposition 1.5. Let $D: H^2(W;\mathbf{Z}_2) \to H_2(W,M;\mathbf{Z}_2)$ be the Poincare duality isomorphism. Then

$$[V_1']_2 + \cdots + [V_{(p-1)/2}']_2 = D(w_2(W)), \quad \text{where } w_2(W) \text{ is}$$

the second Stiefel-Whitney class of W.

Also, we can repeat all these arguments used above for the irregular p-fold dihedral cover for the full regular 2p-fold dihedral cover. In fact this 2p covering space of $\hat{W} \to D^4$ is also a 2-fold covering space of W^4 branched along V_0'; in particular, $\partial\hat{W} = \hat{M} \to M$ is a 2-fold covering space of M branched along α_0. Thus, the branch set in $\hat{M}$ (resp., $\hat{W}$) is the union of p circles (resp., 2-manifolds) $\hat{\alpha}_0,\ldots,\hat{\alpha}_p$ (resp., $\hat{V}_0,\ldots,\hat{V}_p$) which are disjoint (resp., intersect in one common circle $\hat{\beta}$). Here the circles are indexed by the convention $T^i\hat{\alpha}_0 = \hat{\alpha}_i$, where T is the generator of $Z_p \subset D_{2p}$ regarded as the group of covering translations.

Remark 1.6. Notice that in the regular 2p-fold dihedral covering space $\hat{W}$ has $w_2(\hat{W}) = 0$; for, it is a p-fold cyclic (branched) cover of a manifold with zero second Stiefel-Whitney class, the 2-fold cyclic cover of D^4 branched along the Seifert surface V (see [CS1]). On the other hand arguments similar to those used above show that the Poincare dual of $w_2(\hat{W}^4)$ is given by $\sum_{i=0}^{p-1} [\hat{V}_i] \in H_2(\hat{W},\partial\hat{W};\mathbf{Z}_2)$ which hence equals zero. (This can be checked in other ways.)

2. Linking numbers and characteristic classes

Note that for $M_\alpha = \partial W^4$ (resp. $\hat{M}_\alpha = \partial \hat{W}^4$) the irregular (resp., regular) p-fold (resp., 2p-fold) dihedral cover of S^3 as above, as $\hat{M}_\alpha \to \hat{M}_\alpha$ is a 2-fold cyclic branched cover [CS1], $H_1(M_\alpha;\mathbf{Z}_2) = 0$ if and only if $H_1(\hat{M}_\alpha;\mathbf{Z}_2) = 0$. In this section, we assume $H_1(M_\alpha;\mathbf{Z}_2) = 0$; hence the intersection form

$$\begin{array}{ccc} H_2(\hat{W};\mathbf{Z}_2) \times H_2(\hat{W};\mathbf{Z}_2) & \to & \mathbf{Z}_2 \\ (x,\ y) & \to & [x,y] \end{array}$$

is, by standard Poincare duality, a nonsingular symmetric bilinear pairing. Letting T (resp. ϕ) denote, as before, an element of order p (resp., 2) in D_{2p}, the covering translation group of $\hat{M} \to S^3$, we introduce a new bilinear pairing on $H_2(\hat{W};\mathbf{Z}_2)$ with values in $\mathbf{Z}_2[\mathbf{Z}_p]$:

$$\langle x,y\rangle = \sum_{i=0}^{p-1} [T^{-i}x,\phi y]T^i$$

__Lemma 2.1__. __This pairing is bilinear over__ $\mathbf{Z}_2[\mathbf{Z}_p]$ __and symmetric over__ $\mathbf{Z}_2[\mathbf{Z}_p]$ __and nondegenerate__.

__Proof__: To see that it is __symmetric__ (not Hermitian) note that

$$\begin{aligned} [T^{-i}x,\phi y] &= [\phi y,T^{-i}x] \\ &= [y,\phi T^{-i}x] \\ &= [y,T^{i}\phi x] \\ &= [T^{-i}y,\phi x] \end{aligned}$$

To check bilinarity note that

$$\begin{aligned} \langle Tx,y\rangle &= \sum [T^{-i}(Tx),\phi y]T^i \\ &= \sum [T^{-(i-1)}x,\phi y]T^i \\ &= \sum ([T^{-j}x,\phi y]T^j)T \\ &= \langle x,y\rangle T \end{aligned}$$

The nondegeneracy of this pairing follows from that of the intersection pairing.

Now we need some facts about symmetric forms over $\mathbf{Z}_2[\mathbf{Z}_p]$, p odd.

__Proposition 2.2__. __Let__ $p \times p \xrightarrow{\langle\ ,\ \rangle} \mathbf{Z}_2[\mathbf{Z}_p]$ __be a nonsingular symmetric bilinear pairing on the finitely generated__ $\mathbf{Z}_2[\mathbf{Z}_p]$ __module__ P. __Then there is a unique element__ $\alpha \in P$ __satisfying__

$$\langle x,\alpha\rangle^2 = \langle x,x\rangle\ ,\ x \in P\ .$$

__Notation__. α is called the characteristic element of P.

Proof: Consider $L(x) = \langle x,x\rangle^{1/2}$, $x \in P$. This is well-defined as $\mathbf{Z}_2[\mathbf{Z}_p]$ is a product of finite fields of characteristic 2. Moreover, as $L\colon P \to \mathbf{Z}_2[\mathbf{Z}_p]$ is easily seen to be linear, there is a unique $\alpha \in P$ with

$$L(x) = \langle x,\alpha\rangle \quad , \quad x \in P .$$

Recall that $\mathbf{Z}_2[\mathbf{Z}_p] = \oplus F_j$ where each F_j is a field of characteristic 2 and $F_0 \cong \mathbf{Z}_2$. Let e_j denote the multiplication identity of F_j; note that $e_0 = 1 + T^1 + T^2 + \cdots + T^{p-1}$ in $\mathbf{Z}_2[\mathbf{Z}_p]$. Correspondingly, a $\mathbf{Z}_2[\mathbf{Z}_p]$ module P decomposes naturally as

$$P = \oplus(P \otimes_{\mathbf{Z}_2[\mathbf{Z}_p]} F_j) .$$

Proposition 2.3. For $\alpha \in P$, the characteristic element of a symmetric bilinear form on the finitely generated $\mathbf{Z}_2[\mathbf{Z}_p]$ module P

$$\langle\alpha,\alpha\rangle = \sum e_j \operatorname{rank}_{F_j} (P \otimes_{\mathbf{Z}_2[\mathbf{Z}_p]} F_j) .$$

This follows immediately from the corresponding fact over each field F_j , which is easy as such forms decompose into 1-dimensional forms.

We use this to study $\hat{W}^4$. Let $A = [\hat{V}_0] \in H_2(\hat{W},\hat{M};\mathbf{Z}_2) \cong H_2(\hat{W};\mathbf{Z}_2)$.

Proposition 2.4. $A \in H_2(\hat{W};\mathbf{Z}_2)$ is the characteristic element of the pairing $\langle x,y\rangle$.

Lemma 2.5.

$$[x,\phi T^j x] = [T^{i/2}x,A] .$$

Proof of Proposition 2.4:

$$\begin{aligned} \langle A,x\rangle^2 &= \left(\sum [T^{-i/2}x,A]T^{i/2}\right)^2 \\ &= \left(\sum [x,\phi T^{-i}x]T^{i/2}\right)^2 , \quad \text{by the lemma} \\ &= \sum [x,\phi T^{-i}x]T^{i} \qquad \text{in } \mathbf{Z}_2[\mathbf{Z}_p] \\ &= \sum [T^{i}x,\phi x]T^{i} \\ &= \langle x,x\rangle \end{aligned}$$

Proof of Lemma 2.5: Consider the 2-fold covering maps $g_i\colon \hat{W} \to \hat{W}/\phi T^i$; for $i=0$, write $g = g_0\colon \hat{W} \to (\hat{W}/\phi T^0) = W$. As in the dihedral group D_{2p} , $\phi T^{i/2} = T^{i/2}\phi T^i$, there is a commutative diagram:

$$\begin{array}{ccc} \hat{W} & \xrightarrow{T^{i/2}} & \hat{W} \\ {\scriptstyle \phi T^i}\downarrow & & \downarrow{\scriptstyle \phi} \\ \hat{W} & \xrightarrow{T^{i/2}} & \hat{W} \end{array}$$

From this there is a homeomorphism $h_i\colon \hat{W}/\phi T^i \to \hat{W}/\phi$ and a commutative diagram

$$\begin{array}{ccc} \hat{W} & \xrightarrow{T^{i/2}} & \hat{W} \\ {\scriptstyle g_i}\downarrow & & \downarrow{\scriptstyle g} \\ \hat{W}/\phi T^i & \xrightarrow{h_i} & \hat{W}/\phi \end{array}$$

Now, as noted above, $w_2(\hat{W}) = 0$ and hence,

$$[x,x] = 0$$

and thus

$$[x,\phi T^i x] = [x,(i + \phi T^i)x]$$

and using transfers,

$$\begin{aligned} [x,\phi T^i x] &= [g_{i_*}(x), g_i(x)] \\ &= [g_*(T^{i/2}x)\ ,\ g_*(T^{i/2}x)] \\ &= [g_*(T^{i/2}x),\ \sum_{i=1}^{(p-1)/2} [V_i]]\ , \end{aligned}$$

as $w_2(W) = \sum_{i=1}^{(p-1)/2} [V_i]$ by Proposition 1.6. Thus,

$$\begin{aligned} [x,\phi T^i x] &= [T^{i/2}x,\ (1+\phi) \sum_{i=1}^{(p-1)/2} [\hat{V}_i]] \\ &= [T^{i/2}x,\ \sum_{i=1}^{p-1} T^i A] \end{aligned}$$

But as noted in Remark 1.7, $\sum_{i=0}^{p-1} T^i A = 0$. Hence,

$$[x,\phi T^i x] = [T^{i/2}x,A]\ .$$

For P a module over $\mathbf{Z}_2[\mathbf{Z}_p]$, let $[P]$ denote the class represented by P in $R(\mathbf{Z}_p)$, the representation ring of $\mathbf{Z}_p$ over the field $\mathbf{Z}_2$. Notice that $G = \hat{W} \cup_{\partial\hat{W}} \{\text{cone on } \partial\hat{W}\}$ has a natural action of D_{2p} with one fixed point, the cone point, and satisfies, as M is a mod 2-homology sphere, mod 2 Poincare duality.

Hence, $[H_2(\hat{W};\mathbf{Z}_2)] = \sum_{i=0}^{4} [H_i(G;\mathbf{Z}_2)]$ in $R(\mathbf{Z}_p) \otimes \mathbf{Z}_2$. Moreover $\sum_{i=0}^{4} [H_i(G;\mathbf{Z}_2)] = {}_{i=0}^{4} [C_i(G;\mathbf{Z}_2)]$ in $R(\mathbf{Z}_p) \otimes \mathbf{Z}_2$ for $C_i(G;\mathbf{Z}_2)$ the cellular chain groups of a cellular decomposition of G. However, the action of $\mathbf{Z}_p$ on G is free outside a 2-manifold [1/] in $\hat{W}$ and the cone point. Hence in $R(\mathbf{Z}_p) \otimes \mathbf{Z}_2$

$$[H_2(\hat{W};\mathbf{Z}_2)] = k[\mathbf{Z}_2[\mathbf{Z}_p]] \oplus [\mathbf{Z}_2] \text{ , some } k \text{ .}$$

Moreover, as $\hat{W}$ is a 2-fold branched cover of W along V_0, $\chi(\hat{W}) = 2\chi(W) - \chi(V)$ is odd, and hence $\chi(G) \equiv 0 \pmod 2$. Hence, $[H_2(\hat{W};\mathbf{Z}_2)] = [\mathbf{Z}_2[\mathbf{Z}_p]] \oplus [\mathbf{Z}_2]$ in $R(\mathbf{Z}_p) \otimes \mathbf{Z}_2$. Thus, from Propositions 2.4 and 2.3 we conclude that

$$\begin{aligned} \langle A,A\rangle &= 1 + e_0 \\ &= 1 + (1 + T + \cdots + T^{p-1}) \\ &= T + T^2 + \cdots + T^{p-1} \text{ .} \end{aligned}$$

Going back to the definition of the pairing $\langle A,A\rangle$ this says:

Corollary 2.7. *In $H_2(\hat{W},\partial\hat{W};\mathbf{Z}_2)$ the intersection number of $[\hat{V}_0]$ with $[\hat{V}_i]$ is odd.*

Proof of Theorem II: As $\hat{V}_i$ and $\hat{V}_0$ intersect just in the circle β, and this circle of intersections can be removed by pushing one class away from the other, the intersection of $[\hat{V}_i]$ and $[\hat{V}_0]$ is evidently given by the linking numbers of $\partial\hat{V}_i = \hat{\alpha}_i$ and $\partial\hat{V}_0 = \hat{\alpha}_0$ in W. Thus Theorem II, and also Theorem I, follow from Corollary 2.7.

BIBLIOGRAPHY

[BS] Bankwitz, C., and Schumann, H. G., Uber Viergeflechte. Abh. Math. Sem. Univ. Hamburg 10, 263-284 (1934).

[B1] Burde, G., Verschlingungsinvarianten von Knoten und Verkettungen mit zwei Brucken, Math. Zeit., 145, 235-242 (1975).

[B2] ————, On branched coverings of S^3, Canad. J. Math. 23, 84-89 (1971).

[CS1] Cappell, S. E., and Shaneson, J. L., Cyclic branched covering spaces, Knots groups and 3-manifolds; Papers dedicated to the memory of R. H. Fox, ed. by L. P. Neuwirth, Ann. of Math. Studies 84, pp. 165-173, Princeton Univ. Press, 1975.

[CS2] ————, Invariants of 3-manifolds, Bull. Amer. Math. Soc. 81, 559-562 (1975).

[CG] Casson, A., and Gordon, C., Cobordism of classical knots, mimeographed notes, Orsay, 1975.

[1/] This 2-manifold looks like 2 copies of the Seifert surface of joined together along their boundary. To see this pass first to the 2-fold cover of D^4 and then up to $\hat{W}$.

[F1] Fox, R. H., A quick trip through knot theory, Topology of 3-Manifolds and Related Topics (Proc. Univ. of Georgia Conf. 1961), ed. by M. K. Fort, Jr., Prentice-Hall (1962), 120-167.

[F2] ————, Construction of simply connected 3-manifolds, Topology of 3-Manifolds and Related Topics (Proc. Univ. of Georgia Conf. 1961), ed. by M. K. Fort Jr., Prentice-Hall, (1962), 213-216.

[F3] ————, Metacyclic invariants of knots and links, Canad. J. Math. 22, 193-201 (1970).

[Hi] Hilden, H., Every closed orientable 3-manifold is a 3-fold branched covering space of S^3, Bull. Amer. Math. Soc. 80, 1243-1244 (1974).

[HM] Hartley, R., and Murasugi, K., Covering linkage invariants, Canad. Jour. Math. 29, 1312-1339 (1977).

[H] Hartley, R., Homology invariants, Canad. Jour. Math. 30, 655-670 (1978).

[Li] Litherland, R. A., A formula for the Casson-Gordon invariants of a knot, to appear.

[Mo] Montesinos, M., A representation of closed, orientable 3-manifolds as 3-fold branched covers of S^3, Bull. Amer. Math. Soc. 80, 845-846 (1974).

[Pe1] Perko, K., On dihedral covering spaces of knots, Inventiones Math. 34, 77-82 (1976).

[Pe2] ————, On covering spaces of knots, Glasnik Mat. 9, 141-145 (1974).

[Re] Reidemeister, K., Knoten Theorie, Springer-Verlag, 1932.

COURANT INSTITUTE OF MATHEMATICAL SCIENCES
NYU
NEW YORK, NY 10012

DEPARTMENT OF MATHEMATICS
RUTGERS UNIVERSITY
NEW BRUNSWICK, NJ 08903