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A Splitting Theorem for Manifolds

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Preface

This paper studies codimension one submanifolds of manifolds through the development of general splitting theorems. These results are applicable to the study and classification of manifolds with infinite fundamental group; they can be used in decomposing such manifolds into manifolds with simpler fundamental groups. A subsequent paper will apply this to studying higher signatures of manifolds.

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Introduction

We shall be concerned with the following situation. Let Y be a connected closed $(n+1)$ dimensional manifold (or Poincaré complex) and X an n-dimensional connected closed submanifold (or sub-Poincaré complex), $j\colon X\subset Y$, $n\geqq 4$, with $\pi_1(X)\to\pi_1(Y)$ injective. Moreover, we assume X satisfies any one of the following three equivalent conditions:

(1) X cuts some neighborhood of itself in Y into two components.

(2) The normal bundle of X in Y is trivial.

(3) $j*\omega_1(Y)=\omega_1(X)$. Here ω_1 denotes the first Stiefel-Whitney class. These conditions are trivially satisfied if both X and Y are orientable.

Now, let W be a differentiable, piecewise-linear (P.l.) or topological manifold and $f\colon W\to Y$ a homotopy equivalence. The map f is said to be "splittable", or more precisely "splittable along X", if it is homotopic to a map, which we continue to call f, transverse regular to X (whence $f^{-1}(X)$ is an n-dimensional submanifold of Y), with the restriction of f to $f^{-1}(X)\to X$, and hence also to $f^{-1}(Y-X)\to(Y-X)$, being homotopy equivalences. If $(Y-X)$ has 2 components this means that f restricts to a homotopy equivalence to each. In the present paper, for a differentiable (respectively; P.L., topological) manifold, "submanifold" means a differentiable (resp; P.L. locally-flat, topological locally-flat) submanifold.

h-Splitting Problem. When is W h-cobordant to a manifold W' with the induced homotopy equivalence $f'\colon W'\to Y$ splittable along X?

s-Splitting Problem. When is $f\colon W\to Y$ splittable along X?

Corresponding to the number of components of $Y-X$, these problems have two cases. Let $G=\pi_1(Y)$ and $H=\pi_1(X)$; as we assumed $\pi_1(X)\to\pi_1(Y)$ injective, we have $H\subset G$. The following discussion and the methods of the present paper, also apply to relative splitting problems.

Case A. $Y-X$ has two components. In this case, let Y_1 and Y_2 denote the closures in Y of the two components of $Y-X$, so that $Y=Y_1\cup_X Y_2$. Set $G_i=\pi_1(Y_i)$, $i=1,2$; the inclusion $X\subset Y_i$, induces $\xi_i\colon H\to G_i$ with, as $H\subset G$, ξ_i an inclusion, $i=1,2$. By Van Kampen's theorem, G is the free product with amalgamation $G=G_1 *_H G_2$.

Case B. $Y-X$ has one component. In this case, let Y' denote the manifold or Poincaré complex with boundary obtained by cutting Y' along X; that is, the boundary of Y' is $X_1 \cup X_2$, $X_1 \cong X_2 \cong X$ and Y is obtained from Y' by identifying X_1 with X_2. Set $J = \pi_1(Y)$. Corresponding to the inclusions $X_i \subset Y'$, there are two maps ξ_i: $H \to J$ which are injective as $\pi_1(X) \subset \pi_1(Y)$. (To be more precise about basepoints, choose $p \in X$ and correspondingly $p_i \in X_i$, $i=1,2$. Let γ be a path in Y' from p_1 to p_2. There is the obvious inclusion $H = \pi_1(X_1, p_1) \to \pi_1(Y', p_1) = J$. Using the identification $[\gamma]$: $\pi_1(Y', p_2) \to \pi_1(Y', p_1)$ induced from γ, we get another inclusion $H = \pi_1(X_2, p_2) \to \pi_1(Y', p_2) \xrightarrow{[\gamma]} \pi_1(Y', p_1) = J$. The loop γ in Y, represents $t \in G$.) Then from two applications of Van Kampen's theorem we get $G \cong J *_H \{t\}$ where we have:

Definition. For ξ_i: $H \to J$, $i=1,2$, two injective group homomorphisms, let

$$J *_H \{t\} = Z * J / \langle \{t^{-1} \xi_1(u) t \xi_2(u)^{-1} | u \in H\} \rangle$$

where Z is an infinite cyclic group generated by t. As usual $\langle \{P\} \rangle$ denotes the smallest normal subgroup containing $\{P\}$.

This $J *_H \{t\}$ notation [W 1] is concise. Note, however, that the group obtained depends on the inclusions ξ_i.

In applying the results of the present paper to studying manifolds of a given homotopy type, it is useful to note that if Y is a manifold of dimension greater than 4, and if $\pi_1(Y) \cong G_1 *_H G_2$ (respectively $\pi_1(Y) \cong J *_H \{t\}$), then there exists a codimension one submanifold X of Y enjoying all the properties described in the discussion of case A (resp. case B) above. This follows easily from methods of I § 3 below.

Case B was first considered in the setting of the fibration problem, the determination of which high-dimensional manifolds fiber over a circle. Stallings [St 1] obtained a result on this problem for three-dimensional manifolds. In high dimensions this problem was solved by Browder and Levine [BL] [B 2] for $G=Z$. The fibration problem is related to the problem of deciding when an open high dimensional manifold is the interior of some closed manifold. This was solved in the simply connected case by Browder, Levine and Livesay [BLL] and in general by Siebenmann [S 1]. Siebenmann's result implies a splitting theorem for certain open manifolds. A related result was obtained by Novikov [N] and applied by him to prove the topological invariance of rational Pontryagen classes. The high-dimensional fibration problem was solved by Farrell [F]. The related splitting theorem of Farrell and Hsiang solved the case B when $G = Z \times_\alpha H$, $n \geqq 5$ [FH 1]. In the notation introduced above, this corresponds to both ξ_1 and ξ_2 surjective, and $G = Z \times H$ corresponds to $\xi_1 = \xi_2$ being surjective. In a special case, Shaneson extended Farrell-Hsiang splitting to $n=4$ with $G=Z$ [S 3]. The result of [FH 1] was used by Shaneson and by Wall in obtaining a formula for the Wall groups of $Z \times H$ and in computing the Wall groups of free abelian groups [S 2] [W 2].

Case A was solved by Browder [B 1] for Y_1, Y_2 and X all simply connected, $n \geqq 5$. As a consequence of the development of relative non-simply connected surgery theory [W 2], Wall showed that the problem could always be solved in case $H = G_1$, and hence $G_2 = G$, $n \geqq 5$. R. Lee made an important advance when he solved the problem for the case n even and greater than five with $H=0$ and G without 2-torsion [L 1].

All the above splitting theorems for compact manifolds are special cases of the *general high-dimensional splitting results*, Theorems 1 and 2, stated below. We treat cases A and B with the same geometric method. A further refinement, when X is simply connected, is presented in Theorem 3. Corollary 4 applies this to obtain a homotopy-theoretic criteria for a manifold to be a connected sum. Theorem 5, Theorem 3 and Corollaries 4 and 6 extend some of our splitting results to the case $n=4$. Corollary 6 restates the theorem of Farrell and Hsiang, together with an extension in some cases to $n=4$.

The results of this paper lead to Mayer-Vietoris sequences for Wall surgery groups and to the computation of the Wall groups of many infinite groups including free groups, fundamental groups of closed two-manifolds etc. [C4] [C5]. A general rational splitting principle will be used in [C4] in showing, for a very large class of fundamental groups, the Novikov higher signature conjecture. Using the results of the present paper and special low-dimensional methods, a stable splitting theorem for the case $n=4$ was proved by J. Shaneson and the author [CS1].

Since the appearance of this paper in preprint form, we have found examples showing that restrictions of the kind used below in some cases on fundamental groups in order to obtain splitting theorems are necessary. See our examples of "non-splitting" in [C5] [C6]. A description of the extension of the methods of the present paper and a general manifold classification scheme for the fundamental groups in which there is an obstruction to splitting is given in [C8] [C9]. Corresponding general results on the Wall surgery groups of any generalized free product of finitely presented groups are announced in [C7].

We introduce some algebraic notation:

Definition. A subgroup H of a group G is said to be square-root closed in G if, for all $g\in G$, $g^2\in H$ implies $g\in H$.

In [C1] such subgroups were characterized by an equivalent condition called "two-sided subgroup". The formulation of this definition in terms of square-root closed subgroups was suggested to us by C. Miller.

Examples

(1) If H is normal in G, $H\lhd G$, then H is square-root closed in G if and only if G/H has no elements of order 2. In particular, the trivial subgroup is square-root closed in G if and only if G has no elements of order 2.

(2) Any subgroup of a finite group of odd order is square-root closed. In general, a subgroup H of a finite group G is square-root closed if and only if H contains all elements of G of 2-primary order.

(3) H is square-root closed in $G_1 *_H G_2$ if and only if H is square-root closed in both G_1 and G_2.

(4) Given inclusions $\xi_i\colon H\to J$, $i=1,2$, then H is square-root closed in $J *_H \{t\}$ if and only if both $\xi_1(H)$ and $\xi_2(H)$ are square-root closed in J. In particular, (or from (1) above):

(5) H is square-root closed in $Z\times_\alpha H$.

(6) Let G be a free group and H a subgroup generated by a non-square element of G. Then H is square-root closed in G. (This is used in computing in [C3] [C9] the Wall surgery groups of all two-manifolds.)

(7) If H is square-root closed in G, $Z\times H$ is square-root closed in $Z\times G$.

Note that for X an n-dimensional submanifold with trivial normal bundle of the $(n+1)$ dimensional manifold Y with $Y-X$ having two components, $Y=Y_1\cup_X Y_2$ (respectively; one component with $Y=Y'/$(identifying two copies of X) and with $\xi_i\colon \pi_1(X)\to\pi_1(Y')$ the induced maps), $\pi_1(X)\to\pi_1(Y)$ is injective with square-root closed image if and only if this is true for each of the induced maps $\pi_1(X)\to\pi_1(Y_i)$ (resp; $\xi_i\colon \pi_1(X)\to\pi_1(Y')$), $i=1,2$. This follows from example 3 (resp; 4) of square-root closed subgroups above.

We describe first, in Theorem 1, the fundamental groups for which we show that the h or s splitting problems can always be solved. For a group G, $\mathrm{Wh}(G)$ denotes the Whitehead group of G, and $\tilde K_0(G)$ denotes the reduced projective class group of the ring $Z[G]$.

Theorem 1. *Let Y be a closed manifold or Poincaré complex of dimension $n+1$, $n\geq 5$ with $\pi_1(Y)=G$ and X a closed submanifold or sub-Poincaré complex of dimension n of Y with trivial normal bundle and $\pi_1(X)=H\subset G$ a square-root closed subgroup*[1]. *Assume $Y-X$ has two components* (respectively; *one component*) *with fundamental groups G_i and $\xi_i\colon H\to G_i$* (resp; *group J and $\xi_i\colon H\to J$*), *$i=1,2$ the induced maps.*

(i) *If $\xi_{1*}-\xi_{2*}\colon \tilde K_0(H)\to\tilde K_0(G_1)\oplus\tilde K_0(G_2)$* (resp; *$\xi_{1*}-\xi_{2*}\colon \tilde K_0(H)\to\tilde K_0(J)$*) *is injective or even just*

$$H^{n+1}(Z_2;\,\mathrm{Ker}(\tilde K_0(H)\to\tilde K_0(G_1)\oplus\tilde K_0(G_2)))=0$$

$$(\text{resp};\ H^{n+1}(Z_2;\,\mathrm{Ker}(\tilde K_0(H)\xrightarrow{\xi_{1*}-\xi_{2*}}\tilde K_0(J)))=0)$$

then for any homotopy equivalence $f\colon W\to Y$, W a closed manifold, W is h-cobordant to a manifold W' with the induced homotopy equivalence $f'\colon W'\to Y$ splittable.

(ii) *If* $\mathrm{Wh}(G_1)\oplus\mathrm{Wh}(G_2)\to\mathrm{Wh}(G)$ (resp; $\mathrm{Wh}(J)\to\mathrm{Wh}(G)$) *is surjective, then every homotopy equivalence $f\colon W\to Y$, W a closed manifold, is splittable.*

Note that the hypothesis of (i) is always satisfied if one of the inclusions $H\to G_i$ has a retraction. The hypothesis of (ii) is always satisfied for H a member of a class of groups constructed in [W1].

The rings $Z[H]$ and $Z[G]$ acquire, as usual, the involutions determined by $\bar g=\omega(g)\,g^{-1}$, $g\in G\subset Z[G]$, $\omega\colon G\to Z_2=\{\pm 1\}$ the orientation homomorphism. These involutions determine Z_2 actions [M1] on $\mathrm{Wh}(G)$ and $\tilde K_0(H)$, which are referred to in (i) of Theorem 1.

There is a relative form of Theorem 1, in which we begin with a splitting of $\partial W\to\partial Y$ along ∂X and obtain similar results for the problem of extending to a splitting of $W\to Y$ along X.

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[1] Or, assume $H=G_1$ and hence $G_2=G$ and $\mathrm{Wh}(G_2)\to\mathrm{Wh}(G)$ surjective. For simple homotopy equivalence this case is in [W1]

Waldhausen, extending results of [St] [BHS] [FH2] showed that

$$\mathrm{Wh}(G_1 *_H G_2)/\mathrm{Wh}(G_1)\oplus\mathrm{Wh}(G_2) \qquad (\text{resp; } \mathrm{Wh}(J *_H \{t\})/\mathrm{Wh}(J))$$

decomposes as a direct sum of

$$\mathrm{Ker}(\tilde{K}_0(H)\to\tilde{K}_0(G_1)\oplus\tilde{K}_0(G_2)) \qquad (\text{resp; } \mathrm{Ker}(\tilde{K}_0(H)\xrightarrow{\xi_{1*}-\xi_{2*}}\tilde{K}_0(J))$$

and a group of $Z[H]$ linear nilpotent maps [W1]. For the projections to these summands we write

$$\Phi\colon \mathrm{Wh}(G_1 *_H G_2)\to\mathrm{Ker}(\tilde{K}_0(H)\to\tilde{K}_0(G_1)\oplus\tilde{K}_0(G_2))$$

$$\eta\colon \mathrm{Wh}(G_1 *_H G_2)\to\mathrm{Wh}(G_1 *_H G_2)/(\mathrm{Wh}(G_1)\oplus\mathrm{Wh}(G_2)\oplus\mathrm{Ker}(\tilde{K}_0(H)\to\tilde{K}_0(G_1)\oplus\tilde{K}_0(G_2)))$$

$$(\text{resp; } \Phi\colon \mathrm{Wh}(J *_H \{t\})\to\mathrm{Ker}(\tilde{K}_0(H)\xrightarrow{\xi_{1*}-\xi_{2*}}\tilde{K}_0(J))$$

$$\eta\colon \mathrm{Wh}(J *_H \{t\})\to\mathrm{Wh}(J *_H \{t\})/(\mathrm{Wh}(J)\oplus\mathrm{Ker}(\tilde{K}_0(H)\xrightarrow{\xi_{1*}-\xi_{2*}}\tilde{K}_0(J))).$$

For the Z_2 action defined above on $\mathrm{Wh}(G)$ and $\tilde{K}_0(H)$, Lemma II.3 shows that $\Phi(x^*)=-\Phi(x)$, $x\in\mathrm{Wh}(G_1 *_H G_2)$ or $\mathrm{Wh}(J *_H \{t\})$.

For an abelian group C equipped with a Z_2 action, $x\to x^*$, $x\in C$, we make the usual identification $H^k(Z_2;\, C)\cong\{x\in C\,|\,x=(-1)^k x^*\}/\{(x+(-1)^k x^*\,|\,x\in C\}$. If $x\in\mathrm{Wh}(G_1 *_H G_2)$ (resp; $x\in\mathrm{Wh}(J *_H \{t\})$ with $x=(-1)^{k+1}x^*$ in $\mathrm{Wh}(G_1 *_H G_2)/\mathrm{Wh}(G_1)\oplus\mathrm{Wh}(G_2)$ (resp; $\mathrm{Wh}(J *_H \{t\})/\mathrm{Wh}(J)$) we write $\bar{\Phi}(x)$ for the element in

$$H^{k+1}(Z_2;\, \mathrm{Ker}(\tilde{K}_0(H)\to\tilde{K}_0(G_1)\oplus K_0(G_2)))$$

$$(\text{resp; } H^{k+1}(Z_2;\, \mathrm{Ker}(\tilde{K}_0(H)\xrightarrow{\xi_{1*}-\xi_{2*}}\tilde{K}_0(J))))$$

represented by $\Phi(x)$.

Similarly, we write $\bar{\eta}(x)$ for the element in

$$H^k(Z_2;\, \mathrm{Wh}(G_1 *_H G_2)/\mathrm{Wh}(G_1)\oplus\mathrm{Wh}(G_2)\oplus\mathrm{Ker}(\tilde{K}_0(H)\to\tilde{K}_0(G_1)\oplus\tilde{K}_0(G_2)))$$

$$(\text{resp; } H^k(Z_2;\, \mathrm{Wh}(J *_H \{t\})/\mathrm{Wh}(J)\oplus\mathrm{Ker}(\tilde{K}_0(H)\xrightarrow{\xi_{1*}-\xi_{2*}}\tilde{K}_0(J))))$$

represented by $\eta(x)$.

Theorem 2. *Let Y be a closed manifold or Poincaré-complex of dimension $n+1$, $n\geqq 5$ with $\pi_1(Y)=G$, and X a closed submanifold or sub-Poincaré-complex of dimension n of Y with trivial normal bundle and with $\pi_1(X)=H$, $H\subset G$ a square-root closed subgroup. Assume $Y-X$ has two components* (respectively; *one component) with fundamental groups G_1 and G_2* (resp; *group J with $\xi_i\colon H\to J$, $i=1,2$ being the induced maps). Assume given a homotopy equivalence $f\colon W\to Y$, W a closed manifold; denote its Whitehead torsion by $\tau(f)\in\mathrm{Wh}(G)$. Then:*

(i) *W is h-cobordant to a manifold W' with the induced homotopy equivalence $f'\colon W'\to Y$ splittable along X if and only if*

$$\bar{\Phi}(\tau(f))\in H^{n+1}(Z_2;\, \mathrm{Ker}(\tilde{K}_0(H)\to K_0(G_1)\oplus\tilde{K}_0(G_2)))$$

$$(\text{resp; } \bar{\Phi}(\tau(f))\in H^{n+1}(Z_2;\, \mathrm{Ker}(\tilde{K}_0(H)\xrightarrow{\xi_{1*}-\xi_{2*}}\tilde{K}_0(J))))$$

is zero.

(ii) *The map f is splittable along X if and only if the image of $\tau(f)$ in*

$$\mathrm{Wh}(G)/\mathrm{Wh}(G_1)+\mathrm{Wh}(G_2) \quad (\text{resp; } \mathrm{Wh}(G)/\mathrm{Wh}(J))$$

is zero and further, for n odd,

$$\theta(f)\in H^{n+1}(Z_2;\, \mathrm{Wh}(G)/\mathrm{Wh}(G_1)\oplus \mathrm{Wh}(G_2)\oplus(\mathrm{Ker}(\tilde{K}_0(H)\to\tilde{K}_0(G_1)\oplus\tilde{K}_0(G_2)))$$

$$(\text{resp; } H^{n+1}(Z_2;\, \mathrm{Wh}(G)/\mathrm{Wh}(J)\oplus \mathrm{Ker}(\tilde{K}_0(H)\xrightarrow{\xi_{1*}-\xi_{2*}}\tilde{K}_0(J))))$$

is zero.

In (ii) for n odd and $\tau(f)\in\mathrm{Image}\ \mathrm{Wh}(G_1)\oplus\mathrm{Wh}(G_2)\to\mathrm{Wh}(G_1 *_H G_2)$ (resp; $\mathrm{Wh}(J)\to\mathrm{Wh}(J *_H\{t\})$), we define $\theta(f)=\bar{\eta}(x)$, where x is the Whitehead torsion of any h-cobordism, which must from (i) above exist, of W to a split manifold. We show that $\theta(f)$ is a well-defined invariant of the homotopy class of f and assumes for different choices of f all values in the given cohomology group. The group in whose cohomology it takes its value is isomorphic to Waldhausen's group of nilpotent maps. He showed that this group is zero for H a member of a large class of groups including free groups, free abelian groups twisted products of Z, fundamental groups of 2-manifolds, etc., and G_1, G_2, J any groups [W1]. In fact, this group is zero for $Z[H]$ a regular ring, or just a coherent ring of finite global homological dimension. However, his conjecture on the vanishing of this group of nilpotent maps for H a member of a larger class of groups he constructed is not apparently known. However, Theorem 2 is used in an appendix of the present paper to show that, for H square-root closed in G, the odd cohomology of Z_2 with coefficients in this group of nilpotent maps is zero.

There is also a relative form of Theorem 2 in which W is a compact manifold with boundary ∂W, Y a Poincaré complex with boundary ∂Y, and the homotopy equivalence of pairs $f\colon (W,\partial W)\to(Y,\partial Y)$ is split along $\partial X\subset\partial Y$. The obstructions to producing an h-cobordism, fixed on ∂W, of W to a manifold split along X, or of extending the splitting of $\partial W\to\partial Y$ along ∂X to a splitting along X are the same as in the absolute case of Theorem 2.

In an important special case, we weaken both the dimension and the square-root closed restraints.

Theorem 3. *Let $f\colon W\to Y$ be a homotopy equivalence with W a closed $n+1$ dimensional differentiable or P.L.* (resp; *topological*) *manifold and Y a closed $n+1$ dimensional Poincaré complex, $n\geqq 4$. Let X be simply-connected closed codimension 1 sub-Poincaré complex of Y. If $n=4$ assume that X has the homotopy type of a P.L.* (resp; *topological*) *4-dimensional manifold.*

Assume further that

(i) *$\pi_1(Y)$ has no elements*[2] *of order 2*

or

(ii) *$n=2k$, and letting $\omega\colon\pi_1(Y)\to Z_2=\{\pm 1\}$ denote the orientation homomorphism, for each*[2] *$g\in\pi_1(Y)$ with g of order 2, $\omega(g)=(-1)^{k+1}$*

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[2] It suffices to check this for each element g of $\pi_1(Y_1)$ and $\pi_1(Y_2)$ (resp; $\pi_1(Y')$)

or

(iii) *one of the components of* $Y-X$ *is simply connected. Then* f *is splittable along* X.

Theorem 3 applies to both case A and case B. The examples of non-splitting we have constructed in [C5] [C6] (since writing the present paper) show that at least for n even the hypothesis on $\pi_1(Y)$ stated in (ii) or (iii) above are actually always needed.

Note that the hypothesis of (ii) is always satisfied if Y is orientable of dimension $4k+3$. If f is a simple homotopy equivalence, then under the hypothesis of Theorem 3, the split map will induce a simple homotopy equivalence of components. For $n \neq 4$ there is also a relative form of Theorem 3. For $\pi_1(Y)=0$ with $Y-X$ having two components, $n \neq 4$, Theorem 3 was first proved by Browder [B1] and this was extended to the case where only one of the two components of $Y-X$ is assumed simply-connected, $n \neq 4$, by Wall [W2]. The case of Theorem 3 with n even and not 4 with $Y-X$ having two components and $\pi_1(Y)$ without elements of order 2 was proved by R. Lee [L1].

Denote the connected sum of manifolds P and Q of the same dimension by $P \# Q$; we call this a non-trivial connected sum if P and Q are not spheres. Similarly if P and Q are Poincaré complexes of dimension n we define the connected sum $P \# Q$ using the fact that an n-dimensional Poincaré complex, $n \geqq 4$, is homotopy equivalent to a complex with a single n-dimensional cell [W2]. Taking $X=S^n$ in Theorem 3, we obtain the following

Corollary 4. *Connected Sum Homotopy Criteria: A P.L.* (resp; *topological*) *closed manifold* W *of dimension* $n+1$, $n \geqq 4$, *with either*

(i) $\pi_1(W)$ *has no elements of order* 2

or

(ii) *for dimension* $W=2k+1$ *and letting* $\omega: \pi_1(W) \to Z_2 = \{\pm 1\}$ *denote the orientation homomorphism, for each element* $g \in \pi_1(W)$ *with* $g^2=1$, $g \neq 1$, $\omega(g) = (-1)^{k+1}$,

is a non-trivial P.L. (resp; *topological*) *connected-sum if and only if there exist Poincaré complexes,* P, Q, *not homotopy equivalent to spheres, with* W *homotopy equivalent to* $P \# Q$.

Note that the above condition on $\pi_1(W)$ is always satisfied if W is orientable of dimension $4k+3$. In [C5] [C6] we construct examples for n even, when $\pi_1(Y)$ does not satisfy hypothesis (i) or (ii), of manifolds which are homotopy equivalent to nontrivial connected sums but are not themselves connected sums.

The following also extends some of our results to the case $n=4$.

Theorem 5. $f: W \to Y$, $X \subset Y$, $H=\pi_1(X)$ *and* G *as in Theorem* 1[3] *or Theorem* 2 *but with* $n=4$. *In addition assume*

(i) H *is zero or is finite of odd order*

[3] Or as in the footnote to Theorem 1

and

(ii) *if W is a differentiable or P.L.* (resp; *topological*) *manifold assume X has the homotopy type of a P.L.* (resp; *topological*) *four-manifold. Then we have the same conclusion as in Theorems* 1 *and* 2.

More generally, condition (i) may be replaced by $H_2(\pi_1(X); Z_2)=0$ and $[\Sigma X; G/H] \to L^h_5(\pi_1(X), w)$ surjective, $H=PL$ (resp; Top). Thus Theorem 5 also extends to the case W, X topological manifolds with $\pi_1(X)=Z$.

The following important special case of Theorem 2 was proved by Farrell and Hsiang [FH1] for $n>4$. The Farrell fibering theorem [F] could be derived from it. In a special case when $n=4$ with $H=0$, the following was proved by Shaneson [S3]. Here for a group C equipped with an automorphism α we write C^α for $\{x \in C \mid \alpha(x)=x\}$.

Corollary 6. *Let Y be a closed Poincaré complex of dimension* $n+1$, $n \geqq 4$, *with* $\pi_1(Y)=Z \times_\alpha H$ *and with X a sub-Poincaré complex of dimension n of Y with* $\pi_1(X)=H$. *Let* $f\colon W \to Y$ *be a homotopy equivalence, W a closed differentiable or P.L.* (resp; *topological*) *manifold. If* $n=4$, *assume that X has the homotopy type of a closed* 4-*dimensional P.L.* (resp; *topological*) *manifold with* $H=0$ *or H finite of odd order* (resp; *or* $H=Z$). *Then, letting* $\tau(f)$ *denote the Whitehead torsion of f*

(i) *W is h-cobordant to a manifold W′ with the induced homotopy equivalence* $f'\colon W' \to Y$ *splittable along X if and only if* $\bar{\Phi}(\tau(f)) \in H^{n+1}(Z_2; K_0(H)^{\alpha *})$ *is* 0.

(ii) *f is splittable along X if and only if the image of* $\tau(f)$ *in* $\mathrm{Wh}(Z \times_\alpha H)/\mathrm{Wh}(H)$ *is* 0.

When $n>4$, if H is not square-root closed in G, we still construct, if $\bar{\Phi}(\tau(f))=0$, in both case A and case B a normal cobordism of (W, f) to a split homotopy-equivalence. Moreover, the surgery obstruction of this normal cobordism goes to 0 in the "surgery obstruction group" of the ring $Z[\frac{1}{2}][\pi_1(Y)]$. (This leads to general Mayer-Vietoris sequences for Wall groups of $R[G]$, $G=G_1 *_H G_2$ or $G=J *_H \{t\}$, $Z[\frac{1}{2}] \subset R \subset Q$ without the square-root closed restraint.) In the present paper the main technical use made of the square-root closed condition on $H \subset G$ is in concluding, see Lemma II.7, that as a $Z[H]$ bimodule $Z[G] \cong Z[H] \oplus E \oplus \bar{E}$, where the involution on $Z[G]$ sends E to $\bar{E}$ and $\bar{E}$ to E.

We briefly outline the proof of Theorems 1 and 2. First we try to produce a submanifold $f^{-1}(X)$ in W^{n+1} homotopy equivalent to X^n by ambient surgery inside W. Using a procedure related to one used by Waldhausen [W1] in his study of Whitehead torsion of chain complexes, this is carried out in Chapter I to fix up $f^{-1}(X)$ below the middle dimensions. In the middle dimensional range the geometry required to perform ambient surgery by handle exchanges cannot be directly carried out and the effect of such surgeries is more difficult to assess. So for $n=2k$, we go as far as we can up to the middle dimension of $f^{-1}(X)$, and measure the remaining difficulty in terms of certain $Z[H]$-linear nilpotent maps of projective $Z[H]$-modules. Then working **outside** of W, we construct in Chapter II, by a procedure we call the *nilpotent normal cobordism* construction a cobordism of W to a homotopy equivalent split manifold. Of course, we want to replace this cobordism by an s-cobordism or h-cobordism. We compute the obstruction to doing just that, as a surgery obstruction, in terms of the $Z[H]$-linear nilpotent maps. If the subgroup $\pi_1(X)$ is square-root closed in $\pi_1(Y)$, we show algebraically that any surgery obstruction constructed in this way from nilpotent maps is in the image of $L_{n+2}(\pi_1(Y-X)) \to L_{n+2}(\pi_1(Y))$. It can therefore be changed to zero by a further normal cobordism without affecting the splitting[4].

For $n=2k-1$, we could get a weak form of the result of the present paper by crossing with a circle to get into a dimension in which the splitting problem has already been solved. Of course, this employs the observation that if H is square-root closed in G, $Z \times H$ is square-root closed in $Z \times G$. We could then try to split along $S^1 \times X$ and use the Farrell-Hsiang theorem to remove the extraneous circle. However, the obstructions which would arise in this use of the Farrell-Hsiang theorem are difficult to relate to our initial data. Hence this would only suffice to prove Theorem 1 and not Theorem 2.

Instead of this, we use for $n=2k-1$, a direct geometric construction. After working in W to improve $f^{-1}(X)$ below the pair of middle dimensions, we construct an *explicit* geometric splitting of $S^1 \times W$ using the construction already developed for $n=2k$. We then measure *explicitly* the obstruction to removing the circle factor and show it vanishes under the hypothesis of Theorem 2.

The extension of our results from the differentiable and P.L. cases to topological manifolds makes use of topological transversality [KS] and surgery [L2] [KS].

Chapter I: Below the Middle Dimension

§ 1. Ambient Surgery below the Middle Dimension

This chapter begins the proof of Theorem 2 and of its special case, Theorem 1. Notation which will be used repeatedly is introduced in § 2 and § 4. *The square-root closed hypothesis is not used in this chapter.*

Lemma I.1, the main result on ambient surgery below the middle dimension, and Lemma I.2 are stated in the present section. Section 2 reviews well-known material on covering spaces, normal form for elements of $G=G_1 *_H G_2$ or $G=J *_H \{t\}$ (which is less familiar) and corresponding descriptions of $Z[G]$. The derivation of normal form and its geometric meaning is recalled in more detail in the appendix to Chapter I. The technique of handle-exchanging, used to perform ambient surgery below the middle dimension, is recalled in § 3. It is applied there to the special problems encountered in low dimensional surgery.

Section 4 contains computations relating the homology groups of various components of the universal covers of W and Y. *Nilpotent maps* and *upper-triangular filtrations* are used to describe the images of some of these homology groups in each other. Using this, the proof of Lemma I.1 is completed by an induction in § 5.

[4] When $\pi_1(X)$ is not square-root closed in $\pi_1(Y)$ we have since writing the present paper shown how to use these nilpotent maps to define the general codimension 1 splitting obstruction in our UNil groups [C7] [C9]. The surgery obstructions arising from the nilpotent normal cobordism then can be in general reinterpreted as being in the image of a natural map from UNil groups to sorgery groups.

The main result of § 4, Lemma I.10 is derived algebraically from the preceding computations of homology groups. It can also be demonstrated, along the lines suggested in the remarks of Section 4, by a geometric argument which uses a detailed description of universal covers as in the appendix to Chapter I. The uniqueness of normal form, employed in the proof of Lemma I.10, contains implicitly geometric facts about covering spaces.

Lemma I.1. *Let Y be an $n+1$ dimensional closed manifold (or Poincaré complex) and X a codimension one closed submanifold (or sub-Poincaré complex) with trivial normal bundle in Y and with $\pi_1(X)\to\pi_1(Y)$ injective. Let W be an $n+1$ dimensional closed manifold with $f\colon W\to Y$ a homotopy equivalence, $n\geqq 5$.*[5] *Assume given $m<(n-1)/2$; then f is homotopic to a map, which we continue to call f, which is transverse regular to X (whence $f^{-1}X$ is a codimension one submanifold of W) and with the restriction of f to $f^{-1}(X)\to X$ inducing isomorphisms $\pi_i(f^{-1}(X))\to\pi_i(X)$, $i\leqq m$.*

For $m=0$ and $m=1$ this is proved in § 3 and the proof for $m>1$ is completed by induction in § 5. Of course, there is also a relative form of Lemma I.1.

The role of Lemma I.1 as a first step in the proof of Theorem 2 is indicated by Lemma I.2. If f is transverse regular to $X\subset Y$, write M for $f^{-1}(X)$; M is a codimension one submanifold with trivial normal bundle of X. Write $f!\colon M\to X$ for the map obtained by restricting f. Corresponding to the decomposition in case A (respectively; case B) $Y=Y_1\cup_X Y_2$ (resp; $Y=Y'$/identify X_1 with X_2 where $X_1\cong X_2\cong X$) we get decompositions by cutting W along M, $W=W_1\cup_M W_2$ (resp; $W=W'$/identify M_1 with M_2 where $M_1\cong M_2\cong M$). Thus f induces maps $f!\colon W_i\to Y_i$ (resp; $f!\ W'\to Y'$ with $f!\,(M_i)\subset X_i$), $i=1,2$.

Lemma I.2. *Hypothesis as in Lemma I.1*[6]*; then,*

(i) *If $(f!)_*\colon \pi_i(M)\to\pi_i(X)$ is an isomorphism for $0\leqq i\leqq m$, then in case* A *$(f!)_*\colon\pi_i(W_j)\to\pi_i(Y_j)$, $j=1,2$ and in case* B *$(f!)_*\colon\pi_i(W')\to\pi_i(Y')$ are isomorphisms for $0\leqq i\leqq m$.*

(ii) *If $\pi_i(M)\to\pi_i(X)$ is an isomorphism for $0\leqq i\leqq n/2$ then f is split.*

Part (i) of Lemma I.2 will be proved for $m=0$ and $m=1$ in § 3 and the proof of Lemma I.2 will be completed in § 4.

The reader may find it useful when reading this chapter to concentrate on case A, while taking note of the modifications of the notation in case B.

§ 2. Review of Covering Space Theory and Related Algebra

Apart from the introduction of notation for various components of covering spaces of Y and W, the main purpose of this section is to recall the uniqueness of normal form for elements of $G=G_1 *_H G_2$ [K 1] or $G=J *_H \{t\}$ (the geometric facts corresponding to this are described in the appendix to Chapter I), and the corresponding description of the integral group ring of G. For any group H we write $Z[H]$ or just ZH to denote the integral group ring of H.

5 If W is a P.L. or differentiable manifold, this lemma and its proof is valid for $n\geqq 4$

6 Lemma I.2, and its proof, are actually valid for all n

For any connected space V equipped with basepoint $p\in V$, write $\tilde{V}$ for the universal cover of V with covering projection $\pi_V: \tilde{V}\to V$ and let $\tilde{p}$ denote a basepoint of $\tilde{V}$ with $\pi_V(\tilde{p})=p$.

Choose a basepoint for X and Y, $p\in X\subset Y$; recall that $\pi_1(X)\to\pi_1(Y)$ is injective. Let $\hat{Y}$ denote the cover of Y with covering projection $\hat{\pi}_Y: \hat{Y}\to Y$ and with basepoint $\hat{p}$ lying over p and with $(\hat{\pi}_Y)_*(\pi_1(\hat{Y}))=\text{image}(\pi_1(X)\to\pi_1(Y))$. Assume that the basepoint $\tilde{p}\in\tilde{Y}$ has been chosen to lie over $\hat{p}$. The inclusion $X\to Y$ lifts to a unique basepoint-preserving inclusion $X\subset\hat{Y}$. Whenever X is referred to as a submanifold of $\hat{Y}$, this inclusion is meant. Similarly, there is a unique *basepoint-preserving* inclusion $\tilde{X}\subset\tilde{Y}$ lying over $X\subset Y$ or $X\subset\hat{Y}$. While other inclusions of $\tilde{X}$ in $\tilde{Y}$ will be employed, when we write $\tilde{X}\subset\tilde{Y}$, unless otherwise indicated, the unique basepoint-preserving inclusion is meant. Of course $\pi_Y^{-1}(X)$ consists of copies of $\tilde{X}$. (However, unless $\pi_1(X)$ is a normal subgroup of $\pi_1(Y)$, the various components of $\hat{\pi}_Y^{-1}(X)$ may not be homeomorphic.) From the definitions, the preferred inclusions $X\subset\hat{Y}$, $\tilde{X}\subset\tilde{Y}$ induce isomorphisms of fundamental groups.

It is easy to see that $(\tilde{Y}-\tilde{X})$ and $(\hat{Y}-X)$ both have precisely two components. Trivially, they have at most 2 components and if $\tilde{Y}-\tilde{X}$ (respectively; $\hat{Y}-X$) had one component, then $H_1(\tilde{X})\neq H_1(\tilde{Y})$ (resp; $H_1(X)\neq H_1(\hat{Y})$), contradicting $\pi_1(\tilde{X})=\pi_1(\tilde{Y})$ (resp; $\pi_1(X)=\pi_1(\hat{Y})$).

In case A, (respectively; case B) $Y=Y_1\cup_X Y_2$ (resp; $Y=Y'$/identifying X_1 with X_2, $X_1\cong X_2\cong X$) and give Y_i the basepoint $p\in X\subset Y_i$ (resp; give the spaces Y_i', where $Y'\cong Y_i'$ the basepoint $p_i\in X_i\subset Y_i'$ corresponding to $p\in X$) $i=1,2$. Let $\hat{Y}_i$ (resp; $\hat{Y}_i'$) denote the cover of Y_i (resp; Y_i') with basepoint $\hat{p}\in\hat{Y}_i$ (resp; $\hat{p}_i\in\hat{Y}_i'$) lying over p (resp; p_i) and with covering projection $\hat{\pi}_{Y_i}: \hat{Y}_i\to Y_i$ (resp; $\pi_{Y_i'}: \hat{Y}_i'\to Y_i'$) satisfying

$$(\hat{\pi}_{Y_i})_*(\pi_1(\hat{Y}_i))=\text{image}(\pi_1(X)\to\pi_1(Y_i)),$$

$$(\text{resp}; (\hat{\pi}_{Y_i'})_*(\pi_1(\hat{Y}_i'))=\text{image}\,\pi_1(X_i,p_i)\to\pi_1(Y_i',p_i)),\qquad i=1,2.$$

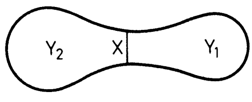
Similarly, construct the basepointed universal covers $(\tilde{Y}_i,\tilde{p})$ (resp; $(\tilde{Y}_i',\tilde{p}_i)$) with $\tilde{p}$ (resp; $\tilde{p}_i$) lying over $\hat{p}$ (resp; $\hat{p}_i$), $i=1,2$. (Note: Of course, in case B, $Y_1'\cong\tilde{Y}_2'$ but this obvious homeomorphism will not preserve basepoints. In general, there may not be a basepoint-preserving homeomorphism of Y_1' and Y_2'.) There are unique basepoint-preserving inclusions $\hat{Y}_i\to\hat{Y}$ (resp; $\hat{Y}_i'\to\hat{Y}$) lying over $Y_i\to Y$ (resp; $Y_i'\to Y$), $i=1,2$. Similarly, there are unique *basepoint preserving* inclusions $\tilde{Y}_i\to\tilde{Y}$ (resp; $\tilde{Y}_i'\to\tilde{Y}$) lying over $Y_i\to Y$ (resp; $Y_i'\to Y$). Thus when we write $\hat{Y}_i\subset\hat{Y}$ (resp; $\hat{Y}'\subset\hat{Y}$) it is always this inclusion that is meant. Similarly, unless stated otherwise, when we write $\tilde{Y}_i\subset\tilde{Y}$ (resp; $\tilde{Y}_i'\subset\tilde{Y}$) this basepoint preserving inclusion is meant, $i=1,2$. Of course, the composites of the preferred, i.e. basepoint preserving, inclusions $X\to\hat{Y}_i\to\hat{Y}$, $\tilde{X}\to\tilde{Y}_i\to\tilde{Y}$ (resp; $X\cong X_i\to\hat{Y}_i'\to\hat{Y}$, $\tilde{X}\cong\tilde{X}_i\to\tilde{Y}_i'\to\tilde{Y}$), $i=1,2$ are again the preferred inclusions.

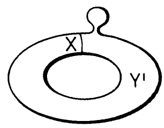
The closure of the component of $\tilde{Y}-\tilde{X}$ containing the interior of $\tilde{Y}_i$ (resp; $\tilde{Y}_i'$) will be called Y_R for $i=1$ and Y_L for $i=2$. Thus $\tilde{Y}=Y_L\cup_{\tilde{X}}Y_R$. Similarly, the closure of the component of $\hat{Y}-\hat{X}$ containing the interior of $\hat{Y}_i$ (resp; $\hat{Y}_i'$) will be called Y_r for $i=1$ and Y_l for $i=2$. Hence $\hat{Y}=Y_l\cup_X Y_r$. The action of $\pi_1(X)$ on $\tilde{Y}$ restricts to actions on $\tilde{X}$, $\tilde{Y}_1$ and $\tilde{Y}_2$ (resp; $\tilde{Y}_1'$ and $\tilde{Y}_2'$), Y_L and Y_R with the quotient spaces being X, $\hat{Y}_1$ and $\hat{Y}_2$ (resp; $\hat{Y}_1'$ and $\hat{Y}_2'$), Y_l and Y_r.

For a subgroup C of a group D, let $[D; C]$ denote the *left* cosets of C in D and let $\overline{[D; C]} = [D; C] - \{C\}$. For $\alpha \in [D; C]$, $g(\alpha)$ denotes some *fixed* choice of an element $g(\alpha) \in \alpha \subset D$. Note that if $C_1 \subset C_2 \subset D$ are groups and $\alpha_1 \in [D; C_1]$, there is a unique element $\alpha_2 \in [D; C_2]$ with $\alpha_1 \subset \alpha_2$. The set $\{g(\alpha) \mid \alpha \in [D; C]\}$ is a basis for the free left $Z[C]$ module structure of $Z[D]$.

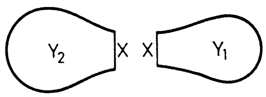
We consider an explicit description of $\tilde{Y}$ in terms of $\tilde{Y}_1$ and $\tilde{Y}_2$ (resp; $\tilde{Y}_1'$ and $\tilde{Y}_2'$), [W 1] the corresponding explicit decomposition of $G_1 *_H G_2$ (resp; $J *_H \{t\}$) in terms of G_1, G_2 (resp; J, ξ_1 and ξ_2) and H, [K] [W 1], and the corresponding explicit description of $Z[G_1 *_H G_2]$ (resp; $Z[J *_H \{t\}]$) in terms of $Z[G_1]$, $Z[G_2]$ (resp; $Z[J]$, ξ_i: $Z[H] \to Z[J]$, $i = 1, 2$) and $Z[H]$ [W 1] [St].

For a subset S of $\tilde{Y}$ and $g \in \pi_1(Y)$, Sg denotes the image of S under the covering translation corresponding to g. Thus adopting the conventions of [W 2], $S(g_1 g_2) = (Sg_1) g_2$, $g_1, g_2 \in G$.[7] Recall that writing $H = \pi_1(X)$, $G_i = \pi_i(Y_i)$ (resp; $J = \pi_1(Y', p_1)$ with ξ_i: $\pi_1(X) \to J$ the two inclusions induced from $X_i \to Y'$) we have $G = \pi_1(Y) = G_1 *_H G_2$ (resp; $J *_H \{t\}$, as explained in more detail below).





Y in case A　　　　Y in case B



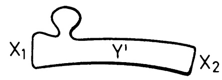


Diagram of $\tilde{Y}$ in case A:

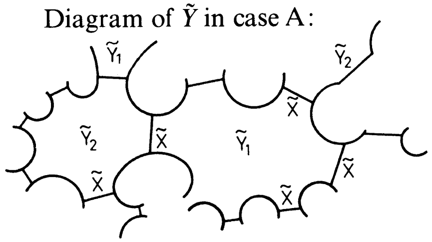
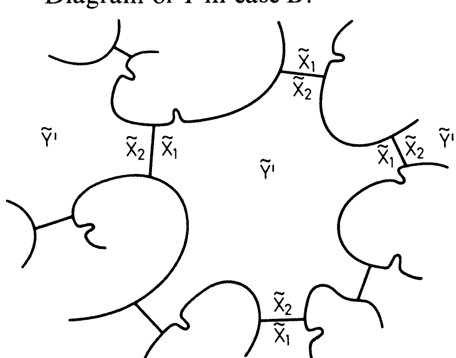


Diagram of $\tilde{Y}$ in case B:



[7] The corresponding convention for multiplication in $\pi_1(X)$ is that for loops α and β, $\alpha\beta$ denotes first tracing the loop β followed by that of α

In case A, recall the preferred embeddings of $\tilde{Y}_1$, $\tilde{Y}_2$, $\tilde{X}$ in $\tilde{Y}$ with $\tilde{Y}_1 \cap \tilde{Y}_2 = \tilde{X}$. Then

(1) $\tilde{Y} = \bigcup_{\alpha\in[G;G_1]} \tilde{Y}_1 g(\alpha) \cup \bigcup_{\alpha\in[G;H]} \tilde{X} g(\alpha) \bigcup_{\alpha\in[G;G_2]} \tilde{Y}_2 g(\alpha)$

Here, for $\alpha\in[G;H]$ and β the unique element of $[G;G_i]$ with $\alpha\subset\beta$, $\tilde{X}g(\alpha)\subset \partial Y_i g(\beta)$, $i=1,2$. In particular

(2) $\partial Y_i = \bigcup_{\alpha\in[G_i;H]} \tilde{X} g(\alpha)$.

Thus, we have the above picture of $\tilde{Y}$. (The reader may find it useful to recall the universal covering space of the figure "eight" with basepoint at the intersection of the two circles. For further details see the appendix to Chapter I.)

In case B, recall the preferred inclusions of $\tilde{X}$, $\tilde{Y}_1'$ and $\tilde{Y}_2'$ in $\tilde{Y}$, and of $\tilde{X}_i$ in $\tilde{Y}_1'$. We have in $\tilde{Y}$, $\tilde{Y}_1'\cap\tilde{Y}_2'=\tilde{X}_1=\tilde{X}_2=\tilde{X}$. Let $J=\pi_1(Y_2',p_1)$ and t a path in $Y_1'=Y'$ from p_1 to p_2. Of course, t represents an element of $\pi_1(Y,p)$. Then, image of $\pi_1(Y_2',p_2)\to\pi_1(Y,p)$ is tJt^{-1}. Setting $H=\pi_1(X)$ we have the inclusions ξ_1: $H=\pi_1(X)=\pi_1(X_1)\to\pi_1(Y_1')=J$, and ξ_2: $H=\pi_1(X)=\pi_1(X_2)\to\pi_1(Y_2')\xrightarrow{c} J$ where $c(x)=t^{-1}xt$. Then $\xi_1(x)=t\xi_2(x)t^{-1}$ and $\pi_1(Y)=G=J *_H\{t\}=Z*J/\{t^{-1}\xi_1(x)t\xi_2(x)^{-1},\ x\in H\}$. Thus

(3) $\tilde{Y} = \bigcup_{\alpha\in[G,J]} Y_1' g(\alpha) = \bigcup_{\beta\in[G,tJt^{-1}]} \tilde{Y}_2' g(\beta)$

(4) with $\tilde{Y}_1' g(\alpha)=\tilde{Y}_2' g(\beta)$ for $\beta=\alpha t^{-1}$.

Moreover, in the decompositions of $\tilde{Y}$ of (3), we have in addition to that of (4) the following further identifications.

Let $\alpha\in[G;\xi_1(H)]=[G;t\xi_2(H)t^{-1}]$, and let β_1 denote the unique element of $[G;J]$ with $\alpha\subset\beta_1$ and hence $\tilde{X}_1 g(\alpha)\subset\tilde{Y}_1' g(\beta_1)$. Similarly, let β_2 denote the unique element of $[G;tJt^{-1}]$ with $\alpha\subset\beta_2$, and hence $\tilde{X}_2 g(\alpha)\subset Y_2' g(\beta)$. Then

(5) $\tilde{X}_1 g(\alpha)$ as a subset of $Y_1' g(\beta_1)$ is identified with

$\tilde{X}_2 g(\alpha)$ as a subset of $\tilde{Y}_2' g(\beta_2)$.

Explicitly, the boundary of $\tilde{Y}_i'$ is given by, $i=1,2$,

(6) $\partial(\tilde{Y}_1') = \bigcup_{\alpha\in[J;\xi_1(H)]} \tilde{X}_1 g(\alpha) \cup \bigcup_{\beta\in[J;\xi_2(H)]} \tilde{X}_2 t g(\beta)$

(7) $\partial(\tilde{Y}_2') = \bigcup_{\alpha\in[J;\xi_2(H)]} \tilde{X}_2 t g(\alpha) t^{-1} \cup \bigcup_{\beta\in[J;\xi_1(H)]} \tilde{X}_1 g(\beta) t^{-1}$

In these two decompositions, every component of $\partial(\tilde{Y}_i')$ labeled $\tilde{X}_1 k$, $k\in G$ lies over X_1 for the covering projection $\tilde{Y}_i'\to\tilde{Y}_i'=Y'$, $i=1,2$. Similarly, every component of $\partial(\tilde{Y}_i')$ labeled $\tilde{X}_2 k$, some k, lies over X_2, $i=1,2$.

(8) Note that from (4) above, $\tilde{Y}_2'=\tilde{Y}_1' t^{-1}$ and thus (7) can be obtained by applying t^{-1} to (6).

In the geometry of Chapters I, II, III, we adopt a convenient slightly different notation to describe case B. As $\xi_1(H)\subset J=\pi_1(Y_1')$ and $t\xi_2(H)t^{-1}\subset tJt^{-1}$, $\xi_1(H)=$

$t\xi_2(H)t^{-1}$ in G, it is natural to simply identify $x\in H$ with $\xi_1(x)=t\xi_2(x)t^{-1}$. Thus we can write $H\subset J$, $H\subset tJt^{-1}$ and the map of $H=\pi_1(X)\to\pi_1(Y)=G$ is the inclusion $H\subset G$. Note that as, $\tilde{Y}_1'\subset\tilde{Y}$, the map of $J=\pi_1(Y_1')\to\pi_1(Y)=G$ also is injective. *Thus in our new notation we have the inclusions*

$$\begin{tikzcd}[column sep=small] & J \arrow[dr, hook] & \\ H \arrow[ur, hook] \arrow[dr, hook] & & G=J*_H\{t\} \\ & tJt^{-1} \arrow[ur, hook] & \end{tikzcd}$$

and Eqs. (6) and (7) can be conveniently rewritten

(9) $\quad \partial(\tilde{Y}_1')=\bigcup_{\alpha\in[J;H]}\tilde{X}_1\,g(\alpha)\cup\bigcup_{\beta\in[tJt^{-1};H]}\tilde{X}_2\,g(\beta)t,$

(10) $\quad \partial(\tilde{Y}_2')=\bigcup_{\alpha\in[tJt^{-1};H]}\tilde{X}_2\,g(\alpha)\cup\bigcup_{\beta\in[J;H]}\tilde{X}_1\,g(\beta)t^{-1}.$

Again note that in (9) and (10) the components of $\partial\tilde{Y}_i'$ labeled $\tilde{X}_jk$ lies over X_j in the projection $\partial\tilde{Y}_i'\to\tilde{Y}'$. Eq. (10) can be obtained, see (4), by applying t^{-1} to both sides of (9).

We recall the unique normal form for elements of $G=G_1*_HG_2$ [K1] and $G=J*_H\{t\}$. This can be derived from a description of $\tilde{Y}$ using trees as recalled and outlined in the appendix to Chapter I. The geometric significance of the normal form is discussed there.

Proposition. *Every element $g\in G$ can be written uniquely in the form $h=hk_1,k_2\dots k_n$ where $h\in H$ and*

(i) *for $G=G_1*_HG_2$, $k_i=g(\alpha)$, for some $\alpha\in\overline{[G_1;H]}\cup\overline{[G_2:H]}$, $1\leqq i\leqq n$ and*

$\{k_i,k_{i+1}\}\not\subset\overline{[G_j;H]}$, $\;j=1,2\quad$ *for* $\;1\leqq i\leqq n-1$ [K1];

(ii) *for $G=J*_H\{t\}$, k_i has the form $g(\alpha)$ or $g(\beta)$ or $g(\gamma)t$ or $g(\delta)t^{-1}$ for $\alpha\in\overline{[J;H]}$, $\beta\in\overline{[tJt^{-1};H]}$, $\gamma\in[tJt^{-1};H]$ $\delta\in[J;H]$, $1\leqq i\leqq n$, and if k_j has the form $g(\alpha)$ or $g(\delta)t^{-1}$* (resp; $g(\beta)$ or $g(\gamma)t$) *then k_{j-1} has the form $g(\beta)$ or $g(\delta)t^{-1}$* (resp; $g(\alpha)$ or $g(\gamma)t$).

Example. If $J=H=tJt^{-1}$ so that $G=Z\times H$, Z generated by t, then $\overline{[J;H]}=\overline{[tJt^{-1};H]}=\emptyset$ and $[J,H]=[tJt^{-1}]=\{\alpha\}$ where $\alpha=H$ and we may take $g(\alpha)=1\in H$. Hence, in this case uniqueness of normal form asserts that every element g of $Z\times H$ can be written uniquely as either $g=ht^i$ or as ht^{-i}, $i\geqq 1$ or as $g=h$, $h\in H$.

Lastly, consider a description of the ring $Z[G]$, $G=G_1*_HG_2$ or $G=J*_H\{t\}$ corresponding to the description of elements of G in the normal form. Note first that a fixed choice of elements $g(\alpha)\in\alpha$ for $\alpha\in\overline{[G_1;H]}\cup\overline{[G_2;H]}$ (resp; $\alpha\in\overline{[J;H]}\cup\overline{[tJt^{-1};H]}$ determines a choice $g(\beta)\in\beta$ for all $\beta\in[G;H]$. In fact, let $g(\beta)$ be the unique element of β which can be written in normal form with $h=1$. This choice of elements $g(\beta)$ provides a basis, called the *normal form basis for the left ZH module structure of ZG.*

Now in case A, define $Z[\widetilde{G_i}]$ (also written $\widetilde{ZG_i}$) to be the additive subgroup of $Z[G_i]$ generated additively by $g\in\{G_i-H\}\subset Z[G_i]$, $i=1,2$. Of course, using the ring inclusion $Z[H]\subset Z[G_i]$, $Z[G_i]$ is a bimodule over $Z[H]$ and, as $\{G_i-H\}\subset G_i$

is invariant under left and right multiplication by H, $Z[\widetilde{G_i}]$ is a $Z[H]$ sub-bimodule of $Z[G_i]$. In fact, as a $Z[H]$ bimodule, $Z[G_i]\cong Z[H]\oplus Z[\widetilde{G_i}]$. The bimodule $Z[\widetilde{G_i}]$ is free as a left $Z[H]$ module; $\{g(\alpha)|\alpha\in\overline{[G_i;H]}\}$ is a basis. Similarly, $Z[\widetilde{G_i}]$ is free as a right $Z[H]$ module (though, if H is not normal in G_i there may not be a set which is simultaneously a left and right basis). Of course, any tensor product over $Z[H]$ of $Z[H]$ bimodules is again a $Z[H]$ bimodule. We claim that as a $Z[H]$ bimodule

$$Z[G_1 *_H G_2]\cong ZH\oplus Z\widetilde{G}_1\oplus Z\widetilde{G}_2\oplus Z\widetilde{G}_1\otimes_{ZH}Z\widetilde{G}_2\oplus Z\widetilde{G}_2\otimes_{ZH}Z\widetilde{G}_1$$
$$\oplus Z\widetilde{G}_1\otimes_{ZH}Z\widetilde{G}_2\otimes_{ZH}Z\widetilde{G}_1\oplus \dots$$

More precisely, let A_i, B_i, C_i, D_i be defined inductively by the following:

$$A_1=Z\widetilde{G}_1,\quad B_1=0,\quad C_1=Z\widetilde{G}_2,\quad D_1=0$$

$$A_{i+1}=Z\widetilde{G}_1\otimes_{ZH}D_i,\quad B_{i+1}=Z\widetilde{G}_1\otimes_{ZH}C_i,\quad C_{i+1}=Z\widetilde{G}_2\otimes_{ZH}B_i$$

$$D_{i+1}=Z\widetilde{G}_2\otimes_{ZH}A_i,\quad i\geqq 1.$$

Obviously for any i, two of the terms A_i, B_i, C_i, D_i are zero. A_i (resp; B_i; C_i; D_i), when not zero, consists of a tensor product with i-terms beginning on the right with ZG_1 (resp; $Z\widetilde{G}_2$; $Z\widetilde{G}_2$; $Z\widetilde{G}_1$), with the terms *alternating* in $Z\widetilde{G}_1$ and $Z\widetilde{G}_2$, and ending on the left with $Z\widetilde{G}_1$ (resp; $Z\widetilde{G}_1$; $Z\widetilde{G}_2$; $Z\widetilde{G}_2$). Clearly such a tensor product exists only for i odd (resp; even; odd; even). The multiplication in ZG induces a map of ZH *bimodules*, which is an isomorphism [St]:

$$Z[G_1 *_H G_2]=ZH\otimes\sum_{i=1}^{\infty}A_i\oplus\sum_{i=1}^{\infty}B_i\oplus\sum_{i=1}^{\infty}C_i\oplus\sum_{i=1}^{\infty}D_i.$$

This is immediate from the uniqueness of normal form for elements of $G_1 *_H G_2$ and examination of the normal form basis for ZG. Note that the normal form basis is a union of bases for $Z[H]$, A_i, B_i, C_i, D_i, $i\geqq 1$. The tensor products A_i, B_i, C_i, D_i will be identified with their images in $Z[G_1 *_H G_2]$.

We give a parallel analysis, in case B, of $Z[J *_H \{t\}]$. Let $\widetilde{Z[J]}$ denote the additive subgroup of $Z[J]$ generated additively by $g\in\{J-H\}\subset Z[J]$. Then, as in case A, as a $Z[H]$ bimodule, $Z[J]\cong Z[H]\oplus\widetilde{Z[J]}$ and $\widetilde{Z[J]}$ is free as a right and as a left module. In fact, as a left $Z[H]$ module, $\{g(\alpha)|\alpha\in\overline{[J;H]}\}$ is a basis for $\widetilde{Z[J]}$. Similarly, use the inclusion of groups $H\subset tJt^{-1}$ to give $Z[tJt^{-1}]$ the structure of a $Z[H]$ bimodule. Letting $Z\widetilde{[tJt^{-1}]}$ be the additive subgroup of $Z[tJt^{-1}]$ additively generated by $g\in\{tJt^{-1}-H\}\subset Z[tJt^{-1}]$, $Z\widetilde{[tJt^{-1}]}$ is a $Z[H]$ bimodule, free as both a left and as a right ZH mocule. A basis for the left module structure is given by $\{g(\beta)|\beta\in\overline{[tJt^{-1};H]}\}$. (Note that in the ξ_1,ξ_2 notation, this bimodule is isomorphic to the $Z[\xi_2(H)]$ bimodule generated by the elements of $(J-\xi_2(H))\subset Z[J]$.)

The additive subgroup $Z[tJ]$ of $Z[J *_H \{t\}]$ generated additively by elements of the form (tg), $g \in J$ is also a $Z[H]$ bimodule. The right module structure is obvious from $H \subset J$ and the left bimodule structure is obvious from $H \subset tJt^{-1}$. (In the ξ_1, ξ_2 notation, as a right module $Z[tJ]$ is isomorphic to the obvious right $Z[\xi_1(H)]$ module structure on $Z[J]$; as a left module it is isomorphic to the left $Z[\xi_2(H)]$ module structure on $Z[J]$). Similarly, $Z[Jt^{-1}]$, the additive subgroup of $Z[J *_H \{t\}]$ generated additively by $Jt^{-1} \subset Z[J *_H \{t\}]$ is a $Z[H]$ sub-bimodule of $Z[J *_H \{t\}]$. (In the ξ_1, ξ_2 notation, the left module structure is isomorphic to the left $Z[\xi_1(H)]$ module structure of $Z[J]$ and the right module structure is isomorphic to the right $Z[\xi_2(H)]$ module structure of $Z[J]$.) Both $Z[tJ]$ and $Z[Jt^{-1}]$ are free as left and as right $Z[H]$ modules. A basis for the left module structure of $Z[tJ]$ is given by $\{g(\gamma)t \mid \gamma \in [tJt^{-1}; H]\}$ and for $Z[Jt^{-1}]$ by $\{g(\delta)t^{-1} \mid \delta \in [J; H]\}$.

Summing up, $\widetilde{Z[J]}$, $\widetilde{Z[tJt^{-1}]}$, $Z[tJ]$, $Z[Jt^{-1}]$ are $Z[H]$ bimodules, free as left and as right modules. Thus their tensor products, over $Z[H]$ are again $Z[H]$ bimodules. Moreover, using the ring structure of $Z[J *_H \{t\}]$, there is an obvious map of any such tensor product to $Z[J *_H \{t\}]$. We claim that as a $Z[H]$ bimodule:

$$\begin{aligned} Z[J *_H \{t\}] \cong Z[H] &\oplus \widetilde{Z[J]} \oplus \widetilde{Z[tJt^{-1}]} \oplus Z[tJ] \oplus Z[Jt^{-1}] \\ &\oplus (Z[tJt^{-1}] \oplus Z[Jt^{-1}]) \otimes_{ZH} (\widetilde{Z[J]} \oplus Z[Jt^{-1}]) \\ &\oplus (\widetilde{Z[J]} \oplus Z[tJ]) \otimes_{ZH} (Z[tJt^{-1}] \oplus Z[tJ]) \\ &\oplus \text{terms with 3 tensor products} \oplus \cdots. \end{aligned}$$

More precisely, define A_i, B_i, C_i, D_i inductively as follows

$$A_1 = \widetilde{ZJ}, \quad B_1 = Z[Jt^{-1}], \quad C_1 = \widetilde{Z[tJt^{-1}]}, \quad D_1 = Z[tJ]$$

$$A_{i+1} = \widetilde{ZJ} \otimes_{ZH} D_i \oplus Z[Jt^{-1}] \otimes_{ZH} A_i$$

$$B_{i+1} = Z[Jt^{-1}] \otimes_{ZH} B_i \oplus \widetilde{ZJ} \otimes_{ZH} C_i$$

$$C_{i+1} = Z[tJ] \otimes_{ZH} C_i \oplus Z[tJt^{-1}] \otimes_{ZH} B_i$$

$$D_{i+1} = Z[tJ] \otimes_{ZH} D_i \oplus Z[tJt^{-1}] \otimes_{ZH} A_i$$

Then,

$$Z[J *_H \{t\}] \cong ZH \oplus \sum_{i=1}^{\infty} A_i \oplus \sum_{i=1}^{\infty} B_i \oplus \sum_{i=1}^{\infty} C_i \oplus \sum_{i=1}^{\infty} D_i$$

as a ZH bimodule.

This is immediate from examination of the normal form basis of $Z[J *_H \{t\}]$. Note that the normal form basis is, as in case A, a union of bases for ZH, A_i, B_i, C_i, D_i, $i \geqq 1$.

The geometric interpretation of the decompositions of $Z[G]$, $G = G_1 *_H G_2$ or $G = J *_H \{t\}$ as a left $Z[H]$ module into the summands $Z[H]$, A_i, B_i, C_i, D_i is described in the appendix to Chapter I.

In both case A and case B, the inductive definition of $A_{i+1}, B_{i+1}, C_{i+1}, D_{i+1}$ is summed up by:

$$A_{i+1}=A_1 D_i+B_1 A_i$$
$$B_{i+1}=B_1 B_i+A_1 C_i$$
$$C_{i+1}=D_1 C_i+C_1 B_i$$
$$D_{i+1}=D_1 D_i+C_1 A_i.$$

From this an easy induction shows that for $i\geqq 1$,

$$A_{i+1}=A_i D_1+B_i A_1=A_i\otimes_{ZH} D_1\oplus B_i\otimes_{ZH} A_1$$
$$B_{i+1}=B_i B_1+A_i C_1=B_i\otimes_{ZH} B_1\oplus A_i\otimes_{ZH} C_1$$
$$C_{i+1}=D_i C_1+C_i B_1=D_i\otimes_{ZH} C_1\oplus C_i\otimes_{ZH} B_1$$
$$D_{i+1}=D_i D_1+C_i A_1=D_i\otimes_{ZH} D_1\oplus C_i\otimes_{ZH} A_1.$$

Example. If $H=J=tJt^{-1}$ so that $J *_H \{t\}=Z\times H$, $A_i=C_i=0$, all i and $D_i=Z[H]t^i$, $i\geqq 1$, $B_i=Z[H]t^{-i}$, $i\geqq 1$.

§ 3. Handle Exchanges and Low Dimensions

Before proving Lemmas I.1 for $m=0$ and $m=1$, f will be made transverse regular to X in the following prescribed manner. *As f is varied by homotopies, where no confusion will result, the new map obtained will continue to be called f.*

Let C' be a finite CW 2-skeleton for $Y-N$, N a tubular neighborhood of $X\subset Y$ and let β be a cellularly embedded arc in Y, intersecting X transversally in one point and with $\beta\cap C'=$endpoints of β. Set $C=C'\cup\beta$ and denote by i the inclusion $i\colon C\to Y$. Letting g denote a homotopy inverse for f, gi is, as $\dim Y>2\dim C$, homotopic to an embedding $h\colon C\to W$. Hence, as $fh\sim fgi\sim i$, by the homotopy extension principle [Sp] f can be varied by a homotopy to achieve $fh=i$. Then f is homotopic, by a homotopy fixed on $h(C)$, to a map transverse regular[8] to X. Write $M=f^{-1}(X)$; M is a codimension one submanifold of W with, as the normal bundle of X in Y is trivial, a trivial normal bundle in W. Below, *as f is varied by homotopies* to obtain maps still transverse to X, we continue to denote $f^{-1}(X)$ by M.

For V a subspace of Y, the restriction of f to $U=f^{-1}(V)$ will usually be denoted by $f\,!\colon U\to V$. The cover of W corresponding to the image of $\pi_1(X)\to\pi_1(Y)\cong\pi_1(W)$ is denoted by $\hat{W}$ and the covering projection by $\hat{\pi}_W\colon \hat{W}\to W$. The induced map covering f will be denoted by $\hat{f}\colon \hat{W}\to\hat{Y}$ and the map induced by f on the unversal covers will be denoted by $\tilde{f}\colon \tilde{W}\to\tilde{Y}$. For the maps induced by restrictions of $\tilde{f}$ and $\hat{f}$ on $\tilde{f}^{-1}(S)\to S$, $S\subset\tilde{Y}$ or $\hat{f}^{-1}(T)\to T$, $T\subset\hat{Y}$, we write $\tilde{f}\,!$ and $\hat{f}\,!$ respectively.

Recall the notation W_1, W_2 in case A and W' in case B employed in the statement of Lemma I.2 and the maps $f\,!$ on these spaces. Similarly, define $W_R=\tilde{f}^{-1}(Y_R)$, $W_L=\tilde{f}^{-1}(Y_L)$, $W_r=\tilde{f}^{-1}(Y_r)$, $W_l=\tilde{f}^{-1}(Y_l)$ and in case A $\hat{W}_i=\hat{f}^{-1}(\hat{Y}_i)$, and in case B $\hat{W}_i'=\hat{f}^{-1}(\hat{Y}_i')$. It will be convenient in case B to employ a slight terminological

[8] In the topological case, this uses [KS]

abuse and refer to Y' as a subspace of Y; this is of course possible as Y' may be identified with the complement of a tubular neighborhood of X in Y. Similar remarks apply to $W' \subset W$.

We proceed to make the map $f!: M \to X$ increasingly connected. The submanifold M will be modified ambiently in W by surgeries corresponding to "handle exchanges". This procedure is based upon the following essentially well known technique:

Lemma I.3. *Handle-exchanging*

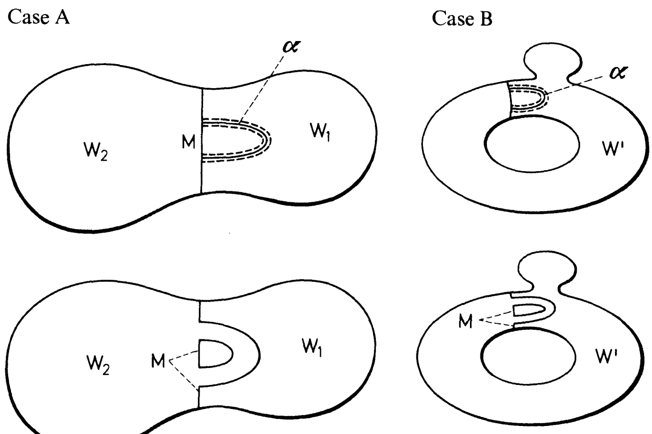
(i) *Let* $\alpha \in \pi_i(W_j, M)$ (resp; $\pi_i(W', M_j)$) *with* $(f!)_*(\alpha) = 0$ *in* $\pi_i(Y_j, M)$ (resp; $\pi_i(Y', X_j)$), $j = 1$ or 2. *Then if* $2i < n+1$, α *can be represented by an embedding* $\alpha: (D^i, S^{i-1}) \times D^{n+1-i} \to (W_j, M)$ (resp; (W', M_j)).

(ii) *Given an embedding* $\alpha: (D^i, S^{i-1}) \times D^{n+1-i} \to (W_j, M)$ (resp; (W', M_j)) $j = 1$ *or* 2. *Let T denote a neighborhood of* $M \cup (\mathrm{image}(\alpha))$. *Then f is homotopic to a map f', by a homotopy fixed outside of T, with* $f'^{-1}(Y_k) = W_k \cup \mathrm{image}(\alpha)$ $k \neq j$ (resp; *with, letting W'' be the manifold obtained by cutting Y along* $f'^{-1}(X)$,

$$W'' = (W' - \mathrm{interior}(\mathrm{image}(\alpha)) \cup_{S^{i-1} \times D^{n+1-i}} D^i \times D^{n+1-i}.$$

Here the map $S^{i-1} \times D^{n+1-i} \to M_K \subset (W' - \mathrm{interior}(\mathrm{image}(\alpha))$ *corresponds, under the identification of M_k with M_j, $k \neq j$, to $\partial\alpha$.) In particular,* $f'^{-1}(X) = M'$, *where M' is obtained from M by a surgery on the restriction of α to* $\partial\alpha: S^{i-1} \times D^{n+1-i} \to M$.

In the differentiable case, the corners of α in part (ii) of Lemma I.3 should be rounded [CF].



Handle-exchanging in W

Proof of Lemma I.3. We briefly outline this standard exercise in general position, for part (i), and in the homotopy extension principle, for part (ii). [FH1] [W2]. The class $\alpha \in \pi_i(W_j, M)$ is represented by an embedding $\alpha: (D^i, S^{i-1}) \to (W_j; M)$ as $2i < \dim W = n+1$. Moreover as the normal bundle of $\alpha(D^i)$ is trivial the map extends to an embedding $\alpha\colon (D^i, S^{i-1}) \times D^{n+1-i} \to (W_j, M)$. As $f\alpha$ is, as a map of pairs, null-homotopic, we have a null-homotopy, relative to the boundary, of the restriction of f to image (α). Extending this by the homotopy extension principle, we construct f'. (resp; the proof in case B is similar; we omit further details.)

Proof of Lemma I.1 for $m=0$. As f is transverse to $X \subset Y$ with W, X, Y compact, M has a finite number of components. We show how, if M has more than one component, to reduce the number of them. It suffices, using Lemma I.3, to construct an arc in case A $\alpha\colon (I, \partial I) \to (W_j, M)$, (resp; in case B, $\alpha\colon (I, \partial I) \to (W', M_j)$), $j=1$ or 2, α joining two components of M and with $f!(\alpha) \in \pi_1(Y_j, X)$ (resp; $\pi_1(Y', X_j)$) the trivial element. Varying α by a homotopy, we may further assume that $\alpha(I) \cap h(C) = \emptyset$. Clearly, a handle exchange on α reduces the number of components of M and we may assume that we still have $fh=i$.

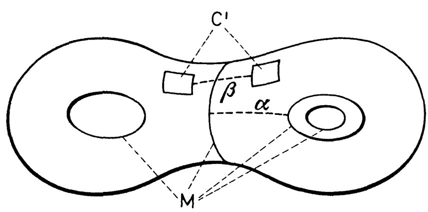


Diagram of W

To construct an arc α with the prescribed properties, it will be convenient to describe the decomposition of $W-M$ into components by means of a tree T. Let T have one vertex for each component of $W-M$ and one edge for each component of M; the incidence relation is defined by having an edge corresponding to a component M_0 of M join the vertices corresponding to the components of $W-M$ whose closures contain M_0.

The graph T is connected. To see this, observe that there is easily constructed embedding $T \to W$ sending each vertex v to a point in the component of $W-M$ corresponding to v; as there is also a retraction $W \to T$ and W is connected, T is connected. (In a sense, T is a kind of "dual cell" complex to $M \subset W$.)

Corresponding to the component of M which intersects $h(\beta)$, there is an edge $\bar{d}$ of T. (Note that in case B, from the construction of C' and β, the two endpoints of $\bar{d}$ are the same point.) As $h(C')$ is contained in the components of $W-M$ corresponding to the endpoints of $\bar{d}$, the fundamental groups of these two (resp; in case B, one) components go onto $\pi_1(Y_1)$ and $\pi_1(Y_2)$ (resp; $\pi_1(Y')$).

If T has more than one edge, that is if M has more than one component, there is in particular, as T is connected, an edge d' with $d' \neq \bar{d}$, $d' \cap \bar{d}$ contains a vertex v_0 of $\bar{d}$ and d'.

In case A let V denote the closure of the component of $W-M$ corresponding to v_0; let α' be an arc in V connecting the component of M corresponding to d' to the component of M corresponding to $\bar{d}$. Then, from the construction of $h(C')$ and $\bar{d}$, $\pi_1(V) \xrightarrow{f!_*} \pi_1(Y_j)$, $j=1$ or 2 is surjective. As $\pi_1(Y_j) \to \pi_1(Y_j, X)$ is also surjective, replacing α' by α, the sum of α' and a loop representing an appropriately chosen element of $\pi_1(V)$, we may assume that $f!_*(\alpha)\in\pi_1(Y_j; X)$ is trivial. This completes the proof of Lemma I.1 for $m=0$ in case A.

In case B, let V denote the component of W' corresponding to v_0. Recall the notation W', M_1, M_2 of §1. From the construction of C', C and $\bar{d}$, there corresponds to $\bar{d}$ a component $\overline{M}_1$ in $\partial V\cap M_1$ and $\overline{M}_2$ in $\partial V\cap M_2$. Moreover, d' corresponds to at least one component M' of ∂V, $\partial V\subset M_1\cup M_2$. Say that $M'\subset M_j$, $j=1$ or 2; let α' be an arc in V from M' to $\overline{M}_j$. From the construction of $h(C')$, $\pi_1(V)\to\pi_1(Y')$ is surjective and as $\pi_1(Y')\to\pi_1(Y', X_j)$ is surjective, by replacing α' by α, the sum of α' and a loop representing an appropriately chosen element of $\pi_1(V)$, we may assume that $(f!_*)(\alpha)\in\pi_1(Y_j, X)$ is trivial.

Proof of Lemma I.2, part (i), for $m=0$. If M is connected, conclude that W_1 and W_2 (resp; W') are connected by examining a part of the Mayer-Vietoris sequence of

$$W=W_1\cup_M W_2 \qquad (\text{resp; } W=W'/M_1=M_2):$$

$$H_0(M)\to H_0(W_1)\oplus H_0(W_2)\to H_0(W) \qquad (\text{resp; and in case B},$$

$$\begin{tikzcd}
H_1(W) \arrow[r] \arrow[d, "\cong"] & H_0(M) \arrow[r] \arrow[d, "\cong"] & H_0(W') \arrow[r] & H_0(W)) \\
H_1(Y) \arrow[r] & H_0(X) & &
\end{tikzcd}$$

Proof of Lemma I.1 for $m=1$. By Lemma 1 for $m=0$, we may assume that M is connected. As f is a homotopy equivalence, the induced map $f!\colon M\to X$ is of degree one [B3]. Hence, $f!_*\colon \pi_1(M)\to\pi_1(X)$ is surjective. (This standard fact about degree 1 maps is proved by observing that $f!$ factors through the cover of X corresponding to $f!_*(\pi_1(M))\subset\pi_1(X)$.) We need the following standard result to complete the argument.

Lemma I.4. *Let $\phi\colon G\to H$ be an epimorphism of groups with G a finitely generated and H a finitely presented group; then* $\operatorname{Ker}\phi$ *is the normal closure of a finitely generated subgroup.*

Proof. Let $g_1,\dots,g_r$ be generators for G and $h_1,\dots,h_s$ generators for H with $w_j(h_1,\dots,h_s)$, $1\leqq j\leqq t$ words in $h_1,\dots,h_s$ which generate the relations of H. As ϕ is surjective, choose $h_i'\in G$ with $\phi(h_i')=h_i$, $1\leqq i\leqq s$. As $\{h_i\}$ generates H, we can write $\phi(g_j)=v_j(h_1,\dots,h_s)$, $1\leqq j\leqq r$ where v_j are words in $\{h_i\}$. Now let K be the subgroup of G generated by the finite set $\{w_i(h_1',\dots,h_s'), v_j(h_1',\dots,h_s')g_j^{-1}\}$ $1\leqq i\leqq t$, $1\leqq j\leqq r$.

Clearly $K\subset\operatorname{Ker}\phi$ and the argument will be completed by showing that the projection $p\colon G\to G/\langle K\rangle$ is $\sigma\phi$, σ a map $H\to G/\langle K\rangle$. Let $\sigma(h_i)=p(h_i')$; as $\sigma(w_i(h_1,\dots,h_s))=p(w_i(h_1',\dots,h_s'))$, σ is a well-defined homomorphism. Moreover, $\sigma\phi(g_i)=\sigma v_j(h_1,\dots,h_s)=v_j(h_1',\dots,h_s')=p(g_i)$ and hence $p=\sigma\phi$.

As $\pi_1(M)$ and $\pi_1(X)$ are fundamental groups of compact manifolds (or possibly X a compact Poincaré complex), they are finitely presented groups. Hence, by Lemma I.4, there are elements $\alpha_1,\dots,\alpha_t\in\operatorname{Ker}(f!_*\colon\pi_1(M)\to\pi_1(X))$ with $\operatorname{Ker}(f!_*)$ $=\langle\alpha_1,\dots,\alpha_t\rangle$ in $\pi_1(M)$.

From the diagram induced by f

$$\begin{tikzcd} \pi_2(W,M) \arrow[r, "\partial"] & \pi_1(M) \arrow[r] \arrow[d] & \pi_1(W) \arrow[d] \\ & \pi_1(X) \arrow[r] & \pi_1(Y) \end{tikzcd}$$

as $\pi_1(X)\to\pi_1(Y)$ is injective, $\exists\beta_1\in\pi_2(W,M)$ with $\partial\beta_1=\alpha_1$. Represent β_1 by an embedded disc $D\subset W$ which meets M transversally. Then $D\cap W$ consists of a finite union of disjoint circles in D. We give a procedure for reducing the number of these circles, while replacing M with $M'=f^{-1}(X)$, where $\pi_1(M')$ is a quotient of $\pi_1(M)$.

Choose an innermost circle among the circles of $D\cap M\subset D$; that is, a circle bounding a disc D' in D with (interior $D')\cap M=\emptyset$. The disc D' represents a class in $\pi_2(W_i,M)$ (resp; in case B, $\pi_2(W',M_i)$), $i=1$ or 2. Moreover, in the diagram induced by f

Case A

$$\begin{tikzcd} \pi_2(W_i) \arrow[r] \arrow[d] & \pi_2(W_i,M) \arrow[r] \arrow[d] & \pi_1(M) \arrow[d] \\ \pi_2(Y_i) \arrow[r] & \pi_2(Y_i,X) \arrow[r] & \pi_1(X) \end{tikzcd}$$

Case B

$$\begin{tikzcd} \pi_2(W') \arrow[r] \arrow[d] & \pi_2(W',M_i) \arrow[r] \arrow[d] & \pi_1(M) \arrow[d] \\ \pi_2(Y') \arrow[r] & \pi_2(Y',X_i) \arrow[r] & \pi_1(X) \end{tikzcd}$$

the map $f!_*\colon \pi_2(W_i)\to\pi_2(Y_i)$ (resp; $\pi_2(W')\to\pi_2(Y')$) is, as $fh(C')=i(C')\subset Y-N$, *surjective*. Hence, by an elementary diagram chase, there is a 2-disc D'' in (W_i,M) (resp; (W',M_i)) with $\partial D''=\partial D'$ in M and $f!_*(D'')\in\pi_2(Y_i,X)$ (resp; $\pi_2(Y',X_i)$) is *trivial*. Now the disc $\bar{D}=(D-D')\cup D''$ has the same boundary as D. Perform using Lemma I.3 a handle-exchange on D'' to obtain M' by a surgery on M. As $\dim D''=2$, it is easy to see that $\pi_1(M')$ is a quotient of $\pi_1(M)$. Note that as $\dim D+\dim C=4<\dim W$, all these discs can be chosen to not intersect C; hence we still have $fh=i$. Moreover, $D''\cap M'=\emptyset$ and thus, $\bar{D}\cap M'$ has one less component than $D\cap M$.

Proceeding in this manner, after eliminating all the components of (interior $D)\cap M$, the above procedure produces a disc with boundary α_1 on which a handle-exchange, again as above varying the disc, can be performed. Thus, handle-exchanges can be performed to "kill" the classes α_1, α_2, etc. Finally, we construct a homotopy of f to a map f' with $\pi_1(f'^{-1}(X))\to\pi_1(X)$ an isomorphism.

Proof of Lemma I.2, part (i) for $m=1$. By Lemma 2 for $m=0$ and the hypothesis M, W_1, W_2 (resp; W') are connected. Now we show that $f!_*\colon\pi_1(W_j)\to\pi_1(Y_j)$,

$j=1,2$ (resp; $\pi_1(W')\to\pi_1(Y')$) are isomorphisms. These homomorphisms, as they are induced from the degree 1 maps $f!$: $W_j\to Y_j$ (resp; $f!$: $W'\to Y'$), are surjective. Moreover, in the commutative diagram

$$\begin{array}{ccc} \pi_1(M) & \longrightarrow & \pi_1(W) \\ \downarrow & & \downarrow \\ \pi_1(X) & \longrightarrow & \pi_1(Y) \end{array}$$

all the maps other than $\pi_1(M)\to\pi_1(W)$ are already known to be injective; hence $\pi_1(M)\to\pi_1(W)$ is injective. Hence $\pi_1(M)\to\pi_1(W_i)$ (resp; $\pi_1(M_i)\to\pi_1(W')$), $i=1,2$ are injective. Then, by Van Kampen's Theorem [Sp], $\pi_1(W)=\pi_1(W_1)*_{\pi_1(M)}\pi_1(W_2)$ (resp; $\pi_1(W)=\pi_1(W')*_{\pi_1(M)}\{t\}$) and in particular $\pi_1(W_i)\to\pi_1(W)$ (resp; $\pi_1(W')\to\pi_1(W)$) is injective. But this factors through $\pi_1(W_j)\to\pi_1(Y_j)$ (resp; $\pi_1(W')\to\pi_1(Y')$) which is therefore also injective. We conclude that $\pi_1(W_i)\cong\pi_1(Y_i)$ (resp; $\pi_1(W')=\pi_1(Y')$).

From this point onwards, we may assume that M, W_1 and W_2 (resp; W') are connected and $\pi_1(M)\to\pi_1(X)$, $\pi_1(W_j)\to\pi_1(Y_j)$, $j=1,2$ (resp; $\pi_1(W')\to\pi_1(Y')$) are isomorphisms. Thus, we may now write $\tilde W_i=\tilde f^{-1}(\tilde Y_i)$ (resp; $\tilde W_i'=\tilde f^{-1}(\tilde Y_i')$). In fact *the analyses of the universal cover $\tilde Y$ of Y, made in Section 2 now applies to the universal cover of W, $\tilde W$*. For example, the boundary of $\tilde W_i$ (resp; $\tilde W_i'$) is described by equations similar to (2) of § 1 (resp; (9), (10) of § 1).

§ 4. Homology Computations

For a connected space V with basepoint $p\in V$, $\tilde V$ denotes the universal cover of V, π_V: $\tilde V\to V$ the covering projection, p some fixed choice of a basepoint in $\tilde V$ with $\pi_V(\tilde p)=p$. Standard notions about cell complexes (and Poincaré complexes) will first be recalled. (See [W2] for further details.) If V is a cell complex, $\tilde V$ has a unique covering cell structure. If T is a subcell complex of V, $\pi_V^{-1}(T)$ is a subcell complex of $\tilde V$. The chain complex of cellular chains of $\tilde V$, modulo the cellular chains of $\pi_V^{-1}(T)$ is denoted $C_*(V,T)$. As usual, using the action induced from the covering translations, this is a complex of free based right $Z\pi_1 V$ modules. Here $Z\pi_1 V$ denotes the integral group ring of $\pi_1 V$. If V is a (compact) differentiable, P.l. or even topological [KS] manifold, it has the structure of a (finite) cell complex, (and even that of a simple Poincaré complex [W2]). If T is a submanifold of V, the inclusion of T in V may be taken to be cellular. In particular, for V, T compact manifolds, or Poincaré complexes, $C_*(V,T)$ consists of finitely generated $Z\pi_1 V$ modules.

The manifold structure of V determines a homomorphism, explicitly given by the first Stiefel-Whitney class of V, $\pi_1 V\xrightarrow{w} Z_2=\{\pm 1\}$. As usual, this is used to define a conjugation on $Z\pi_1 V$ by the formula $\bar g=w(g)\,g^{-1}$, $g\in\pi_1 V\subset Z\pi_1 V$. For B a right $Z\pi_1 V$ module, define homology and cohomology with coefficients in B by

$$H^i(V,T;B)=H_i(\mathrm{Hom}_{Z\pi_1 V}(C_*(V,T),B))$$
$$H_i^t(V,T;B)=H_i(C_*(V,T)\otimes_{Z\pi_1 V}B).$$

In the second of these expressions, B is given the structure of a left $Z\pi_1 V$ module structure by the formula $\lambda c = c\bar{\lambda}$, $\lambda \in Z\pi_1 V$, $c \in B$. (For a discussion of Poincaré duality in this general setting, see, for example, [W2].) An important special case is $B = Z\pi_1 V$, in which case we may omit explicit reference to B.

Assume now that T is a connected submanifold, or subcell complex, of V. A submanifold can in particular be taken to be a subcell complex. It will often be convenient to give T and V the same basepoint. If $\pi_1(T) \to \pi_1(V)$ is injective, there is a unique basepoint preserving inclusion $\tilde{T} \to \tilde{V}$ lying over the inclusion $T \to V$. Of course, $\pi_V^{-1}(T)$ consists, if $\pi_1(T) \neq \pi_1(V)$, of many copies of $\tilde{T}$. In fact, observing that $C_*(\pi_V^{-1} T)$ is, using the covering translations corresponding to $\pi_1 V$, a right $Z\pi_1 V$ module, there is an isomorphism of right $Z\pi_1 V$ chain complexes

$$C_*(\pi_V^{-1} T) = C_*(T) \otimes_{Z\pi_1 T} Z\pi_1 V.$$

Here $Z\pi_1 V$ is given the structure of a left $Z\pi_1 T$ module using the inclusion $Z\pi_1 T \to Z\pi_1 V$. Moreover, as $Z\pi_1 V$ is a free $Z\pi_1 T$ module and denoting the homology of $C_*(\pi_V^{-1} T)$ by $H_i^t(T; Z\pi_1 V)$, we have the isomorphism of $Z\pi_1 V$ modules

$$H_i^t(T; Z\pi_1 V) \cong H_i^t(T; Z\pi_1 T) \otimes_{Z\pi_1 T} Z\pi_1 V.$$

Now let T be a submanifold of V, S a submanifold of U, U and V connected manifolds. Let $g: U \to V$ be a map, assumed proper if these spaces are not compact, and boundary preserving if these spaces have boundary, of *degree one*. Then if g is transverse regular to $T \subset V$ and $S = g^{-1}(T)$, the induced maps

$$H_i(U, S; Z\pi_1 V) \to H_i(V, T; Z\pi_1 V)$$

is *surjective* for all i [W2]. Denote

$$K_i(U, S) = \text{Kernel}\,(H_i^t(U, S; Z\pi_1 V) \to H_i^t(V, T; Z\pi_1 V))$$

and

$$K_i(S; Z\pi_1 V) = \text{Kernel}\,(H_i^t(\pi_V^{-1} S; Z\pi_1 V)) \to H_i^t(T; Z\pi_1 V)).$$

The notation $K_i(U, S)$ omits a reference to g and to the image of g, which will usually be apparent from the context. The group $K_i(U, S)$ can be described as the $(i+1)$ homology group of the quadrad

$$\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr, phantom, "\emptyset"] & T \arrow[d] \\ U \arrow[r] & V \end{tikzcd}$$

[W2].

Proof of Lemma I.2; Part (i). For $m = 0, 1$ Lemma 2 was demonstrated in §3. Under the hypothesis of part (i) by Lemma I.4, $K_j(M) = 0$, $j \leqq m$. We show that this implies $K_j(W_1)$, $K_j(W_2)$ (resp $K_j(W')$) are zero, $j \leqq m$, and, by the Hurewicz theorem this will complete the argument.

Corresponding to the decomposition, similar to that expressed in (1) of §2 (resp; (3) of §2) for $\tilde{Y}$, of $\tilde{W}$ there is a Mayer-Vietoris exact sequence of ZG modules

$$\cdots \to K_{i+1}(W) \to K_i(M; Z\pi_1 W)$$
$$\to K_i(W_1; Z\pi_1 W) \oplus K_i(W_2; Z\pi_1 W) \to K_i(W) \to \cdots$$

(resp; in case B, $\cdots \to K_{i+1}(W) \to K_i(M; Z\pi_1 W) \to K_i(W'; Z\pi_1 W) \to K_i(W) \to \cdots$). But as f is a homotopy equivalence, $K_i(W)=0$ for all i. Also,

$$K_i(M; Z\pi_1 W) = K_i(M) \otimes_{Z\pi_1 M} Z\pi_1 W$$
$$K_i(W_j; Z\pi_1 W) = K_i(W_j) \otimes_{Z\pi_1 W_j} Z\pi_1 W, \quad j=1,2$$
$$(\text{resp; } K_i(W'; Z\pi_1 W) = K_i(W') \otimes_{Z\pi_1(W')} Z\pi_1 W).$$

Hence, we have the isomorphisms of ZG modules

(1) $K_i(M) \otimes_{ZH} ZG \cong K_i(W_1) \otimes_{ZG_1} ZG \oplus K_i(W_2) \otimes_{ZG_2} ZG$

(1) (resp; $K_i(M) \otimes_{ZH} ZG \cong K_i(W') \otimes_{ZJ} ZG$).

As ZG is a free module over ZG_1, ZG_2 (resp; ZJ) clearly if $K_i(M)=0$, then $K_i(W_1)=K_i(W_2)=0$ (resp; $K_i(W')=0$).

Proof of Lemma I.2, Part (ii). By Lemma I.2, part (i), W_1 and W_2 (resp; W') are connected with $\pi_1(W_1)=\pi_1(Y_1)$, $\pi_1(W_2)=\pi_1(Y_2)$ (resp; $\pi_1(W')=\pi_1(Y')$). Moreover, by Poincaré duality ([W 2]) if $K_i(M)=0$ for $i \leqq n/2$, then $K_i(M)=0$ for $i \leqq n$. Then from part (i) of Lemma 2 $K_i(W_j)=0$, $j=1,2$ (resp; $K_i(W')=0$), for all i and hence by the Whitehead theorem, f is split.

Thus, to prove Lemma I.1 it suffices to produce an inductive procedure for, given M connected with $\pi_1(M) \to \pi_1(X)$ an isomorphism and with $K_i(M)=0$, $i \leqq j-1$, varying M to further achieve $K_j(M)=0$, $j<(n-1)/2$. This will require a more detailed description of the groups $K_j(W_i)$ (resp; $K_j(W')$) than that of Eq. (1) above. *For the remainder of this section j is some fixed integer*, $1<j$. Later on in this section j will be further restricted to satisfy $K_i(M)=0$, $i<j$.

Recall that the group $\pi_1(M)$, acting as a group of covering translations, acts on $\tilde{M}$, W_R, W_L, $\tilde{W}=W_R \cup_{\tilde{M}} W_L$ with the quotient spaces being M, W_r, W_l, $\hat{W}=W_r \cup_M W_l$. Corresponding to this decomposition of $\hat{W}$, we therefore have the Mayer-Vietoris sequence of $Z\pi_1 M$ modules

$$\cdots \to K_{i+1}(\hat{W}) \to K_i(M) \to K_i(W_l) \oplus K_i(W_r) \to K_i(\hat{W}) \to \cdots.$$

But as $\hat{f}$ is a homotopy equivalence, $K_i(\hat{W})=K_i(W)=0$, for all i and hence there is the isomorphism, induced by inclusions of spaces, of $Z\pi_1 M$ modules

$$K_j(M) \xrightarrow{\cong} K_j(W_l) \oplus K_j(W_r), \quad \text{all } j.$$

Consider now the exact sequence of the pair $(\hat{W}, M)$. As $K_i(\hat{W})=0$ all i, this reduces to the isomorphism, for all j, $K_j(M) \cong K_{j+1}(\hat{W}, M)$. But, by excision, $K_{j+1}(\hat{W}, M) \cong K_{j+1}(W_r, M) \oplus K_{j+1}(W_l, M)$. Thus there are isomorphisms of ZH modules

(2) $K_{j+1}(W_l, M) \oplus K_j(W_r, M) \cong K_j(M) \cong K_j(W_r) \oplus K_j(W_l)$.

Moreover the composite maps $K_{j+1}(W_l, M) \to K_j(W_l)$ and $K_{j+1}(W_r, M) \to K_j(W_r)$ are, as the composites of consecutive maps in the exact sequences of the pairs (W_v, M), $v=r, l$, *zero*. In particular, $K_{j+1}(W_l, M) \to K_j(W_r)$ and $K_{j+1}(W_r, M) \to K_j(W_l)$ are isomorphisms of ZH modules and the two decompositions of $K_j(M)$ in (2) coincide.

Define:

$$Q = \text{image}\,(K_{j+1}(W_l, M) \to K_j(M)) = \text{Ker}\,(K_j(M) \to K_j(W_l))$$
$$P = \text{image}\,(K_{j+1}(W_r, M) \to K_j(M)) = \text{Ker}\,(K_j(M) \to K_j(W_r)).$$

Thus $K_j(M) = P \oplus Q$.

Recall the description in Eq. (2) (resp; 9 and 10) of §2 of $\partial \tilde{W}_1$ and $\partial \tilde{W}_2$ (resp; $\partial \tilde{W}_1'$ and $\partial \tilde{W}_2'$) in case A (resp; case B). Correspondingly, we have the decompositions of the complements of the interiors of $\tilde{W}_1$ and $\tilde{W}_2$ (resp; W_1' and W_2')

$$\tilde{W}\text{-int}\,\tilde{W}_1 = \bigcup_{\alpha \in [G_1 H]} W_L\, g(\alpha)$$
$$\tilde{W}\text{-int}\,\tilde{W}_2 = \bigcup_{\beta \in [G_2,\, H]} W_R\, g(\beta)$$
$$(\text{resp; } \tilde{W}\text{-int}\,\tilde{W}_1' = \bigcup_{\alpha \in [J;\, H]} W_L\, g(\alpha) \cup \bigcup_{\beta \in [tJt^{-1};\, H]} W_R\, g(\beta)\, t$$
$$\tilde{W}\text{-int}\,\tilde{W}_2' = \bigcup_{\alpha \in [tJt^{-1};\, H]} W_R\, g(\alpha) \cup \bigcup_{\beta \in [J;\, H]} W_L\, g(\beta)\, t^{-1}).$$

Hence, we have the identifications[9]

$$C_*(\tilde{W}\text{-int}\,\tilde{W}_1) \cong C_*(W_L) \otimes_{ZH} ZG_1 \qquad \text{of } ZG_1 \text{ modules}$$
$$C_*(\tilde{W}\text{-int}\,\tilde{W}_2) \cong C_*(W_R) \otimes_{ZH} ZG_2 \qquad \text{of } ZG_2 \text{ modules}$$
$$(\text{resp; } C_*(\tilde{W}\text{-int}\,\tilde{W}_1') \cong C_*(W_L) \otimes_{ZH} ZJ \oplus C_*(W_R) \otimes_{ZH} Z[tJ]$$
$$C_*(\tilde{W}\text{-int}\,\tilde{W}_2') \cong C_*(W_R) \otimes_{ZH} Z[tJt^{-1}] \oplus C_*(W_L) \otimes_{ZH} Z[Jt^{-1}]).$$

Correspondingly, as ZG_1 and ZG_2 (resp; ZJ, $Z[tJt^{-1}]$, $Z[tJ]$, $Z[Jt^{-1}]$) are free ZH modules

(3) $K_j(\tilde{W}\text{-int}\,\tilde{W}_1) \cong K_j(W_L) \otimes_{ZH} ZG_1 \cong P \otimes_{ZH} ZG_1$

isomorphisms of ZG_1 modules;

(4) $K_j(\tilde{W}\text{-int}\,\tilde{W}_2) \cong K_j(W_R) \otimes_{ZH} ZG_2 \cong Q \otimes_{ZH} ZG_2$

isomorphisms of ZG_2 modules;

(5) $$(\text{resp; } K_j(\tilde{W}\text{-int}\,\tilde{W}_1') \cong K_j(W_L) \otimes_{ZH} ZJ \oplus K_j(W_R) \otimes_{ZH} Z[tJ]$$
$$\cong P \otimes_{ZH} ZJ \oplus Q \otimes_{ZH} Z[tJ]$$

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[9] Geometrically, these identifications can be seen directly by considering, for example, $\tilde{W}$-int $\tilde{W}_1$ as $\bar{\pi}^{-1}(\overline{W} - W_1)$ where $\overline{W}$ is the cover of W with $\pi_1(\overline{W}) = G_1$ and $\bar{\pi}: \tilde{W} \to \overline{W}$ is the covering projection

isomorphisms of ZJ modules;

(6) and $K_j(\tilde{W}\text{-int}\,\tilde{W}_2') \cong K_j(W_R) \otimes_{ZH} Z[tJt^{-1}] \oplus K_j(W_L) \otimes_{ZH} Z[Jt^{-1}]$

$$\cong Q \otimes_{ZH} Z[tJt^{-1}] \oplus P \otimes_{ZH} Z[Jt^{-1}],$$

isomorphisms of $Z[tJt^{-1}]$ modules.

Eq. (6) can also be obtained by applying t^{-1} to (5).)

Corresponding to the decomposition $\tilde{W} = \tilde{W}_i \cup_{\partial \tilde{W}_i} (\tilde{W}\text{-int}\,\tilde{W}_i)$, where $\partial \tilde{W}_i = \pi_{W_i}^{-1}(M)$ (resp; $\tilde{W} = \tilde{W}_i' \cup_{\partial \tilde{W}_i'} (\tilde{W} - \tilde{W}_i')$, where $\partial \tilde{W}_i' = \pi_{W_i}^{-1}(M_1 \cup M_2)$) there are Mayer-Vietoris exact sequences

$$\cdots \to K_j(\partial \tilde{W}_i) \to K_j(W_i) \oplus K_j(\tilde{W}\text{-int}\,\tilde{W}_i) \to K_j(W) \to \cdots$$

(resp; $\cdots \to K_j(\partial \tilde{W}_i') \to K_j(\tilde{W}_i') \oplus K_j(\tilde{W}\text{-int}\,\tilde{W}_i') \to K_j(W) \to \cdots$). As $K_s(W)=0$ for all s, there are isomorphisms

(7) $K_j(\partial \tilde{W}_i) \xrightarrow{\cong} K_j(W_i) \oplus K_j(\tilde{W}\text{-int}\,\tilde{W}_i)$,

(8) (resp; $K_j(\partial \tilde{W}_i') \xrightarrow{\cong} K_j(W_i') \oplus K_j(\tilde{W}\text{-int}\,\tilde{W}_i')$).

But from Eq. (2) (resp; (9) and (10)) of § 2

(9) $K_j(\partial \tilde{W}_i') \cong K_j(M) \otimes_{ZH} ZG_i$

$$\cong (P \oplus Q) \otimes_{ZH} ZG_i$$

$$\cong P \otimes_{ZH} ZG_i \oplus Q \otimes_{ZH} ZG_i,$$

(10) (resp; $K_j(\partial \tilde{W}_1') \cong K_j(M) \otimes_{ZH} ZJ \oplus K_j M \otimes_{ZH} Z[tJ]$

$$\cong (P \oplus Q) \otimes_{ZH} ZJ \oplus (P \oplus Q) \otimes_{ZH} Z[tJ]$$

$$\cong (P \otimes_{ZH} ZJ \oplus Q \otimes_{ZH} Z[tJ]) \oplus (Q \otimes_{ZH} ZJ \oplus P \otimes_{ZH} Z[tJ]),$$

(11) and $K_j(\partial \tilde{W}_2') \cong K_j(M) \otimes_{ZH} Z[tJt^{-1}] \oplus K_j(M) \otimes_{ZH} Z[Jt^{-1}]$

$$\cong (P \oplus Q) \otimes_{ZH} Z[tJt^{-1}] \oplus (P \oplus Q) \otimes_{ZH} Z[Jt^{-1}]$$

$$\cong (Q \otimes_{ZH} Z[tJt^{-1}] \oplus P \otimes_{ZH} Z[Jt^{-1}])$$

$$\oplus (P \otimes_{ZH} Z[tJt^{-1}] \oplus Q \otimes_{ZH} Z[Jt^{-1}]).$$

These computations lead to:

Lemma I.5. *In case A,*

(12) $K_j(W_1) \cong Q \otimes_{ZH} ZG_1$ *as a* ZG_1 *module*

(13) $K_j(W_2) \cong P \otimes_{ZH} ZG_2$ *as a* ZG_2 *module*

and in case B,

(14) $K_j(W_1') \cong Q \otimes_{ZH} ZJ \oplus P \otimes_{ZH} Z[tJ]$ *as a* ZJ *module*

(15) $K_j(W_2') \cong P \otimes_{ZH} Z[tJt^{-1}] \oplus Q \otimes_{ZH} Z[Jt^{-1}]$

as a $Z[tJt^{-1}]$ *module.* (Eq. (15) is just obtained by applying t^{-1} to both sides of (14), see Remark 1 below.)

Proof. We show in detail only (12), which is proved by substituting (9) for $i=1$ and (3) in (7). Eq. (13) is proved by similarly substituting (9) for $i=2$ and (4) in (7). Similarly (14) is shown by substituting (10) and (5) in (8) for $i=1$ and (15) is shown by substituting (11) and (6) in (8) for $i=2$.

We concentrate on Eq. (12). First by substituting using (9) and (3) in Eq. (7) we get the isomorphism

$$(16)\quad (P\otimes_{ZH} ZG_1)\oplus(Q\otimes_{ZH} ZG_1)\xrightarrow{\cong} K_j(W_i)\oplus K_j(\tilde{W}\text{-int}\,\tilde{W}_1)$$
$$=K_j(W_1)\oplus P\otimes_{ZH} ZG_1.$$

By showing that one of the "components" of the isomorphism of (16) is the zero map and another is $1_{P\otimes_{ZH}ZG_1}$ we will obtain (12). This isomorphism of (16) is induced from the inclusion

$$\bigcup_{\alpha\in[G_1;H]} \tilde{M}\,g(\alpha)\to W_1 \quad\text{and}\quad \bigcup_{\alpha\in[G_1;H]} \tilde{M}\to \tilde{W}\text{-int}\,\tilde{W}_1.$$

But from (2) above, the map

$$Q\otimes_{ZH} ZG_1\to K_j(\tilde{W}\text{-int}\,\tilde{W}_1)=P\otimes_{ZH} ZG_1$$

is zero and the map

$$P\otimes_{ZH} ZG_1\to K_j(\tilde{W}\text{-int}\,\tilde{W}_1)=P\otimes_{ZH} ZG_1$$

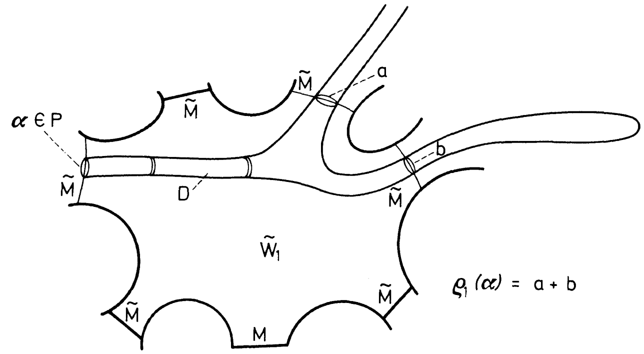
is the identity. Hence, from (16) the map $Q\otimes_{ZH} ZG_1\to K_j(W_1)$ is an isomorphism.

Remark 1. Eq. (15) can be derived directly from (14) by recalling (8) of §2 $\tilde{W}_2'=\tilde{W}_1' t^{-1}$, and applying t^{-1} to both sides of Eq. (14). That this identification of $K_j(W_2')$ with $P\otimes_{ZH} Z[tJt^{-1}]\oplus Q\otimes_{ZH} Z[jt^{-1}]$ is the same as that of (15) follows because (15) is derived from equations, each of which is geometrically seen to be obtained by applying t^{-1} to the corresponding equation for $\tilde{W}_1'$.

Remark 2. We briefly outline the geometric meaning of the next few lemmas. The map $P\otimes_{ZH} ZG_1\to K_j(W_1)$, which is a "component" of the isomorphism of (16) and corresponds geometrically to

$$\bigoplus_{\alpha\in[G_1;H]} (K_{j+1}((W_R, M)\,g(\alpha)))\to \bigoplus_{\alpha\in[G_1;H]} K_j(M\,g(\alpha))\xrightarrow{\ i\ } K_j(W_1)$$

where i is induced from inclusions, need not necessarily be zero. A class in P "dies" in W_R but does not necessarily "die" immediately in W_1. This situation will be studied below in greater detail for j the smallest integer with $K_j(M)\neq 0$, $j\geqq 2$. We shall show that if $\alpha\in K_j(M)$ goes to zero in $K_j(W_1)$, then α is the boundary of some class (not unique!) β in $\mathrm{Ker}(\pi_{j+1}(W_1, M)\to\pi_{j+1}(Y_1, X))$ and Lemma I.3 can be used to perform a handle exchange on β. Hence, we will need to determine when a class $\alpha\in K_j(M)$ goes to zero in $K_j(W_1)$ or $K_j(W_2)$. Of course, if α goes to zero in $K_j(W_1)$, it goes to zero in $K_j(W_r)$ and thus $\alpha\in P$. Therefore, we are concerned with finding a useful description of the map $P\to K_j(W_1)$ and similarly $Q\to K_j(W_2)$ (resp; in case B, $P\to K_j(W_1')$ and $Q\to K_j(W_2')$). For example, if these maps were zero a handle exchange could be performed, if $j<(n-1)/2$, to produce an ambient surgery in W on any class of P or of Q. The maps ρ_1 and ρ_2 are introduced below to describe these maps of P and Q. As the proofs given are formulated algebraically, the reader may find it helpful to keep in mind the following picture. A class $\alpha \in P$ bounds a disc D in W_R (as j is the smallest index for which $K_j(M) \neq 0$) but in general $D \not\subset \tilde{W}_1$. Hence D^{j+1} may cross the boundary of $\tilde{W}_1$. The class $\rho_1(\alpha)$ described formally below measures how the disc D crosses $\partial \tilde{W}_1$. In fact, $\rho_1(\alpha)$ is a sum of classes in $K_j(\partial \tilde{W}_1 - \tilde{M})$ as in the following picture:



For the remainder of this section j is assumed to be an index for which $K_i(M) = 0$ for $i < j$ and $j > 1$.

Lemma I.6. *$K_j(M)$, P and Q are finitely generated ZH modules.*

Proof. $P \oplus Q \cong K_j(M)$. But $K_j(M)$ is the first non-vanishing homology kernel map of the degree one map $M \to X$. A standard argument [W2] shows that such a group is finitely generated.

Now we consider in detail the map of ZH modules,

$$P \to K_j(M) \to K_j(W_1) \cong Q \otimes_{ZH} ZG_1 \cong Q \otimes_{ZH} ZH \oplus Q \otimes_{ZH} \widetilde{ZG}_1.$$

Lemma I.7. (i) *In case* A *the map* $P \to K_j(M) \to K_j(W_1) \cong Q \otimes_{ZH} ZG_1$ *is given by* $P \xrightarrow{\rho_1} Q \otimes_{ZH} \widetilde{ZG}_1 \subset Q \otimes_{ZH} ZG_1$ *where the inclusion* $Q \otimes_{ZH} \widetilde{ZG}_1 \subset Q \otimes_{ZH} ZG_1$ *is induced from* $\widetilde{ZG}_1 \subset ZG_1$. *The map* $Q \to K_j(M) \to K_j(W_2) \cong P \otimes_{ZH} ZG_2$ *is given by* $Q \xrightarrow{\rho_2} P \otimes_{ZH} \widetilde{ZG}_2 \subset P \otimes_{ZH} ZG_2$.

(ii) *In case* B, *the map* $P \to K_j(M) \to K_j(W_1') \cong Q \otimes_{ZH} ZJ \oplus P \otimes_{ZH} Z[tJ]$ *is given by* $P \xrightarrow{\rho_1} Q \otimes_{ZH} \widetilde{ZJ} \oplus P \otimes_{ZH} Z[tJ] \subset Q \otimes_{ZH} ZJ \oplus P \otimes_{ZH} Z[tJ]$. *The map*

$$Q \to K_j(M) \to K_j(W_2') \cong P \otimes_{ZH} Z[tJt^{-1}] \oplus Q \otimes_{ZH} Z[Jt^{-1}]$$

is given by $Q \xrightarrow{\rho_2} P \otimes_{ZH} Z\widetilde{[tJt^{-1}]} \oplus Q \otimes_{ZH} Z[Jt^{-1}]$.

(iii) *In particular,* $x \in K_j(M)$ *goes to zero in* $K_j(W_1)$ (resp; $K_j(W_2)$; $K_j(W_1')$; $K_j(W_2')$) *if and only if* $x \in P$ (resp; Q; P; Q) *and* $\rho_1(x) = 0$ (resp; $\rho_2(x) = 0$; $\rho_1(x) = 0$; $\rho_2(x) = 0$).

Proof. We consider, for example, the map from P. The lemma then is immediate from the definition of ρ_1 and ρ_2, the identification of $K_{j+1}(W_1)$ with $Q \otimes_{ZH} ZG_1$ (resp; $K_{j+1}(W')$ with $Q \otimes_{ZH} \widetilde{ZJ} \oplus P \otimes_{ZH} Z[tJ]$ in case B) and the commutative diagram in case A

$$\begin{tikzcd} P \arrow[r, "\cong"] \arrow[dr] & K_{j+1}(W_r, M) \arrow[r] \arrow[d, "\partial"] & K_{j+1}(W_r, W_1) \arrow[d] \\ & K_j(M) \arrow[r] & K_j(W_1) \end{tikzcd}$$

(resp; and in case B, the same diagram with W_1 replaced by W').

Note, that if $x \in K_j(M)$ goes to zero in $K_j(W_1)$, then as $\tilde{W}_1 \subset W_R$, x goes to zero in $K_j(W_R)$ and hence $x \in \operatorname{Ker}(K_j(\tilde{M}) \to K_j(W_R)) = P$.

Example. In case B if $J = H = tJt^{-1}$ and $G = Z \times H$, which was considered by Farrell and Hsiang, $Z[\widetilde{J}] = Z[\widetilde{tJt^{-1}}] = 0$ and hence in that case

$$\rho_1\colon P \to Q \otimes_{ZH} Z[tH]$$

$$\rho_2\colon Q \to P \otimes_{ZH} Z[Ht^{-1}].$$

Now in case A by applying $\otimes_{ZH} ZG$ for $G = G_1 *_H G_2$ to $P \xrightarrow{\rho_1} Q \otimes_{ZH} \widetilde{ZG}_1$ and as multiplication in ZG induces a map $\widetilde{ZG}_1 \otimes_{ZH} ZG \to ZG$, we get an extension ρ_1' of ρ_1, $\rho_1'\colon P \otimes_{ZH} ZG \to Q \otimes_{ZH} ZG$. The map ρ_1' is an extension of ρ_1, indeed the unique ZG linear extension of ρ_1 as the following diagram commutes:

$$\begin{tikzcd} P \arrow[rr, "\rho_1"] \arrow[d, equal] & & Q \otimes_{ZH} \widetilde{ZG}_1 \arrow[dd, phantom, "\cap"] \\ P \otimes_{ZH} ZH \arrow[d, phantom, "\cap"] & & \\ P \otimes_{ZH} ZG \arrow[rr, "\rho_1'"] & & Q \otimes_{ZH} ZG \end{tikzcd}$$

Extend ρ_1' to a map, still called ρ_1', of $(P \oplus Q) \otimes_{ZH} ZG$

$$(P \oplus Q) \otimes_{ZH} ZG \xrightarrow{\rho_1'} Q \otimes_{ZH} ZG \subset (P \oplus Q) \otimes_{ZH} ZG$$

by setting it equal to 0 on $Q \otimes_{ZH} ZG$ and the given map ρ_1' on $P \otimes_{ZH} ZG$. Similarly, extend $Q \xrightarrow{\rho_2} P \otimes_{ZH} \widetilde{ZG}_2$ to $\rho_2'\colon Q \otimes_{ZH} ZG \to P \otimes_{ZH} ZG$ and hence, extending trivially, to the ZG linear map

$$\rho_2'\colon (P \oplus Q) \otimes_{ZH} ZG \to (P \oplus Q) \otimes_{ZH} ZG.$$

Using a parallel construction in case B for $G = J *_H \{t\}$, apply $\otimes_{ZH} ZG$ to both sides of $P \xrightarrow{\rho_1} Q \otimes_{ZH} \widetilde{ZJ} \oplus P \otimes_{ZH} Z[tJ]$ and using the multiplication in ZG to define maps $Z\tilde{J} \otimes_{ZH} ZG \to ZG$ and $Z[tJ] \otimes_{ZH} ZG \to ZG$ we get $\rho_1'\colon P \otimes_{ZH} ZG \to Q \otimes_{ZH} ZG \oplus P \otimes_{ZH} ZG$. The map ρ_1' is an extension, indeed the unique extension to a ZG linear map, of ρ_1 in that the following diagram commutes:

$$\begin{array}{ccc}
P & \xrightarrow{\rho_1} & Q\otimes_{ZH}\widetilde{ZJ}\oplus P\otimes_{ZH}Z[tJ] \\
\| & & \\
P\otimes_{ZH}ZH & & \cap \\
\cap & & \\
P\otimes_{ZH}ZG & \xrightarrow{\rho_1'} & Q\otimes_{ZH}ZG\oplus P\otimes_{ZH}ZG
\end{array}$$

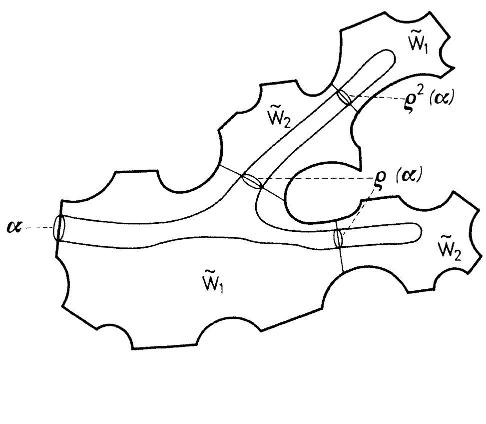
We extend ρ_1' to a map of $(P\oplus Q)\otimes_{ZH}ZG\to(P\oplus Q)\otimes_{ZH}ZG$ by setting $\rho_1'(Q\otimes_{ZH}ZG)=0$. Similarly, we extend ρ_2 to $\rho_2'\colon Q\otimes_{ZH}ZG\to P\otimes_{ZH}ZG\oplus Q\otimes_{ZH}ZG$ and, extending further trivially to $\rho_2'\colon(P\oplus Q)\otimes_{ZH}ZG\to(P\oplus Q)\otimes_{ZH}ZG$.

Now define $\rho\colon(P\oplus Q)\otimes_{ZH}ZG\to(P\oplus Q)\otimes_{ZH}ZG$ a ZG linear map, in both case A and case B by the formula $\rho=\rho_1'+\rho_2'$.

Remark. We briefly explain the geometric interest of the map ρ. As was observed in a remark above, for an element $x\in P\subset K_j(M)$ bounding a disc D in W_R, $\rho_1(x)$ "represents" in case A, the homology class obtained by "$D\cap(\partial\tilde{W}_1-\tilde{M})$" as a class in $K_j(\partial\tilde{W}_1-\tilde{M})$. Then for a class $x\in K_j(M)$ bounding a disc D in $\tilde{W}$, $\rho(x)$ can be thought of as the class "represented" by

$$y=\text{“}D\cap\partial(\tilde{W}_1\cup_{\tilde{M}}\tilde{W}_2)\text{”}\quad\text{in } K_j(\partial(\tilde{W}_1\cup_{\tilde{M}}\tilde{W}_2)).$$

Note that the disc D gives us a "cycle" "$D\cap(\tilde{W}-\partial(\tilde{W}_1\cup_{\tilde{M}}\tilde{W}_2))$" with boundary "$D\cap\partial(\tilde{W}_1\cup_{\tilde{M}}\tilde{W}_2)$". Now we can use this cycle to "compute" $\rho(y)$ by the same method by which D with boundary x was used to compute $\rho(x)$. Proceeding inductively we use D, or rather pieces of it, to "compute" $\rho^s(x)$, $s\geqq 0$. But D, being compact, intersects only finitely many copies of $\tilde{W}_1$ and $\tilde{W}_2$ in $\tilde{W}$. Hence, we expect that $\rho^s(x)=0$ for s sufficiently large.



We call the finite filtrations of ZH modules

$$P=P_0\supset P_1\supset P_2\supset\cdots\supset P_r=0$$

$$Q=Q_0\supset Q_1\supset Q_2\supset\cdots\supset Q_r=0$$

an *upper-triangular filtration* of (P, Q) if

(i) each ZH submodule P_i and Q_i is finitely generated, $i \geqq 0$, and

(ii) in case A,

$$\rho_1(P_i) \subset Q_{i+1} \otimes_{ZH} \widetilde{ZG}_1$$

and

$$\rho_2(Q_i) \subset P_{i+1} \otimes_{ZH} \widetilde{ZG}_2, \qquad i \geqq 0.$$

In case B,

$$\rho_1(P_i) \subset Q_{i+1} \otimes_{ZH} \widetilde{ZJ} \oplus P_{i+1} \otimes_{ZH} Z[tJ]$$

and

$$\rho_2(Q_i) \subset P_{i+1} \otimes_{ZH} Z\widetilde{[tJt^{-1}]} \oplus Q_{i+1} \otimes_{ZH} Z[Jt^{-1}].$$

Lemma I.8. (i) *The map ρ is nilpotent;* i.e. $\exists N \geqq 0$ with $\rho^N(x) = 0$ *for all* $x \in (P \oplus Q) \otimes_{ZH} ZG$.

(ii) *There exists an upper-triangular filtration of* (P, Q).

Before examining the proof of Lemma I.8 the reader may wish to read §5 to see how it leads to the completion of the proof of Lemma I.1. The proof of Lemma I.8 given below derives it as a formal consequence of homology computations demonstrated above. It may also be proved by referring back again directly to the geometric situation.

Proof of Lemma I.8. Using Lemma I.9 below, the proof of Lemma I.8 is reduced to showing that $I + \rho$ is an isomorphism for I the identity map of $(P \oplus Q) \otimes_{ZH} ZG$. We proceed to prove this.

First consider case A. Recall the ZG isomorphism of (1) above in case A

(1) $b\colon K_j(M) \otimes_{ZH} ZG \xrightarrow{\cong} K_j(W_1) \otimes_{ZG_1} ZG \oplus K_j(W_2) \otimes_{ZG_2} ZG$

which is induced, as a ZG linear map, from the ZH linear maps induced from $M \subset W_1$, $M \subset W_2$, i.e. from $K_j(M) \to K_j(W_1)$ and $K_j(M) \to K_j(W_2)$. Moreover, $K_j(M) = P \oplus Q$ and hence

(18) $K_j(M) \otimes_{ZH} ZG = (P \oplus Q) \otimes_{ZH} ZG$,

(12) and from Lemma 5 $K_j(W_1) \cong Q \otimes_{ZH} ZG_1$,

(19) $K_j(W_1) \otimes_{ZG_1} ZG \cong Q \otimes_{ZH} ZG_1 \otimes_{ZG_1} ZG = Q \otimes_{ZH} ZG$

(13) and lastly from Lemma 5 $K_j(W_2) \cong P \otimes_{ZH} ZG_2$,

(20) $K_j(W_2) \otimes_{ZG_2} ZG \cong P \otimes_{ZH} ZG_2 \otimes_{ZG_2} ZG = P \otimes_{ZH} ZG$.

Combining (19) and (20) we get

(21) $K_j(W_1) \otimes_{ZG_1} ZG \oplus K_j(W_2) \otimes_{ZG_2} ZG \cong (P \oplus Q) \otimes_{ZH} ZG$.

Then substituting for the left hand side of (1) using (18) and for the right hand side of (1) using (21), the map b induces an isomorphism, which we continue to denote by b,

(22) $b\colon (P\oplus Q)\otimes_{ZH} ZG \xrightarrow{\cong} (P\oplus Q)\otimes_{ZH} ZG.$

We claim $b=I+\rho$; in particular, $I+\rho$ is therefore an isomorphism. This will be shown by examining each of the four components of the map b corresponding to the decomposition of $(P\otimes_{ZH} ZG)\oplus(Q\otimes_{ZH} ZG)$ into the given two summands. First, as the component of (b) of (1)

$$P\otimes_{ZH} ZG \to K_j(W_1)\otimes_{ZG_1} ZG$$

is induced from the inclusion $P\to K_j(M)\to K_j(W_1)$, as is the isomorphism of (13), it follows that the component map $P\otimes_{ZH} ZG\to P\otimes_{ZH} ZG$ of the map b of (22) is $1_{P\otimes_{ZH} ZG}$. Similarly the component map $Q\otimes_{ZH} ZG\to Q\otimes_{ZH} ZG$ of the map (b) of (22) is $1_{Q\otimes_{ZH} ZG}$. However, the component $P\otimes_{ZH} ZG\to Q\otimes_{ZH} ZG$ of the map (b) of (22) is induced from

$$P\to K_j(M)\to K_j(W_1)=Q\otimes_{ZH} ZG$$

and hence is given by ρ_1 on P. Similarly the component $Q\otimes_{ZH} ZG\to P\otimes_{ZH} ZG$ of the map (b) of (22) is induced from

$$Q\to K_j(M)\to K_j(W_2)=P\otimes_{ZH} ZG_2$$

and hence agrees with ρ_2 on Q. Thus on

$$P\oplus Q=(P\oplus Q)\otimes_{ZH} ZH\subset(P\oplus Q)\otimes_{ZH} ZG,$$

$b=I+\rho$. But as both b and $I+\rho$ are ZG linear, $b=I+\rho$ on all of $(P\oplus Q)\otimes_{ZH} ZG$.

We similarly show in case B that $I+\rho$ is an isomorphism. Recall from earlier in this section the form of the isomorphism (1) in case B, with $G=J*_H\{t\}$

(1) $b\colon K_j(M)\otimes_{ZH} ZG \xrightarrow{\cong} K_j(W_1')\otimes_{ZJ} ZG.$

(14) From Lemma 5, $K_j(W_1')\cong Q\otimes_{ZH} ZJ\oplus P\otimes_{ZH} Z[tJ]$ and hence

$$(23)\quad \begin{aligned} K_j(W_1')\otimes_{ZJ} ZG &\cong (Q\otimes_{ZH} ZJ\oplus P\otimes_{ZH} Z[tJ])\otimes_{ZJ} ZG\\ &= Q\otimes_{ZH} ZG\oplus P\otimes_{ZH} ZG\end{aligned}$$

making the identification of $\otimes_{ZH} Z[tJ]\otimes_{ZJ} ZG$ with $\otimes_{ZH} ZG$ as $tG=G$. Then substituting using (8) on the left-hand side and (23) on the right hand side of (1), we obtain the isomorphism

$$b\colon (P\oplus Q)\otimes_{ZH} ZG \xrightarrow{\cong} (P\oplus Q)\otimes_{ZH} ZG.$$

We claim as in case A, that $b=I+\rho$. It suffices to check this on $P\oplus Q$ as b and $I+\rho$ are ZG linear. Since the map b of (1) is given by a Mayer-Vietoris sequence, it is given on $K_j(M)$ by

$$\begin{aligned} P\oplus Q=K_j(M)&\to K_j(W_1')\oplus K_j(W_2')=K_j(W_1')\oplus K_j(W_1' t^{-1})\\ &=(Q\otimes_{ZH} ZJ\oplus P\otimes_{ZH} Z[tJ])\oplus(P\otimes_{ZH} Z[tJt^{-1}]\oplus Q\otimes_{ZH} Z[Jt^{-1}]).\end{aligned}$$

(See (14), (15) and Remark 1 after Lemma I.5.) We consider the components of this map, which are maps of P and of Q to $(Q\otimes_{ZH}ZJ\oplus P\otimes_{ZH}Z[tJ]$ and to $(P\otimes_{ZH}Z[tJt^{-1}]\oplus Q\otimes_{ZH}Z[Jt^{-1}])$. The map $Q\subset K_j(M)\to K_j(W_1')$ is induced from the inclusion of $\tilde{M}\subset\partial(\tilde{W}_1')$ and the map of (14)

$$(Q\otimes_{ZH}ZJ\oplus P\otimes_{ZH}Z[tJ])\to K_j(W_1')$$

was also produced by restricting the map of $K_j(\partial\tilde{W}_1')\to K_j(W_1')$ induced from $\partial\tilde{W}_1'\subset\tilde{W}_1'$. Therefore we have the commutative diagram

$$\begin{tikzcd}
 & Q \arrow[dl, hook'] \arrow[dr, equal] & & \\
K_j(M) \arrow[ddr] & & Q\otimes_{ZH}ZH \arrow[dr, hook] & \\
 & & & Q\otimes_{ZH}ZJ\oplus P\otimes_{ZH}Z[tJ] \arrow[dll, "\cong"] \\
 & K_j(W_1') & &
\end{tikzcd}$$

and hence the component map $Q\to K_j(M)\to K_j(W_1')\cong Q\otimes_{ZH}ZJ\oplus P\otimes_{ZH}Z[tJ]$ is just the inclusion $Q=Q\otimes_{ZH}ZJ\subset Q\otimes_{ZH}ZJ\oplus P\otimes_{ZH}Z[tJ]$. Similarly the component map $P\to P\otimes_{ZH}Z[tJt^{-1}]\oplus Q\otimes_{ZH}Z[Jt^{-1}]$ is just the inclusion $P=P\otimes_{ZH}ZH\subset P\otimes_{ZH}Z[tJt^{-1}]\oplus Q\otimes_{ZH}Z[Jt^{-1}]$. On the other hand, the component map $P\subset K_j(M)\to K_j(W_1')=Q\otimes_{ZH}ZJ\oplus P\otimes_{ZH}Z[tJ]$ was the definition of the map ρ_1 and the component map

$$Q\subset K_j(M)\to K_j(W_2')=P\otimes_{ZH}Z[tJt^{-1}]\oplus Q\otimes_{ZH}Z[Jt^{-1}]$$

was the definition of ρ_2. Hence on $P\oplus Q$, $b=I+\rho$.

The remainder of this section contains no further geometry and proves two algebraic lemmas employed in the proof of Lemma I.9.

Lemma I.9. *Let P, Q be finitely generated $Z[H]$ modules and $\rho: (P\oplus Q)\otimes_{ZH}ZG\to(P\oplus Q)\otimes_{ZH}ZG$ a ZG linear map, where*

case A: $G=G_1 *_H G_2$; *case* B: $G=J *_H \{t\}$

satisfying

(1) *$I+\rho$ is an isomorphism, I the identity map of $(P\oplus Q)\otimes_{ZH}ZG$*

(2) *case* A: $\rho(P)\subset Q\otimes_{ZH}\widetilde{ZG}_1$, $\rho(Q)\subset P\otimes_{ZH}\widetilde{ZG}_2$,

case B: $\rho(P)\subset Q\otimes_{ZH}\widetilde{ZJ}\oplus P\otimes_{ZH}Z[tJ]$

$\rho(Q)\subset P\otimes_{ZH}Z\widetilde{[tJt^{-1}]}\oplus Q\otimes_{ZH}Z[Jt^{-1}]$.

Then (i) *ρ is nilpotent and*

(ii) *Writing ρ_1 for the restriction of ρ to P and ρ_2 for the restriction of ρ to Q, (P, Q) has an upper-triangular filtration.*

Proof of Part (i) of Lemma I.9. Write, for $\alpha\in[G;H]$, $Z[\alpha]$ to denote the left ZH module generated by $g\in\alpha\subset G\subset Z[G]$. Then as

$$G=\bigcup_{\alpha\in[G;H]}\alpha,$$

as a left ZH module

$$Z[G]=\bigoplus_{\alpha\in[G;H]}Z[\alpha]$$

and

$$(P\oplus Q)\otimes_{ZH}ZG=\sum_{\alpha\in[G;H]}P\otimes_{ZH}Z[\alpha]\oplus\sum_{\alpha\in[G;H]}Q\otimes_{ZH}Z[\alpha].$$

Write P_α for $P\otimes_{ZH}Z[\alpha]$ and Q_α for $Q\otimes_{ZH}Z[\alpha]$, $K_\alpha=P_\alpha\oplus Q_\alpha$ and set

$$K=\bigoplus_{\alpha\in[G;H]}K_\alpha=(P\oplus Q)\otimes_{ZH}ZG.$$

For S a subset of $[G;H]$ write K_S for $\bigoplus_{\alpha\in S}K_\alpha$. Write K_e for $K_{\{H\}}=P\oplus Q$. Now define $T(i)\subset[G;H]$ by $T(0)=\{H\}$ and $T(i)=\{\alpha\in[G;H]\,|\,\alpha\subset A_i, B_i, C_i \text{ or } D_i\}$, $i\geqq 1$. From the description of A_i, B_i, C_i and D_i in §2, $[G;H]$ is the disjoint union of the $T(i)$, $i\geqq 0$.

Now we show that $\rho^s(K_e)\subset K_{T(s)}$. The proof of Part (i) of Lemma I.9 is then completed by Lemma I.12, with $T=[G;H]$. In fact, we show somewhat more and will demonstrate that in both case A and case B, for $s\geqq 1$,

(24) $\rho^s(P)\subset P\otimes_{ZH}D_s\oplus Q\otimes_{ZH}A_s$,

(25) $\rho^s(Q)\subset Q\otimes_{ZH}B_s\oplus P\otimes_{ZH}C_s$.

For $s=1$, this is just a restatement in terms of the A_i, B_i, C_i, D_i notation introduced in §2 of the hypothesis. Assuming these formulas for s, we inductively derive them for $s+1$:

$$\begin{aligned}\rho^{s+1}(P)&=\rho(\rho^s(P))\subset\rho(P\otimes_{ZH}D_s\oplus Q\otimes_{ZH}A_s)\\&\subset\rho(P)D_s\oplus\rho(Q)A_s\\&\subset(PD_1\oplus QA_1)D_s+(QB_1\oplus PC_1)A_s\\&\subset P(D_1D_s\oplus C_1A_s)\oplus Q(A_1D_s\oplus B_1A_s)\\&\subset PD_{s+1}\oplus QA_{s+1},\quad\text{from the definition of } D_{s+1}\text{ and } A_{s+1}\\&\subset P\otimes_{ZH}D_{s+1}\oplus Q\otimes_{ZH}A_{s+1}.\end{aligned}$$

Similarly,

$$\begin{aligned}\rho^{s+1}(Q)&=\rho(\rho^s(Q))\subset\rho(Q\otimes_{ZH}B_s\oplus P\otimes_{ZH}C_s)\\&\subset\rho(Q)B_s\oplus\rho(P)C_s\\&\subset(QB_1+PC_1)B_s\oplus(PD_1+QA_1)C_s\\&\subset Q(B_1B_s+A_1C_s)\oplus P(C_1B_s+D_1C_s),\\&\subset QB_{s+1}\oplus PC_{s+1},\quad\text{from the definition of } B_{s+1}\text{ and } C_{s+1}\\&\subset Q\otimes_{ZH}B_{s+1}\oplus P\otimes_{ZH}C_{s+1}.\end{aligned}$$

Lemma I.10. *Let T be a set equipped with a transitive action of a group G. Assume given corresponding to each element $d \in T$, an abelian group K_d with $K = \bigoplus_{d \in T} K_d$ a finitely generated ZG module and with $K_d g \subset K_{dg}$ for $g \in G \subset ZG$. Assume given a decomposition of T into the disjoints sets $T(n)$, $n \geqq 0$ with $T(0) = \{e\}$, $e \in T$. If $\rho: K \to K$ is a ZG linear map satisfying*

(i) *$I + \rho$ is an isomorphism, I the identity map of K,*

(ii) $\rho^s(K_e) \subset \bigoplus_{d \in T(s)} K_d, \quad s \geqq 1.$

Then ρ is a nilpotent map.

Proof. First observe that as $K_{dg} \subset (K_{dg}) g^{-1} g \subset K_d g$, $K_{dg} = K_d g$. Then, for $d' \in T$, writing $d' = eg$, for some $g \in G$,

$$\rho^s(K_{d'}) = \rho^s(K_{eg}) = \rho^s(K_e) g \subset \bigoplus_{d \in T(s)} (K_d) g \subset \bigoplus_{d \in T(s)g} K_d,$$

and T is the *disjoint* union of the sets $T(s)g$, $s \geqq 0$. Hence, we can define $\chi = (I - \rho - \rho^2 - \rho^3 - \cdots): \bigoplus_{d \in T} K_d \to \prod_{d \in T} K_d$. Let $\emptyset: \bigoplus_{d \in T} K_d \to \prod_{d \in T} K_d$ denote the usual inclusion. Clearly $\chi(I + \rho) = \emptyset$ and in particular, as $I + \rho$ was given in (i) as invertible, $\mathrm{Image}(\chi) \subset \mathrm{Image}(\emptyset)$ and thus $\chi(K_d) \subset \mathrm{Image}(\emptyset)$. Hence for $x \in K_d$, $d \in T$, only finitely many terms of $\rho^1(x), \rho^2(x), \rho^3(x), \ldots$ are non-zero. It follows that for any $x \in K$, for n_0 sufficiently large $\rho^{n_0}(x) = 0$, if K is generated over ZG by $\{z_i\}_{1 \leqq i \leqq r}$ with $\rho^{n_i}(z_i) = 0$, then letting $n = \max n_i$, $1 \leqq i \leqq r$, $\rho^n = 0$.

Proof[10] *of Part (ii) of Lemma I.9.* Recall the normal form basis for the left ZH module structures of A_i, B_i, C_i, D_i, ZG, defined in § 2. Correspondingly we have decompositions,

$$P \otimes_{ZH} D_s \oplus Q \otimes_{ZH} A_s = \sum_{\substack{\delta \in [G;H] \\ \delta \subset D_s}} P \otimes_{ZH} ZHg(\delta) \oplus \sum_{\substack{\alpha \in [G;H] \\ \alpha \subset A_s}} Q \otimes_{ZH} ZHg(\alpha)$$

$$Q \otimes_{ZH} B_s \oplus P \otimes_{ZH} C_s = \sum_{\substack{\beta \in [G;H] \\ \beta \subset B_s}} Q \otimes_{ZH} ZHg(\beta) \oplus \sum_{\substack{\gamma \in [G;H] \\ \gamma \subset C_s}} P \otimes_{ZH} ZHg(\gamma).$$

Here the elements $g(\alpha)$, $g(\beta)$, $g(\gamma)$, $g(\delta)$ are given by the normal form basis. There are obvious isomorphisms

$$\varepsilon_v: Q \otimes_{ZH} ZHg(v) \to Q, \quad v = \delta, \beta$$
$$\varepsilon_v: P \otimes_{ZH} ZHg(v) \to P, \quad v = \alpha, \gamma$$

and these maps are trivially extended to

$$P \otimes_{ZH} (D_s \oplus C_s) \oplus Q \otimes_{ZH} (A_s \oplus B_s)$$

by setting them equal to zero on the other summands.

[10] This algebraic argument can be interpreted geometrically using the geometric interpretation of normal form outlined in the appendix to Chapter I

From (24) and (25), for $x \in P$ we get

$$(26)\quad \rho^s(x) = \sum_{\substack{\delta \in [\overline{G;H}] \\ \delta \subset D_s}} \varepsilon_\delta(\rho^s(x))\, g(\delta) + \sum_{\substack{\alpha \in [\overline{G;H}] \\ \alpha \subset A_s}} \varepsilon_\alpha(\rho^s(x))\, g(\alpha)$$

and for $x \in Q$,

$$(27)\quad \rho^s(x) = \sum_{\substack{\beta \in [\overline{G;H}] \\ \beta \subset B_s}} \varepsilon_\beta(\rho^s(x))\, g(\beta) + \sum_{\substack{\gamma \in [\overline{G;H}] \\ \gamma \subset C_s}} \varepsilon_\gamma(\rho^s(x))\, g(\gamma).$$

Define $P_0 = P$, $Q_0 = Q$, and for $s > 0$, $P_s =$ the ZH submodule generated by $\varepsilon_\delta(\rho^s(P))$, for $\delta \in [\overline{G;H}]$, $\delta \subset D_s$ and by $\varepsilon_\gamma(\rho^s(Q))$ for $\gamma \in [\overline{G;H}]$, $\gamma \subset C_s$, $Q_s =$ the ZH submodule generated by $\varepsilon_\alpha(\rho^s(P))$, for $\alpha \in [\overline{G;H}]$, $\alpha \subset A_s$ and by $\varepsilon_\beta(\rho^s(Q))$ for $\beta \in [\overline{G;H}]$, $\beta \subset B_s$.

As P and Q are finitely generated, say with generating sets $\{p_1, \dots, p_r\}$, $\{q_1, \dots, q_v\}$ respectively and from (26) and (27) for only finitely many δ, $\varepsilon_\delta(P_i)$ are non-zero, $1 \leqq i \leqq r$, and similarly for α, β, γ we conclude that P_s and Q_s are finitely generated. Moreover, as $\rho^s = 0$ for s sufficiently large, $P_s = Q_s = 0$ for s sufficiently large.

Next, we check that in case A, $\rho_1(P_i) \subset Q_{i+1} \otimes_{ZH} Z\widetilde{G}_1$ and in case B, $\rho_1(P_i) \subset Q_{i+1} \otimes_{ZH} Z\widetilde{J} \oplus P_{i+1} \otimes_{ZH} Z[tJ]$. By the same method, we can also check that in case A, $\rho_2(Q_i) \subset P_{i+1} \otimes_{ZH} Z\widetilde{G}_2$, and in case B

$$\rho_2(Q_i) \subset P_{i+1} \otimes_{ZH} Z[\widetilde{tJt^{-1}}] \oplus Q_{i+1} \otimes_{ZH} Z[Jt^{-1}].$$

Taking $s = i$ in the definition of P_s, P_i is generated by $\varepsilon_\delta(\rho^i(P))$, $\varepsilon_\gamma(\rho^i(Q))$; hence, it suffices to check that (i) $\rho_1(\varepsilon_\delta(\rho^i(P)))$ and (ii) $\rho_1(\varepsilon_\gamma(\rho^i(Q)))$ are included in $Q_{i+1} \otimes_{ZH} Z\widetilde{G}_1$ in case A, and included in $Q_{i+1} \otimes_{ZH} Z\widetilde{J} \oplus P_{i+1} \otimes_{ZH} Z[tJ]$ in case B. We check only (ii), by examining Eq. (27); (i) follows by the same discussion applied to (26). Taking Eq. (27) for $s = i$, and applying ρ to both sides, we get for $x \in Q$,

$$\begin{aligned}(28)\quad \rho^{i+1}(x) = \rho(\rho^i(x)) &= \sum_{\substack{\gamma \in [\overline{G;H}] \\ \gamma \subset C_i}} \rho_1(\varepsilon_\gamma(\rho^i(x)))\, g(\gamma) + \sum_{\substack{\beta \in [\overline{G;H}] \\ \beta \subset B_i}} \rho_2(\varepsilon_\beta(\rho^i(x)))\, g(\beta) \\
&= \sum_{\substack{\gamma \in [\overline{G;H}] \\ \gamma \subset C_s}} \sum_{\substack{\delta' \in [\overline{G;H}] \\ \delta' \subset D_1}} \varepsilon_{\delta'}(\rho_1(\varepsilon_\gamma(\rho^i(x))))\, g(\delta')\, g(\gamma) \\
&\quad + \sum_{\substack{\gamma \in [\overline{G;H}] \\ \gamma \subset C_s}} \sum_{\substack{\alpha' \in [\overline{G;H}] \\ \alpha' \subset A_1}} \varepsilon_{\alpha'}(\rho_1(\varepsilon_\gamma(\rho^i(x))))\, g(\alpha')\, g(\gamma) \\
&\quad + \cdots + \cdots, \quad \text{from (26) and (27).}\end{aligned}$$

But the elements $g(\delta')\, g(\gamma)$ together with the similarly constructed elements $g(\gamma')\, g(\beta)$, $\gamma' \in [G;H]$, $\gamma' \subset C_1$, are the normal form basis for C_{i+1}; similarly $g(\alpha')\, g(\gamma)$ and the similarly constructed $g(\beta')\, g(\beta)$, $\beta' \in [\overline{G;H}]$, $\beta' \subset B_1$ are the normal form basis for B_{i+1}. Hence Eq. (28) is Eq. (27) for $s = i+1$. In particular,

$$\varepsilon_{\delta'}(\rho_1(\varepsilon_\gamma(\rho^i(x)))) \in P_{i+1}, \quad \varepsilon_{\alpha'}(\rho_1(\varepsilon_\gamma(\rho^i(x)))) \in Q_{i+1}$$

and thus

$$\varepsilon_{\delta'}(\rho_1(y)) \in P_{i+1}, \quad \varepsilon_{\alpha'}(\rho_1(y)) \in Q_{i+1} \qquad \text{for } y = \varepsilon_\gamma(\rho^i(x)) \subset P_i.$$

But

$$\rho_1(y) = \sum_{\substack{\delta' \in [G;H] \\ \delta' \in D_1}} \varepsilon_{\delta'}(\rho_1(y))\, g(\delta') + \sum_{\substack{\alpha' \in [G;H] \\ \alpha' \in A_1}} \varepsilon_{\alpha'}(\rho_1(y))\, g(\alpha')$$

and hence $\rho_1(y) \in P_{i+1} \otimes_{ZH} D_1 \oplus Q_{i+1} \otimes_{ZH} A_1$. In case A, this asserts that

$$\rho_1(\varepsilon_\gamma(\rho^i(P)) \subset Q_{i+1} \otimes_{ZH} \widetilde{ZG}_1$$

and in case B that

$$\rho_1(\varepsilon_\gamma(\rho^i(P))) \subset Q_{i+1} \otimes_{ZH} \widetilde{ZJ} \oplus P_{i+1} \otimes_{ZH} Z[tJ]$$

as was to be shown.

Lastly, we check that $P_i \supset P_{i+1}$, and similarly we get $Q_i \supset Q_{i+1}$. P_{i+1} is generated by $\varepsilon_\delta(\rho^{i+1}(P))$ and $\varepsilon_\gamma(\rho^{i+1}(Q))$, γ, δ as above. We check for example, that $\varepsilon_\gamma(\rho^{i+1}(Q)) \subset P_i$; the corresponding argument for $\varepsilon_\delta(\rho^{i+1}(P))$ is entirely similar. Now, $g(\gamma) \in C_{i+1}$ and $C_{i+1} = D_i \otimes_{ZH} C_1 \oplus C_i \otimes_{ZH} B_1$. From the construction of the normal form basis for C_{i+1}, we have that $g(\gamma)$ is of the form $g(\delta)\, g(\gamma')$ or $g(\tilde{\gamma})\, g(\beta')$ for $\delta, \gamma', \tilde{\gamma}, \beta' \in [G;H]$, $\delta \subset D_i$, $\gamma' \subset C_1$, $g(\tilde{\gamma}) \subset C_i$, $g(\beta') \subset B_1$. Say, for example, that $g(\gamma_0) = g(\delta_0)\, g(\gamma'_0)$.

For $x \in Q$,

$$\begin{aligned}
(29) \quad \rho^{i+1}(x) = \rho^i(\rho^1(x)) &= \rho^i\Big(\sum_{\substack{\gamma' \in [G;H] \\ \gamma' \subset C_1}} \varepsilon_{\gamma'}(\rho_1(x))\, g(\gamma') + \sum_{\substack{\beta' \in [G;H] \\ \beta' \subset B_1}} \varepsilon_{\beta'}(\rho_1(x))\, g(\beta')\Big) \\
&= \sum_{\substack{\gamma' \in [G;H] \\ \gamma' \subset C_1}} \sum_{\substack{\delta \in [G;H] \\ \delta \subset D_i}} \varepsilon_\delta(\rho^i(\varepsilon_{\gamma'}\, \rho(x)))\, g(\delta)\, g(\gamma') \\
&\quad + \sum_{\substack{\gamma' \in [G;H] \\ \gamma' \subset C_1}} \sum_{\substack{\alpha \in [G;H] \\ \alpha \subset A_i}} \varepsilon_\alpha(\rho^i(\varepsilon_{\gamma'}\, \rho(x)))\, g(\alpha)\, g(\gamma') \\
&\quad + \cdots + \cdots, \qquad \text{from (26).}
\end{aligned}$$

Eq. (29) is again an expression with the coefficients in $Z[G]$ being members of the normal form basis and thus is essentially (27) for $s = i+1$. In particular, for $x \in Q$,

$$\begin{aligned}
\varepsilon_{\gamma_0}(\rho^{i+1}(x)) &= \varepsilon_{\delta_0}(\rho^i(\varepsilon_{\gamma'_0}(x))) \\
&\in \varepsilon_{\delta_0}(\rho^i(P)) \\
&\in P_i \text{ as was to be shown.}
\end{aligned}$$

§ 5. Completion of Proof of Lemma I.1

Proceeding inductively, we assume that Lemma I.1 has already been verified for $m = j-1 \geqq 1$, $m \leqq \frac{n-1}{2}$. We show Lemma I.1 for $m = j$. By the inductive hypothesis, we may assume that M is connected, $\pi_1(M)=\pi_1(X)$, and that j is the smallest integer for which $K_j(M)\neq 0$. Hence the entire discussion of §4, in particular Lemma I.8, is applicable.

Now let

$$P=P_0\supset P_1\supset\cdots\supset P_t=0$$

$$Q=Q_0\supset Q_1\supset\cdots Q_t=0$$

be upper-triangular filtrations of (P,Q), $K_j(M)=P\oplus Q$. These, by Lemma I.8, certainly exist. Let r be the number of non-zero terms in the sequence $\{P_0, Q_0, P_1, Q_1, \ldots, P_n, Q_n\}$. In this section, we construct a map f', homotopic to f, with f' transverse regular to X, and writing $M'=f'^{-1}(X)$, M' connected with $\pi_1(M')=\pi_1(X)$, $K_i(M')=0$, $i<j$ and $K_j(M')=P'\oplus Q'$ with (P',Q') having an upper triangular filtration with at most $r-1$ non-zero terms. By decreasing induction on r, the proof of Lemma I.1 will then be complete.

Let s be the largest index for which $P_s\oplus Q_s\neq 0$. Say, for example, $P_s\neq 0$. Then $\rho_1(P_s)=0$. Let $z_1, z_2, \ldots, z_v$ be a finite set of generators of P_s as a ZH module. By Lemma I.9, $z_i\in\operatorname{Ker}(K_j(M)\to K_j(W_1)$ in case A and $z_i\in\operatorname{Ker}(K_j(M_1)\to K_j(W_1')$ in case B.

Consider the diagram in case A

$$\begin{tikzcd}
\pi_{j+2}(\emptyset) \arrow[r] \arrow[dd] & \pi_{j+1}(X,M) \arrow[r] \arrow[d, "h"', "\cong"] & \pi_{j+1}(Y_1,W_1) \arrow[d, "h"', "\cong"] \\
 & K_j(M) \arrow[r] \arrow[d, phantom, "\bigcap"] & K_j(W_1) \\
\pi_{j+1}(W_1,M)\to\pi_j(M) \arrow[r] \arrow[d] & H_j(M;Z\pi_1 M) & \\
\pi_{j+1}(Y_1,X) & &
\end{tikzcd}$$

where $\emptyset$ is the quadrad,

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & W_1 \arrow[d] \\
X \arrow[r] & Y_1
\end{tikzcd}$$

and in case B, the diagram

$$\begin{tikzcd}
\pi_{j+2}(\emptyset) \arrow[r] \arrow[dd] & & \pi_{j+1}(X_1, M_1) \arrow[r] \arrow[d, "\cong" ', "h"] & \pi_{j+1}(Y_1', W_1') \arrow[d, "\cong" ', "h"] \\
& & K_j(M) \arrow[r] \arrow[d, phantom, "\cap"] & K_j(W_1) \\
\pi_{j+1}(W_1', M_1) \arrow[r] \arrow[d] & \pi_j(M) \arrow[r] & H_j(M; Z\pi_1 M) & \\
\pi_{j+1}(Y_1, X_1) & & &
\end{tikzcd}$$

where $\emptyset$ is the quadrad

$$\begin{tikzcd}
M_1 \arrow[r] \arrow[d] & W_1' \arrow[d] \\
X_1 \arrow[r] & Y_1'
\end{tikzcd}$$

Here the upper rows and left-hand columns are exact, and h is the Hurewicz isomorphism (see §4). From this commutative diagram in case A (resp; case B) z_i can be lifted to $z_i' \in \pi_{j+1}(X, M)$ (resp; $\pi_{j+1}(X_1, M_1)$), and as z_i' goes to zero in $\pi_{j+1}(Y_1, W_1)$, (resp; $\pi_{j+1}(Y_1', W_1')$, it can be lifted to an element z_i'' in $\pi_{j+2}(\emptyset)$. But, letting y_i denote the image of z_i'' in $\pi_{j+1}(W_1, M)$, (resp; $\pi_{j+1}(W_1', M_1)$), we have that $y_i \in \operatorname{Ker}(\pi_{j+1}(W_1, M) \to \pi_{j+1}(Y_1, X))$, (resp; $\operatorname{Ker}(\pi_{j+1}(W_1', M_1) \to \pi_{j+1}(Y_1', X_1))$ and y_i is represented by $(D^{j+1}, S^j) \xrightarrow{y_i} (W_1, M)$, (resp; (W_1', M_1)) with $y_i(S^j)$ representing $z_i \in K_j(M) \subset H_j(M; Z\pi_1 M)$. As $j < \frac{(n-1)}{2}$ handle exchanges can be performed on the classes y_i, by Lemma I.3, to obtain f' homotopic to f and with $M' = f'^{-1}(X)$ produced from M by surgeries on embedded spheres representing z_i. Then, as this is surgery below the middle dimension, a standard argument [W 2], (which is essentially reproduced in the proof of Lemma I.11), shows that M' is connected, $\pi_1(M') = \pi_1(X)$ and $K_i(M') = 0$, $i < j$,

$$K_j(M') \cong K_j(M)/\{z_i\}_{1 \le i \le v} = (P \oplus Q)/\{z_i\} = P/\{z_i\} \oplus Q.$$

We will need to be more explicit in our description of $K_j(M')$. Using the decomposition $K_j(M') = P' \oplus Q'$ defined in §4, and writing ρ_1', ρ_2' for the associated maps defined in §4 we have:

Lemma I.11. *$P' = P/\{z_i\}$, $Q' \cong Q$ and in case* A *diagrams*

$$\begin{tikzcd}
P \arrow[r, "\rho_1"] \arrow[d] & Q \otimes_{ZH} \widetilde{ZG}_1 \arrow[d, "\cong" '] \\
P' \arrow[r, "\rho_1'"] & Q' \otimes_{ZH} \widetilde{ZG}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
Q \arrow[r, "\rho_2"] \arrow[d, "\cong"] & P \otimes_{ZH} \widetilde{ZG}_2 \arrow[d] \\
Q' \arrow[r, "\rho_2'"] & P' \otimes_{ZH} \widetilde{ZG}_2
\end{tikzcd}$$

commute.

In case B, *the diagrams*

$$\begin{array}{ccc} P & \xrightarrow{\rho_1} & Q \otimes_{ZH} \widetilde{ZJ} \oplus P \otimes_{ZH} Z[tJ] \\ \downarrow & & \downarrow \\ P' & \xrightarrow{\rho_1'} & Q' \otimes_{ZH} \widetilde{ZJ} \oplus P' \otimes_{ZH} Z[tJ] \end{array}$$

and

$$\begin{array}{ccc} Q & \xrightarrow{\rho_2} & P \otimes_{ZH} Z[\widetilde{tJt^{-1}}] \oplus Q \otimes_{ZH} Z[Jt^{-1}] \\ \downarrow & & \downarrow \\ Q' & \xrightarrow{\rho_2'} & P' \otimes_{ZH} Z[\widetilde{tJt^{-1}}] \oplus Q' \otimes_{ZH} Z[Jt^{-1}] \end{array}$$

commute.

Using Lemma I.11 we complete the proof of Lemma I.1. Let $P_i' = \text{Image}\,(P_i \to P')$, $Q_i' = \text{Image}\,(Q_i \to Q')$. Then from Lemma I.13,

$$P' = P_0' \supset P_1' \supset P_0' \supset \cdots \supset P_t'$$
$$Q' = Q_0' \supset Q_1' \supset Q_2' \supset \cdots \supset Q_t'$$

is easily seen to be an upper triangular filtration of (P', Q') with $K_j(M') = P' \oplus Q'$. But as $P_s' = P_s/\{z_i\} = 0$, this filtration has at most $r-1$ non-zero terms, as was to be shown.

Proof of Lemma I.11. Let $X \times I \subset Y_1$ with $X \times 0 = X$. Let C be the cobordism, $C \subset W_1$, formed by attaching handles corresponding to y_i to M so that $\partial C = M \cup M'$. They by Lemma I.3, using handle exchanges f is homotopic, by a homotopy fixed on W_2, and restricting to a homotopy from W_1 to Y_1 to f' with $f'(C) \subset X \times I$, $f'(M') \subset X \times 1$ and $f'(W - C) \subset Y_1 - X \times I$. Set $C_0 = C - M'$.

As $j < (n-1)/2$, $K_i(M) = K_i(M') = K_i(C) = 0$ for $i < j$. Moreover, the inclusion $M' \to C$ induces isomorphisms $K_j(M') \cong K_j(C)$ and the inclusion $M \to C$, as C is formed by attaching $(j+1)$ dimensional discs to M, induces an isomorphism $K_j(M)/\{z_i\} \cong K_j(C)$.

Corresponding to the decompositions

$$\begin{array}{ll} \tilde{W} = W_L \cup_{\tilde{M}} W_R, & \hat{W} = W_l \cup_M W_r \\ \tilde{W} = (W_L \cup \tilde{C}) \cup_{\tilde{M}'} (W_R - \tilde{C}_0), & \hat{W} = (W_l \cup C) \cup_M (W_r - C_0) \\ \tilde{W} = (W_L \cup_{\tilde{M}} \tilde{C}) \cup_{\tilde{C}} (W_R), & \hat{W} = (W_l \cup_M C) \cup_C W_r \end{array}$$

there are Mayer-Vietoris decompositions. Comparing these, and recalling that $K_i(\tilde{W}) = 0$ for all i,

$$\begin{tikzcd}
K_j(M) \arrow[r, "\cong"] \arrow[d] & K_j(W_l) \arrow[d] & \oplus K_j(W_r) \arrow[d, "="] \\
K_j(C) \arrow[r, "\cong"] & K_j(W_l \cup_M C) & \oplus K_j(W_r) \\
K_j(M') \arrow[r, "\cong"] \arrow[u, "\cong"] & K_j(W_l \cup_M C) \arrow[u, "\cong"] & \oplus K_j(W_r - C_0) \arrow[u, "\cong"]
\end{tikzcd}$$

Here the vertical maps are induced from the inclusions of spaces. The lower three vertical maps are isomorphisms as $K_j(M') \to K_j(C)$ is. Moreover, $K_j(W_L \cup_M C) \cong K_j(W_l)/\{z_i\}$ as C is formed by attaching handles to the classes z_i. In particular, decomposing $K_j(M') = P' \oplus Q'$, we get that

$$Q' \cong K_j(W_r - C_0) \cong K_j(W_r) \cong Q, \quad \text{and}$$
$$P' \cong K_j(W_l \cup_M C) \cong K_j(W_l)/\{z_i\} = P/\{z_i\}_{1 \leqq i \leqq v}.$$

Now, corresponding to the decompositions of $\tilde{W}$, using

$$W = (W_2 \cup_M C) \cup_{M'} (W_1 - C_0)$$

there are the isomorphisms discussed in §4

$$K_j(W_1 \cup_M C) \cong Q' \otimes_{ZH} ZG_1 \quad \text{and} \quad K_j(W_2 - C_0) \cong P' \otimes_{ZH} ZG_2.$$

The maps of $P' \otimes_{ZH} ZG_2$ to $K_j(W_2 - C_0)$ and of $Q' \otimes_{ZH} ZG_2$ to $K_j(W_1 \cup_M C)$ are induced from the inclusions of

$$\partial(\widetilde{W_2 - C_0}) = \bigcup_{\alpha \in [G_2; H]} \tilde{M}' g(\alpha) \to (\widetilde{W_2 - C_0}) \quad \text{and}$$

$$\partial(\widetilde{W_1 \cup_M C}) = \bigcup_{\alpha \in [G_1; H]} \tilde{M}' g(\alpha) \to (\widetilde{W_1 \cup_M C}).$$

Similarly, using $W = (W_2 \cup_M C) \cup_C W_1$ we get

$$Q' \otimes_{ZH} ZG_2 \cong K_j(W_l \cup_M C) \otimes_{ZH} ZG_2 \cong K_j(W_2 \cup_M C),$$

with the maps again induced from the inclusions of subspaces.

Thus, we have the commutative diagrams

$$\begin{tikzcd}
P \cong K_{j+1}(W_r, M) \arrow[d] \arrow[rrr, bend left, "\rho_1"] & \subset K_j(M) \arrow[r] \arrow[d] & K_j(W_1) \arrow[d] & Q \otimes_{ZH} ZG_1 \arrow[l, "\cong"'] \arrow[d] \\
P' \cong K_{j+1}(W_r, C) & \subset K_j(C) \arrow[r] & K_j(W_1) & Q' \otimes_{ZH} ZG_1 \arrow[l, "\cong"'] \\
P' \cong K_{j+1}(W_r - C_0, M') \arrow[u, "\cong"] \arrow[rrr, bend right, "\rho_1'"'] & \subset K_j(M') \arrow[r] \arrow[u, "\cong"] & K_j(W_1 - C_0) \arrow[u, "\cong"] & Q' \otimes_{ZH} ZG_1 \arrow[l, "\cong"'] \arrow[u, "\cong"]
\end{tikzcd}$$

and

$$
\begin{tikzcd}
Q \cong K_{j+1}(W_l, M) \arrow[r, phantom, "\subset"] \arrow[d] \arrow[rrr, bend left, "\rho_2"] & K_j(M) \arrow[r] \arrow[d] & K_j(W_2) \arrow[d] & P \otimes_{ZH} ZG_2 \arrow[l, "\cong"'] \arrow[d] \\
Q' \cong K_{j+1}(W_l \cup_M C, C) \arrow[r, phantom, "\subset"] & K_j(C) \arrow[r] & K_j(W_2 \cup_M C) & P' \otimes_{ZH} ZG_2 \arrow[l, "\cong"'] \\
Q' \cong K_{j+1}(W_L \cup_M C, M') \arrow[r, phantom, "\subset"] \arrow[u, "\cong"] \arrow[rrr, bend right, "\rho_2'"'] & K_j(M') \arrow[r] \arrow[u, "\cong"] & K_j(W_2 \cup_M C) \arrow[u] & P' \otimes_{ZH} ZG_2 \arrow[l, "\cong"'] \arrow[u]
\end{tikzcd}
$$

Hence the lemma has been proved.

Appendix to Chapter I

Geometry of Normal Form for $G_1 *_H G_2$ and $J *_H \{t\}$

In this appendix we study in more detail the universal cover of Y and interpret geometrically the normal form for elements of $G_1 *_H G_2$, which is familiar, and of $J *_H \{t\}$. The use of trees for this purpose, standard for $G = G_1 *_H G_2$ and suggested also for $G = J *_H \{t\}$ in [W 1] is technically refined below by using oriented trees.

Recall, a tree is a connected and simply connected graph. The tree T_Y, used to describe $\tilde{Y}$ and $\tilde{Y} = \pi_Y^{-1}(X)$, is defined as follows. To each connected component of $\tilde{Y} - \pi_Y^{-1}(X)$ there corresponds a unique vertex of T_Y and two vertices are connected if the closures of the corresponding components of $\tilde{Y} - \pi_Y^{-1}(X)$ have nontrivial intersection. Such an intersection is a connected component of $\pi_Y^{-1}(X)$ and hence each edge corresponds to a connected component of $\pi_Y^{-1}(X)$. That T_Y is actually a tree follows immediately from the connectedness and simple-connectedness of $\tilde{Y}$. (In fact, it is easy to construct an embedding $T_Y \to \tilde{Y}$ with the image of each vertex lying in the corresponding component of $\tilde{Y} - \pi_Y^{-1}(X)$ and with a retraction $\tilde{Y} \to T_Y$.) For an edge d of T_Y, write X_d to denote the corresponding component of $\pi_Y^{-1}(X)$. The edge corresponding to $\tilde{X} \subset \tilde{Y}$ will be denoted d_0 and will be called the base edge.

Given a tree T with some edge d_0 of T called the base edge, define inductively

$$C_0(T) = \{d_0\}$$

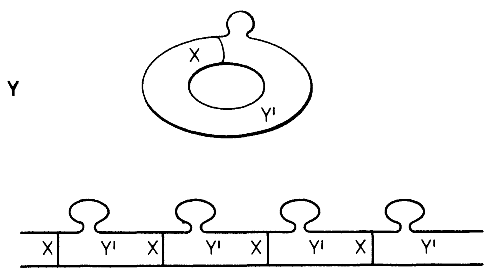
$$C_{n+1}(T) = \{d \text{ an edge of } T \mid \exists \text{ edge } d' \in C_n,\ d \cap d' \neq \emptyset,\ d \notin C_n,\ d \notin C_{n-1}\}.$$

Letting $|T|$ denote the set of *edges* of T, as T is a tree, $|T|$ is the *disjoint* union of the sets $C_n(T)$, $n \geqq 0$.

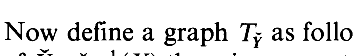
A tree will be called oriented if every edge is given an orientation. No compatibility condition on the orientations of different edges is assumed. The orientation of an edge is described by stating in what direction it is "pointing"; i.e., by ordering its endpoints.

In deriving uniqueness of normal form, it is useful to orient the tree T_Y by the following procedure. In case A, let e: $T_Y \to S$ be the unique graph map (sending vertex to vertex, edge to edge, preserving incidence relations) of T_Y to the graph S, consisting of two vertices v_2 and v_1 with one edge d connecting v_2 to v_1, and with $e(v)=v_1$, for v the vertex of T_Y corresponding to $\tilde{W}_1$. Now orient d to point towards v_1 and give T_Y the unique orientation making e orientation preserving. Thus, T_Y has been oriented so that each edge points to its endpoint corresponding to a component of $\pi_Y^{-1}(Y_1)$.

In case B, intuitively we want to orient T_Y so that the edge d points to the vertex corresponding to the component of $Y-\pi_Y^{-1}(X)$ for which X_d is a boundary component lying over $X_1 \subset Y'$. Precisely, to orient T_Y in case B, let $\check{Y}$ denote the cover of Y, $\check{Y}=\bigcup_{n\in Z} Y'(n)/X_2'(n)=X_1'(n-1)$, $Y'(n)\cong Y'$. The infinite cyclic group Z acts on $\check{Y}$ with, for $m\in Z$, $m(Y'(n))=Y'(n+m)$, and $Y=\check{Y}/Z$. This is the usual "paving-stone" normal covering space of Y. (If $G=Z\times_\alpha H$, $\check{Y}=\hat{Y}$.) Write $\check{\pi}\colon \check{Y}\to Y$, $\overset{*}{\pi}\colon \tilde{Y}\to \check{Y}$ for the covering maps. Thus, $\check{\pi}\overset{*}{\pi}=\pi_Y$.

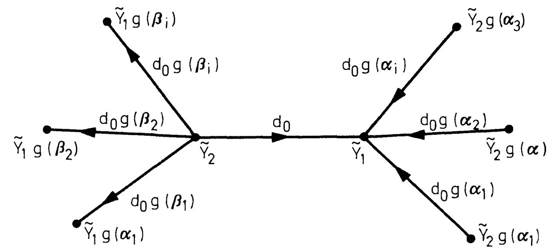


Now define a graph $T_{\check{Y}}$ as follows. Corresponding to each connected component of $\check{Y}-\check{\pi}^{-1}(X)$ there is one vertex of $T_{\check{Y}}$ and two of these are joined by an edge if the closures of the corresponding components of $\check{Y}-\check{\pi}^{-1}(X)$ have non-trivial intersection. Thus, each edge corresponds to a connected component of $\check{\pi}^{-1}(X)$. Orient each edge to point from the component corresponding to $Y(n-1)$ to that corresponding to $Y(n)$. Hence, $T_{\check{Y}}$ is simply a line with all edges pointed in the same direction.

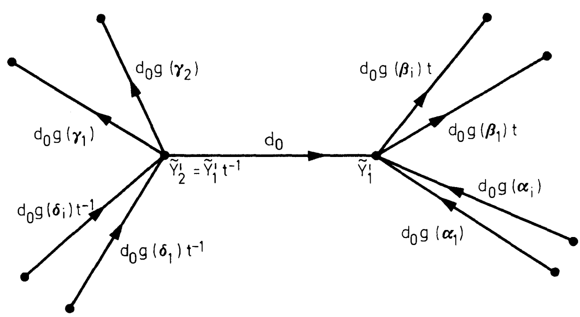


The action of the group of covering translations Z on $\check{Y}$ induces an orientation-preserving action of Z on $T_{\check{Y}}$. Now, let $e\colon T_Y \to T_{\check{Y}}$ be a graph map with for $d\in |T_Y|$, $e(d)$ an edge corresponding to $\overset{*}{\pi}(X_d)$. Now give T_Y the unique orientation making e orientation-preserving.

The tree T_Y may, following [W 1], be described algebraically and we indicate how this is done; we also describe the orientations algebraically. The reader may find the following schematic diagrams, which describe a neighborhood of d_0 in T_Y, helpful.



$\alpha_i \in [\overline{G_1; H}]$, $\beta_i \in [\overline{G_2; H}]$, d_0 corresponds to $\tilde{X} \subset \tilde{Y}$.
I. Diagram of neighborhood of d_0 in T_Y in case A. (See Eq. (2) of § 1.)



$\alpha_i \in [\overline{J, H}]$, $\beta_i \in [tJt^{-1}; H]$, $\gamma_i \in [\overline{tJt^{-1}; H}]$, $\delta \in [J; H]$, d_0 corresponds to $\tilde{X} \subset \tilde{Y}$.
II. Diagram of neighborhood of d_0 in T_Y in case B. (See Eq. (8), (9), (10) of § 1.)

The action of the group G on $\tilde{Y}$ and on $\pi_Y^{-1}(X)$ induces an action of G on T_Y. This action is clearly transitive on $|T_Y|$ with isotropy subgroup $H = \pi_1(X) \subset \pi_1(Y) = G$. Moreover, this action is orientation-preserving. This is trivial in case A, and in case B follows from the action of the covering translation group Z on $T_{\bar{Y}}$ being orientation-preserving.

Following [W 1], the (oriented) tree T_Y will be described algebraically as follows. In case A, the set of vertices of T_Y corresponds to $[G; G_1] \cup [G; G_2]$ and the set of edges of T_Y corresponds to $[G; H]$. (See (1) of § 1.) An edge d corresponding to $\alpha \in [G; H]$ joins the vertex v_1 corresponding to $\beta_1 \in [G; G_1]$ to the vertex v_2 corresponding to $\beta_2 \in [G; G_2]$ if $\alpha \subset \beta_1$ and $\alpha \subset \beta_2$. The edge d is oriented to point towards v_1. The action of G on T_Y is, in this formulation, the obvious one induced from the action of G on $[G; G_1]$, $[G; G_2]$, $[G; H]$. Note that as this action is transitive on $[G; H]$, that is, on the set of edges of T_Y, a "neighborhood" of any edge d of T_Y is carried, by an *orientation-preserving* graph map sending d to d_0 to the neighborhood of d_0 described in schematic diagram *I*.

We briefly recall how this is used to obtain uniqueness of normal form for elements of $G = G_1 *_H G_2$. If $d \in |T_Y|$ with $d \neq d_0$, as T_Y is a tree, there exists a *unique* series of edges $(d_1, d_2, \ldots, d_s)$, $d_s = d$ with d_i intersecting d_{i+1}, $0 \leqq i \leqq s-1$ in one point. Then, using the action of G on T_Y and the description of the oriented neighborhood of d_0, it follows that d can be described by

(1) $d=d_0\, g(\alpha_1)\, g(\alpha_2) \dots g(\alpha_s)$, $\quad \alpha_i \in \overline{[G_1; H]} \cup \overline{[G_2; H]} \quad 1 \leqq i \leqq s$,

$\{\alpha_j, \alpha_{j+1}\} \not\subset [G_k; H], \quad k=1,2, \; 1 \leqq j \leqq s-1.$

In fact, $d_{i+1} = d_i\, g(\alpha_{i+1})$, $0 \leqq i \leqq s-1$.

Conversely, using the fact that every edge of T_Y has a neighborhood isomorphic, by an orientation-preserving isomorphism, to that described above of d_0, it follows that for any edge d written in the form of (1) above, the series $(d_1, \dots, d_s)$ defined by $d_{i+1} = d_i\, g(\alpha_{i+1})$, $0 \leqq i \leqq s-1$ has d_i intersecting d_{i+1} in one point, $0 \leqq i \leqq s-1$. The point of this is that we have shown that each $d \in |T_Y|$, $d \neq d_0$, can be written *uniquely* in the form of Eq. (1). As G acts on $|T_Y|$ with isotropy subgroup H, it follows that any element $g \in G$ can be written uniquely as

$$g = h\, g(\alpha_1)\, g(\alpha_2) \dots g(\alpha_s), \quad \alpha_i \in \overline{[G_1; H]} \cup \overline{[G_2; H]} \quad 1 \leqq i \leqq s,$$

$$\{\alpha_j, \alpha_{j+1}\} \not\subset [G_k; H], \quad k=1,2, \; 1 \leqq j \leqq s-1, \; h \in H.$$

This is the standard *uniqueness* of *normal* form for elements of $G_1 *_H G_2$. Note that for g written in this form, $d_0\, g \in C_s(T_Y)$.

Now the tree T_Y will be described algebraically in case B. In that case, the set of vertices of T_Y corresponds to $[G; J]$ (see (3) of § 1) and the set of edges, $|T_Y|$, corresponds to $[G; H]$ (see (5) of § 1). An edge d corresponding to $\alpha \in [G; H]$ joins v_1 corresponding to $\beta_1 \in [G; J]$ to v_2 corresponding to $\beta_2 \in [G; J]$, and points towards v_1, if and only if $\alpha \subset \beta_1$, $t^{-1} \alpha \subset \beta_2$ (see (5) and (8) of § 1). With this description of T_Y, the action of G on T_Y is induced from the action of G on $[G; J]$ and $[G; H]$. This action is transitive on $[G; H]$ and hence on $|T_Y|$; thus as in case A, each edge is carried by an orientation preserving graph map to the neighborhood of d_0, described in schematic diagram II. For $d \in |T_Y|$, constructing $(d_1, \dots, d_s)$ and arguing as in case A, it follows that every element $g \in G = J *_H \{t\}$ has the following *unique normal form* [W 1]:

$$g = h k_1 k_2 \dots k_s, \quad k_i = g(\alpha_i) \text{ or } g(\beta_i) \text{ or } g(\gamma_i) t \text{ or } g(\delta) t^{-1}$$

$$\text{for } \alpha_i \in \overline{[J; H]}, \quad \beta_i \in [t J t^{-1}; H], \quad \gamma_i \in [t J t^{-1}; H]$$

$\delta_i \in [J; H]$ and if k_{j+1} is of the form $g(\alpha_{j+1})$ or $g(\delta_{j+1}) t^{-1}$ (respectively $g(\beta_{j+1})$ or $g(\gamma_{j+1}) t$) then k_j is of the form $g(\beta_j)$ or $g(\delta_j) t^{-1}$ (resp; $g(\alpha_j)$ or $g(\gamma_j) t$. As in case A, for g in this form, $d_0\, g \in C_s(T_Y)$.

From the uniqueness of normal form, we have that in case A, for $G_i \subset G_1 *_H G_2$, $i=1,2$, $H = G_1 \cap G_2$ and in case B, for J, $t J t^{-1} \subset J *_H \{t\}$, $H = J \cap t J t^{-1}$ in G.

From the discussion of normal form in both case A and case B it is obvious that the summand of $Z[G]$, $A_i \oplus B_i \oplus C_i \oplus D_i$, defined in § 1, is generated as a left $Z[H]$ module by $\{g \in G \mid d_0\, g \in C_i(T_Y)\}$.

The reader may check, though we do not explicitly use this in the present paper, that for $g \in G_1 *_H G_2$ or $g \in J *_H \{t\}$

$$g \in A_i \text{ for some } i \text{ iff } Y_L\, g \subset Y_R, \quad Y_R\, g \not\subset Y_L$$

$$g \in B_i \text{ for some } i \text{ iff } Y_R\, g \subset Y_R, \quad Y_L\, g \not\subset Y_L$$

$g \in C_i$ for some i iff $Y_R g \subset Y_L$, $\quad Y_R g \not\subset Y_R$

$g \in D_i$ for some i iff $Y_R g \subset Y_R$, $\quad Y_L g \not\subset Y_L$

$g \in H$ if $Y_R g \subset Y_R$ and $Y_L g \subset Y_L$.

Chapter II: The Odd-Dimensional Case

§ 1. The Nilpotent Normal Cobordism Construction

This chapter completes the proofs of Theorems 1 and 2 for $n=2k$ and also develops material that will be used in Chapter III to complete the argument for $n=2k+1$. *Without using the square-root closed* hypothesis on $H \subset G$, this section constructs, if $n=2k>4$ and $\bar{\phi}(\tau(f))=0$ by a procedure we call the *nilpotent normal cobordism construction* a normal cobordism of W to a homotopy equivalent split manifold. The intersection form of this $2k+2$ dimensional normal cobordism (denoted λ in the terminology of [W2, Chapter V]) is computed in Lemma II.6 in terms of the maps ρ_1, ρ_2 and ρ defined in Chapter I, § 4. The self-intersection form (μ in the notation of [W2, Chapter V]) of this normal cobordism could also be studied by similar methods. However, for the application we make of Lemma II.6 to the square-root closed case, this is not needed and is not discussed below.

In § 2 Lemma II.10, which uses the square-root closed assumption on $H \subset G$, is used to show that the surgery obstruction of the nilpotent normal cobordism constructed in § 1 lies in a certain subgroup of the surgery group of G. This leads directly to the construction under the assumptions of Part (i) of Theorems 1 and 2 of an h-cobordism, and under the assumptions of Part (ii) of Theorems 1 and 2 of an s-cobordism, of W to a split manifold.

For P finitely generated projective right module over the integral group-ring ZD of a group D, $[P]$ denotes the element of $\tilde{K}_0(D)$, the reduced projective class group of the ring ZD, represented by P.

Lemma II.1. *Let $n=2k$, W a closed manifold and Y a closed manifold (or Poincaré complex) of dimension $n+1$. Assume given X a closed submanifold (or sub-Poincaré complex) of dimension n of Y with trivial normal bundle and with $H=\pi_1(X) \to \pi_1(Y)$ injective. Assume further that $f: W \to Y$ is a homotopy equivalence transverse regular to X with, writing $M=f^{-1}(X)$, M connected and $\pi_1(M) \to \pi_1(X)$ an isomorphism and $K_i(M)=0$, $i<k$. Then letting $K_k(M)=P \oplus Q$ denote the decomposition of ZH modules defined in* I.4,

(i) *$K_k(M)$ is a stably free ZH module and $[P]=-[Q]$. Moreover, in case* A

$$[P] \in \operatorname{Ker}(\tilde{K}_0(H) \to \tilde{K}_0(G_1) \oplus \tilde{K}_0(G_2));$$

in case B,

$$[P] \in \operatorname{Ker}(\tilde{K}_0(H) \xrightarrow{\xi_{1*}-\xi_{2*}} \tilde{K}_0(J)).$$

(ii) *Any finite set of elements of P* (respectively; Q) *can be represented by embedded disjoint framed spheres in M for $k>2$. The intersection pairing* [W2;

Chapter V] *of* $K_k(M)$ *is trivial when restricted to* P (resp; Q) and $Q \cong P^*$. *Thus,* $[P] = -[P^*]$.

(iii) *If* $[P]=0$, f *is homotopic to a map* f' *with* $M' = f'^{-1}(X) \to X$ *k-connected and with, writing* $K_k(M') = P' \oplus Q'$ *for the decomposition of* I.4, P' *and* Q' *free* ZH *modules.*

Thus from (ii), if P is a free module P and Q are, in the terminology of [W2; Chapter V] subkernels of the Hermitian form defined on $K_k(M)$ by intersections and self-intersection. This can be used to show that if $[P]=0$ the surgery problem of $M \to X$ induced by the homotopy equivalence f can be solved to obtain a homotopy equivalence.

Proof of Lemma II.1; Part (i). As $K_i(M)=0$, $i<n/2=k$, $n=$dimension M, $K_k(M)$ is a stably free finitely generated module [W2; Chapter V]. Hence its summands P and Q are finitely generated projective modules and $[P] = -[Q]$. Moreover, in case A from Lemma I.5 the image of $[P]$ under the map $\tilde{K}_0(H) \to \tilde{K}(G_2)$ is $[K_k(W_2)]$ and the image of $[Q]$ in $\tilde{K}_0(G_1)$ is $[K_k(W_1)]$. In particular, $K_k(W_j)$ are projective ZG_j modules for $j=1,2$. But $K_i(W_j)=0$ for $i \neq k$ by Lemma I.5. Hence as the chain complex of (Y_j, W_j) is a free finitely generated ZG_j complex whose only non-zero homology group is the projective module $K_k(W_j)$, by a standard argument [W2] $K_k(W_j)$ is stably free.

Similarly, in case B,

$$(\xi_{1_*} - \xi_{2_*})[P] = \xi_{1_*}[P] + \xi_{2_*}[Q] = [P \otimes_{ZH} ZJ \oplus Q \otimes_{ZH} Z[tJ]].$$

(Recall from I.1 that as a left ZH module $Z[tJ]$ is isomorphic to the left $Z[\xi_2(H)]$ module structure of $Z[J]$.) Again by Lemma I.5

$$[P \otimes_{ZH} ZJ \oplus Q \otimes_{ZH} Z[tJ]] = [K_k(W')]$$

and the rest of the argument for case B is exactly as in case A.

Proof of Part (ii). Recall from the definition of P and Q in I.4, any element $v \in P \subset K_k(M)$ is represented by $\partial\alpha$ for some $\alpha \in K_{k+1}(W_r, M)$. In particular, for a finite set $\{V_i\}$ $1 \leqq i \leqq s$, of elements of P, there are $\{\alpha_i\}$ with $\partial\alpha_i = V_i$ $1 \leqq i \leqq s$. Now as $K_{j+1}(W_r, M) \subset K_j(M) = 0$ for $j<k$ and, see I §1, $\pi_1(M) = \pi_1(W_r)$, $\{\alpha_j\}$ is represented by maps which we denote simply α_j: $(D^{k+1}, S^k) \to (W_r, M)$. But as $\pi_1 M = \pi_1 W_r$, the standard piping argument (see for example [W2; p. 41]) shows that $\{\alpha_i\}$ can be represented by immersions with $\{\partial\alpha_i\}$ framed embedded and disjoint spheres.

Thus the non-singular intersection form of $K_k(M)$ [W2; Chapter V] is trivial when restricted to P, or to Q, and hence its adjoint [W2; p. 44] induces the isomorphisms $P \cong Q^*$, $Q \cong P^*$.

Proof of Part (iii). If $[P]=0$, then by performing trivial ambient surgeries on $M \subset W$ to stabilize P, it may be assumed free. As $Q \cong P^*$, Q will then also be free.

The map ϕ is defined and Lemma II.2(i) is proved in [W1; §5]. Part (ii) of Lemma II.2 just quotes for the case of manifolds a result of [W1; §6]; there the modules P and Q are defined by an analogous procedure to that used in I.4 in the general setting of a CW-complex splitting problem.

Lemma II.2. (i) *In case A,*

$$\phi\colon \mathrm{Wh}(G_1 *_H G_2) \to \mathrm{Ker}(\tilde{K}_0(H) \to \tilde{K}_0(G_1) \oplus \tilde{K}_0(G_2))$$

(*resp; in case* B, $\phi\colon \mathrm{Wh}(J *_H \{t\}) \to \mathrm{Ker}(\tilde{K}_0(H) \xrightarrow{\xi_{1*}-\xi_{2*}} \tilde{K}_0(J))$ *is surjective.*

(ii) *Hypothesis as in Lemma* II.1. Then $[P]=\phi(\tau(f))$, $\tau(f)$ *the Whitehead torsion of* f.

Of course Lemma II.2 and the following result, Lemma II.3, are not needed if $\tilde{K}_0(H) \to \tilde{K}_0(G_1) \oplus \tilde{K}_0(G_2)$ (resp; $\tilde{K}_0(H) \xrightarrow{\xi_{1*}-\xi_{2*}} \tilde{K}_0(J)$) are injective. Lemma II.3 describes the behavior of the usual Z_2 action on Whitehead groups and projective class groups in Waldhausen's exact sequence for $\mathrm{Wh}(G)$, $G=G_1 *_H G_2$ or $G=J *_H \{t\}$.

Lemma II.3. *Let* $H \subset G_i$ (resp; $\xi_i\colon H \to J$) *be inclusions of groups,* $i=1, 2$, *and set* $G=G_1 *_H G_2$ (resp; $G=J *_H \{t\}$. *Assume given homomorphisms* $w_H\colon H \to Z_2$ *and* $w_i\colon G_i \to Z_2$ (resp; $w_J\colon J \to Z_2$ *and* $w_Z\colon Z \to Z_2$, Z *generated by* t) *with* w_i *restricting to* w_H (resp; *with* $w_J \xi_i = w_H$) *for* $i=1, 2$. *Let* $w\colon G \to Z_2$ *denote the unique extension of* w_1 *and* w_2 (resp; w_J *and* w_Z) *to* G. *Let* $x \to x^*$ *denote the* Z_2 *action on* $\mathrm{Wh}(G)$ *and on* $\tilde{K}_0(H)$ *determined by the involutions of* ZH *and* ZG *defined in the usual way using* w_H *and* w. *Then for* $x \in \mathrm{Wh}(G)$, $\phi(x^*) = -\phi(x)^*$.

The proof of Lemma II.3 is technical and is deferred to the end of § 1. Lemma II.4 is also trivial if $\tilde{K}_0(H) \to K_0(G_1) \oplus K_0(G_2)$ (resp; $\tilde{K}_0(H) \xrightarrow{\xi_{1*}-\xi_{2*}} \tilde{K}_0(J)$) is injective.

Lemma II.4. *Let* Y *be a closed manifold or Poincaré complex of dimension* $n+1$, $n \geqq 4$, *with* $\pi_1(Y)=G$, $G=G_1 *_H G_2$ (resp; $G=J *_H \{t\}$). *Then if*

$$\bar{\phi}(\tau(f)) \in H^{n+1}(Z_2; \mathrm{Ker}(\tilde{K}_0(H) \to \tilde{K}_0(G_1) \oplus \tilde{K}_0(G_2)))$$

$$(\mathrm{resp}; \in H^{n+1}(Z_2; \mathrm{Ker}(\tilde{K}_0(H) \xrightarrow{\xi_{1*}-\xi_{2*}} \tilde{K}_0(J)))$$

is zero, there is an h-cobordism $(V; W', W)$ *with, writing* $f' \to Y$ *for the induced homotopy equivalence,* $\phi(\tau(f'))=0$.

Proof. Observe first that as f is a homotopy equivalence of closed $(n+1)$ dimensional manifolds, $\tau(f)=(-1)^n \tau(f)^*$. Hence by Lemma II.3, $\phi(\tau(f))= (-1)^{n+1} \phi(\tau(f))^*$ and thus determines an element $\bar{\phi}(\tau(f))$ of

$$H^{n+1}(Z_2; \mathrm{Ker}(\tilde{K}_0(H) \to \tilde{K}_0(G_1) \oplus \tilde{K}_0(G_2)))$$

$$(\mathrm{resp}; H^{n+1}(Z_2; \mathrm{Ker}(\tilde{K}_0(H) \xrightarrow{\xi_{1*}-\xi_{2*}} \tilde{K}_0(J)))).$$

Now suppose $\bar{\phi}(\tau(f))=0$. Then there

$$\exists \alpha \in \mathrm{Ker}(\tilde{K}_0(H) \to \tilde{K}_0(G_1) \oplus \tilde{K}_0(G_2)) \qquad (\mathrm{resp}; \mathrm{Ker}(\tilde{K}_0(H) \xrightarrow{\xi_{1*}-\xi_{2*}} \tilde{K}_0(J)))$$

with $\phi(\tau(f))=v+(-1)^{n+1} v^*$. Choose, using Lemma II.2 (i) $\beta \in \mathrm{Wh}(G)$ with $\phi(\beta)=v$ and let $(V; W, W')$ be an h-cobordism with torsion β [M 1]. Then

$$\tau(f')=\tau(f)+(-\beta+(-1)^{n+1} \beta^*)$$

and hence $\phi(\tau(f'))=\phi(\tau(f))+(-v+(-1)^n v^*)=0$ by Lemma II.3.

We proceed to construct if $\phi(\tau(f))=0$, and hence also by Lemma II.4 if $\bar{\phi}(\tau(f))=0$, a normal cobordism T of W to a split manifold homotopy equivalent to Y and compute the intersection form of T. This geometric construction will be referred to as the *nilpotent normal cobordism* construction.

If $\phi(\tau(f))=0$, by Lemma II.2(ii) and Lemma I.1(iii) we may assume that P and Q are free ZH modules. Let I' denote the closed interval $[-2, 2]$ and let $M \times I'$ denote a tubular neighborhood of $M \times 0 = M \subset W$ constructed so that extending the lift of M to $\hat{W}$ to a lift of $M \times I'$ we get $M \times (-2) \subset W_l$, $M \times 2 \subset W_r$. (In case A, this last condition is equivalent to $M \times (-2) \subset W_2$, $M \times 2 \subset W_1$). Let $\{\varepsilon_i\}$ $1 \leqq i \leqq d$ denote a fixed choice of a basis for the free ZH module P and let $\{\phi_i\}$ $1 \leqq i \leqq d$ denote a dual basis for Q under the intersection pairing λ of $K_k(M)$. Choose $\{e_i\}$ $1 \leqq i \leqq d$ disjoint framed embedded spheres in M representing $\{\varepsilon_i\}$, and $\{f_i\}$ $1 \leqq i \leqq d$ disjoint framed embedded spheres in M representing $\{\phi_i\}$ $1 \leqq i \leqq d$. Clearly, from the given intersection data on $\{\varepsilon_i\}$ and $\{\phi_i\}$ we may assume that $e_i \cap f_j = \phi$, $i \neq j$, and e_i and f_i, intersect in one point, $1 \leqq i \leqq d$.

Performing surgery on the spheres $\{e_i\}$ $1 \leqq i \leqq d$ representing a basis of the subkernel P of $K_k(M)$ produces a normal cobordism C_p of M to a manifold M_p, M_p homotopy equivalent to X. [W2; Chapter V]. Similarly, performing surgery on the spheres $\{f_i\}$ $1 \leqq i \leqq d$ representing a basis of the subkernel Q of $K_k(M)$ produces a normal cobordism C_Q of M to a manifold M_Q, M_Q homotopy equivalent to X.

Let I denote the interval $[0, 1]$. Attaching $C_p \times [-2, -1]$ along

$$M \times [-2, -1] \times 1 \subset M \times I' \times 1 \subset W \times 1 \subset W \times I$$

and similarly attaching $C_Q \times [1, 2]$ to

$$M \times [1, 2] \times 1 \subset M \times I' \times 1 \subset W \times 1 \subset W \times I$$

produces an $(n+2)$ dimensional manifold T. One component of ∂T is just $W = W \times 0$ and the other is denoted by $\mathring{W}$. See the diagram.

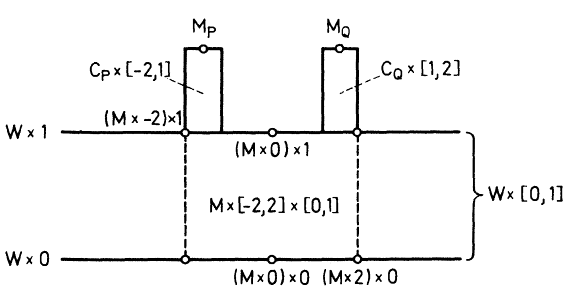


Diagram of the construction of T

Using the standard normal cobordism extension lemma [B4], the normal map induced by the homotopy equivalence f: $W \to Y$ extends to a normal map

$$F: T = W \cup C_p \times [-2, -1] \cup C_Q \times [1, 2] \to Y \times I.$$

Of course covering bundle maps are part of the structure of this normal map F [B3] [W2] though they are not explicitly indicated in our notation. Write $\mathring{f}\colon \mathring{W}\to Y$ for the restriction of F to $\mathring{W}\subset T$. There is an obvious inclusion $C_p\cup_M C_Q\subset \mathring{W}$ and the restriction of $\mathring{f}$, $\mathring{f}!\colon C_p\cup_M C_Q\to X$ is a homotopy equivalence as, up to homotopy, $C_p\cup_M C_Q$ is obtained by attaching cells to spheres representing a basis of $P\oplus Q=K_k(M)$.

Lemma II.5. *The map $\mathring{f}\colon \mathring{W}\to Y$ constructed above is a homotopy equivalence and is homotopic to a split map.*

Remark. The explicit computation performed below of the intersection form of $K_{k+1}(T)$ and the observation that it is a non-singular form leads to another proof of Lemma II.5.

Proof of Lemma II.5. In case A, from the construction of T, see the diagram,

$$\mathring{W}=(W_2\cup_M C_p)\cup_{M_p}(C_p\cup_M C_Q)\cup_{M_Q}(C_Q\cup_M W_1)$$

and $M_p\to X$, $M_Q\to X$, $C_p\cup_M C_Q\to X$ are homotopy equivalences. Thus the Mayer-Vietoris sequence of $\mathring{W}$ gives just

$$K_j(\mathring{W})=K_j(W_2\cup_M C_p;\, ZG)\oplus K_j(W_1\cup_M C_Q;\, ZG).$$

But $K_j(W_2\cup_M C_p;ZG)=K_j(W_2\cup_M C_p)\otimes_{ZG_2}ZG$ and as $K_j(W_2)=0$, $j\neq k$, and by Lemma I.5 $K_k(W_2)=P\otimes_{ZH}ZG_2$ and as C_p is formed, up to homotopy, by attaching cells to a basis of P, $K_k(W_2\cup_M C_p)=0$. Similarly $K_k(W_1\cup_M C_p)=0$ and thus $K_j(\mathring{W})=0$ for all j.

Similarly in case B, $\mathring{W}=(C_p\cup_{M_2}W'\cup_{M_1}C_Q)\cup_{M_p\cup M_Q}(C_p\cup_M C_Q)$ and arguing as in case A, it suffices to show that $K_j(C_p\cup_{M_2}W'\cup_{M_1}C_Q)=0$. But $K_j(W')=0$ for $j\neq k$ and by Lemma I.5 $K_k(W')=Q\otimes_{ZH}ZJ\oplus P\otimes_{ZH}Z[tJ]$. But as $C_p\cup_{M_2}W'\cup_{M_1}C_Q$ is formed up to homotopy by attaching on W' cells on spheres representing generators of $Q\subset K_k(M_1)$ and $Pt\subset K_k(M_2)$ the result follows as in case A.

From the construction the map $\mathring{f}$ is clearly homotopic to a map with $\mathring{f}^{-1}(X)=M_p$ and $M_p\to X$ is a homotopy equivalence. Thus $\mathring{f}$ is a splittable homotopy equivalence.

Let L be the $(-1)^{k+1}$ Hermitian pairing to $Z[G]$ defined on $(P\oplus Q)\otimes_{ZH}ZG$, $G=G_1 *_H G_2$ in case A and $G=J *_H \{t\}$ in case B, by

$$L(x,y)=\lambda(x,y)\quad \text{if } x\in P \text{ and } y\in P, \text{ or } x\in Q \text{ and } y\in Q, \text{ or } x\in P \text{ and } y\in Q.$$

Here λ denotes the intersection form of $K_k(M)$.

Thus, $L(x,y)=0$ if $x,y\in P$ or $x,y\in Q$ and

$$L(y,x)=(-1)^{k+1}\overline{L(x,y)}=(-1)^{k+1}\overline{\lambda(x,y)}\quad \text{for } x\in P,\ y\in Q.$$

The form L thus has subkernels $P\otimes_{ZH}ZG$ and $Q\otimes_{ZH}ZG$.

Lemma II.6. *The normal cobordism T constructed above is connected, $\pi_1 T\to \pi_1 Y$ is an isomorphism, $K_i(T)=0$ $i\neq k+1$, and $K_{k+1}(T)\cong(P\oplus Q)\otimes_{ZH}ZG$. With this identification, the intersection form λ_T of $K_{k+1}(T)$ is given by $\lambda_T(x,y)=L((1+\rho+\rho^2+\rho^3+\cdots)x,y)$, ρ the nilpotent map defined in I §4.*

Proof. T is homotopy equivalent to $W\cup_M(C_p\cup_M(C_Q)$ and hence T is connected, $\pi_1 T=\pi_1 Y$ and the Mayer-Vietoris sequence of $W\cup_M(C_p\cup_M C_Q)$ gives

$$\cdots\to K_{k+1}(T)\to K_k(M;ZG)\to K_k(W)\oplus K_k(C_p\cup_M C_Q;ZG)\to\cdots.$$

But as $W\to Y$ and $C_p\cup_M C_Q\to X$ are homotopy equivalences, this exact sequence reduces to the isomorphism

$$K_{k+1}(T)\cong K_k(M;ZG),\qquad K_i(T)=0,\ i\neq k+1.$$

Moreover, $K_k(M;ZG)\cong K_k(M)\otimes_{ZH}ZG=(P\oplus Q)\otimes_{ZH}ZG$ and thus

$$K_{k+1}(T)\cong(P\oplus Q)\otimes_{ZH}ZG.$$

Lastly we compute the intersection form [W2] λ_T of $K_{k+1}(T)$. The map ρ defined in Chapter I.4 was shown there to be nilpotent. Thus, $1+\rho+\rho^2+\rho^3+\cdots$ exists and in fact $1+\rho+\rho^2+\rho^3+\ldots=(1-\rho)^{-1}$. Thus the proof of Lemma II.6 will be completed by showing that

(1) $\lambda_T((1-\rho)x,y)=L(x,y)$ for $x\in P$ $y\in P$, or $x\in Q$ $y\in Q$, or $x\in P$ $y\in Q$.

For $x,y\in P$ or $x,y\in Q$, this reduces to showing that $\lambda_T((1-\rho)x,y)=0$.

We demonstrate (1) geometrically by constructing immersed spheres E_i, F_j representing ε_i and ϕ_j in $K_{k+1}(T)=(P\oplus Q)\otimes_{ZH}ZG$ and spheres $\bar{E}_i, \bar{F}_j$ representing $(1-\rho)\varepsilon_i$ and $(1-\rho)\phi_j$ in $K_{k+1}(T)$. We will count the intersections of $\bar{E}_i$ and $\bar{F}_j$ with E_k and F_h.

From the construction of C_p, the spheres e_i representing ε_i in $K_k(M)\cong(P\oplus Q)$ bound framed embedded disjoint discs (handle-cores) D_{e_i} in C_p; similarly the spheres f_j bound framed embedded disjoint discs F_{f_i} in C_Q. Pushing these slightly in a normal direction, we obtain disjointly embedded discs $D_{e_i'}$ in C_p, $D_{f_j'}$ in C_Q with $\partial D_{e_i'}=e_i'\subset M$, $\partial D_{f_j'}=f_i'\subset M$. Clearly e_i' represents ε_i, f_j' represents ϕ_j and we may assume that the classes $\{e_i\}\{e_i'\}$ are all disjointly embedded, as are $\{f_j\}\{f_j'\}$. However, in M we have $e_i\cap f_j'=e_i'\cap f_j=\phi$ if $i\neq j$ and e_i and f_i', and e_i' and f_i, intersect respectively in one point.

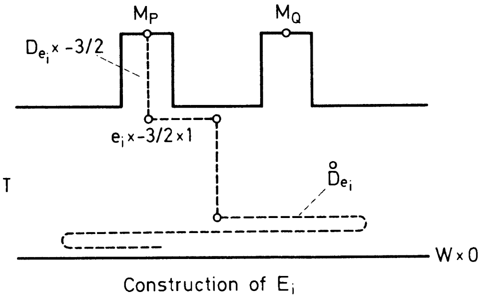
Recall that the class represented by $e_i\times 0\times 0\subset M\times I'\times 0\subset W\times 0\subset T$ is trivial in $W\times 0$ and bounds an immersed disc in $W\times 0\subset T$. Call this disc $\mathring{D}_{e_i}$. We join $\mathring{D}_{e_i}$ and D_{e_i} along e_i to form E_i. Precisely, set

$$E_i=D_{e_i}\times -3/2\cup_{e_i\times -3/2\times 1}e_i\times[-3/2,0]\times 1\cup_{e_i\times 0\times 1}e_i\times 0\times[0,1]$$
$$\cup_{e_i\times 0\times 0}\mathring{D}_{e_i},\qquad i\leqq i\leqq d.$$

Similarly, choose $\mathring{D}_{f_i}$ an immersed disc in $W\times 0$ with $\partial\mathring{D}_{f_i}=f_i\times 0\times 0$ and set

$$F_i=D_{f_i}\times 3/2\cup_{f_i\times 3/2\times 1}\cup f_i\times[3/2,0]\times 1\cup_{f_i\times 0\times 1}f_i\times 0\times[0,1]\cup_{f_i\times 0\times 0}\mathring{D}_{f_i}$$

$1\leqq i\leqq d$; that is,



Construction of E_i

We proceed with the construction of $\bar{E}_i$ and $\bar{F}_i$. First note that, letting $\tilde{e}_i'$ denote a lift of e_i' to $\tilde{M}$, $\tilde{e}_i'$ bounds a disc $\tilde{D}_{e_i'}$ in V where in case A,

$$V=\tilde{W}_1 \cup_{\left(\underset{\alpha\in[\overline{G_1,H}]}{\cup} \tilde{M}g(\alpha)\right)} \Big(\bigcup_{\alpha\in[\overline{G_1,H}]} \tilde{C}_{Qg(\alpha)}\Big)$$

and in case B,

$$V=\tilde{W}_1' \cup_{\left(\underset{\alpha\in[\overline{J,H}]}{\cup} \tilde{M}g(\alpha)\right)} \Big(\bigcup_{\alpha\in[\overline{J,H}]} \tilde{C}_{Qg(\alpha)}\Big) \cup_{\left(\underset{\beta\in[tJt^{-1};\,H]}{\cup} \tilde{M}g(\beta)t\right)} \Big(\bigcup_{\beta\in[tJt^{-1};\,H]} \tilde{C}_p g(\beta)\,t\Big).$$

For, recall from Lemma I.5 that $K_k(\tilde{W}_1)=Q\otimes_{ZH} ZG_1$ (resp; in case B, $K_k(\tilde{W}_1')=Q\otimes_{ZH} ZJ\oplus P\otimes_{ZH} Z[tJ]$) and the image of e_i' representing $\varepsilon_i\in P\subset K_k(M)$ under the map $K_k(M)\to K_k(W_1)$ (resp; $K_k(M)\to K_k(W_1')$) is given, see Lemma I.7, by $\rho(\varepsilon_i)\in Q\otimes_{ZH}\widetilde{ZG}_1$ (resp; $Q\otimes_{ZH}\widetilde{ZJ}\oplus P\otimes_{ZH} Z[tJ]$). Moreover, from the construction of V, clearly

in case A, $K_k(V)=K_k(W_1)/Q\otimes_{ZH}\widetilde{ZG}_1$

and

in case B, $K_k(V)=K_k(W_1')/Q\otimes_{ZH}\widetilde{ZJ}\oplus P\otimes_{ZH} Z[tJ]$.

Thus $\tilde{e}_i'$ bounds an immersed disc $\tilde{D}_{e_i'}$. Now as

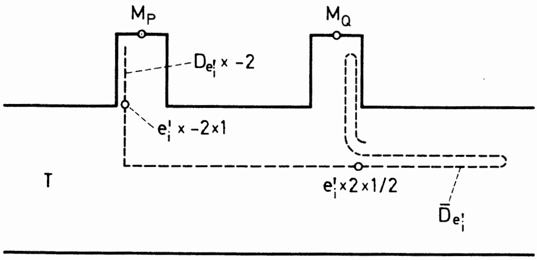
$$R=(W_1\times\tfrac{1}{2}-M\times[0,2)\times\tfrac{1}{2}\cup_{M\times 2\times\frac{1}{2}} M\times 2\times[\tfrac{1}{2},1]\cup_{M\times 2\times 1} C_Q\subset T$$

(resp; in case B,

$$R=(W'\times\tfrac{1}{2}-M\times[0,2)\times\tfrac{1}{2}\cup_{M\times 2\times\frac{1}{2}} M\times 2\times[\tfrac{1}{2},1]\cup_{M\times 2\times 1} C_Q\subset T)$$

is homeomorphic to $W_1\cup_M C_Q$ (resp; $W'\cup_{M_1'} C_Q$) there is an obvious projection map $V\to R$ induced from $V\subset\tilde{R}$. Let $\bar{D}_{e_i'}$ be the image of $\tilde{D}_{e_i'}$ under this map. Thus, we may take $\bar{D}_{e_i'}$ to be an immersed disc with boundary $e_i'\times 2\times\frac{1}{2}$. The sphere $\bar{E}_i$ is formed by joining $D_{e_i'}$ with $\bar{D}_{e_i'}$ along e_i'. Precisely, set

$$\bar{E}_i=D_{e_i'}\times -2\cup_{e_i'\times -2\times 1} e_i'\times -2\times[\tfrac{1}{2},1]\cup_{e_i'\times -2\times\frac{1}{2}} e_i'\times[-2,2]\times\tfrac{1}{2}.$$

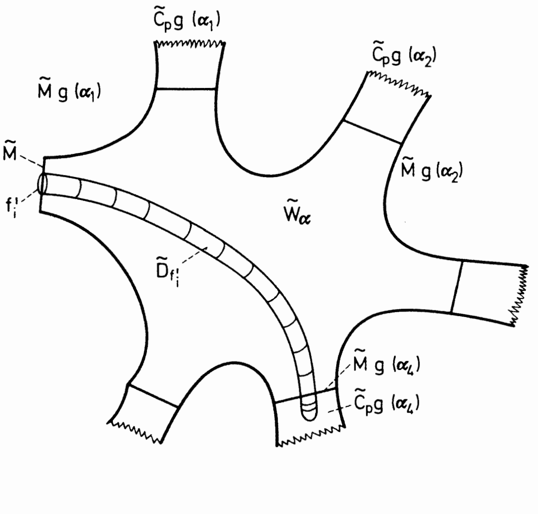


Construction of $\bar{E}_i$

Similarly, construct

$$\tilde{D}_{f'_i} \quad \text{in } \tilde{W}_2 \cup_{\bigcup_{\alpha\in[G_2,H]} \tilde{M} g(\alpha)} \Big(\bigcup_{\alpha\in[G_2,H]} \tilde{C}_p g(\alpha)\Big)$$

$$\Big(\text{resp; } \tilde{W}'_2 \cup_{\bigcup_{\alpha\in[tJt^{-1};H]} \tilde{M} g(\alpha)} \tilde{C}_p g(\alpha) \cup_{\bigcup_{\beta\in[J,H]} \tilde{M} g(\beta) t^{-1}} \tilde{C}_Q g(\beta) t^{-1}\Big)$$



$$D_{f'_i} \quad \text{in } \tilde{W}_2 \cup_{\bigcup_{\alpha\in[G_2,H]} \tilde{M} g(\alpha)} \Big(\bigcup_{\alpha\in[G_2,H]} \tilde{C}_p g(\alpha)\Big)$$

and from this get, as above, $\bar{D}_{f'_i}$ in

$$W_2 \times \tfrac{1}{2} - M \times (-2,0] \times \tfrac{1}{2} \cup_{M\times 2\times -\frac{1}{2}} M \times -2 \times [\tfrac{1}{2}, 1] \cup C_p \times -2$$

$$(\text{resp; } W' \times \tfrac{1}{2} - M \times (-2,0] \times \tfrac{1}{2} \cup_{M\times 2\times -\frac{1}{2}} M \times -2 \times [\tfrac{1}{2}, 1] \cup C_p \times -2)$$

bounding $f'_i \times -2 \times \frac{1}{2}$. Form the sphere $\bar{F}_i$ by joining $D_{f'_i}$ with $\bar{D}_{f'_i}$ along f'_i; precisely, as above, set

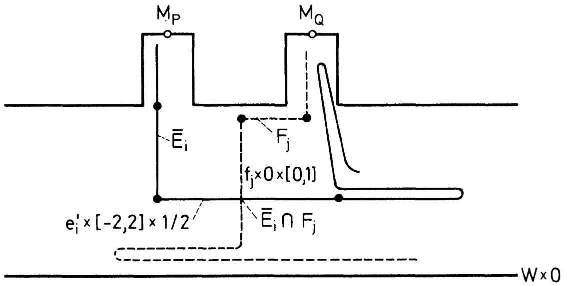
$$\bar{F}_i = D'_{f'_i} \times 2 \cup_{f'_i\times 2\times 1} \cup f'_i \times 2 \times [\tfrac{1}{2}, 1] \cup_{f'_i \times 2 \times \frac{1}{2}} \cup f'_i \times [2, -2] \times \tfrac{1}{2} \cup_{f'_i\times -2\times\frac{1}{2}} \bar{D}_{f'_i}.$$

From the constructions,

$$\bar{E}_i \cap E_j = e_i' \times 0 \times \tfrac{1}{2} \cap e_i \times 0 \times \tfrac{1}{2} = \phi,$$
$$\bar{F}_i \cap F_j = f_i' \times 0 \times \tfrac{1}{2} \cap f_j' \times 0 \times \tfrac{1}{2} = \phi$$

and

$$\bar{E}_i \cap F_j = e_i' \times 0 \times \tfrac{1}{2} \cap f_j \times 0 \times \tfrac{1}{2}.$$



$\bar{E}_i \cap F_j = e_i' \times 0 \times 1/2 \cap f_j \times 0 \times 1/2$

Thus choosing orientations for M and W at a basepoint [W2; Chapter V] to define λ_T, the intersection form on $K_{k+1}(T)$, we get that

$$\lambda_T(\bar{E}_i, F_j) = \lambda(e_i', f_j) = 1, \quad \lambda_T(\bar{E}_i, E_j) = 0, \quad \lambda_T(\bar{F}_j, F_j) = 0,$$

$1 \leqq i \leqq d$, $1 \leqq j \leqq d$. Thus, to complete the proof of (1) it suffices to show that E_i, F_i, $\bar{E}_i$, $\bar{F}_i$ represent ε_i, ϕ_i, $(1-\rho)\varepsilon_i$ and $(1-\rho)\phi_i$ respectively in

$$(P \oplus Q) \otimes_{ZH} ZG \cong K_{k+1}(T).$$

We first check that E_i represents ε_i; an entirely parallel argument checks that F_i represents ϕ_i. As the isomorphism $K_{k+1}(T) \cong K_k(M; ZG) \cong (P \oplus Q) \otimes_{ZH} ZG$ is induced from the Mayer-Vietoris sequence of T, which is homotopy equivalent to $W \cup_M (C_p \cup_M C_Q)$, we have the commutative diagram

$$\begin{tikzcd}
K_{k+1}(T) \arrow[rr] \arrow[d] & & K_k(M; ZG) \cong (P \oplus Q) \otimes_{ZH} ZG \\
K_{k+1}(T, W) \arrow[d, "\text{excision}"] & & \\
K_{k+1}(C_p \cup_M C_Q, M; ZG) \arrow[uurr, "\partial"] & &
\end{tikzcd}$$

Clearly E_i in $K_{k+1}(T)$ goes to (D_{e_i}, e_i) in

$$K_{k+1}(C_p \cup_M C_Q, M; ZH) \subset K_{k+1}(C_p \cup C_Q, M; ZG)$$

and hence to $\varepsilon_i \in K_k(M; ZG)$ representing e_i.

Lastly we verify that $\bar{E}_i$ represents $(1-\rho)\varepsilon_i$ under the isomorphism $K_{k+1}(T)\cong (P\oplus Q)\otimes_{ZH} ZG$; an entirely parallel procedure checks that $\bar{F}_i$ represents $(1-\rho)\phi_i$. From the construction of $\bar{E}_i$ as the union of $D_{e'_i}$ and $\bar{D}_{e'_i}$ along e'_i, where $\bar{D}_{e'_i}$ lifts to $\tilde{D}_{e'_i}$ in V, the sphere $\bar{E}_i$ lifts to a map to $V\cup_{\tilde{M}}\tilde{C}_p$; thus, recalling the definition of V, under the map

$$K_{k+1}(T)\to K_{k+1}(T,W)\xrightarrow{\text{excision}} K_{k+1}(C_p\cup_M C_Q, M; ZG),$$

$\bar{E}_i$ goes to a class in $K_{k+1}(C_p,M)\oplus K_{k+1}(C_Q,M)\otimes_{ZH} ZG_1$ (respectively; in case B, $K_{k+1}(C_p,M)\oplus K_{k+1}(C_Q,M)\otimes_{ZH} ZJ\oplus K_{k+1}(C_p,M)\otimes_{ZH} Z[tJ]$) with the component in $K_{k+1}(C_p,M)$ representing $(D_{e'_i}, e'_i)$. Thus under the following composed map, which is the Mayer-Vietoris isomorphism

$$K_{k+1}(T)\longrightarrow K_{k+1}(C_p\cup_M C_Q, M; ZG)\xrightarrow{\partial} K_k(M;ZG)=(P\oplus Q)\otimes_{ZH} ZG,$$

$\bar{E}_i$ goes to $\varepsilon_i+v\in(P\oplus Q)\otimes_{ZH} ZG$ where in case A, $v\in Q\otimes_{ZH}\widetilde{ZG_1}$ and in case B, $v\in Q\otimes_{ZH}\widetilde{ZJ}\oplus P\otimes_{ZH} Z[tJ]$. Furthermore, from the commutative diagram

$$\begin{tikzcd}
K_{k+1}(V\cup_{\tilde{M}}\tilde{C}_p) \arrow[r] \arrow[d] & K_{k+1}(T) \arrow[d, "\cong"] \arrow[dr] & \\
K_{k+1}(V\cup_{\tilde{M}}\tilde{C}_p, \tilde{C}_p\cup(V-W_1)) \arrow[r] \arrow[d] & K_{k+1}(T, C_p\cup_M C_Q; ZG) \arrow[d, "\cong", "\text{excision}"'] & K_k(M;ZG)=(P\oplus Q)\otimes_{ZH} ZG \\
K_{k+1}(W_1,\partial W_1; ZG_1) \arrow[r] & K_{k+1}(W,M;ZG) \arrow[ur] &
\end{tikzcd}$$

ε_i+v is represented by an element of Image

$$(K_{k+1}(W_1,\partial W_1)\to K_k(\partial W_1))\subset K_k(M:,ZG)$$

(for case B, in the above sentence and diagram, replace W_1 by W'_1). Thus ε_i+v goes to zero under the map $K_k(\partial W_1)\to K_k(W_1)$ (resp; $K_k(\partial W'_1\to K_k(W'_1))$. But as $\varepsilon_i\in P$ and $v\in Q\otimes_{ZH}\widetilde{ZG_1}$ (resp; $Q\otimes_{ZH}\widetilde{ZJ}\oplus P\otimes_{ZH} Z[tJ]$) by Lemma I.7 and the preceding discussion, $v=-\rho_1(\varepsilon_i)$.

Remark. Lemma II.4 can also be demonstrated by a less geometric computation of the adjoint of λ_T in terms of λ and ρ.

Note that if $\bar{\phi}(\tau(f))=0$, W is, by Lemma II.4, h-cobordant to a manifold $\check{W}$ with the induced homotopy equivalence $\check{f}$: $\check{W}\to Y$ satisfying $\phi(\tau(\check{f}))=0$. Then, performing the nilpotent normal cobordism construction of T described above, we obtain a normal cobordism of W to a split homotopy equivalence.

Proof of Lemma II.3. We first prove Lemma II.3 for case A, $G=G_1 *_H G_2$, and then briefly indicate the variation of notation for case *B*, $G=J *_H \{t\}$.

For γ and δ two stable bases of a free $Z[D]$ module, D a group, let $\binom{\gamma}{\delta}$ devote the element of $\mathrm{Wh}(D)$ represented by the automorphism carrying γ to δ. Note that

$\begin{pmatrix}\gamma\\ \delta\end{pmatrix} = -\begin{pmatrix}\delta\\ \gamma\end{pmatrix}$. For P a finitely generated right projective $Z[H]$ module with

$$[P]\in \operatorname{Ker}(\tilde{K}_0(H)\to \tilde{K}_0(G_1)\oplus \tilde{K}_0(G_2)),$$

let $\alpha(P)$ denote an element of a coset of

$$\operatorname{Image}(\operatorname{Wh}(G_1)\oplus \operatorname{Wh}(G_2)\to \operatorname{Wh}(G_1 *_H G_2))$$

defined as follows. Choose a projective module Q with $[P]+[Q]=0$. Thus there is a stable $Z[H]$ basis for $P\oplus Q$ and choose such a basis α_1. We also write α_1 for the induced basis of $(P\oplus Q)\otimes_{ZH} ZG$. However, $P\otimes_{ZH} ZG_1$ is a stably free ZG_1 module and $Q\otimes_{ZH} ZG_2$ is a stably free ZG_2 module. A choice of stable bases for these two modules also determines a choice α_2 of a stable basis for $P\otimes_{ZH} ZG_1 \otimes_{ZG_1} ZG \oplus Q\otimes_{ZH} ZG_2 \otimes_{ZG_2} ZG = (P\oplus Q)\otimes_{ZH} ZG$. Thus we get $\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\in \operatorname{Wh}(G)$, which however, is only well defined, because of the choices[11] made in its construction, modulo Image $(\operatorname{Wh}(G_1)\oplus \operatorname{Wh}(G_2)\to \operatorname{Wh}(G_1 *_H G_2))$. Let $\alpha(P)=\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$.

The argument of [W1; 5.8] shows that every element $x\in \operatorname{Wh}(G)$ has a (not unique) representative of the form[12]

$$x=\alpha(P)\left(\begin{array}{c|c} I & A\\ \hline B & I\end{array}\right) CD$$

where A is a matrix with coefficients in $\widetilde{Z[G_1]}$, B is a matrix with coefficients in $\widetilde{ZG}_2$, C is a matrix with coefficients in ZG_1 and D is a matrix with coefficients in ZG_2. Here $\left(\begin{array}{c|c} I & A\\ \hline B & I\end{array}\right)$ is the matrix form of the element of $\operatorname{Wh}(G)$ given by an element of Waldhausen's *reduced* group of nilpotent maps $\widetilde{\operatorname{Nil}}(H; G_1, G_2)$ and I denotes the identity map. (Note that we are here using the fact that for an element of the *reduced* group $\widetilde{\operatorname{Nil}}(H; G_1, G_2)$ we may choose a representative (P, Q, ρ_1, ρ_2) with, by stabilizing, P and Q free modules of the same rank.) Waldhausen shows that for x in this form, $\phi(x)=[P]$.

Now clearly, as $\operatorname{Wh}(G)$ is abelian,

$$x^* = \alpha(P)^* \left(\begin{array}{c|c} I & \bar{B}^t\\ \hline \bar{A}^t & I\end{array}\right) \bar{C}^t \bar{D}^t$$

where for a matrix $M\in GL(n, Z[G])$, $\bar{M}^t$ denotes the conjugate transpose of M (see [M1]). Moreover, it is easy to see that in $\operatorname{Wh}G$), [M1]

$$\left(\begin{array}{c|c} I & \bar{B}^t\\ \hline \bar{A}^t & I\end{array}\right) = \left(\begin{array}{c|c} I & \bar{A}^t\\ \hline \bar{B}^t & I\end{array}\right).$$

[11] This corrects slightly the statement of [W1, 5.7]

[12] This expression for x indicates the relationship between the results of [W1] and the Higman process of [St]

Also, Lemma I.11 shows that $\left(\begin{array}{c|c} I & \bar{A}^t \\ \hline \bar{B}^t & I \end{array}\right)$ represents an element of Waldhausen's reduced group of nilpotent maps $\widetilde{\mathrm{Nil}}(H; G_1, G_2)$. Writing γ^* to denote a dual basis of N^* for γ a basis for a free module N, we also get

$$\binom{\alpha_2^*}{\alpha_1^*} = -\binom{\alpha_1^*}{\alpha_2^*} = -\alpha(P^*) \text{ modulo } (\mathrm{Image}(\mathrm{Wh}(G_1) \oplus \mathrm{Wh}(G_2) \to \mathrm{Wh}(G_1 *_H G_2)).$$

Hence, writing x^* in the form

$$x^* = \alpha(-[P]^*) \left(\begin{array}{c|c} I & A' \\ \hline B' & I \end{array}\right) C' D'$$

we get $\phi(x^*) = -[P^*]$.

The argument in case B is similar. For

$$[P] \in \mathrm{Ker}(\tilde{K}_0(H) \xrightarrow{\xi_{1_*} - \xi_{2_*}} \tilde{K}_0(J)),$$

let $\alpha(P)$ denote a representative in the coset of $\mathrm{Wh}(J)$ in $\mathrm{Wh}(J *_H \{t\})$ constructed as follows. Again choose Q with $[P] \oplus [Q] = 0$ and choose a stable basis α_1 for $P \oplus Q$ and hence for $(P \oplus Q) \otimes_{ZH} ZG$. Using

$$(\xi_{1_*} - \xi_{2_*})[P] = [P \otimes_{Z[\xi_1 H]} ZJ \oplus Q \otimes_{Z[\xi_2 H]} ZJ],$$

choose a stable basis over ZJ of

$$P \otimes_{Z[\xi_1 H]} ZJ \oplus Q \otimes_{Z[\xi_2 H]} ZJ.$$

This induces a stable basis of

$$P \otimes_{ZH} ZJ \otimes_{ZJ} Z[J *_H \{t\}] \oplus Q \otimes_{Z[\xi_2 H]} ZJ \otimes_{ZJ} Z[J *_H \{t\}].$$

But as there is an obvious isomorphism of

$$Q \otimes_{Z[\xi_2 H]} ZJ \otimes_{ZJ} Z[J *_H \{t\}]$$

with

$$Q \otimes_{ZH} Z[tJt^{-1}] \otimes_{Z[tJt^{-1}]} Z[J *_H \{t\}],$$

this induces a basis α_2 for

$$P \otimes_{ZH} ZJ \otimes_{ZJ} ZG \oplus Q \otimes_{ZH} Z[tJt^{-1}] \otimes_{Z[tJt^{-1}]} Z[G] \cong (P \oplus Q) \otimes_{ZH} ZG.$$

Now set $\alpha(P) = \binom{\alpha_1}{\alpha_2}$, which is well defined modulo $\mathrm{Im}(\mathrm{Wh}(J) \to \mathrm{Wh}(J *_H \{t\}))$. Again, 5.8 of [W 1] shows that every element x of $\mathrm{Wh}(J *_H \{t\})$ can be represented, for some P, by

$$x = \alpha(P) \left(\begin{array}{c|c} I + C & A \\ \hline B & I + D \end{array}\right) E$$

where A (resp; B; C; D; E) is a matrix with entries in ZJ (resp; $Z[\widetilde{tJt^{-1}}]$; $Z[tJ]$; $Z[Jt^{-1}]$; $Z[J]$) and for x in this form, $\phi(x)=[P]$. The remainder of the argument is as in case A.

§ 2. Completion of the Argument for $n=2k$

Lemma II.10 below, the main algebraic result of this section, is used in the completion of the proof for $n=2k$ at the end of this section to analyze the surgery obstruction of the nilpotent normal cobordism construction. Lemma II.7 indicates the role that is played by the square-root closed condition. Lemma II.8 is a necessary technical exercise; the reader may wish to look at its proof only after reading the remainder of the chapter. Lemma II.9 plays a crucial role in the proof of Lemma II.10.

Lemma II.7. *Let H be a square-root closed subgroup of a group D. Let $w: D\to Z_2$ be a homomorphism determining as usual an involution of $Z[D]$. Then as a $Z[H]$ bimodule $Z[D]\cong Z[H]\oplus E\oplus \bar{E}$ where $\bar{E}=\{\bar{x}|x\in E\}$.*

Proof of Lemma II.7. First observe that the only double coset HdH of H in D, $d\in D$, equal to its inverse double coset $(HdH)^{-1}=Hd^{-1}H$ is the trivial double coset[13]; i.e., $d\in H$. For if $h_1dh_2=h_3d^{-1}h_4$, for h_1, h_2, h_3, $h_4\in H$, $d\in D$, then $dh_3^{-1}h_1d=h_4h_2^{-1}\in H$ and thus $(dh_3^{-1}h_1)^2\in H$ and hence as H is square-root closed in D, $dh_3^{-1}h_1\in H$ and $d\in H$.

Then we can construct T a union of double cosets of H in D with $D-H$ the *disjoint* union of T and $T^{-1}=\{x|x^{-1}\in T\}$. Now let E be additively generated by the elements of T.

Using Lemma II.7, if H is square-root closed in G_1 and G_2 (resp; in J and tJt^{-1}) we may write:

(1) $\widetilde{ZG}_1=\mathsf{A}_1\oplus\mathsf{A}_2,\quad \mathsf{A}_2=\bar{\mathsf{A}}_1$

(2) $\widetilde{ZG}_2=\mathsf{B}_1\oplus\mathsf{B}_2,\quad \mathsf{B}_2=\bar{\mathsf{B}}_1$

(resp;

(3) $\widetilde{ZJ}=\mathsf{A}_1\oplus\mathsf{A}_2,\quad \mathsf{A}_2=\bar{\mathsf{A}}_1$

(4) $Z[\widetilde{tJt^{-1}}]=\mathsf{B}_1\oplus\mathsf{B}_2,\quad \mathsf{B}_2=\bar{\mathsf{B}}_1$).

Of course, these decompositions are in general not unique but we choose one and keep it fixed for the remainder of this section.

In I.1, $Z[G]$ for $G=G_1*_HG_2$ (resp; $G=J*_H\{t\}$) was described as a $Z[H]$ bimodule as a sum of $Z[H]$, A_i, B_i, C_i, D_i where each of these last 4 is a tensor product over ZH of $\widetilde{ZG}_1$ and $\widetilde{ZG}_2$ (resp; $\widetilde{ZJ}$, $Z[\widetilde{tJt^{-1}}]$, $Z[tJ]$ and $Z[Jt^{-1}]$. Correspondingly, using the ZH bimodule decompositions (1) and (2) (resp; (3)

[13] In our earlier paper [C 1] and also in [CS 1] this was used as a definition of square-root closedness

and (4)) above, $Z[G]$ may be described in terms of sums of tensor-products of the ZH bimodules A_1, A_2, B_1, B_2 (resp; and $Z[tJ]$ and $Z[Jt^{-1}]$. To make this precise, we introduce the following notation. Let F denote the free associative monoid in case A on the symbols $\alpha_1, \alpha_2, \beta_1, \beta_2$ (resp; and also γ and δ in case B). Let $F_i \subset F$ denote the subset of words of length i, $i \geqq 1$. Call a word $w \in F$ admissible if none of the pairs $\alpha_i \alpha_j$, $\beta_i \beta_j$, $i=1,2$, $j=1,2$ (*resp;* and also $\gamma\alpha_i$, $\beta_i\gamma$, $\delta\beta_i$, $\gamma_i\delta$, $\gamma\delta$, $\delta\gamma$ in case B) occurs as a consecutive pair in w. For example, $\alpha_1\beta_2\alpha_2$ and $\alpha_2\gamma\gamma\beta_1\delta$ are admissible, but $\beta_1\alpha_1\alpha_2\beta_2$ and $\beta_2\alpha_1\delta$ are not admissible. Write F^0 for the subset of admissible words of F and set $F_i^0 = F^0 \cap F_i$. For a word $w \in F^0 \subset F$ in α_1, α_2, β_1 and β_2. (resp; and γ and δ in case B), let $Z[w]$ denote the corresponding tensor product over ZH of the ZH bimodules A_1, A_2, B_1, B_2 (resp; and $Z[tJ]$ and $Z[Jt^{-1}]$ in case B). The precise definition is given by setting:

$$Z[\alpha_1] = \mathsf{A}_1, \qquad Z[\alpha_2] = \mathsf{A}_2$$

$$Z[\beta_1] = \mathsf{B}_1, \qquad Z[\beta_2] = \mathsf{B}_2$$

(resp; and also in case B $Z[\gamma] = Z[tJ]$, $Z[\delta] = Z[Jt^{-1}]$) and for $xw \in F^0$, $Z[xw] = Z[x] \otimes_{ZH} Z[w]$, $x = \alpha_1, \alpha_2, \beta_1, \beta_2$ (resp; or $x = \gamma, \delta$ in case B).

Using the inductive definition of A_i, B_i, C_i, D_i of I.1 we show the following when H is square-root closed in G.

Lemma II.8. *As $Z[H]$ bimodules,*

(i) $Z[G] \cong Z[H] \oplus \sum_{w \in F^0} Z[w]$,

(ii) $A_i \oplus B_i \oplus C_i \oplus D_i \cong \sum_{w \in F_i^0} Z[w]$.

At least for small values of i, the reader may check Part (ii) of this lemma using (1), (2), (3), (4) and the inductive definition of A_i, B_i, C_i, D_i to decompose the left-hand term. The precise proof we give sums up this decomposition process by induction arguments.

Proof of Lemma II.8. Recalling the formula of I.1

$$Z[G] = ZH \oplus \sum_{i=1}^{\infty} (A_i \oplus B_i \oplus C_i \oplus D_i)$$

Part (i) follows immediately from Part (ii), which we proceed to prove. Define subsets of F, $F_i(a)$, $F_i(b)$, $F_i(c)$, $F_i(d)$ for $i \geqq 1$ by the following inductive procedure:

$$F_1(a) = \{\alpha_1, \alpha_2\}, \qquad F_2(c) = \{\beta_1, \beta_2\}.$$

In case A, $F_1(b) = F_1(d) = \phi$.

In case B, $F_1(b) = \{\delta\}$, $F_1(d) = \{\gamma\}$

$$F_{i+1}(a) = \{xy \mid x \in F_1(a), y \in F_i(d) \text{ or } x \in F_1(b), y \in F_i(a)\}$$

$$F_{i+1}(b) = \{xy \mid x \in F_1(b), y \in F_i(b) \text{ or } x \in F_1(a), y \in F_i(c)\}$$

$$F_{i+1}(c) = \{xy \mid x \in F_1(d), y \in F_i(c) \text{ or } x \in F_1(c), y \in F_i(b)\}$$

$$F_{i+1}(d) = \{xy \mid x \in F_1(d), y \in F_i(d) \text{ or } x \in F_1(c), y \in F_i(a)\}.$$

We claim that

$$A_i = \sum_{w\in F_i(a)} Z[w], \quad B_i = \sum_{w\in F_i(b)} Z[w], \quad C_i = \sum_{w\in F_i(c)} Z[w], \quad D_i = \sum_{w\in F_i(d)} Z[w].$$

For $i=1$, this is immediate from (1), (2), (3), (4), and the definitions of A_1, B_1, C_1, D_1 in case A and in case B. For $i>1$, this is shown by an easy induction which compares the inductive definition of A_i, B_i, C_i, D_i with the similar inductive definition of $F_i(a)$, $F_i(b)$, $F_i(c)$, $F_i(d)$. Details are left to the reader.

Hence, it suffices to show that for all i,

$$F_i^0 = F_i(a) \cup F_i(b) \cup F_i(c) \cup F_i(d).$$

(Notice that the terms on the right hand side are disjoint as otherwise A_i, B_i, C_i, D_i would not be disjoint.) It is trivial to check that

$$F_1^0 = F_1(a) \cup F_1(b) \cup F_1(c) \cup F_1(d).$$

Now let w be a word of the shortest length with $w \in F_i^0$ but

$$w \notin F_i(a) \cup F_i(b) \cup F_i(c) \cup F_i(d).$$

As w is admissible we may write, for example, $w = \alpha_2 \beta_1 w'$; the other cases $w = \alpha_2 \beta_2 w' = \beta_1 \alpha_2 w'$ etc. are treated by the same method we employ here. Then $\beta_1 w'$ is admissible and hence, by assumption is in $F_{i-1}(a) \cup F_{i-1}(b) \cup F_{i-1}(c) \cup F'_{i-1}(d)$. But checking definitions of these last 4 sets, only $F_{i-1}(c)$ and $F_{i-1}(d)$ have terms ending on the left in β_1 or β_2. Hence $\beta_1 w' \in F_{i-1}(c) \cup F_{i-1}(d)$; but $\alpha_2 F_{i-1}(c) \subset F_i(b)$, $\alpha_1 F_{i-1}(d) \subset F_i(a)$, whence $w \in F_i(b) \cup F_i(a)$.

Similarly, suppose $w \in F_i(a) \cup F_i(b) \cup F_i(c) \cup F_i(d)$ is a word of smallest length with $w \notin F_i^0$. Then, for example, $w = \alpha_2 \alpha_1 w'$. But $\alpha_1 w'$, as it ends on the left in α_1 or α_2 must be in $F_{i-1}(a)$ or $F_{i-1}(b)$ and examination of the definitions shows that $\alpha_2 F_{i-1}(a) \not\subset F_i(a)$ or $F_i(b)$ or $F_i(c)$ or $F_i(d)$ and similarly for $\alpha_2 F_{i-1}(b)$, contradicting our assumption on w. The other cases, $w = \beta_1 \beta_2 w'$, $w = \gamma \delta w'$ etc. are handled similarly and are left to the reader.

We now define some terms used in stating Lemma II.9, which is used in proving Lemma II.10. Let $Z[F]$ denote the integral monoid-ring of the associative monoid F defined above. Of course, as an additive group

$$(5) \quad Z[F] = Z \oplus \sum_{i \geqq 1} Z[F_i]$$

where $Z[F_i]$ denotes formal linear sums of words of length i. Define an involution $x \to \bar{x}$ on $Z[F]$ by the formulas $\overline{xy} = \bar{y}\bar{x}$, $\overline{x+y} = \bar{x} + \bar{y}$ for $x, y \in F$, $\bar{\alpha}_1 = \alpha_2$, $\bar{\alpha}_2 = \alpha_1$, $\bar{\beta}_1 = \beta_2$, $\bar{\beta}_2 = \beta_1$. (resp; and $\bar{\gamma} = \delta$, $\bar{\delta} = \gamma$). Let I denote the ideal in $Z[F]$ additively generated by the elements of $F - F^0$ and let Λ be the quotient ring $Z[F]/I$. As $I = \bar{I}$, Λ inherits an involution from $Z[F]$. Also as I is homogeneous in the decomposition of (5), we get the additive group decomposition $\Lambda = Z \oplus \sum_{i \geqq 1} \Lambda_i$, where $\Lambda_i = \text{Image}(Z[F_i] \to \Lambda)$. Write $Z[F^0]$ for the additive subgroup of $Z[F]$ consisting of linear sums of elements of F^0 and $Z[F_i^0]$ for the linear sums of elements of F_i^0. The restriction of $Z[F] \to \Lambda$ gives additive involution-preserving isomorphisms

$$Z[F^0] \to \Lambda, \; Z[F_i^0] \to \Lambda_i, \; i \geqq 1.$$

Lemma II.9. *For* $i \geqq 0$, $\exists v_i \in \sum_{j \geqq 1} \Lambda_j$ *with, setting* $t = 1 - (\alpha_1 + \alpha_2 + \beta_1 + \beta_2)$ (resp; *in case* B, $t = 1 - (\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma + \delta)$

$$t - (1 + \overline{V_i})(1 + V_i) \in \sum_{j > i} \Lambda_j.$$

Proof of Lemma II.9. Observe first that for $w \in F^0$, $w \neq \bar{w}$. We check this first in case A. Recall that w is a word in α_1, α_2, β_1 and β_2. If it has an odd number of factors, clearly the middle factors of w and $\bar{w}$ are different. If it has an even number of terms, as w is admissible the factors alternate between α_1 or α_2 and β_1 or β_2. For example, if w with an even number of factors is of the form $w = \alpha_{i_1}, \beta_{i_2}, \alpha_{i_2} \ldots \alpha_{i_{j-1}} \beta_{i_j}$, $\bar{w} = \bar{\beta}_{i_j} \bar{\alpha}_{i_{j-1}} \ldots \bar{\beta}_{i_2} \bar{\alpha}_{i_1}$ and $\bar{\beta}_{i_j} = \beta_1$ or β_2 and so $\bar{w} \neq w$. Similarly, if the first factor on the left of w is β_{i_1}, we get $\bar{w} \neq w$.

In case B, for a word w, let $\varphi(w)$ denote the word obtained by delecting all the factors γ and δ. It is easy to see from the definition of admissibility that if w is admissible, so is $\varphi(w)$. Moreover, if $w = \bar{w}$, then $\varphi(w) = \overline{\varphi(w)}$. Hence if $w = \bar{w}$, $\varphi(w)$ is the null word and hence w is a word in γ and δ. But as w is admissible, it does not contain as consecutive pairs $\gamma\delta$ or $\delta\gamma$. Hence $w = \delta^i$ or $w = \gamma^i$. But also $\bar{\delta}^i = \gamma^i$, $\bar{\gamma}^i = \delta^i$ and as in case A we conclude that $w \neq \bar{w}$.

As an additive group with involution, $Z[F_i^0] \cong \Lambda_i$ and hence from the above demonstration that $w \neq \bar{w}$, we get that if $x \in \Lambda_i$ satisfies $x = \bar{x}$, then there exists $y \in \Lambda_i$ with $x = y + \bar{y}$.

Inductively, define V_i as follows. Set $V_0 = 0$. Assume V_i satisfying the conclusion of the lemma has been defined; we proceed to define V_{i+1}. From

$$t = \bar{t}$$

we get $\quad t - (1 + \overline{V_i})(1 + V_i) = \overline{t - (1 + \overline{V_i})(1 + V_i)}$

and as $\quad t - (1 + \overline{V_i})(1 + V_i) \in \sum_{j > i} \Lambda_j$,

$$t - (1 + \overline{V_i})(1 + V_i) = C + \bar{C} + d, \; C \in \Lambda_{i+1}, \; d \in \sum_{j > i+1} \Lambda_j.$$

Now set $\quad V_{i+1} = V_i + C$

Then,
$$\begin{aligned} t - (1 + \overline{V_{i+1}})(1 + V_{i+1}) &= t - (1 + \overline{V_i} + \bar{C})(1 + V_i + C) \\ &= t - (1 + \overline{V_i})(1 + V_i) - (C + \bar{C}) - (C\bar{C} + \bar{C} V_i + \overline{V_i} C) \\ &= C + \bar{C} + d - (C + \bar{C}) - (C\bar{C} + \bar{C} V_i + \overline{V_i}\, C) \\ &= d - C\bar{C} - \bar{C} V_i - \overline{V_i}\, C \\ &\in \sum_{j > i+1} \Lambda_j. \end{aligned}$$

The main algebraic result of this section used to study the Hermitian pairing λ_T of the nilpotent normal cobordism construction of II § 1, is the following:

Lemma II.10. *Let* $H \subset G_i$, $i = 1, 2$ (resp; $H \subset J$, $H \subset tJt^{-1}$) *be inclusions of* H *as a square-root closed subgroup. Let* w: $G \to Z_2$ *be a homomorphism of* $G = G_1 *_H G_2$ (resp; $G = J *_H \{t\}$) *and also denote the restrictions of* w *to subgroups of* G *by* w. *Let* $x \in L^h_{2k+2}(G, w)$ *be represented by the* $(-1)^{k+1}$ *Hermitian form* (N, ϕ, μ) *where:*

(i) *There are free ZH modules P, Q with $N=(P\oplus Q)\otimes_{ZH} ZG$.*

(ii) *There are ZH linear maps, nilpotent in the sense of* I.4,

$$\rho_1\colon\ P\to Q\otimes_{ZH}\widetilde{ZG}_1$$

$$\rho_2\colon\ Q\to P\otimes_{ZH}\widetilde{ZG}_2$$

$$(\text{resp}; \rho_1\colon\ P\to Q\otimes_{ZH}\widetilde{ZJ}\oplus P\otimes_{ZH} Z[tJ]$$

$$\rho_2\colon\ Q\to P\otimes_{ZH} Z[\widetilde{tJt^{-1}}]\oplus Q\otimes_{ZH} Z[Jt^{-1}]).$$

(iii) *There is a non-singular $(-1)^{k+1}$ Hermitian pairing $L\colon(P\oplus Q)\times(P\oplus Q)\to ZH$ with $L(x,y)=0$ for $x, y\in P$ or $x, y\in Q$ and, letting L also denote the extension of this to a Hermitian pairing $L\colon\ N\times N\to ZG$, with $\phi(x,y)=L((1-\rho)^{-1}x,y)$. Here, ρ denotes the ZG linear map $\rho\colon N\to N$ induced by ρ_1 and ρ_2* (see I.4).

Then, if $\bar{H}$ is a subgroup of H containing all elements of order 2 in H,

$$x\in\operatorname{Image}(L^h_{4k+2}(\bar{H},w)\to L^h_{4k+2}(G,w)).$$

Remark. In the present section we use this only for $\bar{H}=H$.

Proof of Lemma II.10: For a ZG linear map $s\colon N\to N$, let $\bar{s}$ denote the map $\bar{s}\colon N\to N$ satisfying

$$L(sx,y)=L(x,\bar{s}y),\qquad x\,y\in N.$$

We first show that to prove the lemma it suffices to produce a $Z[G]$ linear map $V\colon N\to N$ with $(1-\rho)=\overline{V}V$. Let $e_1,\dots,e_r$ denote a basis for P, and $f_1,\dots,f_r$ a dual basis for Q so that

$$L(e_i,e_j)=0,\quad L(f_i,f_j)=0,\quad 1\leqq i\leqq r,\ 1\leqq j\leqq r$$

$$L(e_i,f_j)=0,\quad i\neq j$$

$$L(e_i,f_i)=1,\quad 1\leqq i\leqq r.$$

Set $e_i'=\overline{V}(e_i)$, $f_i'=\overline{V}(f_i)$, $1\leqq i\leqq r$. Then

$$\begin{aligned}\phi(e_i',e_j')&=\phi(\overline{V}e_i,\overline{V}e_j)=L((1-\rho)^{-1}\overline{V}e_i,\overline{V}e_j)\\&=L((\overline{V}V)^{-1}\overline{V}e_i,\overline{V}e_j)\\&=L(V^{-1}e_i,\overline{V}e_j)\\&=L(VV^{-1}e_i,e_j)\\&=0.\end{aligned}$$

Similarly, $\phi(f_i',f_j')=0$, and also if $i\neq j$ $\phi(e_i',f_j')=L(e_i,f_j)=0$. But, $\phi(e_i',f_i')=L(e_i,f_i)=1$, $1\leqq i\leqq r$.

Moreover, as H is square-root closed in G, H and hence also $\bar{H}$, contain all elements of order 2 in G. Thus, as

$$\mu(e_i')+(-1)^{k+1}\overline{\mu(e_i')}=\phi(e_i',e_i')=0$$

$\mu(e_i')$, and similarly $\mu(f_i')$ take values in $Z[\bar{H}]/\{u-(-1)^{k+1}\bar{u}|u\in Z[\bar{H}]\}$ (see [W2; Chapter V]). Expressing the Hermitian form (N,ϕ,μ) with respect to the basis given by $\{e_1', e_2', \ldots, e_r', f_1', \ldots, f_r'\}$ it is clearly obtained from a Hermitian form over $Z[\bar{H}]$ by just extending coefficients.

It remains only to check the claim that there is a map V with $(1-\rho)=\bar{V}V$. Corresponding to the decompositions (1) and (2) (resp; (3) and (4)) above, we may decompose the ZH linear maps ρ_1 and ρ_2,

$$\rho_1\colon P\to Q\otimes_{ZH}\widetilde{ZG}_1 = Q\otimes_{ZH}\mathsf{A}_1\oplus Q\otimes_{ZH}\mathsf{A}_2$$

$$\rho_2\colon Q\to P\otimes_{ZH}\widetilde{ZG}_2 = P\otimes_{ZH}\mathsf{B}_1\oplus P\otimes_{ZH}\mathsf{B}_2$$

(resp; in case B,

$$\rho_1\colon P\to Q\otimes_{ZH}\widetilde{ZJ}\oplus P\otimes_{ZH}Z[tJ] = Q\otimes_{ZH}\mathsf{A}_1\oplus_{ZH}\mathsf{A}_2\oplus P\otimes_{ZH}Z[tJ]$$

$$\begin{aligned}\rho_2\colon Q&\to P\otimes_{ZH}Z[\widetilde{tJt^{-1}}]\oplus Q\otimes_{ZH}Z[Jt^{-1}]\\ &= P\otimes_{ZH}\mathsf{B}_1\oplus P\otimes_{ZH}\mathsf{B}_2\oplus Q\otimes_{ZH}Z[Jt^{-1}])\end{aligned}$$

and write $\rho_1=\alpha_1\oplus\alpha_2$, $\rho_2=\beta_1\oplus\beta_2$ (resp; $\rho_1=\alpha_1\oplus\alpha_2\oplus\gamma$, $\rho_2=\beta_1\oplus\beta_2\oplus\delta$),

Now as

$$\begin{aligned}L((1-\rho)^{-1}x,y)&=\phi(x,y)=(-1)^{k+1}\,\overline{\phi(y,x)}\\ &=(-1)^{k+1}\,\overline{L((1-\rho^{-1}y,x)}\\ &=(-1)^{k+1}(-1)^{k+1}\,L(x,(1-\rho)^{-1}y)\\ &=L(x,(1-\rho)^{-1}y)\end{aligned}$$

we get $\overline{(1-\rho)}=1-\rho$ and in particular,

$$\bar{\rho}=\rho.$$

Denote the extensions of $\alpha_1\,\alpha_2$, β_1, β_2 (resp; and γ and δ) in the usual way (see the procedure in I.4 for extending ρ_1 and ρ_2) to ZG linear maps of N to itself by the same symbols. Then[14], $\rho=\rho_1'+\rho_2'=\alpha_1+\alpha_2+\beta_1+\beta_2$ (resp; $\rho=\rho_1'+\rho_2'=\alpha_1+\alpha_2+\gamma+\beta_1+\delta$). This decomposition expresses the components of the restriction of the map ρ to

$$\rho\colon P\oplus Q\to P\otimes_{ZH}\mathsf{B}_1\oplus P\otimes_{ZH}\mathsf{B}_2\oplus Q\otimes_{ZH}\mathsf{A}_1\oplus Q\otimes_{ZH}\mathsf{A}_2$$

$$\begin{aligned}(\text{resp; }\rho\colon P\oplus Q&\to P\otimes_{ZH}\mathsf{B}_1\oplus P\otimes_{ZH}\mathsf{B}_2\oplus Q\otimes_{ZH}Z[Jt^{-1}]\\ &\oplus Q\otimes_{ZH}\mathsf{A}_1\oplus Q\otimes_{ZH}\mathsf{A}_2\oplus P\otimes_{ZH}Z[tJ]).\end{aligned}$$

As $\rho=\bar{\rho}$, $\bar{\alpha}_1+\bar{\alpha}_2+\bar{\beta}_1+\bar{\beta}_2=\alpha_1+\alpha_2+\beta_1+\beta_2$ (resp; $\bar{\alpha}_1+\bar{\alpha}_2+\bar{\beta}_1+\bar{\beta}_2+\bar{\gamma}+\bar{\delta}=\alpha_1+\alpha_2+\beta_1+\beta_2+\gamma+\delta$).

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[14] Recall the notation of I.4. The maps ρ_1' and ρ_2' denote the extensions of ρ_1 and ρ_2 to ZG linear maps

Comparing domains and ranges, we get

$$\bar{\alpha}_1=\alpha_2, \quad \bar{\alpha}_2=\alpha_1$$
$$\bar{\beta}_2=\beta_1, \quad \bar{\beta}_1=\beta_2$$

(resp; and also $\bar{\gamma}=\delta$, $\bar{\delta}=\gamma$).

For example, $L(x,\bar{\alpha}_1 y)=L(\alpha_1 x, y)$. Hence, as $\alpha_1 x\in Q\otimes_{ZH}\mathrm{A}_1$, $L(x,\bar{\alpha}_1 y)=0$ for $y\in Q$. Moreover, for $y\in P$, $L(x,\bar{\alpha}_1 y)=L(\alpha_1 x, y)\in \mathrm{A}_1$ for $x\in P$ and thus

$$\bar{\alpha}_1 y\in Q\otimes_{ZH}\bar{\mathrm{A}}_1=Q\otimes_{ZH}\mathrm{A}_2.$$

Now to complete the proof of Lemma II.10 we wish to apply Lemma I.7 with

$$t=1-\rho=1-(\alpha_1+\alpha_2+\beta_1+\beta_2)$$
$$(\text{resp; } t=1-\rho=1-(\alpha_1+\alpha_2+\beta_1+\beta_2+\gamma+\delta)).$$

Note first that if w is a word in $\alpha_1,\alpha_2,\beta_1,\beta_2$ (resp; and γ and δ) which is not admissible, then by comparing domains and ranges we see that the composite map of N to itself represented by w is trivial. For example, $\alpha_1\alpha_2=0$ because

$$\begin{aligned}\alpha_2(N)=\alpha_2(P\oplus Q\otimes_{ZH}ZG)&\subset(\alpha_2(P)+\alpha_2(Q))ZG\\ &\subset\alpha_2(P)ZG\subset(Q\otimes_{ZH}\mathrm{A}_2)ZG\subset Q\otimes_{ZH}ZG\end{aligned}$$

and $\alpha_1(Q)=0$.

Thus, from Lemma II.9, there exist maps $1+V_i$ with $(1-\rho)-(1+\bar{V}_i)(i+V_i)$ a map represented by a sum of words of length greater than i in $\alpha_1,\alpha_2,\beta_1,\beta_2$ (resp; and γ and δ). The proof will therefore be completed by showing that the nilpotency condition on ρ implies that the composite functions obtained by sufficiently long compositions, in any order, of the maps $\alpha_1,\alpha_2,\beta_1,\beta_2$ (resp; and γ and δ) are zero.

As $\rho=\alpha_1+\alpha_2+\beta_1+\beta_2$ (resp; $\rho=\alpha_1+\alpha_2+\beta_1+\beta_2+\gamma+\delta$) $\rho^i=(\alpha_1+\alpha_2+\beta_1+\beta_2)^i$, $i\geqq 1$ (resp; $\rho^i=(\alpha_1+\alpha_2+\beta_1+\beta_2+\gamma+\delta)^i$) and thus $\rho^i=\sum_{w\in F_i} w$. But, as observed above, if w is not an admissible word in $\alpha_1,\alpha_2,\beta_1,\beta_2$ (resp; and γ and δ), the map represented by w is 0.

Hence, (6) $\rho^i=\sum_{w\in F_i^0} w$, $i\geqq 1$. But also, by (24) and (25) of I.4,

$$\rho^i(P\oplus Q)\subset(P\oplus Q)\otimes_{ZH}(A_i\oplus B_i\oplus C_i\oplus D_i)$$

and by Lemma II.8 we get from this

$$(7)\quad \rho^i(P\oplus Q)\subset(P\oplus Q)\otimes_{ZH}\sum_{w\in F_i^0} Z[w].$$

Moreover, beginning with the fact that this is obviously true if w has length 1 the same kind of induction as that used in Section I.5 above shows that

$$(8)\quad w(P\oplus Q)\subset(P\oplus Q)\otimes_{ZH}Z[w],\qquad w\in F_i^0.$$

Hence, from (6), (7), and (8), on $P \oplus Q$,

$$\rho^i = \bigoplus_{w \in F_i^0} w, \qquad i \geqq 1$$

and hence as, by Lemma I.11, $\rho^i = 0$ for i sufficiently large, we get that for i sufficiently large and $w \in F_i^0$, the map

$$w\colon P \oplus Q \to (P \oplus Q) \otimes_{ZH} ZG \quad \text{is zero.}$$

Completion of the Proofs of Theorems 1 and 2 for $n=2k$; Part (i). We first show, *for all n*, the necessity of the condition $\bar{\phi}(\tau(f))=0$. The proof of Lemma II.4 shows that $\phi(\tau(f))=(-1)^{n+1}\,\phi(\tau(f))^*$ and thus $\phi(\tau(f)$ determines an element

$$\bar{\phi}(\tau(f) \in H^{n+1}(Z_2; \operatorname{Ker}(\tilde{K}_0(H) \to \tilde{K}_0(G_1) \oplus \tilde{K}_0(G_2)))$$

$$(\text{resp; } H^{n+1}(Z_2; \operatorname{Ker}(\tilde{K}_0(H) \xrightarrow{\xi_{1*} - \xi_{2*}} \tilde{K}_0(J)))).$$

If there is an h-cobordism with torsion v $(V; W, \mathring{W})$ with the induced homotopy equivalence $\mathring{f}\colon \mathring{W} \to Y$ split along X, then [M 1]

(4) $\tau(f) - \tau(\mathring{f}) = v + (-1)^{n+1}\, v^*.$

Moreover, the split map $\mathring{f}$ induces a Mayer-Vietoris decomposition of the chain complex of the acyclic pair $(Y, \mathring{W})$, from which $\tau(\mathring{f})$ is computed,

Case A.

$$0 \to C_*(X, \mathring{M}) \otimes_{ZH} ZG \to C_*(Y_1, \mathring{W}_1) \otimes_{ZG_1} ZG \otimes C_*(Y_2\, \mathring{W}_2) \otimes_{ZG_2} ZG \\ \to C_*(Y, \mathring{W}) \to 0.$$

Case B.

$$0 \to C_*(X, \mathring{M}) \otimes_{ZH} ZG \xrightarrow{\xi_{1*} - \xi_{2*}} C_*(Y', \mathring{W}') \otimes_{ZJ} ZG \to C_*(Y, \mathring{W}) \to 0$$

and, all the chain complexes in the above sequences being acyclic, and comparing their torsions we get [M 1],

$$\tau(\mathring{f}) \in \operatorname{Image}(\operatorname{Wh}(G_1) \oplus \operatorname{Wh}(G_2) \to \operatorname{Wh}(G))$$

$$(\text{resp; } \tau(\mathring{f}) \in \operatorname{Image}(\operatorname{Wh}(J) \to \operatorname{Wh}(G)).$$

But, then by [W 1],

(5) $\phi(\tau(\mathring{f}))=0.$

Hence from (4) and (5),

$$\tau(f) = v + (-1)^{n+1}\, v^*$$

and hence by Lemma II.3,

$$\phi(\tau(f)) = \phi(v) + (-1)^n\, \phi(v)^*$$

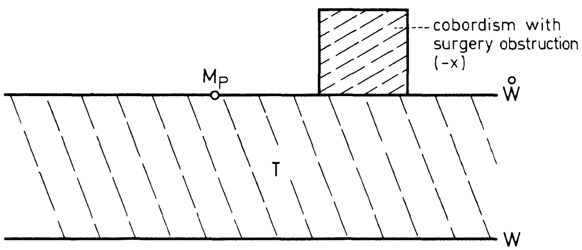
which asserts that in

$$H^{n+1}(Z_2; \operatorname{Ker}(\tilde{K}_0(H) \to \tilde{K}_0(G_1) \oplus \tilde{K}_0(G_2)))$$

$$(\text{resp}; H^{n+1}(Z_2; \operatorname{Ker}(\tilde{K}_0(H) \xrightarrow{\xi_{1*}-\xi_{2*}} \tilde{K}_0(J)))) \qquad \bar{\phi}(\tau(f))=0.$$

To prove sufficiency, assume $\bar{\phi}(\tau(f))=0$. Using Lemma II.4, it suffices to prove the result if $\phi(\tau(f))=0$. In that case, the nilpotent normal cobordism construction of II §1 produced a normal cobordism T of W to a split homotopy equivalence $\mathring{f}\colon \mathring{W} \to Y$. Moreover, the surgery obstruction $x \in L^h_{2k+2}(G, w)$ of this normal cobordism, by Lemma II.6, satisfies the hypothesis of Lemma II.10 for $\bar{H}=H$. Hence, $x \in \operatorname{Image}(L^h_{2k+2}(H, w) \to L^h_{2k+2}(G, w))$.

Furthermore the inclusion $H \to G$ factors through $G_1 = \pi_1(W_1)$, (resp; $J = \pi_1(Y')$). Hence a normal cobordism with surgery obstruction in $L^h_{2k+2}(G_1, w)$ (resp; $L^h_{2k+2}(J, w)$) going to $(-x)$ under the map $L^h_{2k+2}(G_1, w) \to L^h_{2k+2}(G, w)$ (resp; $L^h_{2k+2}(J, w) \to L^h_{2k+2}(G, w)$) can be constructed [W 2] on the homotopy equivalence $\mathring{W}_1 \to Y_1$ (resp; $\mathring{W}' \to Y'$), fixed on $\partial \mathring{W}_1$ (resp; $\partial \mathring{W}'$). Attaching this to T along $\mathring{W}_1$ (resp; $\mathring{W}'$) we get a normal cobordism with 0 surgery obstruction of W to a split homotopy equivalent manifold.



Then, surgery can be performed on this normal cobordism to obtain the required h-cobordism.

Proof of Theorems 1 and 2 for $n=2k$; Part (ii).

The proof of the necessity of

$$\tau(f) \in \operatorname{Im}(\operatorname{Wh}(G_1) \oplus \operatorname{Wh}(G_2) \to \operatorname{Wh}(G))$$

$$(\text{resp}; \tau(f) \in \operatorname{Im}(\operatorname{Wh}(J) \to \operatorname{Wh}(G))$$

is the same as the argument in the first part of the above completion of the proof for part (i) showing $\tau(\mathring{f})$ is in this Image.

Part (i) constructed for $n=2k$ an h-cobordism of W to a split manifold. To complete the proof of part (ii), we show that beginning with the sharper control on torsion given in the hypothesis of part (ii), the geometric constructions employed in the proof of part (i) can be used to give an s-cobordism. Then the s-cobordism theorem [K 3] [M 1] will complete the argument.

To prove this, it will be convenient to use, generalizing the surgery groups $L^s_n(G, w)$ and $L^h_n(G, w)$, surgery groups $L^B_n(G, w)$ where B is a subgroup of $\operatorname{Wh}(G)$

satisfying $B=B^*$ for $B^*=\{x\in \mathrm{Wh}(G)|x^*\in B\}$. The group $L_n^B(G,w)$ is defined precisely as $L_n^s(G,w)$ is defined, but with torsions evaluated in $\mathrm{Wh}(G)/B$. For example, a simple isomorphism modulo B of based free $Z[G]$ modules means an isomorphism with torsion in B. In particular, $L_n^{\{0\}}(G,w)=L_n^s(G,w)$ and $L_n^{\mathrm{Wh}(G)}(G,w)=L_n^h(G,w)$. These generalized Wall groups were introduced in [C1]; see [C3] and also [CS1] for examples of applications and see [R1] for further algebraic generalizations.

Assume now that

$$\tau(f)\in \mathrm{Image}\,(\mathrm{Wh}(G_1)\oplus \mathrm{Wh}(G_2)\to \mathrm{Wh}(G))$$

$$(\text{resp; } \tau(f)\in \mathrm{Image}\,(\mathrm{Wh}(J)\to \mathrm{Wh}(G))$$

so that f is a modulo B simple homotopy equivalence for

$$B=\mathrm{Image}\,(\mathrm{Wh}(G_1)\oplus \mathrm{Wh}(G_2)\to \mathrm{Wh}(G))$$

$$(\text{resp; } B=\mathrm{Image}\,(\mathrm{Wh}(J)\to \mathrm{Wh}(G)).$$

The nilpotent normal cobordism construction of II.1 gave a normal cobordism T of $f\colon W\to Y$ to $\mathring{f}\colon \mathring{W}\to Y$ with $\mathring{f}$ split. Hence, by an argument employed above in the proof of necessity in part (i), $\tau(\mathring{f})\in \mathrm{Im}\,(\mathrm{Wh}(G_1)\oplus \mathrm{Wh}(G_2)\to \mathrm{Wh}(G))$ (resp; $\tau(\mathring{f})\in \mathrm{Im}\,(\mathrm{Wh}(J)\to \mathrm{Wh}(G))$. Thus, $\mathring{f}$ is a modulo B simple homotopy-equivalence and the normal cobordism T has a surgery obstruction x in $L_{2k+2}^B(G,w)$ represented by a modulo B based Hermitian form $(K_{k+1}(T),\phi,\mu)$. Moreover, the basis $\varepsilon_1,\ldots,\varepsilon_r,\phi_1,\ldots,\phi_r$ which corresponds geometrically to the spheres producted from the handle cores $E_1,\ldots,E_r,F_1,\ldots,F_r$ represents the modulo B equivalence class of bases for $K_{k+1}(T)$ [W2].

As in part (i), we complete the argument by showing that

$$x\in \mathrm{Image}\,(L_{2k+2}^h(H,w)\to L_{2k+2}^B(G,w)).$$

For, in that case, by the same argument as that employed in part (i), there is a "modulo B s-cobordism", i.e., an h-cobordism with torsion in

$$B=\mathrm{Image}\,(\mathrm{Wh}(G_1)\oplus \mathrm{Wh}(G_2)\to \mathrm{Wh}(G))$$

$$(\text{resp; } B=\mathrm{Image}\,(\mathrm{Wh}(J)\to \mathrm{Wh}(G)),$$

of W to a split manifold $\check{W}$. Attaching to this h-cobordism an h-cobordism of $\check{W}_1$ and $\check{W}_2$ (resp; of $\check{W}'$) we can obtain an s-cobordism of W to a split homotopy equivalent manifold [M1].

To show that $x\in \mathrm{Image}\,(L_{2k+2}^h(H)\to L_{2k+2}^B(G,w))$, we use the same argument as that employed in the proof of Lemma II.10, but we must now also check that the automorphism induced on $(P\oplus Q)\otimes_{ZH} ZG$ by $(1+V_i)$, for i sufficiently large, is simple modulo B.

To see this, recall from [W1] the construction of Grothendieck groups of *reduced* nilpotent objects (P,Q,ρ_1,ρ_2) with P,Q free ZH modules. From the homotopy equivalence $f\colon W\to Y$, we constructed in I.4 an element of this group with $P\oplus Q=K_k(M)$. Waldhausen [W1] shows, in the general setting of a CW complex splitting problem that if $\tau(f)\in \mathrm{Image}\,(\mathrm{Wh}(G_1\oplus \mathrm{Wh}(G_2)\to \mathrm{Wh}(G))$

(resp; $\tau(f) \in \mathrm{Image}(\mathrm{Wh}(J) \to \mathrm{Wh}(G))$) then this (P, Q, ρ_1, ρ_2) represents the 0-object in the group of nilpotent objects. Moreover, the map $1+V_i$ was constructed formally, when H is square-root closed in G and using Lemma II.9, as a non-commutative polynomial in the components $\alpha_1, \alpha_2, \beta_1, \beta_2$ (resp; and γ and δ) of ρ_1 and ρ_2. As this definition can clearly be extended, using the same formal polynomial V_i, to any object (P, Q, ρ_1, ρ_2), it defines a map $(1+V_i)$ (P, Q, ρ_1, ρ_2) of $(P \oplus Q) \otimes_{ZH} ZG \to (P \oplus Q) \otimes_{ZH} ZG$. Moreover as V_i is a sum of monomials each having 1 or more factors of $\alpha_1, \alpha_2, \beta_1, \beta_2$ (resp; and γ and δ), see Lemma II.9, V_i^r for r large is a sum for maps represented by words of length greater than or equal to r in $\alpha_1, \alpha_2, \beta_1, \beta_2$ (resp; and γ and δ). The argument used in the proof of Lemma II.10 shows that for r sufficiently large, $V_i^r = 0$ and hence $(1+V_i)(P, Q, \rho_1, \rho_2)$ is invertible and thus determines an element of $\mathrm{Wh}(G)$. Hence, inducing addition in $\mathrm{Wh}(G)$ by the direct sum construction, the formal polynomial $1+V_i$ determines a homomorphism

$$1+V_i\colon \widetilde{\mathrm{Nil}}(H; G_1, G_2) \to \mathrm{Wh}(G)$$

$$(\text{resp; } 1+V_i\colon \widetilde{\mathrm{Nil}}(H; J, \xi_1, \xi_2) \to \mathrm{Wh}(G)).$$

This follows immediately from the definition of $\widetilde{\mathrm{Nil}}$ as a Grothendieck group. In particular, for (P, Q, ρ_1, ρ_2), constructed as in our present situation from a homotopy equivalence with

$$\tau(f) \in \mathrm{Im}\,(\mathrm{Wh}(G_1) \oplus \mathrm{Wh}(G_2) \to \mathrm{Wh}(G))$$

$$(\text{resp; } \tau(f) \in \mathrm{Im}\,(\mathrm{Wh}(J) \to \mathrm{Wh}(G))$$

and hence representing the 0-element of $\widetilde{\mathrm{Nil}}(H; G_1, G_2)$ (resp; $\widetilde{\mathrm{Nil}}(H; J, \xi_1, \xi_2)$, $(1+V_i)(P, Q, \rho_1, \rho_2)$ is a simple isomorphism.

Remark. This argument involves choosing i large enough so that for the given P, Q, ρ_1, ρ_2 produced from $f\colon W \to Y$, the map induced from $1+V_i$ satisfies $(1-\rho) = (1+\bar{V}_i)(1+V_i)$ (see the proof of Lemma II.10). In fact, the argument could be made uniform by observing that in the construction of V_{i+1} in the proof of Lemma II.9 $V_{i+1} - V_i$ is a sum of words of length i. Hence for any (P, Q, ρ_1, ρ_2), for i sufficiently large, $(1+V_i)(P, Q, \rho_1, \rho_2) = (1+V_{i+1})(P, Q, \rho_1, \rho_2)$ and we could define $(1+V)(P, Q, \rho_1, \rho_2)$ this may obtained is the "limit" as i gets large.

Remarks. 1) If f is a simple homotopy equivalence, the above argument shows that the nilpotent normal cobordism construction is a normal cobordism of f to a split simple homotopy equivalence. Moreover, for $\bar{H}$ a subgroup of H containing all elements of order 2 in H and H square-root closed in G the above argument shows that for $x \in L^s_{2k+2}(G, \omega)$ denoting the surgery obstruction of the nilpotent normal cobordism, $x \in \mathrm{Image}\,(L^s_{2k+2}(\tilde{H}, \omega) \to L^s_{2k+2}(G, \omega))$.

2) Even if H is not square-root closed in G the surgery obstruction of the nilpotent normal cobordism construction is easily seen to go to zero in the Wall group of the ring $R[G]$, $Z[\frac{1}{2}] \subset R \subset G$. Over the ring $R[G]$ in place of Lemmas II.7, II.8 and II.9 which led to the construction of $I+V_i$ by using square-root closedness to decompose ρ_1, ρ_2 in terms of $\alpha_1, \alpha_2, \beta_1, \beta_2$ (resp; and γ and δ in case B) just use the decomposition $\rho_1 = 1/2\,\rho_1 + 1/2\,\rho_1$, $\rho_2 = 1/2\,\rho_2 + 1/2\,\rho_2$. From this, it follows

using homology surgery theorem [CS 2] that even when $H \subset G$ is not a square-root closed subgroup, if $\Phi(\tau(f))=0$, $n=2k>4$, there is a "homology h-cobordism" of W to a split "homology-equivalent manifold", where homology is taken using local coefficients in $Z[\frac{1}{2}][\pi_1(Y)]$. This can be directly applied to compute the Wall groups of the ring $R[G_1 *_H G_2]$, $Z[\frac{1}{2}] \subset R \subset Q$, even when H is not square-root closed in G_1 and G_2.

3) If $n \geqq 6$ and $H=G_1$, even if H is not square-root closed in G, and in fact even if $H \to G_2$ is not injective, any homotopy equivalence $f\colon W \to Y$ is splittable. For f a simple homotopy equivalence this is Theorem 12.1 of [W 2]. Essentially the same proof, but using the surgery theory associated to the homotopy equivalence problem in place of that for simple homotopy equivalences, shows that W is h-cobordant to a split manifold. But if $H=G_1$, then $G=G_1 *_H G_2=G_2$ and in particular, $\mathrm{Wh}(G_2) \to \mathrm{Wh}(G)$ is surjective. From this, the h-cobordism to a split homotopy equivalence is easily replaced by an s-cobordism to a split homotopy equivalence.

Chapter III: The Even-Dimension Case

§ 1. Splitting in a Covering Space

The present chapter completes the proofs of Theorems 1 and 2 for $n=2k-1$, that is for dimension Y even. The results of Chapters I and II are heavily used below.

In many cases, for example when $\tilde{K}_0(H)$ is zero, the results of the present chapter could be derived quite easily from those of Chapter II. However, in the general case considered here considerable effort is needed, when $n=2k-1$, to establish the relationship between the projective modules arising in the proof and the initial data on the Whitehead torsion of f. This is accomplished in Lemma III.4.

We briefly outline the argument of the present chapter. In § 1, we consider the problem of splitting the map of covering spaces $\hat{f}\colon \hat{W} \to \hat{Y}$ along $X \subset \hat{Y}$. Lemma III.2 shows that the only obstruction is an element of $\tilde{K}_0(H)$. In § 2, this obstruction is shown to depend only on $\tau(f)$. Using the results of § 1, we carefully construct a submanifold $V \subset W \times S^1$, with V a transverse inverse image of $X \times S^1 \subset Y \times S^1$ and with $V \to X \times S^1$ a k connected map. This is the situation which was studied in Lemma II.1 and we perform the nilpotent normal cobordism construction of Chapter II. The main result of § 2, Lemma III.6 shows that when $\Phi(\tau(f))=0$ this produces a nilpotent normal cobordism of $W \times S^1$ to a manifold U, with $\mathring{W} \subset U$ and $\mathring{W} \to Y$ a split homotopy equivalence. In § 3 results of Chapter II are used to replace this normal cobordism by an s-cobordism of $W \times S^1$ to U', with $\mathring{W} \subset U'$. Hence $\mathring{W} \subset W \times S^1$ and we complete the proof of the h-splitting theorem by showing that this implies W and $\mathring{W}$ are h-cobordant. The s-splitting result for $n=2k-1$ is derived from the h-splitting result.

The assumption that H is square-root closed in G is not used in § 1 and § 2. It is used in § 3 in replacing a nilpotent normal cobordism by an s-cobordism. In particular, even if H is not spare-root closed in G the results of § 2 show that if

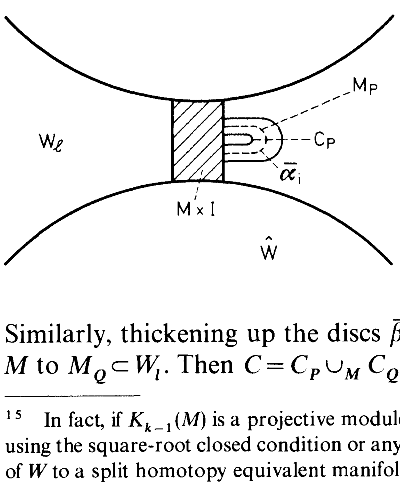
$\Phi(\tau(f))=0$, (W,f) is normally cobordant to a split homotopy equivalence, $n=2k-1$. For $n=2k$ this was proved in Chapter II.

To begin the argument, note that from Lemma I.1, when $n=2k-1$ we may assume that the homotopy equivalence $f\colon W\to Y$ is transverse to $X\subset Y$ with, setting as usual $M=f^{-1}(X)$, $f!\colon M\to X$ inducing isomorphisms of fundamental groups and $K_i(M)=0$, $i\neq k-1, k$. Unfortunatelly, unlike the situation studied in Chapter II, $K_{k-1}(M)$ and $K_k(M)$ will not, in general, be free or even projective modules[15] over $Z[H]$.

Recall the decomposition defined in I §4, $K_{k-1}(M)=P\oplus Q$ where $P\cong K_k(W_r, M)$ and $Q\cong K_{k-1}(W_l, M)$ are by Lemma I.6 finitely generated $Z[H]$ modules. Let $\alpha_1,\dots,\alpha_r$ be a $Z[H]$ module generating set for P and let $\beta_1,\dots,\beta_s$ be a $Z[H]$ module generating set for Q. These elements are represented by disjoint embedded $(k-1)$ dimensional spheres in M^{2k-1}. In the middle-dimension under consideration here, we cannot apply Lemma I.3 to perform handle exchange on $M\subset W$. However, as we shall see below handle-exchanges, on discs with boundaries the spheres $\alpha_1,\dots,\alpha_r$, $\beta_1,\dots,\beta_s$, can be performed on $M\subset\hat{W}$.

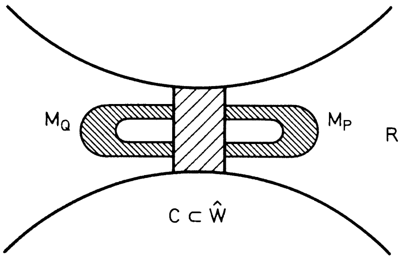
Let $M\times I$, $I=[-1,1]$ be a neighborhood of $M\times 0=M$ in W, lifting to a neighborhood $M\times I$ of M in $\hat{W}$ with $M\times 1\subset W_r$, $M\times -1\subset W_l$. Recall from Chapter I that $K_{i+1}(W_r, M)\oplus K_{i+1}(W_l, M)\cong K_i(M)$ and hence $K_i(W_r, M)=0$ for $i<k$. Hence as $\pi_1(M)=\pi_1(W_r)=\pi_1(W_l)$ by the relative Hurewicz theorem each class of $K_k(W_r, M)$ is represented by an immersed disc $(D^k, S^{k-1})\to(W_r, M)$. In particular, let $\bar{\alpha}_i\colon(D^k, S^{k-1})\to(W_r, M)$ be an immersed disc with $\partial\bar{\alpha}_i=\alpha_i$. Similarly there are immersions $\bar{\beta}_i\colon(D^k, S^{k-1})\to(W_l, M)$ with $\partial\bar{\beta}_i=\beta_i$. As $\pi_1(M)=\pi_1(W_r)$ a standard piping argument (see [Z, Lemma 48], [M2, p. 71], [W2, p. 39]) shows that the immersions $\{\bar{\alpha}_i\}$ $1\leqq i\leqq r$ are regularly homotopic to disjoint embeddings in (W_r, M). Similarly, the immersions $\{\bar{\beta}_i\}$ $1\leqq i\leqq s$ are regularly homotopic to disjoint embeddings in (W_l, M).

Thus, we may assume that $\{\bar{\alpha}_i\}$ $1\leqq i\leqq r$, $\{\bar{\beta}_j\}$ $1\leqq j\leqq s$ are disjointly embedded discs in $(\hat{W}, M)$. Thickening M and the discs $\bar{\alpha}_i\subset W_r$, $1\leqq i\leqq r$, we obtain a cobordism C_P of M to a manifold $M_P\subset W_r$.



Similarly, thickening up the discs $\bar{\beta}_i$, $1\leqq i\leqq s$, produces a cobordism C_Q in W_l of M to $M_Q\subset W_l$. Then $C=C_P\cup_M C_Q\subset\hat{W}$ is a cobordism of M_P to M_Q.

[15] In fact, if $K_{k-1}(M)$ is a projective module and $\tilde{K}_0(H)=0$ it is not too difficult to construct, without using the square-root closed condition or any condition on elements of order 2 in $\pi_1(Y)$, an h-cobordism of W to a split homotopy equivalent manifold. See in this connection the discussion on Farrell-Hsiang splitting in Chapter VI



The map $\hat{f}$: $\hat{W} \to \hat{Y}$ restricts, after being varied by a homotopy, to $\hat{f}$!: $C \to X \times I$, $X \times I$ a neighborhood of X in $\hat{Y}$. Let R = closure of $W_r - C_P$, L = closure of $W_l - C_Q$ in $\hat{W}$.

The following result is proved by a method similar to that used in proving Lemma 4.1 of [FH1].

Lemma III.1. *$K_i(M_P)=0$, $K_i(M_Q)=0$ for $i \neq k-1$, k. $K_{k-1}(M_P)$, $K_{k-1}(M_Q)$, $K_k(M_P)$ and $K_k(M_Q)$ are finitely generated projective $Z[H]$ modules. $K_i(C)=0$ for $i \neq k$ and $K_k(M_P) \oplus K_k(M_Q) \to K_k(C)$ is an isomorphism of stably free $Z[H]$ modules. $K_i(R)=0$, $K_i(L)=0$ for $i \neq k-1$ and $K_{k-1}(M_P) \to K_{k-1}(R)$, $K_{k-1}(M_Q) \to K_{k-1}(L)$ are isomorphisms of $Z[H]$ modules. Also $K_k(M_Q) \to K_k(W_r \cup_M C_Q)$ and $K_k(M_P) \to K_k(W_l \cup_M C_P)$ are isomorphisms, and $K_i(W_r \cup_M C_Q)=0$, $K_i(W_l \cup_M C_P)=0$ for $i \neq k$.*

Proof. We will need to employ in this proof the cohomology cokernel groups $K^i(C)$ and $K^i(C, \partial C)$ associated to the maps $C \to X \times I$. Such groups are discussed in a more general setting in [W2; Chapter II].

First note that up to homotopy, C was produced by attaching k-dimensional cells to a generating set for $P \oplus Q = K_{k-1}(M)$. Hence, $K_j(C)=0$ for $j \neq k$. Moreover, we claim that $K^{k+1}(C; B)=0$, B any $Z[H]$ module, and hence by a standard argument [W2; p. 26] $K_k(C)$ is a stably free $Z[H]$ module. To show the vanishing of $K^{k+1}(C; B)$ recall that by Poincaré duality, $K^{k+1}(C; B) \cong K_{k-1}(C, \partial C; B)$ and we have the exact sequence

$$K_{k-1}(C; B) \to K_{k-1}(C, \partial C; B) \to K_{k-2}(\partial C; B).$$

Moreover, $\partial C = M_P \cup M_Q$ with M_P and M_Q obtained from M by surgeries on $(k-1)$-dimensional spheres and hence as $K_j(M; B)=0$ $j<k-1$, we get $K_j(\partial C; B) = K_j(M_P; B) \oplus K_j(M_Q; B)=0$ for $j<k-1$. Thus, $K_k(C)$ is a stably free finitely generated $Z[H]$ module.

We proceed to relate $K_k(C)$ to $K_k(M_P) \oplus K_k(M_Q) = K_k(\partial C)$. In the exact sequence

$$K_{k+1}(C, \partial C) \to K_k(\partial C) \to K_k(C)$$

by Poincaré duality $K_{k+1}(C, \partial C) \cong K^{k-1}(C)=0$. Hence

$$K_k(\partial C) = K_k(M_P) \oplus K_k(M_Q) \to K_k(C)$$

is injective. To see that this map is also surjective, use the Mayer-Vietories sequence of $\hat{W}=L\cup_{M_Q} C\cup_{M_P} R$; this reduces, as $\hat{W}\to\hat{Y}$ is a homotopy equivalence, to the isomorphisms

(1) $K_i(M_P)\oplus K_i(M_Q)\to K_i(L)\oplus K_i(C)\oplus K_i(R)$.

In particular, $K_k(M_P)\oplus K_k(M_Q)\to K_k(C)$ is also surjective and hence is an isomorphism of $Z[H]$ modules.

To compute $K_i(R)$ and $K_i(L)$, from (1) and $K_i(C)=0$, $i\neq k$, $K_k(C)=K_k(M_P)\oplus K_k(M_Q)$ conclude that $K_i(R)=0$, $K_i(L)=0$ $i\neq k-1$, and $K_{k-1}(M_P)\to K_{k-1}(R)$, $K_{k-1}(M_Q)\to K_{k-1}(L)$ are isomorphisms.

To check that $K_{k-1}(M_p)$ is projective, we show that $K_{k-1}(M_p)\oplus K_{k-1}(M_Q)=K_{k-1}(\partial C)$ is a stably free $Z[H]$ module. As $K_k(\partial C)\to K_k(C)$ was seen above to be an isomorphism and as $K_{k-1}(C)=0$, we have $K_k(C,\partial C)\cong K_{k-1}(\partial C)$ and $K_i(C,\partial C)=0$ for $i\neq k$. Hence by an argument used above, to show that $K_k(C,\partial C)$ is a stably free $Z[H]$ module, it suffices to check that $K^{k+1}(C,\partial C\colon B)=0$ for any $Z[H]$ module B. But, $K^{k+1}(C,\partial C\colon B)\cong K_{k-1}(C\colon B)=0$ by Poincaré duality [W2].

Lastly, these computations and the Mayer-Vietoris sequence of $\hat{W}=L\cup_{M_P}(C_P\cup_M W_l)$

$$\cdots\to K_i(M_P)\to K_i(L)\oplus K_i(C_P\cup_M W_l)\to K_i(\hat{W})\to\cdots$$

show immediately that $K_i(C_P\cup_M W_l)=0$ for $i\neq k$, $K_k(C_P\cup_M W_l)=K_k(M_P)$. Similar remarks apply to $C_Q\cup_M W_r$.

After variation by a homotopy, the map $\hat{f}\colon \hat{W}\to\hat{Y}$ restricts to $f\,!\colon C\to X\times I$, $I=[-1,1]$, $X\times I$ a neighborhood of $X\times 0=X\subset Y$, with $\hat{f}(M_P)=X\times 1$, $\hat{f}(M_Q)=X\times -1$.

Lemma III.2. *If* $[K_k(M_P)]\in\tilde{K}_0(H)$ *is zero, then, after performing further trivial ambient surgeries on* $(k-1)$ *spheres of* M_Q *and still calling the resulting manifold* M_P, $K_{k-1}(M_P)$ *and* $K_k(M_P)$ *will be free* $Z[H]$ *modules. If* $K_{k-1}(M_P)$ *is a free* $Z[H]$ *module, then* $\hat{f}\,!\colon C\to X\times I$ *is homotopy by a homotopy fixed on* ∂C, *to a map, which we continue to denote* $\hat{f}$, *transverse to* X *with* $\hat{f}^{-1}(X)\to X$ *a homotopy equivalence.*

Proof. As $K_{k-1}(M_P)$ and $K_k(M_P)$ are projective and are the only nonzero homology groups of a free $Z[H]$ complex standard arguments show that $[K_{k-1}(M_P)]-[K_k(M_P)]=0$ in $\tilde{K}_0(H)$. As performing trivial surgeries stabilizes $K_{k-1}(M_P)$ and $K_k(M_P)$, the first statement of the lemma is obvious. Note that performing trivial surgeries corresponds to enlarging the generating set $(\alpha_1,\ldots,\alpha_r)$ of P with copies of the zero element.

To see the second statement of the lemma, let $e_1,\ldots,e_v$ be a basis for the free $Z[H]$ module $K_{k-1}(M_P)$. As $K_{k-1}(C)=0$, $K_k(C,M_P)\to K_{k-1}(M_P)$ is surjective. Moreover, as $K_i(C,M_P)=0$ for $i<k$, by the relative Hurewicz theorem there are immersed discs $d_i\colon(D^k,S^{k-1})\to(C,M_P)$ with ∂d_i representing e_i, $1\leqq i\leqq v$. As $\pi_1(C)=\pi_1(M_P)$ a standard piping argument, see [W2; p. 39], used above shows that after variation by regular homotopies the discs $\{d_i\}$ $1\leqq i\leqq v$ may be taken to be disjoint embeddings. Performing handle exchanges on M_P

in C using these embedded discs, we obtain a cobordism C'_P of M_P to M' with $C'_P \subset C$.

We claim that $M' \to X$ is a homotopy equivalence. Actually, it is a standard fact that surgery on a free $Z[H]$ module basis for $K_{k-1}(M_P^{2k-1})$ produces a homotopy equivalence, but for completeness we give in the present case a direct argument. Clearly $\pi_1 M' = \pi_1 X$. Moreover, $C'_P = \partial(R \cup_{M_P} C'_P)$ and as $K_i(R)=0$, $i \neq k-1$ and C'_P is up to homotopy, produced by attaching cells to a basis of $K_{k-1}(M_P)$, $K^j(C'_P) \to K^j(M_P)$ is an isomorphism for $j \neq k-1$. But this map factors through $K^j(C'_P) \to K^j(C'_P)$ which is onto for $j \neq k-1$ as

$$K^{j+1}(C'_P, \partial C'_P) \cong K_{2k-j-1}(C'_P) = 0 \qquad \text{for } j \neq k-1.$$

Hence as $\partial C = M_P \cup M'$, $K^j(M')=0$ for $j \neq k-1$ and using Poincaré Duality, we conclude that $K_i(M')=0$ for all i.

We will show below in Lemma III.4 that $\Phi(\tau(f)) = [K_k(M_P)] \in \tilde{K}_0(H)$ and hence from Lemma III.2 if $\Phi(\tau(f))=0$, $\hat{f}$ is homotopic to a map split along $X \subset \hat{Y}$.

To proceed with the proof in § 2 we will need to be more careful in our choice of a basis for P and Q and correspondingly in the construction of C_P and C_Q. Using Lemma I.8, we may choose finite $Z[H]$ module generating sets $\alpha_1, \ldots, \alpha_u$ for P, $\beta_1, \ldots, \beta_u$ for Q with $\rho_1(\alpha_1)=0$, $\rho_2(\beta_1)=0$, $\rho(\alpha_i)$ and $\rho(\beta_i)$ elements of the $Z[H]$ submodule of $(P \oplus Q) \otimes_{ZH} ZG$ generated by $\alpha_1, \ldots, \alpha_{i-1}$, $\beta_1, \ldots, \beta_{i-1}$, $1 < i \leqq u$. The corresponding construction of C_P and C_Q given above constructed embedded discs $\bar{\alpha}_i$, $\bar{\beta}_i$, $1 \leqq i \leqq u$ with $\partial \bar{\alpha}_i$, $\partial \bar{\beta}_i$ being spheres in M representing α_i and β_i respectively. Note that these embedded discs $\{\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_i, \bar{\beta}_i, \ldots, \bar{\beta}_i\}$ can be chosen so that their images in W are in general position and thus, as dimension $W = 2k$, have only isolated points of self-intersection. (For example, given an embedding $\bar{\alpha}_i$: $(D^k, S^{k-1}) \to (W_r, M)$ by varing the image of $\bar{\alpha}$ in W by a small regular homotopy fixed on S^{k-1} this image may be put in general position. The lift of this homotopy is a regular homotopy of $\bar{\alpha}$ to a map, which if the regular homotopy was chosen sufficiently small, will still be an embedding.)

We may impose one further restraint on the choice of the discs $\bar{\alpha}_i$, $\bar{\beta}_j$. Filter the space C_P by setting

$$C_{P,i} = M \times [0,1] \cup \text{handles obtained by thickening } \bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\alpha}_i$$

so that $C_P = C_{P,u} \supset C_{P,u-1}, \ldots, C_{P,0} = M \times [0,1]$. Similarly, filter C_Q by setting

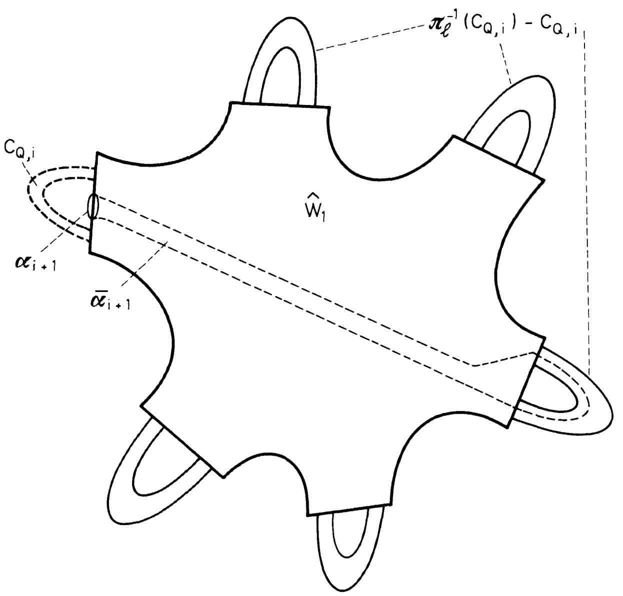
$$C_{Q,i} = M \times [0,-1] \cup \text{handles obtained by thickening } \bar{\beta}_1, \ldots, \bar{\beta}_i.$$

We state the condition first in case A, that is $Y - X$ has 2 components. Note that $W_l \cup_M W_1$ is the covering space of W corresponding to the subgroup $\pi_1(W_1) \subset \pi_1(W)$ and similarly $W_2 \cup_M W_r$ is the covering space of W corresponding to $\pi_1(W_2) \subset \pi_1(W)$. Let π_l: $\hat{W} \to W_l \cup_M W_1$, π_r: $\hat{W} \to W_2 \cup_M W_r$ denote the obvious covering space maps corresponding to $H \subset G_1$ and to $H \subset G_2$. Now we claim that the discs $\bar{\alpha}_i$, $\bar{\beta}_i$ can be chosen inductively so that

(2) $\bar{\alpha}_{i+1} \subset \text{closure}\,(\pi_l^{-1}(W_1 \cup_M C_{Q,i}) - C_{Q,i})$ in $\hat{W}$,

(2) $\bar{\beta}_{i+1} \subset \text{closure}\,(\pi_r^{-1}(W_2 \cup_M C_{P,i}) - C_{P,i})$ in $\hat{W}$.

Also, the disc $\bar{\alpha}_{i+1}$ (resp; $\bar{\beta}_{i+1}$) is to be chosen so that its image represents the zero element of $\pi_k(\hat{Y}_1, X)$ (resp; $\pi_k(\hat{Y}_2, X)$).



Essentially the same arguments as those employed in I § 5 show that the discs $\bar{\alpha}_i$, $\bar{\beta}_i$ may be inductively chosen to satisfy this condition. The point is again, see Lemma I.7, α_i and $\rho(\alpha_i) \in Q \otimes_{ZH} ZH_1$ represent essentially from the definition of ρ the same element of $K_{k-1}(W_1)$. But from our choice of generating sets, $\rho(\alpha_i) \in \{\beta_1, \ldots, \beta_{i-1}\} \otimes_{ZH} \widetilde{ZG}_1$ where $\{\beta_1, \ldots, \beta_{i-1}\}$ is the ZH submodule of Q generated by $\beta_1, \ldots, \beta_{i-1}$ and

$$K_{k-1}(\pi_r^{-1}(W_1 \cup_M C_{Q,i}) - C_{Q,i}) = K_{k-1}(W_1)/\{\beta_1, \ldots, \beta_{i-1}\} \otimes_{ZH} \widetilde{ZG}_1.$$

Hence as in the arguments of Lemma II.6 α_i bounds a disc in $(\pi_r^{-1}(W_1 \cup_M C_{Q,i}) - C_{Q,i})$ and as observed above this disc $\bar{\alpha}_i$ may be taken to be embedded and with its image in W having only point self-intersections.

Note that as $C_{P,i+1}$ is obtained from $C_{P,i}$ by attaching a handle obtained by thickening $\bar{\alpha}_{i+1}$, we may inductively construct $C_{P,i+1}$ so that

$$C_{P,i+1} \subset \text{closure}\,(\pi_l^{-1}(W_1 \cup_M C_{Q,i}) - C_{Q,i}) \quad \text{in } \hat{W}.$$

Similarly, we may assume that

$$C_{Q,i+1} \subset \text{closure}\,(\pi_r^{-1}(W_2 \cup_M C_{P,i}) - C_{P,i}) \quad \text{in } \hat{W}.$$

We now briefly state the corresponding condition which we may impose in case B on the choice of the discs $\bar{\alpha}_i$ and $\bar{\beta}_i$. Note that $W_l \cup_{M_1} W' \cup_{M_2} W_r$, with basepoint the basepoint of M_1, is the covering space of W corresponding to $J \subset J *_H \{t\}$. If we pick as the basepoint of $W_l \cup_{M_1} W' \cup_{M_2} W_r$ the basepoint of M_2, this is the covering space of W corresponding to $tJt^{-1} \subset J *_H \{t\}$. Let

$$\pi_l\colon \hat{W} \to W_l \cup_{M_1} W' \cup_{M_2} W_r, \qquad \pi_r\colon \hat{W} \to W_l \cup_{M_1} W' \cup_{M_2} W_r$$

be the covering projections corresponding to $H \subset J$ and $H \subset tJt^{-1}$ with $\pi_r(M) = M_1$, $\pi_l(M) = M_2$. Then arguing exactly as in case A, we may inductively choose the discs $\bar{\alpha}_{i+1}$, $\bar{\beta}_{i+1}$ so that

(3) $\bar{\alpha}_{i+1} \subset \text{closure}\,(\pi_l^{-1}(C_{Q,i} \cup_{M_1} W' \cup_{M_2} C_{P,i}) - C_{Q,i})$ in $\hat{W}$,

(3) $\bar{\beta}_{i+1} \subset \text{closure}\,(\pi_r^{-1}(C_{Q,i} \cup_{M_1} W' \cup_{M_2} C_{P,i}) - C_{P,i})$ in $\hat{W}$

and correspondingly

$$C_{P,i+1} \subset \text{closure}\,(\pi_l^{-1}(C_{Q,i} \cup_{M_1} W' \cup_{M_2} C_{P,i}) - C_{Q,i}),$$

$$C_{Q,i+1} \subset \text{closure}\,(\pi_r^{-1}(C_{Q,i} \cup_{M_1} W' \cup_{M_2} C_{P,i}) - C_{P,i}) \quad \text{in } \hat{W}.$$

§ 2. A Nilpotent Normal Cobordism on $W \times S^1$

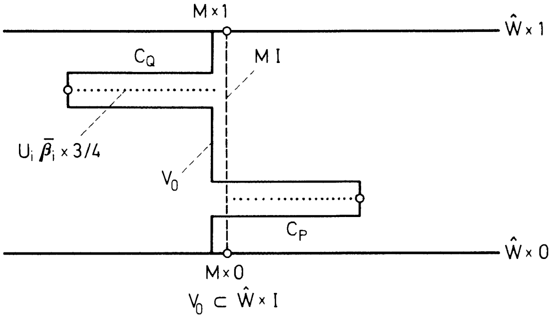
Lemma III.6, which is proved in this section using the results of III § 1, constructs a nilpotent normal cobordism of $W \times S^1$ to a manifold U, with $\mathring{W} \subset U$ and $\mathring{W} \to Y$ a split homotopy equivalence. This, together with the results of Chapter II, is used to complete the argument in § 3.

Recall that the nilpotent normal cobordism construction of II § 1 began with the construction, see Lemma II.1, of a codimension one transverse inverse image on which the homotopy equivalence restricts to a map connected below the middle dimension. Such a submanifold $V \subset W \times S^1$ is explicitly constructed in proving Lemma III.3. The submanifold V is constructed by performing handle-exchanges relative to the boundary on $M \times I \subset W \times I$, $I = [0, 1]$, to obtain $V_0 \subset W \times I$ with $\partial V_0 = M \times 0 \cup M \times 1$. We will then set $V = V_0$/identify $M \times 0$ with $M \times 1$, a codimension one submanifold of $W \times S^1 = W \times I$/identify $W \times 0$ with $W \times 1$. Of course, corresponding to the ambient surgeries on $M \times I \subset W \times I$ there are ambient surgeries on $M \times I \subset \hat{W} \times I$ and it will be convenient to begin with a discussion of surgery in $\hat{W} \times I$.

Clearly, ambient surgery can be performed on the spheres $\alpha_1 \times 1/4$, $\alpha_2 \times 1/4, \ldots, \alpha_u \times 1/4$, $\beta_1 \times 3/4$, $\beta_2 \times 3/4, \ldots, \beta_u \times 3/4$ in $M \times I$, $M \times I \subset \hat{W} \times I$ using handle-exchanges on the discs $\bar{\alpha}_i \times 1/4$, $\bar{\beta}_i \times 3/4$, $1 \leqq i \leqq u$, $I = [0, 1]$. Here $\bar{\alpha}_i$, $\bar{\beta}_i$, $1 \leqq i \leqq u$ are the discs constructed in III § 1. These handle-exchanges, see Lemma I.3, correspond to a homotopy, fixed on the boundary, of $\hat{f} \times 1_I\colon \hat{W} \times I \to \hat{Y} \times I$ to a map g', where

$$g'^{-1}(X \times I) \cong C_P \cup_{M_P} C_P \cup_M C_Q \cup_M C_Q$$
$$\cong C_P \cup_{M_P} C \cup_{M_Q} C_Q.$$

Let $V_0 = C_P \cup_{M_P} C \cup_{M_Q} C_Q$ and $s_0\colon V_0 \to \hat{W} \times I$ this inclusion so that $s_0(V_0) = g'^{-1}(X \times I)$.



We proceed to construct an embedding $V_0 \subset W \times I$. The proof of the following result uses the observation that it is easier to embedded a disc in $W \times I$ than in W.

Lemma III.3. *The map* $f \times 1_I\colon W \times I \to Y \times I, I=[0,1]$, *is homotopic, by a homotopy fixed on* $\partial(W \times I)$, *to a map* $g_0\colon W \times I \to Y \times I$ *transverse to* $X \times I \subset Y \times I$ *and with* $V_0 \cong (g_0)^{-1}(X \times I)$ *and, letting* $s_1\colon V_0 \to \hat{W} \times I$ *denote a lift of*

$$V_0 \cong (g_0)^{-1}(X \times I) \to W \times I,$$

with s_0 *isotopic to* s_1 *by an isotopy fixed on* ∂V_0.

Proof. We begin with some generalities about embedding discs in a "manifold $\times I$". Let L be a $2k$ dimensional manifold, $k>2$, and N a codimension 1 submanifold of L with trivial normal bundle in L. Let $\gamma\colon (D^k, S^{k-1}) \to (L, N)$ be an immersion with $\partial\gamma\colon S^{k-1} \to N$ an embedding, and with γ (interior $D) \cap N = \emptyset$ and γ in general position, that is with only point self-intersections, having inverse images $p_1, \ldots, p_v \in D^k$. Let $p\colon (L, N) \times I \to (L, N)$ denote the projection, $I=[0,1]$. Then the immersion $\gamma_0\colon (D^k, S^{k-1}) \to (L, N) \times I$, $\gamma_0(x) = (\gamma(x), 1/2)$ is easily seen to be regularly homotopy to an embedding γ_1 by a homotopy $\gamma_t\colon (D, S) \to (L, N) \times I$, $0 \leqq t \leqq 1$ fixed on S and with $p\gamma_t = \gamma$. In fact, if $g_1\colon D^k \to [1/2-\varepsilon, 1/2+\varepsilon]$, $0<\varepsilon<1/4$, is a smooth function with $g_1(p_i) \neq g_1(p_j)$ for $p_i \neq p_j$, $1 \leqq i \leqq v$, $1 \leqq j \leqq v$, and $g_t\colon D \to I$, $0 \leqq t \leqq 1$, is a smooth homotopy of g_1 to g_0, $g_0(x) = 1/2$ for $x \in D$, just set $\gamma_t(x) = (\gamma(x), g_t(x))$. We refer to this as a "handle-pushing" procedure.

We proceed to show that in a covering space in which γ lifts to an embedding, the lift of γ_1 is unique up to isotopy. Precisely, assume further that $\pi\colon L' \to L$ is a covering space and that N' is a component of $\pi^{-1}(N)$ and that γ lifts to an *embedding* $\gamma'\colon (D, S) \to (L', N')$, i.e. $\pi\gamma' = \gamma$. Write γ'_0 for the embedding $\gamma'_0\colon (D, S) \to (L', N') \times I$, $\gamma'_0(x) = (\gamma'(x), 1/2)$. Then if γ_1 as above is an embedding, $\gamma_1\colon (D, S) \to (L, N) \times I$ with $p\gamma_1 = \gamma$ and with $\gamma_1(x) = (\gamma(x), 1/2)$ for $x \in S$, then letting γ'_1 denote a lift of γ_1 to $\gamma'_1\colon (D, S) \to (L', N') \times I$, $(\pi \times 1_I)\gamma'_1 = \gamma_1$ and with $(\gamma'_1 \mid S) = (\gamma'_0 \mid S)$, γ'_1 is isotopic to γ'_0 by an isotopy $\gamma'_t\colon (D, S) \to (L', N') \times I$, $0 \leqq t \leqq 1$, with $(\gamma'_t \mid S) = (\gamma'_0 \mid S)$ and with $p'\gamma'_t = \gamma'$, $p'\colon L' \times I \to L'$ the projection. In fact, choosing a smooth homotopy $h_t\colon (D, S) \to I$, $0 \leqq t \leqq 1$, with h_i the composite, for $i=0$ or 1, $D \xrightarrow{\gamma'_i} L' \times I \to I$, set $\gamma'_t(x) = (\gamma'(x), h_t(x))$. Notice that here $\gamma'_1 = (\gamma', h_1)$ because they are both lifts of γ_1 and they coincide on S.

Using this handle-pushing procedure, we describe an inductive process for performing handle exchange on $M \times I \subset W \times I$ using discs which have boundary

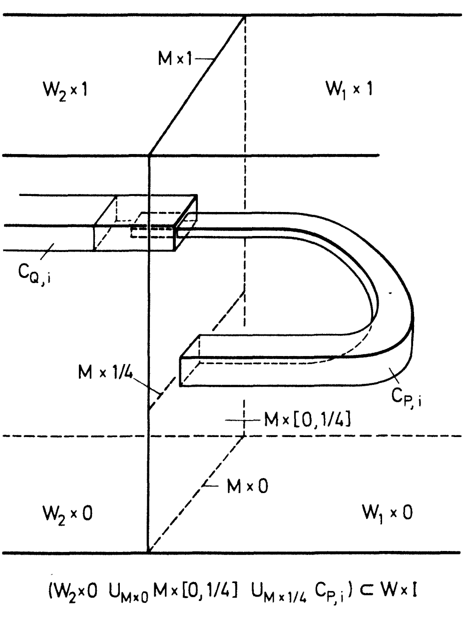
$\alpha_i\times 1/4\subset M\times 1/4$, $\beta_i\times 3/4\subset M\times 3/4$, $1\leqq i\leqq u$. The discs $\bar{\alpha}'_i$ and $\bar{\beta}'_i$ with $\partial\bar{\alpha}'_i=\alpha_i\times 1/4$, $\partial\bar{\beta}'_i=\beta_i\times 3/4$ will be constructed inductively to satisfy $p(\bar{\alpha}'_i)=\hat{\pi}_W(\bar{\alpha}_i)$, $p(\bar{\beta}'_i)=\hat{\pi}_W(\bar{\beta}_i)$, $p\colon W\times I\to W$ the projection and $\hat{\pi}_W\colon \hat{W}\to W$ the covering space map, $\bar{\alpha}_i$, $\bar{\beta}_i$ as in III §1. Recall from the discussion at the end of III §1, that $\hat{\pi}_W(\bar{\alpha}_i)$ and $\hat{\pi}_W(\bar{\beta}_i)$, while not necessarily embedded, will be in general position and have only point self-intersections.

We adapt our notation in the following inductive construction to case A; modifications of the notation for case B are briefly described afterwards. To start the argument, recall from (2) of III §1, $\hat{\pi}(\bar{\alpha}_1)\subset W_1$, $\hat{\pi}(\bar{\beta}_1)\subset W_2$ and $\bar{\alpha}_1$ and $\bar{\beta}_1$ were constructed so that $\hat{\pi}(\bar{\alpha}_1)$ and $\hat{\pi}(\bar{\beta}_1)$ would be immersions with isolated self-intersection points. Therefore, using the handle-pushing procedure described above, $\hat{\pi}(\bar{\alpha}_1)\times 1/4\subset W_1\times I$, $\hat{\pi}(\bar{\beta}_2)\times 3/4\subset W_2\times I$ may be perturbed slightly to get embedded discs $\bar{\alpha}'_i$ and $\bar{\beta}'_i$ with $\partial\bar{\alpha}'_1=\alpha_1\times 1/4$, $\partial\bar{\beta}'_1=\beta_1\times 1/4$, $p(\bar{\alpha}'_1)=\hat{\pi}_W(\bar{\alpha}_i)$, $p(\bar{\beta}'_1)=\hat{\pi}_W(\bar{\beta}_1)$. Now perform handle-exchanges on $M\times I\subset W\times I$ using thickenings of the discs $\bar{\alpha}'_1$, $\bar{\beta}'_1$.

We proceed to the similar general inductive step. Assume that handle-exchanges have been performed on $M\times I\subset W\times I$ using the discs $\bar{\alpha}'_1, \bar{\beta}'_1, \ldots, \bar{\alpha}'_i, \bar{\beta}'_i$, with boundaries $\alpha_1\times 1/4, \beta_1\times 3/4, \ldots, \alpha_i\times 1/4, \beta_i\times 3/4$, and with

(4) $\quad p\bar{\alpha}'_j=\hat{\pi}_W\,\bar{\alpha}_j, \qquad p\bar{\beta}'_j=\hat{\pi}_W\,\bar{\beta}_j, \qquad 1\leqq j\leqq i.$

To construct $\bar{\beta}'_{i+1}$, notice first that $W_2\times 0\cup_{M\times 0} M\times[0,1/4]\cup$(thickenings of the handles $\bar{\alpha}'_1, \bar{\alpha}'_2, \ldots, \bar{\alpha}'_i)\cong W_2\cup_M C_{P,i}$ is included in the component of $W\times I$ which is, after the performance of these first $2i$ handle-exchanges, the inverse image of $Y_2\times I\subset Y\times I$; in fact, a copy of $W_2\times 0\cup_M C_{P,i}$ is included in the boundary of this subspace of $W\times I$. Pushing slightly $M\times[0,1/4]$ into the interior of this inverse image of $Y_2\times I\subset Y\times I$, we can arrange from the inductive assumption



$(W_2\times 0\ \cup_{M\times 0} M\times[0,1/4]\ \cup_{M\times 1/4} C_{P,i})\subset W\times I$

of (4) that the composite map $W_2 \cup_M C_{P,i} \to W \times I \to W$ is the same as

$$W_2 \cup_M C_{P,i} \to \hat{W} \to W.$$

Here the inclusion $W_2 \cup_M C_{p,i}$ is that given in III § 1.

Now as β_{i+1} in M was constructed to be disjoint from $\alpha_1, \ldots, \alpha_i, \beta_1, \ldots, \beta_i$, $\beta_{i+1} \times 3/4$ is trivially seen to be isotopic to the sphere $\beta_{i+1} \times 1/4$ in the manifold obtained from $M \times I$ by performing surgery on

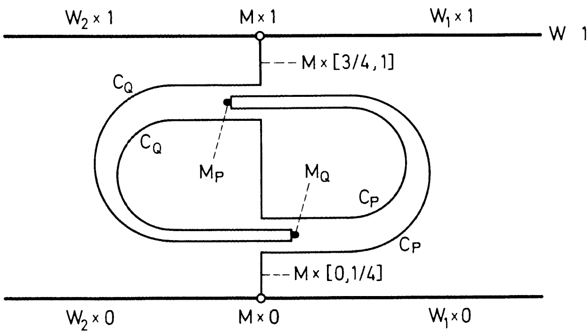
$$\alpha_1 \times 1/4, \ldots, \alpha_i \times 1/4, \qquad \beta_1 \times 3/4, \ldots, \beta_i \times 3/4.$$

In fact we will see easily below that it suffices to construct a disc $\bar{\beta}''_{i+1}$ with boundary $\beta_i \times 1/4$ and with $p\bar{\beta}''_{i+1} = \hat{\pi}_W \bar{\beta}_{i+1}$. But $\beta_{i+1} \times 1/4 \subset M \times 1/4$ is in the image of the embedding of $W_2 \cup_M C_{P,i} \subset W \times I$ constructed above. Moreover, from the (2) of III § 1, the composed map $D^k \xrightarrow{\bar{\beta}_{i+1}} \hat{W} \xrightarrow{\hat{\pi}_W} W$ was $D^k \xrightarrow{\beta} W_2 \cup_M C_{P,i} \to W$ where β is the composite $D^k \xrightarrow{\bar{\beta}_{i+1}} \hat{W} \to$ (covering space of W corresponding to $G_2 \subset G$). Hence, as β was constructed to be in general position with only point self-intersections, we may use the handle-pushing procedure described above to obtain an embedding $\bar{\beta}''_{i+1}$ in a neighborhood of $W_2 \cup_M C_{P,i}$ with $p\bar{\beta}''_{i+1} = \hat{\pi}_W(\bar{\beta}_{i+1})$.

Clearly the effect of these ambient surgeries on $\alpha_1 \times 1/4, \ldots, \alpha_u \times 1/4, \beta_1 \times 3/4, \ldots, \beta_u \times 3/4$ is to produce an embedding $V_0 \subset W$ and the homotopy equivalence $f \times 1_I \colon W \times I \to Y \times I$, from Lemma I.3 and the construction of $\bar{\alpha}_1, \bar{\alpha}_2, \ldots, \bar{\beta}_1, \ldots, \bar{\beta}_u$, can be varied by a homotopy to obtain g_0 homotopic to $f \times 1_{S^1}$, $g_0 \colon W \times I \to Y \times I$ with $g_0^{-1}(X \times I) \cong V_0$. Now recall that the selfintersections of $\bar{\beta}_{i+1}$, and its intersections with all the $\bar{\beta}_j$, are isolated points which may be taken to be outside of $\bar{\beta}_{i+1}(V)$, V a neighborhood of ∂D^{k+1} in D^{k+1}. Let $h \colon D^{k+1} \to \mathring{I} =$ interior I be a smooth map with $h(\partial D^{k+1}) = 3/4$ and $h|(D^{k+1} - V)$ the composite

$$D^{k+1} \xrightarrow{\bar{\beta}''_{i+1}} W \times I \to I.$$

Now set $\bar{\beta}'_{i+1} = (\hat{\pi}_W \bar{\beta}_{i+1}, h)$.



$V_0 \subset W \times I$

Moreover, as $\bar{\alpha}_1, \ldots, \bar{\alpha}_u, \bar{\beta}_1, \ldots, \bar{\beta}_u$ were constructed in III § 1 to be embedded in $\hat{W} \times I$, from the remarks at the beginning of this proof on the uniqueness up to isotopy of the handle-pushing procedure in covering spaces, the lifts of $\bar{\alpha}_i'$ and $\bar{\beta}_i'$ to $\hat{W}\times I$ are isotopic to $\bar{\alpha}_i\times 1/4$ and $\bar{\beta}\times 3/4$, $1\leq i\leq u$. Hence, the lift of $V_0\to W\times I$ to $s_1: V_0\to \hat{W}\times I$ is isotopic to $s_0: V_0\to W\times I$.

This completes the proof of the lemma in case A. The argument in case B is entirely similar and the modifications of the notation are left to the reader. Note that in case B, (3) of III §1 must be used in place of (2) of III §1 and the role of $W_2\times 0\cup_{M\times 0} M\times[0,1/4]$ and of $W_1\times 1\cup_{M\times 1} M\times[3/4,1]$ in the above argument is taken by a copy of W' in $W\times I$ with boundary $M\times 1/4\cup M\times 3/4$. The discs $\bar{\alpha}_1', \bar{\beta}_1', \bar{\alpha}_2', \bar{\beta}_2', \ldots, \bar{\beta}_u'$ in case B are constructed to be disjoint from this copy of W'.

Setting $W\times S^1 = W\times I$/identify $W\times 0$ with $W\times 1$, and $V=V_0$/identify $M\times 0$ with $M\times 1$, we have $V\subset W\times S^1$ and $f\times 1_{S^1}: W\times S^1\to Y\times S^1$ is homotopic to a map g transverse to $X\times S^1$ with $V=g^{-1}(X\times S^1)$. From this and Lemma III.3 we will also deduce the following:

Lemma III.4. $\Phi(\tau(f))=[K_k(M_P)]\in\tilde{K}_0(H)$.

Proof. We employ again, as in the proof of Lemma II.2(ii), the geometric interpretation of $\Phi(\tau(f))$ given in [W1; § 5]. Recall this involves construction of a CW complex splitting problem with torsion $\tau(f)$ and with homology kernel group non-zero in only one dimension. Clearly the homotopy equivalence $f\times 1_{D^2}: W\times D^2\to Y\times D^2$ has Whitehead torsion $\tau(f)$ and we will construct a homotopy of $f\times 1_{D^2}$ to a map e, extending the homotopy constructed above of $f\times 1_{S^1}$ to g, with $e^{-1}(X\times D^2)\cong C\times I$.

In the proof of Lemma III.3, we constructed inductively a series of handle exchanges to produce ambient surgeries on $M\times S^1\subset W\times S^1$. Corresponding to these handle exchanges, we may inductively perform a corresponding exchange in $W\times D^2$ using a neighborhood in $W\times D^2$ of each of these handles in $W\times S^1$. This produces a codimension one submanifold of $W\times D^2$ and from the last part of Lemma III.3 and its proof, the decomposition this induces of $\hat{W}\times D^2$ can be identified with

$$\begin{aligned}&\hat{W}\times D^2\\&=\{(W_l\times D^2\text{-neighborhood }(C_Q\times 3/4))\cup\text{closure of a neighborhood } C_P\times 1/4\}\\&\cup\{(W_r\times D^2\text{-neighborhood }(C_P\times 1/4))\cup\text{closure of a neighborhood } C_Q\times 3/4\}\end{aligned}$$

Here, as in the proof of Lemma III.3, $C_P\times 1/4\subset\hat{W}\times 1/4\subset\hat{W}\times S^1=\partial(\hat{W}\times D^2)$, $C_Q\times 3/4\subset\hat{W}\times 3/4\subset\hat{W}\times S^1=\partial(\hat{W}\times D^2)$. But the removal of a neighborhood of a subcomplex in the boundary of a manifold does not change the homotopy type and thus these two components of $\hat{W}\times D^2$ are homotopy equivalent to $W_l\cup_M C_P$ and $W_r\cup_M C_Q$ respectively. But by Lemma III.1, $K_i(W_l\cup_M C_P)=0$, $K_i(W_r\cup_M C_Q)=0$ for $i\neq k$ and $K_k(W_l\cup_M C_P)$ and $K_k(W_r\cup_M C_Q)$ are projective $Z[H]$ modules. But then by $[W1; § 5]$, $\Phi(\tau(f\times 1_{D^2}))=[K_k(W_l\cup_M C_P)]$. But $\Phi(\tau(f\times 1_{D^2}))=\Phi(\tau(f))$ and by Lemma III.1, $K_k(W_l\cup_M C_P)\cong K_k(M_P)$. Hence, $\Phi(\tau(f))=[K_k(M_P)]$.

Thus, from Lemma III.4 and Lemma III.2 when $\Phi(\tau(f))=0$, we may assume that $K_k(M_P)$ and $K_{k-1}(M_P)$ are free $Z[H]$ modules. As from Lemma III.1, $[K_k(M_P)]=-[K_k(M_Q)]$ when $\Phi(\tau(f))=0$ we may also assume that $K_k(M_Q)$ and $K_{k-1}(M_Q)$ are free $Z[H]$ modules.

The proof of the main result of this section, Lemma III.6 below, involves constructing explicitly the nilpotent normal cobordism of Chapter II on the homotopy equivalence $W \times S^1 \to Y \times S^1$ with $V \subset W \times S^1$, then inverse image of $X \times S^1 \subset Y \times S^1$, taking the role of the codimension one submanifold studied in Lemma II.1. This construction uses a description, provided in Lemma III.5, of the decomposition defined in I § 4 and used in Lemma II.1,

$$K_k(V) = P_V \oplus Q_V$$

where P_V and Q_V are $Z[\pi_1(X \times S^1)]$ modules, $\pi_1(X \times S^1) = H \times Z$. (Warning on notation: Below, we always use P_V and Q_V to denote the summands of $K_k(V)$, instead of the P, Q notation of Lemma II.1; the symbols P, Q are already being used here to denote the summands of $K_{k-1}(M)$.)

Recall from Lemma II.2 that when $K_k(M_P)$ is a free $Z[H]$ module, the map of $C = C_P \cup_M C_Q \xrightarrow{\hat{f}!} X \times [-1, 1]$ is homotopic to a map transverse to $X \times 0$ with the inverse image M' of $X \times 0$ homotopy equivalent to X. Write C'_P for the closure of the component of $C - M'$ which contains M_P and write C'_Q for the closure of the component of $C - M'$ which contains M_Q. Thus,

$$C = C_P \cup_M C_Q = C'_P \cup_{M'} C'_Q.$$

Hence,

$$\begin{aligned} V &= V_0/\text{identify } M \times 0 \text{ with } M \times 1 \\ &= C_P \cup_{M_P} C \cup_{M_Q} C_Q/\text{identify } M \times 0 \text{ with } M \times 1 \\ &= C \cup_{M_Q} C_Q \cup_M C_P/\text{identify the two copies of } M_P \text{ in the boundary} \\ &= C'_P \cup_{M'} C'_Q \cup_{M_Q} C'_Q \cup_{M'} C'_P/\text{identify the two copies of } M_P \text{ in the boundary} \\ &= (C'_Q \cup_{M_Q} C'_Q) \cup_{M'} (C'_P \cup_{M_P} C'_P)/\text{identify the two copies of } M' \text{ in the} \\ &\quad\ \text{boundary.} \end{aligned}$$

Moreover, the map $V \to X \times S^1$ restricts to maps $C'_Q \cup_{M_Q} C'_Q \xrightarrow{h_1} X \times I'$, I' an interval in S^1 and $C'_P \cup_{M_P} C'_P \xrightarrow{h_2} X \times I''$, I'' an interval in S^1. Note that

$$C'_Q \cup_{M_Q} C'_Q \subset \partial(C'_Q \times I')$$

and we construct in the proof of Lemma III.5. (ii) an extension of h_1 to $C'_Q \times I' \to X \times I'$. Thus, we may make use of the map $K_k(C'_Q \cup_{M_Q} C'_Q) \to K_k(C'_Q)$. Similar remarks applied to C_P will produce a map $K_k(C'_P \cup_{M_P} C'_P) \to K_k(C'_P)$. Also, there are obvious maps of $K_k(M_P)$, $K_k(M_Q)$, $K_k(C'_P \cup_{M_P} C'_P)$, $K_k(C'_Q \cup_{M_Q} C'_Q)$ to $K_k(V)$ induced from the inclusions of subspaces in V.

Lemma III.5. *Let V be as above with $K_k(M_P)$ a free $Z[H]$ module.*

(i) $K_i(V) = 0$ *for* $i \neq k$.

(ii) $\operatorname{Ker}(K_k(C'_Q \cup_{M_Q} C'_Q) \to K_k(C'_Q)) \cong K_{k-1}(M_Q)$
$\operatorname{Ker}(K_k(C'_P \cup_{M_P} C'_P) \to K_k(C'_P)) \cong K_{k-1}(M_P)$.

(iii) *The summands P_V and Q_V of $K_k(V)$ are given by*

$$P_V = \mathrm{Ker}(K_k(C'_Q \cup_{M_Q} C'_Q) \to K_k(C'_Q)) \otimes_{Z[H]} Z[H \times Z] \oplus K_k(M_P) \otimes_{Z[H]} Z[H \times Z]$$

$$Q_V = \mathrm{Ker}(K_k(C'_P \cup_{M_P} C'_P) \to K_k(C'_P)) \otimes_{Z[H]} Z[H \times Z] \oplus K_k(M_Q) \otimes_{Z[H]} Z[H \times Z].$$

Note that in Lemma III.5, $Z[H \times Z] = Z[\pi_1(X \times S^1)]$.

We defer the proof of Lemma III.5 to the end of this section and apply it immediately to the proof of Lemma III.6.

Lemma III.6. *For $n = 2k-1 > 3$, if $\Phi(\tau(f)) = 0$, the nilpotent normal cobordism construction can be performed on $f \times 1_{S^1}$: $W \times S^1 \to Y \times S^1$, for the problem of splitting along $X \times S^1 \subset Y \times S^1$, to obtain a normal cobordism of $f \times 1_{S^1}$ to a homotopy equivalence α: $U \to Y \times S^1$ with α transverse to $Y \times p$, $p \in S^1$ and with, setting $\mathring{W} = \alpha^{-1}(Y \times p)$, $\mathring{W} \to Y = Y \times p$ a homotopy equivalence split along $X \subset Y$.*

Proof. From Lemma III.4, Lemma III.2 and Lemma III.1, when $\Phi(\tau(f)) = 0$ we may assume that $K_k(M_P)$, $K_{k-1}(M_P)$, $K_k(M_Q)$, $K_{k-1}(M_Q)$ are free $Z[H]$ modules. Fix a choice of bases for $K_k(M_P)$ and $K_k(M_Q)$. This determines dual bases for $K_{k-1}(M_P)$ and $K_{k-1}(M_Q)$ and hence, by Lemma III.5.(ii), this determines fixed bases for $\mathrm{Ker}(K_k(C'_Q \cup_{M_Q} C'_Q) \to K_k(C'_Q))$ and $\mathrm{Ker}(K_k(C'_P \cup_{M_P} C'_P) \to K_k(C'_P))$. From Lemma III.5.(iii), the basis of $K_k(M_P)$ together with the basis of

$$\mathrm{Ker}(K_k(C'_Q \cup_{M_Q} C'_Q) \to K_k(C'_Q))$$

form a basis for the free $Z[H \times Z]$ module P_V. From Lemma II.1.(ii) P_V is a subkernel of $K_k(V)$ and we may represent this basis of P_V by disjoint *embedded* spheres in V; moreover, we may clearly choose the spheres representing the basis of $K_k(M_P)$ to lie in a neighborhood of M_P in V and those representing the basis of

$$\mathrm{Ker}(K_k(C'_Q \cup_{M_Q} C'_Q) \to K_k(C'_Q))$$

to lie in the interior of $C'_Q \cup_{M_Q} C'_Q \subset V$. Entirely similar remarks apply to Q_V. We show, at the end of the proof of Lemma III.5, that this basis for the subkernel Q_V is dual to that of the subkernel P_V.

Note. The above argument for embedding elements representing a basis for $K_k(M_P)$ in $M_P \times I$ makes strong use of the geometry of the splitting problem. In general, it is not true that k-spheres which come from codimension one submanifolds of a $2k$-dimensional manifold are homotopic to embeddings; the intersection form is easily seen to vanish on such spheres, but the self-intersection form might be non-zero [W 2; § 8]. A more directly geometric argument to embedd these classes of $K_k(M_P)$ in $M_P \times I$ than that given above is the following: From Lemma III.1, $K_{k+1}(R \times I, M_P \times I) \to K_k(M_P \times I)$ is surjective and a standard piping argument [W 2; § 4] shows that elements of $K_{k+1}(R \times I, M_P \times I)$ can be represented by immersed discs with the boundary sphere embedded.

Let $V_0 \times [-2, 2]$ denote a neighborhood of $V_0 \times 0 = V_0$ in $W \times I$; correspondingly we have a neighborhood $V \times [-2, 2]$ of $V \times 0 = V$ in $W \times S^1$. Here, the parameterization of the neighborhood of V is chosen so that letting $V \times [-2, 2]$

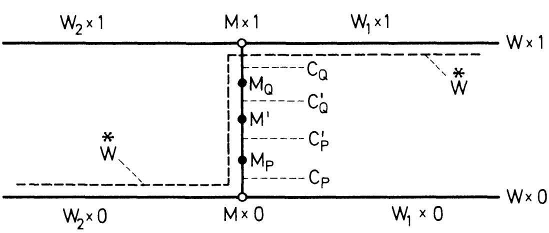
also denote the corresponding neighborhood of V in $\hat{W}\times S^1$, $V\times 2$ (resp; $V\times -2$) is in the component of $\hat{W}\times S^1 - V\times 0$ which is the inverse image under $\hat{g}$ of $Y_r\times S^1$ (resp; $Y_l\times S^1$). We now perform the nilpotent normal cobordism construction of Chapter II using $V\subset W\times S^1$ and the given bases for P_V and Q_V. Precisely, attach handles to spheres, embedded in $(C'_Q\cup_{M_Q} C'_Q)\times 2\subset V\times 2\subset W\times S^1$, representing the basis of $\operatorname{Ker}(K_k(C'_Q\cup_{M_Q} C'_Q)\to K_k(C'_Q))$ and attach handles to spheres, embedded in a neighborhood of $M_P\times 1$ in $V\times 1\subset W\times S^1$, representing the basis for $K_k(M_P)$. Also, attach handles to spheres, embedded in $(C'_P\cup_{M_P} C')\times -2\subset V\times -2\subset W\times S^1$, representing the basis of $\operatorname{Ker}(K_k(C'_P\cup_{M_P} C'_P)\to K_k(C'_P))$ and lastly attach handles to spheres, embedded in a neighborhood of $M_Q\times -1$ in $V\times -1\subset W\times S^1$. This constructs the nilpotent normal cobordism of Chapter II for the problem of splitting $f\times 1_{S^1}\colon W\times S^1\to Y\times S^1$ along $X\times S^1$. Let $\alpha\colon U^{2k+1}\to Y\times S^1$ denote the homotopy equivalence of $(2k+1)$ dimensional manifolds obtained by these surgeries on $W\times S^1$.

To complete the proof of Lemma III.6, we construct a submanifold $\overset{*}{W}\subset W\times S^1$ with $M'\subset\overset{*}{W}$ and with, after varying $f\times 1_{S^1}$ by a homotopy, $\overset{*}{W}$ the inverse image of $Y\times p\subset Y\times S^1$. However, $\overset{*}{W}\to Y\times p$ will not be a homotopy equivalence. But, from $\overset{*}{W}\subset W\times S^1$ we construct $\mathring{W}\subset U$ with $W=\alpha^{-1}(Y\times p)$ and $W\to Y\times p$ a homotopy equivalence split along $X\times p\subset Y\times p$. We proceed with the construction of $\overset{*}{W}\subset W\times I$ and hence of $\mathring{W}\subset W\times S^1$.

Recall that

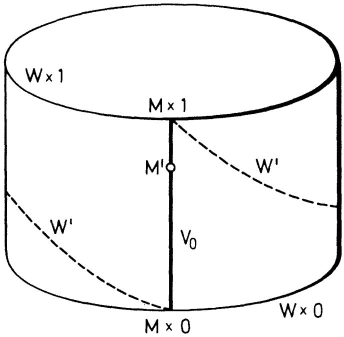
$$V_0=C_P\cup_{M_P} C\cup_{M_Q} C_Q=(C_P\cup_{M_P} C'_P)\cup_{M'}(C'_Q\cup_{M_Q} C_Q)$$

and in particular this gives a fixed embedding $M'\subset V_0$. This $M'\subset V_0$ corresponds to a component of the boundary of $C'_Q\cup_{M_Q} C'_Q$ and of $C'_P\cup_{M_P} C'_P$ in $V=V_0\times I/$ identify $M_0\times 0$ with $M\times 1$.



$\overset{*}{W}\subset W\times I$ in case A

Now in case A, set $\overset{*}{W}=W_2\times 0\cup_M V_0\cup_M W_1\times 1$, as in the diagram. (Resp; Recall that in case B, V_0 was constructed in $W\times I$ disjoint from a copy of W' with for $\partial W'=M_1\cup M_2$, M_1 going to $M\times 0$ and M_2 to $M\times 1$. Now set $\overset{*}{W}=$this copy of $W'\cup_{M\times 0\cup M\times 1} V_0$.)



The map $f\times 1_{S^1}\colon W\times S^1\to Y\times S^1$ is easily varied by a homotopy to obtain $\overset{*}{W}$ as the inverse image of $Y\times p\subset Y\times S^1$, p a fixed choice of a point in S^1. From $M'\subset V_0\subset \overset{*}{W}$, M' is the inverse image of $X\times p$.

Note that $M_P\subset V_0\subset \overset{*}{W}$ and $M_Q\subset V_0\subset \overset{*}{M}$. From the construction of U, U contains the manifold $\mathring{W}$ obtained from $\overset{*}{W}$ by surgery on the basis of $K_k(M_P)$ and $K_k(M_Q)$ and after varying α by a homotopy, $\mathring{W}=\alpha^{-1}(Y\times p)$. As M_P and M_Q are disjoint from $M'\subset W$, we get $M'\cong\alpha^{-1}(X\times p)$. The proof of Lemma III.6 is completed by showing that $\mathring{W}\to Y$ is a homotopy equivalence. Clearly, it suffices to show that $K'_i(\overset{*}{W})=0$ for $i<k$ and that the basis for $K_k(M_P)$ and $K_k(M_Q)$ generate a subkernel of $K_k(\overset{*}{W})$ [W 2; Chapter V]. We will show this in case A; the argument in case B is entirely similar, with apropriate changes of notation.

Recall that $K_i(W_1)=0$, $i\neq k-1$, k, $K_i(W_2)=0$, $i\neq k-1, k$. Then, as C_P was constructed by attaching discs to spheres representing a generating set for P and C_Q was constructed by attaching discs to a generating set for Q, by Lemma I.5, $K_i(W_1\cup_M C_Q)=0$ for $i\neq k$ and $K_i(W_2\cup_M C_P)=0$ for $i\neq k$. Recall also from Lemma III.1 that $K_i(M_P)=0$, $K_i(M_Q)=0$ for $i\neq k-1, k$ and $K_i(C)=0$ for $i\neq k$, $K_k(M_P)\oplus K_k(M_Q)\to K_k(C)$ is injective. Hence the Mayer-Vietoris sequence for

$$\overset{*}{W}\cong(W_1\cup_M C_Q)\cup_{M_Q}(C)\cup_{M_P}(W_2\cup_M C_P)$$

reduces to

$$\begin{aligned}0\to K_k(M_P;Z[G])\oplus K_k(M_Q;Z[G])&\to K_k(W_1\cup_M C_Q;Z[G])\oplus K_k(C;Z[G])\\ &\oplus K_k(W_2\cup_M C_P;Z[G])\to K_k(\overset{*}{W})\to K_{k-1}(M_P;Z[G])\oplus K_{k-1}(M_Q;Z[G])\\ &\to 0.\end{aligned}$$

We show below that $K_k(M_P;\, Z[G])\to K_k(W_2\cup_M C_P;\, Z[G])$ and that

$$K_k(M_Q;\, Z[G])\to K_k(W_1\cup_M C_Q;\, Z[G])$$

are isomorphisms. Hence, as $K_k(M_P)\oplus K_k(M_Q)\to K_k(C)$ is an isomorphism, and $Z(G)$ is a free $Z[H]$, module, this Mayer-Vietoris sequence gives the short exact sequence

$$0 \to K_k(M_P) \otimes_{Z[H]} Z[G] \oplus K_k(M_Q) \otimes_{Z[H]} Z[G]$$
$$\to K_k(\overset{*}{W}) \to K_{k-1}(M_P) \otimes_{Z[H]} Z[G] \oplus K_{k-1}(M_Q) \otimes_{Z[H]} Z[G] \to 0.$$

But then, as $K_k(M_P)$, $K_k(M_Q)$, $K_{k-1}(M_P)$, $K_{k-1}(M_Q)$ are free $Z[H]$ modules, and as by Poincaré duality rank $K_k(M_P) = \text{rank } K_{k-1}(M_P)$, and $K_k(M_Q) = \text{rank } K_{k-1}(M_Q)$, the basis of $K_k(M_P)$ and $K_k(M_Q)$ generate a free $Z[G]$ summand of rank $\frac{1}{2}(\text{rank } K_k(\overset{*}{W}))$ in $K_k(\overset{*}{W})$. Moreover, as from the construction of $\mathring{W} \subset U$ surgery could be performed on the basis of this summand, it is a subkernel of $K_k(\overset{*}{W})$.

It remains only to check the fact used above, $K_k(M_P) \otimes_{Z[H]} Z[G_2] \to K_k(W_2 \cup_M C_P)$ is an isomorphism. An entirely similar argument also shows that $K_k(M_Q) \otimes_{Z[H]} Z[G_2] \to K_k(W_1 \cup_M C_Q)$ is an isomorphism.

Recall, as in Lemma III.1, the construction of $C_P \subset W_r$, $W_r = R \cup_{M_P} C_P$. Clearly,

$$\begin{aligned}
\tilde{W} &= \tilde{W}_2 \cup_{\left(\bigcup_{\alpha \in [G_2, H]} \tilde{M} g(\alpha)\right)} \Big(\bigcup_{\alpha \in [G_2, H]} \tilde{W}_r g(\alpha) \Big) \\
&= \tilde{W}_2 \cup_{\bigcup_{\alpha \in [G_2, H]} \tilde{M} g(\alpha)} \Big(\bigcup_{\alpha \in [G_2, H]} (\tilde{C}_P \cup_{\tilde{M}_P} \tilde{R}) g(\alpha) \Big) \\
&= (\widetilde{W_2 \cup_M C_P}) \cup_{\bigcup_{\alpha \in [G_2, H]} \tilde{M}_P g(\alpha)} \tilde{R} g(\alpha).
\end{aligned}$$

But then as $K_i(W) = 0$, for all i, the Mayer-Vietoris sequence of this decomposition of $\tilde{W}$ gives

$$K_i(M_P) \otimes_{Z[H]} Z[G_2] \xrightarrow{\cong} K_i(W_2 \cup_M C_P) \oplus K_i(R) \otimes_{Z[H]} Z[G_2].$$

From Lemma III.1, $K_k(R) = 0$ and hence

$$K_k(M_P) \otimes_{Z[H]} Z[G_2] \xrightarrow{\cong} K_k(W_2 \cup_M C_P).$$

As $Z[G]$ is a free $Z[G_2]$ module, this gives the isomorphism

$$K_k(M_P; Z[G]) \to K_k(W_2 \cup_M C_P; Z[G]).$$

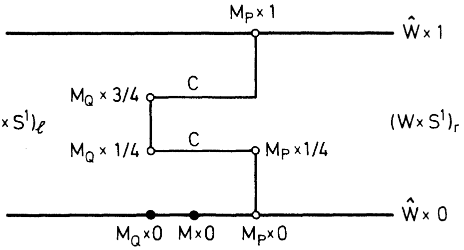
Proof of Lemma III.5. As $V \cong C \cup_{(M_P \cup M_Q)} C$ and by Lemma III.1 $K_i(C) = 0$, $i \neq k$ and $K_i(M_P) = 0$, $K_i(M_Q) = 0$ for $i < k-1$, the Mayer-Vietoris sequence for this decomposition of V shows that $K_i(V) = 0$, $i \leq k-1$. Hence, by Poincaré duality, $K_i(V) = 0$ for $i \neq k$. This proves part (i).

The embedding $V \hookrightarrow W \times S^1$ lifts to an embedding $V \hookrightarrow (\hat{W} \times S^1)$, extending the lift of $M \to W$ to $M \to \hat{W}$. From the last part of Lemma III.3, this embedding can after an isotopy be identified with the embedding $V \hookrightarrow \hat{W} \times S^1 = \hat{W} \times I/$identify $\hat{W} \times 0$ with $\hat{W} \times 1$ given by

$$M_P \times [0, 1/4] \cup_{M_P \times 1/4} C \times 1/4 \cup_{M_Q \times 1/4} M_Q \times [1/4, 3/4] \cup_{M_Q \times 3/4}$$

$C \times 3/4 \cup_{M_P \times 3/4} M_P \times [3/4, 1]$ with $M_P \times 0$ identified with $M_P \times 1$.
Note that

$$\begin{aligned}
C'_Q \cup_{M_Q} C'_Q &\cong C'_Q \times 3/4 \cup_{M_Q \times 3/4} M_Q \times [3/4, 1/4] \cup_{M_Q \times 1/4} C'_Q \times 1/4 \\
&\subset \partial(C'_Q \times [1/4, 3/4])
\end{aligned}$$



$V \subset \hat{W} \times S^1 = \hat{W} \times I/(\hat{W} \times 0 = \hat{W} \times 1)$

and thus $C'_Q \cup_{M_Q} C'_Q \to X$ extends to $C'_Q \times [1/4, 3/4]$. Moreover, as $C'_Q \cup_{M'} C'_P = C$ and $K_i(C)=0$ for $i \neq k$ and $K_i(M')=0$ for all i, Mayer-Vietoris sequences show that $K_i(C'_Q)=0$ for $i \neq k$. Thus, the sequence of the pair $(C'_Q \times [1/4, 3/4], C'_Q \cup_{M_Q} C'_Q)$ gives the exact sequence

$$\begin{aligned}0 &\to K_{k+1}(C'_Q \times [1/4; 3/4], C'_Q \cup_{M_Q} C'_Q) \to K_k(C'_Q \cup_{M_Q} C'_Q)\\ &\to K_k(C'_Q \times [1/4, 3/4])\end{aligned}$$

and thus

$$\operatorname{Ker}(K_k(C'_Q \cup_{M_Q} C'_Q) \to K_k(C'_Q \times [1/4, 3/4]) \cong K_{k+1}(C'_Q \times [1/4, 3/4], C'_Q \cup_{M_Q} C'_Q).$$

But by Poincaré Duality, and as

$$\begin{aligned}&\partial(C'_Q \times [1/3, 3/4]) - \text{interior}\,(C'_Q \cup_{M_Q} C'_Q) = M' \times [1/4, 3/4]\\ &K_{k+1}(C'_Q \times [1/4, 3/4], C'_Q \cup_{M_Q} C'_Q) \cong K^k(C'_Q \times [1/4, 3/4], M' \times [1/4, 3/4])\\ &\qquad \cong K^k(C'_Q, M')\\ &\qquad \cong K_k(C'_Q, M_Q).\end{aligned}$$

But, as above, the Mayer-Vietoris sequence of $C = C'_Q \cup_{M'} C'_P$ shows as $K_i(M')=0$ for all i that $K_k(C) = K_k(C'_Q) \oplus K_k(C'_P)$. Recall also from Lemma III.1, $K_k(C) = K_k(M_P) \oplus K_k(M_Q)$. Hence, $M_P \to C'_P$ and $M_Q \to C'_Q$ induce isomorphisms $K_k(M_P) \to K_k(C'_P)$, $K_k(M_Q) \to K_k(C'_Q)$. Then, the exact sequence of the pair (C'_Q, M_Q) gives as $K_{k-1}(C'_Q)=0$ $K_k(C'_Q, M_Q) \cong K_{k-1}(M_Q)$.

We conclude that

$$\operatorname{Ker}(K_k(C'_Q \cup_{M_Q} C'_Q) \to K_k(C'_Q \times [1/4, 3/4])) \cong K_{k-1}(M_Q).$$

An entirely similar result for M_P completes the demonstration of Part (ii).

To prove part (iii) of Lemma III.5, recall from I § 4 that the decomposition $K_k(V) = P_V \oplus Q_V$ is defined using the decomposition of $\hat{W} \times S^1$, the covering space of $W \times S^1$ corresponding to $\pi_1(X \times S^1) \subset \pi_1(Y \times S^1)$, given by

$$\hat{W} \times S^1 = (W \times S^1)_l \cup_V (W \times S^1)_r.$$

Here we denote by $(W \times S^1)_l$ (resp; $(W \times S^1)_r$) the closure of the component of $\hat{W} \times S^1 - V$ which is the inverse image of $\hat{Y}_l \times S^1$ (resp; $(\hat{Y}_r \times S^1)$) under the lift of the map g.

The definition of P in I § 4 gives

$$P_V = \text{Image}\,(K_{k+1}((W \times S^1)_r, V) \to K_k(V)) \cong K_{k+1}((W \times S^1)_r, V).$$

But, see the above diagram, for R as in Lemma III.1 we may decompose

$$\begin{aligned}(W \times S^1)_r &= R \times [0, 1/4] \cup_{R \times 1/4} ((W_r \cup_M C) \times [1/4, 3/4]) \\ &\quad \cup_{R \times 3/4} R \times [3/4, 1])/\text{identify } R \times 0 = R \times 1 \\ &= H_1 \cup_{((R \cup_M C'_P) \times 1/4 \cup (R \cup_M C'_P) \times 3/4)} H_2\end{aligned}$$

where set

$$\begin{aligned}H_1 &= (R \cup_{M_P} C) \times [1/4, 3/4] \\ H_2 &= (((R \cup_{M_P} C) \times 1/4 \cup_{R \times 1/4} R \times [0, 1/4]) \cup ((R \cup_M C) \times 3/4 \\ &\quad \cup_{R \times 3/4} R \times [3/4, 1]))/\text{identify } R \times 0 = R \times 1.\end{aligned}$$

Correspondingly, decompose

$$V = F_1 \cup_{(M' \times 1/4 \cup M' \times 3/4)} F_2$$

where

$$\begin{aligned}F_1 &= C'_Q \times 1/4 \cup_{M_Q \times 1/4} M_Q \times [1/4, 3/4] \cup_{M_Q \times 3/4} C'_Q \times 3/4 \\ &\cong C'_Q \cup_{M_Q} C'_Q \\ F_2 &= ((M_P \times [0, 1/4] \cup_{M_P \times 1/4} C'_P \times 1/4) \cup (C'_P \times 3/4 \cup_{M_P \times 3/4} \\ &\quad M_P \times [3/4, 1]))/\text{identify } M_P \times 0 = M_P \times 1 \\ &\cong C'_P \cup_{M_P} C'_P.\end{aligned}$$

Note that $F_1 \subset H_1$, $F_2 \subset H_2$. Thus as, see the proof of Lemma III.2, $K_i(W_r \cup_M C'_P, M') = 0$ for all i, the Mayer-Vietoris sequence for $((\hat{W}_r \times S^1)_r V) = (H_1, F_1) \cup (F_2, H_2)$ gives

$$\begin{aligned}P_V \cong K_{k+1}((W_r \times S^1)_r, V) &= K_{k+1}(H_1, F_1; Z[\pi_1(X \times S^1)]) \\ &\quad \oplus K_{k+1}(H_2, F_2; Z[\pi_1(X \times S^1)]).\end{aligned}$$

But by excision,

$$\begin{aligned}K_{k+1}(H_2, F_2; Z[\pi_1(X \times S^1)]) &\cong K_{k+1}(R, M_P; Z[\pi_1(X \times S^1)]) \\ &\cong K_k(M_P; Z[\pi_1(X \times S^1)]) \quad \text{by Lemma III.1} \\ &\cong K_k(M_P) \otimes_{Z[H]} Z[H \times Z].\end{aligned}$$

We now proceed to complete the description of P_V by computing

$$K_{k+1}(H_1, F_1; Z[\pi_1(X \times S^1)]).$$

First note that

$$\begin{aligned}(H_1, F_1) &= (C'_Q \times [1/4, 3/4], C'_Q \times 1/4 \cup_{M_Q \times 1/4} M_Q \times [1/4, 3/4] \cup_{M_Q \times 3/4} C'_Q \times 3/4) \\ &\quad \cup_{(M' \times [1/4, 3/4], M' \times 1/4 \cup M' \times 3/4)} ((R \cup_M C'_P) \times ([1/4, 3/4], (1/4 \cup (3/4)))\end{aligned}$$

and as $K_i(M')=0$, for all i, and from the proof of Lemma III.2, $K_i(R\cup_M C'_P)=0$, for all i, the Mayer-Vietoris sequence of this decomposition gives

$$\begin{aligned}
&K_{k+1}(H_1, F_1; Z[\pi_1(X\times S^1)])\\
&\cong K_{k+1}(C'_Q\times[1/4,3/4], C'_Q\times 1/4\cup_{M_Q\times 1/4} M_Q\times[1/4,3/4]\cup_{M_Q\times 3/4} C'_Q\times 3/4;\\
&\quad Z[\pi_1(X\times S^1)])\\
&\cong K_{k+1}(C'_Q\times[1/4,3/4], C'_Q\times 1/4\cup_{M_Q\times 1/4} M_Q\times[1/3,3/4]\cup_{M_Q\times 3/4} C'_Q\times 3/4)\\
&\quad \otimes_{Z[H]} Z[H\times Z].
\end{aligned}$$

This last expression was described in proving part (ii) above. This completes the computation of P_V and entirely similar methods give the description of Q_V in Lemma III.5.

Notice that from the above description of P_V and Q_V it is easy to obtain an explicit description of the intersection pairing on $K_k(V)$. Let $e_1, \ldots, e_r$ be spheres representing a basis for $K_k(M_P)$ and let $e'_1, \ldots, e'_r$ represent a dual basis for $K_{k-1}(M_P)$. Similarly, let $f_1, \ldots, f_s$ be a basis for $K_k(M_Q)$ and $f'_1, \ldots, f'_s$ a dual basis for $K_{k-1}(M_Q)$. Then, a basis for P_V (resp; Q_V) is given by $e_1, e_2, \ldots, e_r$, $f''_1, f''_2, \ldots, f''_s$ (resp; $f_1, f_2, \ldots, f_s$, $e''_1, e''_2, \ldots, e''_r$); here f''_i (resp; e''_i) is explicitly constructed by the following procedure. The sphere f'_i (resp; e'_i) bounds a disc in C'_Q (resp; C'_p); taking the double of this disc we get the sphere f''_i (resp; e''_i) in $C'_Q\cup_{M_Q} C'_Q$ (resp; $C'_P\cup_{M_P} C'_P$). Clearly this sphere bounds a disc in $C'_Q\times I$, where $C'_Q\cup_{M_Q} C'_Q\subset\partial(C'_Q\times I)$ (resp; $C'_P\times I$, where $C'_P\cup_{MP} C'_P\subset\partial(C'_P\times I)$). As $V=(C'_P\cup_{M_P} C'_P)\cup_{M'\cup M'}(C'_Q\cup_{M_Q} C'_Q)$ the intersection form λ on $K_k(V)=P_V\oplus Q_V$ is easily seen to be given explicitly by

$$\begin{aligned}
&\lambda(e_i, e_j)=\lambda(e_i, f''_j)=\lambda(e_i, f_j)=\lambda(f''_i, f''_j)=\lambda(f''_i, e''_j)=\lambda(f_i, f_j)=\lambda(f_i, e''_j)\\
&\qquad =\lambda(e''_i, e''_j)=0 \quad \text{for all } i, j.\\
&\lambda(e_i, e''_j)=0 \quad \text{for } i\neq j \quad\text{and}\quad \lambda(e_i, e''_i)=1, \quad 1\leqq i\leqq r\\
&\lambda(f_i, f''_j)=0 \quad \text{for } i\neq j \quad\text{and}\quad \lambda(f_i, f''_i)=1, \quad 1\leqq i\leqq s.
\end{aligned}$$

Remark. Even if $H\subset G$ is not a square-root closed subgroup, the normal cobordism of W to $\mathring{W}$ which can be obtained from Lemma III.6 shows that when $\Phi(\tau(f))=0$ and $n>5$, $W\xrightarrow{f} Y$ is normally cobordant to a homotopy equivalence split along $X\subset Y$. For n even this was proved in Chapter II.

§ 3. Completion of the Argument

Proof of Theorem 1 for $n=2k-1$, Part (i). The necessity of the condition $\bar{\Phi}(\tau(f))=0$ was shown for all n in the completion of the proof of part (i) for $n=2k$ in Chapter II.

Assume now that $\bar{\Phi}(\tau(f))=0$. Then by Lemma II.4, W is h-cobordant to a manifold $\check{W}$ for which the induced homotopy equivalence $\check{f}\colon \check{W}\to Y$ satisfies $\Phi(\tau(\check{f}))=0$. Thus, in proving part (i), we may assume without loss of generality that $\Phi(\tau(f))=0$. Then Lemma III.6 constructs a nilpotent normal cobordism of $W\times S^1 \xrightarrow{f\times 1_{S^1}} Y\times S^1$ to $\alpha\colon U\to Y\times S^1$ with $\mathring{W}\subset U$, $\mathring{W}\to Y$ a homotopy equivalence split along X.

But as H is square-root closed in G, $Z\times H$ is square-root closed in $Z\times G$; moreover, H contains all elements of order 2 in $Z\times H$. Now as f is a homotopy equivalence, $f \times 1_{S^1}$: $W \times S^1 \to Y \times S^1$ is a simple homotopy equivalence [KwS]. Hence, using Remark 1 at the end of Chapter II, the surgery obstruction v of the nilpotent normal cobordism of Lemma III.6 is in

Image $(L^S_{2k+2}(H,\omega) \to L^S_{2k+2}(Z \times G, \omega))$.

Hence, as $H \subset G = \pi_1(Y \times S^1 - Y \times p) = \pi_1(U - \mathring{W})$, we may attach a normal cobordism along $U - \mathring{W}$ realizing a surgery obstruction whose image in

$$L^S_{2k+2}(Z \times G, \omega)$$

is $(-v)$ [W2]. This produces a normal cobordism with zero surgery obstruction in $L^S_{2k+2}(Z \times G, \omega)$ of $f \times 1_{S^1}$: $W \times S^1 \to Y \times S^1$ to a simple homotopy equivalences α': $U' \to Y \times S^1$, $\alpha'^{-1}(Y \times p) = \mathring{W}$. Then, $W \times S^1$ is s-cobordant to U' and so $f \times 1_{S^1}$: $W \times S^1 \to Y \times S^1$ is homotopic to a map transverse to $Y \times p$ with $\mathring{W}$ the inverse image of $Y \times p$. Passing to the covering space $W \times R$, $R = (-\infty, \infty)$, we have corresponding to $W \subset W \times S^1$ and $\mathring{W} \subset W \times S^1$ that $W \subset W \times R$ and $\mathring{W} \subset W \times R$. Applying a covering translation of the covering space $W \times R \to W$, we may assume that W and $\mathring{W}$ are embedded disjointly in $W \times R$. Let D denote the compact submanifold of $W \times R$ with $\partial D = W \cup \mathring{W}$. We claim that $(D; W, \mathring{W})$ is an h-cobordism. In fact, we may regard D as $g^{-1}(Y \times [p,q])$ for g: $W \times R \to Y \times R$ a proper homotopy-equivalence property homotopic to $f \times 1_R$ with $g^{-1}(Y \times p) = W$, $g^{-1}(Y \times q) = \mathring{W}$. Then as g is split along $Y \times p$ and $Y \times q$ the methods of Lemma I.2.(i) show that $D \to Y \times [p,q]$ is a homotopy equivalence. This completes the proof of part (i).

Notice that part (ii) of Theorem 1 is an immediate consequence of part (i). If $\mathrm{Wh}(G_1) \oplus \mathrm{Wh}(G_2) \to \mathrm{Wh}(G)$ (resp; $\mathrm{Wh}(J) \to \mathrm{Wh}(G)$) is surjective, it is easy to replace the h-cobordism of W to a split manifold by an s-cobordism to a split manifold. Before completing the proof of part (ii) of Theorem 2 for $n = 2k-1$ we need some preliminary results on h-cobordant splitting problems.

Recall the maps defined in [W1]

$$\mathrm{Wh}(G_1 *_H G_2) \xrightarrow{\Phi} \mathrm{Ker}\,(\tilde{K}_0(G) \to \tilde{K}_0(G_1) \oplus \tilde{K}_0(G_2))$$

$$(\text{resp; } \mathrm{Wh}(J *_H \{t\}) \xrightarrow{\Phi} \mathrm{Ker}\,(K_0(H) \xrightarrow{\xi_{1*} - \xi_{2*}} K_0(J)))$$

$$\mathrm{Wh}(G_1 *_H G_2) \xrightarrow{\eta} \widetilde{\mathrm{Nil}}\,(H; G_1, G_2)$$

$$(\text{resp; } \mathrm{Wh}(J *_H \{t\}) \xrightarrow{\eta} \widetilde{\mathrm{Nil}}\,(H, J; \xi_1, \xi_2))$$

where

$$\begin{aligned} &\mathrm{Wh}(G_1 *_H G_2)/(\mathrm{Wh}(G_1) + \mathrm{Wh}(G_2)) \\ &\qquad \cong \mathrm{Ker}\,(\tilde{K}_0(H) \to \tilde{K}_0(G_1) \oplus \tilde{K}_0(G_2)) \oplus \widetilde{\mathrm{Nil}}\,(H; G_1, G_2) \\ &(\text{resp; } \mathrm{Wh}(J *_H \{t\})/\mathrm{Wh}(J) \\ &\qquad \cong \mathrm{Ker}\,(\tilde{K}_0(H) \xrightarrow{\xi_{1*} - \xi_{2*}} K_0(J)) \oplus \widetilde{\mathrm{Nil}}\,(H, J; \xi_1, \xi_2). \end{aligned}$$

As usual, a fixed homomorphism $G \to Z_2$, restricting to $H \to Z_2$ determines Z_2 actions on $\mathrm{Wh}(G)$ and $\tilde{K}_0(H)$. The following result is a refinement of Lemma II.3.

Lemma III.7. *Let G, H and G_1, G_2* (resp; $\xi_i\colon H\to J$, $i=1,2$) *be as above with* $G=G_1 *_H G_2$ (resp; $G=J *_H \{t\}$). *Then the involution $x\to x^*$ on* $\mathrm{Wh}(G)$ *induces an involution on*

$$\begin{aligned}&\mathrm{Wh}(G)/(\mathrm{Wh}(G_1)+\mathrm{Wh}(G_2))\\ &\qquad\cong \mathrm{Ker}\,(\tilde K_0(H)\to \tilde K_0(G_1)\oplus \tilde K_0(G_2))\oplus \widetilde{\mathrm{Nil}}\,(H;G_1,G_2)\end{aligned}$$

$$\begin{aligned}&(\text{resp; } \textit{on } \mathrm{Wh}(G)/\mathrm{Wh}(J)\\ &\qquad\cong \mathrm{Ker}\,(\tilde K_0(H)\to \tilde K_0(J))\oplus \widetilde{\mathrm{Nil}}\,(H,J;\xi_1;\xi_2))\end{aligned}$$

which restricts to the negative of the usual involution on

$$\mathrm{Ker}\,(\tilde K_0(H)\to \tilde K_0(G_1)\oplus \tilde K_0(G_2))\subset \tilde K_0(H)$$

$$(\text{resp; } \textit{on } \mathrm{Ker}\,(\tilde K_0(H)\xrightarrow{\xi_{1*}-\xi_{2*}} \tilde K_0(J))\subset \tilde K_0(H))$$

and to an involution on $\widetilde{\mathrm{Nil}}\,(H;G_1,G_2)$ (resp; $\widetilde{\mathrm{Nil}}\,(H,J;\xi_1,\xi_2)$).

Proof. This has essentially the same proof as that used to show Lemma II.3.

Remark. Actually, the proof of Lemma II.3 could be used to show a stronger result, not needed below. The splitting

$$\widetilde{\mathrm{Nil}}\,(H;G_1,G_2)\to \mathrm{Wh}(G)\qquad (\text{resp; } \widetilde{\mathrm{Nil}}\,(H,J;\xi_1,\xi_2)\to \mathrm{Wh}(G))$$

commutes with the Z_2-actions.

Lemma III.8. *Let Y be a closed Poincaré complex of dimension $n+1$, $n\geq 5$ with $Y=Y_1\cup_X Y_2$* (resp; $Y=Y'/$*identify X_1 with X_2, $\partial Y'=X_1\cup X_2$*), *X a closed codimension one sub-Poincaré complex of Y. Assume*

$$H=\pi_1(X)\to \pi_1(Y_i)=G_i\qquad (\text{resp; } \xi_i\colon H=\pi_1(X)\to \pi_1(Y)=J)$$

are injective, $i=1,2$. Let W be a closed manifold and $f\colon W\to Y$ a homotopy equivalence split along X. Then, given

$$x\in \mathrm{Ker}\,(\tilde K_0(H)\to \tilde K_0(G_1)\oplus K_0(G_2))\qquad (\text{resp; } x\in \mathrm{Ker}\,(K_0(H)\xrightarrow{\xi_{1*}-\xi_{2*}} K_0(J))$$

with $x=(-1)^n x^*$, there is an h-cobordism $(V;W,\mathring{W})$ with torsion v satisfying $\eta(v)=0$, $\Phi(v)=x$ and with the induced homotopy equivalence $\mathring{f}\colon \mathring{W}\to Y$ split along X.

We defer the proof of Lemma III.8 and use it to show the following.

Lemma III.9. *Let $f\colon W\to Y$ be split as in Lemma III.8. If $(D;W,\overset{*}{W})$ is an h-cobordism with torsion d satisfying $d=(-1)^{n+1}d^*$ in*

$$\mathrm{Wh}(G)/(\mathrm{Wh}(G_1)+\mathrm{Wh}(G_2))\qquad (\text{resp } \textit{in } \mathrm{Wh}(G)/\mathrm{Wh}(J)),\qquad G=\pi_1(Y),$$

and if $\bar\eta(d)$, the element represented by $\eta(d)$ in

$$H^{n+1}(Z_2;\widetilde{\mathrm{Nil}}\,(H;G_1,G_2))\qquad (\text{resp; } \textit{in } H^{n+1}(Z_2;\widetilde{\mathrm{Nil}}\,(H,J;\xi_1,\xi_2))$$

is zero, then the induced homotopy equivalence $\overset{*}{f}\colon \overset{*}{W}\to Y$ is split along X.

Proof. From the hypothesis and Lemma III.3, $\Phi(d)=(-1)^n \Phi(d)^*$. Then, let V be as in Lemma III.8 with $x=(-1)^n \Phi(d)^*$ and let $D'=D\cup_W V$ so that $(D'; \mathring{W}, \overset{*}{W})$ is an h-cobordism with torsion denoted u and with $\mathring{W}\to Y$ split. Then as $W\to \mathring{W}$ splits, it has torsion, see II § 2, in Image

$$((\mathrm{Wh}(G_1)\oplus \mathrm{Wh}(G_2))\to \mathrm{Wh}(G)) \qquad (\text{resp}; (\mathrm{Wh}(J)\to \mathrm{Wh}(G))$$

and the torsion u of $(D'; \mathring{W}, \overset{*}{W})$ is $(-1)^n v^*+d$ in

$$\mathrm{Wh}(G)/(\mathrm{Wh}(G_1)+\mathrm{Wh}(G_2)) \qquad (\text{resp; in } \mathrm{Wh}(G)/\mathrm{Wh}(J)).$$

Then,

$$\Phi(u)=\Phi((-1)^n v^*)+\Phi(d)=(-1)^{n+1} \Phi(v)^*+\Phi(d)=(-1)^{n+1} x^*+\Phi(d)=0$$

and $\eta(u)=\eta((-1)^n v^*)+d=\eta(d)$. In particular, $\Phi(u)=0$ and $\bar{\eta}(u)=0$.

Thus in proving Lemma III.9, we may assume without loss of generality that $\bar{\eta}(d)=0$ and $\Phi(d)=0$. But then d represents the zero element of

$$H^{n+1}(Z_2; \mathrm{Wh}(G))/(\mathrm{Wh}(G_1)+\mathrm{Wh}(G_2))) \qquad (\text{resp; of } H^{n+1}(Z_2; \mathrm{Wh}(G)/\mathrm{Wh}(J)))$$

and hence there is an h-cobordism, relative to the boundary, of D to an h-cobordism $(D''; W, \mathring{W})$ whose torsion represents zero in

$$\mathrm{Wh}(G)/(\mathrm{Wh}(G_1)+\mathrm{Wh}(G_2)) \qquad (\text{resp; } \mathrm{Wh}(G)/\mathrm{Wh}(J)).$$

But then by attaching along $W-f^{-1}(X)$ a further h-cobordism we obtain an s-cobordism of $\mathring{W}$ to a split manifold.

The proof of Lemma III.8, is based on a realization procedure, in the relative case, using an infinite process trick for the Siebenmann obstruction to constructing a boundary for an open manifold.

Proof of Lemma III.8. Let $M=f^{-1}(X)$, where from the hypothesis of Lemma III.8, $M\to X$ is a homotopy equivalence. Let $x\in \tilde{K}_0(H)$ with $x=(-1)^n x^*$. We construct a proper homotopy equivalence of open manifolds $g: (T, \partial T)\to (M\times I\times I-M\times \partial I\times \text{interior } I,\ M\times I\times -1\cup M\times I\times 1)$, $I=[-1,1]$, with $\partial T=M\times I\times -1\cup V$, $(g\,|\,(M\times I\times -1))=1_{M\times I\times -1}$ and $g!: V\to M\times I\times 1$ a proper homotopy equivalence split along $M\times 0\times 1$ and with, setting $N=g^{-1}(M\times 0\times I)$, g transverse to $M\times 0\times 1$ with:

(i) for $n=2k$ $K_i(N)=0$ $i\neq k$ and $x=[K_k(g^{-1}(M\times[0,1]\times I-M\times 1\times(-1,1)))]$

(ii) for $n=2k-1$, $K_i(N)=0$ $i\neq k-1, k$, $K_{k-1}(N)$ and $K_k(N)$ are projective $Z[H]$ modules with $K_k(N)\to K_k(g^{-1}(M\times[0,1]\times I-M\times 1\times(-1,1)))$ an isomorphism and $x=[K_k(N)]$.

We defer the construction of T to the end of the argument.

Now, attach T along $M\times I\times -1\subset T$ to the boundary component $W\times -1$ of $W\times[-2,-1]$. If

$$x\in \mathrm{Ker}\,(\tilde{K}_0(H)\to \tilde{K}_0(G_1)\oplus \tilde{K}_0(G_2))$$
$$(\text{resp}; x\in \mathrm{Ker}\,(\tilde{K}_0(H)\xrightarrow{\xi_{1*}-\xi_{2*}} \tilde{K}_0(J)))$$

from [S 1] the resulting manifold is the interior of an h-cobordism of W to some manifold $\mathring{W}$, $V \subset \mathring{W}$. From the geometric description of η and Φ of [W1; § 5], letting v denote the Whitehead torsion of this h-cobordism, $\eta(v)=0$ and $\Phi(v)=x$. Moreover, as $V \subset \mathring{W}$, $\mathring{W} \to Y$ is split.

It remains only to construct T. For $n=2k-1$, let P be a finitely generated projectiv $Z[H]$ module with $[P]=x$ and $P \oplus P^*$ a free module. Introduce on $P \oplus P^*$ the obvious Hermitian form structure and [W2; § 5] realize this by a normal cobordism C of M to the homotopy equivalent manifold M'. Now to $M \times R \times [-1, 1]$, $R=(\infty, \infty)$ attach along a neighborhood of $M \times n \times 1$ in $M \times R \times 1$, for each $n \in Z \subset R$, the manifold $C \times [-n+\varepsilon, n+\varepsilon]$ to get H. Let P_n denote the copy of P in $K_k(C \times (n+\varepsilon))$ and P_n^* denote the copy of P^* in $K_k(C \times (n-\varepsilon))$. Then attach to H handles corresponding to a basis of $\bigoplus_{n \in Z}(P_n \oplus P_{n+1}^*)$. From the resulting manifold remove all the boundary other than a closed neighborhood of M and one of M'; this is T.

The argument for $n=2k$ is similar. Let $[P]=x$ with $\alpha: P \to P^*$ an isomorphism. Let Q be a finitely generated $Z[H]$ module with $P \oplus Q$ free and $\beta: Q \to Q^*$ an isomorphism. Then $\alpha \oplus 1_Q$ (resp; $1_P \oplus \beta$) extends in the obvious manner to determine an automorphism E (resp; F) of the obvious Hermitian form on

$$(P \oplus Q) \oplus (P^* \oplus Q^*).$$

Let C_E (resp; C_F) denote a realization of this by a normal cobordism of M to M' [W2; § 6], $K_k(C_E) \cong Q^* \cong Q$ (resp; $K_k(C_F) \cong P^* \cong P$). Now attach to $M \times R \times [-1, 1]$ in a neighborhood of each $M \times 2k \times 1$ a copy of $C_E \times [2k-\varepsilon, 2k+\varepsilon]$ and to a neighborhood of each copy of $M \times (2k+1) \times 1$ attach a copy of $C_F \times [2k+1-\varepsilon, 2k+1+\varepsilon]$ just $\cdots \cup (C_E \cup C_F) \cup (C_E \cup C_F) \cup \cdots$ and as $K_k(C_E \cup C_F) \cong Q \oplus P$, is a free module perform surgery on each copy of $K_k(C_E \cup C_F)$. From the resulting manifold remove all the boundary other than a closed neighborhood of M and M' to get T.

Completion of the Proof of Theorem 2 for $n=2k-1$, Part (ii). The necessity of the condition

$$\tau(f) \in \operatorname{Image}((\operatorname{Wh}(G_1) \oplus \operatorname{Wh}(G_2)) \to \operatorname{Wh}(G))$$
$$(\text{resp; } \operatorname{Image}(\operatorname{Wh}(J) \to \operatorname{Wh}(G)))$$

was proved in Chapter II. We defer to the end of the argument the proof that $\theta(f)=\bar{\eta}(u)$, u the Whitehead torsion of an h-cobordism, which must by part (i) exist, of W to a split manifold, depends only on f. Assuming this, clearly $\theta(f)=0$ is a necessary condition for splitting; if f splits, we could use the trivial h-cobordism to define $\theta(f)$.

Assume now that f satisfies $\tau(f) \in \operatorname{Image}(\operatorname{Wh}(G_1) \oplus \operatorname{Wh}(G_2) \to \operatorname{Wh}(G))$ (resp; $\operatorname{Wh}(J) \to \operatorname{Wh}(G)$). Then, as $\Phi(\tau(f))=0$, by part (i) of Theorem 2, there exists an h-cobordism of W to a split manifold. Letting u denote the torsion of this h-cobordism, as $\theta(f)=\bar{\eta}(u)$, from Lemma II.9 if $\theta(f)=0$, f is splittable.

It remains only to check that when $\tau(f) \in \operatorname{Image}(\operatorname{Wh}(G_1) \oplus \operatorname{Wh}(G_2) \to \operatorname{Wh}(G))$ (resp; $\operatorname{Wh}(J) \to \operatorname{Wh}(G)$), $\theta(f)$ is well-defined. Recall that from part (i), with this assumption on $\tau(f)$ there is an h-cobordism $(V_1, W, \mathring{W})$ of W to $\mathring{W}$, $\mathring{W} \xrightarrow{\mathring{f}} Y$ split and let v_1 denote the torsion of this h-cobordism. As $\mathring{f}$ is split,

$$\tau(\mathring{f}) \in \operatorname{Image}(\mathrm{Wh}(G_1) \oplus \mathrm{Wh}(G_2) \to \mathrm{Wh}(G)) \quad (\text{resp}; \mathrm{Wh}(J) \to \mathrm{Wh}(G)),$$

by a result of Chapter II. Hence $v_1 + (-1)^n v_1^* = \tau(\mathring{f}) - \tau(f)$ represents 0 in

$$\mathrm{Wh}(G)/(\mathrm{Wh}(G_1) + \mathrm{Wh}(G_2)) \quad (\text{resp}; \mathrm{Wh}(G)/\mathrm{Wh}(J))$$

and so $\eta(v_1)$ represents an element $\bar{\eta}(v_1)$ in $H^{n+1}(Z_2; \widetilde{\mathrm{Nil}}(H; G_1, G_2))$ (resp; $H^{n+1}(Z_2; \widetilde{\mathrm{Nil}}(H, J; \xi_1, \xi_2))$). Now, let $(V_2; W, \overset{*}{W})$ be another h-cobordism of W to a manifold $\overset{*}{W}$, $\overset{*}{W} \to Y$ split; denote its torsion by v_2. The proof is completed by showing $\bar{\eta}(v_1) = \bar{\eta}(v_2)$.

Using Lemma III.8, exactly as it is used in the first paragraph of the proof of Lemma III.9, we may assume without loss of generality that $\Phi(v_1) = 0$ and $\Phi(v_2) = 0$. But then $(V_1 \cup_W V_2; \mathring{W}, \overset{*}{W})$ is an h-cobordism with torsion q satisfying $\Phi(q) = 0$ and $\eta(q) = v_1 - v_2$. By the relative form of part (i) of Theorem 2, there is an h-cobordism of $V_1' \cup_W V_2'$ to a map *split along* $X \times I$, and so letting z denote the Whitehead torsion of this h-cobordism

$$q = z + (-1)^{n+1} z^* \text{ modulo } (\mathrm{Wh}(G_1) + \mathrm{Wh}(G_2)) \quad (\text{resp}; \mathrm{Wh}(J)).$$

Hence $v_1 - v_2$ represents the zero element of

$$H^{n+1}(Z_2; \widetilde{\mathrm{Nil}}(H; G_1, G_2)) \quad (\text{resp}; H^{n+1}(Z_2; \widetilde{\mathrm{Nil}}(H, J; \xi_1, \xi_2)).$$

Remark. If $H^{2k}(Z_2; \widetilde{\mathrm{Nil}}(H; G_1, G_2))$ (resp; $H^{2k}(Z_2; \widetilde{\mathrm{Nil}}(H, J; \xi_1, \xi_2))$ is not trivial, then $\theta(f)$ takes all values in this group. To realize an element in this group, just construct an h-cobordism, with torsion representing this element, on a split manifold.

Chapter IV: Another Splitting Theorem

§ 1. Another Result when $H = 0$

Theorem 3 will be proved in this section. Note first that for n odd or if $\pi_1(Y)$ has no 2-torsion, Theorem 3 is just a special case of part (ii) of Theorem 1. In case A this uses the fact that $\mathrm{Wh}(G_1) \oplus \mathrm{Wh}(G_2) \to \mathrm{Wh}(G_1 * G_2)$ is surjective [St]; in case B note that for $H = 0$, $J *_H \{t\} = J * Z$ and as $\mathrm{Wh}(Z) = 0$ [BHS], $\mathrm{Wh}(J) \to \mathrm{Wh}(J * Z)$ is surjective.

We give below a proof of Theorem 3 for $n = 2k$, $n \neq 4$. The extension to the case $n = 4$ follows from the method used in the proof of Theorem 5 in Chapter V to derive five-dimensional splitting results from the corresponding high-dimensional results. Note also that as a consequence of the results on Whitehead groups quoted above, it suffices to show that W is h-cobordant to a manifold $\mathring{W}$ for which the induced homotopy equivalence $\mathring{W} \to Y$ is splittable.

We return to the conclusion of Lemma II.1. Recall we may assume that as $n = 2k$ and $H = 0$, $\pi_1 M = 0$, $K_i(M) = 0$ $i < k$ and there is the splitting $K_k(M) =$

$P \oplus Q$, P and Q finitely generated projective $Z[H]$ modules. In the case under consideration here as $H=0$ P and Q are free finitely generated abelian groups. Assume that $K_k(M) \neq 0$. The proof will be completed by constructing a normal cobordism, with zero surgery obstruction, of W to a manifold $\mathring{W}$ with the induced homotopy equivalence $\mathring{f}\colon \mathring{W} \to Y$ transverse to $X \subset Y$ and with $\mathring{M} = \mathring{f}^{-1}(X)$ connected and simply-connected and with $K_i(\mathring{M})=0$ $i<k$, rank $(K_k(\mathring{M}))<$ rank $(K_k(M))$. Repeating this construction, we finally get a normal cobordism with zero surgery obstruction of W to a split homotopy-equivalent manifold and hence W is h-cobordant to a splitting.

To construct $\mathring{W}$, recall from Lemma I.8 that there is an upper-triangular filtration

$$P=P_0 \supset P_1 \supset \cdots \supset P_r = 0,$$
$$Q=Q_0 \supset Q_1 \supset \cdots \supset Q_r = 0.$$

Let s denote the largest number for which $P_s \oplus Q_s \neq 0$. Then clearly $\rho_1(P_s)=0$ and $\rho_2(Q_s)=0$. Thus, ρ_1 or ρ_2 has a non-zero kernel. We will assume that ρ_1 has a non-zero kernel; the argument of ρ_2 has a non-zero kernel is entirely similar. As the Image $(\rho_1) \subset Q \oplus_Z Z[G_1]$ (resp; $Q \oplus_Z Z[J] \oplus P \oplus_Z Z[tJ]$) is a subgroup of a free abelian group, it is also free. Hence Ker (ρ_1) is a direct summand of P and thus P contains a free direct summand of rank 1 generated by an element e with $\rho_1(e)=0$. From Lemma II.1 (ii) we may choose $f \in Q$ with $\lambda(e,f)=1$ for λ the intersection form of $K_k(M)$. The arguments of I § 5 show that we may represent e by $\partial\alpha$ where $\alpha\colon (D^{k+1}, S^k) \to (W_1, M)$ (resp; W', M_1)) is an immersion. As $Q \cong K_{k+1}(W_l, M)$, f may be represented by $\partial\beta$ for β an immersion $\beta\colon (D^{k+1}, S^k) \to (W, M)$ lifting to $\hat{\beta}\colon (D^{k+1}, S^k) \to (W_l, M) \subset (\hat{W}, M)$. From Lemma II.1 (ii) we may assume that $\partial\alpha$ and $\partial\beta$ are embedded spheres, with trivial normal bundles, which intersect in one point. The idea of using such pairs $\partial\alpha$ and $\partial\beta$ is due to Ronnie Lee in [L 1].

We complete the argument first for k odd, that is for (dimension $Y) \equiv 3$ (modulo 4). Let $M \times I \subset W$, $I=[-2,2]$ denote a neighborhood of $M \times 0 = M \subset W$ constructed so that if this inclusion is lifted to $\hat{W}$, $M \times -2 \subset W_l$, $M \times 2 \subset W_r$. (In case A this means just that $M \times -2 \subset W_2$, $M \times 2 \subset W_1$.) Let C_e denote the cobordism obtained by attaching a handle to $\partial\alpha \subset M$. Clearly the map $f!\colon M \to X$ extends to the normal cobordism $C_e \to X$ where $\partial C_e = M \cup M_e$. Similarly, let C_f denote the cobordism obtained by attaching a handle to $\partial\beta \subset M$; again, $f!$ extends to a normal cobordism $C_f \to X$ where $\partial C_f = M \cup M_f$. It is easy to see that M_f is connected and simply-connected, and as M_f was produced by surgery on a free summand of $K_k(M)$, for the induced map $M_f \to X$ $K_i(M_f)=0$ $i<k$, rank $(K_k(M_f))=$ rank $(K_k(M))-2$, [KM].

Now attach $C_f \times [1,2]$ to $W \times I'$, $I'=[0,1]$, along $M \times [1,2] \times 1 \subset W \times 1$ and attach $C_e \times [-2,-1]$ to $W \times I'$ along $M \times [-2,-1] \times 1 \subset W \times 1$. Call the resulting manifold

$$T = W \times I' \cup_{(M \times [-2,-1] \times 1 \cup M \times [1,2] \times 1)} (C_e \times [-2,-1] \cup C_f \times [1,2]).$$

By the normal cobordism extension lemma [B 2], the map f extends to a normal cobordism $F\colon T \to Y$, $\partial T = W \cup \mathring{W}$. We will show that $\mathring{W} \to Y$ is a homotopy equivalence and that T is a normal cobordism with zero surgery obstruction in $L^h_{n+2}(\pi_1(Y), w)$. Arguments similar to those employed in the proof of Lemma II.6, and easy to check directly in the present instance, show that $K_i(T)=0$ for $i<k+1$; also, $K_{k+1}(T)$ is a free $Z[\pi_1(Y)]$ module on two generators E, F, with as $\partial\alpha$ and $\partial\beta$ intersect in one-point and $\rho_1(e)=0$ and $\alpha\subset W_1$, $\lambda_T(E,E)=0$ and $\lambda_T(E,F)=1$, for λ_T the intersection pairing of $K_{k+1}(T)$. But then the Hermitian pairing λ_T is non-singular on $K_{k+1}(T)$ and hence [W 2; Chapter V] $\mathring{W}\to Y$ is a homotopy equivalence.

Now for dimension $Y\equiv 3$ (modulo 4), T has dimension a multiple of 4 and hence under the assumptions about the orientations of elements of order 2 in $\pi_1(Y)$ made in the hypothesis of Theorem 3, $\lambda_T(E,E)=0$ implies $\mu_T(E)=0$ for μ_T the self-intersection form of T. Hence, E generates a subkernel [$W2$; Chapter V] of $K_{k+1}(T)$ and thus the surgery obstruction of this normal cobordism is zero. Lastly, note that the induced homotopy equivalence $\mathring{f}\colon \mathring{W}\to Y$ can be made transverse to $X\subset Y$ with $\mathring{f}^{-1}(X)=M_f$. This completes the argument for k odd.

For the case (dimension $Y)\equiv 1$ (modulo 4), the above argument does not quite work because under the assumptions made about the orientations carried by elements of order 2 in the hypothesis of Theorem 3, $\lambda_T(E,E)=0$ only implies that $\mu_T(E)=0$ or is represented by $\mu_T(E)=1\in Z[\pi_1(Y)]$. Of course, if $\mu_T(E)=0$. The argument is completed as above. If $\mu_T(E)=1$ we indicate a slightly different construction of a normal cobordism T'. Using the method of [W 2; p. 54] vary the embedding of $\partial\alpha\times(-3/2)$ in $M\times[-2,-1]$ to obtain a regular homotopy with a single self-intersection point in $M\times[-2,-1]\times I''$, $I''=[0,1]$, of $\partial\alpha\times(-3/2)$ to a sphere γ embedded in $M\times[-2,-1]\times 1$. Now define D as the manifold obtained by attaching a handle to

$$\gamma\subset M\times[-2,-1]\times I''$$

and set

$$T'=W\cup_{(M\times[-2,-1]\cup M\times[1,2])}D\cup C_f\times[1,2].$$

Clearly $K_i(T')=0$ $i<k+1$, $K_{k+1}(T')$ is generated by E', F' where $\lambda_{T'}(E',E')=0$, $\lambda_{T'}(E',F')=1$ and $\mu_{T'}(E')=1+\mu_T(E)$ in $Z[\pi_1 Y]/\{V+(-1)^k\bar{V}\}$. Thus, either T' or T has zero surgery obstruction.

§ 2. A Remark on $H=0$

Even when the hypothesis on 2-torsion of Theorem 3 is not satisfied, if $H=0$ the method of IV § 1 still provides some useful information for $n=2k$. The constructions described there show that whenever $H=0$, $n=2k$, W is normally cobordant to a split homotopy equivalence by a normal cobordism whose surgery obstruction is in the subgroup of $L^h_{2k+2}(G,w)$ generated by Hermitian forms (V,λ,μ) where

(i) V is a free $Z[G]$ module on 2-generators E, F,

(ii) $\lambda(E,F)=1$, $\lambda(E,E)=0$.

In particular, such elements are easily seen to have order 2 in $L^h_{2k+2}(G, w)$. Thus even when the hypothesis of Theorem 3 is not satisfied, for $H=0$ and $n=2k>4$ there is always a normal cobordism, with surgery obstruction of order 2, of W to a split homotopy-equivalent manifold.

Chapter V: Five-Dimensional Splitting

§ 1. P.1. and Differentiable Splitting for $n=4$

In this section, we prove Theorem 5 in the P.L. and differentiable case. The extension to the topological case is proved in § 2.

Recall the method used in the proof of Theorems 5.1 and 4.1 of [CSl] to derive 5-dimensional stable[16] splitting theorems from high-dimensional splitting theorems. In the proofs of 5.1 and 4.1 of [CS 1] the stabilization procedure was employed at two points in the argument. We show here that under the hypothesis of Theorem 5 in both cases in [CS 1] in which the stabilization procedure is used it can be avoided.

Stabilization is used the first time in the stable-splitting results of [CS 1] to get around the difficulty that it is not in general known f a normal map with zero surgery obstruction to a 4-dimensional Poincaré-complex X is normally cobordant to a homotopy equivalence. But, by [S 3] when $\pi_1(X)=0$ and in general by [W 2; § 16], this is always the case if X has the homotopy type of a closed P.L. 4-manifold and $H_2(\pi_1(X); Z_2)=0$.

Stabilization is used a second time in the proof of the stable splitting result of [CS 1] to get around the difficulty that it is not in general known if given a homotopy equivalence f: $M^4 \to X$, M a closed 4-dimensional manifold, there is a normal cobordism of f realizing a given element of $L^h_5(\pi_1(X), w)$. However, this problem does not arise if $L^h_5(\pi_1(X), w)=0$. More generally, if $[\Sigma M; G/PL] \to L^h_5(\pi_1(X), w)$ is surjective, every element of $L^h_5(\pi_1(X), w)$ is realized by a normal cobordism of f to itself.

Note that if H is a finite group of odd order, $H_2(H; Z_2)=0$ and $L^h_5(H)=0$ [B4]. This completes the proof of our 5-dimensional splitting theorem in the P.L. or differentiable case.

The proof of our h-splitting result in dimension 5 is very similar to the above argument for 5-dimensional splitting and we only give an outline of the argument. Note first that if $\bar{\phi}(\tau(f))=0$, Lemma II.4 shows that W is h-cobordant to a manifold $\mathring{W}$ for which the induced homotopy equivalence $\mathring{f}$: $\mathring{W} \to Y$ satisfies $\phi(\tau(\mathring{f}))=0$. Thus, we may assume without loss of generality that $\phi(\tau(f))=0$. In that case, the proof of our 5-dimensional h-splitting result proceeds with the same adaptation of the methods of [CS 1] indicated above but with the exact sequence (5.2) of [CS 1] replaced by that of Theorem 8 of [C 3] and with the exact sequence of p. 525 of [CS 1] replaced by that of Theorem 9 of [C 3].

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[16] In stable splitting problems we permit X to be replaced by the manifold $X \# k(S^2 \times S^2)$ obtained by performing trivial ambient surgeries on $X \subset Y$. The reader should be warned that this is called S-splitting in [CS 1], not to be confused with the s-splitting problem of the present paper

§ 2. Topological Splitting for $n=4$

The proof of Theorem 5 in the topological case follows that given above in the P.L. or smooth cases. Note first that the arguments of [W 2; Chapter 16] are easily modified to cover the topological case with G/Top [KS] replacing G/PL.

However, the argument employed in [CS 1, p. 517–519] uses transversality in showing that (W, f) is normally cobordant, after stabilizing X, to a normal map $g\colon Q \to Y$ with g transverse to $X \subset Y$ with $g^{-1}(X) \to X$ a homotopy equivalence. (Here g need not be a homotopy equivalence.) As topological transversality is not known in this dimension, we give, under the hypothesis of Theorem 5, a different construction of g without stabilizing X. First observe that the homotopy equivalence $f\colon W \to Y$ induces a lift of the Spivak normal bundle $\nu_Y\colon Y \to BG$ to a map $\xi\colon Y \to B\,\mathrm{Top}$ [B 2] [KS].

$$\begin{tikzcd} & B\,\mathrm{Top} \arrow[d, "v"] \\ Y \arrow[ur, "\xi"] \arrow[r, "\nu_Y"'] & BG \end{tikzcd}$$

The composite map $\nu_Y j\colon X \to BG$, $j\colon X \to Y$ the inclusion, is easily seen to be the Spivak normal bundle for X and hence ξj is a lift to $B\,\mathrm{Top}$ of the spivak normal bundle of X. By the topological analogue of [W 2; Chapter 16] if X has the homotopy type of a closed 4 dimensional topological manifold and if $H_2(\pi_1(X), Z_2)=0$, this lift of the Spivak normal bundle of X is, homotopic, by a homotopy $h_t\colon X \to B\,\mathrm{Top}$, $t \in [0,1]$, $v h_t j = \nu_y j$, to a map $h_1\colon X \to B\,\mathrm{Top}$ induced from a homotopy equivalence $h\colon M \to X$, M a closed 4-dimensional topological manifold. By the homotopy extension principle, h_t can be extended to a homotopy $H_t\colon \to B\,\mathrm{Top}$ with $v H_t = \nu_y$. Then as H_1 extends h_1, using 5-dimensional topological transversality [KS] the normal map $h\colon M \to X$ extends to a normal map $g\colon Q \to Y$. From the construction (Q, g) and (W, f) represent homotopic lifts of ν_y to $B\,\mathrm{Top}$ and hence are, suing 6-dimensional topological transversality [KS], normally cobordant.

The remainder of the argument follows the adaptation of [CS 1] indicated in V § 1.

Chapter VI: Some Remarks on $G = Z \times_\alpha H$

The Farrell-Hsiang splitting theorem, which is part (ii) of Corollary 6 for $n \neq 4$, is a special case of Theorem 2. This is immediate for $n=2k$. For $n=2k-1$, we also need that for $\xi_i\colon H \to J$ isomorphisms, $i=1,2$, so that $G = Z \times_\alpha H$, $\alpha = \xi_2^{-1}\xi_1$, there is a decomposition

$$\widetilde{\mathrm{Nil}}(H, J; \xi_1, \xi_2) = \widetilde{\mathrm{Nil}}(H, \alpha) \oplus \widetilde{\mathrm{Nil}}(H, \alpha)$$

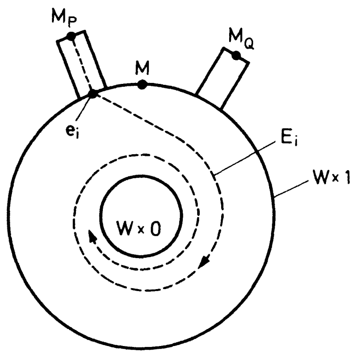
with the involution switching both copies of $\widetilde{\mathrm{Nil}}(H, \alpha)$. This is proved in [FH 2]; it also follows easily from the general description of $\widetilde{\mathrm{Nil}}(H, J; \xi_1, \xi_2)$ of [W 1] and the methods of Lemma II.3 and III.7. From this formula,

$$H^n(Z_2;\ \widetilde{\mathrm{Nil}}(H, J;\ \xi_1, \xi_2))=0$$

for all $n>1$ when ξ_1 and ξ_2 are isomorphisms and thus in this case the invariant $\theta(f)$ takes values in the zero-group.

For $n=4$, Corollary 6 is essentially a special case of Theorem 5.

Actually, the nilpotent normal cobordism construction of II § 1 could be used to obtain another very quick proof of Farrell-Hsiang splitting for $n=2k$, avoiding all the algebra of II § 2 and the explicit computation of the intersection form (λ) in Lemma II.6. A basis for the summand $P\otimes_{ZH}Z[G]$ of $K_{k+1}(T)$ is represented by spheres winding around T in the same direction and hence their mutual intersections are zero. See the diagram below. By exercising greater care in the construction and using the fact that H contains all elements of order 2 in $Z\times_\alpha H$, or alternatively adapting the method of handling self-intersections in the last part of IV § 1, we could also arrange for the self-intersections to vanish.



Construction of E_i (see Lemma II.6) in the Farrell-Hsiang case

For $n=2k-1$, note that when $\xi_1\colon H\to J$ is an isomorphism, the surgeries used to construct $M_P\subset\hat{W}$ can be performed in W. Thus, in that case, if $\Phi(\tau(f))=0$ may assume that $K_i(M)=0$ for $i<k-1$ and $K_{k-1}(M)$ is a free module. This idea is used in [FH2].

Actually, in the general splitting problem for $n=2k-1>3$, $H\subset G_i$ (resp; $\xi_i\colon H\to J$ injective), if $\Phi(\tau(f))=0$ and $K_i(M)=0$, $i<k-1$ and $K_{k-1}(M)$ is a free module it is not hard to see that, even if H is not square closed in $G=G_1 *_H G_2$ (resp; $G=J *_H \{t\}$), W is h-cobordant to a split manifold. Let C_P and C_Q be as in Chapter III. Then, attaching $C_P\times I$ and $C_Q\times I$ to $W\times 1\subset W\times[0,1]$, we obtain a normal cobordism of W to $\overset{*}{W}$; here $\overset{*}{W}$ is as in the proof of Lemma III.6. Recall the construction of the split homotopy equivalence $\mathring{W}\to Y$ in the proof of Lemma III.6 by performing surgery on a basis for $K_k(M_P)\oplus K_k(M_Q)\subset K_k(\overset{*}{W})$. Thus, we get a normal cobordism T of W to $\mathring{W}$ where from the Mayer-Vietoris sequence of

the decomposition up to homotopy of T,

$$T=(W_1 \cup_M C_Q \cup \text{handles attached to a basis of } K_k(M_Q))$$
$$\cup_M (W_2 \cup_M C_P \cup \text{handles attached to a basis of } K_k(M_P))$$
$$(\text{resp; } T=(W' \cup_{M_1} C_Q \cup_{M_2} C_P) \cup (\text{handles attached to a basis of}$$
$$K_k(M_P) \oplus K_k(M_Q))/\text{identify } M_1 \text{ with } M_2$$

and Lemma III.1, it follows that $K_i(T)=0$, $i<k$ and $K_k(T) \cong K_{k-1}(M; ZG)$. Then, as $K_k(T)$ is a free $Z[G]$ module and T is of dimension $2k+1$, surgery can be performed on T to replace it by an h-cobordism of W to a split manifold.

Note that in the above 3-paragraphs on the Farrell-Hsiang splitting theorem, we actually only used $\xi_1 \colon H \to J$ an isomorphism and $\xi_2 \colon H \to J$ *injective*. This slight extension of Farrell-Hsiang splitting is not always covered by Theorem 2; i.e. the conclusion of Theorems 1 and 2 is valid whenever $\xi_1 \colon H \to J$ is an isomorphism and $\xi_2 \colon H \to J$ is injective, even if $\xi_2(H)$ is not square-root closed in J. This can be applied to compute the surgery groups of some groups containing infinitely divisible elements.

Example. Let G be the group generated by α and β with the one relation $\alpha\beta\alpha^{-1}=\alpha^P$, $p \neq 0$. Then $G=J *_H \{t\}$ where $J \cong Z$, $H \cong Z$, ξ_1 is just 1_Z and ξ_2 is multiplication by p. When $\pi_1(Y)=G$, $\pi_1(X)=H$, dimension $X \geqq 5$, the Farrell-Hsiang splitting theorem applies only for $p=\pm 1$ and Theorem 2 gives a splitting theorem for p odd. The above remark, however, gives a splitting theorem for all p. Note that when $p \neq \pm 1$, α is infinitely divisible in G.

Chapter VII: Square Root Closed Subgroups

Of the seven examples presented in the introduction of squareroot closed subgroups, only (3), (4) and (6) are not immediately obvious. The present chapter verifies these three examples.

The following example is used in computing the surgery groups of all the fundamental groups of 2-manifolds, and of many three-manifolds, by using splitting theorems.

Proposition VII.1. *Let G be a free group and H a subgroup of G generated by a non-square element of G. Then H is square-root closed in G.*

Proof. Let h generate H. If $g \in G$ with $g^2 \in H$, then the subgroup H' generated by $\{h, g\}$ is, as it is a subgroup of G, a free-group, and since its abelianization is a finite extension of H, H' is infinite cyclic. But as $H \subset H'$ is a subgroup of index 1 or 2, either $H=H'$ and hence $g \in H$, or h is a square in H'.

Of course, it is trivial to check if a given word in a free group is a square element.

Example. Let M be a connected 2-dimensional manifold, with M not RP^2, $S^1 \times S^1$, the Moebius band or the Klein bottle. Then from Prop. VII.1, $\pi_1(M)=F_1 *_H F_2$ where F_1, F_2 are free groups and $H=0$ or Z is square-root closed in F_1 and F_2 and hence, by Prop. VII.2 below, is square-root closed in $\pi_1(M)$.

Proposition VII.2. *Let ξ_i: $H \to G_i$* (resp; *ξ_i: $H \to J$) $i=1,2$ be inclusions of groups. Then H is square-root closed in $G_1 *_H G_2$* (resp; *in $J *_H \{t\}$) if and only if $\xi_i(H)$ is square-root closed in G_i* (resp; *J) $i=1,2$.*

Proof. Trivially, if H is square-root closed in G, then $\xi_i(H)$ is square-root closed in G_i (resp; J) $i=1,2$. The other implication is not hard to show using covering space theory or group theory. The following argument's only virtue is that it avoids introducing further notation.

Assume $\xi_i(H)$ is square-root closed in G_1 (resp; J) $i=1,2$. Recall that this is essentially all that is used in proving Lemma II.8. Moreover, from the proof of Lemma II.8, for each $g \in \{G-H\}$, $G=G_1 *_H G_2$ (resp; $J *_H \{t\}$) is an element of $Z[w]$ for some $w \in F^0$. Also, as $Z[w]$ is a $Z[H]$ bimodule, if $g \in Z[w]$, $HgH \subset Z[w]$. But then from the first paragraph of the proof of Lemma II.9, if $g \in Z[w]$, $g^{-1} \notin Z[w]$. In particular, for all $g \in \{G-H\}$, $HgH \cap Hg^{-1}H = \emptyset$ and hence $g^2 \in H$ implies $g \in H$.

Appendix I: The Z_2 Action on $\mathrm{Wh}(G_1 *_H G_2)$ and $\mathrm{Wh}(J *_H \{t\})$

The splitting theorem of the present paper has implications for the calculation of Whitehead groups. In many geometric applications of Whitehead groups and in computing surgery groups, it suffices to compute just the "symmetries modulo norms", i.e. $H^*(Z_2; Wh(G))$. We analyze here the contribution to this group coming from the group of nilpotet maps of [W1].

Proposition A.1. *Let H and G_1, G_2* (resp; *J) be finitely presented groups with ξ_i: $H \to G_i$* (resp; *ξ_i: $H \to J$) monomorphisms, $i=1,2$, and ω_i: $G_i \to Z_2$* (resp; *ω: $J \to Z_2$, ω_Z: $Z \to Z_2$) homomorphisms, $i=1,2$, with $\xi_1\omega_1 = \xi_2\omega_2$* (resp; *$\omega\xi_1 = \omega\xi_2$) determining* (see Lemma III.7 *involutions on $\widetilde{\mathrm{Nil}}(H; G_1, G_2)$* (resp; *$\widetilde{\mathrm{Nil}}(H, J; \xi_1, \xi_2)$). Then if $\xi_i(H)$ is square-root closed in G_i* (resp; *J) $i=1,2$,*

$$H^{2k+1}(Z_2; \widetilde{\mathrm{Nil}}(H; G_1, G_2)) = 0, \qquad k>0.$$

(resp; $H^{2k+1}(Z_2; \widetilde{\mathrm{Nil}}(H, J; \xi_1, \xi_2)) = 0$).

Conjecture. Under the same hypothesis, the even cohomology of Z_2 with coefficients in $\widetilde{\mathrm{Nil}}$ should also vanish.

We briefly outline the proof of this proposition.

Proof of *Proposition.* Recall that Theorem 2(ii) for n odd was formally derived from Theorem 2(i) and that in the process we obtained an obstruction to splitting in $H^{2k}(Z_2; \widetilde{\mathrm{Nil}}(H; G_1, G_2))$ (resp; $H^{2k}(Z_2; \widetilde{\mathrm{Nil}}(H, J; \xi_1, \xi_2))$) and we observed that every element of this group was in fact realized in this manner. An entirely parallel argument shows that we could define and realize an obstruction to splitting in $H^{2k+1}(Z_2; \widetilde{\mathrm{Nil}}(H; G_1, G_2))$ (resp; $H^{2k+1}(Z_2; \widetilde{\mathrm{Nil}}(H, J; \xi_1, \xi_2))$. As no such obstruction to splitting for n even arose in the proof of Theorem 2, we conclude that this group is zero.

The above proposition is particularly useful for relating surgery groups defined relative to subgroups of the Whitehead group.

Corollary A 2. *Let H, square-root closed in G_1, G_2* (resp; J), *be as in Prop. A.1. Then if*

$$H^{2k+1}(Z_2, Wh(G_1) \underset{Wh(H)}{\oplus} Wh(G_2))$$

and

$$H^{2k}(Z_2; \operatorname{Ker}(\tilde{K}_0(H) \to \tilde{K}_0(G_1) \oplus \tilde{K}_0(G_2)) = 0$$
$$(\text{resp}; H^{2k+1}(Z_2; Wh(J)/((\xi_{1_*} - \xi_{2_*}) Wh(H))) = 0$$

and

$$H^{2k}(Z_2; \operatorname{Ker}(\tilde{K}_0(H) \xrightarrow{\xi_{1*} - \xi_{2*}} \tilde{K}_0(J))) = 0), \qquad k > 0,$$

then

$$H^{2k+1}(Z_2; Wh(G_1 *_H G_2)) = 0$$
$$(\text{resp}; H^{2k+1}(Z_2; Wh(J *_H \{t\})) = 0).$$

Proof. This result is immediate from the description of $Wh(G)$, $G = G_1 *_H G_2$ or $G = J *_H \{t\}$ of [W 1], Prop. A.1. above, and Lemma II.3.

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