Localization for Exact Categories

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Abstract. This paper extends the notion of localization to exact categories and provides a homotopy fiber sequence which relates the K-theory of the category and its localization.

Contents

Introduction

In this work, it is given a new proof to the Mixed Localization Theorem by Levine, [**Lev83**][Appendix], and hence, as a special case, to Quillen's Localization Theorem for abelian categories, [**Qui72**][pp. 113-116]. In the proof given by Levine the calculations are carried out using the kind of techniques already used by Quillen for his theorem. This forces, somehow, the existence of an ambient abelian category inside of which computations make sense. In our context this is not absolutely necessary and most of the proofs are done in a abelian-free enviroment. Moreover, the algebraic calculations are given in terms of Waldhausen's description of K-theory instead of the Q-construction. This would allow in the future to state the Main Theorem, 7.0.63 on page 51, in a more general setting.

Chapter 1 describes the basic results on simplicial sets and homotopy theory on nerves of small categories necessary for Chapter 2.

In Chapter 2 Waldhausen's version of K-theory is developed. It is based on the concept of of category with cofibrations and weak equivalences. This chapter also lists the most used theorems that allow to compare the K-theories of related categories.

In Chapter 3 it is introduced the concept of exact category. Exact categories are treated under a dual point of view: Bass' as subcategories of abelian categories and Quillen's as additive categories satisfying some extra axioms. Though our preferred point of view is Quillen's, Bass' one is very useful at the time of computations.

The localization axioms are described in Chapter 4. The axioms are described in terms of the categories involved in the localization, $\mathcal{A} \subseteq \mathcal{U}$. In some sense, they are intrinsic. Then it is defined the exact category $\mathcal{A}^{-1}\mathcal{U}$ as the calculus of fractions, two sided!, on $\mathcal{U}$ by the class of morphisms in $\mathcal{U}$ which 'have' kernel and cokernel belonging to $\mathcal{A}$, whenever they exist. Calculus of fractions is briefly described on appendix A. Some other technical results about the behavior of morphisms in the localized category are given in this chapter. At the end of this chapter 4 a universal property for the localizing functor is shown.

In Chapter 5, we construct the exact category of finite chain complexes on another given exact category. It is shown it has a natural structure as category with cofibrations and weak equivalences.

Chapter 6 contains an slightly improved version of [**TT90**, Theorem 1.11.7] which identify the K-theories of an exact category and of the exact category of its finite chain complexes. This improvement has already appeared in [**CP97**, Proposition 6.1].

The Main Theorem is set in chapter 7. Given exact categories $\mathcal{A} \subseteq \mathcal{U}$ satisfying the localization axioms, 4.0.35, then there exists an exact category $\mathcal{A}^{-1}\mathcal{U}$, the localized category, and a homotopy fibration

$$K(\mathcal{A}) \to K(\mathcal{U}) \to K(\mathcal{A}^{-1}\mathcal{U}) \ .$$

At the end, we describe a three step program for the proof of the theorem. Step ii) is already done in Chapter 5.

Steps i) and iii) are done in Chapter 8. The proof of i) is a simple application of the approximation theorem, 2.0.25. This allows to identify the K-theory of $\mathcal{A}^{-1}\mathcal{U}$ with the K-theory of $C(\mathcal{U})$, which is equivalent to that of $\mathcal{U}$, with a new set of weak equivalences. These new set of weak equivalences is the reflected class of quasi-isomorphisms in $C(\mathcal{A}^{-1}\mathcal{U})$. At this point, the generic fibration lemma, 2.0.23, can be applied to

$$qC(\mathcal{U}) \to \overline{q}C(\mathcal{U})$$

obtaining $qC(\mathcal{U})^{\overline{q}}$ as homotopy fiber. Step iii) identifies in K-theory terms $qC(\mathcal{U})^{\overline{q}}$ and $qC(\mathcal{A})$. This is done in two ways. The first one consists on taking a chain complex in $qC(\mathcal{U})^{\overline{q}}$ and 'deformed' it into one in $qC(\mathcal{A})$. Using repeatedly the additivity theorem, 2.0.20, and results on domination of chain complexes by [**CP95**] or [**Ran92**] show the result.

The second one is based on Thomason's proof of 6. We notice the simple fact that the Euler characteristic of a chain complex in $qC(\mathcal{U})^{\overline{q}}$, chain complexes with homology in $\mathcal{A}$, must lie in $\mathcal{A}$ like for those in $C(\mathcal{A})$. This allows us to show that $qC(\mathcal{A})$ is the homotopy cofiber of $iC(\mathcal{U})^{q} \to iC(\mathcal{U})^{\overline{q}}$, which on the other was $qC(\mathcal{U})^{\overline{q}}$.

Finally Chapter 9 shows the equivalence between the Mixed Localization Theorem ,9.0.2, and the Main Theorem, 7.0.63. Also, as a special case, implies Quillen's Localization Theorem, 9.0.6.

Appendix B describes the Gabriel-Quillen embedding i of an exact category $\mathcal{E}$ into the abelian category $\mathcal{A}b(\mathcal{E})$ of left exact functors $F : \mathcal{E}^{op} \to \mathbb{Z}-$modules given by $i(E) = \mathrm{Hom}_{\mathbb{Z}}(\quad, E)$. This result allows the double point of view of exact categories.

On appendix C are developed the techniques which permit to replace an exact category $\mathcal{E}$ by a slightly larger one $\overline{\mathcal{E}}$ with some advantages and no consequences at the K-theory level. We define the idempotent completion $\mathcal{E}^{\wedge}$ of the exact category $\mathcal{E}$ and its restricted version $\overline{\mathcal{E}}$, the full sub category of $\mathcal{E}^{\wedge}$ with objects having its stable isomorphism class in $K_0(\mathcal{E})$. $\overline{\mathcal{E}}$ has the advantage of satisfying property C.0.24 and still being k-theoretically equivalent to $\mathcal{E}$. This fact is used very often all along the text.

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M.

Part 1

Foundations

CHAPTER 1

Simplicial Sets

Let Δ be the category of finite ordered sets. The objects are $[n] = \{0 < 1 < \ldots < n\}$ for each n in $\mathbb{N} \cup \{0\}$ and morphisms are monotone maps $\alpha : [q] \longrightarrow [n]$ with $0 \leq i \leq j \leq q$ and $\alpha(i) \leq \alpha(j)$. We have the following basic monotone maps:

1. monotone monic maps: $\delta^i_n : [n-1] \longrightarrow [n]$ given by $\delta^i_n(j) = \begin{cases} j & j < i \\ j+1 & j \geq i \end{cases}$
2. monotone epic maps: $\delta^i_n : [n+1] \longrightarrow [n]$ given by $\delta^i_n(j) = \begin{cases} j & j \leq i \\ j-1 & j > i \end{cases}$

with the relations

$$\begin{array}{llllll} \delta^j\delta^i = \delta^i\delta^{j-1} & i < j \ ; & \sigma^j\delta^i = \delta^i\sigma^{j-1} & i < j \ ; & \sigma^j\delta^i = \delta^{i+1}\sigma^j & i > j+1 \\ \sigma^j\sigma^i = \sigma^i\sigma^{j+1} & i \leq j \ ; & & & \sigma^i\delta^i = 1 = \sigma^i\delta^{i+1}. & \end{array}$$

In fact, these basic monotone maps are the building blocks of any monotone map.

Theorem 1.0.1. *Every monotone map function $\alpha : [q] \longrightarrow [n]$ is composite:*

$$\alpha = \delta^{i_1} \cdots \delta^{i_s} \sigma^{j_1} \cdots \sigma^{j_t}$$

where $i_1 > i_2 > \cdots > i_s$ represents the elements not in the image of α and $j_1 < \cdots < j_t$ represents the repetitions $\alpha(j_k) = \alpha(j_{k+1})$.

Denote by $\nabla(n)$ the standard affine simplex given by the convex hull of $A_i = \{0, \ldots, 0, \overset{i)}{1}, 0, \ldots, 0\}$ for $i = 0, \ldots, n$ in $\mathbb{R}^{n+1} \subseteq \mathbb{R}^\infty$. Let $\overset{\circ}{\nabla}(n) = \{\sum t_i A_i / 1 \neq t_i \neq 0 \text{ and } \sum t_i = 1\} \subset \nabla(n)$. Given any monotone map $\alpha : [q] \longrightarrow [n]$, it can be realized as $|\alpha| : \nabla(q) \longrightarrow \nabla(n)$ in the obvious way, such that $|\alpha\beta| = |\alpha||\beta|$.

The next two results will be helpful on future calculations.

Lemma 1.0.2. $\forall x \in \nabla(n)$ *there is exactly one $q \leq n$ and only one $u \in \overset{\circ}{\nabla}(q)$ and one injective monotone map $[q] \longrightarrow [n]$ such that $x = |\alpha|(u)$.*

Lemma 1.0.3. $\alpha : [q] \longrightarrow [n]$ *is determined by the value of $|\alpha|$ on just one $x \in \overset{\circ}{\nabla}(n)$.*

Definition 1.0.4. A simplicial set X is a functor

$$X : \Delta^{\text{op}} \longrightarrow \text{Sets}$$

from the category Δ^{op} which is the opposite of the category of finite ordered sets. Alternatively, we can see X as a collection sets $\{X_n\}_{n=0}^\infty$, where $X_n = X([n])$ is called the set of n-simplices of X, with structure maps

$$d_i = X(\delta_i) : X_n \longrightarrow X_{n-1} \quad i = o, \ldots, n+1 \quad \text{faces,}$$
$$s_i = X(\sigma_i) : X_n \longrightarrow X_{n+1} \quad i = o, \ldots, n \quad \text{degeneracies}$$

satisfying

$$d_i d_j = d_{j-1} d_i \qquad \text{for } i < j$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{for } i < j \\ 1 & \text{for } i = j, j+1 \\ s_j d_{i-1} & \text{for } i > j+1 \end{cases}$$

$$s_i s_j = s_{j+1} s_i \qquad \text{for } i \le j$$

We call a simplex x degeneracy if it can be written as $x = s_i y$ for some i. If $x = d_i y$, it is called a face.

Hence, given a monotone map $\alpha : [q] \longrightarrow [n]$ there is an induced $X(\alpha) : X_n \longrightarrow X_q$. In general, for $\alpha \in \text{Mor}(\Delta^{op})$ we shall write $\alpha^\star$ for $X(\alpha) : X_. \to X_.$ specifying no dimensions.

A simplicial map $f : X \longrightarrow Y$ is then a natural transformation between two functors X and Y, or alternatively, a simplicial map is a collection of functions $f = \{f_i : X_i \longrightarrow Y_i\}$ commuting with the d_i and the s_i.

We will say A is a simplicial subset of X, denoted by $A \subset X$, if $A([n]) \subseteq X([n])$ for all n and $\forall \alpha \in \text{Mor}(\Delta^{op})$ $\alpha^\star(A) \subseteq A$, i.e. $d_i(A_n) \subseteq A_{n-1}$ and $s_i(A_n) \subseteq A_{n-1}$ for all n. A simplicial set L is a subsimplicial set of X if for each $n \in \mathbb{N}$ $L_n = L([n]) \subset X_n = X([n])$ and L is closed under d_i and s_i.

If $\Sigma \subset X$ is a simplicial subset, we define $Span(\Sigma) = \bigcup_{\alpha \in \text{Mor}(\Delta^{op})} \alpha^\star(\Sigma)$. In other words, $Span(\Sigma)$ is the simplicial set formed by all the faces and degeneracies able to be obtained out of Σ. The n-skeleton of X is $X^n = Span(X_n)$. If $X = X^n$ we say $dim X \le n$.

Moreover, we can characterize the simplices.

LEMMA 1.0.5. *If x and y are degenerate then $[dx = dy \Longrightarrow x = y]$.*

THEOREM 1.0.6. *Every simplex in a simplicial set X is canonically $x = \beta y$ where β is epic and y is nondegenerate.*

REMARK 1.0.7. Iterating the process described above, define simplicial simplicial sets as functors $X : \Delta^{op} \to SimplicialSets$, where $SimplicialSets$ is the category of simplicial sets. Theses are called bisimplicial sets, since they can be identified with functors $X : \Delta^{op} \times \Delta^{op} \to Sets$. Via the diagonal map $\Delta^{op} \to \Delta^{op} \times \Delta^{op}$ we can associate to the bisimplicial sets a simplicial set, the diagonal simplicial set with $(diag\ X_{..})_n = X_{nn}$. Similarly multi-simplicial sets can be defined.

1.1. Some basic simplicial sets

Let us see some examples of simplicial sets. One way to produce simplicial sets is the following, by products.

DEFINITION 1.1.1. Given simplicial sets X and Y we shall say $X \times Y$ is the simplicial set with n-simplices $(X \times Y)_n = X_n \times Y_n$ and faces and degeneracies defined as follows:

$$d_i(x, y) = (d_i x, d_i y) \text{ and } s_i(x, y) = (s_i x, s_i y)$$

There is still another way to produce simplicial sets.

DEFINITION 1.1.2. An abstract simplicial set K is an ordered finite set $\{A_0, \ldots, A_n\}$, the vertices of K, certain subsets of it and all the subsets of these, the simplices of K. We associate to K the simplicial set ΣK with $\Sigma K([n]) = (\Sigma K)_n$ the set of those monotone maps from $[n]$ to $\{A_0, \ldots, A_n\}$ such that the vertices in the image lie on some simplex in K.

Next it is the well-known prototype of simplex.

DEFINITION 1.1.3. For each $n \in \mathbb{N}$, the standard n-simplex $\Delta([n])$ is the simplicial set defined by $\Delta([n])_q = \{0 \leq a_0 \leq \cdots \leq a_q \leq n; a_i \in \mathbb{Z}^+\}$ with $d_i(a_0, \ldots, a_q) = (a_0, \ldots, \hat{a_i}, \ldots, a_q)$ and $s_i(a_0, \ldots, a_q) = (a_0, \ldots, a_{i-1}, a_i, a_i, a_{i+1}, \ldots, a_q)$. In other words, $\Delta([n]) = \Sigma((0, 1, \ldots, n))$, where $(0, 1, \ldots, n) \in \Delta([n])_n$ is the abstract ordered n-simplex, which we will denote by $[n]$ as well. We can also say $\Delta([n]) = Span([n])$.

But also $\Delta([n])$ can be described as $\mathrm{Hom}_\Delta(-, [n]) : \Delta^{op} \to Sets$. Then by the Yoneda Lemma there is a natural bijection between the simplicial maps $\Delta([n]) \to Y$ and Y_n, the set of n-simplices of Y. In particular, for each monotone map $\alpha : [q] \to [n]$ corresponds a unique q-simplex in $\Delta([n])$, namely $\alpha([q])$.

$\Delta([n])$ contains the simplicial subset boundary of $\Delta([n])$, $\dot{\Delta}([n])$. The boundary is just $(\Delta([n]))^{n-1}$, the $(n-1)$-skeleton of $\Delta([n])$. It also contains the i-th horn for each $i = 0, \ldots, n$ $\dot{\Lambda}([n]) = Span(d_0([n]), d_1([n]), \ldots, \hat{d_i}([n]), \ldots, d_n([n]))$. An specially important simplicial set is $\Delta([1])$ which we denote by I. The following is an useful notation for the simplices in the product of a simplicial set by I:

$$(\Delta([n]) \times I)_q \ni \left((a_0, \ldots, a_q), \underbrace{(0, 0, \ldots, 0, \overset{i)}{1}, \ldots, 1)}_{q} \right) := (a_0, a_1, \ldots, a_i, a'_{i+1}, \ldots, a'_q)$$

where d_i would mean to drop the i-th term and s_i to repeat the i-th term.

Other basic simplicial sets are those given by the singular functor.

DEFINITION 1.1.4. The singular functor is

$$S. : \mathrm{Top} \longrightarrow \mathrm{Simplicial\ Sets}$$
$$X \longmapsto S.X$$

where Top is the category of topological spaces and continuous maps, $SimplicialSets$ is the category of simplicial sets and simplicial maps. $S.X$ is defined by $S.X([n]) = S_nX = \mathrm{Hom}_{Top}(\nabla(n), X)$. It is verified that if $Y \subset X$, then $S.Y \subset S.X$, $S.(X \cup Y) = (S.X \cup S.Y)$ and $S.(X \times Y) = S.X \times S.Y$.

1.2. Homotopy

DEFINITION 1.2.1. A simplicial map $H : X \times I \longrightarrow Y$ is called a simplicial homotopy between $F = i_0X$ and $G = i_1Y$. We shall say $F \xrightarrow{\sim} G$, is simplicially homotopic to G. Unfortunately this is not an equivalence relation (unless Y is a Kan set, see remark bellow 1.2.2). Following Peter May's,[**May67**], description H can be seen as a collection of functions

$$\forall q \quad h_q = \{h_i : X_q \longrightarrow Y_{q+1}\} \;\; 0 \leq i \leq q$$

satisfying:

(i) $d_0 h_0 = F$, $d_{q+1} h_q = G$,

(ii)

$$\begin{aligned} d_i h_j &= h_{j-1} d_i && i < j \\ d_{j+1} h_{j+1} &= d_{j+1} h_j && \\ d_i h_j &= h_j d_{i-1} && i > j+1 \end{aligned}$$

(iii)

$$\begin{aligned} s_i h_j &= h_{j+1} s_i && i \leq j \\ s_i h_j &= h_j s_{i+1} && i > j \end{aligned}$$

REMARK 1.2.2. X is a Kan set if it satisfies the Kan condition:

$$\forall f \text{ simplicial map } \forall q \ \forall i \ \exists \widetilde{f} \ \text{ such that } \quad \begin{array}{ccc} \Lambda^i[q] & \xrightarrow{f} & X \\ \big\downarrow & \nearrow_{\widetilde{f}} & \\ \Delta([q]) & & \end{array} .$$

Given (A, X) and (B, Y) pairs of simplicial sets:

(i) when Y is Kan then homotopy $X \longrightarrow Y$ is an equivalence relation,

(ii) $(X, A) \longrightarrow (Y, B)$ with Y and B being Kan, then homotopy is an equivalence relation.

1.3. Geometric Realization Functor

Given X a simplicial set $X_0, X_1, \ldots$, endowed with the discrete topology:

$$\overline{X} = X_0 \times \nabla(0) \sqcup X_1 \times \nabla(1) \sqcup \cdots \sqcup X_n \times \nabla(n) \sqcup \ldots$$

with the disjoint union topology,

U is open in $\overline{X}$ if and only if $U \cap \{x_q\} \times \nabla(q)$ is open for any $x_q \in X_q$ $\forall q$.

Given $(x, t) \in \overline{X}$ with $x \in X_q$, $t \in \nabla(q)$ and α a monotone map, declare $(\alpha^\star, t) \sim (x, |\alpha| t)$ related. This relation generates an equivalence relation $\simeq$. (x, t) is said to be regular if x is non-degenerate and $t \in \overset{\circ}{\nabla}(q)$. The geometric realization of the simplicial set X is the topological space:

$$|X| := \overline{X} / \simeq \; = \coprod X_n \times \nabla(n) / (\alpha^\star x, t) \simeq (x, |\alpha|(t)).$$

Given $x \in X_q$, the characteristic map of x is

$$\begin{aligned} \chi_x : \nabla(q) &\longrightarrow |X| \\ t &\longmapsto (x, t). \end{aligned}$$

The topology on $|X|$ happens to be the finest topology making all χ_x continuous. In fact, it is enough to check the continuity on generating sets since $\chi_{\alpha^\star(x)} = \chi_x \circ |\alpha|$.

LEMMA 1.3.1. *If Σ generates X, then the realization of X*

$$|X| = \bigcup_{y \in \Sigma} \chi_y(\nabla(dim\ y))$$

THEOREM 1.3.2. *If $\overset{\circ}{x} = \{|x, t| \in |X| \ |t \in \overset{\circ}{\nabla}(q)\} \subseteq X$ then $|X| = \bigcup_{x \ non\text{-}deg} \overset{\circ}{x}$ as sets.*

PROOF. For the proof it is enough to check the existence of a set map $\Phi : \overline{X} \to \overline{X}$ satisfying:

1. $\forall(x,t)$, $\Phi(x,t)$ is regular,
2. if (x,t) is regular then $\Phi(x,t)=(x,t)$,
3. $\Phi(x,t)\sim(x,t)$ and
4. if $(x,t)\sim(y,s)$ then $\Phi(x,t)=\Phi(y,s)$.

Given (x,t), $t=|\alpha|(u)$ where α is $1-1$ and u is interior, then $(x,t)\sim(\alpha^\star,u)$. There exists β onto and y non-degenerate such that $\alpha^\star x=\beta^\star y$. We can define $\Phi(x,t)=(y,|\beta|u)$. $|\beta|u$ is interior, since u is interior and β is epi. Property 1 and 2 are trivially satisfied.

To check properties 3 and 4 it is enough to see that $\Phi(\gamma^\star x,t)=\Phi(x,|\gamma|t)$. Let $t=|\alpha|u$ with α $1-1$ and u interior. By lemma 1.0.2, $\alpha^\star\gamma^\star x=\beta_1^\star y_1$ where β_1 is onto and y_1 non-degenerate. Then $(y_1,|\beta_1|u)=\Phi(\gamma^\star x,t)$. By theorem 1.0.1, $\gamma\alpha=\delta\sigma$ where σ is onto and δ is $1-1$. Now, $|\gamma|(t)=|\gamma\alpha|(u)=|\delta\sigma|(u)=|\delta||\sigma|(u)$ hence $|\sigma|(u)$ is an interior point. We can write $\delta^\star x=\beta_2^\star y_2$ with β_2 onto and y_2 non-degenerate, then $\Phi(x,|\gamma|t)=y_2,|\beta_2|(|\sigma|(u)))$. We have $\sigma^\star(\beta_2^\star y_2)=\sigma^\star\delta^\star x=\alpha^\star\gamma^\star x=\beta_1^\star y_1$; γ, β_2, β_1 are surjective, y_1, y_2 are interior points. Hence $\beta_2\sigma=\beta_1$ and $y_1=y_2$ because there is a unique way to write a simplex as an epimorphism and a non-degenerate simplex. We are done. □

A simplicial map $f:X\to Y$ induce an obvious morphism $\overline{f}:\overline{X}\to\overline{Y}$, which respects the equivalence relation $\simeq$ and hence inducing a map $|f|:|X|\to|Y|$. Actually, $|..|$ is a functor,the geometric realization functor from *SimplicialSets* to *Top*. See 1.3.9 bellow for further details.

Following Goerss and Jardine's book, [**GJ**], there is a quick way to define the realization functor. Regard the simplex category $\Delta\downarrow X$ of a simplicial set X. The objects in $\Delta\downarrow X$ are maps $\sigma:\Delta([n])\to X$, or simplices of X, 1.1.3. Morphisms in $\Delta\downarrow X$ are commutative diagrams of simplicial maps

$$\begin{array}{ccc}\Delta([n]) & \xrightarrow{\ \theta\ } & \Delta([m]) \\ & {\scriptstyle\beta}\searrow \quad \swarrow{\scriptstyle\alpha} & \\ & X & \end{array}.$$

Recall 1.0.2 and 1.0.3, θ is induced by a monotone map $\theta:[m]\to[n]$.

Lemma 1.3.3 ([**GJ**],chapter I, Lemma 2.1). *There is an isomorphism*

$$X\simeq\varinjlim_{\substack{\Delta([n])\to X\\ in\Delta\downarrow X}}\Delta([n]).$$

Then the realization is defined as the following colimit

$$|X|=\varinjlim_{\substack{\Delta([n])\to X\\ \text{in}\Delta\downarrow X}}|\Delta([n])|$$

in the category Top. $|X|$ is functorial, since any simplicial map $f:X\to Y$ induces an obvious functor $f_\star:\Delta\downarrow X\to\Delta\downarrow Y$. Moreover,

Proposition 1.3.4 ([**GJ**],chapter I, Proposition 2.2). *The realization functor is left adjoint to the singular functor in the sense that there is an isomorphism*

$$\mathrm{Hom}_{Top}(|X|,Y)\cong\mathrm{Hom}_{SiSets}(X,S.Y)$$

which is natural in simplicial sets X and topological spaces Y.

Next we relate some properties of the geometric realization functor. First,

THEOREM 1.3.5. *$\chi_{[n]} : \nabla(n) \to |\Delta([n])|$ is a homeomorphism.*

PROOF.

1. $\chi_{[n]}$ is injective.
   Let $\chi_{[n]}(t_1) = \chi_{[n]}(t_2)$, $t_i = |\alpha_i|(u_i)$ where α_i is epic and u_i is interior. Now $\chi_{[n]}(t_i) = |[n], t_i| = |[[n], |\alpha_i|(u_i)| = |\alpha_i^\star[n], u - i||$, but it is a regular representation hence $u_1 = u_2$ and $\alpha_1^\star([n]) = \alpha_2^\star([n])$. That means $\alpha_1 = \alpha_2$ and therefore $t_1 = t_2$.
2. χ is continuous by definition.
3. χ is epic since $\Delta(n)$ is generated by $[n]$.
4. $|\Delta(n)|$ is Hausdorff and compact.

$\chi_{[n]}$ is closed hence it is an homeomorphism. □

From now on, we can identify $\nabla(n)$ and $|\Delta(n)|$. Then if $\chi_x : \delta(n) \to |X|$ is the characteristic map for $x \in X$

$$\begin{aligned} |\chi_x| : \nabla(n) = |\Delta(n)| &\longrightarrow |X| \\ t &\longmapsto |(x, t)|. \end{aligned}$$

It is also verified that

THEOREM 1.3.6. *$f : X \to Y$ is $1-1$ if and only if $|f| : |X| \to |Y|$ is $1-1$.*

PROOF.
$|x_1, t_1|$, $|x_2, t_2| \in |X|$ x_i non-degenerates and t_i interior points.

$|f(x_1), t_1| = |f|(|x_1, t_1)| = |f||(x_2, t_2)| = |f(x_2), t_2|$. f is $1-1$, hence $f(x_i)$ are non-degenerate since x_i is non-degenerate.

$f(x_i) = s_j y$ implies $f(d_j x_i) = d_j f(x_i) = y$ and hence $f(s_j d_j x_i) = s_j y = f(x_i)$. Being f $1-1$ implies that $x_i = s_j d_j x_i$.

$\Leftarrow$

$|f|$ is $1-1$. $f(x) = f(y)$; $t \in \overset{\circ}{\nabla}$; $|f|(|x, t|) = |f(x), t| = |f(y), t| = |f|(|y, t|)$ would imply $|x, t| = |y, t|$. Let $x = \beta^\star z$ and $y = \gamma^\star z'$ with β and γ monotone. On the other hand, $\overset{\circ}{z}{}' \ni \chi_{z'}(|\gamma, t|) = |y, t| = |x, t| = \chi_z(|\beta|, t) \in \overset{\circ}{z}$ but $|Y|$ is a disjoint union, hence $z = z'$. Since χ_z is a homeomorphism $|\beta|t = |\gamma|t$ with t interior which implies $\beta = \gamma$. Then $x = y$. □

PROPOSITION 1.3.7 ([**FP90**] Proposition 4.3). *If A is a simplicial subset of X, then $|A| \subset |X|$ is a closed subset.*

PROOF. Let $U \subseteq |A|$ closed. Regard the characteristic map for x

$$\begin{aligned} \chi_x : \nabla(n) &\longrightarrow |X| \\ (x, t) &\longmapsto |x, t| = |y, t| \end{aligned}$$

with $y \in A$. Then for α monic, $x = \alpha^\star y$, $t = |\alpha| s$. This means that $\chi_x^{-1}(U) = \bigcup_{\alpha^\star x \in A} |\alpha|(\chi_{\alpha^\star x}^{-1}(U))$. This union is finite. So it is closed iff each $|\alpha|(\chi_{\alpha^\star x}^{-1}(U))$ is closed, which is true. □

Moreover,

PROPOSITION 1.3.8 ([**GJ**],chapter I, Proposition 2.3). *$|X|$ is a CW-complex for each simplicial set X.*

PROOF. We will apply a result by a JHC Whitehead. We need $\emptyset \neq S_0, \dots, S_n$ where $S_0 = \{P^0, P^1, \dots, P^n\}$ with the discrete topology; let P^{n-1} and $S_n \neq \emptyset$.

$$\begin{array}{ccc} S_n \times \overset{\bullet}{\nabla}(n) & \longrightarrow & P^{n-1} \\ \big\downarrow & \text{p.o.} & \big\downarrow \\ S_n \times \nabla(n) & \longrightarrow & P^n \end{array}$$

Let S_n be the set of non-generate n-simplices in X. Then $|X^0| = S_0$ clearly (having the discrete topology).

Let's assume that $|X^{n-1}| = |X|^{n-1}$. For every non-degenerate $x \in X_n$, we can look at

$$\phi_x : \overset{\bullet}{\nabla}(n) \longrightarrow |X^{n-1}| = |X|^{n-1}.$$

We have $h : S_n \times \nabla(n) \overset{\bullet}{\bigcup} |X^{n-1}| \twoheadrightarrow |X^n|$, h factors through $|X|^n$ clearly.

$$\begin{array}{ccc} h : S_n \times \nabla(n) \overset{\bullet}{\bigcup} |X^{n-1}| & \xrightarrow{continuous} & |X^n| \\ {\scriptstyle quotient}\ \big\downarrow {\scriptstyle p} & \nearrow {\scriptstyle continuous; onto 1-1} & \\ |X|^n & & \end{array}$$

We need to see it is open.

$U \subseteq |X|^n$ open; it is enough to see that $\chi_x^{-1}(U)$ for x non-degenerate n-simplex. $\chi_x^{-1}(U) = \{x\} \times \nabla(n) \cap p^{-1}(U)$ which is open. $|X|$ has the union topology. $|X| = X_n \times \nabla(n)/\sim$ but the topology is given by the characteristic maps. This implies that $|X^n| \hookrightarrow |X|$ is continuous. ☐

REMARK 1.3.9. In particular $|X|$ is a compactly generated Hausdorff space, and so the realization functor takes values in the category $CGHauss$ of all such. We shall interpret $|..|$ as a functor from $SimplicialSets$ to $CGHauss$.

In this category we have the following result.

THEOREM 1.3.10. *Let X and Y be simplicial sets and $p_i : X \times Y \longrightarrow X, Y$ the corresponding projections. Then*

$$p = |p_1| \times |p_2| : |X \times Y| \longrightarrow |X| \times_k |Y|$$

is a homeomorphism.

In this way, we avoid problems, since $|X| \times |Y|$ is not homeomorphic to $|X \times Y|$ in general.

REMARK 1.3.11. $|X| \times_k |Y|$ is the finest topology making all the maps $\nabla(h) \times \nabla(l) \xrightarrow{\chi_x \times \chi_y} |X| \times |Y|$ continuous.

LEMMA 1.3.12. *For $X = \Delta(n)$, $Y = \Delta(q)$ the theorem above is true.*

1.4. Nerves

Definition 1.4.1. Given a small category $\mathcal{C}$, define nerve of $\mathcal{C}$, or classifying space, as the simplicial set $N.\mathcal{C} : \Delta^{op} \to Sets$, where $(N\mathcal{C})_n$ is the set of functors $[n] \to \mathcal{C}$, i.e. an element of $(N\mathcal{C})_n$ is a sequence of n composable morphisms in $\mathcal{C}$.

$$N_q\mathcal{A} = \{A_0 \to A_1 \to \cdots \to A_q\}$$

The degeneracies d_i are given by composition and the faces s_i by inserting identities. This simplicial set is called the nerve of $\mathcal{C}$, or classifying space, and its geometric realization is denoted by $B\mathcal{C}$. In the literature $N.\mathcal{C}$ is also denoted by $B\mathcal{C}$, see [**GJ**].

1.4.2. It is clear that a functor $F : \mathcal{A} \to \mathcal{B}$ between small categories induces a simplicial map $N.F : N.\mathcal{A} \to N.\mathcal{B}$ between the nerves. Moreover, a natural transformation $t : F \to G$ induces a simplicial homotopy $Nt : N.\mathcal{A} \times I \to N.\mathcal{B}$ by regarding the ladder

$$\begin{array}{ccccccc}
G(A_0) & \longrightarrow & G(A_1) & \longrightarrow & \cdots & \longrightarrow & G(A_q) \\
\uparrow{\scriptstyle t(A_0)} & & \uparrow{\scriptstyle t(A_1)} & & & & \uparrow{\scriptstyle t(A_q)} \\
F(A_0) & \longrightarrow & F(A_1) & \longrightarrow & \cdots & \longrightarrow & F(A_q)
\end{array}$$

for each q-simplex and hence,

$$Nt(A_0 \to \cdots \to A_q, (0, \ldots, \overset{i}{\overbrace{01}}, \ldots, 1)) =$$

$$\begin{array}{ccccccccc}
 & & & & & G(A_{i-1}) & & & \\
 & & & & \nearrow & & \searrow & & \\
= (F(A_0) \longrightarrow F(A_1) & \longrightarrow & \cdots & \longrightarrow & F(A_{i-1}) & \longrightarrow & G(A_i) & \longrightarrow \cdots \longrightarrow & G(A_q)) \\
 & & & & \searrow & & \nearrow & & \\
 & & & & & F(A_i) & & &
\end{array}$$

which is truly a simplicial homotopy. There has been then introduced certain idea of homotopy in the category of small categories. We can say $\mathcal{C}$ is contractible if the identity functor on $N.\mathcal{A}$ is homotopic to a constant functor.

From the discussion above, it is clear that $N.(\mathcal{A} \times \mathcal{B}) = N.\mathcal{A} \times N.\mathcal{B}$ and hence $B(\mathcal{A} \times \mathcal{B}) \simeq B(\mathcal{A}) \times B(\mathcal{B})$ is a homeomorphism.

Example 1.4.3. If G is a group, let EG be the nerve of the category formed by an object for each $g \in G$ and a map for each pair of non-equal elements. EG is contractible since there is a natural transformation from the identity functor $id : EG \to EG$ to the trivial functor, which sends every morphism to the identity. This natural transformation induces a homotopy between the identity map on $|EG|$ and a contraction. Now let BG be the nerve of the category having a sole object $\star$ and an endomorphism $g : \star \to \star$ for each element g of G. Its geometric realization $|BG|$, generally denoted by BG as well, is an Eilenberg-Mac Lane space of the form $K(G, 1)$. There is an obvious simplicial map $\pi : EG \to BG$ induced by a functor.

Example 1.4.4. If $\mathcal{A}$ is a small category having either an initial or a final object then $N.\mathcal{A}$ is contractible.

Assume, for example, $\mathcal{A}$ has a final object $\star$, then $id_{\mathcal{A}}$ is homotopic to

$$\begin{array}{rl} c_\star : \mathcal{A} \longrightarrow \mathcal{A} & \\ A \longmapsto c_\star(A) & \star \end{array}$$

For each q, there are ladders

$$\begin{array}{ccccccc} \star & = & \star & = & \cdots & = & \star \\ \uparrow & & \uparrow & & & & \uparrow \\ A_0 & \longrightarrow & A_1 & \longrightarrow & \cdots & \longrightarrow & A_q \end{array}$$

which define a simplicial homotopy $H : N.\mathcal{A} \times I \to N.\mathcal{A}$. This homotopy could be seen as the collection:

$$\forall q \quad h_i : N_q\mathcal{A} \longrightarrow N_{q+1}\mathcal{A} \qquad i = 0, \ldots, q$$

$$\begin{array}{l} h_0 : A_0 \longrightarrow \star = \star = \cdots = \star = \star \\ h_1 : A_0 \longrightarrow A_1 \longrightarrow \star = \cdots = \star = \star \\ \vdots \\ h_q : A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_q \longrightarrow \star \end{array}$$

clearly $d_0h_0 = c_\star$ and $d_{q+1}h_q = id_{\mathcal{A}}$. Similarly for $\mathcal{A}$ having an initial object.

1.5. Notions of Homotopy Theory

As we saw in 1.2, we cannot reflect our concepts about homotopy from Top to $SiSets$ completely. On the other hand, we still have that the composed functors $|S. - |$ and $S.|..|$ preserve homotopies, homotopy equivalences and contractibility, see corollary 4.3.19 in [**FP90**], and that the geometric realization functor commutes with products. This is enough for introducing a notion of homotopy. Hence here is a bit more of terminology.

Definition 1.5.1. A simplicial map $f : X \to X'$ is called a weak homotopy equivalence if $|f|$ is a honest homotopy equivalence. A bisimplicial map $X.. \to Y..$, in general a multi-simplicial map, is a weak homotopy equivalence whenever $(diag\, X..) \to (diag\, Y..)$. is, recall 1.0.7. A simplicial set is said weakly contractible if its geometric realization is contractible.

1.5.2. As we did before, we can transfer our homotopy ideas further towards the categorical level. Two functors, F and G, will be said homotopic if $|N.F| \simeq |N.G|$, F will be a homotopy equivalence if $|N.F|$ is a homotopy equivalence and $\mathcal{A}$ will be said contractible if $|N.\mathcal{A}|$ is contractible.

Lemma 1.5.3 (Realization Lemma in [**Wal78**]). *Let $X_{..} \to Y_{..}$ be a map of bisimplicial sets. Suppose that for every n, the map $X_{.n} \to Y_{.n}$ is a homotopy equivalence. Then $X_{..} \to Y_{..}$ is a homotopy equivalence.*

Definition 1.5.4. We say a map is constant if it factors through a terminal object.

Definition 1.5.5. A sequence of maps of topological spaces $A \to B \to C$ is called a fibration up to homotopy if the composed map $A \to C$ is constant, and the resulting map from A to the homotopy theoretic fiber of $B \to C$ is a homotopy equivalence.

A sequence of multi-simplicial sets will be called a fibration up to homotopy if the sequence of geometric realizations is.

1.5.6. Now, we can extend the notions of homotopy theory to the category of small categories.

A functor $F : \mathcal{C} \to \mathcal{C}'$ between small categories will be called a homotopy equivalence whenever $N.F : N.\mathcal{C} \to N.\mathcal{C}'$ is a weak homotopy equivalence. A natural transformation of F, induces a functor $\mathcal{C} \times [1] \to \mathcal{C}'$ and ,as we saw in 1.4.2, this one induces a simplicial homotopy $H : N.(\mathcal{C} \times [1]) = N.\mathcal{C} \times N.[1] \to N.\mathcal{C}'$. Clearly $N.[1] = I$ and $|N.\mathcal{C} \times I| \cong |N.\mathcal{C}| \times |I| = |N.\mathcal{C}| \times I$. Therefore we end up with a homotopy $|H| : |N.\mathcal{C}| \times I \to |N.\mathcal{C}'|$ in Top. In particular F is an equivalence of categories of of it has an adjoint, it is a homotopy equivalence. Using this chain of constructions the ideas of homotopy at the category level, the small one, fit perfectly with the regular notions of homotopy theory for topological spaces.

1.5.7. Next we give some criteria to detect homotopy equivalences and fibrations up to homotopy. Further details in [**Wal78**].

Let $F : \mathcal{C} \to \mathcal{C}'$ be a map of small categories and X' an object in $\mathcal{C}'$.

a) Denote by $F|X'$, the left fiber of F over X', the category whose objects are the pairs (X, x) with X an object in $\mathcal{C}$ and $x : F(X) \to X'$ is a morphism in $\mathcal{C}'$; and whose morphisms from (X, x) to (Y, y) is a map $f : X \to Y$ such that $x = y \circ F(y)$.
b) A morphism $m : X' \to Y'$ in $\mathcal{C}'$ induces a functor $F|m : F|X' \to F|Y'$.
c) Dually it can be defined the right fiber of F over X'.

THEOREM 1.5.8 (Theorem A). *Let $F : \mathcal{C} \to \mathcal{C}'$ be a map of small categories. Suppose that for every X' object in $\mathcal{C}'$ the category $F|X'$ is contractible. Then F is a homotopy equivalence.*

THEOREM 1.5.9 (Theorem A by Quillen). *Let $F : \mathcal{C} \to \mathcal{C}'$ a map of small categories and X' an object in $\mathcal{C}'$. Let $F|X$ be the fiber of F over X', i.e. the category with objects (X, x), $F(X) \overset{x}{\to} X'$, and morphisms $f : X \to Y$ such that*

$$\begin{array}{ccc} F(X) & \xrightarrow{x} & X' \\ {\scriptstyle F(f)}\downarrow & \nearrow_{y} & \\ F(Y) & & \end{array} .$$

If for every object X' in $\mathcal{C}'$, $F|X'$ is contractible then F is a homotopy equivalence.

LEMMA 1.5.10. *Let $X_{..} \to Y_{..} \to Z_{..}$ be a sequence of bisimplicial sets so that $X_{..} \to Z_{..}$ is constant. Suppose that $X_{.n} \to Y_{.n} \to Z_{.n}$ is a fibration up to homotopy, for every n. Suppose further that $Z_{.n}$ is connected, for every n. Then $X_{..} \to Y_{..} \to Z_{..}$ is a fibration up to homotopy.*

THEOREM 1.5.11 (Theorem B). *Let $F : \mathcal{C} \to \mathcal{C}'$ be a map of small categories. Suppose that for every morphism $m : X' \to Y'$ in $\mathcal{C}'$, the map $F|m : F|X' \to F|Y'$ is a homotopy equivalence. Then for every X' object in $\mathcal{C}'$, the square*

$$\begin{array}{ccc} F|X' & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow{\scriptstyle F} \\ id_{\mathcal{C}}|X' & \longrightarrow & \mathcal{C}' \end{array}$$

is homotopy Cartesian. (If $\mathcal{C}'$ is connected, then $B(F|X') \simeq hom fiber(B\mathcal{C} \xrightarrow{F} B\mathcal{C}'))$. Dually, left fibers can be replaced by right fibers.

Remark 1.5.12. A commutative diagram of topological spaces

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{\sigma} & D \end{array}$$

is homotopy Cartesian if the map from A to the homotopy theoretic fiber product $C \times_D D^I \times_D B$ is a homotopy equivalence, whereas D^I denotes the space of maps from I to D.

CHAPTER 2

Categories with cofibrations and weak equivalences

In this section we present a quick review of Waldhausens K-theory of a small category with cofibrations and weak equivalences [**Wal85**]. One example to keep in mind is an additive category where the cofibrations are inclusions of direct summands up to isomorphism, and the weak equivalences are the isomorphisms. Another example is the category of finite chain complexes in an additive category with cofibrations the degreewise inclusions of direct summands and weak equivalences the homotopy equivalences. If, in this example, we take the weak equivalences to be the isomorphisms, we get an example of an exact category (since exact sequences are only degreewise split exact). In addition we recall the basic tools which will allow us to decide when two categories have isomorphic K-theory.

Given any small category $\mathcal{C}$ with some extra structure described below, Waldhausen assigns functorially to $\mathcal{C}$ a topological space $K(\mathcal{C})$, which we call K-theory of $\mathcal{C}$. The homotopy groups are defined to be the K-groups of $\mathcal{C}$. This extends the classical definitions of K-groups of a ring R by taking $\mathcal{C}$ to be the additive category of finitely generated projective modules over R, with cofibrations inclusions of direct summands, and weak equivalences isomorphisms, see below for the meaning of these terms.

Definition 2.0.13. [**Wal85**, Sections 1.1 and 1.2] A small category $\mathcal{C}$ with a zero object is said to be a category with cofibrations and weak equivalences if it has two distinguished subcategories, $\text{co}\,\mathcal{C}$ and $w\mathcal{C}$, satisfying the following axioms:
a) *co*$\mathcal{C}$ axioms.

cof 1: Isomorphisms in $\mathcal{C}$ are cofibrations.
cof 2: For every $A \in \mathcal{C}$, $* \to A$ is a cofibration.
cof 3: Cofibrations admit cobase change:
- **a:** If $A \to B$ is a cofibration and $A \to C$ any map, then the push out exists in $\mathcal{C}$.
- **b:** $C \to C \bigcup_A B$ is a cofibration.

b) $w\mathcal{C}$ axioms.

weq 1: Isomorphisms in $\mathcal{C}$ are weak equivalences.
weq 2: (Gluing Lemma) If in the commutative diagram

$$\begin{array}{ccccc} B & \longleftarrow & A & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ B' & \longleftarrow & A' & \longrightarrow & C' \end{array}$$

the horizontal arrows on the left are cofibrations and all three vertical arrows are in $w\mathcal{C}$ then

$$B\bigcup_A C \to B'\bigcup_A C$$

is in $w\mathcal{C}$.

The two following axioms may, or may not, be satisfied by $\mathcal{C}$.

Saturation axiom: If a, b are composable maps in $\mathcal{C}$ and if two of a, b, ab are in $w\mathcal{C}$ then so is the third.

Extension axiom: Let

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & B/A \\ \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & B'/A' \end{array}$$

be a map of cofibration sequences ($B/A = * \bigcup_A B$). If $A \to A'$ and $B/A \to B'/A'$ are in $w\mathcal{C}$ then $B \to B'$ is in $w\mathcal{C}$ as well.

Having fixed $co\mathcal{C}$ and $w\mathcal{C}$, we have a simplicial category:

$$\begin{aligned} S.\mathcal{C} : \Delta^{op} &\longrightarrow (cat) \\ [n] &\longmapsto S_n\mathcal{C} \end{aligned}$$

where $S_n\mathcal{C}$ is the category with objects composable cofibrations:

$$* \to A_1 \to A_2 \to \cdots \to A_n$$

with chosen quotients $A_{i,j} = A_i/A_j$, $1 \leq i \leq j \leq n$. We always have $A_{i,i} = *$. The morphisms in the category $S_n\mathcal{C}$ are maps $A_i \to B_i$ commuting with the cofibration sequences. $S.\mathcal{C}$ is a simplicial category as follows: the degeneracy maps are given by inserting identities, and the boundary maps d_i by omitting the index i, for $1 \leq i \leq n$. If d_0 were given by extending this recipe and omitting $*$, then the construction would give the nerve of the category $\mathrm{co}\,\mathcal{C}$, which is contractible since there is an initial object. Instead, d_0 prescribes taking all the quotients by A_1, hence the necessity for including a choice of quotients from the beginning. The category $S_n\mathcal{C}$ is a category with cofibrations and weak equivalences, by defining a map $A \to A'$ to be a cofibration if

$$A_j \longrightarrow A'_j \quad \text{and} \quad A'_j \bigcup_{A_j} A_{j+1} \longrightarrow A'_{j+1}$$

are cofibrations in $\mathcal{C}$ for all j. An arrow $A \to A'$ is defined to be a weak equivalence if the arrow $A_{i,j} \longrightarrow A'_{i,j}$ is a weak equivalence for each pair $i \leq j$. We thus have that $S.$ is a functor from categories with cofibrations and weak equivalences to simplicial categories with cofibrations and weak equivalences. For more details about this see sections 1.1, 1.2 and 1.3 in [**Wal85**].

We can think of:

$$\begin{aligned} wS.\mathcal{C} : \Delta^{op} &\longrightarrow (cat) \\ [n] &\longmapsto wS_n\mathcal{C} \end{aligned}$$

as a bisimplicial set by taking the nerve of $wS_n\mathcal{C}$

Definition 2.0.14. [**Wal85**, Section 1.3] The *Algebraic K-theory* of the category with cofibrations $\mathcal{C}$, with respect to the category of weak equivalences $w\mathcal{C}$ is given by the pointed space

$$K(\mathcal{C}) = \Omega|wS.\mathcal{C}|.$$

The K-groups of $\mathcal{C}$ are the homotopy groups of $K(\mathcal{C})$

$$K_*\mathcal{C} = \pi_*(\Omega|wS.\mathcal{C}|)\ (= \pi_{*+1}|wS.\mathcal{C}|).$$

Actually K-theory can be described as a spectrum rather than just a space. The $S.$-construction extends namely, by naturality, to simplicial categories with cofibrations and weak equivalences. In particular it thus applies to $S.\mathcal{C}$ to produce a bisimplicial category with cofibrations and weak equivalences, $S.S.\mathcal{C} = S.^{(2)}\mathcal{C}$. Again the construction extends to bisimplicial categories with cofibrations and weak equivalences and so on. Therefore we get a spectrum whose n'th space is $|wS.^{(n)}\mathcal{C}|$ The structural maps are defined as the adjoint of the map $\Sigma|w\mathcal{C}| \to |wS.\mathcal{C}|$ which is given as the inclusion of the 1-skeleton in the $S.$-construction, see [**Wal85**, page 329].

It turns out that this spectrum is an Ω-spectrum beyond the first term (the additivity theorem 2.0.20 below is needed to prove this). As the spectrum is connective (the n-th term is (n-1)-connected) an equivalent assertion is that in the sequence

$$|w\mathcal{C}| \to \Omega|wS.\mathcal{C}| \to \Omega^2|wS.S.\mathcal{C}| \to \cdots$$

all maps except the first are homotopy equivalences. Hence K-theory of $\mathcal{C}$ could equivalently be defined as the infinite loop space

$$\Omega^\infty|wS.^{(\infty)}\mathcal{C}| = \varinjlim_n \Omega^n|wS.^{(n)}\mathcal{C}|$$

We will refer to any of the three versions as the K-theory of $\mathcal{C}$ and denote it as $K(\mathcal{C})$. If it is necessary to emphasize the category of weak equivalences $w\mathcal{C}$ used to define the K-theory of $\mathcal{C}$, we will write $K(w\mathcal{C})$ instead of $K(\mathcal{C})$, by a slight abuse of notation.

Now we recall criteria that determine when two categories have homotopy equivalent K-theories. Some extra structure is required on the category. It is necessary to have a notion of cylinder in order to define some kind of homotopy theory.

Definition 2.0.15. [**Wal85**, section 1.1] A functor $F : C \to C'$, between categories with cofibrations and weak equivalences is said to be *exact* if F preserves all relevant structures. Such a functor induces in a natural way a map

$$wS.F : wS.\mathcal{C} \to wS.\mathcal{C}'$$

and therefore a map between the K-theories.

2.0.16. The properties of the product and the realization functor ensure that, given a map (simplicial homotopy)

$$H : X \times I \to Y$$

where X and Y are simplicial sets, there is an induced homotopy

$$H : |X| \times I \to |Y|$$

between $|F| = |H_{|X|\times\{0\}}|$ and $|G| = |H_{|X|\times\{1\}}|$. This applies, in particular, to our case when X and Y are the $S.$-constructions of categories $\mathcal{C}$ and $\mathcal{C}'$.

Therefore we have a notion of homotopy between functors. To see more about this we refer the reader to [**Wal78**, Section 5, Notions of homotopy theory].

Definition 2.0.17. [**Wal85**, Section 1.6] A category $\mathcal{C}$ with cofibrations and weak equivalences has a cylinder functor if there is a functor

$$T : \operatorname{Ar}\mathcal{C} \to \operatorname{Diag}\mathcal{C}$$

where $\operatorname{Ar}\mathcal{C}$ is the category of arrows of $\mathcal{C}$ and $\operatorname{Diag}\mathcal{C}$ is the category of diagrams in $\mathcal{C}$.

$$T(f : A \to B) \quad \equiv \quad \begin{array}{ccccc} A & \xrightarrow{i_1} & T(f) & \xleftarrow{i_2} & B \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle id} & \\ & & B & & \end{array}$$

satisfying:

Cyl 1: Front and back inclusion assemble to an exact functor

$$\begin{aligned} \operatorname{Ar}\mathcal{C} &\to F_1\mathcal{C} \\ (f : A \to B) &\to (A \vee B \rightarrowtail T(f)) \end{aligned}$$

where $F_1\mathcal{C}$ is the full subcategory of $\operatorname{Ar}\mathcal{C}$ whose objects are the cofibrations in $\mathcal{C}$.

Cyl 2: $T(* \to A) = A$, for every $A \in C$ and projection and back inclusion are the identity on A.

There is an additional axiom that is often satisfied:

Cylinder axiom: The projection $T(f) \to B$ is in $w\mathcal{C}$ for every $f : A \to B$.

Definition 2.0.18. [**Wal85**, section 1.3] A *cofibration sequence* of exact functors $\mathcal{C} \to \mathcal{C}'$ is a sequence of natural transformations $F' \to F \to F''$ having the property that for every $A \in \mathcal{C}$ $F'(A) \to F(A) \to F''(A)$ is a cofibration sequence in $\mathcal{C}'$.

One of the basic tools is the additivity theorem [**Wal85**, Theorem 1.4.2 and Proposition 1.3.2], see also [**McC93**]. To state it we need a definition.

Definition 2.0.19. [**Wal85**, section 1.1] Given a category with cofibrations and weak equivalences $\mathcal{C}$ and subcategories with cofibrations and weak equivalences $\mathcal{A}$ and $\mathcal{B}$, we define the extension category $E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ to be the category with objects cofibrations $A \to C \to B$ in $\mathcal{C}$ where A is an object of $\mathcal{A}$, B an object of $\mathcal{B}$ and C an object of $\mathcal{C}$. This is a category with cofibrations and weak equivalences in an obvious manner as a subcategory of $S_2\mathcal{C}$. We shall denote $E(\mathcal{C}, \mathcal{C}, \mathcal{C})$ as $E(\mathcal{C})$.

We can now state the additivity Theorem

Theorem 2.0.20. *The maps*

$$|wS.F| \text{ and } |wS.(F' \vee F'')| \tag{2.2.0.20.1}$$

are homotopic.

This statement is equivalent to either of the following statements:

(i) The map

$$wS.E\,(\mathcal{A},\mathcal{C},\mathcal{B}) \longrightarrow wS.\mathcal{A} \times wS.\mathcal{B} \qquad A \to C \to B \longmapsto (A,B) \tag{2.2.0.20.2}$$

is a homotopy equivalence.

(ii) The map

$$wS.E\,(\mathcal{C}) \longrightarrow wS.\mathcal{C} \times wS.\mathcal{C} \qquad A \to C \to B \longmapsto (A,B) \tag{2.2.0.20.3}$$

is a homotopy equivalence.

(iii) The two maps

$$wS.E\,(\mathcal{C}) \longrightarrow wS.\mathcal{C} \qquad A \to C \to B \longmapsto C,\ A \vee B \tag{2.2.0.20.4}$$

are homotopic.

Let us see how the K-theories of a category and a subcategory relate to each other.

Definition 2.0.21. Let $\mathcal{A}$ be an exact subcategory of the exact category $\mathcal{B}$. $\mathcal{A}$ is said to be cofinal in $\mathcal{B}$ if $0 \to A' \to B \to A'' \to 0$ is exact in $\mathcal{B}$ with A' and A'' in $\mathcal{A}$, then so is B, and if for each B in $\mathcal{B}$ there is a B' in $\mathcal{B}$ so that $B \oplus B'$ is isomorphic to an object in $\mathcal{A}$. (For simplicity we will assume $\mathcal{A}$ is isomorphism closed in $\mathcal{B}$. This does not change the K-theory of $\mathcal{A}$).

The next theorem is known as the cofinality theorem.

Theorem 2.0.22. [**Sta85**, Theorem 2.1] *Let $\mathcal{A}$ be cofinal in $\mathcal{B}$ and $G = K_0(\mathcal{B})/K_0(\mathcal{A})$. Then there is a fibration sequence up to homotopy*

$$K(iS.\mathcal{A}) \to K(iS.\mathcal{B}) \to BG.$$

Notice $w\mathcal{A} = i\mathcal{A}$ and $w\mathcal{B} = i\mathcal{B}$, where i denotes the isomorphisms, the minimal possible choice.

In general, given a category $\mathcal{C}$ we will fix the cofibrations and then look at the interplay of the two K-theories defined by two different notions of weak equivalences. Let $\mathcal{C}$ be a category with cofibrations equipped with two categories of weak equivalences, one finer than the other, $v\mathcal{C} \subset w\mathcal{C}$. Let $\mathcal{C}^w$ denote the full subcategory with cofibrations of $\mathcal{C}$ given by the objects A in $\mathcal{C}$ having the property $* \to A$ is in $w\mathcal{C}$. It inherits weak equivalences:

$$v\mathcal{C}^w = \mathcal{C}^w \cap v\mathcal{C} \qquad w\mathcal{C}^w = \mathcal{C}^w \cap w\mathcal{C}$$

Now recall the generic fibration lemma.

Lemma 2.0.23. [**Wal85**, Theorem 1.6.4] *If $\mathcal{C}$ has a cylinder functor, and the coarse category of weak equivalences $w\mathcal{C}$ satisfies the cylinder axiom, saturation axiom and extension axiom, then the square:*

$$\begin{array}{ccc} vS.\mathcal{C}^w & \longrightarrow & wS.\mathcal{C}^w \ (\simeq *) \\ \downarrow & & \downarrow \\ vS.\mathcal{C} & \longrightarrow & wS.\mathcal{C} \end{array}$$

is homotopy Cartesian, and the upper right term is contractible.

Next we recall the approximation theorem, a sufficient condition for an exact functor $F\colon \mathcal{A} \to \mathcal{B}$ to induce a homotopy equivalence $wS.\mathcal{A} \to wS.\mathcal{B}$.

Definition 2.0.24. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor of categories with cofibrations and weak equivalences. We say it has the approximation property if it satisfies:

App 1: An arrow in $\mathcal{A}$ is a weak equivalence in $\mathcal{A}$ if and only if its image in $\mathcal{B}$ is a weak equivalence in $\mathcal{B}$.

App 2: Given any object A in $\mathcal{A}$ and any map $x\colon F(A) \to B$ in $\mathcal{B}$ there exists a cofibration $a\colon A \to A'$ in $\mathcal{A}$ and a weak equivalence $x' : F(A') \to B$ in $\mathcal{B}$ such that

$$\begin{array}{ccc} F(A) & \xrightarrow{x} & B \\ {\scriptstyle F(a)}\downarrow & \nearrow_{x'} & \\ F(A') & & \end{array}$$

commutes.

The approximation theorem says:

Theorem 2.0.25. [**Wal85**, Theorem 1.6.7] *Let $\mathcal{A}$ and $\mathcal{B}$ be categories with cofibrations and weak equivalences. Assume $w\mathcal{A}$ and $w\mathcal{B}$ satisfy the saturation axiom. Suppose $\mathcal{A}$ has a cylinder functor that satisfies the cylinder axiom. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor having the approximation properties. Then $w\mathcal{A} \to w\mathcal{B}$ and $wS.\mathcal{A} \to wS.\mathcal{B}$ induce homotopy equivalences.*

Remark 2.0.26. There is a sutile refinement of 2.0.25 in [**TT90**, page 263-264]. Instead of require *App 2* it is required:

App 2': Given any object A in $\mathcal{A}$ and any map $x\colon F(A) \to B$ in $\mathcal{B}$ there exists a morphism $a\colon A \to A'$ in $\mathcal{A}$ and a weak equivalence $x' : F(A') \to B$ in $\mathcal{B}$ such that

$$\begin{array}{ccc} F(A) & \xrightarrow{x} & B \\ {\scriptstyle F(a)}\downarrow & \nearrow_{x'} & \\ F(A') & & \end{array}$$

commutes.

App 2' may look weaker than *App 2*, but it actually implies the stronger in presence of the rest of the hypothesis of the theorem 2.0.25. Given $x = x'F(a)$ as in *App 2'*, apply the cylinder functor to $a : A \to A'$ to factor $a = a''a'$ with $A \rightarrowtail A'' = T(A)$, and a'' the weak equivalence $A'' = T(a) \xrightarrow{\sim} A'$. Then $x'' = x'F(a'') : F(A'') \xrightarrow{\sim} B$ is a weak equivalence, $a' : A \rightarrowtail A''$ is a cofibration, and $x = x''F(a')$. Hence *App 2'* implies *App 2*.

CHAPTER 3

Exact Category

DEFINITION 3.0.27. An additive category $\mathcal{U}$is a small category with a zero object 0, where $\mathrm{Hom}_{\mathcal{U}}(U,V)$ is abelian for all objects U, V. Moreover, composition is bilinear with respect to this operation. Finite products and coproducts exist in such category are isomorphic. We call them direct sum and write it as $U \oplus V$.

DEFINITION 3.0.28. An abelian category $\mathcal{A}$ is an additive category in which each morphism has a kernel and a cokernel, every monic arrow is a kernel , and every epi a cokernel. (Recall that $f : B \to C$ is called monic, or monomorphism, if $fe_1 \neq fe_2$ for every $e_1 \neq e_2 : A \to B$; it is called epi, or epimorphism, if $g_1 f \neq g_2 f$ for every $g_1 \neq g_2 : C \to D$).

3.0.29. In an abelian category, we call a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ exact if $\ker g$ equals $\mathrm{Im}\, f \equiv \{B \to \mathrm{coker}\, f\}$. A longer sequence is exact if it is exact at all places. By short exact sequence in an abelian category $\mathcal{A}$ we mean an exact sequence of the form: $0 \to A' \to A \to A'' \to 0$.

DEFINITION 3.0.30. An additive functor $T : \mathcal{U} \to \mathcal{V}$ between additive categories are those which satisfy $T(f + f') = T(f) + T(f')$ for any pair of morphisms $f, f' : U \to V$. Moreover, such a functor satisfies $T(U \oplus V) = T(U) \oplus T(V)$.

For further details we refer to [**Lan71**] pages 192-193.
Following [**Qui72**] we define an exact category as follows.

DEFINITION 3.0.31. An exact category $\mathcal{U}$ is an additive category together with a choice of a class of sequences $\{ E_1 \rightarrowtail E_2 \twoheadrightarrow E_3 \}$ said to be exact. This determines two classes of morphisms:

1. admissible epimorphisms: $E_1 \rightarrowtail E_2$
2. admissible monomorphisms: $E_2 \twoheadrightarrow E_3$

The exact category is to satisfy the following axioms:

a) The class of admissible monomorphisms is closed under composition and is closed under cobase change by pushout along an arbitrary map $E_1 \to E_1'$.
b) Dually, the class of admissible epimorphisms is closed under composition and under base change by pull-back along an arbitrary map $E_3' \to E_3$.
c) Any sequence isomorphic to an exact sequence is exact, and any split sequence

$$E \rightarrowtail E \oplus F \twoheadrightarrow F$$

is to be exact.
d) In any sequence $E_1 \rightarrowtail E_2 \twoheadrightarrow E_3$, the map $E_1 \rightarrowtail E_2$ is a kernel for $E_2 \twoheadrightarrow E_3$, and $E_2 \twoheadrightarrow E_3$ is a cokernel for $E_1 \rightarrowtail E_2$.

e) Let $E \xrightarrow{i} F$ be a morphism in $\mathcal{U}$ which has a cokernel in $\mathcal{U}$. If there exists a map $F \xrightarrow{k} G$ such that $E \rightarrowtail^{ki} G$ is an admissible monomorphism, then $E \rightarrowtail^{i} F$ is itself an admissible monomorphism.

f) Dually, if $F \xrightarrow{i} E$ has a kernel in $\mathcal{U}$, and if there exists a $G \xrightarrow{k} F$ such that $G \overset{ik}{\twoheadrightarrow} E$ is an admissible epimorphism, then $F \twoheadrightarrow E$ is an admissible epimorphism.

Most of exact categories satisfy an stronger version of e) and f) , C, namely: If $f : E \to F$ is a morphism in $\mathcal{U}$, and there is a morphism $s : F \to E$ which splits f so $fs = 1_F$, then f is an admissible epimorphism $E \twoheadrightarrow F$.

This property and its dual are satisfied, in presence of e) and f), if $\mathcal{U}$ as exact category is idempotent complete, see appendix A of [**TT90**] for more details or the appendix C.

3.0.32. There is an alternative, and equivalent, description of an exact category due to Bass, see [**Bas68**]. We follow here Weibel's version, [**Wei**].

An exact category is a pair $(\mathcal{C}, \mathcal{E})$, where $\mathcal{C}$ is an additive category and $\mathcal{E}$ is a family of sequences in $\mathcal{C}$ satisfying that there is a full embedding of $\mathcal{C}$ as a full subcategory of an abelian category $\mathcal{A}$ so that

(i) $\mathcal{E}$ is the class of all short sequences in $\mathcal{C}$ which are exact in $\mathcal{A}$;

(ii) $\mathcal{C}$ is closed under extensions in the sense that if $0 \to B \xrightarrow{i} C \xrightarrow{j} D \to 0$ is exact sequence in $\mathcal{A}$ with $B, D \in \mathcal{C}$ then $C \in \mathcal{C}$.

$\mathcal{E}$ is the class of short exact sequences of $\mathcal{C}$. A morphism which occurs as the monomorphism i (resp. as the epimorphism j) in some sequence in $\mathcal{E}$will be called admissible monomorphism (resp. admissible epimorphism).

As in Quillen's definition, there is a property which many exact categories satisfy: closure under kernels of surjections in $\mathcal{A}$. $\mathcal{C}$ is closed under kernels of surjections in $\mathcal{A}$ provided that whenever a morphism $\varphi : B \to C$ in $\mathcal{C}$ is an epimorphism in $\mathcal{A}$ then $\ker \varphi \in \mathcal{C}$. Again, if $\mathcal{C}$ is idempotent complete then it is closed under kernels of surjections in $\mathcal{A}$. Also, again see appendix A of [**TT90**] for more details or the appendix B.

This double point of view of the concept of exact category will be helpful: Quillen's version in order to check the exactness of a category, Bass' to make calculations with morphisms. We will use mainly Quillen's definition but keeping in mind that via the Gabriel-Quillen embedding the exact category $\mathcal{U}$ can be seen as a full subcategory of an abelian category $\mathcal{A}b(\mathcal{U})$ of the left exact functors $\mathcal{U}^{op} \to \mathbb{Z} - modules$. Once more see appendix A of [**TT90**] for a complete description or appendix B for a fast review.

3.0.33. $\mathcal{U}$ is a category with cofibrations and weak equivalences.

$\mathcal{U}$ as an exact category has a natural structure of category with cofibrations and weak equivalences. Let co$\mathcal{U}$ be the admissible monomorphisms and let $w\mathcal{U}$ be the isomorphisms in $\mathcal{U}$. Let us check Waldhausen's axioms:

a)$co\mathcal{U}$ axioms.

cof 1: Isomorphisms are in $co\mathcal{U}$ since they are admissible monomorphisms.

cof 2: $\forall U \in \mathcal{U}\ * \to U$ is a cofibration since we always have the following split sequence in $\mathcal{U}$

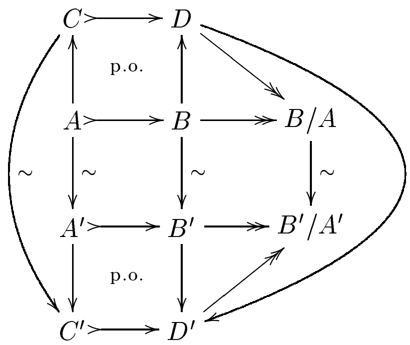
$$* \rightarrowtail * \oplus A \twoheadrightarrow A\ .$$

cof 3: Cofibrations admit cobase change by axiom a) for exact categories.

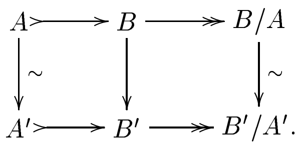
b)$w\mathcal{U}$ axioms.

weq 1: Isomorphisms are in $w\mathcal{U}$ by definition.

weq 2: Gluing Lemma is easy to check by chasing in the following diagram:



3.0.34. The saturation axiom, 2.0.13, is satisfied trivially since we are dealing with isomorphisms. The extension axiom can be verified by chasing in the diagram:



Recall that in an exact category $\mathcal{U}$ a morphism is an isomorphism if and only if it has both kernel and cokernel and both are trivial, i.e. the zero object.

CHAPTER 4

Localization for Exact Categories

4.0.35. Let $\mathcal{U}$ be an exact category and $\mathcal{A} \subseteq \mathcal{U}$ an exact subcategory of $\mathcal{U}$. We shall say that $\mathcal{A}$ localizes $\mathcal{U}$, if for each U in $\mathcal{U}$ there are families of exact sequences

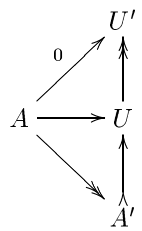
$$\{ A \rightarrowtail U \twoheadrightarrow U' \ , \ U'' \rightarrowtail U \twoheadrightarrow A' \}$$

with A and A' in $\mathcal{A}$ verifying:

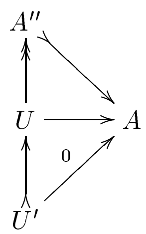
(i) $\mathcal{A}$ is a Serre subcategory, i.e. the exact sequences are closed in $\mathcal{A}$:

For $A' \rightarrowtail A \twoheadrightarrow A''$,$A \in \mathcal{A}$ if and only if $A', A'' \in \mathcal{A}$.

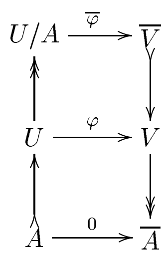
(ii) $\forall A \to U$, $\exists$



(iii) $\forall U \to A$, $\exists$

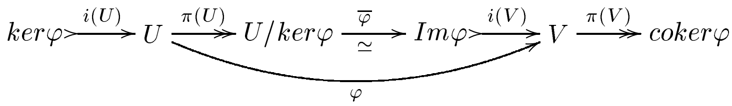


Definition 4.0.36. A morphism $U \overset{\varphi}{\to} V$ in $\mathcal{U}$ will be called a $\mathcal{A}^{-1}$-isomorphism if there are exact sequences in $\mathcal{U}$ corresponding to U and V such that



and $\overline{\varphi}$ is an isomorphism.

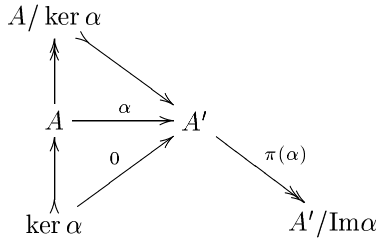
Remark 4.0.37. If φ is an $\mathcal{A}^{-1}$-isomorphism, then φ has kernel and cokernel, both in $\mathcal{A}$. The diagram above can be seen as



by which $\pi(U)$ and $i(V)$ are again $\mathcal{A}^{-1}$-isomorphisms.

Corollary 4.0.38. *Morphisms in $\mathcal{A}$ are $\mathcal{A}^{-1}$-isomorphisms.*

Proof. Let $A \xrightarrow{\alpha} A'$ be a morphism in $\mathcal{A}$. Then by axiom ii), 4.0.35, there is an a diagram like the following:

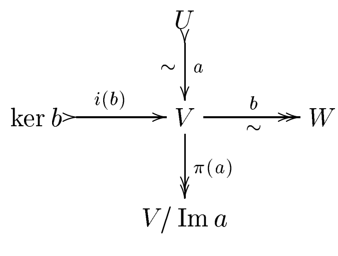


We have that α can be factored as $\overline{\alpha}\pi(\alpha)$ with $\pi(\alpha)$ an admissible epimorphism and $\overline{\alpha}$ an admissible monomorphism. In fact, we have shown that every morphism in $\mathcal{A}$ has kernel and cokernel, in $\mathcal{A}$, and hence is an $\mathcal{A}^{-1}$-isomorphism. □

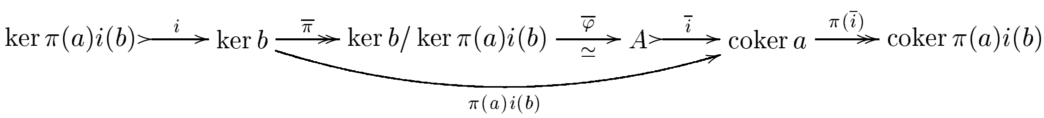
Remark 4.0.39. In fact Corollary 4.0.38 implies that $\mathcal{A}$ is an abelian category inside of $\mathcal{U}$. Abelian categories are additive categories where all morphisms have kernels and cokernels, see [**Wei**].

Let us check the set of $\mathcal{A}^{-1}$-isomorphisms verify the axioms A.0.8.

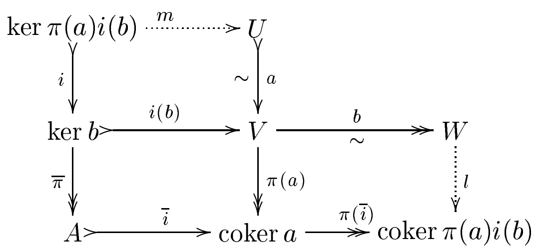
4.0.40. Let a, b be $\mathcal{A}^{-1}$-isomorphisms where a is a monomorphism and b is an epimorphism. We have then the following diagram:



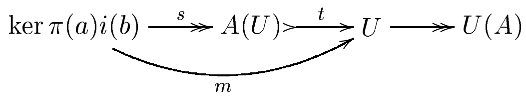
By 4.0.38, $\ker b \xrightarrow{\pi(a)i(b)} \operatorname{coker} a$ is factorized like,



Hence the lower left corner is completed to



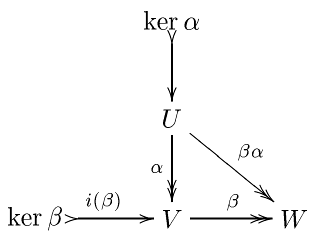
Since $\pi(a)i(b) = \bar{i}\bar{\pi}$, we have that $\pi(a)i(b)i = 0$ which implies $i(b)i$ factors through a. Let us say$i(b)i = am$. m is obviously a monomorphism in $\mathcal{U}$. Actually m is an admissible monomorphism. By property 4.0.38, we have



where t is an admissible monomorphism and hence a monomorphism. Since m is a monomorphism as well, this implies s is a monomorphism. s is also an epimorphism. This means s is an isomorphism and hence m is an admissible monomorphism. We can perform this last step by using the ambient abelian category in which the exact category $\mathcal{U}$ may be embedded. See chapter I. In the ambient abelian category a morphism is an isomorphism if and only if it is an epimorphism and a monomorphism.

Call $\overline{V} = \operatorname{coker} m$. Dually, $\pi(\bar{i})\pi(a) = lb$ and by a dual argument to the one above l is an admissible epimorphism. Call $\overline{W} = \ker l$. Clearly , there is a morphism $\overline{V} \to \overline{W}$. Once more by regarding $\mathcal{U}$ embedded in an ambient abelian category, it can be shown by chasing in the diagram that such morphism is in fact an isomorphism.

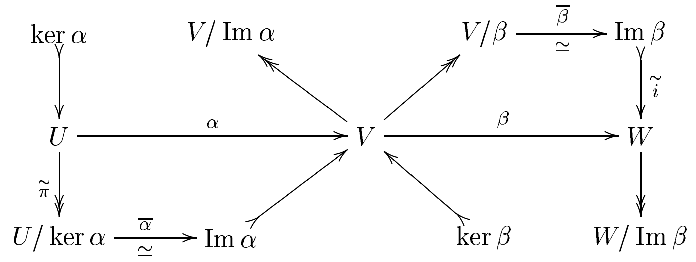
4.0.41. Let $\alpha : U \longrightarrow V$, $\beta : V \longrightarrow W$ be two admissible epimorphisms which are $\mathcal{A}^{-1}$-isomorphisms as well, i.e. their kernels are in $\mathcal{A}$.



By 3.0.31 there exists A pull back for $i(\beta)$ and α which is an object in $\mathcal{A}$ for being an extension of $\ker \beta$ and $\ker \alpha$, 4.0.35. By 3.0.31 $\beta\alpha$ is an admissible epimorphism. Pull back properties imply that $\overline{i(\beta)}$ is the kernel for $\beta\alpha$. By uniqueness of kernels $\overline{i(\beta)}$ is an admissible monomorphism. Hence $\beta\alpha$ is an admissible epimorphism which is also an $\mathcal{A}^{-1}$-isomorphism.

4.0.42. Dually to 4.0.41 given α and β admissible monomorphisms and $\mathcal{A}^{-1}$-isomorphism, it can be shown $\beta\alpha$ is of the same kind.

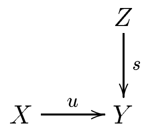
4.0.43. Let $\alpha : U \longrightarrow V$, $\beta : V \longrightarrow W$ be two $\mathcal{A}^{-1}$-isomorphisms. By definition of $\mathcal{A}^{-1}$-isomorphism there exists the diagram that follows for $\beta\alpha$:



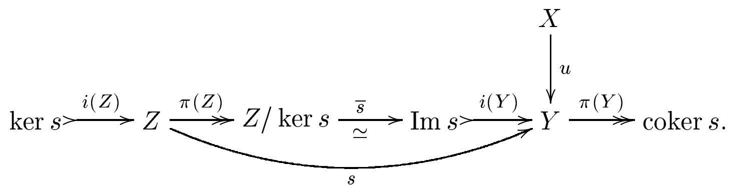
Applying 4.0.40, 4.0.41 and 4.0.42 in combination to the diagram above we have that $\beta\alpha$ is again an $\mathcal{A}^{-1}$-isomorphism.

This finishes the verification of axiom MS1, 4.0.35.

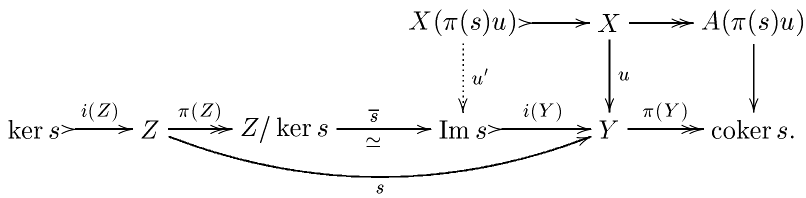
4.0.44. Lets us check $MS2$. Given



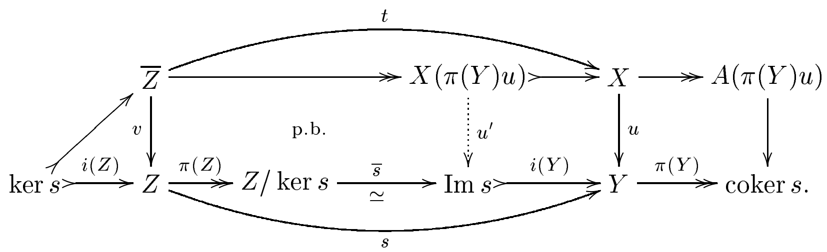
with s an $\mathcal{A}^{-1}$-isomorphism. This diagram can be rewritten to



Applying property property iii) of 4.0.35 to $\pi(Y)u$, we find an exact sequence for X factoring $\pi(Y)u$.



Since $\pi(Y)u = 0$, there is morphism $u' : X(\pi(Y)u) \longrightarrow \operatorname{Im} s$. By axiom b) for exact categories, 3.0.31, we can pullback u' along $\ker s \xrightarrow{i(Z)} Z \xrightarrow{\pi(Z)} Z/\ker s$. We complete the diagram to

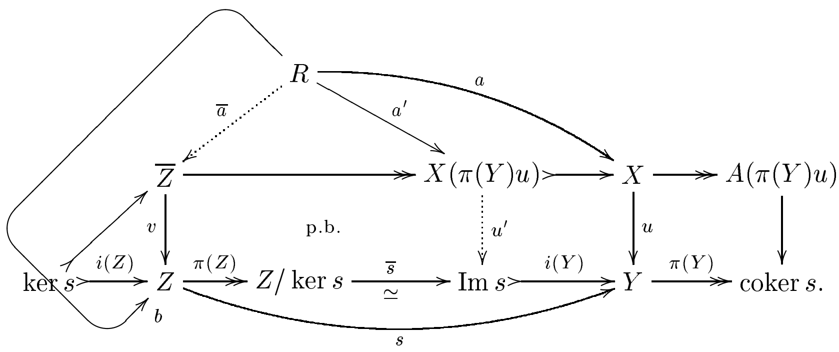


Let $W = \overline{Z}$ and v, t be the morphisms indicated in the diagram above. It is clear that t is an $\mathcal{A}^{-1}$-isomorphism.

We have two remarks to do:

REMARK 4.0.45. (i) If u is a monomorphism so is u' and hence v.
(ii) If s is a monomorphism (resp. epimorphism) besides being $\mathcal{A}^{-1}$-isomorphism, t would be a monomorphism (resp. epimorphism). In the case that s is a monomorphism we shall say that X can be replaced by the "smaller" W to make the diagram commutative.

REMARK 4.0.46. Notice that the object $\overline{Z}$ obtained after completing the diagram in MS2, satisfies the properties of a pull-back. Hence we will call to complete such a diagram to pull backwards. Given morphisms $a : R \to X$ and $b : R \to Z$ such that $ua = sb$, there is a morphism $\overline{a} : R \to \overline{Z}$ making the whole diagram commutative.

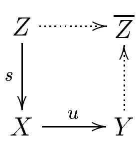


Since $au = sb$ then $\pi(Y)au = \pi(Y)sb = 0$. This means $R \xrightarrow{a} X \twoheadrightarrow A(\pi(Yu)$ is zero and hence a factors through $X(\pi(Y)u)$, call it a'. It satisfies $i(Y)u'a' = ua = sb = i(Y)\overline{s}\pi(Z)b$. The morphism $i(Y)$ is a monomorphism, then $u'a' = \overline{s}\pi(Z)b$. The diagram now is a truly pull-back in $\mathcal{U}$. We have then the existence of $\overline{a}$, our dotted arrow.

The proof for $MS2'$ is dual to the one for $MS2$, with dual remarks.

REMARK 4.0.47. (i) If u is an epimorphism so is u' and hence v.
(ii) If s is a monomorphism (resp. epimorphism) besides being $\mathcal{A}^{-1}$-isomorphism, t would be a monomorphism (resp. epimorphism). In the case that s is a epimorphism we shall say that X can be replaced by the "smaller" W to make the diagram commutative.

REMARK 4.0.48. Dually to remark 4.0.46, the object $\overline{Z}$ completing the diagram in MS2':



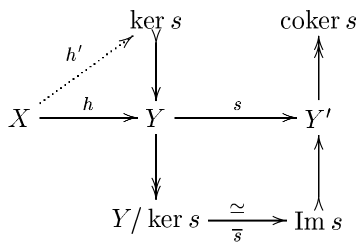
satisfies the properties of a push-out. We will call to push-forward to complete this kind of diagram. The proof is also dual to that of remark 4.0.46.

4.0.49. In order to check $MS3$ under our hypothesis, notice that since $\mathcal{U}$ is an exact category, hence additive $MS3$ can be restated as:

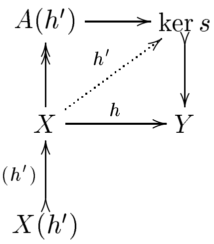
Given $h : X \longrightarrow Y$, are equivalent

(a') $\exists s : Y \to Y'$ in S such that $sh = 0$.
(b') $\exists t : X' \to X$ in S such that $ht = 0$.

(a')⇒(b'). Given

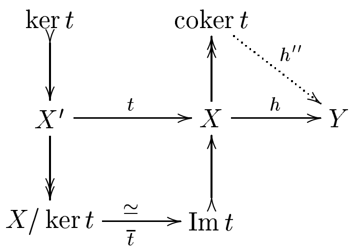


Since $hs = 0$, h factors through $\ker s$. By 4.0.35, we find an exact sequence for X and h':

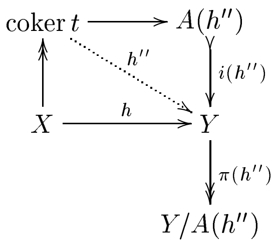


Let $X' = X(h')$ and $t = i(h')$, clearly $i(h')$ is an $\mathcal{A}^{-1}$-isomorphism and $i(h')h = 0$.

(a')⇐(b'). Given



Since $ht = 0$, h factors through $\operatorname{coker} t$ which is in $\mathcal{A}$. By property iii) of 4.0.35 we find an exact sequence for Y and h''.



Let $Y' = Y/A(h'')$ and $s = \pi(h'')$, clearly $\pi(h'')$ is an $\mathcal{A}^{-1}$-isomorphism and $h\pi(h'') = 0$.

DEFINITION 4.0.50. We define the localized category $\mathcal{A}^{-1}\mathcal{U}$ as the category with objects $obj\mathcal{A}^{-1}\mathcal{U} = obj\mathcal{U}$ and morphisms

$$\operatorname{Hom}_{\mathcal{A}^{-1}\mathcal{U}}(U,V) = \varinjlim_{I_U, J_V} \operatorname{Hom}_{\mathcal{U}}(U',V')$$

where $I_U = \{U' \overset{\sim}{\to} U\}$ and $J_V = \{V' \overset{\sim}{\to} V\}$ are the categories described in appendix A. In other words, $\mathcal{A}^{-1}\mathcal{U}$ is the localization of $\mathcal{U}$ with respect to the set of $\mathcal{A}^{-1}$-isomorphisms.

4.0.51. Hence by A $\mathcal{A}^{-1}\mathcal{U}$ is a category, in fact a small category since it has the same set of objects $\mathcal{U}$ has.

Let us check it is an additive category, [**Har66**]. We will only give an sketch of a proof. For further details we recommend [**Wei94**] page 383. There is an obvious zero object, and the sum of objects will be given by the sum in $\mathcal{U}$.

Given f,$g \in \operatorname{Hom}_{\mathcal{A}^{-1}\mathcal{U}}(U,V)$, let f be represented by

$$\begin{array}{ccccc} U & \xleftarrow{\sim} & U' & & \\ & & \downarrow{\scriptstyle f'} & & \\ & & V' & \xleftarrow{\sim} & V \end{array}$$

and g represented by

$$\begin{array}{ccccc} U & \xleftarrow{\sim} & U'' & & \\ & & \downarrow{\scriptstyle g''} & & \\ & & V'' & \xleftarrow{\sim} & V. \end{array}$$

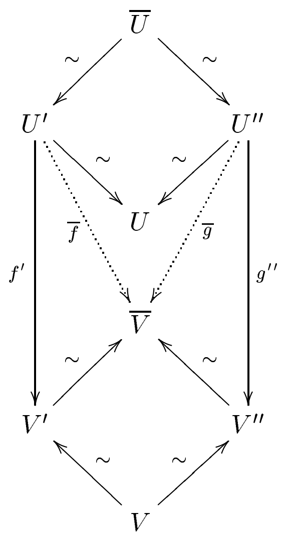
Let $\overline{U}$ and $\overline{V}$ be, using calculus of fractions A, such that

$$\begin{array}{ccccc} & & \overline{U} & & \\ & \overset{\sim}{\swarrow} & & \overset{\sim}{\searrow} & \\ U' & & \# & & U'' \\ & \underset{\sim}{\searrow} & & \underset{\sim}{\swarrow} & \\ & & U & & \end{array} \quad \text{and} \quad \begin{array}{ccccc} & & \overline{V} & & \\ & \overset{\sim}{\nearrow} & & \overset{\sim}{\nwarrow} & \\ V' & & \# & & V'' \\ & \underset{\sim}{\nwarrow} & & \underset{\sim}{\nearrow} & \\ & & V & & \end{array}$$

commute.

We obtain the following commutative diagram:

(4.4.0.51.1)



The morphisms $\overline{f}$,$\overline{g} \in \mathrm{Hom}_{\mathcal{U}}(\overline{U}, \overline{V})$ are new representatives for f and g respectively. We define $f+g \in \mathrm{Hom}_{\mathcal{A}^{-1}\mathcal{U}}(U, V)$ to be represented by $\overline{f}+\overline{g} \in \mathrm{Hom}_{\mathcal{U}}(\overline{U}, \overline{V})$. We would need to show this definition is good. It is clear from diagram 4.4.0.51.1 that by taking enough "pushouts" and "pull-backs" we would get that if f and f' are equivalent and, g and g' are equivalent then so are $f+g$ and $f'+g'$.

In a similar fashion the rest of the axioms for an additive category can be checked.

An exact sequence $0 \to M \to N \to P \to 0$ in $\mathcal{A}^{-1}\mathcal{U}$ will be declared to be short exact sequence in $\mathcal{A}^{-1}\mathcal{U}$ if it is $\mathcal{A}^{-1}$-isomorphic to a short exact sequence $\overline{M} \rightarrowtail \overline{N} \twoheadrightarrow \overline{P}$ in $\mathcal{U}$. We mean there is a commutative diagram in $\mathcal{A}^{-1}\mathcal{U}$ as the following one:

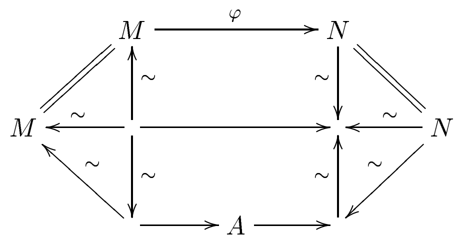
$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{f} & N & \xrightarrow{g} & P & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \\
0 & \longrightarrow & \pi(\overline{M}) & \longrightarrow & \pi(\overline{N}) & \longrightarrow & \pi(\overline{P}) & \longrightarrow & 0
\end{array}$$

where u, v and w are isomorphisms in $\mathcal{A}^{-1}\mathcal{U}$.

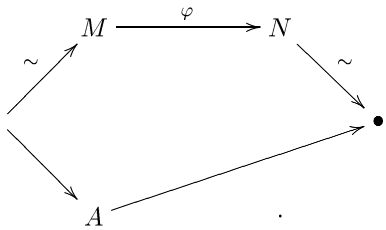
Before checking the axioms 3.0.31 for $\mathcal{A}^{-1}\mathcal{U}$ we are going to prove some easy facts about morphisms in the localized category which may make things easier. Let φ and ψ be morphisms in $\mathcal{U}$.

4.0.52 (a). It is verified that $\pi(\varphi) = 0$ if and only if φ factors through $\mathcal{A}$.

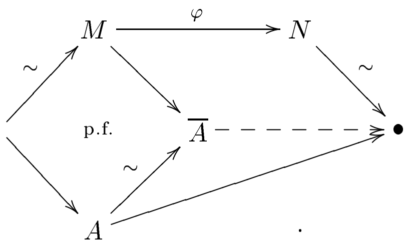
It is clear that if φ factors through $\mathcal{A}$ then $\pi(\varphi) = 0$. On the other hand, that $\pi(\varphi) = 0$ means that $\pi(\varphi)$ factors through the zero object $\star$ in $\mathcal{A}^{-1}\mathcal{U}$. The zero object $\star$ in $\mathcal{A}^{-1}\mathcal{U}$ is represented by any object A in $\mathcal{A}$. We have then the next diagram in $\mathcal{U}$:



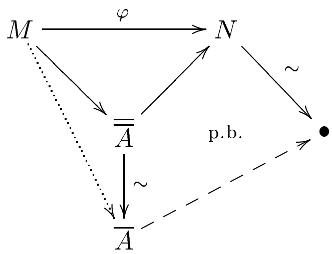
The upper half of the diagram might be rewritten then as



Take push forward, 4.0.48, on the left,

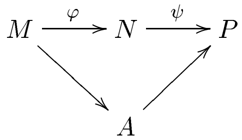


and now the pull-back, 4.0.46, on the upper right. This shows φ factors through an object $\overline{\overline{A}}$ in $\mathcal{A}$.

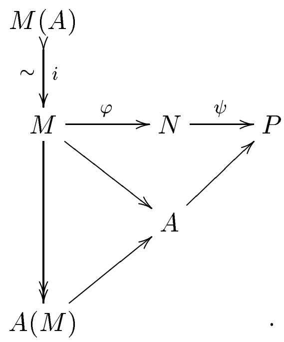


4.0.53 (b). Moreover, it is verified if $\pi(\psi\varphi) = 0$ then there is a representative φ' for φ, $\pi(\varphi) = \pi(\varphi')$, such that $\psi\varphi' = 0$ in $\mathcal{U}$.

If $\pi(\psi\varphi) = 0$, then by 4.0.52 there is an object A in $\mathcal{A}$ through which $\psi\varphi$ factors.



Now, apply the localization axiom iii), 4.0.35, to the morphism from M to A,



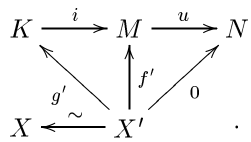
Call $\varphi' : M(A) \to M$ to φi. Clearly $\psi\varphi' = 0$.

Dually, it can be shown that in the given conditions there is a representative ψ' for ψ, $\pi(\psi') = \pi(\psi)$ with $\psi'\varphi = 0$.

4.0.54 (c). Kernels and cokernels persist in $\mathcal{A}^{-1}\mathcal{U}$. Let $M \stackrel{u}{\to} N$ be a morphism in $\mathcal{U}$ and $K \stackrel{i}{\to} M$ its kernel.

It is clear that $\pi(i)$ is a monomorphism in $\mathcal{A}^{-1}\mathcal{U}$. Let now $\overline{f} : M \to X$ be a morphism in $\mathcal{A}^{-1}\mathcal{U}$ such that $\pi(u)\overline{f} = 0$ in $\mathcal{A}^{-1}\mathcal{U}$. We shall show there is a morphism $\overline{g} : X \to X$ with $\pi(i)\overline{g} = \overline{f}$ in $\mathcal{A}^{-1}\mathcal{U}$.

Since $\pi(u)\overline{f} = 0$ then by 4.0.53 there is a representative f' for $\overline{f}$, i.e. $\pi(f') = \overline{f}$, satisfying that $uf' = 0$. Hence f' factors through $K \stackrel{i}{\to} M$, i.e. $f' = ig'$ for $g' : X' \to K$. Call $g = \pi(g')$. Clearly $\pi(i)g = \overline{f}$ in $\mathcal{A}^{-1}\mathcal{U}$.

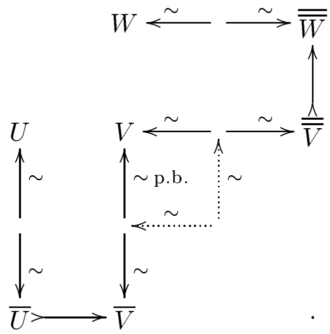


Apply a ‘dual’ argument for the ‘respective’ case of the cokernels.

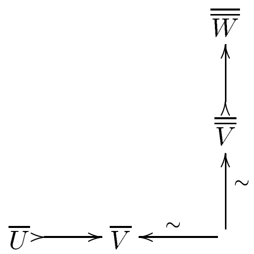
This result implies that if φ has kernel and $\ker\varphi \in \mathcal{A}$, resp. cokernel and $\operatorname{coker}\varphi \in \mathcal{A}$, then φ is a monomorphism, resp. an epimorphism, in $\mathcal{A}^{-1}\mathcal{U}$.

4.0.55. Let us check the axiom a) 3.0.31 for $\mathcal{A}^{-1}\mathcal{U}$. First, we shall see the composition of admissible monomorphisms is again an admissible monomorphism. For these computations it will be made use of the closure condition under kernels of epimorphisms, see 3.0.32, for $\mathcal{U}$. We can assume this condition since it has no K-theoretical repercussions, see C.4.3 on page 90.

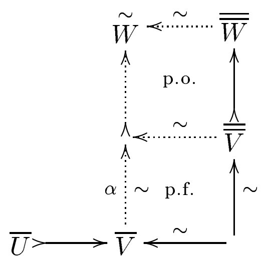
Let $a : U \to V$ and $b : V \to W$ two admissible monomorphisms in $\mathcal{A}^{-1}\mathcal{U}$, which we may represent as follows in $\mathcal{U}$:



After the indicated pull-back and some simplifications, it becomes



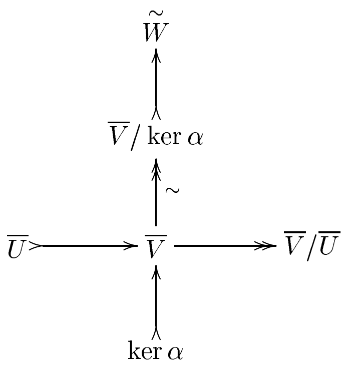
Then we take push-forward and apply axiom a) 3.0.31 to the second admissible monomorphism, as indicated bellow,



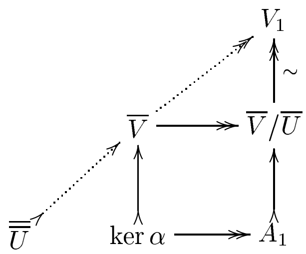
Since α is an $\mathcal{A}^{-1}$-isomorphism, there is a decomposition like

$$\ker\alpha \rightarrowtail U \twoheadrightarrow \overline{V}/\ker\alpha \xrightarrow[\simeq]{\overline{\alpha}} \operatorname{Im}\alpha \rightarrowtail V \twoheadrightarrow \operatorname{coker}\alpha$$

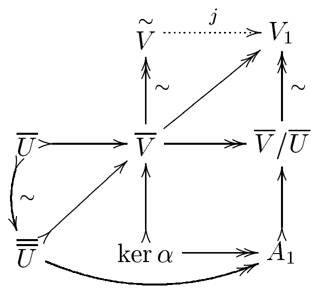
and composition of admissible monomorphisms is closed in $\mathcal{U}$, the diagram will be reduced to



Apply axiom (ii) of 4.0.35 to the lower right corner, which will decomposed as

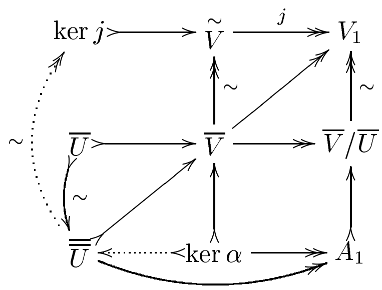


Admissible epimorphisms are closed under composition, hence the doted arrow in the diagram above is an admissible epimorphism. Call $\overline{\overline{\overline{U}}}$ to its kernel. There is then an induced morphism from $\overline{U}$ to $\overline{\overline{\overline{U}}}$ whose cokernel is A_1. Hence by axiom e) of 3.0.31 this morphism is an admissible monomorphism. Moreover it is an $\mathcal{A}^{-1}$-isomorphism.

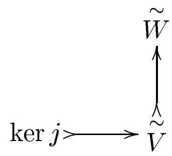


On the other hand, there is an induced morphism $j : \tilde{V} \to V_1$, which is obviously an epimorphism. Here we apply the closeness condition to $\mathcal{U}$, having then the existence of $\ker j$. Now j is an admissible epimorphism by axiom f), 3.0.31.

The existence of this j induces a morphism from $\ker \alpha$ to $\overline{\overline{\overline{U}}}$ whose cokernel happens to be $\ker j$. Hence these morphisms, by axiom e), 3.0.31, is an admissible monomorphism. Moreover the morphism from $\overline{\overline{\overline{U}}}$ to $\ker j$ is an $\mathcal{A}^{-1}$-isomorphism besides being an admissible epimorphism.



We have obtained the next morphisms

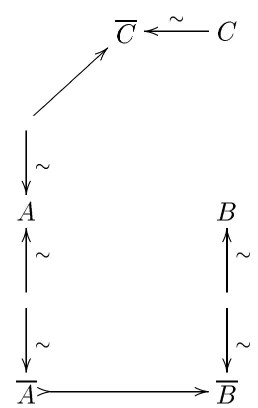


in $\mathcal{U}$, whose composition is a admissible monomorphism in $\mathcal{U}$. $\ker j$ and $\widetilde{W}$ are isomorphic in $\mathcal{U}^{-1}\mathcal{A}$ to U and W respectively through chains of $\mathcal{A}^{-1}$-isomorphisms. We obtained our purpose then.

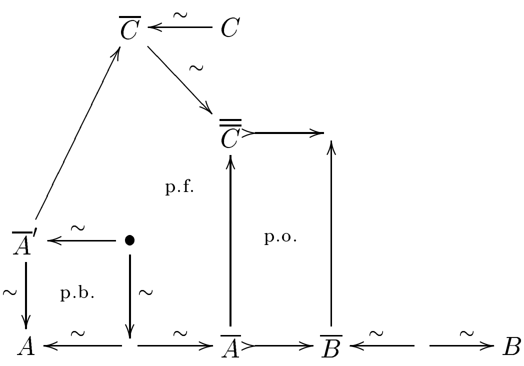
Using dual arguments it can be sown that admissible epimorphisms in $\mathcal{A}^{-1}\mathcal{U}$ are closed under composition.

Let us check that admissible monomorphisms are closed under cobase change by push out along arbitrary morphisms.

Let $a : A \to B$ be an admissible monomorphism in $\mathcal{A}^{-1}\mathcal{U}$ and $c : A \to C$ be any other morphism in the localized category. After choosing representatives for these morphisms, we have in $\mathcal{U}$:

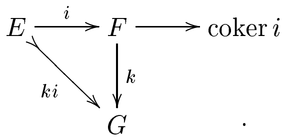


Below, in the diagram, we sketch the steps to perform. First get the pull back out of A, then push forward from this newly pull-backed object. In this way, it is obtained an object $\overline{\overline{C}}$ and a morphism from $\overline{A}$ to $\overline{\overline{C}}$ along which we push out the admissible monomorphism which represent a. That finishes the proof.

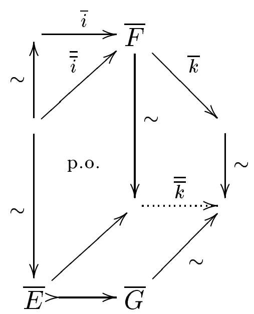


A dual argument would work for the base change of admissible epimorphisms in $\mathcal{A}^{-1}\mathcal{U}$ by pull-back along arbitrary morphisms.

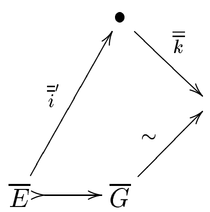
4.0.56. Finally let us check axioms e) and f) of 3.0.31. Let the following diagram be in $\mathcal{A}^{-1}\mathcal{U}$:



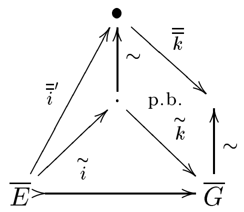
Choosing the right representatives for E, F, G, say $\overline{E}$, $\overline{F}$ and $\overline{G}$ respectively, and possibly others; it is possible to construct the outer part of the next diagram in $\mathcal{U}$ after some simplifications and after taking a few push-forwards and pull-backs.



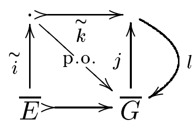
Call $\overline{\overline{i}}$ the composite in the upper left corner and take push-out along $\overline{\overline{i}}$, then $\overline{k}$ would factor through the push-out.



Now take pull-back along $\overline{\overline{k}}$, $\overline{\overline{i}}'$ factors through the pull-back.

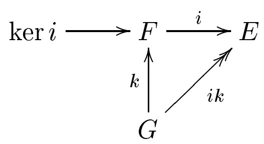


Hence we arrive to a situation similar to the one in axiom e) but in $\mathcal{U}$. We are only left to prove $\widetilde{i}$ has a cokernel. Apply axiom a) of exact category, 3.0.31, to $\overline{E} \rightarrowtail \overline{G}$ along $\widetilde{i}$.

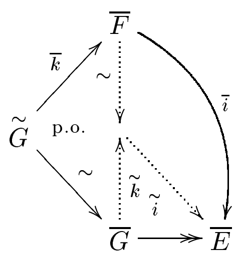


But the existence of $\widetilde{k}$ implies that of a morphism l which splits j, i.e. $lj = 1_{\overline{G}}$. Result C.0.25 implies j is an admissible monomorphism, hence it has cokernel. Since the diagram is a push-out then it is also the kernel for $\widetilde{i}$. By property 4.0.54 this implies i has cokernel in $\mathcal{A}^{-1}\mathcal{U}$ and it is isomorphic in $\mathcal{A}^{-1}\mathcal{U}$ to coker $\widetilde{i}$.

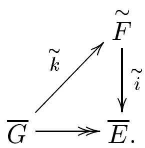
For the axiom f) we do similar calculations. Given a diagram



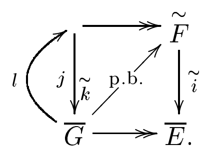
in $\mathcal{A}^{-1}\mathcal{U}$. Choosing representatives for these morphisms in $\mathcal{U}$ and after simplification as above we may have a commutative diagram like the following one:



Take push-forward along $\overline{k}$ through which, as indicated above, $\overline{i}$ factors. We arrive to



Now apply axiom b) of 3.0.31 to $\overline{G} \twoheadrightarrow \overline{E}$:

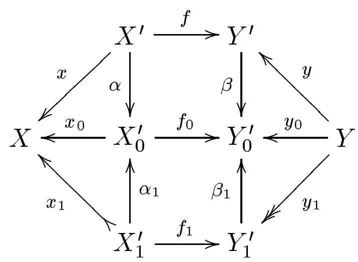


The existence of $\widetilde{k}$ implies there is a morphism l which splits j, i.e. $jl = 1_{\overline{G}}$. Property C.0.24 implies j is an admissible epimorphism and therefore has a kernel, say $\ker j$. By property 4.0.54 this implies that i has a kernel in $\mathcal{A}^{-1}\mathcal{U}$ which is isomorphic in $\mathcal{A}^{-1}\mathcal{U}$ to $\pi(\ker \widetilde{i})$. We are done since we have shown an admissible epimorphism isomorphic in $\mathcal{A}^{-1}\mathcal{U}$ to i.

We have then checked on $\mathcal{A}^{-1}\mathcal{U}$ all the axioms for an exact category, 3.0.31. Hence, the localizing functor π is exact.

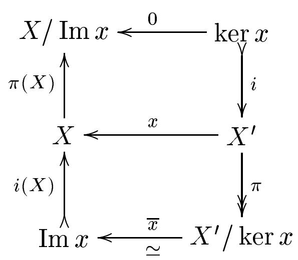
Since our localization is respect to the $\mathcal{A}^{-1}$- isomorphisms we can obtain a few more properties of the kind in A. These have been recollected from [**Bas68**]

vi) Let $X \xleftarrow{x} X' \xrightarrow{f} Y' \xleftarrow{y} Y$, with x and y $\mathcal{A}^{-1}$- isomorphisms. Then we can construct the commutative diagram.

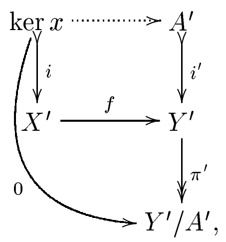


where x_1 is a admissible monomorphism and y_1 is an admissible epimorphism.

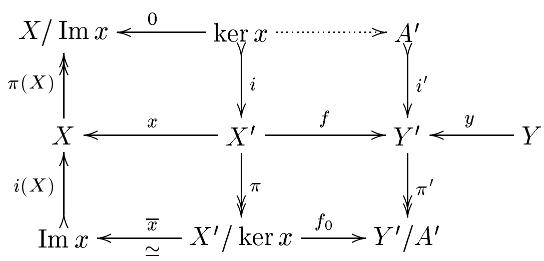
Proof. Since x is an $\mathcal{A}^{-1}$-isomorphisms then by definition of $\mathcal{A}^{-1}$-isomorphism,



where $\ker x$, $X/\operatorname{Im} x$ are in $\mathcal{A}$ and $\overline{x}$ is an isomorphism. Now, $fi : \ker x \to Y$ by ii) of 4.0.35, factors as

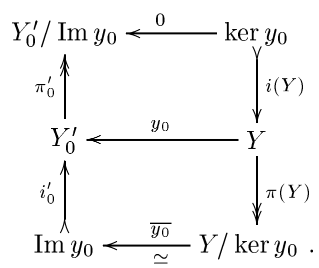


so $\pi' f i = 0$, hence $\pi' f$ factors through $X'/\ker x$ via π. Call that morphism f_0.

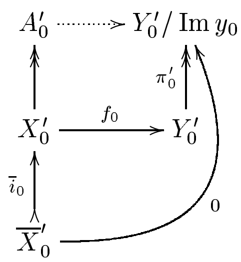


Call $y_0 = \pi' y$, which is an $\mathcal{A}^{-1}$-isomorphism by saturation. Denote by $X'_0 = X'/\ker x$ and $Y'_0 = Y'/A'$.

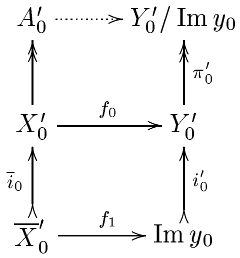
We have also by definition the following diagram for y_0,



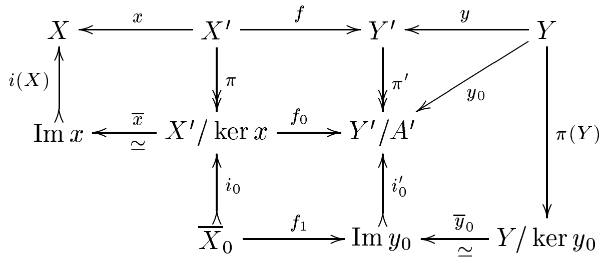
Apply the axiom ii), to $\pi'_0 f_0 : X'_0 \to Y'_0/\operatorname{Im} y_0$.



So $\pi_0' f_0 \bar{i}_0 = 0$, hence $f_1 \bar{i}_0$ factors through i_0', call it f_1.

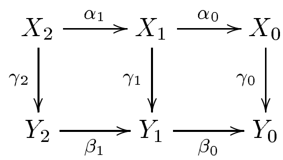


The final diagram is then,

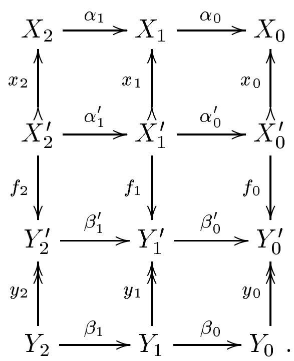


Let $X_1' = \overline{X}_0$, $Y_1' = \operatorname{Im} y_0$, $X_0' = X'/\ker x$, $Y_0' = Y'/A'$, $\alpha = \pi$, $\beta = \pi'$, $y_1 = \overline{y}_0 \pi(Y)$, $\beta_1 = i_0'$, $\alpha_1 = i_0$, $x_1 = \overline{x} i_0$ and $x_0 = \overline{x}$. These choices satisfy the required conclusions. □

vii) Let



be a commutative diagram in $\mathcal{A}^{-1}\mathcal{U}$. Then there is a commutative diagram as follows in $\mathcal{U}$:



Here the verticals represent γ_2, γ_1 and γ_0 respectively, the x's are admissible monomorphisms, and the y's are admissible epimorphisms. In particular, if $\alpha_0\alpha_1$ and $\beta_0\beta_1$ are zero then so also are $\alpha_0'\alpha_1'$ and $\beta_0'\beta_1'$. To construct such a diagram we start with vertical representations of the γ's so that the x's are admissible monomorphisms and the y's are admissible epimorphisms, using vi) above. In order to complete the construction we shall replace the initial choices of the X's and

Y's by "smaller" ones. Recall 4.0.45 and 4.0.47. For an X_i' this means a smaller subobject of X_i such that the inclusion into X_i is still in an $\mathcal{A}^{-1}$ isomorphism. For the Y_i's this means a smaller quotient of Y_i such that the projection from Y_i is still an $\mathcal{A}^{-1}$-isomorphism.

Step1 Make X_1' and Y_1' smaller so that α_0' and β_1' exist making the upper right and lower left rectangles, respectively commute.

Step2 Make X_2' and Y_0' smaller so that α_0' and β_0' exist, making the upper left and lower right rectangles commute.

Step3 Make Y_0' still smaller so that the middle right rectangle commute.

Step4 Make Y_1' smaller so that the middle rectangle commutes.
It is easily seen that all of the above reductions are possible, and that each step leaves intact the conditions achieved by the previous ones.

We obtain the following result:

THEOREM 4.0.57. *Let $\mathcal{U}$ be an exact category and $\mathcal{A}$ exact subcategory of $\mathcal{A}$ satisfying the axioms 4.0.35. Then there is an exact category $\mathcal{A}^{-1}\mathcal{U}$ and an exact localizing functor $\pi : \mathcal{U} \to \mathcal{A}^{-1}\mathcal{U}$ satisfying the following universal property:*

Given an exact functor $T : \mathcal{U} \to \mathcal{V}$ such that $TA \cong 0$ for all A in $\mathcal{A}$, there is a unique functor $U : \mathcal{A}^{-1}\mathcal{U} \to \mathcal{V}$ such that $T = U \circ \pi$. Moreover $\mathcal{U}$ is exact.

PROOF. The proof follows after applying the universal property for calculus of fractions,A.0.9, to our situation, since $\mathcal{A}^{-1}\mathcal{U}$ is $\mathcal{U}$ localized with respect to the $\mathcal{A}^{-1}$-isomorphisms. □

CHAPTER 5

The category $C(\mathcal{U})$ and its structures

Given an exact category $\mathcal{U}$, we can define the category of finite chain complexes in $\mathcal{U}$, where objects are:

$$C_{\#} : 0 \to C_r \xrightarrow{d} C_{r-1} \xrightarrow{d} \cdots \to C_l \to 0$$

such that $d^2 = 0$, i. e. d^2 factors through the zero object. Morphisms are chain maps $f : C_{\#} \to D_{\#}$, collection of morphisms $f = \{f_r : C_r \to D_r\}$ such that $d_D f = f d_C$. We shall denote this category by $C(\mathcal{U})$. A chain homotopy in $\mathcal{U}$

$$e : f \simeq f' : C \to D$$

is a collection of morphisms $\{e : C_r \to D_{r+1}\}$ such that $d_D e + e d_C = f' - f : C_r \to D_r$. A chain equivalence is a chain map $f : C \to D$ which admits a chain homotopy inverse, that is, a chain map $g : D \to C$ such that

$$\exists h : gf \simeq 1 : C \to C \text{ and } k : fg \simeq 1 : D \to D \ .$$

As we have already seen in chapter 3 the exact category $\mathcal{U}$ can be thought fully embedded in an abelian category, $\mathcal{A}b(\mathcal{U})$, via the Gabriel-Quillen embedding, see appendix B. From this point of view, we are able to talk about homology of chain complexes in $\mathcal{U}$. Obviously this homology takes values in $\mathcal{A}b(\mathcal{U})$.

Given $C_{\#}$ with differential d denote: the kernel of $d_n : C_n \to C_{n-1}$ by $Z_n(C) = Z_n$, the n-cycles of C; and the image of $d_{n+1} : C_{n+1} \to C_n$ by $B_n(C) = B_n$, the n-boundaries of C. Since dd=0, $B_n \to Z_n \to C_n$ are monomorphisms in $\mathcal{A}b(\mathcal{U})$ for all n. The n^{th}-homology of C is the quotient, in $\mathcal{A}b(\mathcal{U})$, $H_n C = Z_n / B_n$.

A chain map $f : C_{\#} \to D_{\#}$ is called a quasi-isomorphism if the induced maps $f_\star : H_n C \to H_n D$ are all isomorphisms in $\mathcal{A}b(\mathcal{U})$. If $f : C_{\#} \to D_{\#}$ is null-homotopic, i.e. homotopic to the zero chain map, then the maps $f_\star : H_n C \to H_n D$ are zero. Hence if f and g are homotopic, they induce the same maps $H_n C \to H_n D$. It is clear now that chain homotopy equivalences are quasi-isomorphisms. A complete description on these topics is given in [**Wei94**, chapter 1].

5.0.58. $C(\mathcal{U})$ is an exact category.

The short exact sequences in $C(\mathcal{U})$ will be the short sequences of chain complexes which degree-wise are shot exact sequences in $\mathcal{U}$. It is clear that $C(\mathcal{U})$ wit this family of short exact sequences is an exact category.

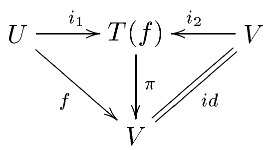
5.0.59. Moreover $C(\mathcal{U})$ is a category with cofibrations and weak equivalences. The cofibrations will be those chain maps which degree-wise are admissible monomorphisms. The weak equivalences will be the quasi-isomorphisms. The cofibrations and weak equivalences axioms, 2.0.13, can be checked rapidly using simple constructions.Also the saturation and extension axioms, 2.0.13, are easy to check: use long exact sequences in homology associated to the short exact sequences of complexes and the 5-lemma, see [**Wei94**, pages 10-15] for further details.

5.0.60. $C(\mathcal{U})$ has a cylinder functor and satisfies the cylinder axiom, 2.0.17.

Given $f : U \to V$ a morphism, let $T(f)$ be the chain complex $(T(f))_p = U_p \oplus U_{p-1} \oplus V_p$ with boundary

$$d_p \equiv \begin{pmatrix} d_U & -1 & 0 \\ 0 & -d_U & 0 \\ 0 & f & d_V \end{pmatrix}.$$

We have the following diagram:



where j_1 and j_2 are the obvious inclusions as direct summands. Degree-wise π is defined as:

$$\pi_p \equiv (f, 0,\ 1).$$

It is easy to check that *Cyl 1* and *Cyl 2* are satisfied. The cylinder axiom also holds. To see this, we show that π is a chain homotopy equivalence and hence a quasi-isomorphism, in this case a weak equivalence. The homotopy inverse is the natural inclusion

$$i_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Degree-wise, all is given by the following matrices:

$$\pi_p = (f, 0,\ 1) \quad d_p = \begin{pmatrix} d & -1 & 0 \\ 0 & -d & 0 \\ 0 & f & d \end{pmatrix} \quad \Gamma_p = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check now that $\Gamma_p d_{p+1} + d_{p+2}\Gamma_{p+1} = i_2\pi_{p+1} - 1$ and $\pi i_2 = 1$.

CHAPTER 6

K-theory of Chain Complexes

Let $\mathcal{U}$ be an exact category. $\mathcal{U}$ has a natural structure as a category with cofibrations and weak equivalences,3. Admissible monomorphims are cofibrations and isomorphisms are weak equivalences. $\mathcal{U}$ is fully embedded in $C(\mathcal{U})$, the category of finite chain complexes with the degree-wise monomorphisms chain maps as cofibrations and the quasi-isomorphisms as weak equivalences. $C(\mathcal{U})$ can be given another structure as category with cofibtrations and weak equivalences. Denote by $C(\mathcal{U})_\star$ the category of finite chain complexes in $\mathcal{U}$ where the cofibrations are the degree-wise split monomorphisms as weak equivalences, still, are the quasi-isomorphisms. $C(\mathcal{U})_\star$ also verifies the saturation, extension and cylinder axioms, 2.0.13 and 2.0.17. We have the following theorem in [**TT90**] which relates the K-theory of $\mathcal{U}$ and of $C(\mathcal{U})$.

THEOREM 6.0.61. [**TT90**, Theorem 1.11.7]
Given $\mathcal{U}$ an exact category, let $C(\mathcal{U})$ be its category of finite chain complexes. Assume $\mathcal{U}$ and $C(\mathcal{U})$ are given the usual 'structures' of categories with cofibrations and weak equivalences as explained in sections 3.0.33 and 5.0.59. Then the embedding $\mathcal{U} \hookrightarrow C(\mathcal{U})$, as chain complexes of length 1, induces a homotopy equivalence of K-theory spectra.

Moreover, the exact functor induced by the identity, $C(\mathcal{U})_\star \hookrightarrow C(\mathcal{U})$, induces also a homotopy equivalence of K-theory spectra.

PROOF. For the proof of the first part see [**TT90**, Theorem 1.11.7, pages 279-282] or the proof to [**CP97**, Proposition 6.1]. In [**CP97**], though the proof is stated in terms of additive categories with homotopy equivalences as weak equivalences, the proof goes through for exact categories with the quasi-isomorphisms as weak equivalences, under a minor change in one step.

It has to be remind that, as in Thomason's proof, $\mathcal{U}$ is required to verify the closure condition under kernels of epimorphisms in $\mathcal{A}b(\mathcal{U})$. For this, it is enough to hold property C.0.24, see B.0.19[b)]. This can be solved as we already did in the proof of [**CP97**, Proposition 6.1].

If $\mathcal{U}$ does not satisfy the property (P) then $\overline{\mathcal{U}}$ does, see C.1.6. Moreover, $\mathcal{U}$ is cofinal in $\overline{\mathcal{U}}$ and $K(\mathcal{U}) \to K(\overline{\mathcal{U}})$ is a homotopy equivalence by the cofinality theorem since it is an isomorphism on K_0, see section C.3.2. Similarly $iC(\mathcal{U}) \to iC(\overline{\mathcal{U}})$ and $iC(\mathcal{U})^q \to iC(\overline{\mathcal{U}})^q$ are cofinal inclusions which are easily seen to induce isomorphisms on K_0, hence induce homotopy equivalences in K-theory by the cofinality theorem.

Consider the diagram

$$\begin{array}{ccccccccc}
K(iC(\mathcal{U})^q) & \longrightarrow & K(iC(\mathcal{U})) & \longrightarrow & K(qC(\mathcal{U})) & & & \hookleftarrow & \\
\downarrow & & \downarrow & & \downarrow & & & & \\
K(iC(\overline{\mathcal{U}})^q) & \longrightarrow & K(iC(\overline{\mathcal{U}})) & \longrightarrow & K(qC(\overline{\mathcal{U}})) & \simeq & K(\overline{\mathcal{U}}) & \simeq & K(\mathcal{U}).
\end{array}$$

The top and bottom row are fibrations by the generic fibration lemma 2.0.23. The two homotopy equivalences at the bottom are consequences of theorem 6.0.61 and cofinality 2.0.22. We have just argued that the vertical arrows on the left and in the middle are homotopy equivalences , hence we can conclude that the vertical arrow on the right is a homotopy equivalence. The right hand side diagram commutes and therefore the theorem holds for arbitrary exact categories.

For the second statement, follow the proof to [**TT90**, Proposition 1.9.2]. Simply apply the Approximation theorem, 2.0.25, under the conditions in remark 2.0.26 to the inclusion $C(\mathcal{U})_\star \hookrightarrow C(\mathcal{U})$. Condition *App 1* is obviously satisfied and *App 2′* is trivially satisfied with $a = x$, $x' = 1$. ☐

REMARK 6.0.62. The second part of the theorem allows us to work, whenever necessary, under the 'simpler' conditions of $C(\mathcal{U})_\star$ instead of those in $C(\mathcal{U})$.

Part 2

Main Theorem

CHAPTER 7

Setting the Main Theorem

In this part we shall prove:

THEOREM 7.0.63 (Main Theorem). *Given $\mathcal{U}$, an exact category, $\mathcal{A}$ an exact subcategory satisfying axioms 4.0.35, and let $\mathcal{A}^{-1}\mathcal{U}$ be the associated localized exact category, see chapter 4, then*

$$\mathcal{A} \longrightarrow \mathcal{U} \longrightarrow \mathcal{A}^{-1}\mathcal{U}$$

is a fibration, up to homotopy, of K-theory spectra.

Given the exact category $\mathcal{A}$ of $\mathcal{U}$ under the conditions of the Main Theorem, 7.0.63, we have the localization functor $\pi : \mathcal{U} \longrightarrow \mathcal{A}^{-1}\mathcal{U}$. Consider the associated category of finite chain complexes $C(\mathcal{A}^{-1}\mathcal{U})$ with its usual structure, see 5. The localization functor induces the following diagram of categories:

$$\begin{array}{ccc} C(\mathcal{U}) & \xrightarrow{C(\pi)} & C(\mathcal{A}^{-1}\mathcal{U}) \\ \uparrow & & \uparrow \\ \mathcal{U} & \xrightarrow{\pi} & \mathcal{A}^{-1}\mathcal{U}. \end{array}$$

We will use three different classes of weak equivalences on $C(\mathcal{U})$:

a) i, isomorphisms;
b) q, quasi-isomorphisms, morphisms inducing isomorphisms on homology; recall the definition in chapter 5; it will also be used q for the isomorphisms in $C(\mathcal{A}^{-1}\mathcal{U})$;
c) $\overline{q}$, quasi-isomorphisms in $C(\mathcal{A}^{-1}\mathcal{U})$ reflected on $C(\mathcal{U})$ via π, i.e. those chain morphisms in $C(\mathcal{U})$ which once considered in $C(\mathcal{A}^{-1}\mathcal{U})$ are quasi-isomorphims, induce isomorphisms on homology, in $\mathcal{A}^{-1}\mathcal{U}$! More easily, dimension-wise the induced homology maps in $\mathcal{A}b(\mathcal{U})$ have kernel and cokernel in $\mathcal{A}$.

Having fixed the cofibrations in $C(\mathcal{U})$, we will denote by $wC(\mathcal{U})$ the choice of weak equivalences made for $C(\mathcal{U})$, where w might be: i, q, $\overline{q}$ or any other class. Applying Waldhausen's generic fibration lemma, 2.0.23 on page 19, to the following functor induced by the identity:

$$qC(\mathcal{U}) \to \overline{q}C(\mathcal{U})$$

we obtain as homotopy fiber the K-theory of the category $qC(\mathcal{U})^{\overline{q}}$. The theorem, 7.0.63, will follow once we apply the following results:

i) The K-theory of $\overline{q}C(\mathcal{U})$ is homotopy equivalent to that of $qC(\mathcal{A}^{-1}\mathcal{U})$ and hence, by 6.0.61 on page 47, to that of $\mathcal{A}^{-1}\mathcal{U}$.

ii) he K-theory of $qC(\mathcal{U})$ is homotopy equivalent to that of $\mathcal{U}$. This is given by 6.0.61 on page 47 directly.
iii) The K-theory of $qC(\mathcal{U})^{\overline{q}}$ is homotopy equivalent to that of $\mathcal{A}$.

REMARK 7.0.64. It will be repeatedly used along the proof the property that $\mathcal{U}$ is closed under taking kernels of surjections in the abelian category $\mathcal{A}b(\mathcal{U})$. We can assume $\mathcal{U}$ satisfies this property. By C.4.3 in case $\mathcal{U}$ does not satisfy the property, it can be replaced by $\overline{\mathcal{U}}$ with no K-theoretical consequences.

CHAPTER 8

Proof of the Main Theorem

8.1. Proof of i)

The localization functor$\pi : \mathcal{U} \to \mathcal{A}^{-1}\mathcal{U}$, induces an exact functor:

$$qC(\mathcal{U}) \xrightarrow{C(\pi)} qC(\mathcal{A}^{-1}\mathcal{U}),$$

which, obviously, factors as follows

$$\begin{array}{ccc} qC(\mathcal{U}) & \longrightarrow & \overline{q}C(\mathcal{U}) \\ & \searrow^{C(\pi)} & \vdots\, R \\ & & qC(\mathcal{A}^{-1}\mathcal{U}) \end{array} .$$

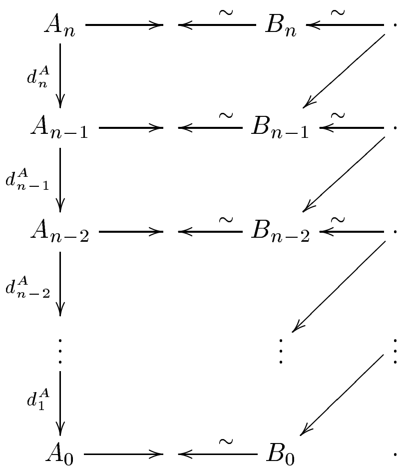
PROPOSITION 8.1.1. *The functor defined above $R : \overline{q}C(\mathcal{U}) \to qC(\mathcal{A}^{-1}\mathcal{U})$ induces a homotopy equivalence in K-theory.*

PROOF. For the proof we will apply the Approximation theorem, 2.0.25, by checking the conditions *App 1* and *App 2'* in 2.0.26. $\overline{q}C(\mathcal{U})$ has cylinder functor from $C(\mathcal{U})$ and satisfies the cylinder and saturation axiom since $\overline{q}$ is the reflection of the quasi-isomorphisms q in $C(\mathcal{A}^{-1}\mathcal{U})$ which satisfy those axioms, see 5.

App 1 is trivially hold by definition of $\overline{q}$.

Let us check *App 2'*. Let $A \in C(\mathcal{U})$, $B \in C(\mathcal{A}^{-1}\mathcal{U})$ and $x : A \to B$ a chain morphism in $C(\mathcal{A}^{-1}\mathcal{U})$. A chain morphism between chain complexes which degreewise is given by $\mathcal{A}^{-1}$-isomorphisms will be called chain $\mathcal{A}^{-1}$-isomorphism. These are isomorphisms in $C(\mathcal{A}^{-1}\mathcal{U})$ and hence are weak equivalences. The morphism x is given by a collection $\{x_r : A_r \to B_r\}_{0\leq r\leq n}$ of morphisms in $\mathcal{A}^{-1}\mathcal{U}$. (We are assuming that $A_r = 0 = B_r$ for $r < 0$ and $r > n$). These morphisms x_r can be represented in $\mathcal{U}$ by diagrams $\{A_r \to \overset{\sim}{\leftarrow} B_r\}$ for all r, see property iii) in appendix A. Since B is a chain complex in $\mathcal{A}^{-1}\mathcal{U}$, its boundaries $d_r^B : B_r \to B_{r-1}$ can also be represented by diagrams $\{B_r \overset{\sim}{\leftarrow}\to B_{r-1}\}$ for all r. The complete diagram representing $x : A \to B$ in $\mathcal{U}$ may look like the following:

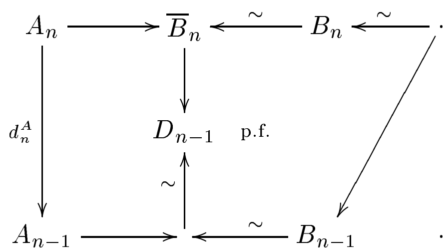
(*)



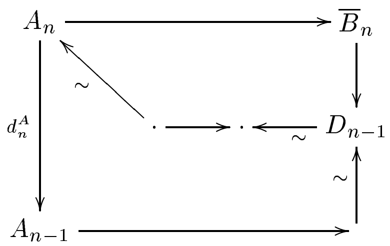
Each of the diagrams in (*) are commutative diagrams in $\mathcal{A}^{-1}\mathcal{U}$, as morphisms from A_r to B_{r-1}, for each r.

In a first step, we will replace B by a chain complex $\overline{B}$ in such a way that the resulting morphism $\overline{x} : A \to \overline{B}$ is realized by commutative squares in $\mathcal{U}$.

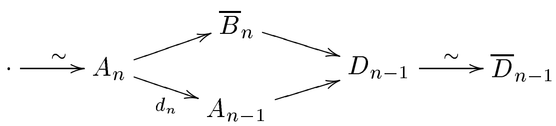
Take push-forward at the upper right corner of the square at dimension n.



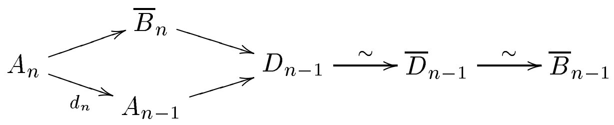
The resulting square at the left half is a gain a commutative diagram in $\mathcal{A}^{-1}\mathcal{U}$ from A_n to D_{n-1}. By definition, see appendix A, this commutativity is expressed in $\mathcal{U}$ by the existence of a diagram in $\mathcal{U}$ like the following:



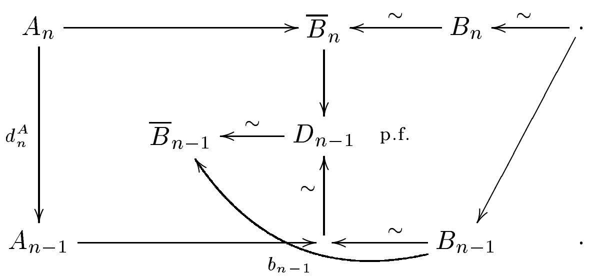
which can be rewritten as



By MS3, appendix A, there exists an $\mathcal{A}^{-1}$-isomorphism $\overline{D}_{n-1} \overset{\sim}{\to} \overline{B}_{n-1}$ such that

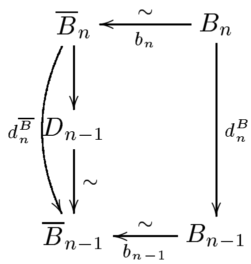


is commutative in $\mathcal{U}$. In the original diagram



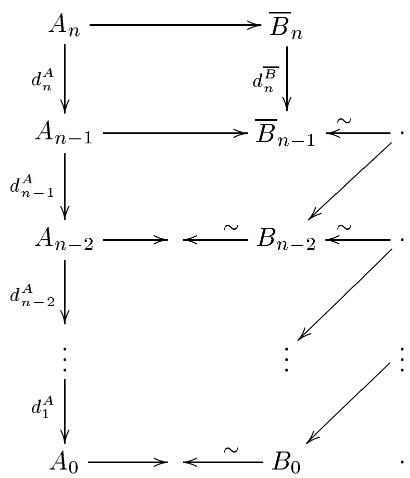
now the left square is also commutative in $\mathcal{U}$. B_n is $\mathcal{A}^{-1}$-isomorphic to $\overline{B}_n$ via b_n and B_{n-1} is $\mathcal{A}^{-1}$-isomorphic to $\overline{B}_{n-1}$ via, let us say b_{n-1}.

Notice that



is commutative in $\mathcal{A}^{-1}\mathcal{U}$. The diagram (*) can be substituted by

(**)



The process described above can be repeated for the next dimension:

$$\begin{array}{ccccc}
A_{n-1} & \longrightarrow & \overline{B}_{n-1} & \overset{\sim}{\longleftarrow} & \cdot \\
\downarrow{\scriptstyle d^A_{n-1}} & & & \swarrow & \\
A_{n-2} & \longrightarrow \;\; \overset{\sim}{\longleftarrow} & B_{n-2} & & \cdot
\end{array}$$

Notice that $\overline{B}_{n-1}$ and all the rest of data in higher dimensions is kept. The output is a new diagram:

$$\begin{array}{ccccc}
A_{n} & \longrightarrow & \overline{B}_{n} & & \\
\downarrow{\scriptstyle d^A_{n}} & & \downarrow{\scriptstyle d^{\overline{B}}_{n}} & & \\
A_{n-1} & \longrightarrow & \overline{B}_{n-1} & & \\
\downarrow{\scriptstyle d^A_{n-1}} & & \downarrow{\scriptstyle d^{\overline{B}}_{n-1}} & & \\
A_{n-2} & \longrightarrow & \overline{B}_{n-2} & \overset{\sim}{\longleftarrow} & \cdot \\
\downarrow{\scriptstyle d^A_{n-2}} & & & \swarrow & \\
\vdots & & \vdots & & \vdots \\
\downarrow{\scriptstyle d^A_{1}} & & & \swarrow & \\
A_{0} & \longrightarrow \;\; \overset{\sim}{\longleftarrow} & B_{0} & & \cdot
\end{array}$$

with the two upper squares commutative in $\mathcal{U}$.

This process can be repeated down to dimension zero. We shall end up with the following diagram:

(***)

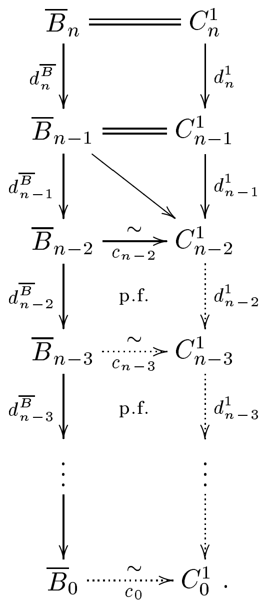
$$\begin{array}{ccc}
A_{n} & \xrightarrow{\overline{x}_n} & \overline{B}_{n} \\
\downarrow{\scriptstyle d^A_{n}} & & \downarrow{\scriptstyle d^{\overline{B}}_{n}} \\
A_{n-1} & \xrightarrow{\overline{x}_{n-1}} & \overline{B}_{n-1} \\
\downarrow{\scriptstyle d^A_{n-1}} & & \downarrow{\scriptstyle d^{\overline{B}}_{n-1}} \\
\vdots & & \vdots \\
\downarrow{\scriptstyle d^A_{1}} & & \downarrow{\scriptstyle d^{\overline{B}}_{1}} \\
A_{0} & \xrightarrow{\overline{x}_0} & \overline{B}_{0}
\end{array}$$

where each of the diagrams is strictly commutative in $\mathcal{U}$.

On one hand, we have described $\overline{B} = \{\overline{B}_n \to \overline{B}_{n-1} \to \dots \overline{B}_0\}$, a new chain complex in $\mathcal{A}^{-1}\mathcal{U}$ which is isomorphic in the localized category to B via a chain $\mathcal{A}^{-1}$-isomorphism $b : B \to \overline{B}$. The diagram (***) represents a chain morphism $\overline{x} : A \to \overline{B}$ in $C(\mathcal{A}^{-1}\mathcal{U})$, which by the construction described above clearly factors as $\overline{x} = bx$.

As a second step we shall replace the chain complex $\overline{B} \in C(\mathcal{A}^{-1}\mathcal{U})$ we have just obtained by a 'strict' chain complex in $C(\mathcal{U})$ to which will be chain $\mathcal{A}^{-1}$-isomorphic.

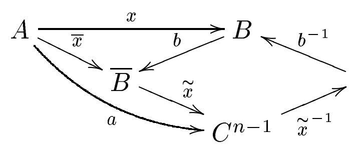
Since $\overline{B}$ is in $C(\mathcal{A}^{-1}\mathcal{U})$, $d^{\overline{B}}_{r-1}d^{\overline{B}}_r = 0$ in $\mathcal{A}^{-1}\mathcal{U}$ for all r. Let us start at the top dimension n. $d^{\overline{B}}_{n-1}d^{\overline{B}}_n = 0$ in $\mathcal{A}^{-1}\mathcal{U}$, by 4.0.53, implies there is an $\mathcal{A}^{-1}$-isomorphism $c^1_{n-2} : \overline{B}_{n-2} \xrightarrow{\sim} C^1_{n-2}$ such that $c^1_{n-2}d^{\overline{B}}_{n-1}d^{\overline{B}}_n = 0$ in $\mathcal{U}$. Now, take successive push-forward using c^1_{n-2} and its descendents in order to complete the diagram to a chain complex:



So C^1_{n-3} is the push-forward to $d^{\overline{B}}_{n-3}$ and c_{n-2}, C^1_{n-4} the push-forward to $d^{\overline{B}}_{n-4}$ and c_{n-3} and so on till dimension zero. We have $\overline{B}$ chain $\mathcal{A}^{-1}$-isomorphic to C^1 which satisfies $d^1_{n-1}d^1_n = 0$, strictly in $\mathcal{U}$; the rest as in $\overline{B}$. Now, $d^1_{n-2}d^1_{n-3} = 0$ in $\mathcal{A}^{-1}\mathcal{U}$. By the same reasons, as above, there is an $\mathcal{A}^{-1}$-isomorphism $c^2_{n-3} : C^1_{n-3} \xrightarrow{\sim} C^2_{n-3}$ such that $c^2_{n-3}d^1_{n-2}d^1_{n-3} = 0$ in $\mathcal{U}$. And, as before, take successive push-forward using c^2_{n-3} and its descendents down to dimension zero in order to define C^2_r and d^2_r for $r \leq n-4$. Call $C^2_r = C^1_r$, $d^2_r = d^1_r$ for $r \geq n-2$; C^2_{n-3}, $d^2_{n-2} = c^2_{n-3}d^1_{n-2}$. This new chain complex $C^2 \in C(\mathcal{A}^{-1}\mathcal{U})$ verifies $d^2_{r-1}d^2_r = 0$ in $\mathcal{U}$ for $r \geq n-1$ and $\overline{B} \underset{c_{n-2}}{\xrightarrow{\sim}} C^1 \underset{c_{n-3}}{\xrightarrow{\sim}} C^2$ is an $\mathcal{A}^{-1}$-isomorphism.

Continuing with this process, we shall have finally a chain complex C^{n-1} with $d^{n-1}_{r-1}d^{n-1}_r = 0$ in $\mathcal{U}$ for $r \geq 2$. In other words, $C^{n-1} \in C(\mathcal{U})$, and, moreover there is a chain of $\mathcal{A}^{-1}$-isomorphisms , $\overline{B} \xrightarrow{\sim} C^1 \xrightarrow{\sim} \ldots \xrightarrow{\sim} C^{n-1}$, between $\overline{B}$ and C^{n-1}. Call this morphism $\widetilde{x} : \overline{B} \xrightarrow{\sim} C^{n-1}$. Notice, that for the construction of C^{n-1} we have used operations in $\mathcal{U}$ using commutative diagrams. Hence $A \xrightarrow{\overline{x}} \overline{B} \xrightarrow{\widetilde{x}} C^{n-1}$ is a morphism in $C(\mathcal{U})$ and $\widetilde{x}\overline{x} = \widetilde{x}bx$. Since $\widetilde{x}$ and b are $\mathcal{A}^{-1}$-isomorphism in $C(\mathcal{A}^{-1}\mathcal{U})$ so is $\widetilde{x}b$, therefore a weak equivalence. Hence there exists its inverse $(\widetilde{x}b)^{-1} : C^{n-1} \to B$ in $C(\mathcal{A}^{-1}\mathcal{U})$. Call $a = \widetilde{x}\overline{x} = \widetilde{x}bx : A \to C^{n-1}$. Then in $C(\mathcal{A}^{-1}\mathcal{U})$ it is true that $(\widetilde{x}b)^{-1}a = b^{-1}\widetilde{x}^{-1}\widetilde{x}bx = x$. With this the verification of

App 2' is finished.



□

8.2. Proof of iii)

In this section we shall try to identify the K-theory of $qC(\mathcal{U})^{\overline{q}}$ with that of $\mathcal{A}$.

8.2.1. Domination. Before showing item iii), we recall the definition of domination of chain complexes, see [**Ran92**] or for further details.

Let $\mathcal{A} \subseteq \mathcal{U}$ a full subcategory of an additive category $\mathcal{U}$. A chain complex U in $\mathcal{U}$ is $\mathcal{A}$-dominated if there is a chain complex A in $\mathcal{A}$ and chain maps $i : U \to A$ and $r : A \to U$ so that ri is homotopic to the identity. We need the following result by 8.2.1.

Lemma 8.2.1. [**CP95**, Lemma 4.8] *Let $\mathcal{A}$ be a full subcategory of $\mathcal{U}$, $U_{\#}$ an $\mathcal{A}$-dominated chain complex in $\mathcal{U}$. Let K be the inverse image of $K_0(\mathcal{U})$ under the induced map $K_0(\mathcal{A}^{\wedge}) \longrightarrow K_0(\mathcal{U}^{\wedge})$, and let $\mathcal{U}^{\wedge K}$ be the full subcategory with objects $U \oplus (A,p)$, $[(A,p)] \in K$.*

Then the induced chain complex in $\mathcal{U}^{\wedge K}$ under the inclusion $\mathcal{U} \to \mathcal{U}^{\wedge K}$ is chain homotopy equivalent to a chain complex in $\mathcal{A}^{\wedge K}$.

Let the full subcategory of $C(\mathcal{U})$ with $\mathcal{A}$-dominated objects be denoted by $C(\mathcal{U})^{\mathcal{A}}$. The lemma 8.2.1 would just say that $C(\mathcal{U})^{\mathcal{A}}$ and $\mathcal{A}^{\wedge K}$ are K-theoretically homotopy equivalent.

Under the hypothesis in the Main Theorem, 7.0.63 on page 51, $\mathcal{A}$ is an abelian category, see 4.0.39 on page 26, and hence idempotent complete. The lemma 8.2.1 reduces then to the homotopy equivalence between the K-theories of $C(\mathcal{U})^{\mathcal{A}}$ and $C(\mathcal{A})$.

8.2.2. For integers $a \leq b$, let $[C(\mathcal{U})]_a^b$ be the full subcategory of those complexes C in $C(\mathcal{U})$ such that $C_i = 0$ for $i \leq a-1$ and for $i \geq b+1$.

Hence $\mathcal{U} = [C(\mathcal{U})]_0^0$ and $C(\mathcal{U})$ is the direct colimit of $[C(\mathcal{U})]_a^b$ as b goes to $+\infty$ and a goes to $-\infty$.

Set $w[C(\mathcal{U})]_a^b = wC(\mathcal{U}) \cap [C(\mathcal{U})]_a^b$, where w is the class of weak equivalences, (whatever is the chosen one: isomorphisms, quasi-isomorphisms, etc) and $co[C(\mathcal{U})]_a^b = coC(\mathcal{U}) \cap [C(\mathcal{U})]_a^b$. Then $w[C(\mathcal{U})]_a^b$ is a category with cofibrations and weak equivalences.

Similarly, $wC(\mathcal{U})^{\overline{q}} = \lim\limits_{\substack{b \to +\infty \\ a \to -\infty}} w[C(\mathcal{U})^{\overline{q}}]_a^b$. Recall $C(\mathcal{U})^{\overline{q}}$ is the full subcategory of objects in $C(\mathcal{U})$ which are $\overline{q}$-contractible. This means the objects C in $C(\mathcal{U})^{\overline{q}}$ are acyclic in $\mathcal{A}^{-1}\mathcal{U}$, i.e. $H_r(C) \in \mathcal{A}$ for all r.

For each pair of integers $a \leq b$, consider $[C(\mathcal{U})_0^{\overline{q}}]_a^b$, the full subcategory of those complexes C in $[C(\mathcal{U})^{\overline{q}}]_a^b$ such that $H_a C = 0$. As above, $w[C(\mathcal{U})_0^{\overline{q}}]_a^b$ becomes a category with cofibrations and weak equivalences by inheritance.

From now on, the class of weak equivalences w are fixed to be the isomorphisms i, unless otherwise denoted.

8.2.3. For each pair of integers $a \leq b$, there is a 'natural' exact functor

$$\Psi : [C(\mathcal{U})^{\overline{q}}]_a^b \longrightarrow E\left([C(\mathcal{U})_0^{\overline{q}}]_a^b, [C(\mathcal{U})^{\overline{q}}]_a^b, [C(\mathcal{U})^{\overline{q}}]_a^a \cong \mathcal{A}\right)$$

which induces a homotopy equivalence on K-theory. In general, $E(\mathcal{A}, \mathcal{C}, \mathcal{B})$ is the category of cofibration sequences $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{C}$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$ for $\mathcal{A}$ and $\mathcal{B}$ subcategories of the $\mathcal{C}$, see [**Wal85**][page 325] or 2.0.19.

In fact, Ψ is an equivalence of categories. Given C in $[C(\mathcal{U})^{\overline{q}}]_a^b$, Ψ associates the following extension

$$C \equiv \left\{ \begin{array}{c} C_b \\ \downarrow{\scriptstyle \partial_b} \\ C_{b-1} \\ \downarrow \\ \vdots \\ \downarrow \\ C_{a+1} \\ \downarrow{\scriptstyle \partial_{a+1}} \\ C_a \end{array} \right\} \longrightarrow \Psi(C) \equiv \{\, C' \rightarrowtail C \twoheadrightarrow C'' \,\} \equiv \left\{ \begin{array}{ccccc} C_b & = & C_b & \longrightarrow & \star \\ \downarrow{\scriptstyle \partial_b} & & \downarrow{\scriptstyle \partial_b} & & \| \\ C_{b-1} & = & C_{b-1} & \longrightarrow & \star \\ \downarrow & & \downarrow & & \| \\ \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow \\ C_{a+1} & = & C_{a+1} & \longrightarrow & \star \\ \downarrow & & \downarrow{\scriptstyle \partial_{a+1}} & & \downarrow \\ \operatorname{Im} \partial_{a+1} & \rightarrowtail & C_a & \twoheadrightarrow & H_a C \end{array} \right\}$$

It is clear that $C'$is in $[C(\mathcal{U})_0^{\overline{q}}]_a^b$ and C'' in $[C(\mathcal{U})^{\overline{q}}]_a^a \cong \mathcal{A}$.

Remark 8.2.2. In order to have Ψ properly defined it is necessary to show the existence of the following factorization in $\mathcal{U}$:

(8.8.2.2.1)
$$\begin{array}{ccccccc} \ker \partial_{a+1} & \rightarrowtail & C_{a+1} & \xrightarrow{\partial_{a+1}} & C_a & \twoheadrightarrow & H_a C \\ & & & \searrow \quad \operatorname{Im} \partial_{a+1} \quad \nearrow & & & \end{array}$$

We will justify this later on 8.2.4.

The inverse to Ψ, call it Ψ^{-1}, is given by projecting the extension $C' \rightarrowtail C \twoheadrightarrow C''$ onto the middle term C. By the Additivity Theorem,2.0.20, the map induced on K-theory by Ψ^{-1} is homotopic to the sum of the maps induced on K-theory by sending $C' \rightarrowtail C \twoheadrightarrow C''$ to C' and C'', call it $\tilde{\Psi}^{-1}$. Hence the composition

$\Psi^{-1}\Psi(C) = Id(C) = C$ is homotopic to

$$\overset{\sim}{\Psi}^{-1}\psi(C) \equiv \left\{ \begin{array}{ccc} C_b & & \\ \downarrow{\scriptstyle \partial_b} & & \\ C_{b-1} & & \\ \downarrow & & \\ \vdots & & \\ \downarrow & & \\ C_{a+1} & & \\ \downarrow & \searrow^{0} & \\ \operatorname{Im}\partial_{a+1} & \oplus & H_a C \end{array} \right\} .$$

8.2.4. On the other hand, by the Additivity Theorem, 2.0.20, the categories

$$E\left([C(\mathcal{U})_0^{\overline{q}}]_a^b, [C(\mathcal{U})^{\overline{q}}]_a^b, [C(\mathcal{U})^{\overline{q}}]_a^a \cong \mathcal{A}\right) \cong [C(\mathcal{U})_0^{\overline{q}}]_a^b \times \mathcal{A}$$

are K-theoretically equivalent. This together with the equivalence of categories Ψ and 8.2.3 yields the existence of a functor

$$\overline{\Psi} : [C(\mathcal{U})^{\overline{q}}]_a^b \overset{\cong}{\to} [C(\mathcal{U})_0^{\overline{q}}]_a^b \times \mathcal{A} \ ,$$

given by

$$\overline{\Psi}(C) \equiv (C', C'') \equiv \left(\left\{ \begin{array}{c} C_b \\ \downarrow{\scriptstyle \partial_b} \\ C_{b-1} \\ \downarrow \\ \vdots \\ \downarrow \\ C_{a+1} \\ \downarrow \\ \operatorname{Im}\partial_{a+1} \end{array} \right\}, H_a C \right)$$

induces a homotopy equivalence on K-theory.

8.2.4.1. For each pair of integers $a \le b$, there is a 'natural' functor

$$\Delta : [C(\mathcal{U})_0^{\overline{q}}]_a^b \longrightarrow E\left([C(\mathcal{U})^{\overline{q}}]_{a+1}^b, [C(\mathcal{U})_0^{\overline{q}}]_a^b, [C(\mathcal{U})^{q}]_a^{a+1} \cong \mathcal{U}\right)$$

which induces a homotopy equivalence on K-theory. Actually, Δ is an equivalence of categories. Given C in $[C(\mathcal{U})_0^{\overline{q}}]_a^b$,

$$C \equiv \left\{ \begin{array}{c} C_b \\ \downarrow{\scriptstyle \partial_b} \\ C_{b-1} \\ \downarrow \\ \vdots \\ \downarrow \\ C_{a+1} \\ \downarrow{\scriptstyle \partial_{a+1}} \\ C_a \end{array} \right\} \longrightarrow \Delta(C) \equiv \{\, \overline{C}' \rightarrowtail C \twoheadrightarrow \overline{C}'' \,\} \equiv \left\{ \begin{array}{ccccc} C_b & = & C_b & \longrightarrow & \star \\ \downarrow{\scriptstyle \partial_b} & & \downarrow{\scriptstyle \partial_b} & & \| \\ C_{b-1} & = & C_{b-1} & \longrightarrow & \star \\ \downarrow & & \downarrow & & \| \\ \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow \\ C_{a+2} & = & C_{a+2} & \longrightarrow & \star \\ \downarrow & & \downarrow{\scriptstyle \partial_{a+2}} & & \downarrow \\ \ker \partial_{a+1} & \rightarrowtail & C_{a+1} & \twoheadrightarrow & C_a \\ \downarrow & & \downarrow{\scriptstyle \partial_{a+1}} & & \|{\scriptstyle \sim} \\ \star & \rightarrowtail & C_a & = & C_a \end{array} \right\}.$$

REMARK 8.2.3. The extension $\Delta(C)$can be given thanks to the existence of the short exact sequence $\ker \partial_{a+1} \rightarrowtail C_{a+1} \overset{\partial_{a+1}}{\twoheadrightarrow} C_a$. This existence is argumented in 8.2.4, in the same way that it will be done for 8.2.2.

The inverse to Δ, say Δ^{-1}, is given by the projection onto the middle term C of $\overline{C}' \rightarrowtail C \twoheadrightarrow \overline{C}''$. By the Additivity Theorem, 2.0.20, the map induced on K-theory by Δ^{-1} is homotopic to the sum of the maps on K-theory induced by sending $\overline{C}' \rightarrowtail C \twoheadrightarrow \overline{C}''$ to $\overline{C}'$ and $\overline{C}''$, call it $\tilde{\Delta}^{-1}$. Then $\Delta^{-1}\Delta(C)$ is homotopic to

$$\tilde{\Delta}^{-1}\Delta(C) \equiv \left\{ \begin{array}{ccc} C_b & & \\ \downarrow{\scriptstyle \partial_b} & & \\ C_{b-1} & & \\ \downarrow & & \\ \vdots & & \\ \downarrow & & \\ C_{a+2} & & \\ \downarrow & & \\ \ker \partial_{a+1} & \oplus & C_a \\ {\scriptstyle 0}\downarrow & \swarrow{\scriptstyle 1} & \\ C_a & & \end{array} \right\}.$$

8.2.5. On the other hand, by the Additivity Theorem, 2.0.20, the following two categories

$$E\left([C(\mathcal{U})^{\overline{q}}]^b_{a+1}, [C(\mathcal{U})^{\overline{q}}_0]^b_a, [C(\mathcal{U})^{\overline{q}}]^{a+1}_a \cong \mathcal{U}\right) \cong [C(\mathcal{U})^{\overline{q}}]^b_{a+1} \times \mathcal{U}$$

have homotopy equivalent K-theories. This together with 8.2.4.1 yields a functor

$$\overline{\delta} : [C(\mathcal{U})^{\overline{q}}_0]^b_a \xrightarrow{\cong} [C(\mathcal{U})^{\overline{q}}]^b_{a+1} \times \mathcal{U}$$

given by

$$\overline{\Delta}(C) \equiv \left(\overline{C}', \overline{C}''\right) \equiv \left(\left\{ \begin{array}{c} C_b \\ \Big\downarrow{\scriptstyle \partial_b} \\ \vdots \\ \Big\downarrow \\ C_{a+2} \\ \Big\downarrow \\ \ker \partial_{a+1} \\ \Big\downarrow \\ \star \end{array} \right\}, C_a \right)$$

which induces a homotopy equivalence on K-theory.

8.2.6. Combining 8.2.4 and 8.2.5 we can conclude that for each pair of integers $a \leq b$ there is a functor inducing homotopy equivalence on K-theory

$$[C(\mathcal{U})^{\overline{q}}]^b_a \xrightarrow{\cong} \prod_a^b \mathcal{A} \times \prod_{a+1}^b \mathcal{U} \ ,$$

given by $C \mapsto \left(\prod_a^b H_i C, \prod_{a+1}^b \operatorname{Im} \partial_i \right)$. Hence

$$C(\mathcal{U})^{\overline{q}} \cong \lim_{\substack{b \to +\infty \\ a \to -\infty}} \left(\prod_a^b \mathcal{A} \times \prod_{a+1}^b \mathcal{U} \right).$$

8.2.7. Combining 8.2.3 and 8.2.4.1, the identity functor on $[C(\mathcal{U})^{\overline{q}}]_a^b$ is homotopic to the functor defined by

$$C \equiv \left\{ \begin{array}{c} C_b \\ \downarrow{\scriptstyle \partial_b} \\ \vdots \\ \downarrow \\ C_a \end{array} \right\} \mapsto \left\{ \begin{array}{ccc} C_b & & \\ \downarrow{\scriptstyle \partial_b} & & \\ \vdots & & \\ \downarrow & & \\ C_{a+2} & & \\ \downarrow & & \\ \ker \partial_{a+1} & & \operatorname{Im} \partial_{a+1} \\ & \swarrow{\scriptstyle 1} & \\ \operatorname{Im} \partial_{a+1} & \oplus H_a C & \end{array} \right\}.$$

Iterating this process, we have that the identity functor on $C(\mathcal{U})^{\overline{q}}$ is homotopic to the functor ρ described below

$$C \equiv \left\{ \begin{array}{c} C_b \\ \downarrow{\scriptstyle \partial_b} \\ \vdots \\ \downarrow \\ C_a \end{array} \right\} \overset{\rho}{\mapsto} \left\{ \begin{array}{ccccc} & & H_b C \oplus & \oplus \operatorname{Im} \partial_b & \\ & \swarrow{\scriptstyle 1} & & & \\ \vdots & & \vdots & & \vdots \\ & \swarrow{\scriptstyle 1} & & & \\ \ker \partial_{a+1} & \oplus & H_{a+1} C & \oplus \operatorname{Im} \partial_{a+1} & \\ & \swarrow{\scriptstyle 1} & & & \\ \operatorname{Im} \partial_{a+1} & \oplus H_a C & & & \end{array} \right\}.$$

8.2.8. Applying the Generic Fibration lemma, 2.0.23, to

$$iC(\mathcal{U})^{\overline{q}} \xrightarrow{\pi} qC(\mathcal{U})^{\overline{q}}$$

where π is the exact functor induced by the identity, we obtain that the homotopy fiber is identified with the subcategory $i[C(\mathcal{U})^{\overline{q}}]^q = iC(\mathcal{U})^q$ of $iC(\mathcal{U})^{\overline{q}}$.

Since we are talking of spectra, it can also be said that

$$iC(\mathcal{U})^{\overline{q}} \xrightarrow{\pi} qC(\mathcal{U})^{\overline{q}}$$

is the homotopy cofiber map for $iC(\mathcal{U})^q \to iC(\mathcal{U})^{\overline{q}}$. By 8.2.5, π is homotopic to $\pi \circ \rho$. But $\pi \circ \rho$ factors through the subcategory $C(\mathcal{U})^{\mathcal{A}}$ of $C(\mathcal{U})^{\overline{q}}$. Recall that $C(\mathcal{U})^{\mathcal{A}}$ is the full subcategory of $\mathcal{A}$-dominated objects in $C(\mathcal{U})$. Notice that for C in $C(\mathcal{U})^{\overline{q}}$, $\rho(C)$ is clearly homotopy equivalent to a chain complex in $\mathcal{A}$, hence $\mathcal{A}$-dominated.

Moreover,

$$iC(\mathcal{U}) \longrightarrow iC(\mathcal{U})^{\overline{q}} \xrightarrow{\pi \circ \rho} qC(\mathcal{U})^{\mathcal{A}}$$

is null-homotopic. Then $qC(\mathcal{U})^{\mathcal{A}}$ is the homotopy cofiber to $iC(\mathcal{U})^q \to iC(\mathcal{U})^{\overline{q}}$ and the inclusion

$$qC(\mathcal{U})^{\mathcal{A}} \longrightarrow qC(\mathcal{U})^{\overline{q}}$$

induces a homotopy equivalence on K-theory. But by [**CP95**][Lemma 4.8], under the right conditions on which we are, see below, $C(\mathcal{U})^{\mathcal{A}}$ is equivalent to $C(\mathcal{A})$.

Hence $qC(\mathcal{A}) \longrightarrow qC(\mathcal{U})^{\overline{q}}$ is a homotopy equivalence on K-theory as required.

Remark 8.2.4. In 8.2.2 and 8.2.3, we only need the existence of the kernels for epimorphisms in $\mathcal{A}b(\mathcal{U})$ in order to obtain the required exact sequences. Because of C.4.3 on page 90, we can assume without K-theoretical consequences that $\mathcal{U}$ satisfies the closure property mentioned in 3.0.32: closed under kernels of epimorphisms in $\mathcal{A}b(\mathcal{U})$.

8.3. Alternative proof for iii)

This section describes an alternative to the arguments in section 8.2.

Thomason's proof of Theorem 1.11.7, shows that the inclusion $\mathcal{E} \hookrightarrow C(\mathcal{E})$ induces a homotopy equivalence of K-theories

$$iK(\mathcal{E}) \xrightarrow{\cong} qK(C(\mathcal{E}))\ .$$

The proof can be described by a diagram like the following:

$$(8.8.3.0.1) \qquad \lim_{\substack{b\to+\infty \\ a\to-\infty}} \left\{ \begin{array}{ccc} i[C(\mathcal{E})^q]_a^b & \overset{\varphi}{\cong} & \prod_{a+1}^{b} i\mathcal{E} \\ \downarrow & & \downarrow j \\ i[C(\mathcal{E})]_a^b & \overset{\psi}{\cong} & \prod_{a}^{b} i\mathcal{E} \\ \downarrow & & \downarrow \chi \\ qC(\mathcal{E}) & \xleftarrow[i]{} & \mathcal{E} \end{array} \right\} .$$

The diagram is homotopically commutative for each pair of integers $a \leq b$ and the final result is obtained at the limit. The exact functors heuristically work as follows:

i) $\varphi(C) = (\operatorname{Im} \partial_{a+1}, \dots, \operatorname{Im} \partial_b)$, which induces homotopy equivalence;
ii) $\psi(C) = (C_a, \dots, C_b)$, which also induces homotopy equivalence;
iii) $j\varphi(C) = (\operatorname{Im} \partial_{a+1}, \operatorname{Im} \partial_{a+1} \oplus \operatorname{Im} \partial_{a+2}, \dots, \operatorname{Im} \partial_{b-1} \oplus \operatorname{Im} \partial_b, \operatorname{Im} \partial_b)$;
iv) $\chi(C_a, \dots, C_b) = \sum_a^b (-1)^k C_k$ and
v) i is the inclusion as a chain complex of length 0.

The idea behind is that $C(\mathcal{E}) = \lim_{\substack{b\to+\infty \\ a\to-\infty}} i[C(\mathcal{E})]_a^b$, for weak equivalences w.

In our context, regard the diagram above for the exact subcategory $\mathcal{A}$. Under the hypothesis of the Main Theorem, 7.0.63 on page 51, the subcategory $\mathcal{A}$ is abelian and hence satisfies the conditions required for the proof of the theorem 6.0.61 by

Thomason. For our purposes it is enough to show that for integers $a \leq b$ there exists the following homotopy commutative diagram:

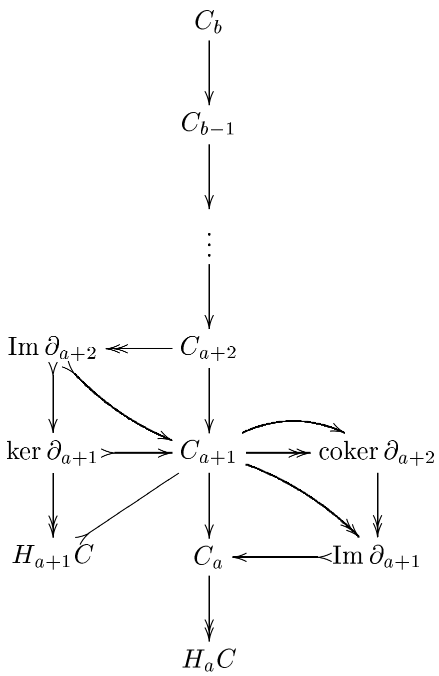
(8.8.3.0.2)
$$\begin{array}{ccc} i[C(\mathcal{A})]_a^b & \xrightarrow{\psi} & \prod_a^b i\mathcal{A} \\ \Big\downarrow q & \nearrow \Sigma & \\ \prod_a^b i\mathcal{A} \times \prod_{a+1}^b i\mathcal{A} & & \end{array}.$$

The exact functors are defined as follows:

i) ψ the same one as above;
ii) $q(C) = (H_aC, H_{a+1}C, \ldots, H_bC) \times (\operatorname{Im}\partial_{a+1}, \ldots, \operatorname{Im}\partial_b)$ and
iii) $\Sigma(C) = (H_aC \oplus \operatorname{Im}\partial_{a+1}, \operatorname{Im}\partial_{a+1} \oplus H_{a+1}C \oplus \operatorname{Im}\partial_{a+2}, \ldots, \operatorname{Im}\partial_b \oplus H_bC)$.

We already know by the proof for 6.0.61 on page 47 ψ induces a homotopy equivalence. Now, we justify the definition of the rest of the functors and the commutativity of the diagram.

In a first step, in order to define an iterative construction, we aim for the existence of the following diagram:



By the Additivity Theorem, 2.0.20, and the existence of the short exact sequence

$$\operatorname{Im}\partial_{a+1} \rightarrowtail C_a \twoheadrightarrow H_aC$$

can be deduced that the projection sending C to C_a is homotopic to the sum of the maps induced on K-theory by sending C to $\operatorname{Im}\partial_{a+1}$ and to H_aC.

Again, by the Additivity Theorem and the existence of short exact sequence

$$\ker\partial_{a+1} \rightarrowtail C_{a+1} \twoheadrightarrow \operatorname{Im}\partial_{a+1}$$

the projection sending C to C_{a+1} is homotopic to the sum of the maps induced on K-theory by sending C to $\operatorname{Im}\partial_{a+1}$ and to $\ker\partial_{a+1}$. Now, using the short exact sequence

$$\operatorname{Im}\partial_{a+2} \rightarrowtail \ker\partial_{a+1} \twoheadrightarrow H_{a+1}C$$

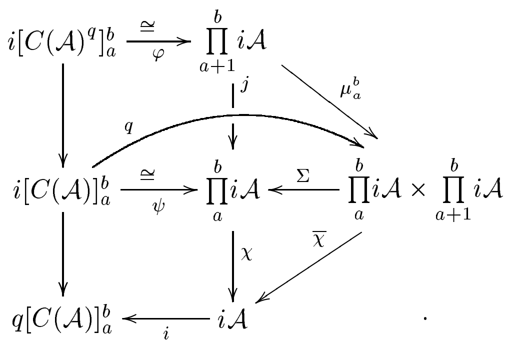
we can say that that the map induced on K-theory by sending C to $\ker\partial_{a+1}$ is homotopic to the sum of the maps induced on K-theory by sending C to $\operatorname{Im}\partial_{a+2}$ and to $H_{a+1}C$. Then, the map induced on K-theory by sending C to C_{a+1} is homotopic to the sum of the maps induced on K-theory by sending C to $\operatorname{Im}\partial_{a+1}$, to $H_{a+1}C$ and to $\operatorname{Im}\partial_{a+2}$.

This process can be continued up to dimension b. Moreover, this process would show that q induces a homotopy equivalence on K-theory.

Hence ψ would factor, on K-theory, up to homotopy, as $\Sigma \circ q$, i.e. the diagram 8.8.3.0.2 homotopy commutes.

Denote by $\overline{\chi}$ the map induced on K-theory by the composition of the exact functors $\chi \circ \Sigma$. The diagram 8.8.3.0.1 on page 64, forgetting for a moment the limit to the infinities, in our particular case is:

(8.8.3.0.3)



Since 8.8.3.0.2 is commutative, up to homotopy, then the homotopy fibers of $\chi \circ \psi$ and of $\chi \circ \Sigma \circ q = \overline{\chi} \circ q$, which is $iC(\mathcal{A})^q (\xrightarrow[\varphi]{\cong} \lim\limits_{\substack{b\to+\infty \\ a\to-\infty}} \prod_{a+1}^{b} i\mathcal{A})$, must be the same. Recall that when dealing with spectra homotopy fiber sequences and homotopy cofiber sequences are the same.

In particular, on K-theory, we have that the alternated sum of objects in a chain complex (the Euler Characteristic) induces a map which is homotopic to the alternated sum of the homology of that chain complex.

Hence the homotopy fiber of $\overline{\chi} \circ q : iC(\mathcal{A}) \to i\mathcal{A}$ is $iC(\mathcal{A})^q$, i.e.

$$iC(\mathcal{A})^q \longrightarrow iC(\mathcal{A}) \xrightarrow{q \circ \overline{\chi}} i\mathcal{A}$$

is also a homotopy fibration.

Let us come back to our problem. We have the following diagram

(8.8.3.0.4)
$$\begin{array}{ccccc}
iC(\mathcal{A})^q & \longrightarrow & iC(\mathcal{U})^q & = & iC(\mathcal{U})^q \\
\downarrow & & \downarrow & & \downarrow \\
iC(\mathcal{A}) & \longrightarrow & iC(\mathcal{U})^{\overline{q}} & \longrightarrow & iC(\mathcal{U}) \\
\downarrow & & \downarrow & & \downarrow \\
qC(\mathcal{A}) & \longrightarrow & qC(\mathcal{U})^{\overline{q}} & \longrightarrow & qC(\mathcal{U}) \\
\simeq \uparrow & & & & \simeq \uparrow \\
i\mathcal{A} & & \hookrightarrow & & i\mathcal{U}
\end{array}$$

where the vertical sequences are homotopy fibrations. Notice that $iC(\mathcal{U})^q = i\left[C(\mathcal{U})^{\overline{q}}\right]^q$. Based on the discussion above, the construction in the earlier section, 8.1, and the proof for Thomason's result, 6.0.61 or 8.8.3.0.1 on page 64, we can change the diagram 8.8.3.0.4, up to limits when $b \to +\infty$ and $a \to -\infty$, by the next one:

$$\begin{array}{ccccc}
\prod_{a+1}^{b} i\mathcal{A} & \longrightarrow & \prod_{a+1}^{b} i\mathcal{U} & \overset{j_{\mathcal{U}}}{\searrow} & \\
\downarrow \mu_a^b & & \downarrow \lambda_a^b & & \\
\prod_{a}^{b} i\mathcal{A} \times \prod_{a+1}^{b} i\mathcal{A} & \longrightarrow & \prod_{a}^{b} i\mathcal{A} \times \prod_{a+1}^{b} i\mathcal{U} & \longrightarrow & \prod_{a}^{b} i\mathcal{U} \\
\downarrow \overline{\chi} & & \downarrow \overline{\overline{\chi}} & & \downarrow \chi_{\mathcal{U}} \\
i\mathcal{A} & \hookrightarrow & \text{cofiber } \lambda_a^b & \longrightarrow & i\mathcal{U}
\end{array}$$

Since $\overline{\overline{\chi}}$, which gives the cofiber to λ_a^b, clearly factors through $i\mathcal{A}$, the cofiber of λ_a^b is the same as he cofiber for μ_a^b which K-homotopically is $i\mathcal{A}$. This would finish the proof for iii).

REMARK 8.3.1. Along this section, we have given for good the existence of certain short exact sequences. The reason is the same that the one in 8.2.4: $\mathcal{U}$ is assumed to satisfy closure property in 3.0.32 by application of C.4.3.

CHAPTER 9

Applications

The Main Theorem, 7.0.63, is equivalent to the Mixed Localization Theorem by Levine, [**Lev83**][Appendix A]. The setting and the proof are though completely new.

THEOREM 9.0.2 (Mixed Localization Theorem). [**Lev83**]*[pages 171-174] Let $\mathcal{A}$ be a Serre subcategory of an abelian category $\mathcal{M}$. Let H be an exact subcategory of $\mathcal{M}$ containing $\mathcal{A}$ as an exact subcategory. We assume the following condition:*

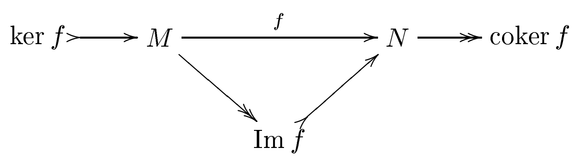
(*) If N' is in $\mathcal{M}$, N in $\mathcal{H}$, and $u : N' \to N$ a monomorphism in $\mathcal{M}$ then u is admissible in $\mathcal{H}$, i.e. $\operatorname{coker} u \in \mathcal{H}$*. Similarly, if $u' : N \to N'$ is an epimorphism in $\mathcal{M}$, then* $\ker u \in \mathcal{H}$*.*

Then $\mathcal{H}/\mathcal{A}$ is an exact subcategory of $\mathcal{M}/\mathcal{A}$, and $BQ\mathcal{A} \to BQ\mathcal{H} \to BQ\mathcal{H}/\mathcal{A}$ is a fibration. In particular, there is a long exact sequence

$$\cdots \to K_0(\mathcal{A}) \to K_0(\mathcal{H}) \to K_0(\mathcal{H}/\mathcal{A}) \to 0 \ .$$

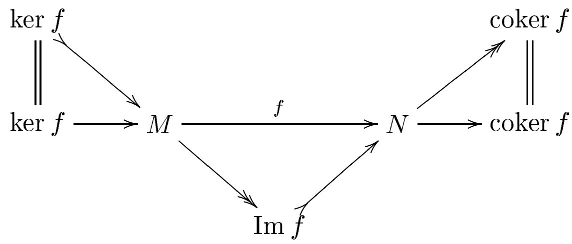
REMARK 9.0.3. a) Waldhausen shows in [**Wal85**] that for an exact category $\mathcal{E}$ with the usual cofibrations and weak equivalences, $BQ\mathcal{E}$ is homotopy equivalent to $K(i\mathcal{E})$. The sequence $BQ\mathcal{A} \to BQ\mathcal{H} \to BQ\mathcal{H}/\mathcal{A}$ is just our $K(i\mathcal{A}) \to K(i\mathcal{H}) \to K(i\mathcal{A}^{-1}\mathcal{H})$.

b) At first, it may look that the definitions of $\mathcal{A}^{-1}\mathcal{H}$ in this text and that of Levine, [**Lev83**][Appendix A], denote it as $\mathcal{H}/\mathcal{A}$ are different. But in both cases it is done calculus of fractions on $\mathcal{H}$ by the same class of morphisms. In case of $\mathcal{H}/\mathcal{A}$, the class is of those morphisms f in $\mathcal{H}$ whose kernel and cokernel in $\mathcal{M}$, the ambient abelian category, are objects in $\mathcal{A}$. In case of $\mathcal{A}^{-1}\mathcal{H}$, they are those f for which exist a factorization in $\mathcal{H}$



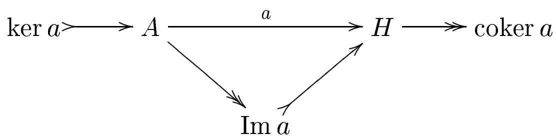
such that $\ker f$ and $\operatorname{coker} f$ are in $\mathcal{A}$. Clearly this second kind are part of those of the first kind.

Let us see now the other inclusion. Under our hypothesis, if f has $\ker f$ and $\operatorname{coker} f$, computed in $\mathcal{M}$, in $\mathcal{A}$ then Axioms (ii) and (iii) imply the existence of the next diagram:



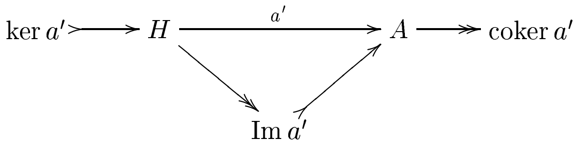
a ker-coker factorization. Notice that actually the diagrams given by Axioms (ii) and (iii) are unique, because of the uniqueness, up to isomorphisms, of the ker-coker factorizations in abelian categories. Hence f is also of the second kind.

9.0.4. Given a morphism $a : A \to H$, it has a unique ker-coker factorization as morphism of the abelian category $\mathcal{M}$.



Since $\mathcal{A}$ is Serre, $\ker a$ and $\operatorname{Im} a$ are objects in $\mathcal{A}$. By condition (*), $\operatorname{coker} a$ is in $\mathcal{H}$. So $\operatorname{Im} a \rightarrowtail H \twoheadrightarrow \operatorname{coker} a$ is exact in $\mathcal{H}$. Axiom (ii) is satisfied.

Similarly, given a morphism $a' : H \to A$, it has a unique ker-coker factorization as a morphism in the abelian category $\mathcal{M}$.



Since $\mathcal{A}$ is Serre, $\operatorname{coker} a'$ and $\operatorname{Im} a'$ are in $\mathcal{A}$. Because of condition (*) $\ker a'$ is in $\mathcal{H}$. Hence $\ker a' \rightarrowtail H \twoheadrightarrow \operatorname{Im} a'$ is exact in $\mathcal{H}$. Axiom (iii) is satisfied.

Under the conditions of 9.0.2 $\mathcal{A}$ localizes $\mathcal{H}$, therefore by 7.0.63:

$$K(i\mathcal{A}) \to K(i\mathcal{H}) \to K(i\mathcal{A}^{-1}\mathcal{H})$$

is a fibration up to homotopy. We have the conclusion of 9.0.2.

9.0.5. In particular, 7.0.63 implies Quillen's Localization Theorem, [**Qui72**][page 113].

THEOREM 9.0.6 (Localization). *Let $\mathcal{B}$ be a Serre subcategory of $\mathcal{A}$, abelian category, let $\mathcal{A}/\mathcal{B}$ be the associated quotient abelian category,* [**Gab62**] *and* [**Swa68**], *and let $e : \mathcal{B} \to \mathcal{A}$, $s : \mathcal{A} \to \mathcal{B}/\mathcal{A}$ denote the canonical functors. Then there is a long exact sequence*

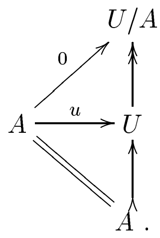
$$\cdots \to K_0(\mathcal{A}) \to K_0(\mathcal{H}) \to K_0(\mathcal{H}/\mathcal{A}) \to 0 \ .$$

Since $\mathcal{A}$ is abelian, and $\mathcal{B}$ is Serre it is immediate that $\mathcal{B}$ localizes $\mathcal{A}$. This can be seen as for 9.0.2 in 9.0.4.

9.0.7. As we have mentioned in the remark above, the factorizations that Axioms (ii) and (iii) provide are actually unique. This means that 9.0.2 implies 7.0.63.

Let$\mathcal{A} \subseteq \mathcal{U}$ be as in 7.0.63. Then let $H = \mathcal{A}b(\mathcal{U})$ be the abelian category given by the Gabriel-Quillen embedding theorem. It is only left to check condition (*) to be under the hypothesis of 9.0.2. But this is verified because of the uniqueness we have mentioned above.

If $u : A \to U$ is a monomorphism in $\mathcal{M}$, then the diagram given by Axiom (ii) must be



Hence $\operatorname{coker} u = U/A$ is in $\mathcal{U}$. Similarly for $u' : U \to A$ an epimorphism in $\mathcal{U}$.

APPENDIX A

Calculus of Fractions

This appendix is a collection of results from and [**Bas68**][Chapter 8,paragraph 5].

Definition A.0.8. Let $\mathcal{C}$ be a category and S a collection of arrows in $\mathcal{C}$. S is called a multiplicative system if it satisfies the following axioms:

MS1 If f, g are in S then the composition fg exists and is in S. For any object X in $\mathcal{C}$, id_X is in S.

MS2 Any diagram

$$\begin{array}{ccc} & & Z \\ & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{u} & Y \end{array}$$

in $\mathcal{C}$ with s in S can be completed to

$$\begin{array}{ccc} W & \xrightarrow{v} & Z \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{u} & Y \end{array}$$

with t in S.

MS3 If $f, g : X \to Y$ are morphisms in $\mathcal{C}$, the following are equivalent:

(a) $\exists s : Y \to Y'$ in S such that $sf = sg$.

(b) $\exists t : X' \to X$ in S such that $ft = gt$.

Or if it satisfies the axioms MS1, MS3 and MS2', the dual to MS2,

MS2' Any diagram

$$\begin{array}{ccc} & & Z \\ & & \uparrow{\scriptstyle s} \\ X & \xleftarrow{u} & Y \end{array}$$

in $\mathcal{C}$ with s in S can be completed to

$$\begin{array}{ccc} W & \xleftarrow{v} & Z \\ {\scriptstyle t}\uparrow & & \uparrow{\scriptstyle s} \\ X & \xleftarrow{u} & Y \end{array}$$

with t in S.

Definition A.0.9. Let $\mathcal{C}$ be a category and S a multiplicative system. The localization of $\mathcal{C}$ with respect to S (or calculus of fractions with respect to S) is a category $\mathcal{C}_S$ (or $S^{-1}\mathcal{C}$), together with a functor

$$Q : \mathcal{C} \longrightarrow S^{-1}\mathcal{C}$$

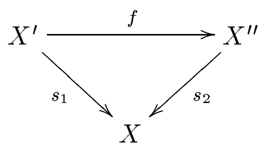
such that:

a) $Q(s)$ is an isomorphism for every $s \in S$.
b) Any functor $F : \mathcal{C} \longrightarrow \mathcal{D}$, such that $F(s)$ is an isomorphism for any $s \in S$, factors through Q.

PROPOSITION A.0.10. *$S^{-1}\mathcal{C}$ (or $\mathcal{C}_S$) can be obtained as follows:*

$$Obj S^{-1}\mathcal{C} = Obj\mathcal{C} \quad \mathrm{Hom}_{S^{-1}\mathcal{C}}(X,Y) = \varinjlim_{I_X} \mathrm{Hom}_{\mathcal{C}}(X',Y)$$

where I_X is the category whose objects are morphisms $s : X' \longrightarrow X$ in S and whose morphisms are commutative diagrams



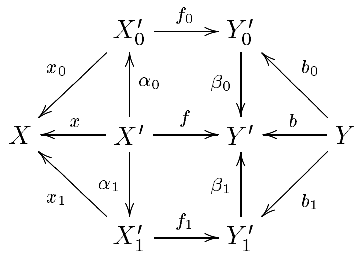
Furthermore, if $\mathcal{C}$ is an additive category, so is $S^{-1}\mathcal{C}$.

REMARK A.0.11. $\mathrm{Hom}_{S^{-1}\mathcal{C}}(X,Y)$ can also be obtained as $\varinjlim_{I_Y} \mathrm{Hom}_{\mathcal{C}}(X,Y')$ where $J_Y = \{s : Y \to Y'/s \in S\}$.

REMARK A.0.12. In case S verifies both MS2 and MS2' $\mathrm{Hom}_{S^{-1}\mathcal{C}}(X,Y)$ could be obtained as $\varinjlim_{I_X J_Y} \mathrm{Hom}(X',Y')$. Hence if $\overline{f} : X \to Y$ is a morphism in $S^{-1}\mathcal{C}$ it can be represented by a diagram

$$X \xleftarrow{x} X' \xrightarrow{f} Y' \xleftarrow{y} Y$$

in $\mathcal{C}$, which we will call a representation of $\overline{f}$, with $x, y \in S$. Moreover, if
$X \xleftarrow{x_i} X'_i \xrightarrow{f_i} Y'_i \xleftarrow{y_i} Y \quad i = 0, 1$
are two representations of $\overline{f}$, then there is a commutative diagram



whose middle row also represents $\overline{f}$. Note that then $\alpha_i, \beta_i \in S$ $i = 0, 1$.

The two basic facts about π other than those in the theorem-definition A.0.9 are:
i) Q is bijective on objects.
ii) Every morphism in $S^{-1}\mathcal{C}$ has a representation as we said above.

We shall now derive some further properties.

iii) Let $X \xleftarrow{x} X' \xrightarrow{f} Y' \xleftarrow{y} Y$ represent $\overline{f}$. Then we can form the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f_A} & Y'' \\ {\scriptstyle x}\uparrow & & \uparrow{\scriptstyle y'} \\ X' & \xrightarrow{f} & Y' \\ {\scriptstyle x''}\uparrow & & \uparrow{\scriptstyle y} \\ X'' & \xrightarrow[f_B]{} & Y \end{array}$$

by using MS2 and MS2'. Since $x, y \in S$ it follows $x', y' \in S$. Consequently we have new representations,

$$X \xrightarrow{f_A} Y'' \xleftarrow{b'b} Y$$

and

$$X \xleftarrow{xx'} X'' \xrightarrow{f_B} Y$$

iv) Suppose $X \xrightarrow{\overline{f}} Y \xrightarrow{\overline{g}} Z$ are two morphisms in $S^{-1}\mathcal{C}$ with representations

$$X \xleftarrow{x} X' \xrightarrow{f} Y' \xleftarrow{y} Y$$

and

$$Y \xleftarrow{y_1} X' \xrightarrow{g} Z' \xleftarrow{z} Z,$$

respectively. Then we construct the diagram

$$\begin{array}{ccccccc} X & \xleftarrow{x} & X' & \xrightarrow{f} & Y' & & \\ & & {\scriptstyle x'}\uparrow & & \uparrow{\scriptstyle y} & & \\ & & X'' & \xrightarrow{f_B} & Y & \xrightarrow{g_B} & Z'' \\ & & & & \uparrow{\scriptstyle y_1} & & \uparrow{\scriptstyle z'} \\ & & & & Y_1 & \xrightarrow{g} & Z' & \xleftarrow{z} & Z \end{array}$$

as in iii) and it is seen that

$$X \xleftarrow{xx'} X'' \xrightarrow{g_B f_B} Z'' \xleftarrow{z'z} Z$$

represents $\overline{gf}$.

APPENDIX B

The Gabriel-Quillen embedding

This appendix is extracted from [**TT90**][Appendix A]. We only list the main results.

THEOREM B.0.13. *Let $\mathcal{E}$ be a small exact category. Then there is an abelian category $\mathcal{A}b(\mathcal{E})$, and a fully faithful exact functor $i : \mathcal{E} \longrightarrow \mathcal{A}b(\mathcal{E})$ that reflect exactness. Moreover $\mathcal{E}$ is closed under extensions in $\mathcal{A}b(\mathcal{E})$.*

$\mathcal{A}b(\mathcal{E})$ may be canonically chosen to be the category of left exact functors $\mathcal{E}^{op} \longrightarrow \mathbb{Z}$-modules, and $i : \mathcal{E} \longrightarrow \mathcal{A}b(\mathcal{E})$ to be the Yoneda embedding $i(E) = \mathrm{Hom}_{\mathcal{E}}(\quad, E)$.

Let $\mathcal{B}$ be the abelian category of additive functors $F : \mathcal{E}^{op} \longrightarrow \mathbb{Z}$-modules, where $\mathbb{Z}$-modules is the category of abelian groups. Limits and colimits exist in $\mathcal{B}$, and are formed pointwise, so $\left(\varprojlim F_\alpha\right)(E) = \varprojlim(F_\alpha(E))$, etc. Then it is clear that direct colimits in $\mathcal{B}$ are exact, i.e., Grothendieck's axiom AB5 holds.

Also, $\mathcal{B}$ has a set of generators consisting of the functors $yE = \mathrm{Hom}(\quad, E)$ for E in $\mathcal{E}$. The Yoneda embedding $y : \mathcal{E} \longrightarrow \mathcal{B}$ is fully faithful by the Yoneda lemma. Thus $\mathcal{B}$ is a Grothendieck abelian category as is well-known.

DEFINITION B.0.14. Let $G : \mathcal{E}^{op} \longrightarrow \mathbb{Z}$-modules be an object of $\mathcal{B}$. One says G is separated if for all admissible epimorphisms $E \twoheadrightarrow F$ in $\mathcal{E}$, the induced map $G(F) \longrightarrow G(E)$ is a monomorphism. One says G "left exact" if for all admissible epimorphisms $E \twoheadrightarrow F$ in $\mathcal{E}$, then B.B.0.14.1 is a difference kernel, where the maps D are induced by the two projections $p : E \times_F E \longrightarrow E$:

$$\text{(B.B.0.14.1)} \qquad G(F) \longrightarrow G(E) \underset{d^1=G(p_1)}{\overset{d^0=G(p_0)}{\rightrightarrows}} G(E \times_F E)$$

Thus $G(F)$ is the kernel of $d^0 - d^1 : G(E) \longrightarrow G(E \times_F E)$.

B.0.15. Let $\mathcal{A}b(\mathcal{E})$ be the full subcategory of $\mathcal{B}$ consisting of the "left exact" functors $\mathcal{E}^{op} \longrightarrow \mathbb{Z}$-modules. Let $j_\star : \mathcal{A}b(\mathcal{E}) \longrightarrow \mathcal{B}$ be the inclusion. It is verified that $j_\star$ has a left adjoint $j^\star$ so that $j^\star j_\star = 1_{\mathcal{A}b(\mathcal{E})}$. Then $\mathcal{A}b(\mathcal{E})$ will be a Grothendieck abelian category such that $j^\star$ is an exact functor, and $j_\star$ is left exact (in the covariant abelian sense that $j_\star$ preserves kernels).

B.0.16. The Yoneda embedding $y : \mathcal{E} \longrightarrow \mathcal{B}$ factors through $\mathcal{A}$, so $y = j_\star \dot{i}$ for a functor $i : \mathcal{E} \longrightarrow \mathcal{A}$.

Then it can be shown that:

PROPOSITION B.0.17. *The Yoneda functor $i : \mathcal{E} \longrightarrow \mathcal{A}b(\mathcal{E})$ of B.0.16 is fully faithful and exact.*

The next two lemmas are crucial for the restricted completions in C.2.

LEMMA B.0.18. *Let* $e : E \longrightarrow F$ *be a map in* $\mathcal{E}$. *Then* $i(e)$ *is an epimorphism in* $\mathcal{A}b(\mathcal{E})$ *if and only if there is a* $k : E' \longrightarrow E$ *in* $\mathcal{E}$ *such that* $ek : E' \twoheadrightarrow F$ *is an admissible epimorphism.*

More generally, for any A *in* $\mathcal{A}b(\mathcal{E})$ *and* E *in* $\mathcal{E}$, *a map* $e : A \longrightarrow i(F)$ *in* $\mathcal{A}b(\mathcal{E})$ *is an epimorphism in* $\mathcal{A}b(\mathcal{E})$ *if and only if there is a* $k : i(E') \longrightarrow A$ *(i.e. a* $k \in A(E')$*) such that* $ek : E' \twoheadrightarrow F$ *is an admissible epimorphism in* $\mathcal{E}$.

LEMMA B.0.19. *(a) The embedding* $i : \mathcal{E} \to \mathcal{A}b(\mathcal{E})$ *reflects exactness.*

(b) If $\mathcal{E}$ *satisfies the extra axiom C.0.24 on the next page, and if* e *is a map in* $\mathcal{E}$ *such that* $i(e)$ *is an epimorphism in* $\mathcal{A}b(\mathcal{E})$, *then* e *is an admissible epimorphism in* $\mathcal{E}$.

LEMMA B.0.20. $\mathcal{E}$ *is closed under extensions in* $\mathcal{A}b(\mathcal{E})$.

LEMMA B.0.21. *Let* $f : \mathcal{E} \longrightarrow \mathcal{E}'$ *be an exact functor between exact categories. Then* f *preserves push outs along an admissible mono, and* f *preserves pullbacks along an admissible epi.*

PROPOSITION B.0.22. *Let* $f : \mathcal{E} \longrightarrow \mathcal{E}'$ *be an exact functor between exact categories. Let* $i : \mathcal{E} \longrightarrow \mathcal{A}b(\mathcal{E})$ *and* $i' : \mathcal{E}' \longrightarrow \mathcal{A}(\mathcal{E}')$ *be the Gabriel-Quillen embeddings into the categories of "left exact" functors.*

Then there is a right exact additive functor $f^\star : \mathcal{A}b(\mathcal{E}) \longrightarrow \mathcal{A}b(\mathcal{E}')$ *extending* f *in that* $f^\star i \cong i'$. *This* $f^\star$ *has an additive left exact functor right adjoint functor* $f_\star : \mathcal{A}b(\mathcal{E}') \longrightarrow \mathcal{A}b(\mathcal{E})$.

THEOREM B.0.23. *Let* $\mathcal{E}$ *be an exact category. Then*

a) There is a Karoubian C additive category $\mathcal{E}'$ *and a fully faithful additive functor* $f : \mathcal{E} \longrightarrow \mathcal{E}'$ *such that any additive functor from* $\mathcal{E}$ *to a Karoubian additive category factors uniquely-up-to-natural-isomorphism through* $\mathcal{E} \longrightarrow \mathcal{E}'$.

b) Every object in $\mathcal{E}'$ *is a direct summand in* $\mathcal{E}'$ *of an object in* $\mathcal{E}$. *We say a sequence in* $\mathcal{E}'$ *is exact if and only if it is a direct summand of an exact sequence in* $\mathcal{E}$. *This makes* $\mathcal{E}'$ *an exact category. This inclusion functor* $f : \mathcal{E} \longrightarrow \mathcal{E}'$ *is exact and reflects exactness, and* $\mathcal{E}$ *is closed under extensions in* $\mathcal{E}'$.

c) $K(\mathcal{E})$ *is a covering spectrum of* $K(\mathcal{E}')$, *in fact* f *induces an isomorphism of Quillen* K*-groups* $K_n(\mathcal{E}) \xrightarrow{\cong} K_n(\mathcal{E}')$ *for* $n \geq 1$, *and a monomorphism* $K_0(\mathcal{E}) \subseteq K_0(\mathcal{E}')$.

APPENDIX C

Exact Categories and Idempotent Completions

Many exact categories satisfy a stronger version of axioms e) and f), see 3, namely:

C.0.24. If $f : E \to F$ is a morphism in $\mathcal{E}$, and there is a morphism $s : F \to E$ which splits f so $fs = 1_F$, then f is an admissible epimorphism $E \twoheadrightarrow F$.

Remark C.0.25. In an exact category $\mathcal{E}$, the presence of C.0.24 implies its dual.

We shall see that assuming $\mathcal{E}$ having this property does not harm in K-theoretic terms. Next we digress about additive categories $\mathcal{U}$, but the results extend to exact categories $\mathcal{E}$.

Definition C.0.26. An additive category satisfies property (P) if given maps $f : E \to F$ and $s : F \to E$ such that $fs = 1_F$, then there is an object G and an isomorphism $E \cong F \oplus G$ under which f becomes projection on the first factor.

Property (P) obviously implies C.0.24. Hence, if property (P) is hold the exact category $\mathcal{E}$ would be closed under kernels of surjections in $\mathcal{A}b(\mathcal{E})$, see 3.0.32 and appendix B or [**TT90**][Lemma A.7.16 b)].

We do not wish to assume our categories satisfy property (P), and one of the aims of this section is to be able to replace an additive category by an additive category which does satisfy property (P) without changing its K-theory. This is obtained by considering suitable subcategories of the idempotent completion of an additive category.

The idempotent completion of an additive category $\mathcal{U}$, denoted $\mathcal{U}^\wedge$, is the additive category with objects (U,p) with $p = p^2 : U \to U$ and morphisms $f : (U,p) \to (V,q)$ satisfying $f = qfp : U \to V$. The identity morphism of (U,p) in $\mathcal{U}^\wedge$ is represented by p. We get an embedding of additive categories:

$$\mathcal{U} \hookrightarrow \mathcal{U}^\wedge$$

sending U to $(U,1)$ which is full and cofinal. The morphisms $f : (U,1) \to (V,1)$ in $\mathcal{U}^\wedge$ are precisely those in $\mathcal{U}$, and for every (U,p) in $\mathcal{U}^\wedge$

$$(U,p) \oplus (U,1-p) \underset{\binom{p}{1-p}}{\overset{(p,1-p)}{\rightleftarrows}} (U,1)$$

are isomorphisms expressing (U,p) as a direct summand of $(U,1)$. By the cofinality theorem 2.0.22 we have a fibration up to homotopy:

$$K(wS.\mathcal{U}) \to K(wS.\mathcal{U}^\wedge) \to B\pi$$

where $\pi = K_0(\mathcal{U}^\wedge)/K_0(\mathcal{U})$. In particular, this implies

$$K_0(\mathcal{U}) \rightarrowtail K_0(\mathcal{U}^\wedge).$$

Lemma C.0.27. *Property (P) holds for $\mathcal{U}^\wedge$.*

Proof. Let

$$(U,p) \underset{s}{\overset{r}{\rightleftarrows}} (V,q) \tag{C.0.28}$$

be such that $rs = q$. We have also

$$qrp = r \quad p^2 = p \quad psq = s \quad s^2 = s.$$

Now $(sr)(sr) = s(rs)r = sqr = (psq)qr = psq^2r = psqr = (psq)r = sr$ so (U, sr) makes sense in $\mathcal{U}^\wedge$ and moreover it is an idempotent for (U,p). Since $\mathcal{A}^\wedge$ is complete by definition we have

$$(U,p) \cong (U, p - sr) \oplus (U, sr). \tag{C.C.0.28.1}$$

Moreover

$$(U,sr) \underset{s}{\overset{r}{\rightleftarrows}} (V,q)$$

are isomorphic by those morphisms and therefore r in C.0.28 is an admissible epimorphism. □

C.0.29. The isomorphism C.C.0.28.1 is true by the following argument.

If $q\colon (U,p) \to (U,p)$ is such that $q^2 = q$, also $pqp = q$, then

$$(pqp)(pqp) = (pqp)$$

thus by the properties of the idempotent completion we have

$$(U,p) \cong (U,pqp) \oplus (U, p - pqp)$$

where the isomorphisms are given by the matrices

$$\begin{pmatrix} pqp \\ p - pqp \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} pqp, & p - pqp \end{pmatrix}.$$

$\mathcal{U}^\wedge$ is Karoubian or equivalently idempotent complete. We say that an additive category E is Karoubian if whenever $p\colon E \to E$ such that $p^2 = p$ then there is an isomorphism $E \cong E' \oplus E''$ under which p corresponds to the endomorphism $1 \oplus 0$. It is easy to see that the category $\mathcal{U}^\wedge$ satisfies a stronger property than property (P).

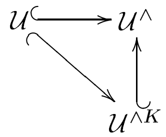
Lemma C.0.30. *If an exact category $\mathcal{U}$ is Karoubian, it satisfies the extra axiom C.0.24.*

The proof can be seen in [**TT90**][Lemma A.6.2].

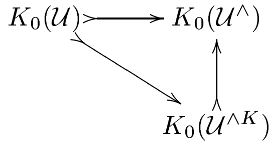
C.1.

Now let $\mathcal{A}$ be a full subcategory of the additive category $\mathcal{U}$.

DEFINITION C.1.1. Let $K \subset K_0(\mathcal{A}^\wedge)$ be the inverse image of $K_0(\mathcal{U})$ under the map $K_0(\mathcal{A}^\wedge) \to K_0(\mathcal{U}^\wedge)$. We shall denote the full subcategory of $\mathcal{U}^\wedge$ with objects $U \oplus (A,p)$, where $[(A,p)] \in K$ by $\mathcal{U}^{\wedge K}$. Notice that $\mathcal{A}^\wedge$ is embedded in $\mathcal{U}^\wedge$, $\mathcal{A}^\wedge \hookrightarrow \mathcal{U}^\wedge$.



$\mathcal{U}$ is cofinal in $\mathcal{U}^{\wedge K}$ and in $\mathcal{U}^\wedge$ hence $\mathcal{U}^{\wedge K}$ is cofinal in $\mathcal{U}^\wedge$. We thus obtain a diagram of monomorphisms



where the images of $K_0(\mathcal{U})$ and $K_0(\mathcal{U}^{\wedge K})$ in $K_0(\mathcal{U}^\wedge)$ are the same. Hence

$$K_0(\mathcal{U}) \to K_0(\mathcal{U}^{\wedge K})$$

is an isomorphism and therefore $\mathcal{U}$ and $\mathcal{U}^{\wedge K}$ have homotopy equivalent K-theories, by the cofinality theorem.

In a more general setting we give the following definition.

DEFINITION C.1.2. Given $\mathcal{U}$ an additive category and K a subgroup of $K_0(\mathcal{U})$. Let $\mathcal{U}^{\wedge K}$ be the full subcategory of $\mathcal{U}^\wedge$ with objects (U,p) so that its stable isomorphism class lies in K. When $K = K_0(\mathcal{U})$ we denote $\mathcal{U}^{\wedge K_0(\mathcal{U})}$ by $\overline{\mathcal{U}}$.

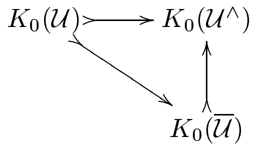
EXAMPLE C.1.3. If $\mathcal{U}$ is the category of finitely generated free R-modules for some ring R, then $\mathcal{U}^\wedge$ is equivalent to the category of finitely generated projective R-modules, and $\overline{\mathcal{U}}$ is equivalent to the category of finitely generated stably free R-modules.

REMARK C.1.4. Notice the following

(i) The category $\overline{\mathcal{U}}$ can be seen in terms of the first definition as $\mathcal{U}^{\wedge K_0(\mathcal{U})}$ by taking the trivial subcategory $\mathcal{A} = \mathcal{U}$.
(ii) Using the same notation for $\mathcal{U}^{\wedge K}$ in the two definitions above will not cause confusion since in one situation $K \subset K_0(\mathcal{A}^\wedge)$ and in the other $K \subset K_0(\mathcal{U}^\wedge)$.

LEMMA C.1.5. *The inclusion $\mathcal{U} \subset \overline{\mathcal{U}}$ induces isomorphism in K-theory*

PROOF. The category $\mathcal{U}$ is cofinal in $\overline{\mathcal{U}}$ and therefore



is a commutative diagram where all arrows are monomorphisms. By the same argument as above, $K_0(\mathcal{U}) \cong K_0(\overline{\mathcal{U}})$ is an isomorphism. Again, by the cofinality theorem 2.0.22, $\mathcal{U}$ and $\overline{\mathcal{U}}$ have homotopy equivalent K-theories. □

LEMMA C.1.6. *The category $\overline{\mathcal{U}}$ satisfies property (P) and hence C.0.24.*

PROOF. We can use an argument similar to the one used above for $\mathcal{U}^\wedge$. If we have the diagram in $\overline{\mathcal{U}}$

$$(U,p) \underset{s}{\overset{r}{\rightleftarrows}} (V,q)$$

with $rs = q$, it is also a map in $\mathcal{U}^\wedge$ and thus

$$(U,p) \cong (U, p - sr) \oplus (U, sr)$$

and

$$(U, sr) \cong (V, q).$$

But (U,p) and (V,q) are in $\overline{\mathcal{U}}$ so by the properties of K_0 and the definition of $\overline{\mathcal{U}}$ we conclude (U, sr) and $(U, p - sr)$ are in $\overline{\mathcal{U}}$. Hence r is an admissible epimorphism and $\overline{\mathcal{U}}$ satisfies the property (P). □

C.2.

Let $\mathcal{E}$ be an exact category. Then the idempotent completion $\mathcal{E}^\wedge$ of $\mathcal{E}$ can be given an structure of exact category so that the inclusion $\mathcal{E} \hookrightarrow \mathcal{E}^\wedge$ is exact and reflect exactness. Moreover $\mathcal{E}$ is closed under extensions in $\mathcal{E}^\wedge$.

We will say a sequence in $\mathcal{E}^\wedge$ is exact if and only if it is a direct summand of an exact sequence in $\mathcal{E}$. In other words, $A \to B \to C$ a sequence in $\mathcal{E}$ is exact if and only if there are objects $A', B' \in \mathcal{E}^\wedge$ so that $A \oplus A' \rightarrowtail A' \oplus B \oplus C' \twoheadrightarrow C \oplus C'$ is an exact sequence in $\mathcal{E}$.

C.2.1. We follow once more [**TT90**][Appendix A, pages 407-408]. To show that $\mathcal{E}^\wedge$ is an exact category consider the Gabriel-Quillen embedding $\mathcal{E} \hookrightarrow \mathcal{A}b(\mathcal{E})$. This induces a fully faithful functor between the idempotent completions, $\mathcal{E}^\wedge \hookrightarrow \mathcal{A}b(\mathcal{E})^\wedge$. By definition of exact sequence in the completion, and the fact $\mathcal{E} \hookrightarrow \mathcal{A}b(\mathcal{E})$ preserves and reflects exactness, the induced functor $\mathcal{E}^\wedge \hookrightarrow \mathcal{A}b(\mathcal{E})^\wedge$ preserves and reflects exact sequences. But as $\mathcal{A}b(\mathcal{E})$ already has images of idempotents, $\mathcal{A}b(\mathcal{E})^\wedge$ is equivalent to the abelian category $\mathcal{A}b(\mathcal{E})$.

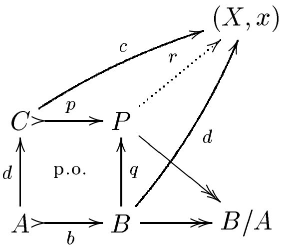
We claim that $\mathcal{E}^\wedge$ is closed under extensions in $\mathcal{A}b(\mathcal{E})$. For let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathcal{A}b(\mathcal{E})$ with A and C in $\mathcal{E}^\wedge$. Then there are $A', B' \in \mathcal{E}^\wedge$ so that $A' \oplus A$ and $C \oplus C'$ are isomorphic to objects of $\mathcal{E}$. The sequence $0 \to A' \oplus A \to A' \oplus B \oplus C' \to C \oplus C' \to 0$ is exact in $\mathcal{A}b(\mathcal{E})$, and shows that $A' \oplus B \oplus C'$ is isomorphic to an object in $\mathcal{E}$, since $\mathcal{E}$ is closed under extensions in $\mathcal{A}b(\mathcal{E})$, see appendix B. Thus B is a summand of an object in $\mathcal{E}$, hence is isomorphic to the image of an idempotent in $\mathcal{E}$, and hence is isomorphic to an object of $\mathcal{E}^\wedge$. This proves the claim. Now, $\mathcal{E}^\wedge$ is an exact category by 3.0.32. As the functors $\mathcal{E} \hookrightarrow \mathcal{A}b(\mathcal{E})$ and $\mathcal{E}^\wedge \hookrightarrow \mathcal{A}b(\mathcal{E})^\wedge$ preserve and reflect exactness, so does the functor $\mathcal{E} \hookrightarrow \mathcal{E}^\wedge$.

C.2.2. Alternatively to C.2.1 C.2 can shown in an abelian-free enviroment. Before that, a clear fact.

LEMMA C.2.3. *The embedding $\mathcal{E} \hookrightarrow \mathcal{E}^\wedge$ preserves push outs along admissible monomorphisms and pull backs along admissible epimorphisms.*

Proof. We will just show the first part, the second is dually proven.

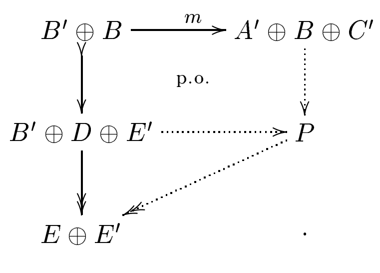
Let us have the following push out diagram in $\mathcal{E}$ embedded in the usual way in $\mathcal{E}^\wedge$.



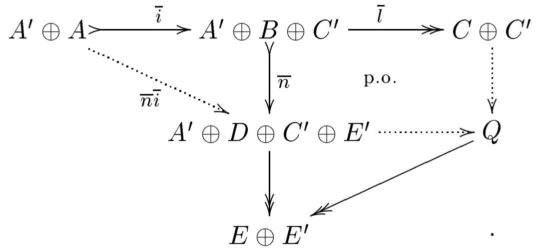
Let also (X, x) be an object in $\mathcal{E}^\wedge$ and $c : C = (C, 1) \to (X, x)$ and $d : B = (B, 1) \to (X, x)$ be morphisms in $\mathcal{E}^\wedge$ verifying that $ca = db$ in $\mathcal{E}^\wedge$. Then, by definition of completed category, $c = xc$ and $d = xd$ therefore $db = xdb = xca = ca$. Since P is push out in $\mathcal{E}$, there is a unique morphism $r : P \to X$ so that $rp = c$ and $rq = d$. But r also verifies the universal property for xc and xd, $xca = xdb$. Hence $xr = r$, i.e. r is a morphism in $\mathcal{E}^\wedge$, and P is a push out in $\mathcal{E}^\wedge$. □

C.2.4. Let us check the axioms 3.0.31 on page 21.

Axiom a). Let $A \xrightarrow{i} B$ and $B \xrightarrow{j} C$ be admissible monomorphisms corresponding to the exact sequences $A \xrightarrow{i} B \xrightarrow{l} C$ and $B \xrightarrow{j} C \xrightarrow{k} E$ in $\mathcal{E}^\wedge$. There are objects A', B', C', E' in $\mathcal{E}^\wedge$ so that $A' \oplus A \rightarrowtail^{\bar{i}} A' \oplus B \oplus C' \xrightarrow{\bar{l}} C \oplus C'$ and $B' \oplus B \rightarrowtail^{\bar{j}} B' \oplus D \oplus E' \xrightarrow{\bar{k}} E \oplus E'$ are exact sequences in $\mathcal{E}^\wedge$. Let $m : B' \oplus B \to A' \oplus B \oplus C'$ be the morphism defined as the identity on B and the zero morphisms on the rest, $m = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1_B & 0 \\ 0 & 0 & 0 \end{pmatrix}$ as a matrix. Since $B' \oplus B$ and $A' \oplus B \oplus C'$ are isomorphic to objects in $\mathcal{E}^\wedge$ and the embedding $\mathcal{E} \hookrightarrow \mathcal{E}^\wedge$ is full and faithful, the morphism m corresponds to a morphism in $\mathcal{E}^\wedge$. Take the following push out along m:

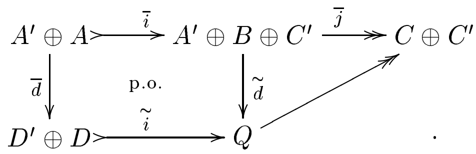


This diagram has P as a push out in the category $\mathcal{E}$, but the object $A' \oplus D \oplus C' \oplus E'$ also satisfies the universal property in the larger category $\mathcal{E}^\wedge$. Since the embedding $\mathcal{E} \hookrightarrow \mathcal{E}^\wedge$ is full;l and faithful, it preserves push outs and pull backs. The object $A' \oplus D \oplus C' \oplus E'$ is isomorphic to P, an object in $\mathcal{E}$. Moreover $A' \oplus B \oplus C' \rightarrowtail A' \oplus D \oplus C' \oplus E' \twoheadrightarrow E \oplus E'$ is exact in $\mathcal{E}$. Now the push out along $\overline{l} : A' \oplus B \oplus C' \twoheadrightarrow C \oplus C'$.

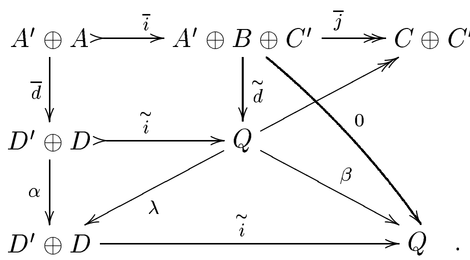


Then Q is an object in $\mathcal{E}$ from which clearly splits off $C' \oplus E'$ as objects in $\mathcal{E}^\wedge$. Hence Q can be written in $\mathcal{E}^\wedge$ as $R \oplus C' \oplus E'$. Then $\overline{n}\overline{i}$ is an admissible monomorphism in $\mathcal{E}$ and Q is its cokernel. Therefore $A' \oplus A \overset{\overline{n}\overline{i}}{\rightarrowtail} A' \oplus D \oplus C' \oplus E' \twoheadrightarrow Q = R \oplus C' \oplus E'$ is exact in $\mathcal{E}$. The sequence $A \overset{ji}{\to} D \to R$ can be recognized as a direct summand of the exact sequence above. This proves that the admissible monomorphisms are closed under composition. This is the first part of Axiom a).

Now, let $A \overset{i}{\to} B \overset{j}{\to} C$ be an exact sequence in $\mathcal{E}^\wedge$ and $A \overset{d}{\to} D$ be any morphism in $\mathcal{E}$. There are objects A', C' and D' in $\mathcal{E}^\wedge$ so that $A' \oplus A \overset{\overline{i}}{\rightarrowtail} A' \oplus B \oplus C' \overset{\overline{j}}{\twoheadrightarrow} C \oplus C'$ is exact in $\mathcal{E}$ and $D' \oplus D$ is isomorphic to an object in $\mathcal{E}$. Consider $\overline{d} : A' \oplus A \to D' \oplus D$ the corresponding morphism in $\mathcal{E}$ given by the matrix description in $\mathcal{E}^\wedge$, $\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$. We now take push out in $\mathcal{E}$ along $\overline{d}$:



Consider $\alpha : D' \oplus D \to D' \oplus D$ given by the matrix $\alpha + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. α is an idempotent in $\mathcal{E}$ and obviously the object $(D' \oplus D, \alpha)$ correspond to D' in $\mathcal{E}^\wedge$. Regard the following morphisms $\alpha\tilde{i} : D' \oplus D \to Q$, α and the zero morphisms from $A' \oplus B \oplus C'$ to Q and to $D' \oplus D$. By the universal property for the push outs, there are morphisms $\beta : Q \to Q$ and $\lambda : Q \to D' \oplus D$, making all the following squares commutative:



Notice that since $\alpha^2 = \alpha$, $\beta^2 = \beta$. It is left to show (Q,β) is isomorphic to $(D' \oplus D, \alpha) = D'$ in $\mathcal{E}^\wedge$. For that regard the isomorphisms

$$(D' \oplus D, \alpha) \underset{\alpha\gamma}{\overset{\tilde{i}\,\alpha}{\rightleftarrows}} (Q, \beta)$$

in $\mathcal{E}^\wedge$.

Similarly, let $\Delta : A' \oplus B \oplus C' \to A' \oplus B \oplus C'$ be given by $\Delta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
This defines an idempotent in $\mathcal{E}$. Regard the following morphisms Δ, $\tilde{d}\Delta$: $A' \oplus B \oplus C' \to Q$ and the zero morphisms from $D' \oplus D$ to $D' \oplus D$ and to Q. As before, the properties of the push out define morphisms $\rho : Q \to Q$ and $l : Q \to A' \oplus B \oplus C'$ respectively, with $\rho^2 = \rho$, $\Delta = l\tilde{d}$ and $\rho = \tilde{d}l$. Then,

$$(Q, \rho) \underset{\tilde{d}\Delta}{\overset{l\rho}{\rightleftarrows}} (A' \oplus B \oplus C', \Delta) = C'$$

define isomorphisms in $\mathcal{E}^\wedge$. Since the idempotents ρ and β are orthogonal, $(\rho+\beta)$ is also an idempotent of Q. Moreover, it can be shown that $(Q, (\rho+\beta))$ is isomorphic to $(D' \oplus D, \alpha) \oplus (A' \oplus B \oplus C', \Delta) = D' \oplus C'$. This means Q splits as $D' \oplus R \oplus C'$ and in the exact sequence $D' \oplus D \rightarrowtail D' \oplus R \oplus C' \twoheadrightarrow C \oplus C'$, the sequence $D \to R \to C$ can be seen as a summand. This shows that the exact sequences are closed under cobase change by push out along an arbitrary map. This is the second part of Axiom a).

Axiom b). The proof for Axiom a) dualizes for Axiom b).

Axiom c). This axiom is satisfied clearly because of the definition of an exact sequence in $\mathcal{E}^\wedge$.

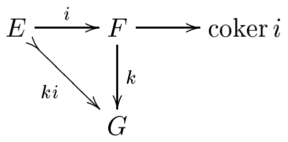
Axiom d). Let $A \xrightarrow{i} B \xrightarrow{j} C$ be exact in $\mathcal{E}^\wedge$. Let us see i is a kernel for j. Let $k : D \to B$ be a morphism in $\mathcal{E}^\wedge$ such that $jk = 0$ in $\mathcal{E}^\wedge$. There are objects A', C' and D' in $\mathcal{E}^\wedge$ so that $A' \oplus A \overset{\bar{i}}{\rightarrowtail} A' \oplus B \oplus C' \overset{\bar{j}}{\twoheadrightarrow} C \oplus C'$ is exact in $\mathcal{E}$ and $D \oplus D'$ is isomorphic to an object in $\mathcal{E}$. Let $\overline{k} : D \oplus D' \to A' \oplus B \oplus C'$ be the corresponding morphism in $\mathcal{E}$ to the one given by the matrix $\begin{pmatrix} 0 & 0 \\ k & 0 \\ 0 & 0 \end{pmatrix}$ in $\mathcal{E}^\wedge$. In the same way, $\bar{i}$ can be described by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & i \\ 0 & 0 \end{pmatrix}$ and $\bar{j}$ by $\begin{pmatrix} 0 & j & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
Then $\overline{jk} = 0$, and hence there is a morphism $\bar{l} : D \oplus D' \to A' \oplus A$ so that $\overline{il} = \overline{k}$. The morphism $\bar{l}$ may be described by a matrix $\begin{pmatrix} l_1 & l_3 \\ l_2 & l_4 \end{pmatrix}$ in $\mathcal{E}^\wedge$.

$$\overline{k} = \begin{pmatrix} 0 & 0 \\ k & 0 \\ 0 & 0 \end{pmatrix} = \overline{il} = \begin{pmatrix} 1 & 0 \\ 0 & i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_3 \\ l_2 & l_4 \end{pmatrix} = \begin{pmatrix} l_1 & l_3 \\ il_2 & il_4 \\ 0 & 0 \end{pmatrix}$$

In particular, the morphism $l_2 : D \to A$ satisfies that $il_2 = k$ in $\mathcal{E}^\wedge$. Hence i is a kernel for j.

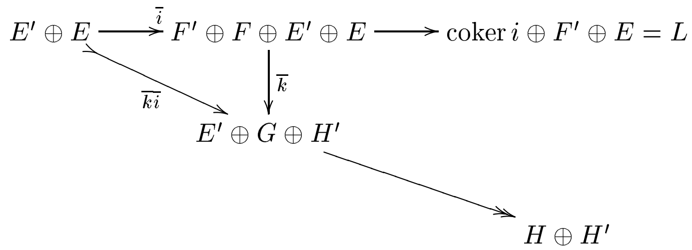
Dually, it can be shown that j is a cokernel for i.

Axiom e). Let the following diagram be in $\mathcal{E}^\wedge$.

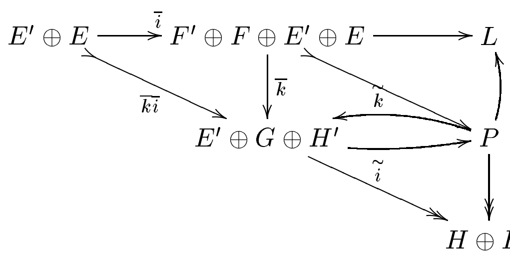


We have to show that the horizontal line at the top is short exact in $\mathcal{E}^\wedge$.

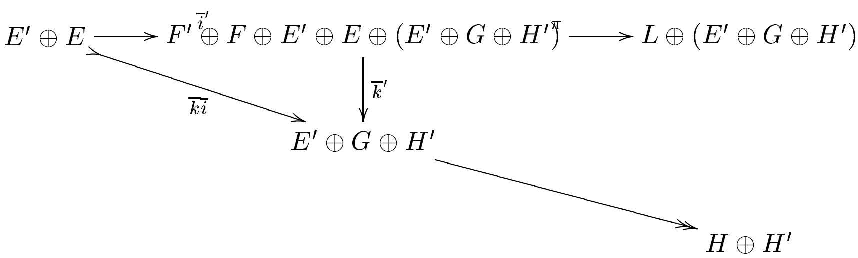
Let us call H the cokernel for ki. There are objects E', H', F' and K' in $\mathcal{E}^\wedge$ so that $E' \oplus E \rightarrowtail E' \oplus G \oplus H' \twoheadrightarrow H \oplus H'$ is exact in $\mathcal{E}$ and $F' \oplus F$ and $K' \oplus \operatorname{coker} i$ are isomorphic to objects in $\mathcal{E}$. In the same spirit as above, it can be described the following diagram:



The object we have called L is clearly a cokernel for $\bar{i}$ but so far in $\mathcal{E}^\wedge$. Take push out in $\mathcal{E}$ along $\bar{i}$.



This diagram is also a push out diagram in $\mathcal{E}^\wedge$ by C.2.3. The existence of $\overline{k}$ induces $\tilde{k}$, as already indicated above, which splits $\tilde{i}$: $\tilde{k}\tilde{i} = 1_{E'\oplus G\oplus H'}$. Hence P splits as $P = (E' \oplus G \oplus H') \oplus L$ in $\mathcal{E}^\wedge$. Therefore $E' \oplus G \oplus H') \oplus L$ is isomorphic to an object in $\mathcal{E}$. It can be described then the following diagram:

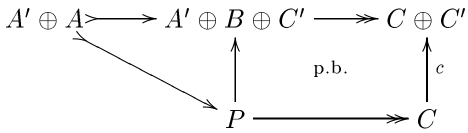


In this last diagram π is a cokernel for $\bar{i}'$. We are now able to apply Axiom e) in $\mathcal{E}$ to this diagram. Then the horizontal top line is short exact in $\mathcal{E}$. There it can be identified as a summand our original sequence $E \xrightarrow{i} F \to \operatorname{coker} i$, and hence it is exact.

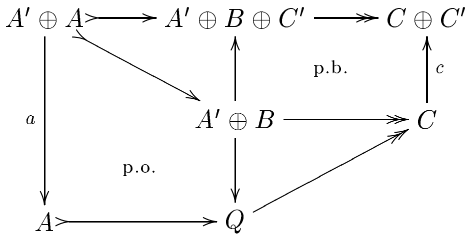
Axiom f). The proof for Axiom e) dualizes for Axiom f).

C.2.5. the exact category $\mathcal{E}$ is closed in $\mathcal{E}^\wedge$ under extensions.

Let $A \to B \to C$ an exact sequence in $\mathcal{E}^\wedge$, where A and C are isomorphic to objects in $\mathcal{E}$. As the sequence is exact in $\mathcal{E}^\wedge$, there are objects A', C' in $\mathcal{E}^\wedge$ so that $A' \oplus A \rightarrowtail A' \oplus B \oplus C' \twoheadrightarrow C \oplus C'$ is an exact sequence in $\mathcal{E}$. Since C is isomorphic to an object in $\mathcal{E}$, let $c : C \to C \oplus C'$ the corresponding morphism in $\mathcal{E}$ to the inclusion into the first factor, $c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Take pull back along c in $\mathcal{C}$.



As pull backs made in $\mathcal{E}$ persist in $\mathcal{E}^\wedge$, C.2.3, P can be seen ,as a pull back in $\mathcal{E}^\wedge$, as $A' \oplus B$. Again, as A is isomorphic to an object in $\mathcal{E}$, we can regard the morphism projecting onto A from $A' \oplus A$ as representing a morphism $a = (0, 1)$ in $\mathcal{E}$. Take now push out in $\mathcal{E}$ along a in the diagram above.



Once more, by C.2.3, push out in $\mathcal{E}$ persist in $\mathcal{E}^\wedge$. Then Q can be seen as a push out in $\mathcal{E}^\wedge$, where it is simply B. Hence B is isomorphic to an object in $\mathcal{E}$. This proves our claim.

C.3.

In section C.1, we have seen that $\mathcal{E}^\wedge$ can be given an exact structure so that $\mathcal{E}$ is fully embedded as an exact subcategory of its idempotent completion. Then the full subcategory $\overline{\mathcal{E}}$, C.1.2 on page 81, of $\mathcal{E}^\wedge$ can also be seen as an exact subcategory of $\mathcal{E}^\wedge$. $\mathcal{E}$ is also fully embedded in $\overline{\mathcal{E}}$. Recall lemma C.1.6, $\overline{\mathcal{E}}$ satisfies property (P) and hence C.0.24.

Lemma C.3.1. *The exact category $\mathcal{E}$ is cofinal in $\mathcal{E}^\wedge$.*

Proof. In C.2.5 and in C.2.1, we have seen that $\mathcal{E}$ is closed under extensions in $\mathcal{E}^\wedge$. By definition of $\mathcal{E}^\wedge$, see C, given (U, p) in $\mathcal{E}^\wedge$, then $(U, p) \oplus (U, 1-p)$ is isomorphic to $(U, 1)$, an object in $\mathcal{E}$. Hence we satisfy the cofinality conditions, 2.0.22. There is then a fibration up to homotopy:

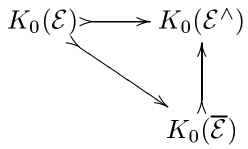
$$K(i.S.\mathcal{E}) \to K(i.S.\mathcal{E}^\wedge) \to B\pi$$

where $\pi = K_0(\mathcal{E}^\wedge)/K_0(\mathcal{E})$. $\square$

Moreover:

LEMMA C.3.2. *The inclusion of exact categories $\mathcal{E} \hookrightarrow \overline{\mathcal{E}}$ induces isomorphisms in K-theory.*

PROOF. From the proof above, it is clear that $\mathcal{E}$ is also cofinal in $\overline{\mathcal{E}}$. As for the proof of C.1.5, there is a commutative diagram



where all the arrows are monomorphisms. The images of $K_0(\mathcal{E})$ and $K_0(\overline{\mathcal{E}})$ in $K_0(\mathcal{E}^\wedge)$ are the same. Therefore $K_0(\mathcal{E}) \to K_0(\overline{\mathcal{E}})$ is an isomorphism. The cofinality theorem 2.0.22 on page 19 implies that $\mathcal{E}$ and $\mathcal{E}^\wedge$ have homotopy equivalent K-theories. □

C.4.

If $\mathcal{A}$ localizes $\mathcal{U}$, then $\mathcal{A}$ also localizes $\overline{\mathcal{U}}$ and $\mathcal{U}^\wedge$ as well. For this, we will need the next two results.

LEMMA C.4.1. *Let $A \overset{i}{\rightarrowtail} B \overset{j}{\twoheadrightarrow} C$ be an exact short sequence in $\mathcal{E}$ so that there is an isomorphism $\varphi : B \to V \oplus W$ in $\mathcal{E}^\wedge$ such that $\varphi i = \begin{pmatrix} a \\ 0 \end{pmatrix}$ with $a : A \to V$ a morphism in $\mathcal{E}^\wedge$. Then W can be splitted off, i.e. there are: an object R , a morphism $\pi : V \to R$ and compatible isomorphisms so that the given short exact sequence is isomorphic to*

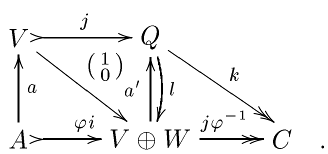
$$A \overset{\varphi i}{\rightarrowtail} V \oplus W \xrightarrow{\begin{pmatrix} \pi & 0 \\ 0 & 1_W \end{pmatrix}} R \oplus W \quad .$$

Hence

$$A \overset{a}{\rightarrowtail} V \overset{\pi}{\twoheadrightarrow} R$$

is an exact sequence in $\mathcal{E}^\wedge$.

PROOF. Given the exact sequence and the isomorphism φ, then $A \overset{\varphi i}{\rightarrowtail} V \oplus W \overset{j\varphi^{-1}}{\twoheadrightarrow} C$ is also exact in $\mathcal{E}^\wedge$. Take push out in $\mathcal{E}^\wedge$ along a. Notice that $\begin{pmatrix} 1 \\ 0 \end{pmatrix} : V \to V \oplus W$ induces an splitting of the the push out $l : Q \to V \oplus W$.



Since $\mathcal{E}^\wedge$ is idempotent complete, there is an isomorphism $\Psi : Q \to R \oplus (V \oplus W)$ so that $l\Psi^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the projection onto the $V \oplus W$ factor, and $\Psi a' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$, the injection into the second factor. This makes a' an admissible monomorphism with with cokernel isomorphic to R. Since the diagram is a push out, R is also a cokernel for a, and hence by Axiom e) for exact categories a is an admissible monomorphism with R as cokernel. In other words

$$A \overset{a}{\rightarrowtail} V \overset{\pi}{\twoheadrightarrow} R$$

is exact in $\mathcal{E}^\wedge$. Notice that $\begin{pmatrix} \pi & 0 \\ 0 & 1_W \end{pmatrix} : V \oplus W \to R \oplus W$ is also a cokernel for φi. Therefore $R \oplus W$ and C must be isomorphic in the completed category. The original sequence is then isomorphic to

$$A \overset{\varphi i}{\rightarrowtail} V \oplus W \overset{\left(\begin{smallmatrix} \pi & 0 \\ 0 & 1_W \end{smallmatrix}\right)}{\twoheadrightarrow} R \oplus W \quad .$$

□

There is also a dual lemma with a dual proof.

LEMMA C.4.2. *Let* $A \overset{i}{\rightarrowtail} B \overset{j}{\twoheadrightarrow} C$ *be an exact short sequence in* $\mathcal{E}$ *so that there is an isomorphism* $\varphi : B \to V \oplus W$ *in* $\mathcal{E}^\wedge$ *such that* $j\varphi^{-1} = \begin{pmatrix} b \\ 0 \end{pmatrix}$ *with* $b : V \to C$ *a morphism in* $\mathcal{E}^\wedge$. *Then* W *can be splitted off, i.e. there are: an object* T, *a morphism* $r : T \to V$ *and compatible isomorphisms so that the given exact sequence is isomorphic to*

$$T \oplus W \overset{\left(\begin{smallmatrix} r & 0 \\ 0 & 1_W \end{smallmatrix}\right)}{\rightarrowtail} V \oplus W \overset{j\varphi^{-1}}{\twoheadrightarrow} C \quad .$$

Hence

$$T \overset{b}{\rightarrowtail} V \overset{\pi}{\twoheadrightarrow} C$$

is an exact sequence in $\mathcal{E}^\wedge$.

In order to verify the localization axioms, 4.0.35 on page 25, for $\overline{\mathcal{U}}$ and $\mathcal{U}^\wedge$, we have to select certain family of short exact sequences.

Axiom (i) is satisfied since $\mathcal{A}$ is closed in $\mathcal{U}$ under extensions, because it localizes, and $\mathcal{U}$ is closed in $\overline{\mathcal{U}}$ and in $\mathcal{U}^\wedge$, see C.2.1 on page 82 or C.2.5 on page 87.

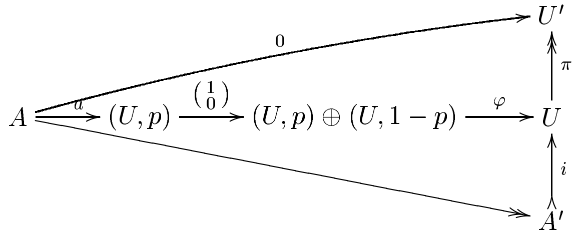
Axioms (ii) and (iii) are dual. We will only check Axiom (ii).

Let $a : A = (A, 1) \to (U, p)$ be a morphism in either $\overline{\mathcal{U}}$ or $\mathcal{U}^\wedge$ from an object A in $\mathcal{A}$. Then $(U, p) \oplus (U, 1-p)$ is isomorphic to U. Let us say $\varphi : (U, p) \oplus (U, 1-p) \to U$ is the isomorphism. Consider the composition:

$$A \xrightarrow{a} (U, p) \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} (U, p) \oplus (U, 1-p) \xrightarrow{\varphi} U \quad .$$

This is a morphism in $\mathcal{U}$, because $\mathcal{U}$ is fully embedded in $\mathcal{U}^\wedge$. Since $\mathcal{A}$ localizes, there is a short exact sequence associated to this morphism such that

(*)



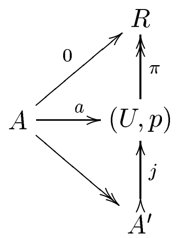
this diagram is commutative. Since φ is an isomorphism, then

$$A' \xrightarrow{\varphi^{-1} i} (U,p) \oplus (U, 1-p) \xrightarrow{\pi\varphi} U'$$

is also exact in $\overline{\mathcal{U}}$ or in $\mathcal{U}^\wedge$, depending on the case. From the diagram *, it is clear that i factors through (U,p) and hence $\varphi^{-1} i$ can be written as $\begin{pmatrix} j \\ 0 \end{pmatrix}$ for some $j : A' \to (U,p)$. Then by C.4.1, the vertical short exact sequence in * may be written as

$$A' \xrightarrow{\binom{j}{0}} (U,p) \oplus (U, 1-p) \xrightarrow{\left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right)} R \oplus (U, 1-p)$$

for some R in $\overline{\mathcal{U}}$ or in $\mathcal{U}^\wedge$, depending on the case. Hence $A' \xrightarrow{j} (U,p) \xrightarrow{\pi} R$ is exact in $\overline{\mathcal{U}}$ or in $\mathcal{U}^\wedge$, depending on the case, and



is commutative in $\overline{\mathcal{U}}$ or in $\mathcal{U}^\wedge$. Axiom (ii) is satisfied. Similar considerations can be made for Axiom (iii).

C.4.3. By lemma C.3.2, $\mathcal{U}$ and $\overline{\mathcal{U}}$ have the 'same' K-theory, $\mathcal{A}$ localizes $\overline{\mathcal{U}}$, see C.4 on page 88, and $\overline{\mathcal{U}}$ satisfies property (P), C.1.6 or C.3, therefore satisfies C.0.24 and hence it is closed under kernels of epimorphisms in $\mathcal{A}b(\mathcal{U})$, see B.0.19[b)]. This last property is required along the text in many places. By these results, we can replace from the beginning the exact category $\mathcal{U}$ by $\overline{\mathcal{U}}$, its restricted completion, to facilitate the computations without changing the final results.

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