CHAPTER 13

Stable Homotopy and Iterated Loop Spaces

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1. Introduction

Homology theory has been a very effective tool in the study of homotopy invariants for topological spaces. An important reason for this is the fact that it is often easy to compute homology groups. For instance, if one is given a finite simplicial complex, computing its homology becomes a straightforward problem in the linear algebra of finitely generated free modules over the integers. More generally, homology groups admit long exact Mayer–Vietoris sequences, which describe the homology, $H_*(X)$, of a space X which is a union of open subsets U and V in terms of $H_*(U)$, $H_*(V)$, and $H_*(U \cap V)$. In addition, under quite general circumstances when $A \subseteq X$ is a closed subspace, there is a long exact sequence

$$\cdots \longrightarrow \tilde{H}_{i+1}(X/A) \longrightarrow \tilde{H}_i(A) \longrightarrow \tilde{H}_i(X) \longrightarrow \tilde{H}_i(X/A) \longrightarrow \tilde{H}_{i-1}(A) \longrightarrow \cdots$$

where X/A denotes the result of identifying A to a point. Iterated applications of these long exact sequences are quite effective in computing the homology of many spaces.

Homotopy groups are much more difficult to compute. For instance, there are no finite CW-complexes except for the classifying spaces of certain infinite groups, for example bouquet of circles or compact closed surfaces, whose homotopy groups are known completely. The difficulty in carrying out this calculation can be traced in part to the nonexistence of an excision theorem for homotopy groups, and the consequent nonexistence of long exact Mayer–Vietoris sequences and long exact sequences of cofibrations.

It turns out to be possible, using a theorem of Freudenthal [17], to modify the homotopy groups a bit via a process of stabilization, so as to allow excision. The stabilization procedure goes as follows. For any space X, we have a homomorphism $\sigma : \pi_i(X) \longrightarrow \pi_{i+1}(\Sigma X)$, where ΣX denotes the suspension of X. σ applied to an element in $\pi_i(X)$ is obtained by suspending a representing map, and identifying ΣS^i with S^{i+1}. One can repeat this process and obtain a directed system

$$\cdots \longrightarrow \pi_{i+k}(\Sigma^k X) \longrightarrow \pi_{i+k+1}(\Sigma^{k+1} X) \longrightarrow \cdots$$

whose direct limit is defined to be $\pi_i^s(X)$, the i-th stable homotopy group of X. Freudenthal's theorem is that this limit is actually attained at a finite stage, in fact at $k = i$. A consequence of Freudenthal's theorem is that given a cofibration sequence $A \longrightarrow X \longrightarrow X/A$, one obtains a long exact sequence

$$\cdots \longrightarrow \pi_{i+1}^s(X/A) \longrightarrow \pi_i^s(A) \longrightarrow \pi_i^s(X) \longrightarrow \pi_i^s(X/A) \longrightarrow \pi_{i-1}^s(A) \longrightarrow \cdots$$

of stable homotopy groups, just as one would in the case of homology. For this reason, π_*^s is referred to as a generalized homology theory, since it now satisfies all the Eilenberg–Steenrod axioms for a homology theory with the exception of the dimension axiom, which identifies the value of the theory on a point. The generalized homology theory property is quite useful. It permits the construction of the Adams spectral sequence [3] and its variants, which are effective computational methods for stable homotopy groups. For instance, they have allowed the calculation of stable homotopy groups in a far larger range of dimensions than is currently possible for unstable groups.

The stabilization procedure described above for homotopy groups can also be carried out on the level of spaces, rather than groups. For any based space X, let ΩX denote the loop space of X, i.e. the set of based maps from the circle to X, equipped with the compact open topology (see [27]). Then suspension gives rise to maps $\sigma : \Omega^k \Sigma^k X \longrightarrow \Omega^{k+1} \Sigma^{k+1} X$, and hence homomorphisms

$$\pi_i(\sigma) : \pi_i(\Omega^k \Sigma^k X) \longrightarrow \pi_i(\Omega^{k+1} \Sigma^{k+1} X).$$

Via the standard adjoint identification $\pi_i(\Omega^k X) \cong \pi_{i+k}(X)$, we obtain a homomorphism $\pi_{i+k}(\Sigma^k X) \longrightarrow \pi_{i+k+1}(\Sigma^{k+1} X)$, which is easily seen to be equal to the map in the directed system defining $\pi_i^s(X)$. Freudenthal's theorem can now be interpreted as a statement about the connectivity of the inclusion $\Omega^k \Sigma^k X \longrightarrow \Omega^{k+1} \Sigma^{k+1} X$.

It has turned out to be possible to obtain very detailed information about the spaces $\Omega^k \Sigma^k X$. In fact one can give an explicit description of $H_*(\Omega^k \Sigma^k X)$ as a functor of $H_*(X)$, and produce explicit combinatorial constructions which are homotopy equivalent to the spaces $\Omega^k \Sigma^k X$. This line of work began with the James construction [19] for the case $k = 1$, and was extended to the case of all k by Milgram [24]. An alternate version, based on Boardman's "little cubes", was worked out by J.P. May [22]. Barratt and Eccles [6] developed a simplicial version for the limiting case $k = \infty$, and J. Smith [30] gave a simplicial version valid for all k.

The case $k = \infty$, i.e. $\lim_k \Omega^k \Sigma^k X$, is usually denoted $Q(X)$. It is called an "infinite loop space" since it is a k-fold loop space for all $k \geqslant 0$. Of course infinite loop spaces need not arise only in this way. What one needs are spaces Z_k, $k = 0, 1, 2, \ldots$, and identifications $Z_k \simeq \Omega Z_{k+1}$. The collection of spaces $\{Z_k\}_{k \geqslant 0}$ forms a *spectrum*. It turns out that a spectrum determines a generalized homology theory in the above sense. The spectrum $\{Q(S^k)\}_{k \geqslant 0}$ determines stable homotopy theory. Other spectra determine well known generalized homology theories such as K-theory, the various bordism theories, and of course ordinary singular homology theory.

The theory of iterated loop spaces described above can be used to give a structure on a space which assures that the space is the zeroth space in some spectrum. The relevant structure turns out to be a homotopy theoretic version of an abelian group structure. In particular, topological abelian groups are always infinite loop spaces. This result is J.P. May's "recognition principle" for the case $k = \infty$. It in turn allows the construction of spectra and hence generalized homology theories [11] out of category theoretic data, specifically from categories with a coherently commutative and associative sum operation. The category of finite sets gives stable homotopy theory under this construction.

In this chapter we discuss these ideas. The second section outlines the general homotopy theoretic information we will need. The third section gives a proof of Freudenthal's theorem and the generalized homology theory property of stable homotopy. Section 4 studies Spanier–Whitehead duality, which can be though of as a space level version of Lefschetz duality. Section 5 contains the James construction as well as results of Adams and Hilton [1] and Adams [2] concerning the structure of loop spaces of general spaces (not necessarily suspensions). In Section 6 we give a detailed discussion of double loop spaces. This serves to motivate and clarify the work in the following chapter, and the case $k = 2$ contains all the essential difficulties that occur for arbitrary k. Section 7 contains an extended discussion of all the models mentioned above for $\Omega^k \Sigma^k X$. Finally, in Section 8 we sketch May's recognition principle as well as Segal's Γ-space version, and describe the necessary category theoretic data for constructing spectra.

2. Prerequisites

We summarize some basic material from homotopy theory which we will be using. We assume the reader has the standard knowledge of homology theory, as well as of the definitions and elementary properties of homotopy groups.

2.1. *Basic homotopy theory*

Recall that the Hurewicz homomorphism $h_n : \pi_n(X,*) \longrightarrow H_n(X)$ is given by $h_n([f]) = H_n(f)(i_n)$ where i_n is the standard generator for $H_n(S^n)$. Throughout this paper $[x]$ will denote the equivalence class of x in various contexts. It should not create confusion.

DEFINITION 2.1.1. A space X is said to be *n-connected* if $\pi_i(X) = 0$ for $i \leqslant n$. A map $f : X \longrightarrow Y$ is said to be n-connected if $\pi_i(f)$ is an isomorphism for $i \leqslant n$ and $\pi_{n+1}(f)$ is surjective. A pair (X, Y) is said to be n-connected if the inclusion $Y \longrightarrow X$ is.

THEOREM 2.1.1 (Hurewicz, Absolute case). *If $\pi_n(X,*) = 0$ for $0 < n < N$, and X is connected, then $H_n(X) = 0$ for $0 < n < N$, and h_N is an isomorphism if $N \geqslant 2$. If $N = 1, h_N$ is just abelianization. Note that this also implies that if X is simply connected and $H_n(X) = 0$ for $0 < n < N$, then $\pi_n(X,*) = 0$ for $0 < n < N$.*

We shall also need the relative form of this theorem. First, recall the notion of the homotopy group (or set if $n = 1$) of a pair (A, B).

DEFINITION 2.1.2. Let (A, B) be a pair of spaces, i.e. B is a subspace of A. Then by $\pi_n(A, B)$, we mean the set of homotopy classes of maps of the standard n-cube which carry the boundary into B (and the bottom face to the basepoint). This is a set if $n = 1$, a (perhaps non-abelian) group if $n = 2$, and an abelian group if $n \geqslant 3$. We have a relative Hurewicz homomorphism $h_n(A, B) : \pi_n(A, B) \longrightarrow H_n(A, B)$defined in the obvious way.

We can now formulate the relative version of the Hurewicz theorem.

THEOREM 2.1.2 (Hurewicz, Relative form). *Suppose A and B are connected, $N \geqslant 2$ and $\pi_n(A, B) = 0$ for $0 < n < N$. Then, if $N \geqslant 3$, $H_n(A, B) = 0$ for $0 < n < N$ and $h_N : \pi_N(A, B) \longrightarrow H_N(A, B)$ is an isomorphism, and if $N = 2$, $h_2 : \pi_2(A, B) \longrightarrow H_2(A, B)$ is abelianization.*

COROLLARY 2.1.1 (Whitehead). *Let X and Y be CW complexes, and let $f : X \longrightarrow Y$ be a continuous map*[1]. *If X and Y are simply connected, and $H_n(f)$ is an isomorphism*

[1] Actually, all our maps are continuous so from here on we will simply call them maps.

for $0 < n < N$, *with* $N \geqslant 3$, *then* $\pi_n(f)$ *is an isomorphism for* $0 < n < N - 1$. *Conversely, if* $\pi_n(f)$ *is an isomorphism for* $0 < n < N$, *then* $H_n(f)$ *is an isomorphism for* $0 < n < N - 1$.

We also record the following standard result about CW complexes. Recall that a continuous map $f : X \longrightarrow Y$ is said to be a weak equivalence if $\pi_n(f)$ is an isomorphism for all n.

THEOREM 2.1.3. *Let* X *and* Y *be CW complexes, and suppose* $f : X \longrightarrow Y$ *is a weak equivalence. Then* f *is a homotopy equivalence. Also, suppose* X *and* Y *are simply connected, and suppose* $H_n(f)$ *is an isomorphism for all* n. *Then* f *is a homotopy equivalence.*

2.2. *Hurewicz fibrations*

We recall some parts of the theory of Hurewicz fibrations.

DEFINITION 2.2.1. A map $p : E \longrightarrow B$ is a Hurewicz fibration if for every pair of spaces (X, Y), and every commutative diagram

$$\begin{array}{ccc} X \times \{0,1\} \cup Y \times I & \longrightarrow & E \\ \downarrow & & \downarrow p \\ X \times I & \xrightarrow{H} & B \end{array}$$

there is a map $\bar{H} : X \times I \longrightarrow E$ making both triangles commute. If $F = p^{-1}(b)$, for a point $b \in B$, and B is path connected, we obtain a long exact sequence on homotopy groups,

$$\cdots \longrightarrow \pi_i(F) \longrightarrow \pi_i(E) \longrightarrow \pi_i(B) \longrightarrow \cdots .$$

Equivalently, $\pi_i(E, F) \longrightarrow \pi_i(B, b)$ is an isomorphism.

For us, a fibration will mean a Hurewicz fibration. In the case of a path-connected base space B, it follows directly from this definition that if b_0 and b_1 are points of B, then $p^{-1}(b_0)$ and $p^{-1}(b_1)$ are homotopy equivalent.

DEFINITION 2.2.2. Let X be a space. By a space over X we mean a space E together with a reference map $E \xrightarrow{r} X$. If $E_1 \xrightarrow{r_1} X$ and $E_2 \xrightarrow{r_2} X$ are spaces over X, then a map over X from $(E_1, r_1)$to (E_2, r_2) is a map $f : E_1 \longrightarrow E_2$ so that $r_1 = r_2 \circ f$.

For any space (E, r) over X, we have the space $(E \times I, r \circ p_E)$, over X, where $p_E : E \times I \longrightarrow E$ is the projection. With this construction, *homotopies over* X are defined in the evident way, as are homotopy equivalences.

- Note that a map f over X from (E_1, r_1) to (E_2, r_2) gives rise to a map $Cyl(f)$ from the mapping cylinder $Cyl(r_1)$ on r_1 to the mapping cylinder $Cyl(r_2)$ on r_2, and an

induced map $C(f)$ from the mapping cone $C(r_1)$ on r_1 to the mapping cone $C(r_2)$ on r_2.
– If f is a homotopy equivalence over X, then $C(f)$ is a homotopy equivalence.

DEFINITION 2.2.3. Let X and Y be spaces, and suppose (E, r) is a space over Y. Then if $f : X \longrightarrow Y$ is a continuous map, the pullback $f^*(E, r)$ is the space (f^*E, f^*r) over X, defined by letting f^*E be the subspace of $X \times E$ given by

$$f^*E = \{(x, e) \mid f(x) = r(e)\},$$

and letting f^*r be the composite $f^*E \longrightarrow X \times E \xrightarrow{\pi_X} X$. If r is a fibration, then so is f^*r.

The pullback operation has an important homotopy invariance property when r is a fibration.

PROPOSITION 2.2.1. *Suppose (E, r) is a space over Y, with r a fibration. Let $f, g : X \longrightarrow Y$ be homotopic continuous maps. Then $f^*(E, r)$ and $g^*(E, r)$ are homotopy equivalent spaces over X.*

PROOF. Let H be a homotopy from f to g, and consider the space $H^*(E, r)$ over $X \times I$. $i_0^* H^*(E, r) \cong f^*(E, r)$ and $i_1^* H^*(E, r) \cong g^*(E, r)$ as spaces over X. The homotopy lifting property for fibrations applied to the canonical homotopy from i_0 to i_1 gives a map α from $i_0^* H^*(E, r)$ to $i_1^* H^*(E, r)$ of spaces over X, and similarly we obtain a map $\beta : i_1^* H^*(E, r) \longrightarrow i_0^* H^*(E, r)$, also over X. We must show that $\alpha\beta$ and $\beta\alpha$ are homotopic to the identity over X. Consider $\beta\alpha$. From the way in which α and β were constructed, it is clear that there is a map $h : i_0^* H^*(E, r) \times I \longrightarrow H^*(E, r)$, so that the composite $H^*r \circ h$ is equal to $g \circ (i_0^* H^*r \times id)$, where $g : X \times I \longrightarrow X \times I$ is given by $g(x, t) = (x, 2t)$ for $0 \leqslant t \leqslant \frac{1}{2}$, and $g(x, t) = (x, 2 - 2t)$ for $\frac{1}{2} \leqslant t \leqslant 1$, and so that $h \mid i_0^* H^*(E, r) \times 0$ is the inclusion, and $H \mid i_0^* H^*(E, r) \times 1$ is $\beta\alpha$ composed with the inclusion. In view of the fact that there is an evident homotopy from g to the constant homotopy $\hat{g}^*, \hat{g}^*(x, t) = (x, 0)$, we may use the homotopy lifting property again to obtain a map $\hat{h}$ from $i_0^* H^*(E, r) \times I \longrightarrow H^*(E, r)$, so that $\hat{h} \mid i_0^* H^*(E, r) \times 0 \cup i_0^* H^*(E, r) \times 1 = h \mid i_0^* H^*(E, r) \times 0 \cup i_0^* H^*(E, r) \times 1$ and so that $H^*r \circ \hat{h} = \hat{g} \circ (i_0^* H^*r \times id)$. $\hat{h}$ is now the required homotopy over X from the identity on $i_0^* H^*(E, r)$ to $\beta\alpha$. The procedure works similarly for $\alpha\beta$. □

COROLLARY 2.2.1. *Let X be a space, and let (E, r) be a space over X, with r a fibration. Suppose X is contractible. Then for any $x \in X$, (E, r) is homotopy equivalent over X to the space $(X \times r^{-1}(x), \pi_X)$ over X.*

PROOF. This is an easy application of the preceding result. □

PROPOSITION 2.2.2. *Let X be a CW-complex, and let A be a subcomplex. Let Y be a space. Let $F(X, Y)$ denote the space of maps from X to Y, with the compact open topology (see [27]), then the restriction map $F(X, Y) \longrightarrow F(A, Y)$ is a Hurewicz fibràtion. Moreover, the inverse image of the constant map from A to Y is identified with $F(X/A, Y)$.*

PROOF. This fact follows directly from the fact that inclusions of subcomplexes of CW-complexes have the *homotopy extension property*, which is a dual condition to the homotopy lifting property characterizing Hurewicz fibrations. It states that if we are given a map $f : X \longrightarrow Y$ and a homotopy $H : A \times I \longrightarrow Y$ so that $H \mid A \times 0 = f \mid A$, then there exists an extension $\hat{H} : X \times I \longrightarrow Y$ so that $\hat{H} \mid X \times 0 = f$ and $\hat{H} \mid A \times I = H$. That this property holds in the case of the inclusion of a subcomplex of a CW-complex is proved in [21]. □

REMARK. Generally, a map having the homotopy extension property is referred to as a cofibration.

2.3. *Serre fibrations*

A Serre fibration has the same definition as an Hurewicz fibration except the spaces X and Y are restricted to being finite polyhedral complexes. These are particularly useful when we are dealing with mapping spaces $X^Y = \{f : Y \longrightarrow X \mid f \text{ continuous }\}$ which are assumed, as in 2.2.2, to have the compact-open topology.

Given any continuous map $f : Y \longrightarrow X$ we have the associated Serre fibration $E^{\bar{X}}_{Y,\bar{X}} \xrightarrow{\pi} X$ where $\bar{X}$ is the mapping cone of f, $E^C_{A,B}$ is the space of paths in C that start in A, end in B and $\pi : E^C_{A,B} \longrightarrow B$ is projection onto the endpoint. The fiber of π over the point x is the subspace $E^{\bar{X}}_{Y,x}$ and we have the commutative diagram

$$\begin{array}{ccccc} f^{-1}(x) & \hookrightarrow & Y & \xrightarrow{f} & X \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle =} \\ E^{\bar{X}}_{Y,x} & \hookrightarrow & E^{\bar{X}}_{Y,\bar{X}} & \xrightarrow{\pi} & X \end{array} \qquad (2.1)$$

where i includes $y \in Y$ as the constant path at y.

2.4. *Quasifiberings*

DEFINITION 2.4.1. A continuous map $f : Y \longrightarrow X$ is a quasifibration if and only if, for all $x \in X$, the map i above restricted to $f^{-1}(x)$ is a weak homotopy equivalence.

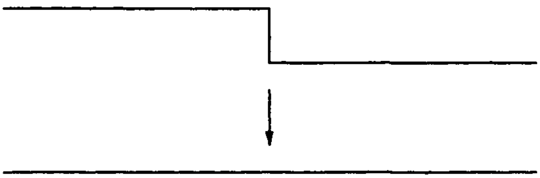
Using the 5-lemma this is equivalent to the condition

LEMMA 2.4.1. *$f : Y \longrightarrow X$ is a quasifibration if and only if, for all $x \in X$ and $y \in f^{-1}(x)$, the induced map of homotopy groups*

$$f_* : \pi_*(Y, f^{-1}(x), y) \longrightarrow \pi_*(X, x)$$

is an isomorphism.

Basically, it turns out that the difference between quasifibrations and Hurewicz fibrations is that with an Hurewicz fibration one can lift homotopies "on the nose", however, in a quasifibration, the weak equivalence condition limits the homotopies to finite cell complexes and homotopies can be lifted, but only "up to a homotopy". A good example to keep in mind is the map



(2.2)

which is a quasifibration but not an Hurewicz fibration.

One has notions of the equivalence of two quasifibrations, principal quasifibrations, and the equivalence of principal quasifibrations similar to those for bundles. However, the construction of "associated quasifibrations" is more difficult.

DEFINITION 2.4.2. (i) Two quasifibrations, $E \xrightarrow{p} B$ and $E' \xrightarrow{p'} B'$ are said to be equivalent if there are weak homotopy equivalences $f : E \longrightarrow E'$, $\bar{f} : B \longrightarrow B'$ so that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{\bar{f}} & B' \end{array} .$$

(ii) A quasifibration $p : E \longrightarrow B$ is a (left)-principal M-quasifibration if M is an associative, unitary H-space and there is a map $\mu : M \times E \longrightarrow E$ so that

a) $\mu(mm', e) = \mu(m, \mu(m', e))$ for all $m, m' \in M$, $e \in E$, (associative action).

b) $\mu(1, e) = e$ all $e \in E$ where $1 \in M$ is the unit, (unitary action).

c) $p(\mu(m, e)) = p(e)$ for all $e \in E$, $m \in M$, (fiber preserving).

d) $\mu(-, e) : M \longrightarrow p^{-1}p(e)$ is a weak homotopy equivalence for each $e \in E$.

(iii) Two principal M-quasifibrations $p : E \longrightarrow B$ and $p' : E' \longrightarrow B'$ are called structurally equivalent if they are equivalent via $f, \bar{f}$ where f preserves the M-structure.

The best references for the structure of quasifibrations are [16], [15], [32] and we summarize the results from [16, §2] that we will need in the sequel now.

The main tool for constructing lifts up to homotopy is the following result.

LEMMA 2.4.2. *Let $p : F \longrightarrow U$ be continuous, $V \subset U$ and $G = p^{-1}(V)$. Let K be an*

r-cell ($r \geqslant 0$) and assume that for all $x \in U$, $y \in p^{-1}(x)$ we have

$$\begin{cases} p_r : \pi_r(F,G,y) \longrightarrow \pi_r(U,V,x) & \textit{is a monomorphism,} \\ p_{r+1} : \pi_{r+1}(F,G,y) \longrightarrow \pi_{r+1}(U,V,x) & \textit{is an epimorphism.} \end{cases}$$

Then p has the following homotopy lifting property: suppose that we are given three maps

(i) $\bar{H} : (K \times I, K \times 1) \to (U,V)$,
(ii) $h : (K \times 0 \cup \partial K \times I, \partial K \times 1) \to (F,G)$,
(iii) $d : ((K \times 0 \cup \partial K \times I) \times I, (\partial K \times 1) \times I) \to U,V)$

with $d(z,t,0) = \bar{H}(z,t)$, $d(z,t,1) = p \circ h(z,t)$ for all $z \in K$, $t \in K$. Then there is a map

$$H : (K \times I, K \times 1) \longrightarrow (F,G)$$

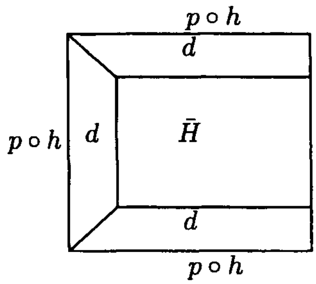
with $H|K \times 0 \cup \partial K \times 1$ equal to h, and a homotopy

$$D : (K \times I \times I, K \times 1 \times I) \longrightarrow (U,V)$$

filling in d in the sense that

$$\begin{aligned} D|(K \times 0 \cup \partial K \times 1) \times I &= d, \\ D(z,t,0) &= \bar{H}(z,t), \\ D(z,t,1) &= p \circ H(z,t). \end{aligned}$$

PROOF. h defines an element $a \in \pi_r(F,G)$ with $f(a) = 0 \in \pi_r(U,V)$ using $\bar{H}$ and d to construct the trivializing homotopy.



(2.3)

But since we assume that f_* on $\pi_r(F,G,y)$ is a monomorphism, it follows that $a = 0$, and there is a trivializing homotopy

$$H' : (K \times I, K \times 1) \longrightarrow (F,G)$$

with $H|(K \times 0) = h$. Adding the image of H' to the map in fig. 2.3, we have a map $H'' : (K \times I \times 0 \cup \partial(K \times I) \times I) \to U$ with $H''|\partial(K \times I) \times 1$ contained in V. H'' in turn defines an element $\gamma \in \pi_{r+1}(U, V, x)$ which may not be zero. However, we are free to modify the homotopy H' by any element $\beta \in \pi_{r+1}(F, G, y)$, and this will change γ to $\gamma + p_*(\beta)$. Consequently, since p_{r+1} is onto, we can assume γ represents 0 and the existence of the desired homotopy follows. □

We now give some geometric conditions which will guarantee that a map f is a quasifibration.

DEFINITION 2.4.3. Let $f : X \longrightarrow Y$ be a continuous map, and $U \subset Y$ be any subset. We say that U is distinguished for f if $f : f^{-1}(U) \longrightarrow U$ is a quasifibration.

LEMMA 2.4.3. *Suppose that $f : X \longrightarrow Y$ is a continuous map. Suppose that $Y' \subset Y$ is distinguished for f with $X' = f^{-1}(Y')$. Suppose that there are deformations*

$$\begin{aligned} D : I \times (X, X') &\longrightarrow (X, X'), \\ d : I \times (Y, Y') &\longrightarrow (Y, Y') \end{aligned}$$

so that $D_0 = id$, $d_0 = id$, $im(D_1) \subset X'$, $im(d_1) \subset Y'$, $f \circ D_1 = d_1 \circ f$, and finally, for every $x \in X$, $D_{1*} : \pi_*(f^{-1}(x)) \longrightarrow \pi_*(f^{-1}(d_1(x)))$ is an isomorphism. Then Y is distinguished for f, i.e. f is a quasifibration.

PROOF. d_1 and D_1 are deformations so d_{1*} and D_{1*} induce homotopy equivalences. Now, from the induced maps of pairs $(X, f^{-1}(y) \longrightarrow (X', f^{-1}(d_1(y)))$ and the five-lemma we have that $\pi_*(X, f^{-1}(y)) \cong \pi_*(X', f^{-1}(d_1(y)))$. But since Y' is distinguished for f we know $\pi_*(X', f^{-1}(y')) \cong \pi_*(Y', y')$, and d_1 shows that these groups are isomorphic to $\pi_*(Y, y)$. □

Perhaps the most important method of showing that f is a quasifibration is the following result.

THEOREM 2.4.1. *Let $f : X \longrightarrow Y$ be a continuous map, and suppose that there is a family $\mathcal{Y}$ of distinguished open sets for f, $U_i \subset Y$ with the following two properties:*

- *The sets $U_i \in \mathcal{Y}$ cover Y.*
- *For every pair $U_i, U_j \in \mathcal{Y}$ and $y \in U_i \cap U_j$ there is a $U_y \in \mathcal{Y}$ with $y \in U_y \subset U_i \cap U_j$.*

Then Y is distinguished for f.

(The idea of the proof is to modify the standard proof of (polyhedral) homotopy lifting for f if $\mathcal{Y}$ was a family of open sets for which $f^{-1}(U_i) = Y_i \times U_i$, i.e. the map has a local product structure. One covers the homotopy on the base by distinguished neighborhoods, and then refines the polyhedral decomposition so that each polygon has the form $P_i \times [a, b]$ and is contained in one of the distinguished neighborhoods. One then constructs the extension over skeleta, one cell at a time. The only difference here is that the lifting is not exact but involves a second homotopy. The homotopy extension lemma above provides the necessary tool.)

2.5. *Associated quasifibrations*

For ordinary (local product) fibrations one can associate a (left)-principal fibration to any fibration $F \longrightarrow E \longrightarrow B$, which we can write $H \longrightarrow \mathcal{E} \overset{p}{\longrightarrow} B$ with fiber a subgroup $H \subset Aut(F)$, the group of homeomorphisms of F. Then given any Y with H-action $Y \times H \longrightarrow Y$, there is an associated fibration

$$Y \longrightarrow Y \times_H \mathcal{E} \longrightarrow B.$$

However, for quasifibrations this construction may not always result in a quasifibration. For one thing, since M is not a group in general, the operation $\times_M$ is not directly an equivalence relation. For another, even taking the associated equivalence relation, the local structure may be sufficiently bad that the map of the quotient to B is not a quasifibration.

The problem was studied by Stasheff in [32] and he introduced a classifying space construction there which made sense of the notion of associated quasifibrations. Basically, given a left M-space E, and a right M-space X, he constructs a space $E(X, M, E)$ with the following properties:

- $E(X, M, E)$ is natural in all three variables. For example, if $h : X \longrightarrow X'$ is a map of right M-spaces then there is an induced map

  $$E(h, 1, 1) : E(X, M, E) \longrightarrow E(X', M, E)$$

  and similarly for the other variables which satisfy the expected naturality properties. Also, if the maps are weak homotopy equivalences, then the resulting maps of $E(X, M, E)$ are also.
- $E(X, M, M) \simeq X$, $E(M, M, E) \simeq E$.
- If $E \longrightarrow B$ is a principal quasifibering then $E(M, M, E) \longrightarrow E(*, M, E)$ is a principal quasifibering which is structurally equivalent to $E \longrightarrow B$.
- If $E \longrightarrow B$ is a principal quasifibering, then $E(X, M, E) \longrightarrow E(*, M, E)$ is a quasifibering with fiber X.

An important example to keep in mind is the loop-path Serre fibration

$$\Omega X \longrightarrow E^X_{*,X} \overset{p_1}{\longrightarrow} X.$$

These spaces are constructed as a limit over n of spaces constructed from the products $\sigma^n \times X \times M^n \times E$ by introducing the equivalence relation

$$(t, x, m_2, \ldots, m_{n+1}, e) \sim (t, x', m'_2, \ldots, m'_{n+1}, e')$$

where $m_i m_{i+1} = m'_i m'_{i+1}$ if $t_i = 0$, the t_i are barycentric coordinates for the simplex σ^n and in this relation $x = m_1$, $e = m_{n+2}$. One must be a bit careful with the topologies here. In particular Stasheff, following [15], gives the quotients a topology just strong enough for certain maps to be continuous. However, one can use the compactly generated topology in the quotient, and this will work as well. (For a complete study of the properties of this topology see [33].)

The construction has the property that it is graded and $E_n - E_{n-1}$ is a product $Y \times \mathrm{Int}(\sigma^n) \times N^n \times X$ where $N = M - *$, and that there is a neighborhood U_n of E_{n-1} in E_n together with a deformation retraction, D, of U_n onto E_{n-1} so that for any point

$$(x, t, n_2, \ldots, n_{n+1}, y) \in U \cap (E_n - E_{n-1})$$

$D_1(x, t, n_2, \ldots, n_{n+1}, y)$ lies in a product neighborhood $E_j - E_{j-1}$ for a unique j and there has the form $(xm_1, t', n'_2, \ldots, n'_{j+1}, m_2 y)$ with m_1, m_2 independent of x, y. In particular each fiber $X \times Y$ is mapped by a translation of the form $(x, y) \mapsto (xm_1, ym_2)$ and if we assume that the actions $M \times Y \longrightarrow Y$, $X \times M \longrightarrow X$ give rise to weak homotopy equivalences $y \mapsto my$, $x \mapsto xm$ for all $m \in M$ then the results of Dold and Thom above show that the construction gives a quasifibration.

There is one more property of these spaces which will be useful to us. If X also has a left N-action, then the space $E(X, M, E)$ becomes a left N-space from the action on passing to quotients. (The compactly generated topology again seems better here than Stasheff's original topology.)

3. The Freudenthal suspension theorem

The computation of homotopy groups is a notoriously difficult problem. Even for spheres, our knowledge is quite spotty compared with what might have been expected over forty years ago, when work on them began in earnest. An important simplification was made by Freudenthal, who proved his famous suspension theorem, which asserts that for $k < n$ the suspension homomorphism $\sigma : \pi_{n+k}(S^n) \longrightarrow \pi_{n+k+1}(S^{n+1})$ is an isomorphism. On can therefore compute the value of infinitely many homotopy groups of spheres by computing one stable group, i.e. one group of the form $\pi_{n+k}(S^n)$, $k < n$.

Let Σ denote the reduced suspension functor. For any based space (X, a), we may define a suspension homomorphism

$$\sigma : \pi_i(X, x) \longrightarrow \pi_{i+1}(\Sigma X, x)$$

and consequently, a directed system of groups $\{\pi_{i+l}(\Sigma^l X, x)\}_{l \geqslant 0}$ by the requirement that $\sigma[f] = [\Sigma f]$. $\varinjlim_l \pi_{i+l}(\Sigma^l X, x)$ is now an abelian group valued functor of spaces, which we denote by $\pi_i^s(X, x)$. It will follow from Freudenthal's result that this system eventually stabilizes, i.e. that for sufficiently large l, the suspension homomorphism $\pi_{i+l}(\Sigma^l X, x) \longrightarrow \pi_{i+l+1}(\Sigma^{l+1} X, x)$ is an isomorphism. It also turns out that the graded group valued functor $\pi_*^s(-)$ is a generalized homology theory in X. This means that many of the methods used to compute integral homology so successfully also apply to stable homotopy theory; the only obstacle is that one cannot compute its value on a point.

In this section we will outline proofs of these fundamental results. We will assume that the reader is familiar with the standard theory of Hurewicz fibrations, presented in Section 2.2.

LEMMA 3.1. *Let $p : E \longrightarrow B$ be a Hurewicz fibration, where B is a path connected CW complex with preferred base point b. Suppose further that B is obtained from a subcomplex B_0 by attaching a single n-cell along a based map $f : S^{n-1} \longrightarrow B$, so $B = B_0 \cup e^n$. Finally, suppose $F = p^{-1}(b)$ is k-connected. Then the map of pairs $(E, p^{-1}(B_0)) \longrightarrow (B, B_0)$ induces isomorphisms on H_j for $j \leqslant n+k$.*

PROOF. Let $f^n \subseteq e^n$ denote the closed disc of radius $\frac{1}{2}$ centered at the origin. It is clear that B_0 is a deformation retract of $B - \mathring{f}^n$. It is therefore a direct consequence of the homotopy lifting property that $p^{-1}(B_0)$ is also a deformation retract of $p^{-1}(B - \mathring{f}^n)$. Consequently, the inclusions $(B, B_0) \longrightarrow (B, B - \mathring{f}^n)$ and $(E, p^{-1}(B_0)) \longrightarrow (E, p^{-1}(B - \mathring{f}^n))$ induce isomorphisms on relative homology. It therefore suffices to show that the homomorphism $H_j(E, p^{-1}(B - \mathring{f}^n)) \longrightarrow H_j(B, B - \mathring{f}^n)$ is an isomorphism when $0 \leqslant j \leqslant n+k$. Let ∂f^n denote the boundary of f^n. It is a direct consequence of the excision theorem for homology that the inclusions $(f^n, \partial f^n) \longrightarrow (B, B - \mathring{f}^n)$ and $(p^{-1}(f^n), p^{-1}(\partial f^n)) \longrightarrow (E, p^{-1}(B - \mathring{f}^n))$ induce isomorphisms on relative homology, H_i, for all i. It consequently suffices to show that the homomorphism

$$H_j(p^{-1}(f^n), p^{-1}(\partial f^n)) \longrightarrow H_j(f^n, \partial f^n)$$

is an isomorphism for $0 \leqslant j \leqslant n+k$.

Let $v \in f^n$ denote the center of the ball. Note that since B is path connected, it follows from the fact that F is k-connected that $p^{-1}(v)$ is. Since f^n is a contractible space, we have a homotopy equivalence over X from $p^{-1}(f^n)$, with the restriction of p as reference map, to $f_n \times p^{-1}(v)$, with projection on the first factor as reference map. It now follows that it suffices to show that the projection homomorphism

$$H_j(f_n \times p^{-1}(v), \partial f_n \times p^{-1}(v)) \longrightarrow H_j(f_n, \partial f_n)$$

is an isomorphism for $0 \leqslant j \leqslant n+k$. But this follows from the Künneth formula and the Hurewicz theorem. □

COROLLARY 3.1. *Suppose, as before, that we have a Hurewicz fibration $E \xrightarrow{p} B$, where B is a CW complex equipped with a preferred base point $b \in B$. Suppose that F is k-connected and B is n-connected. Then the natural map of pairs $(E, F) \longrightarrow (B, b)$ induces isomorphisms on H_j for $0 \leqslant j \leqslant n+k+1$.*

PROOF. It is standard homotopy theory that there is a based homotopy equivalence $(B, b) \xrightarrow{\phi} (B', b')$, where B' is a CW complex with a unique 0-cell b', and which has no l-cells for $0 \leqslant l \leqslant n$. By pulling back E along a homotopy inverse to ϕ, we obtain from Proposition 2.2.1 an equivalent fibration E' over B'.

We are therefore free to suppose that B has b as unique 0-cell, and that B has no l-cells for $0 \leqslant l \leqslant n$. Let $B^{(t)}$ denote the t-skeleton of B. We will show inductively that the homomorphisms $H_j(p^{-1}(B^{(l)}), F)) \longrightarrow H_j(B^{(l)}, b)$ are isomorphisms for $0 \leqslant j \leqslant n+k+1$, and all l. For $l = 0$, this is trivial since $B^{(0)} = 0$ and therefore $p^{-1}(B^{(0)}) = F$, so both target and source of the homomorphisms in question are trivial groups. Now suppose the result is known for l, and we attempt to show that $H_j(p^{-1}(B^{(l+1)}), F) \longrightarrow H_j(B^{(l+1)}, b)$ is an isomorphism for $0 \leqslant j \leqslant n+k+1$. Consider the following commutative diagram

$$\begin{array}{ccccc}
H_{j+1}(p^{-1}(B^{(l+1)}), p^{-1}B^{(l)}) & \longrightarrow & H_j(p^{-1}(B^{(l)}), F) & \longrightarrow & H_j(p^{-1}(B^{l+1}), F) \\
\downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \psi_3} \\
H_{j+1}(B^{(l+1)}, B^{(l)}) & \longrightarrow & H_j(B^{(l)}, b) & \longrightarrow & H_j(B^{l+1}, b)
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & H_j(p^{-1}(B^{(l+1)}), p^{-1}(B^{(l)})) & \longrightarrow & H_{j-1}(p^{-1}(B^{(l)}), F) \\
 & \downarrow{\scriptstyle \psi_4} & & \downarrow{\scriptstyle \psi_5} \\
\longrightarrow & H_j(B^{(l+1)}, B^{(l)}) & \longrightarrow & H_{j-1}(B^{(l)}, b)
\end{array}$$

It is just an induced map of homology long exact sequences induces by p. Suppose $l < n$. Then, since $B^{(l)} = B^{(l+1)} = b$, it follows directly from this sequence that $H_j(p^{-1}(B^{(l+1)}), F) = H_j(B^{(l+1)}, b) = 0$, which gives the result in this case. If $l = n$, then ψ_2 and ψ_5 are both isomorphisms since their domains and images are trivial groups. On the other hand, ψ_1 and ψ_5 are isomorphisms by Lemma 3.1. The five lemma now shows that ψ_3 is an isomorphism. Finally, if $l > n$, then ψ_2 and ψ_5 are isomorphisms by the inductive hypothesis, and ψ_1 and ψ_4 are again isomorphisms by 3.1. This gives the result. □

We now wish to use these results to give proofs of Freudenthal's theorem and of the generalized homology theory property of π_*^s. Let $i : (Y, y_0) \longrightarrow (X, x_0)$ be a based cofibration, let $Cyl(i)$ and $C(i)$ denote the reduced mapping cylinder and reduced mapping cone construction on i, respectively. Thus, $Cyl(i) = Y \times [0,1] \cup X/ \simeq$, where $\simeq$ is the equivalence relation generated by $(y, 0) \simeq i(y)$, and $(y_0, t) \simeq x_0$ for all t, and $C(i) = Cyl(i)/\text{Image}(Y)$. Let E denote the space of maps $\phi : [0,1] \longrightarrow C(i)$ such that $\phi(0) \in X$, with the compact open topology. We have a projection map $p : E \longrightarrow C(i)$, given by $p(\phi) = \phi(1)$; it is a Hurewicz fibration. Let F denote the fibre over x_0 of p; thus, F is the space of maps $\phi : [0,1] \longrightarrow C(i)$ such that $\phi(1) = x_0$ and $\phi(0) \in X$. We now define a map $\lambda : Y \longrightarrow F$ by $\lambda(y) = \psi_Y$, where $\psi_Y(t) = [y, 1-t]$. Let $j : F \longrightarrow E$ be the inclusion; note that the composite $j \circ \lambda$ is homotopic, rel y_0, to the map $\mu : Y \longrightarrow E$ which sends y to the constant path with values $i(y)$. The homotopy is given by $H(s, y) = [y, 1 - st]$. Of course, μ extends to a map $\bar{\mu} : X \longrightarrow E$, which sends x to the constant path with value x. We therefore have a map $Y \times [0,1] \longrightarrow X \longrightarrow E$, which is H on $Y \times [0,1]$ and is $\bar{\mu}$ on X, and which respects the equivalence relation defining $Cyl(i)$. Since the map restricts to λ on the image of $Y \times 0$, we have a map of pairs $(Cyl(i), Y) \longrightarrow (E, F)$. Further, the composite $(Cyl(i), Y) \longrightarrow (E, F) \longrightarrow (C(i), x_0)$ is just the identification map $Cyl(i) \longrightarrow C(i)$, which shrinks Y to a point.

THEOREM 3.1. *Let X, Y, and i be as above. Suppose that Y is k-connected and $C(i)$ is l-connected, with $k > 0$, $l > 1$. Then the map $\lambda : Y \longrightarrow F$ induces isomorphisms on π_j for $0 \leqslant j \leqslant k + l$.*

PROOF. We know by Lemma 3.1 and Corollary 3.1 that the homomorphism $H_j(E,F) \longrightarrow H_j(C(i), x_0)$ is an isomorphism for $0 \leqslant j \leqslant k + l + 1$. By the above description of the composite

$$\big(Cyl(i), Y\big) \longrightarrow (E,F) \longrightarrow \big(C(i), x_0\big),$$

and the excision theorem, we conclude that

$$H_j\big(Cyl(i), Y\big) \longrightarrow H_j(E,F) \longrightarrow H_j\big(C(i), x_0\big)$$

is an isomorphism for all j, and hence that $H_j(Cyl(i), Y) \longrightarrow H_j(E,F)$ is an isomorphism for $0 \leqslant j \leqslant k + l + 1$. Now consider the commutative diagram below

$$\begin{array}{ccccccccc}
H_{j+1}(Cyl(i)) & \longrightarrow & H_{j+1}(Cyl(i), Y) & \longrightarrow & H_j(Y) & & & & \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & & & \\
H_{j+1}(E) & \longrightarrow & H_{j+1}(E,F) & \longrightarrow & H_j(F) & & & & \\
& & & \longrightarrow & H_j(Cyl(i)) & \longrightarrow & H_j(Cyl(i), Y) & & \\
& & & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} & & \\
& & & \longrightarrow & H_j(E) & \longrightarrow & H_j(E,F) & &
\end{array}$$

It is easy to check that the map $Cyl(i) \longrightarrow E$ is a homotopy equivalence, so α and δ are isomorphisms for $j \leqslant k + l$. β and ε are also isomorphisms, from the above discussion. The five lemma now shows that γ is an isomorphism. It follows easily from the long exact homotopy sequence of the fibration $F \longrightarrow E \longrightarrow C(i)$ that F is simply connected. Therefore, the relative Hurewicz theorem asserts that $\pi_j(Y) \longrightarrow \pi_j(F)$ is an isomorphism for $0 \leqslant j \leqslant k + l$. □

Let X be any connected CW complex. Define a based map $J : X \longrightarrow \Omega\Sigma X$, where ΣX denotes the reduced suspension of X by $x \mapsto [x]$ where $[x](t) = [x,t] \in \Sigma X$.

THEOREM 3.2 (Freudenthal). *If X is k connected then the homomorphism*

$$\pi_i(J) : \pi_i(X) \longrightarrow \pi_i(\Omega\Sigma X)$$

is an isomorphism for $0 \leqslant i < 2k + 1$.

PROOF. Apply Theorem 3.1 to the inclusion $X \hookrightarrow CX$; λ in this case is J. □

COROLLARY 3.2. *Let $\sigma : \pi_i(X,*) \longrightarrow \pi_{i+1}(\Sigma X,*)$ be the suspension homomorphism. Suppose X is k-connected and $i < 2k+1$. Then σ is an isomorphism.*

PROOF. Standard adjointness identifies $\pi_{i+1}(\Sigma X,*)$ with $\pi_i(\Omega\Sigma X,*)$; it is not hard to see that after this identification, σ corresponds to $\pi_i(J)$. □

We now prove the cofibration property.

THEOREM 3.3. *Let $i : Y \longrightarrow X$ be a cofibration and let $C(i)$ denote its reduced mapping cone. Then there is a long exact sequence*

$$\cdots \longrightarrow \pi^s_{i+1}(C(i)) \longrightarrow \pi^s_i(Y) \longrightarrow \pi^s_i(X) \longrightarrow \pi^s_i(C(i)) \longrightarrow \pi^s_{i-1}(Y) \longrightarrow \cdots$$

PROOF. Consider the map $\Sigma^k i : \Sigma^k Y \longrightarrow \Sigma^k X$. From the definitions, it is clearly seen that $\Sigma^k C(i)$ is naturally homeomorphic to $C(\Sigma^k i)$. Let $E(\Sigma^k i)$ denote the space of maps $\phi : [0,1] \longrightarrow C(\Sigma^k i)$ with $\phi(0) \in \Sigma^k X$; as before, the map $p : E(\Sigma^k i) \longrightarrow C(\Sigma^k i)$ is a fibration and we let $F(\Sigma^k i)$ denote the inverse image of the basepoint. There is an evident map $\Sigma F(\Sigma^k i) \longrightarrow F(\Sigma^{k+1} i)$.

We therefore obtain a directed system of groups $\{\pi_{i+k}(F(\Sigma^k i))\}$. It now follows from the long exact sequences of the fibrations $F(\Sigma^k i) \longrightarrow E(\Sigma^k i) \longrightarrow C(\Sigma^k i)$ that we have a long exact sequence

$$\cdots \longrightarrow \varinjlim_k \pi_{i+k+1}\big(C(\Sigma^k i)\big) \longrightarrow G_i \longrightarrow \pi^s_i(X) \longrightarrow \varinjlim_k \pi_{i+k}\big(C(\Sigma^k i)\big) \longrightarrow \cdots$$

From the identification $\Sigma^k C(i) \simeq C(\Sigma^k i)$, we see that $\lim_k \pi_{i+k}(C(\Sigma^k i)) \cong \pi^s_i(C(i))$. On the other hand, there are maps $\Sigma^k Y \longrightarrow F(\Sigma^k i)$ which give a homomorphism of directed systems of abelian groups

$$\{\pi_{i+k}(\Sigma^k Y)\}_{k\geqslant 0} \longrightarrow \{\pi_{i+k}(F(\Sigma^k i))\}_{k\geqslant 0}$$

and hence a homomorphism

$$\pi^s_i(Y) \longrightarrow \varinjlim_k \pi_{i+k}\big(F(\Sigma^k i)\big) = G_i.$$

Theorem 3.1 now shows that for sufficiently large k, $\pi_{i+k}(\Sigma^k Y) \longrightarrow \pi_{i+k}(F(\Sigma^k i))$ is an isomorphism, hence so is the homomorphism $\pi^s_i(Y) \longrightarrow G_i$. This gives the required result. □

We obtain a corollary concerning the homology of iterated loop spaces.

COROLLARY 3.3. *Consider the iterated loop space $\Omega^k S^N$ where $k < N$. We have the map $S^{N-k} \xrightarrow{\lambda} \Omega^k S^N$, adjoint to the standard identification $\Sigma^k S^{N-k} \xrightarrow{=} S^N$. Then λ induces isomorphisms on H_j for $j < 2(N-k-1)$.*

PROOF. Consider $\pi_j(\lambda) : \pi_j(S^{N-k}) \longrightarrow \pi_j(\Omega^k S^N) \cong \pi_{j+k}(S^N)$. $\pi_j(\lambda)$ is identified with the k-fold suspension homomorphism, which is an isomorphism if $j < 2(N-k) - 1$ by Corollary 3.2. Thus, by the Whitehead theorem $H_j(\lambda)$ is an isomorphism if $j < 2(N-k) - 2$, which is the required result. □

4. Spanier–Whitehead duality

4.1. *The definition and main properties*

Let X be a based finite complex. One may consider the function space of based maps $X \longrightarrow S^N$, $F(X, S^N)$, as usual in the compact open topology. This space does not have the homotopy type of a finite complex. However, for N sufficiently large, there is a finite complex Y and a map $Y \longrightarrow F(X, S^N)$ which induces isomorphisms on homotopy groups in dimensions less than $2N - 2k$. One could also state the result as follows. We have natural suspension maps $\Sigma F(X, S^N) \longrightarrow F(X, S^{N+1})$, and hence a directed system of abelian groups $\{\pi_{i+k}(F(X, S^{N+k}))\}_{k \geqslant 0}$. We also have maps $\Sigma^k Y \longrightarrow F(X, S^{N+k})$, and these maps are compatible with respect to suspensions. This gives a homomorphism of abelian groups

$$\varinjlim_k \{\pi_{i+k}(\Sigma^k Y)\}_{k \geqslant 0} \longrightarrow \varinjlim_k \pi_{i+k}(F(X, S^{N+k})). \tag{4.1}$$

The statement will be that this homomorphism is in fact an isomorphism. This theorem and the general development is due to Spanier and Whitehead; see [31].

To study this situation, we first consider any two based CW complexes X and Y. Let S_*X and S_*Y denote the complexes of singular chains on X and Y respectively. We have the evaluation map $e : X \wedge F(X,Y) \longrightarrow Y$. Therefore we have a chain map $S_*e : S_*(X \wedge F(X,Y)) \longrightarrow S_*Y$. Let $\sigma : S_*(X) \otimes S_*(F(X,Y) \longrightarrow S_*(X \times F(X,Y))$ be any chain inverse to the Alexander–Whitney homomorphism, e.g., the shuffle homomorphism. $S_*e \circ \sigma$ is now a homomorphism $S_*(X) \otimes S_*(F(X,Y) \longrightarrow S_*(Y)$ and we may take its adjoint

$$S_*(F(X,Y)) \xrightarrow{\alpha(X,Y)} Hom(S_*(X), S_*(Y)).$$

Now let $Y = S^N$, and fix a generating cocycle c for $H^N(S^N) \cong \mathbf{Z}$. c now gives a chain map which we also call c from $C_*(S^N)$ to the chain complex D_* with $D_i = 0$ when $i \neq N$, and $D_N \cong \mathbf{Z}$, and c induces an isomorphism on H_N. $c \circ \alpha(X, S^N)$ is now a homomorphism from $S_*(F(X, S^N)$ to $Hom(S_*(X), D_*)$, and $H_i(Hom(S_*(X), D_*)) \cong H^{N-i}(X)$, as contravariant functors in X.

THEOREM 4.1.1. *Let X be a finite complex of dimension i. Then $c \circ \alpha(X, S^N)$ induces an isomorphism on H_j for $0 < j < 2N - 2i - 2$.*

PROOF. We first study the situation where X is an i-sphere. In this case,

$$H_*(F(X,S^N)) \cong H_*(\Omega^i S^N) \cong H_*(S^{N-i})$$

for $* < 2(N-i-1)$. In the range in question then, we are only required to verify that $c \circ \alpha(X,S^N)$ induces an isomorphism $H_{N-i}(F(X,S^N)) \cong \mathbf{Z}$. But from the definitions, this is equivalent to the assertion that

$$H_i(S^i) \otimes H_{N-i}(F(S^i,S^N)) \longrightarrow H_N(S^i \times F(S^i,S^N)) \xrightarrow{H_N(e)} H_N(S^N)$$

is a perfect pairing. Note further that if the composite

$$\beta : S^i \wedge S^{N-i} \longrightarrow S^i \wedge F(S^i,S^N) \xrightarrow{e} S^N$$

is the standard identification, then the homomorphism

$$H_i(S^i) \otimes H_{N-i}(S^{N-i}) \longrightarrow H_N(S^i \wedge S^{N-i}) \xrightarrow{\beta} H_N(S^N)$$

yields a perfect pairing. This gives the result for spheres in view of Corollary 3.3.

To deal with a general complex, we work by induction on the dimension i. The case $i = 0$ is trivial. Suppose the result is known for complexes of dimension $< i$, and consider an i-dimensional complex X. Let $X^{(i-1)}$ denote the $(i-1)$-skeleton. Then we have a fibration

$$\begin{array}{ccc} \prod_{\alpha\in A} \Omega^i S^N & \longrightarrow & F(X,S^N) \\ & & \downarrow \\ & & F(X^{(i-1)},S^N) \end{array}$$

where A is an indexing set for the collection of i-cells in X, and the vertical arrow is restriction to the $i-1$ skeleton. $F(X^{(i-1)},S^N)$ is $(N-i)$-connected and $\Omega^i S^N$ is $N-i-1$ connected, so, by Corollary 3.1, we have exact sequences

$$\begin{aligned} H_{j+1}(F(X^{(i-1)},S^N)) &\to H_j\Big(\prod_{\alpha\in A} \Omega^i S^N\Big) \to H_j(F(X,S^N)) \\ &\to H_j(F(X^{(i-1)},S^N)) \to H_j\Big(\prod_{\alpha\in A} \Omega^i S^N\Big) \end{aligned}$$

for $j < 2(N-i)-1$. These exact sequences map to the corresponding long exact sequences

$$\begin{aligned} H^{N-j-1}(X^{i-1}) &\to H^{N-j}\Big(\bigvee_{\alpha\in A} S^i\Big) \to H^{N-j}(X) \\ &\to H^{N-j}(X^{i-1}) \to H^{N-j-1}\Big(\bigvee_{\alpha\in A} S^i\Big) \end{aligned}$$

associated to the pair (X, X^{i-1}). The five lemma and the inductive hypothesis now give the result. □

Now, suppose we have two based finite complexes X and Y, with a map $X \wedge Y \xrightarrow{D} S^N$. Consider the composite

$$C_*(X) \otimes C_*(Y) \longrightarrow C_*(X \wedge Y) \longrightarrow C_*(S^N)$$

where the left hand arrow is the same chain inverse to the Alexander–Whitney map which we chose earlier. We therefore obtain an adjoint chain map

$$C_*(Y) \xrightarrow{\Delta} Hom(C_*(X), C_*(S^N)).$$

We say the map D is an *S-duality map* if Δ is a chain equivalence, i.e. induces an isomorphism on homology, and we refer to Y as an S-dual to X.

PROPOSITION 4.1.1. *Suppose $D : X \wedge Y \longrightarrow S^N$ is an S-duality map. Consider the adjoint map $adj(D) : Y \longrightarrow F(X, S^N)$. Then, if X is i-dimensional $adj(D)$ induces an isomorphism on H_j for $j < 2N - 2i - 2$, and hence on π_j for $j < 2N - 2i - 3$.*

PROOF. We have the following commutative diagram of chain complexes

$$\begin{array}{ccccc} C_*(X) \otimes C_*(Y) & \longrightarrow & C_*(X \wedge Y) & \longrightarrow & C_*(S^N) \\ \downarrow{\scriptstyle l_1} & & \downarrow{\scriptstyle l_2} & & \downarrow{\scriptstyle =} \\ C_*(X) \otimes C_*(F(X, S^N)) & \longrightarrow & C_*(X \wedge F(X, S^N)) & \longrightarrow & C_*(S^N) \end{array}$$

where l_1 is the chain map $C_*(id) \otimes C_*(adj(D))$ and l_2 is $C_*(id \wedge adj(D))$. Therefore, we have another commutative diagram

$$\begin{array}{ccc} C_*(Y) & \longrightarrow & Hom(C_*(S), C_*(S^N)) \\ \downarrow & & \downarrow{\scriptstyle =} \\ C_*(F(X, S^N)) & \longrightarrow & Hom(C_*(X), C_*(S^N)) \end{array}$$

where the upper horizontal arrow induces isomorphisms on H_j for all j, and the lower horizontal arrow induces isomorphisms on H_j for $j < 2(N - i) - 2$. The result is now immediate. □

4.2. *Existence and construction of S-duals*

We must address the question of whether or not there exists an S-dual for a given finite complex X and some N. We first examine what happens when we attach one cell.

PROPOSITION 4.2.1. *Suppose we have an S-duality $X \wedge Y \longrightarrow S^N$, and a map $f : S^d \longrightarrow X$. Let $X' = X \cup_f e^{d+1}$. Suppose further that $dim(Y) < 2(N-d) - 1$. Then there is a finite based complex Y', of dimension $\leqslant max(dim(Y)+1, N-d+1)$ and an S-duality $X' \wedge Y' \longrightarrow S^N$.*

PROOF. We first consider the sequence of maps

$$Y \longrightarrow F(X, S^N) \xrightarrow{F(f,S^N)} \Omega^d S^N \cong F(S^d, S^N) \longleftarrow S^{N-d}.$$

Here the left arrow is the adjoint to the original S-duality, and the right one is the adjoint to the identification $S^d \wedge S^{N-d} \xrightarrow{\sim} S^N$. Since $dim(Y) < 2(N-d) - 1$, there is a map ϕ, $Y \xrightarrow{\phi} S^{N-d}$ which makes the diagram commute up to homotopy. Equivalently, we have a commutative diagram

$$\begin{array}{ccccc} & & Y & \longrightarrow & F(X,S^N) \\ & & \downarrow & & \downarrow{\scriptstyle F(f,S^N)} \\ S^{N-d} & \simeq & Cyl(\phi) & \longrightarrow & F(S^d,S^N) \end{array}$$

where $Cyl(\phi)$ is the mapping cylinder of ϕ and the left vertical map is the inclusion on one end of the cylinder. Now consider the diagram

$$\begin{array}{ccccc} Y & \longrightarrow & F(X,S^N) & \longrightarrow & \Omega F(X,S^{N+1}) \\ \downarrow & & \downarrow{\scriptstyle F(f,S^N)} & & \downarrow{\scriptstyle \Omega F(f,S^{N+1})} \\ Cyl(\phi) & \longrightarrow & F(S^d,S^N) & \longrightarrow & \Omega F(S^d,S^{N+1}) \\ & & & & \downarrow \\ & & & & F(X \cup_f e^{d+1}, S^{N+1}) \end{array}$$

where the right hand vertical sequence is the fibration sequence obtained via Proposition 2.2.2 by applying $F(-, S^{N+1})$ to the inclusion

$$X \cup_f e^{d+1} \longrightarrow X \cup_f e^{d+1} \cup CX \simeq \Sigma S^d \simeq S^{d+1}.$$

Since the composite

$$\Omega F(X, S^{N+1}) \longrightarrow \Omega F(S^d, S^{N+1}) \longrightarrow F(X \cup_f e^{d+1}, S^{N+1})$$

is null homotopic, the map $Y \longrightarrow F(X \cup_f e^{d+1}, S^{N+1})$ is null homotopic, and therefore the composite $Cyl(\phi) \longrightarrow \Omega F(S^d, S^{N+1}) \longrightarrow F(X \cup_f e^{d+1}, S^{N+1})$ extends over $Cyl(\phi) \cup CY$. We therefore have a commutative diagram

$$\begin{array}{ccccc}
Y & \longrightarrow & F(\Sigma X, S^{N+1}) & & (= \Omega F(X, S^{N+1})) \\
\downarrow & & \downarrow & & \\
Cyl(\phi) & \longrightarrow & F(S^{d+1}, S^{N+1}) & & (= \Omega F(S^d, S^{N+1})) \\
\downarrow & & \downarrow & & \\
Cyl(\phi) \cup CY & \xrightarrow{\alpha} & F(X \cup_f e^{d+1}, S^{N+1}) & &
\end{array}$$

Let $Y' = Cyl(\phi) \cup cY$, with a map $X' \wedge Y' \xrightarrow{D'} S^{N+1}$ given as the adjoint of α. We claim D' is an S-duality map. To see this, it is only required to show that the associated maps

$$H_i(Y') \longrightarrow H^{N+1-k}(X')$$

are isomorphisms. But this follows from the 5-lemma and the following diagram of long exact sequences:

$$\begin{array}{ccccccccc}
\to & H_k(Y) & \to & H_k(Cyl(\phi)) & \to & H_k(Y') & & & \\
 & \downarrow & & \downarrow & & \downarrow & & & \\
\to & H^{N+1-k}(X) & \to & H^{N+1-k}(S^d) & \to & H^{N+1-k}(X') & & & \\
 & & & \to & H_{k-1}(Y) & \to & H_{k-1}(Cyl(\phi)) & \to & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & \to & H^{N+2-k}(X) & \to & H^{N+2-k}(S^d) & \to &
\end{array}$$

□

REMARK. S-duals are also unique in the following sense. Suppose we have a finite complex X, and S-duality maps $D : X \wedge Y \longrightarrow S^N$ and $D' : X \wedge Y' \longrightarrow S^{N'}$. Suppose $N' \geqslant N$. Then, for sufficiently large l there is a homotopy equivalence $\Sigma^{N'-N+l} Y \xrightarrow{\theta} \Sigma^l Y'$. Furthermore, it is characterized by the requirement that

$$\begin{array}{ccc}
X \wedge \Sigma^{N'-N+l} Y & \xrightarrow{\Sigma^{N'-N+l}} & S^{N'+l} \\
 & {\scriptstyle id \wedge \theta} \searrow \quad \swarrow {\scriptstyle \Sigma^l D'} & \\
 & X \wedge \Sigma^l Y' &
\end{array}$$

commutes up to homotopy.

It is also possible to describe the S-dual in a very concrete fashion. Let X be a finite CW complex. It is well known that it is possible to embed X in Euclidean space, $\mathbf{R}^N$, and from now on we view X as a subspace of $\mathbf{R}^N$. Let Y denote the complement

$\mathbf{R}^N - X$. For any pair of distinct points $v, w \in \mathbf{R}^N$, let $l(v,w) : \mathbf{R} \longrightarrow \mathbf{R}^N$ be given by $l(v,w)(t) = (1-t)v + tw$. Notice that since v and w are distinct, if we view S^N as the one point compactification of $\mathbf{R}^N$, then $l(v,w)$ defines a loop in S^N. l may therefore be viewed as a map from $E \subseteq \mathbf{R}^N \times \mathbf{R}^N$, $E = \{(v,w) \mid v \neq w\}$, to ΩS^N. Let $i : X \longrightarrow \mathbf{R}^N$ and $j : Y \longrightarrow \mathbf{R}^N$ be inclusions, then $X \times Y \xrightarrow{i \times j} \mathbf{R}^N \times \mathbf{R}^N$ factors through E, and we call the composite $X \times Y \longrightarrow E \xrightarrow{l} \Omega S^N$ the (preliminary) duality map, $\hat{D}$. Since X is compact, X is contained in some ball, B, in $\mathbf{R}^N$. Choose a basepoint y for Y outside that ball. Observe that $\hat{D}|X \times y$ extends over $B \times y$, since $y \notin B$. Since B is contractible, we obtain an extension $\tilde{D}$ from $X \times Y \cup C(X \times y)$ to ΩS^N. $X \times Y \cup C(X \times y)$ is homotopy equivalent to $X \times Y/(X \times y)$, which, in turn, is homeomorphic to $X_+ \wedge Y$, where X_+ denotes X with a disjoint basepoint added. Let $D : \Sigma X_+ \wedge Y \longrightarrow S^N$ denote the adjoint.

THEOREM 4.2.1. *D is an S-duality map.*

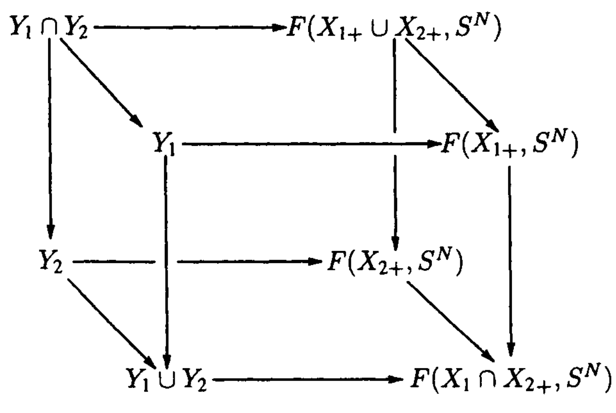
PROOF. For any finite subcomplex $X \subset \mathbf{R}^N$, with $Y = \mathbf{R}^N - X$, let D_X denote the map constructed above. (Here, a point y is chosen once and for all, and will be contained in the complements of all the subcomplexes we deal with.) We will show that if D_{X_1}, D_{X_2}, and $D_{X_1 \cap X_2}$ are S-duality maps for subcomplexes X_1 and X_2 of $\mathbf{R}^N$ which are contained in a ball which does not contain y, then $D_{X_1 \cup X_2}$ is also an S-duality map. Let $Y_i = \mathbf{R}^N - X_i$. Note that we have a pullback square of fibrations

$$\begin{array}{ccc} F(X_1 \cup X_{2+}, S^N) & \longrightarrow & F(X_{1+}, S^N) \\ \big\downarrow & & \big\downarrow \\ F(X_{2+}, S^N) & \longrightarrow & F(X_1 \cap X_{2+}, S^N) \end{array}$$

We suppose, for the moment, that N is sufficiently large that the natural maps

$$C_*(F(X_1 \cup X_{2+}, S^N)) \longrightarrow Hom(\tilde{C}_*(X_1 \cup X_{2+}), \mathbf{Z})$$

and $C_*(F(X_{i+}, S^N)) \longrightarrow Hom(\tilde{C}_*(X_{i+}), \mathbf{Z})$ induce isomorphism on homology for $* \leqslant N$. Note also that from the definitions, we have a commutative diagram



and therefore a commutative diagram

$$\begin{array}{ccccc} C_*(Y_1 \cap Y_2) & \longrightarrow & C_*(Y_1) \oplus C_*(Y_2) & \longrightarrow & C_*(Y_1 \cup Y_2) \\ \downarrow & & \downarrow & & \downarrow \\ C^*(X_1 \cup X_{2+}) & \longrightarrow & C^*(X_{1+}) \oplus C^*(X_{2+}) & \longrightarrow & C^*(X_1 \cap X_{2+}) \end{array}$$

This gives rise to a commutative diagram of Mayer–Vietoris sequences, which in the relevant range is

$$\begin{array}{cccccc} \longrightarrow & H_{d+1}(Y_1 \cup Y_2) & \longrightarrow & H_d(Y_1 \cap Y_2) & \longrightarrow \\ & \downarrow & & \downarrow & \\ \longrightarrow & H^{N-d-1}(X_1 \cap X_{2+}) & \longrightarrow & H^{N-d}(X_1 \cup X_{2+}) & \longrightarrow \end{array}$$

$$\begin{array}{cc} H_d(Y_1) \oplus H_d(Y_2) & \longrightarrow \\ \downarrow & \\ H^{N-d}(X_{1+}) \oplus H^{N-d}(X_{2+}) & \longrightarrow \end{array}$$

where the vertical arrows are all adjoints to the duality maps.

Since $D_{X_1 \cap X_2}$, D_{X_1}, and D_{X_2} all induce isomorphisms on homology, so does $D_{X_1 \cup X_2}$. To obtain a proof of the required result, we must now show that the result holds for a single point. But for a single point, the complement has the homotopy type of S^{N-1}, and the map $S^0 \wedge S^{N-1} \xrightarrow{D} \Omega S^N$ is easily seen to be equal to the map $J : S^{N-1} \longrightarrow \Omega S^N$ from Section 3, whose adjoint is the identity map of S^N. This gives the result. □

If one wants to give a duality map for X itself (rather than for X_+), one must only adjoin the point at infinity to Y. More generally, let $X_1 \subseteq X_2$ be an inclusion of subcomplexes of $\mathbf{R}^N$, and let $Y_1 \supseteq Y_2$ denote the complements.

COROLLARY 4.2.1. *In the above situation, there is an S-duality map*

$$D : \Sigma(X_2/X_1 \wedge Y_1/Y_2) \longrightarrow S^N$$

When X is a compact closed manifold, we obtain the following geometric description. See [5] and [31].

COROLLARY 4.2.2 (Spanier, Atiyah). *Let X be a compact closed smooth manifold, and suppose X is smoothly embedded in $\mathbf{R}^N$. Let N denote the normal bundle to the embedding, and let $\tau(N)$ denote its Thom complex. Then there is an S-duality map $X \wedge \tau(N) \longrightarrow S^N$.*

PROOF. Let $B(X)$ denote a small tubular neighborhood of X. Via the exponential map on the normal bundle, it is homeomorphic to the open unit disc bundle of N. If $Y = \mathbf{R}^N - X$, we have the S-duality map

$$X \wedge \Sigma Y \longrightarrow \Sigma(X \wedge Y) \longrightarrow S^N$$

But, $\mathbf{R}^N/Y$ is naturally homotopy equivalent to ΣY, since it is homotopy equivalent to the mapping cone on the inclusion $Y \longrightarrow \mathbf{R}^N$, and $\mathbf{R}^N$ is contractible. On the other hand, let $\overline{B}$ denote the closure of B; then $\mathbf{R}^N/Y$ is homeomorphic to $\overline{B}/\partial\overline{B}$, which in turn is homeomorphic to the quotient of the closed unit disc bundle of N by the unit sphere bundle. This is the definition of the Thom complex of N. □

5. The construction and geometry of loop spaces

To understand stabilization a bit better it is useful to be able to compute the homology of loop spaces, and in particular loop spaces of suspensions. This was first carried out by I.M. James for the case of $\Omega\Sigma X$. Soon afterwards J.F. Adams and P. Hilton constructed a model for ΩX when X is any simply connected CW complex with one zero cell and no one cells.[2] In both cases explicit models for the loop spaces were constructed. Later developments, particularly the construction of the Eilenberg–Moore spectral sequences made these original constructions less compelling for homology calculations but nonetheless, the geometry of ΩX reveals a great deal about the structure of X, so explicit constructions still play a vital role in the theory.

Both the James and Adams–Hilton models had a multiplicative structure and were even free associative monoids with unit. In fact more was true, each was a CW complex and the multiplication was cellular, so that the cellular chain complex was a tensor algebra with one generator in each dimension $(n-1)$ for each cell in dimension n of X. However, while in the James model for $\Omega\Sigma X$, the boundary map was explicitly determined by the boundary map for ΣX, in the Adams–Hilton model the boundary map was not determined at all initially. In a following paper Adams determined the boundary map for their construction in the case where X is a *simplicial complex* with the 1-skeleton collapsed to a point.

This work was of seminal importance in the theory and, though, as indicated, we can today replace most of it using the techniques of Eilenberg–Moore and classifying space theory, in this section we will describe the techniques and results of James, Hilton and Adams, much in the spirit in which they had originally been developed.

5.1. *The space of Moore loops*

It will first be necessary to describe a space homotopy equivalent to the usual loop space, the space of "Moore loops", $\Omega^M(X,*)$. Let $F(\mathbf{R},X)$ denote the space of all maps $\phi : \mathbf{R} \longrightarrow X$, in the compact open topology. Let $\Omega^M(X,*) \subseteq F(\mathbf{R},X) \times [0,\infty)$

[2] The construction given here is first described in the proof of Theorem 2.1 of [1]. However, the actual geometric construction is secondary to their objectives there. What they do is to construct a chain map of the cellular chain complex of this model into the singular cubical complex of $\Omega^M(Y)$ and show, by chain level arguments, that the resulting embedding induces isomorphisms in homology.

In later work S.Y. Husseini directly constructs this model for $\Omega^M(Y)$ as a special case of his general notion of a "relation in r-variables, $M_r(X)$", [18].

denote the subspace of all pairs (ϕ, r) for which $\phi(0) = *$ and for which $\phi(t) = *$ for all $t \geqslant r$. Note that the standard loop space $\Omega(X,*)$ can be identified with the subspace of all pairs of the form $(\phi, 1)$ with $\phi(t) = *$ for $t \geqslant 1$.

PROPOSITION 5.1.1. *$\Omega(X,*)$ is a deformation retract of $\Omega^M(X,*)$.*

PROOF. First consider $\tilde{\Omega}(X,*) \subseteq \Omega^M(X,*)$, the subspace of all (ϕ, t) with $t \geqslant 1$. A deformation retraction, H, of $\Omega^M(X,*)$ to $\tilde{\Omega}(X,*)$ is given by the following formulae.

$$H(s,(\phi,r)) = (\phi, r+s) \quad \text{when } r+s \leqslant 1,$$
$$H(s,(\phi,r)) = (\phi, 1) \quad \text{when } r \leqslant 1 \text{ and } r+s \geqslant 1,$$
$$H(s,(\phi,r)) = (\phi, r) \quad \text{when } r \geqslant 1.$$

Now we give a deformation retraction G from $\tilde{\Omega}(X,*)$ to $\Omega(X,*)$ by the formula

$$G(s,(\phi,r)) = (\phi_s, (1-s)r + s),$$

where

$$\phi_s(t) = \phi\left(\frac{r}{((1-s)r+s)}t\right).$$

This gives the required deformation retraction. □

We now remark that $\Omega^M(X,*)$ is actually a topological monoid, where the multiplication is given by $(\phi, r)\cdot(\psi, s) = (\phi * \psi, r+s)$ and

$$\begin{cases} \phi * \psi(t) = \phi(t) & \text{when } 0 \leqslant t \leqslant r, \\ \phi * \psi(t) = \psi(t-r) & \text{when } r \leqslant t \leqslant r+s, \\ \phi * \psi(t) = * & \text{when } t \geqslant r+s. \end{cases} \tag{5.1}$$

The point $(*, 0)$, where $*$ denotes the constant loop with value 0, is the identity element.

5.2. *Free topological monoids*

We now discuss the construction of the free monoid on a based topological space. First, if we have a based set $(X,*)$, recall that the free monoid on $(X,*)$ consists of all the "words" in X, with $*$ set to the identity. Formally this can be described as

$$\coprod_{n\geqslant 0} X^n/\sim, \tag{5.2}$$

where $\sim$ is the equivalence relation generated by all relations of the form

$$(x_1,\ldots,x_{i-1},*,x_{i+1},\ldots,x_n) \simeq (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n). \tag{5.3}$$

Multiplication is now just juxtaposition of words. This construction can now be applied equally well to based topological spaces, since one can construct the quotient space associated to an equivalence relation. Let the resulting construction be denoted by $M(X,*)$. It has the following universality property.

PROPOSITION 5.2.1. *Let $(X,*)$ be a based space, and let $f : X \longrightarrow M$ be any map to a topological monoid, M, with $f(*) = e$. Then there is a unique homomorphism $\bar{f} : M(X,*) \longrightarrow M$ of topological monoids so that the composite*

$$(X,*) \longrightarrow M(X,*) \xrightarrow{\bar{f}} M$$

is equal to f.

REMARK. When dealing with quotient spaces and products there is sometimes trouble, since the quotient of a product is not usually a product, even if only one of the two spaces is quotiented. However, with the compactly generated topology this difficulty is avoided, and we always assume that we are using this topology from now on. (See the remarks at the end of 2.5.)

5.3. *The James construction*

Let $(X,*)$ be any based space. Recall the definition of the "James map",

$$J : (X,*) \longrightarrow \Omega(\Sigma X,*), \qquad J(x)(t) = [t,x] \in \Sigma X.$$

If we compose this map with the inclusion into $\Omega^M(\Sigma X,*)$, we obtain a map, J, which does not carry the basepoint to the identity. Let

$$\hat{X} = X \coprod [0,1]/\simeq,$$

where $\simeq$ is generated by $1 \simeq *$, and define an extension $\hat{J}$ of J to $\hat{X}$ by $\hat{J}(s) = (*,s)$, where $0 \leqslant s \leqslant 1$, and $*$ denotes the constant map with value $*$. Of course, if X is a CW-complex, then X and $\hat{X}$ are based homotopy equivalent. This now becomes a pointed map if we let 0 be the basepoint for $\hat{X}$. Since we have a based map $\hat{J} : \hat{X} \longrightarrow \Omega^M(\Sigma X,*)$, we obtain a homomorphism $\bar{J} : M(\hat{X},0) \longrightarrow \Omega^M(\Sigma X,*)$. The theorem of James is that this map is a homotopy equivalence when X is a connected CW-complex.

Before proving this theorem we need to do some preliminary work on the homology of both spaces involved. For simplicity, we will consider homology with field coefficients ($\mathbf{F}_p$, p a prime, or $\mathbf{Q}$). For any topological monoid M, the homology groups of M form a graded, associative algebra with unit via

$$H_*(M) \otimes H_*(M) \cong H_*(M \times M) \xrightarrow{H_*(\mu)} H_*(M) \tag{5.4}$$

where $\mu : M \times M \longrightarrow M$ is the multiplication map. Thus the graded groups $H_*(M(\hat{X},0))$ and $H_*(\Omega^M(X,*))$ have the structure of graded rings, and this additional structure will be quite useful in describing the homology.

We recall the notion of the *tensor algebra* of a vector space V, $T(V)$. If V is a graded vector space, $T(V)$ obtains a natural grading where $v_1 \otimes \cdots \otimes v_n$ has grading $\sum_{i=1}^n \alpha_i$ if v_i has grading α_i. The tensor algebra has the universal property that if V is a graded vector space and $V \xrightarrow{\lambda} A$ is a map from a graded vector space into a graded algebra, then λ extends uniquely to a homomorphism of graded algebras $\Lambda : T(V) \longrightarrow A$.

Now consider $M(\hat{X},0)$; it is filtered by subspaces $M_n(\hat{X},0)$, where $M_n(\hat{X},0)$ is the image of $\hat{X}^n$ in $M_n(\hat{X},0)$. Thus $M_n(\hat{X},0)$ consists of the "words of length less than or equal to n" in the free monoid on $(\hat{X},0)$. From the definition of the equivalence relation defining $M(\hat{X},0)$ it is clear that the subquotient $M_n(\hat{X},0)/M_{n-1}(\hat{X},0)$ is homeomorphic to the smash product

$$\underbrace{\hat{X} \wedge \cdots \wedge \hat{X}}_{n \text{ times}}.$$

The Künneth formula now tells us that

$$\tilde{H}_*(\hat{X} \wedge \cdots \wedge \hat{X}) \cong \bigotimes_{i=1}^{n} \tilde{H}_*(\hat{X})$$

where the tensor product denotes tensor product of graded vector spaces. Let us now examine the collapse map

$$M_n(\hat{X},0) \longrightarrow M_n(\hat{X},0)/M_{n-1}(\hat{X},0).$$

We claim that it is surjective on homology. To see this, note that we have a map $X^n \longrightarrow M_n(\hat{X},0)$, given as the composite of the inclusion $X^n \longrightarrow \hat{X}^n$ with the identification map $\hat{X}^n \longrightarrow M_n(\hat{X},0)$. The composite

$$X^n \longrightarrow M_n(\hat{X},0) \longrightarrow M_n(\hat{X},0)/M_{n-1}(\hat{X},0)$$

is the equivalence $X^n \longrightarrow \hat{X}^n$ composed with the collapse of the product to the smash product. The Künneth formula shows that this is surjective, hence the result. We conclude that

$$H_*(M(\hat{X},0)) \cong \mathbf{F}_p \oplus \bigoplus_{i=1}^{n} \otimes_n \tilde{H}_*(X).$$

Now, the inclusion $\hat{X} \longrightarrow M(\hat{X},0)$ induces a map of graded vector spaces

$$\tilde{H}_*(\hat{X}) \rightarrow H_*(M(\hat{X},0)),$$

and hence a homomorphism of graded algebras $\Lambda : T(\tilde{H}_*(\hat{X})) \longrightarrow H_*(M(\hat{X},0))$.

PROPOSITION 5.3.1. *Λ is an isomorphism of graded algebras.*

PROOF. For any graded vector space V, let $T_n(V) = V \otimes \cdots \otimes V$. It now follows from the above analysis that under Λ, $T_n(\tilde{H}(X))$ has image in $H_*(M_n(\hat{X},0))$, and that it surjects to

$$\tilde{H}_*(M_n(\hat{X},0)/M_{n-1}(\hat{X},0)) \cong \bigotimes_{i=1}^{n} \tilde{H}_*(X).$$

Since we have a surjective map of isomorphic vector spaces, it is an isomorphism, and hence Λ is an isomorphism. □

We must now perform a similar analysis for $H_*(\Omega^M(\Sigma\hat{X},0)) \cong H_*(\Omega\Sigma X,*)$. Note that $\Omega\Sigma$ is equipped with its own loop sum operation μ, defined by $\mu(\phi,\psi) = \phi * \psi$, where $\phi*\psi(t) = \phi(2t)$ for $0 \leqslant t \leqslant 1/2$, and $\phi*\psi(t) = \psi(2t-1)$ for $1/2 \leqslant t \leqslant 1$. μ is not associative, but is homotopic to the restriction of the multiplication map on $\Omega^M(X,*)$ to $\Omega(X,*)$ and is therefore homotopy associative. In particular, μ gives $H_*(\Omega\Sigma X)$ the structure of an associative graded algebra. Let E denote the space of maps $\phi : [0,1] \longrightarrow \Sigma X$ with $\phi(0) = *$. The evaluation map $p : E \longrightarrow \Sigma X$, $p(\phi) = \phi(1)$ is a Hurewicz fibration, and the fibre over the point $*$ is clearly homeomorphic to the standard loop space $\Omega(\Sigma X,*)$. Let C_+X denote the image of $[\frac{1}{2},1] \times X$ in ΣX, and similarly C_-X will be the image of $[0,\frac{1}{2}] \times X$. Both these spaces are contractible, and their intersection is X. By Corollary 2.2.1, it follows that $p^{-1}(C_+X)$ (respectively $p^{-1}(C_-X)$) is homotopy equivalent as a space over C_+X (respectively C_-X) to $C_+X \times \Omega\Sigma X$ (respectively $C_-X \times \Omega\Sigma X$). We obtain explicit homotopy equivalences as follows. Let $H_\pm : C_\pm X \times I \longrightarrow C_\pm X$ be the standard deformation retraction of $C_\pm X$ to $*$. Define maps $\theta_\pm p^{-1}(C_\pm X) \longrightarrow C_\pm X \times \Omega\Sigma X$ by setting $\theta_\pm(\phi) = (p(\phi),\psi_\pm)$, where $\psi_\pm(t) = \phi(2t)$ for $0 \leqslant t \leqslant \frac{1}{2}$ and $\psi_\pm(t) = H_\pm(p(\phi),2t-1)$ for $\frac{1}{2} \leqslant t \leqslant 1$. One readily checks that these are homotopy equivalences over $C_\pm X$. When we restrict $\theta_\pm$ to $X \subseteq C_\pm X$, we obtain two distinct homotopy equivalences

$$p^{-1}(X) \xrightarrow{\theta_\pm} X \times \Omega\Sigma X.$$

We also define homotopy inverses $\eta_\pm$ to $\theta_\pm$ over X as follows. $\eta_\pm(x,\phi) = (x,\xi_\pm)$, where $\xi_\pm(t) = 2t$ for $0 \leqslant t \leqslant \frac{1}{2}$ and $\xi_\pm(t) = H(x,2-2t)$ for $y \leqslant t \leqslant 1$. Consider the composite $\theta_- \circ \eta_+ : X \times \Omega\Sigma X \longrightarrow X \times \Omega\Sigma X$. It is given by $\theta_- \circ \eta_+(x,\phi) = (x,\zeta)$, where ζ is described by the following formulae:

$$\begin{cases} \zeta(t) = \phi(4t) & \text{for } 0 \leqslant t \leqslant \frac{1}{4}, \\ \zeta(t) = H_+(x,2-4t) & \text{for } \frac{1}{4} \leqslant t \leqslant \frac{1}{2}, \\ \zeta(t) = H_-(x,2t-1) & \text{for } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Note that after suitable reparameterization, $\theta_-\eta_+$ becomes equal to the composite

$$X \times \Omega\Sigma X \xrightarrow{\alpha} X \times \Omega\Sigma X \times \Omega\Sigma X \xrightarrow{\beta} X \times \Omega\Sigma X$$

where $\alpha(x, \phi) = (x, J(s), \phi)$ and $\beta(x, \phi_1, \phi_2) = (x, \mu \circ (\phi_1, \phi_2))$. Here $J : X \longrightarrow \Omega\Sigma X$ is the James map and μ is the loop sum multiplication on $\Omega\Sigma X$. Now consider the Mayer–Vietoris sequence for the covering of E by $p^{-1}(C_+X)$ and $p^{-1}(C_-X)$. It has the form

$$\begin{array}{ccccc} & & H_*(\Omega\Sigma X) \cong H_*(p^{-1}(U_-)) & & \\ & \nearrow^{g} & & \searrow & \\ \cdots \longrightarrow H_*(X \times \Omega\Sigma X) & & \oplus & & H_*(E) \longrightarrow \cdots \\ & \searrow_{f} & & \nearrow & \\ & & H_*(\Omega\Sigma X) \cong H_*(p^{-1}(U_+)) & & \end{array}$$

If we identify $p^{-1}(X)$ with $X \times \Omega\Sigma X$ via θ_+, then f is just the homomorphism induced by projection. g, on the other hand, is given by the composite

$$H_*(X \times \Omega\Sigma X) \xrightarrow{J\times 1} H_*(\Omega\Sigma X \times \Omega\Sigma X) \xrightarrow{\mu} H_*(\Omega\Sigma X).$$

If we identify $H_*(X \times \Omega\Sigma X)$ with $H_*(X) \otimes H_*(\Omega\Sigma X)$, then the map is given by

$$H_*(X) \otimes H_*(\Omega\Sigma X) \xrightarrow{H_*(J)\otimes Id} H_*(\Omega\Sigma X) \otimes H_*(\Omega\Sigma X) \xrightarrow{H_*(\mu)} H_*(\Omega\Sigma X).$$

Since $H_*(E)$ is trivial we conclude that the map

$$\tilde{H}_*(X \times \Omega\Sigma X) \xrightarrow{(g,f)} \tilde{H}_*(\Omega\Sigma X) \oplus \tilde{H}_*(\Omega\Sigma X)$$

is an isomorphism of graded vector spaces. Further,

$$\tilde{H}_*(X \times \Omega\Sigma X) \cong \left[\tilde{H}_*(X) \otimes H_*(\Omega\Sigma X)\right] \oplus \tilde{H}_*(\Omega\Sigma X),$$

and f is just the projection on the second factor. It follows that the map $\tilde{H}_*(X) \otimes H_*(\Omega\Sigma X) \longrightarrow \tilde{H}_*(\Omega\Sigma X)$ is an isomorphism. Therefore, if we let $V_* = \tilde{H}_*(X)$ and A_* be the algebra $H_*(\Omega\Sigma X)$, and let $\bar{A}$ denote the ideal of positive dimensional elements, then $V_* \otimes A_* \longrightarrow \bar{A}_*$ is an isomorphism. We claim this characterizes A_* completely.

PROPOSITION 5.3.2. *Let A_* be a graded algebra with A_0 a field, and let $\bar{A}_*$ denote the ideal*

$$\bigotimes_{i=1}^{\infty} A_i.$$

Let $V_* \xrightarrow{i} \bar{A}_*$ be a map of graded vector spaces. Suppose the multiplication map $V_* \otimes A_* \longrightarrow \bar{A}_*$ is an isomorphism of graded vector spaces. Then the algebra homomorphism $\hat{\imath} : T_*(V) \longrightarrow A_*$ which restricts to i on V_* is an isomorphism.

PROOF. We first show that $\hat{\imath}$ is surjective. $\hat{\imath}$ is clearly an isomorphism in dimension 0. We now proceed by induction. Consider any $\alpha \in A_n$, and suppose it is known that all elements in A_{n-1} are in the image of $\hat{\imath}$. Since $V_* \otimes A_* \longrightarrow \bar{A}_*$ is an isomorphism, any homogeneous element α can be written in the form $\sum v_i \otimes \alpha_i$, where $v_i \in V_*$ and $\alpha_i \in A_*$. Since the v_i's all have grading greater than 0, the α_i's all have grading less than n and hence are in the image of $\hat{\imath}$. The v_i's are clearly in the image of $\hat{\imath}$, so therefore is α. To prove injectivity, we observe that $\hat{\imath}$ is an isomorphism in dimension 0. Now consider an element τ of minimal grading n on which $\hat{\imath}$ vanishes. Since τ is of positive grading, it lies in the image of $V \otimes T(V)$ in $T(V)$, i.e. $\tau = \Sigma v_i \otimes t_i$, where each t_i has grading less than n. Therefore, $\sum v_i \otimes \hat{\imath}(t_i) \neq 0$ in $V_* \otimes A_*$. But since the multiplication map $V_* \otimes A_* \longrightarrow \bar{A}_*$ is an isomorphism, we conclude that $\hat{\imath}(\tau) \neq 0$, which is a contradiction. □

COROLLARY 5.3.1. *Let $J : X \longrightarrow \Omega\Sigma X$ be the James map. Then the natural homomorphism $T(\tilde{H}_*(X)) \longrightarrow H_*(\Omega\Sigma X)$ is an isomorphism of graded algebras.*

COROLLARY 5.3.2. *If X is a connected CW complex, the map*

$$\hat{J} : M(\hat{X}, 0) \longrightarrow \Omega^M(\Sigma X, 0)$$

induces an isomorphism on homology groups. Hence, $\hat{J}$ is a homotopy equivalence.

PROOF. The homology statement is clear since we have a commutative diagram

$$\begin{array}{ccc} & T(\tilde{H}_*(\hat{X})) & \\ \swarrow & & \searrow \\ H_*(M(\hat{X},0)) & \xrightarrow{H_*(\hat{J})} & H_*(\Omega^M(\Sigma X,0)) \end{array}$$

where we have proved that both diagonal arrows are isomorphisms.

This shows that $H_*(\hat{J})$ induces isomorphism on $H_*(\ ;\mathbf{Q})$ and $H_*(\ ;\mathbf{F}_p)$. The universal coefficient theorem then gives the result for $H_*(\ ;\mathbf{Z})$. The relative Hurewicz theorem now gives the result for homotopy groups. $M(\hat{X},0)$ has a natural cell structure coming from the cell structures on the products $\hat{X}^n$, so $M(\hat{X},0)$ is a CW complex. By a theorem of Milnor, [25], $\Omega^M(\Sigma X,0)$ has the homotopy type of a CW complex. Theorem 2.1.3 now applies. □

5.4. *The Adams–Hilton construction for ΩY*

We now build a model for $\Omega^M Y$ where Y is a simply connected CW complex but not necessarily a suspension.

The model for the construction we are about to present is James' result above. Note that $J(X) \simeq \Omega\Sigma X$ is a free, associative, unitary monoid with a natural CW decomposition provided that the base point $*$ is a vertex[3], coming from the natural decomposition of X^n as a product CW complex. Thus, $J(X)$ has the following three properties:

- every element $v \in J(X)$ has a unique expression $v = *$ or $v = x_1 x_2 \cdots x_n$, $x_i \in X - *$ for $1 \leqslant i \leqslant n$,
- $x_1 \cdots x_n$ is contained in a unique cell of $J(X)$, the cell $C_1 \times C_2 \times \cdots \times C_n$ where $x_i \in \mathrm{Int}(C_i)$, $1 \leqslant i \leqslant n$, so in particular, no indecomposable cell contains decomposable points,
- the cell complex has the form of a tensor algebra $T(C_\#(X))$, where the subcomplex $C_\#(X)$ is exactly the indecomposables, and the generating cells in dimension i are in 1–1 correspondence with the cells in dimension $i+1$ of ΣX.

THEOREM 5.4.1 (Adams–Hilton). *Let Y be a CW complex with a single vertex and no 1-cells:*

$$Y = * \cup e_1^2 \cup e_2^2 \cup \cdots \cup e_r^2 \cup e_1^3 \cup \cdots.$$

Then there is a model for $\Omega^M(Y)$ which is a free associative CW monoid, with $*$ the only vertex, the generating cells $f_1^i, \ldots, f_r^i, \ldots$ in dimension i are in 1–1 correspondence with the $(i+1)$-dimensional cells of Y and it satisfies condition (2) above. (For (3) there is no reason to assume that ∂ of an indecomposable cell consists only of indecomposable terms.)

PROOF. The proof essentially goes by noting the way in which the loop space changes as we add cells to our space Y.

In particular, the 2-skeleton,

$$sk_2(Y) = * \cup e_1^2 \cup e_2^2 \cup \cdots \cup e_r^2 \simeq \bigvee S^2 = \Sigma \bigvee S^1,$$

is a suspension and the theorem is James' result. So what we need is a device for doing an inductive step.

DEFINITION 5.4.1. Let M be an associative, unitary monoid with base point the identity, and suppose that $f : X \longrightarrow M$ is a based map. Then the prolongation $P(M, f, cX)$ is the associative, unitary monoid

$$\coprod_{n=1}^{\infty} (M \cup_f cX)^n / \sim$$

with multiplication induced by juxtaposition, and where $\sim$ is the equivalence relation

$$(x_1, \ldots, x_n) \sim (x_1, \ldots, \widehat{x_i}, x_i x_{i+1}, \ldots, x_n)$$

if and only if both x_i and x_{i+1} are contained in M or one of x_i, x_{i+1} is the unit $*$.

[3] Using the compactly generated topology so that products behave well.

$P(M, f, cX)$ has obvious universality properties: it is universal for maps of $M \cup_f cX$ into associative unitary monoids, which are multiplicative on M. Additionally, if X is a sphere S^n, f is cellular, and M has a CW multiplication, then $P(M, f, cX)$ has a CW multiplication, and $C_\#(P(M, f, cX))$ has the form $T(A, e^{n+1})$ where A is the CW complex of M.

Now, we suppose that a principal M-quasifibering has been constructed $M \longrightarrow E \longrightarrow B$ with E contractible which is sufficiently structured that we can build the associated principal $P(M, f, cX)$ quasifibering over B by just replacing the fiber M by $P(M, f, cX)$, so we have, by a minor abuse of notation, the quasifibering

$$P(M, f, cX) \longrightarrow P(M, f, cX) \times_M E \longrightarrow B.$$

This extends to a quasifibering

$$\{P(M, f, cX) \times_M E\} \cup \{P(M, f, cX) \times c(cX)\}/\sim \longrightarrow B \cup c\Sigma X \qquad (5.5)$$

where $\sim$ is the identification $(p, 0, \{t, x\}) \sim (p, t, f(x))$ where $(t, f(x))$ is the track of the contracting homotopy in E on the image of $f(x) \in M$.

The base of this quasifibration is $B \cup_{\Sigma f} c\Sigma X$ and it is not hard to show that the total space is again contractible if say X is a sphere S^n, $n \geqslant 1$, and f is cellular. This can be verified by using the contracting homotopy in $C_\#(E)$ together with the obvious contraction of the new cell e^{n+1} in the new part to build a contraction on the entire cellular chain complex. Moreover, in our situation it will also be direct to check that the resulting quasifibering has sufficient structure that we can again build an associated principal quasifibration from it.

We now proceed with the construction, starting with the trivial $M = *$ over $*$. The next step attaches e^1's, one for each 2-cell of Y via the unique map $f : \bigvee S^0 \longrightarrow *$. The resulting quasifibering has the form $J(\bigvee S^1) \cup J(\bigvee S^1) \times c(\bigvee S^1)$ where

$$(x_1 \cdots x_r, 1, x) \sim x_1 \cdots x_r \cdot x, \quad (x_1 \cdots x_r, 0, x) \sim x_1 \cdots x_r,$$

and $(x_1 \cdots x_r, t, *) \sim x_1 \cdots x_r$ as well. The base is, of course, $sk_2(Y) \simeq \bigvee S^2$.

At each stage, the space $P(M, f, cX)$ has the homotopy type of $\Omega^M(B \cup c\Sigma X)$ where the attaching map is $\Sigma f : \Sigma X \longrightarrow B$. Consequently, assuming that B is the homotopy type of $sk_i(Y)$, we can assume $\Sigma X = \bigvee S^i$, one sphere for each $(i+1)$-cell in Y, with Σf restricted to S^i_j the j-th attaching map, and the base for $P(M, f, cX)$, using the construction above has the homotopy type of $sk_{i+1}(Y)$. (It should be noted that the attaching maps in M are uniquely determined since the total space of the quasifibration at the $(i-1)^{st}$ stage is assumed to be contractible, and that the images of the traces of the contraction on f in the base will be the attaching maps for $sk_{i+1}(Y)$.) □

This is the *Adams–Hilton model* for $\Omega^M X$. Of course, since the prolongation construction is universal it is not always the most efficient way to build a model for the loop space.

EXAMPLE 5.1. $\mathbf{CP}^2 = S^2 \cup e^4$ where the attaching map is the classical Hopf map $h : S^3 \longrightarrow S^2$. There is a fibration $S^1 \longrightarrow S^5 \longrightarrow \mathbf{CP}^2$, and hence, taking loops, a fibration

$$\Omega^M S^5 \longrightarrow \Omega^M \mathbf{CP}^2 \longrightarrow S^1.$$

We claim that this fibration splits up to homotopy type as the product $\Omega^M(S^5) \times S^1$. From the long exact sequence of homotopy groups for the fibration, we see that

$$\pi_1(\Omega^M \mathbf{CP}^2) \longrightarrow \pi_1(S^1) = \mathbf{Z}$$

is an isomorphism. Consequently, mapping $S^1 \longrightarrow \Omega^M \mathbf{CP}^2$ so as to represent a generator of $\pi_1(\Omega^M \mathbf{CP}^2)$, and using the homotopy lifting property, we can map $S^1 \longrightarrow \Omega^M \mathbf{CP}^2$ so that the composite $S^1 \longrightarrow \Omega^M \mathbf{CP}^2 \longrightarrow S^1$ is the identity. Now, using the multiplication in Ω^M, we have a map of the product $(\Omega^M S^5) \times S^1 \longrightarrow \Omega^M \mathbf{CP}^2$ which gives the asserted homotopy equivalence.

This shows $\Omega^M \mathbf{CP}^n$ has the homotopy type of a CW complex with one cell in each dimension congruent 0 and 1 mod 4 and no other cells. Furthermore, the fact that the bottom circle splits off implies that the boundary map in the cellular chain complex is identically zero.

On the other hand, the Adams–Hilton theorem gives as a model for $\Omega^M \mathbf{CP}^2$ the prolongation $P(\Omega^M S^2, \Omega h, e^4)$ which has a cell decomposition given by the prolongation of

$$\left(e^1 \cup e^2 \cup e^3 \cup \cdots\right) \cup_{\Omega(h)} f^3.$$

Thus, P has cells of the form

$$e^{i_0} \times f^{3k_0} \times e^{i_1} \times f^{3k_1} \times \cdots.$$

This cellular decomposition of $\Omega^M \mathbf{CP}^2$ is much bigger than the one obtained above by splitting off the circle and therefore there must be a massive number of nontrivial boundary maps here. For example, $e^2 = e^1 * e^1$ so $\partial(e^2) = 0$, but since $H_2(\Omega^M \mathbf{CP}^2) = 0$ we must have $\partial(f^3) = e^2$. Using the multiplication in the cell complex this boundary map now determines all the boundary maps.

EXAMPLE 5.2. We know from James' construction that

$$\Omega S^{n+1} = S^n \cup e^{2n} \cup e^{3n} \cup e^{4n} \cup e^{5n} \cup \cdots.$$

Thus, the Adams–Hilton construction implies that, for $n \geqslant 2$, there is a cell decomposition

$$C_\#(\Omega^2 S^{n+1}) = T\left[e^{n-1}, e^{2n-1}, e^{3n-1}, e^{4n-1}, e^{5n-1}, \cdots\right].$$

The results in Sections 6 and 7 determine the boundary maps which are quite complex and begin to reflect some of the deeper structure of S^{n+1}. For example, it turns out that

$$\partial\left(e^{2n-1}\right) = 2\left[S^{n-1}\right] * \left[S^{n-1}\right].$$

In general the examples above show that it is quite difficult to understand the boundary maps in the cell decomposition provided by the Adams–Hilton theorem. However, in the special case that Y is given as a *simplicial complex* with no edges, and consequently only one vertex Adams built an explicit model with an explicit ∂ map and we discuss his results next.

5.5. *The Adams cobar construction*

To compute the boundary in the chain complex of $AH(X)$ for general X is a major problem in homotopy theory. (If one knows how to do this sufficiently well it gives as a special case reasonable algorithms for determining the $\pi_*^s(S^0)$ for example.) For certain special types of complexes this has been done, though, and here we follow J.F. Adams, [2], and assume that X is, in fact, an ordered simplicial complex with the 1-skeleton collapsed to the base point $*$. This is actually only a weak restriction on X since we have

LEMMA 5.5.1. *Let X be a connected, locally finite simplicial complex with $\pi_1(X) = 0$, then there is a finite 2-dimensional subcomplex, $C_2 \subset X$, containing the entire 1-skeleton, $sk_1(X)$, with $\tilde{H}_*(C_2, \mathbf{Z}) \equiv 0$ and the quotient map $p : X \longrightarrow X/C_2$ is a homotopy equivalence.*

PROOF. $sk_1(X)$ has the homotopy type of a wedge of circles, $\bigvee_1^m S^1$, and there is a cofibering

$$X \longrightarrow X/sk_1(X) \xrightarrow{w} \bigvee_1^m S^2.$$

Since $\tilde{H}_i(X) = 0$ for $i = 0, 1$, the homology long exact sequence for the cofibering implies that

$$w_* : H_2(X/sk_1(X); \mathbf{Z}) \longrightarrow H_*(\Sigma sk_1(X); \mathbf{Z})$$

is onto. On the other hand, a basis for $H_2(X/sk_1(X); \mathbf{Z})$ can be chosen which consists only of the Hurewicz images of the fundamental classes of embeddings, $\phi(\sigma^2/\partial\sigma^2) \longrightarrow X/sk_1(X)$, where the σ^2 run over a subset of the 2-simplexes of X. Consequently the same is true for $im(w_*)$. That is to say, there are m 2-simplexes $\sigma_1^2, \ldots, \sigma_m^2$ in $sk_2(X)$ so that

$$sk_1(X) \cup \bigcup_1^m \sigma_j^2 = C_2$$

has trivial reduced homology. Now, $\pi_1(C_2)$ need not be zero, so C_2 need not be contractible. However, in the cofibering

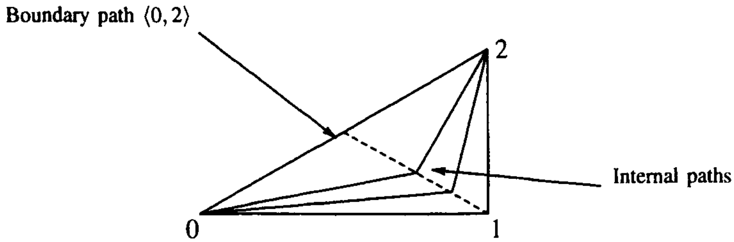
$$C_2 \longrightarrow X \xrightarrow{p} X/C_2$$

we must have $\pi_1(X/C_2) = 0$, since C_2 contains the entire 1-skeleton of X. Consequently, p is a homotopy equivalence. □

The simplicial structure of X/C_2 is sufficiently rigid to allow us to systematically compute the boundary map ∂ in $AH(X/C_2)$. Similarly, we will consider the problem when X is given as a cubical complex.

In both cases the idea is to make an explicit model consisting of paths from an initial to a final vertex in the simplex or the cube, via acyclic models types of techniques to describe each generating $n-1$ cell in $AH(X)$ for every cell $e^n_\alpha \subset X$.

To begin consider the ordered triangle $\langle 0,1,2\rangle$



(5.6)

Fig. 1

The paths we construct will start at the vertex $\langle 0\rangle$ and end at $\langle 2\rangle$. To begin we consider paths along the boundary. There are two ways of moving along edges from 0 to 2. The first path, which we denote $\langle 0,1\rangle * \langle 1,2\rangle$, moves linearly along the bottom edge from 0 to 1 and then from 1 to 2. The second path, which we denote $\langle 0,2\rangle$ moves linearly along the hypotenuse from 0 to 2. Now let l be the line connecting 1 to the midpoint[4] of the path from 0 to 2. For each $t \in l$ there is the straight line path from 0 to t to 2 and this gives a one parameter family of paths from 0 to 2 connecting the two edge paths, $\langle 0,1\rangle * \langle 1,2\rangle$ and $\langle 0,2\rangle$. If we order the vertices of X, then each 2-simplex is linearly identified with $\langle 0,1,2\rangle$ and we can use this identification to associate to each 2-simplex σ^2 a 1-simplex in the path space on σ^2.

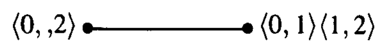


Fig. 2

Moreover, since, by assumption, X has only a single vertex, these paths actually are all in $\Omega^M X$, and we have constructed a correspondence from the 2-cells of X to 1-cells in $\Omega^M X$. In the Adams–Hilton construction, what was important to show that the cells there were "correct", was that the evaluation map

$$eval : \left(I \times e^{n-1}, \partial I \times e^{n-1}\right) \longrightarrow \left(sk_n(X), sk_{n-1}(X)\right)$$

[4] The notation is chosen to emphasize that this path is actually the composition of two paths, the first from 0 to 1 and the second from 1 to 2. Its length is 2, so we are naturally working here in the space of Moore loops.

have degree one to the corresponding cell in X. In the case here the evaluation map is explicit and evidently of degree one.

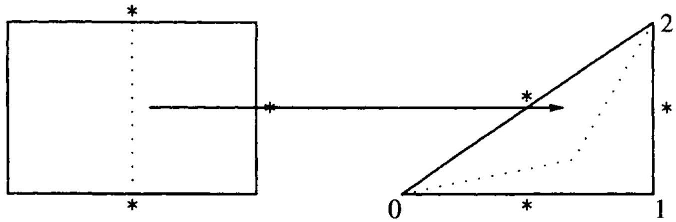


Fig. 3

(5.7)

To continue we need to study the analogous construction for higher dimensional simplexes. Thus, consider the tetrahedron $\langle 0,1,2,3\rangle$

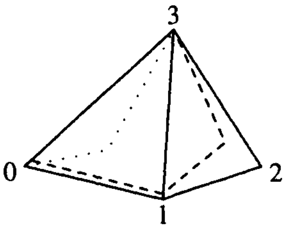


Fig. 4

(5.8)

To begin, we know how to fill in paths along the two faces $\langle 0,1,3\rangle$ and $\langle 0,2,3\rangle$ containing both vertices 0 and 3 by using the previous construction for σ^2. Moreover, along their intersection $\langle 0,3\rangle$, the paths agree. On the face $\langle 1,2,3\rangle$ we know how to construct paths from 1 to 3, and to construct paths from 0 to 3 we simply compose with the path $\langle 0,1\rangle$! Thus, here the paths are of the form $\langle 0,1\rangle * \varphi_t$. Moreover, the boundary paths are $\langle 0,1\rangle * \langle 1,3\rangle$ which is also a boundary path for the paths in $\langle 0,1,3\rangle$, and $\langle 0,1\rangle * \langle 1,2\rangle * \langle 2,3\rangle$.

Also, the paths in $\langle 0,2,3\rangle$ have boundary paths $\langle 0,3\rangle$ which is already accounted for, and $\langle 0,2\rangle * \langle 2,3\rangle$. Note that this implies that we should fill in the paths along the final face $\langle 0,1,2\rangle$ so that they have the form $\varphi_t * \langle 2,3\rangle$.

Thus we have extended the construction above to fill in paths from 0 to 3 along all four of the faces of the tetrahedron using four intervals connected together in the form of the boundary of the square, and, since the map is degree one on each face, it clearly gives a degree one map, on evaluation

$$eval : \left(I \times \partial I^2, \partial I \times \partial I^2\right) \longrightarrow \left(\partial\sigma^3, \{0,3\}\right).$$

Now, contracting σ^3 to $\langle 0,3\rangle$ extends our construction of paths to a three-dimensional analog of the previous construction,

$$h_2 : \left(I^2, \partial I^2\right) \longrightarrow \left(E_{0,3}^{\sigma^3}, E_{0,3}^{\partial\sigma^3}\right)$$

which is again degree one on evaluation,

$$eval : \left(I^3, \partial\right) \longrightarrow \left(\sigma^3, \partial\right)$$

by filling in the following diagram

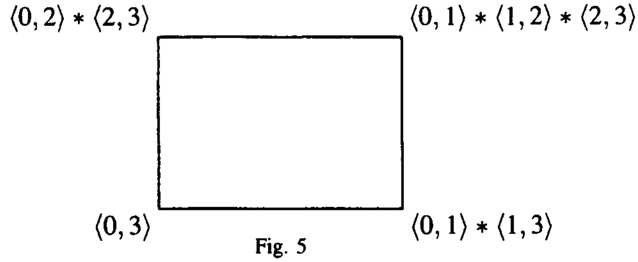


Fig. 5

(5.9)

With these preliminary constructions in mind, we can describe the general case.

THEOREM 5.5.1 (Adams). *For each positive integer n, $n = 2, 3, \ldots$, there is a map*

$$p_n : I^{n-1} \longrightarrow E_{0,n}^{\sigma^n}$$

so that $p_n|\partial I^{n-1}$ has image contained in $E_{0,n}^{\partial\sigma^n}$ and the evaluation map

$$eval(p_n) : \left(I^n, \partial I^n\right) \longrightarrow \left(\sigma^n, \partial\sigma^n\right)$$

has degree one. Moreover, the p_n fit together in the sense that p_n restricted to the boundary consists of maps of the form $p_j * p_{n-j-1}$ where $*$ represents juxtaposition of paths.

PROOF. The proof is by induction. To begin, we assume the p_j are defined for $j \leqslant n-1$, and, since we have already constructed the maps for $n = 1, 2, 3$, we might as well assume $n \geqslant 4$.

Each point in $\partial(I^{n-1})$ can be regarded as an n-tuple $(t_1, \ldots, t_n)$ where at least one of the t_i's is either zero or one. We can assign to every vertex the edge path in $\partial\sigma^n$ from 0 to $n-1$ given by

$$\left\langle 0, , \ldots, i_1 \right\rangle * \left\langle i_1, \ldots, \widehat{k_1 - 1}, \widehat{k_1}, \ldots, i_2 \right\rangle * \cdots$$

where we have cut $\langle 0, \ldots, n\rangle$ at every i_j where $t_{i_j} = 0$ by inserting a $\ldots, i_j\rangle\langle i_j, \ldots$ and dropped the vertices corresponding to every $t_{k_j} = 1$. Once again, we can fill in this map over the faces of I^{n-1}, so that, over the face $J_{j_1,\ldots,j_s}^{i_1,\ldots,i_r}$ (where $t_{i_r} = 0$ while $t_{j_s} = 1$), we have $p_{l_1} * p_{l_2} * \cdots * p_{l_{r+1}}$.

To be precise, p_{l_t} maps to the paths on the face $\langle i_t, \ldots, \widehat{j_s}, \ldots, i_{t+1}\rangle \subset \sigma^n$, from i_t to i_{t+1}, where the j_s are deleted from $\{i_t, i_t + 1, \ldots, i_{t+1} - 1, t_{t+1}\}$ for each j_s with $i_t < j_s < i_{t+1}$. Clearly, this definition is consistent and defines p_n on ∂I^{n-1}. Moreover, evaluation is degree one on each of the $n-1$ faces J_l, $1 \leqslant l \leqslant n-1$, as well as on the faces J^1 and J^{n-1} and their images lie in distinct $(n-1)$ faces of σ^n, by the inductive assumptions and the construction. Also, the images of the remaining faces all lie in the $(n-2)$ skeleton of σ^n. Hence, it follows that evaluation has degree one on ∂I^{n-1}, and hence, from the 5-lemma, also has degree one for p_n. This completes the inductive step. □

Thus, we see that the boundary of the cell I^{n-1} corresponding to the simplex $\sigma^n \subset X$ is a union of products of lower dimensional cells under the loop sum operation in the Moore loop space, as well as a piece corresponding to the original boundary of σ^n. Formally, on the complex

$$T(X) = T\left(e_1^1, \ldots, e_r^1, \ldots, e_\alpha^{n-1}, \ldots\right),$$

remembering that $\partial(\sigma^n) = \sum(-1)^i F_i(\sigma^n)$, we have

$$\partial\left(e^{n-1}\right) = \sum(-1)^i e\left(F_i(\sigma^n)\right) + \sum_{j=2}^{n-2} e\left(f_j(\sigma^n)\right) e\left(l_j(\sigma^n)\right). \tag{5.10}$$

Here

- $f_j(\sigma^n) = \langle 0, \ldots, j\rangle$ is the map on the front j face,
- $l_j(\sigma^n) = \langle j, \ldots, n\rangle$ is the map on the back $n-j$ face.

The second term in (5.7) formally corresponds to the the Alexander diagonal approximation, A, which is given on simplices as

$$A : \sigma^n \longrightarrow \sum_{j=0}^{n} f_j(\sigma^n) \otimes l_j(\sigma^n)$$

and induces a chain map on simplicial complexes and singular complexes:

$$A_* : C_*(X) \longrightarrow C_*(X) \otimes C_*(X),$$

so $A\partial = \partial^{\otimes} A$. It is also easy to check

PROPOSITION 5.5.1. *The Alexander chain map is coassociative. That is,*

$$(A \times 1) \circ A = (1 \times A) \circ A.$$

Moreover, A is chain homotopic to the diagonal map

$$C_*(X) \xrightarrow{\Delta} C_*(X) \otimes C_*(X)$$

in the singular chain complex of X.

Dualizing, we have

PROPOSITION 5.5.2. *The dual Alexander map*

$$A^* : C^*(X) \otimes C^*(X) \longrightarrow C^*(X)$$

is an associative cochain map. Moreover, the induced pairing on cohomology $H^*(X;\mathbf{F}) \otimes H^*(X;\mathbf{F}) \longrightarrow H^*(X;\mathbf{F})$ is just the cup product.

Summarizing, we have identified the second summand in 5.7,

$$\sum_{j=2}^{n-2} e\big(f_j(\sigma^n)\big)e\big(l_j(\sigma^n)\big), \tag{5.11}$$

and at the cohomology level, it is directly tied in to the cup product. In particular, if the cup product structure for $H^*(X;\mathbf{F})$ is nontrivial, then this piece *must* be present.

PROPOSITION 5.5.3. *If X is a suspension, say $X = \Sigma Z$, then all cup products in $H^*(X;\mathbf{A})$ are zero.*

PROOF. Consider the homotopy of the diagonal map

$$\Sigma X \xrightarrow{\Delta} \Sigma X \times \Sigma X$$

defined by

$$H\big(\tau, \{t, x\}\big) = \big(\{\overline{(\tau+1)t}, x\}, \{\underline{(1+\tau)t - \tau}, x\}\big)$$

where $\underline{m} = m$ if $m \geqslant 0$, and is 0 if $m \leqslant 0$, while $\overline{m} = m$ if $0 \leqslant m \leqslant 1$ and is 1 if $m \geqslant 1$. When $\tau = 1$ it has image contained in $\Sigma X \vee \Sigma X \subset \Sigma X \times \Sigma X$, and the result follows. □

This partially explains why we can replace the general Adams–Hilton model by the James model for ΩX in case X is a suspension.

To actually compute we note that given a j-cell σ^j in X we have constructed a $j-1$ cell $e(\sigma^j)$ in $AH(X)$. We denote the dual cochain by $|\sigma^j|$. Thus, given a product cell

$$I^{j_1-1} \times I^{j_2-1} \times \cdots \times I^{j_l-1}$$

in $AH(X)$ we label the dual cochain, which is of dimension $\sum j_i - l$ by

$$|\sigma^{j_1}|\sigma^{j_2}|\cdots|\sigma^{j_l}|.$$

Thus, dualizing 5.7 we can write the coboundary map

$$\begin{aligned} \delta\big(|\sigma^{j_1}|\cdots|\sigma^{j_l}|\big) &= \sum(-1)^t|\sigma^{j_1}|\cdots|\delta(\sigma^{j_t})|\cdots|\sigma^{j_l}| \\ &= \sum(-1)^s|\sigma^{j_1}|\cdots|\sigma^{j_t} \cup \sigma^{j_t}|\cdots|\sigma^{j_l}|. \end{aligned} \tag{5.12}$$

Here

- $|\delta(\sigma^{j_t})|$ is shorthand for the coboundary on $|\sigma^{j_t}|$, the dual of $e(\sigma^{j_t})$,
- $|\sigma^{j_t} \cup \sigma^{j_t}|$ is shorthand for the cup product on the obvious dual co-cells,
- the sign in the second sum is given by setting s equal to the number of bars plus the sum of the dimension of the cells that are passed over.

This is just the *Bar construction* on the associative chain algebra $C^{\#}(X)$! We can filter $C^{\#}(AH(X))$ by the number of bars describing a (dual) cell and (5.12) shows that $\delta\mathcal{F}_i(AH(X)) \subset \mathcal{F}_i(AH(X))$. Consequently, we obtain a spectral sequence for computing $H^*(\Omega X;\mathbf{F})$, with E_2-term

$$Ext_{H^*(X;\mathbf{F})}(\mathbf{F},\mathbf{F}), \tag{5.13}$$

where the ring structure on $H^*(X;\mathbf{F})$ is obtain from the cup product. As an example $H^*(\mathbf{CP}^\infty;\mathbf{F}) = \mathbf{F}[b]$, a polynomial algebra on a two dimensional generator, and

$$Ext_{\mathbf{F}[p]}(\mathbf{F},\mathbf{F}) = E(|b|),$$

the exterior algebra on a one dimensional generator. Hence, in this case the spectral sequence collapses. But the spectral sequence does not always collapse, and the higher differentials measure the difference between the information given by the chain level Alexander diagonal approximation and the cup product.

REMARK. One other reason for the close connection between the diagonal map and ΩX is the fact that the fiber of the Serre fibration

$$X \xrightarrow{\Delta} X \times X$$

is ΩX.

6. The structure of second loop spaces

In Section 5 we showed that for a connected CW complex with no one cells one may produce a CW complex, with cell complex given as the free monoid on generating cells, each in one dimension less than the corresponding cell of X, which is homotopy equivalent to ΩX. To go further one should study similar models for double loop spaces, and more generally for iterated loop spaces.

In principle this is direct. Assume X has no i-cells for $1 \leqslant i \leqslant n$ then we can iterate the Adams–Hilton construction of Section 5 and obtain a cell complex which represents $\Omega^n X$. However, the question of determining the boundaries of the cells is very difficult as we already saw with Adams' solution of the problem in the special case that X is a simplicial complex with $sk_1(X)$ collapsed to a point. It is possible to extend Adams' analysis to $\Omega^2 X$, but as we will see there will be severe difficulties with extending it to higher loop spaces except in the case where $X = \Sigma^n Y$.

6.1. *Homotopy commutativity in second loop spaces*

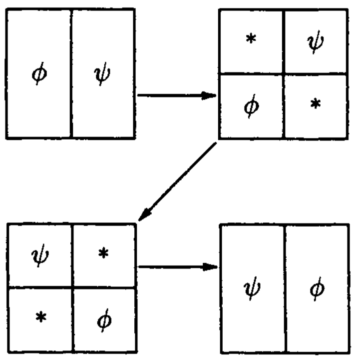
Given a based CW complex X, elements in Ω^2X can be thought of as maps from I^2 to X, so that $\partial(I^2)$ is sent to the base point. There are two notions of loop sum in Ω^2X; we consider the one coming from the loop structure in the first variable, and call it μ; thus

$$\mu(\phi,\psi)(s,t) = \begin{cases} \phi(2s,t) & \text{if } 0 \leqslant s \leqslant \frac{1}{2}, \\ \psi(2s-1,t) & \text{if } \frac{1}{2} \leqslant s \leqslant 1. \end{cases}$$

It is typically shown in first year topology that $\pi_2(X)$ is abelian for any complex X. From the usual adjointness considerations this is equivalent to the assertion that $\pi_0(\Omega^2X)$ is abelian. This suggests that μ itself should, in some sense, be commutative, at least up to homotopy. The formal version of this statement is that if we let

$$T : \Omega^2X \times \Omega^2X \longrightarrow \Omega^2X \times \Omega^2X$$

be the twist map, $T(\phi_1,\phi_2) = (\phi_2,\phi_1)$, then $\mu \circ T$ is homotopic to μ. The homotopy, $\mathcal{H}$, is given by the following figure.



(6.1)

Thus, two fold loop spaces are "homotopy commutative". One might now guess that $\Omega^2\Sigma^2X$ should be homotopy equivalent to the free commutative monoid on X, as $\Omega\Sigma X$ is equivalent to the free monoid on X. This naive guess fails, however, as one can see from the Dold–Thom theorem, which asserts that if $SP^\infty(X)$ denotes the infinite symmetric product on X (i.e. the free abelian monoid), then $\pi_*(SP^\infty(X)) \cong H_*(X)$. Thus, $\pi_*(SP^\infty(S^2)) = 0$ for $* > 2$, while $\pi_3(\Omega^2\Sigma^2S^2) = \pi_5(S^4) = \mathbf{Z}/2$, generated by the double suspension of the Hopf map $\eta : S^3 \longrightarrow S^2$.

It turns out that there are "degrees" of homotopy commutativity which must be encoded in our models, and that Ω^2X is, in a sense, minimally homotopy commutative and Ω^kX becomes more and more highly homotopy commutative as k goes to infinity. But even within the second loop space there are levels of homotopy commutativity which must be distinguished. For example there are two ways of using homotopy commutativity to pass from $a * b * c$ to $c * b * a$. We have

$$a*b*c \mapsto b*a*c \mapsto b*c*a \mapsto c*b*a,$$
$$a*b*c \mapsto a*c*b \mapsto c*a*b \mapsto c*b*a$$

corresponding to the relation $(1,2)(2,3)(1,2) = (2,3)(1,2)(2,3) = (1,3)$ in the symmetric group S_3. Gluing together the three homotopies above give two maps $\psi, \Psi : [0,3] \times \left(\Omega^2 X\right)^3 \longrightarrow \Omega^2 X$ where

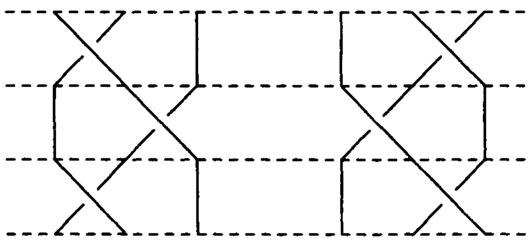
$$\psi\left(0 \times \left(\Omega^2 X\right)^3\right) = \Psi\left(0 \times \left(\Omega^2 X\right)^3\right), \quad \psi\left(3 \times \left(\Omega^2 X\right)^3\right) = \Psi\left(3 \times \left(\Omega^2 X\right)^3\right),$$

and hence a map $\Theta : D \times (\Omega^2 X)^3 \longrightarrow \Omega^2 X$ where D is the boundary of a *hexagon*, $C(2)$ and the map on each interval represents one of the homotopies.

LEMMA 6.1.1. *The map Θ may be filled in so as to give a map $A_2 : C(2) \times (\Omega^2 X)^3 \longrightarrow \Omega^2 X$ which agrees with Θ on $D \times (\Omega^2 X)^3$.*

PROOF. Note that $a * b * c$ is a map of I^2 to X with ∂I^2 mapping to $*$ and three smaller rectangles specified on which the map is, respectively a, b, and then c. What we did in the original homotopy of commutation was shrink these rectangles and move them past each other, then increase their size. So what we do is to shrink them even smaller and slide them past each other in an appropriate way so as to move from the first homotopy to the second. We can specify the motion by specifying the centers and sizes of the rectangles and then moving the centers.

The following diagram shows the movement of the respective centers in I^2 as we move from the $a * b * c$ to $c * b * a$ in the three stages indicated and in the two distinct manners indicated. The first is $\sigma_1\sigma_2\sigma_1$ and the second is $\sigma_2\sigma_1\sigma_2$ where σ_1 exchanges the first and second while σ_2 exchanges the second and third.



$\sigma_1\sigma_2\sigma_1$ $\sigma_2\sigma_1\sigma_2$ (6.2)

The two homotopies are described in (6.2), but, as asserted, (6.2) also makes it clear that the first can be deformed to the second without introducing any self intersections and without moving the points at the top or bottom of the two "braids". This deformation fills in the hexagon. □

In the next section we generalize this construction and extend the ideas of Section 5 to create a good model for the second loop space. Additionally, the point of view developed in the analysis here, in Section 7 becomes the key to developing good models for $\Omega^n \Sigma^n X$ for all n.

6.2. *The Zilchgon model for* $\Omega^2 X$

Adams replaced simplices, σ^n, by cubes, I^{n-1}, in building an explicit model for the Adams–Hilton construction of ΩX when X is a simplicial complex with its one skeleton collapsed to a point. It is natural to try to generalize this. Thus, suppose that Y is a *cubical* CW complex where the one skeleton has again been collapsed to a point. It is certainly possible to find combinatorial cells $C(n-1)$ which will replace each I^n in Y in building an explicit model for the Adams–Hilton construction. If this can be done in a sufficiently natural manner then, for X is a simplicial complex with $sk_2(X)$ collapsed to a point, this would give an explicit construction for $\Omega^2 X$. This, in fact turns out to be possible and we describe the construction now.

We begin by looking at the edge paths starting at $(0,\dots,0) \in I^n$ and ending at $(1,\dots,1)$. An edge has the form $(\varepsilon_1,\dots,\varepsilon_r,t,\varepsilon_{r+2},\dots,\varepsilon_n)$ where each ε_i is either a zero or a one. Then, we can specify the edge path by specifying which coordinates are moved in which order. So $E(1)E(3)E(2)$ for I^3 would mean the path which first moves the first coordinate, then goes from $(1,0,0)$ to $(1,0,1)$ by using the third coordinate, and finally goes from $(1,0,1)$ to $(1,1,1)$ using the second coordinate. It follows that these edge paths are indexed by the elements in the symmetric group S_3, and for I^n, by the symmetric group S_n. So we look for a polyhedron of dimension $n-1$ with vertices indexed by S_n to model paths in I^n.

We now introduce a family of combinatorial cells which do just this, the Zilchgons, (also called permutahedra by combinatorialists), $C(n)$. This will allow us to build explicit models for ΩX where X is a cubical complex with $sk_1(X) \sim *$ or $\Omega^2 X$ where X is a simplicial complex with $sk_2(X) \sim *$. But any attempt to continue this process will require many different combinatorial cells in each dimension $\geqslant 2$.

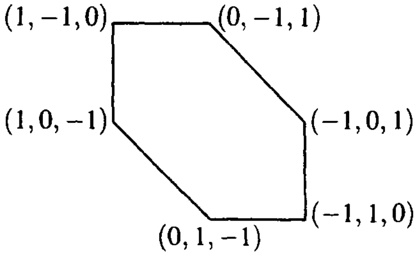
Let $e = (1,2,\dots,n) \in \mathbf{R}^n$ and let $C(n-1)$ be the convex hull of the translates of e by the usual permutation action of the symmetric group S_n on $\mathbf{R}^n$. Note that the convex hull spanned by a set S is the set of points

$$\left\{ \sum t_i s_i \mid 0 \leqslant t_i,\ \sum t_i = 1,\ s_i \in S \right\}.$$

In particular $C(n-1) \subset A^{n-1}$ where A^{n-1} is the $(n-1)$ dimensional affine plane in $\mathbf{R}^n$ with equation

$$\sum_{i=1}^{n} x_i = n(n+1)/2.$$

EXAMPLE 6.1. $C(1)$ is the line segment from $(1,2)$ to $(2,1)$ in $\mathbf{R}^2$ while $C(2)$ is the convex hull spanned by the six points $(1,2,3)$, $(1,3,2)$, $(2,1,3)$, $(2,3,1)$, $(3,1,2)$, and $(3,2,1)$, or projecting onto the plane through the origin parallel to the plane $x+y+z=6$, with coordinates $(-1,0,1)$, $(-1,1,0)$, $(0,-1,1)$, $(0,1,-1)$, $(1,-1,0)$ $(1,0,-1)$.



The space $C(2)$

(6.3)

It will turn out that $C(1)$ represents the homotopy of commutativity, while $C(2)$ represents the homotopy of $\sigma_1\sigma_2\sigma_1$ to $\sigma_2\sigma_1\sigma_2$ discussed in the last section. The higher dimensional $C(r)$'s will give all the possible ways, involving $r+1$ loops, of homotopy commuting the homotopies of commutation in the previous constructions involving fewer loops.

We now show that $C(n-1)$ is topologically a closed $(n-1)$ ball in $\mathbf{R}^{n-1}$ with boundary given as the union of products of lower dimensional $C(j)$'s.

LEMMA 6.2.1. *Let $\sigma \in S_n$ be the cycle $(1,2,3,\ldots,n)$, then the n vectors $e, \sigma(e), \ldots, \sigma^{n-1}(e)$ are linearly independent in A^{n-1} and consequently span an embedded $n-1$ dimensional simplex there.*

PROOF. It suffices to show that the $n-1$ vectors

$$\sigma^i(e) - e = (\underbrace{n-i,\ldots,n-i}_{i \text{ times}}, \underbrace{-i,\ldots,-i}_{n-i \text{ times}})$$

are linearly independent for $1 \leqslant i \leqslant n-1$. But this is clear by looking at the last $n-1$ columns of the array. □

COROLLARY 6.2.1. *$C(n-1)$ is topologically a closed $n-1$ disk D^{n-1} with boundary S^{n-2}.*

PROOF. $C(n-1)$ is certainly closed and convex. It is also compact since it is contained in the cube $[0,n]^n$. The lemma above shows that it has a nonempty interior, so, by a standard result it is topologically a closed disk. □

Actually more is true. $C(n-1)$ is a polyhedron with faces determined as the convex hulls of subsets of the points $\{\sigma(1,\ldots,n) \mid \sigma \in S_n\}$. (This is a general property of the convex hulls of finite point sets.) We now determine these faces and show that they are closely connected with certain subgroups of $\mathcal{S}_n$.

LEMMA 6.2.2. *Let $H_r = \mathcal{S}_r \times \mathcal{S}_{n-r} \subset \mathcal{S}_n$, $1 \leqslant r \leqslant n-1$, be the subgroup preserving the first r and the last $n-r$ coordinates. Then the convex hull of the points $\sigma(e)$, $\sigma \in H_r$ is an $n-2$ dimensional face of $C(n-1)$ and, as a polyhedron, is isomorphic to $C(r-1) \times C(n-r-1)$.*

PROOF. Consider the map

$$p_r : \mathbf{R}^n \longrightarrow \mathbf{R}^+, \quad p_r(\boldsymbol{h}) = \sum_1^r h_j,$$

where h_j is the j-th coordinate of $\boldsymbol{h}$. Then for every point $\boldsymbol{h}$ of

$$C(n-1)\ p_r(\boldsymbol{h}) \geqslant \frac{r(r+1)}{2}.$$

Moreover, equality occurs if and only if $\boldsymbol{h}$ is contained in the convex hull generated by the points $\sigma(e)$, $\sigma \in H_r$. It follows that this polyhedron is contained in the topological boundary of $C(n-1)$. Finally, as the two subgroups $\mathcal{S}_r$ and $\mathcal{S}_{n-r}$ act independently and on disjoint sets of coordinates the remainder of the lemma is clear. □

Note that $e = (1, 2, 3, \ldots, n)$ is the intersection of $C(n-1)$ and the hyperplanes, $K_r = \{\boldsymbol{h} \mid p_r(\boldsymbol{h}) = r(r+1)/2\}$,

$$e = C(n-1) \cap K_1 \cap K_2 \cap \cdots \cap K_{n-1}$$

and, since faces of faces are faces, e is a vertex of $C(n-1)$. All the vertices of $C(n-1)$ are contained among the elements $\sigma(e)$, $\sigma \in \mathcal{S}_n$, since $C(n-1)$ is the convex hull spanned by the points $\sigma(e)$. But the symmetric group, $\mathcal{S}_n$, acts as a group of transformations on $C(n-1)$, taking faces to faces. It follows that the vertices of $C(n-1)$ are in 1–1 correspondence with the elements of $\mathcal{S}_n$ and are precisely the vectors $\sigma(e)$.

Similarly, for each r with $1 \leqslant r \leqslant n-1$ we have distinct faces of $C(n-1)$ corresponding to the cosets of $\mathcal{S}_r \times \mathcal{S}_{n-r}$ in $\mathcal{S}_n$. We now describe coset representatives for the cosets of $\mathcal{S}_r \times \mathcal{S}_{n-r} \subset \mathcal{S}_n$, which thus label the $(n-2)$ faces of $C(n-1)$ which we have found so far.

Let $(j_1, j_2, \ldots, j_r)$, $j_i \geqslant 1$, $\sum j_i = n$, be an ordered partition of n. Define

$$shuff(j_1, j_2, \ldots, j_r)$$

as the set of $\sigma \in \mathcal{S}_n$ so that $\sigma(i) < \sigma(j)$ whenever i and j belong to the same block in the partition, i.e. when there is a k so that

$$\sum_1^k j_s < i < j \leqslant \sum_1^{k+1} j_s, \quad 0 \leqslant k < r.$$

When $r = 2$ this corresponds to an ordinary shuffle of a deck of cards and likewise gives representatives for the cosets of $S_{j_1} \times S_{n-j_1}$ in S_n. For larger r it corresponds to breaking the deck into r pieces and then successively shuffling them together, and gives coset representatives for the cosets of $S_{j_1} \times \cdots \times S_{j_r}$ in S_n.

We note the straightforward but important

LEMMA 6.2.3. *Let $s \in shuff(j_1, j_2)$ and $s' \in shuff(j_1 + j_2, j_3, \ldots, j_r)$. Then the composite $s's \in shuff(j_1, j_2, j_3, \ldots, j_r)$ where $s \in S_{j_1+j_2}$ and $S_{j_1+j_2}$ is embedded in S_n with*

$$n = \sum_1^r j_s$$

as the subgroup fixing the last $n - (j_1 + j_2)$ points.

LEMMA 6.2.4. *The collection of all the $n-2$ dimensional faces of $C(n-1)$ consists of those elements enumerated above in 1–1 correspondence with the union of the $(r, n-r)$ shuffles, $1 \leqslant r \leqslant n-1$.*

PROOF. The proof is by induction. Note to begin with that the interiors of the $(n-2)$ dimensional faces in the lemma are disjoint since they lie in distinct hyperplanes. Now consider an $(n-3)$-face of one of these subcomplexes. By the inductive assumption it has the form $\sigma(C(l-1) \times C(r-l-1)) \times C(n-r-1)$ or $C(r-1) \times \sigma(C(s-1) \times C(n-r-s-1))$ since $\partial A \times B = (\partial A) \times B \sqcup A \times (\partial B)$.

Assume the face is of the first type. It can be uniquely written as the face of an appropriate shuffle of $C(l-1) \times C(n-l-1)$, and in the second case it is uniquely the face of an appropriate shuffle of $C(r+s-1) \times C(n-r-s-1)$. Thus, each $n-3$ face is incident to precisely two of the $n-2$ dimensional faces listed and it follows that the sum of these faces forms a closed cycle mod(2). But this implies that we have enumerated all the $n-2$ dimensional faces and completes the proof. □

COROLLARY 6.2.2. *The complete set of faces of $C(n-1)$ is indexed by ordered pairs consisting of first an ordered partition of n $p = (j_1, \ldots, j_w)$ $(\sum j_i = n)$ and a $(j_1, \ldots, j_w)$ shuffle s. Such a face has dimension $n - w$.*

EXAMPLE 6.2. $C(3)$ has as its faces 4 copies of $C(2) \times 1$, 4 copies of $1 \times C(2)$ and 6 copies of $C(1) \times C(1)$. It has 36 edges corresponding to 12 copies each of $1 \times 1 \times C(1)$, $1 \times C(1) \times 1$, and $C(1) \times 1 \times 1$. Finally, it has 24 vertices. It can be realized by taking the tetrahedron, T, and cutting out 6 small tetrahedra about the six vertices of T.

REMARK. The lowest dimensional faces of $C(n)$ which do not have a fixed coordinate, i.e. are not translates of a face corresponding to a partition with one or more 1's in it, such as $\mathcal{S}_{n-3} \times \mathcal{S}_1 \times \mathcal{S}_2$, correspond to

$$\begin{cases} \underbrace{\mathcal{S}_2 \times \cdots \times \mathcal{S}_2}_{n/2 \text{ times}} & \text{if } n \text{ is even} \\ \underbrace{\mathcal{S}_2 \times \cdots \times \mathcal{S}_2}_{[n/2] \text{ times}} \times \mathcal{S}_3 & \text{if } n \text{ is odd.} \end{cases}$$

Hence they have the form $I^{n/2}$ or $I^{[n/2]} \times C(2)$. This leads to "stabilization" results in constructions which use Zilchgons.

Let $b_n \in C(n-1)$ be the barycenter,

$$b_n = \underbrace{\left(\frac{n+1}{2}, \ldots, \frac{n+1}{2}\right)}_{n \text{ times}}.$$

Then, for $\boldsymbol{h} \in C(n-1)$, $\boldsymbol{h} \neq b_n$, there are unique points $\boldsymbol{v} \in \partial C(n-1)$, $t \in [0,1)$, so that $\boldsymbol{h} = tb_n + (1-t)\boldsymbol{v}$. Suppose that a map

$$\tilde{\phi} : \partial\big(C(n-1)\big) \longrightarrow E_{\mathbf{0},\mathbf{1}}^{\partial I^n}$$

is defined so that the image consists of linearly parameterized, piecewise linear paths. Then $\tilde{\phi}$ can be extended to $\phi : C(n-1) \longrightarrow E_{\mathbf{0},\mathbf{1}}^{I^n}$ by the rule

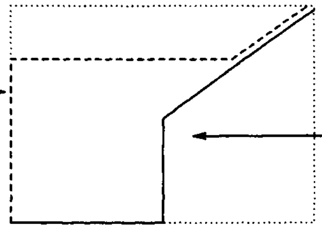
$$\begin{aligned} &\phi\big(tb_n + (1-t)\boldsymbol{v}\big)(\tau) \\ &\quad = \begin{cases} (1-t)\boldsymbol{v}(\frac{\tau}{1-t}) & \tau < (1-t)l(\phi(\boldsymbol{v})), \\ (\tau - (1-t)(l(\phi(\boldsymbol{v})) - 1))\mathbf{1} & (1-t)l(\phi(\boldsymbol{v})) \leqslant \tau \\ & \text{and } \tau \leqslant t + (1-t)l(\phi(\boldsymbol{v})), \\ \mathbf{1} & \tau \geqslant t + (1-t)l(\phi(\boldsymbol{v})). \end{cases} \end{aligned} \tag{6.4}$$

Note that $l(\phi((1-t)\boldsymbol{v} + tb_n) = t + (1-t)l(\phi(\boldsymbol{v}))$, and that the path is again linearly parameterized and piecewise linear. Indeed, it is the original path, but in the smaller cube, $[0,(1-t)]^n$, and then the diagonal path from the diagonal point $(1-t)^n$ to 1^n.

Now, let us suppose that $\phi_j : C(j-1) \longrightarrow E_{\mathbf{0},\mathbf{1}}^{I^j}$ is defined for all $j < n$. We define $\tilde{\phi}_n : \partial C(n-1) \longrightarrow E_{\mathbf{0},\mathbf{1}}^{\partial I^n}$ by

$$\tilde{\phi}_n |\ \sigma\big(C(j_1 - 1) \times \cdots \times C(j_r - 1)\big) \longrightarrow \sigma\left(\phi_{j_1} * \cdots * \phi_{j_r}\right) \tag{6.5}$$

where $*$ denotes juxtaposition of paths and $\sigma \in shuff(j_1, \ldots, j_r)$.



Sample paths in I^2

These two steps combine to define $\phi_n : C(n-1) \longrightarrow E_{0,1}^{I^n}$ for all $n \geqslant 1$ so that

(i) $eval(\phi_n) : (I \times C(n-1), \partial(I \times C(n-1))) \longrightarrow (I^n, \partial I^n)$ has degree one.

(ii) $\phi_n \mid \partial C(n-1)$ consists of two parts, the first, on the cells

$$shuff(1, n-1)C(n-2), \quad shuff(n-1, 1)C(n-2),$$

which corresponds to ∂I^n, and the second, on the

$$shuff(r, n-r)(C(r-1) \times C(n-r-1)), \quad 2 \leqslant r \leqslant n-2,$$

which corresponds to

$$\Delta(I^n) = \sum_{\sigma \in shuff(r,n-r)} \sigma I^r \times I^{n-r},$$

the usual chain approximation to the diagonal on I^n,

(iii) The paths in $\phi_n(C(n-1))$ are piecewise linear, and linearly parameterized, and have the property that over each linear segment there is a subset of $\mathcal{W} = \{1, 2, \ldots, n\}$ and the points of the segment have the form $(\varepsilon_1, \ldots, t_{w_1}, \ldots, t_{w_2}, \ldots)$. More precisely, the i^{th} coordinate is either 0 or 1 if $i \notin \mathcal{W}$, and is t if $i \in \mathcal{W}$.

This allows us to iterate the Ω construction, as promised to construct $\Omega^2 X$ when X is a simplicial complex with $sk_2(X)$ collapsed to a point.

REMARK. This was the original motivation of the second author when, in 1964, he first constructed the $C(n)$'s. When he told W. Browder about the construction, Browder suggested that it might be possible to modify it to study $\Omega^n \Sigma^n X$ since there are huge numbers of "cubes" in $J(\Sigma^{n-1} X)$, $n \geqslant 2$. (See the discussion in the next section.)

In order to push this suggestion through, the second author had to introduce degeneracies into the $C(n)$'s and construct systematic methods of reparameterizing paths to account for the effects of the *base point identifications* introduced in the James model.

In the writeup of these results in [24] only the construction of $\Omega^n\Sigma^n X$ was discussed however, and in the interim several students have written theses pointing out the connection with $\Omega^2 X$.

6.3. *The degeneracy maps for the Zilchgon models*

We now describe the degeneracy maps $d_i : C(n-1) \longrightarrow C(n-2)$. First, there are "degeneracy" maps for the symmetric groups, $d_i : \mathcal{S}_n \longrightarrow \mathcal{S}_{n-1}$, $1 \leqslant i \leqslant n$ defined by

$$d_i(\sigma)(j) = \begin{cases} \sigma(j) & \text{if } j < \sigma^{-1}(i),\ \sigma(j) < i, \\ \sigma(j+1) & \text{if } j \geqslant \sigma^{-1}(i),\ \sigma(j) < i, \\ \sigma(j) - 1 & \text{if } j < \sigma^{-1}(i),\ \sigma(j) > i, \\ \sigma(j+1) - 1 & \text{if } j \geqslant \sigma^{-1}(i),\ \sigma(j+1) > i. \end{cases} \tag{6.6}$$

If one writes σ as the array

$$\begin{pmatrix} 1 & 2 & \cdots & \sigma^{-1}(i) & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & i & \cdots & \sigma(n) \end{pmatrix}$$

then d_i deletes the $\sigma^{-1}(i)$ column and reindexes to get an element in $\mathcal{S}_{n-1}$.

These correspond to the maps

$$p_i : I^n \longrightarrow I^{n-1}, \quad p_i(t_1, \dots, t_n) = (t_1, \dots, \widehat{t_i}, t_{i+1}, \dots, t_n),$$

that deletes the i-th coordinate. The image of an edge path under p_i is an edge path in I^{n-1}, *at least as a point set*, though the parameterization is changed, since, when we come to what should have been movement along the i-th coordinate the path stays fixed in the image.

Note that if $\sigma \in shuff(j_1, \dots, j_r)$ and i belongs to the block j_k, then $d_i(\sigma) \in shuff(j_1, \dots, j_k - 1, \dots, j_r)$, where, if $j_k = 1$, we simply delete that block. It follows that if $\sigma_1(e), \dots, \sigma_r(e)$ are contained in a face, $\sigma(C(j_1 - 1) \times \cdots \times C(j_r - 1))$, of $C(n-1)$ then $d_i(\sigma_1(e)), \dots, d_i(\sigma_r(e))$ are contained in the face

$$d_i(\sigma)\big(C(j_1 - 1) \times \cdots \times C(j_k - 2) \times \cdots \times C(j_r - 1)\big).$$

Now, by mapping b_m's to $b_{m'}$'s and extending linearly, we have geometric maps

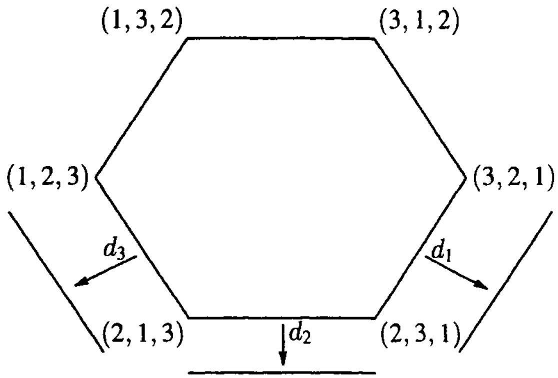
$$d_i : C(n-1) \longrightarrow C(n-2), \quad 1 \leqslant i \leqslant n,$$

which satisfy the usual condition for degeneracies:

$$d_i d_j = \begin{cases} d_j d_{i-1} & \text{if } i > j, \\ d_j d_i & \text{if } j \geqslant i. \end{cases}$$

When we compose with the ϕ_n, and use p_i, collapsing the i-th coordinate, as the corresponding degeneracy on I^n, we obtain that $\phi_{n-1} d_i(w)$ is a path which has as its image *the same point set* as $p_i\phi_n(w)$, but the parameterizations are different. However, that is easily handled since we have the following result.

LEMMA 6.3.1. *The space of nondecreasing maps of the unit interval onto itself is convex and so is the subspace of piecewise linear maps.*



(6.7)

The degeneracies for $C(2)$

6.4. *The Zilchgon models for iterated loop spaces of iterated suspensions*

To explain these models consider again the James model, $M(\Sigma X, 0)$,

$$M(\Sigma X,0)=\coprod_{k=1}^{\infty}(\Sigma X)^k/\sim.$$

Since $\sim$ collapses the fat wedge,

$$W_n(\Sigma X)=\{(y_1,\dots,y_n)\in(\Sigma X)^n \mid y_i=* \text{ for some } i,\ 1\leqslant i\leqslant n\},$$

onto $(\Sigma X)^{n-1}$ it follows that we have subspaces

$$M_n(\Sigma X,0)=\coprod_{k=1}^{n}(\Sigma X)^k/\sim$$

and

$$\Omega^2\Sigma^2X\simeq\Omega^M M(\Sigma X,0)=\lim_{n\to\infty}\big(\Omega M_n(\Sigma X,0)\big).$$

On the other hand, $(\Sigma X)^n = I^n\times X^n/\mathcal{R}$ where $\mathcal{R}$ is a relation on $\partial(I^n)\times X^n\cup I^n\times W_n(X)$ which has the property that $((0,\dots,0),(x_1,\dots,x_n))$ and $((1,\dots,1),(x_1,\dots,x_n))$ are both identified with $(*,\dots,*)$.

From this we get a map

$$C(n-1)\times X^n \longrightarrow \Omega^2\Sigma^2 X \tag{6.8}$$

by simply using the map $C(n-1)\longrightarrow E^{I^n}_{0,1}$ constructed in (6.4), (6.5). Inductively, we can assume that we have used this construction to build $\Omega^M M_s(\Sigma X,0)$ for $s<n$, and we can use Lemma 6.3.1 together with the map in (6.8) to obtain the following model for $\Omega^M M_n(\Sigma X,0)$:

$$\hat{J}_{2,n}(X)\simeq \Omega M_n(\Sigma X,0)=P\big(\hat{J}_{2,n-1}(X),f,C(n-1)\times X^n\big) \tag{6.9}$$

where $P(-,-,-)$ is the Prolongation functor introduced in Definition 5.4.1 with the obvious modification that we are identifying a subspace of $C(n-1)\times X^n$, $\partial(C(n-1))\times X^n\cup C(n-1)\times W_n(X)$, with a piece of $\hat{J}_{2,n-1}(X)$. The introduction here of models for the loop spaces $\Omega(M_s(\Sigma X,0))$ is similar to some of Husseini's ideas in [18].

EXAMPLE 6.3. $\hat{J}_{2,1}(X)=M(X,0)$, the James construction on X. Then $\hat{J}_{2,2}(X)$ is obtained by adjoining $I\times X^2$ where we have the identifications

$$\begin{aligned}(0,x_1,x_2) &\sim (x_1,x_2)\in \hat{J}_{2,1}(X),\\ (1,x_1,x_2) &\sim (x_2,x_1)\in \hat{J}_{2,1}(X),\\ (t,x_1,*) &\sim x_1,\\ (t,*,x_2) &\sim x_2.\end{aligned}$$

Thus we can think of $\hat{J}_{2,2}(X)$ as the free gadget which makes $M(X,0)$ homotopy commutative.

$\hat{J}_{2,3}(X)$ is obtained from $\hat{J}_{2,2}(X)$ by adjoining $C(2)\times X^3$ where we make the identifications

$$\begin{aligned}\big(\sigma(C(1)\times C(0)),(x_1,x_2,x_3)\big) &\sim \big\{C(1)\times\big(x_{\sigma^{-1}(1)},x_{\sigma^{-1}(2)}\big)\big\} * x_{\sigma^{-1}(3)},\\ \big(\sigma(C(0)\times C(1)),(x_1,x_2,x_3)\big) &\sim x_{\sigma^{-1}(1)} * \big\{C(1)\times\big(x_{\sigma^{-1}(2)},x_{\sigma^{-1}(3)}\big)\big\},\\ \big(v,(*,x_2,x_3)\big) &\sim \big\{d_1(v),(x_2,x_3)\big\}\in C(1)\times X^2,\end{aligned}$$

and similarly in the case when x_2 or x_3 is the basepoint $*$.

The general case should now be clear,

$$\Omega^2\Sigma^2X\simeq J_2(X)=\coprod_{k=1}^{\infty} C(k-1)\times X^k/\mathcal{R} \tag{6.10}$$

where $\mathcal{R}$ identifies points of $\partial(C(k-1))\times X^k$ with products

$$C(j_1-1)\times X^{j_1} * \cdots * C(j_r-1)\times X^{j_r}$$

where the coordinates are shuffled according to the shuffle associated with the face, and, on $W_k(X)$ makes identifications using the degeneracies on $C(k-1)$.

To go further, note that (6.10) allows us to write

$$J_2(X) = \lim_{n \to \infty} (J_{2,n}(X))$$

where

$$J_{2,n}(X) = \coprod_{k=1}^{n} C(k-1) \times X^k / \mathcal{R}^5,$$

and

$$J_{2,n}(\Sigma X) = J_{2,n-1}(\Sigma X) \cup C(k-1) \times I^k \times X^k / \mathcal{R}'. \tag{6.11}$$

Once more we can use prolongation to iteratively build models for $\Omega^M(J_{2,n}(\Sigma X))$. Here the piece that is added at stage k is the product $C(k-1) \times C(k-1) \times X^k$. However, when we make identifications they are a bit more complex than those at the previous level: on a face in $\partial(C(k-1)) \times C(k-1) \times X^k$, we act on the second $C(k-1)$ and the X^k by the shuffle associated with the face, however, on a face $C(k-1) \times \partial(C(k-1)) \times X^k$, we must use degeneracies on the first $C(k-1)$ to project it onto an appropriate product $C(j_1-1) \times \cdots \times C(j_r-1)$. Finally, on $C(k-1) \times C(k-1) \times W_k(X)$ we use the appropriate $d_s \times d_s$ on $C(k-1) \times C(k-1)$.

At this stage we have seen all the steps needed to define the general construction

$$J_n(X) = \coprod_{k=1}^{\infty} \underbrace{C(k-1) \times \cdots \times C(k-1)}_{(n-1) \text{ times}} \times X^k / \mathcal{R} \tag{6.12}$$

which gives a model for $\Omega^n \Sigma^n X$ for any connected CW complex X.

REMARK. The explicitness of this model allows us to make chain level calculations to study the homology of $\Omega^n \Sigma^n X$. In particular, it is not hard to see that at each step passing from $\Omega^{n-k} \Sigma^n X$ to $\Omega^{n-k+1} \Sigma^n X$ the *cotor*-spectral sequence of 5.13 collapses and we obtain an effective method for determining $H_*(\Omega^n \Sigma^n X; \mathbf{F})$ for any $n > 0$ and any connected CW complex X. Further discussion of the actual results will be given in 7.3.

[5] It should be noted that the decompositions of $J_2(X)$ via the $J_{2,n}(X)$ and the $\hat{J}_{2,n}$ are quite distinct.

7. The structure of iterated loop spaces

Given any map into an iterated loop space, $f : X \longrightarrow \Omega^n(X)$, it factors through an n-fold loop map in the following way:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & \Omega^n Y \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle \Omega^n Adj^n(f)} & \\ & \Omega^n \Sigma^n Y & \end{array}$$

where $i : X \longrightarrow \Omega^n \Sigma^n X$ is the usual inclusion:

$$i(x)(t_1, \dots, t_n) = \{t_1, \dots, t_n, x\} \in \Sigma^n X = I^n \times X/\sim .$$

Thus, the structure of the category of n-fold loop spaces and n-fold loop maps is closely reflected by the properties of the spaces $\Omega^n \Sigma^n X$, which play a role here analogous to the role of Eilenberg–MacLane spaces for ordinary spaces and maps.

It was conjectured in the 1950's that the homology of $\Omega^n \Sigma^n X$ should depend in a functorial way only on $H_*(X)$, and these homology classes will represent *homology operations* in the category. In this section we discuss the explicit construction of small models for the spaces $\Omega^n \Sigma^n X$ much as was done in Section 6, but here the models have better naturality properties which make aspects of the structure of $\Omega^n \Sigma^n X$ more transparent, in particular the proof of the conjecture above. They also allow a convenient passage to the limit, $Q(X) = \Omega^\infty \Sigma^\infty X$, under the natural inclusions

$$\begin{aligned} &i_n : \Omega^n \Sigma^n X \hookrightarrow \Omega^{n+1} \Sigma^{n+1} X, \\ &\big(i_n(f)(t_1, \dots, t_n, t_{n+1}) = \{t_1, f(t_2, \dots, t_{n+1})\}\big). \end{aligned}$$

An important feature of these models is that they permit the explicit description of $H_*(\Omega^k \Sigma^k X, \mathbf{F})$ as a functor of $H_*(X, \mathbf{F})$, where $\mathbf{F}$ is a field. This description is implicit in [24] but is carried out in detail in [14]. It turns out that if one considers the category of spaces which are k-fold loop spaces and maps which are k-fold loop maps, the $\mathbf{F}_p$-homology groups admit certain operations, some of which are stable and yield operations on infinite loop spaces (Dyer–Lashof operations) and some which are not (Browder operations), and the homology groups $H_*(\Omega^k \Sigma^k X; \mathbf{F}_p)$ can roughly be described as a free Hopf algebra on $H_*(X; \mathbf{F}_p)$ over an algebra involving these operations. The reader should see [14] for precise formulations and proofs of these results.

We have looked at Milgram's original Zilchgon model in Section 6.4. The models we will discuss now together with their various advantages are the May–Milgram configuration space model, Barratt–Eccles simplicial model for $Q(X)$, and J. Smith's unstable versions of the Barratt–Eccles construction.

7.1. *Boardman's little cubes*

In order to describe these models efficiently we will introduce some terminology.

DEFINITION 7.1.1. Let Γ^0 denote the category whose objects are the sets $\mathbf{n} = \{1, 2, \dots, n\}$ for $n = 0, 1, 2 \dots$, and where the morphisms from $\mathbf{m}$ to $\mathbf{n}$ are the injective maps.

For $n = 0$, this is understood to mean the empty set $\varnothing$, and it is understood that for every object $\mathbf{n}$ of Γ^0, there is a unique morphism from $\varnothing$ to $\mathbf{n}$ and that for every $n > 0$, the set of morphisms from $\mathbf{n}$ to $\varnothing$ is empty.

DEFINITION 7.1.2. An $\mathcal{O}$-space will be a contravariant functor from Γ^0 to the category of topological spaces.

This is the same as saying that an $\mathcal{O}$-space is a family of spaces X_n, $n \geqslant 0$, so that for each n, X_n is acted on by the symmetric group $\mathcal{S}_n$, and where for each n, we have maps $\delta_i : X_n \longrightarrow X_{n-1}$, $1 \leqslant i \leqslant n$, so that $\delta_i\delta_j = \delta_j\delta_{i+1}$ if $i \geqslant j$, and so that for any permutation $\sigma \in \mathcal{S}_{n+1}$, $\delta_i\sigma = \hat{\sigma}\delta_{\sigma^{-1}(i)}$, where $\hat{\sigma}$ is characterized by the equations in (6.6).

DEFINITION 7.1.3. If $\underline{C} = \{C_n\}_{n\geqslant 0}$ is an $\mathcal{O}$-space, and X is a based CW complex, we define $\underline{C}[X]$ to be

$$\coprod_{n\geqslant 0} C_n \times X^n / \simeq,$$

where $\simeq$ is the equivalence relation generated by relations of the form

$$\big(\sigma(e), x_1, \dots, x_n\big) \simeq \big(e, x_{\sigma(1)}, \dots, x_{\sigma(n)}\big)$$

and

$$(e, x_1, \dots, x_{i-1}, *, x_{i+1}, \dots, x_n) \simeq (\delta_i e, x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n).$$

Next we need morphisms of $\mathcal{O}$-spaces.

DEFINITION 7.1.4. A morphism of $\mathcal{O}$-spaces is a natural transformation of functors on Γ^{op}.

A morphism $f : \underline{C} \longrightarrow \underline{C}'$ induces a map $\underline{C}[X] \xrightarrow{f[X]} \underline{C}'[X]$.

DEFINITION 7.1.5. An $\mathcal{O}$-space $\underline{C} = \{C_n\}_{n\geqslant 1}$ is said to be free if each C_n is a free $\mathcal{S}_n$-space.

$\underline{C}[X]$ is also equipped with an increasing filtration $F_l\underline{C}[X]$, where

$$F_l\underline{C}[X] = \text{image}\Bigg(\coprod_{0\leqslant n\leqslant l} C_n \times X^n\Bigg). \tag{7.1}$$

REMARK. The definition of an $\mathcal{O}$-space is just a part of J.P. May's definition of an operad [22]; we retain only what is needed to make the construction $\underline{C}[X]$.

EXAMPLE 7.1. Here are some examples.

(A) $\underline{C} = \{C_n\}_{n\geqslant 0}$, $C_n = *$ for all n, where all permutations and all δ_i's are identity maps. This $\mathcal{O}$-space is not free. If X is a based CW complex, $\underline{C}[X] \cong SP^\infty(X)$, the free abelian monoid on X.

(B) $\underline{\mathcal{F}} = \{F_n\}_{n\geqslant 0}$, where F_n is the set of total orderings on the set $\mathbf{n} = \{1, 2, \ldots, n\}$. S_n acts on F_n in an evident way. $\delta_i : F_n \longrightarrow F_{n-1}$ is given by restricting an ordering on $\mathbf{n}$ to an ordering on $\{1, 2, \ldots, i-1, i+1, \ldots, n\}$ and identifying $\{1, 2, \ldots, i-1, i+1, \ldots, n\}$ with $\{1, 2, \ldots, n-1\}$ via the unique order preserving bijection. $\underline{\mathcal{F}}$ is a free $\mathcal{O}$-space. In this case it follows easily from the definitions that $\underline{\mathcal{F}}[X]$ is homeomorphic to the James construction $M(X, *)$ in Section 5.3.

(C) Fix $k \geqslant 1$ and let $\underline{C}(k) = \{C_n(k)\}_{n\geqslant 0}$ be defined as follows. $C_n(k)$ is the space of ordered n-tuples of distinct points in $\mathbf{R}^k$, i.e., $C_n(k) \subset (\mathbf{R}^k)^n$ is the set of n-tuples $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ with $\boldsymbol{x}_i \neq \boldsymbol{x}_j$ if $i \neq j$. S_n acts by permuting the vectors, and δ_i deletes the i-th vector. $\underline{C}(k)$ is a free $\mathcal{O}$-space. In this case it can be shown that $\underline{C}(k)[X]$ is naturally equivalent to $\Omega^k \Sigma^k X$ for connected, based CW complexes X.

(D) Fix $k \geqslant 1$ and $d \geqslant 1$. Let $\underline{C}^d(k) = \{C_n^d(k)\}_{n\geqslant 0}$ be defined as follows. $C_n^d(k)$ will be the space of ordered n-tuples of vectors in $\mathbf{R}^k$ so that no vector occurs more than d times in the n-tuple. If $d = 1$ we are in the situation of (C). If $d > 1$, this is no longer a free operad. It is not known what $\underline{C}^d(k)[X]$ is. The case $d = 2$ has been studied by Karageuezian [20].

We record a useful technical result concerning these constructions. Both results are proved in the context of operads in [22]; the proofs in our setting are identical, and we omit them.

PROPOSITION 7.1.1 ([22, p. 14]). *Let $\underline{C}$ be an $\mathcal{O}$-space. Then the subquotients*

$$F_l\underline{C}[X]/F_{l-1}\underline{C}[X]$$

are homeomorphic to the quotients $C_l \ltimes_{S_l} X^{\wedge(l)}$. (Recall that if X and Y are spaces, and $y \in Y$ then $X \ltimes Y$ denotes the "half smash product" $X \times Y/X \times y$. If X and Y are G-spaces, where G is a group, and G fixes y, then $X \ltimes_G Y$ denotes the orbit space of the diagonal action of G on $X \ltimes Y$.)

We then have

PROPOSITION 7.1.2 ([22, p. 22]). *Let $f : \underline{C} \longrightarrow \underline{C}'$ be a map of $\mathcal{O}$-spaces. Suppose that $f_n : C_n \longrightarrow C_n'$ is a homotopy equivalence for all $n \geqslant 0$, and that both $\underline{C}$ and $\underline{C}'$ are free $\mathcal{O}$-spaces. Then $f[X] : \underline{C}[X] \longrightarrow \underline{C}'[X]$ is a weak equivalence for all based connected CW complexes, X.*

We now wish to describe the relationship between the constructions $\underline{C}(k)[X]$ and the spaces $\Omega^k\Sigma^k X$. We first define a modified version of $\underline{C}(k)$, which we denote $\underline{\tilde{C}}(k)$, and call "Boardman's little cube" $\mathcal{O}$-space.

For any vector $v \in \mathbf{R}^k$ and positive real number R, let $Cu(v, R)$ denote the open k-cube centered at v, $\prod_{i=1}^{n}(v_i - R, v_i + R)$. We define $\tilde{C}_n(k)$ to be the space of ordered n-tuples $((v_1, \ldots, v_n), (R_1, \ldots, R_n))$, for which the cubes $Cu(v_i, R_i)$ are pairwise disjoint. Note that $\tilde{C}_n(k)$ is acted on freely by S_n, by permuting coordinates in both n-tuples.

There is an evident forgetful map $\phi_n : \tilde{C}_n(k) \longrightarrow C_n(k)$,

$$\phi_n\big((v_1, \ldots, v_n), (R_1, \ldots, R_n)\big) = (v_1, \ldots, v_n).$$

These maps ϕ_n assemble into a map $\Phi : \tilde{\underline{C}}(k) \longrightarrow \underline{C}(k)$. Further, ϕ_n admits a section $\sigma_n : C_n(k) \longrightarrow \tilde{C}_n(k)$, defined by

$$\sigma_n(v_1, \ldots, v_n) = \big((v_1, \ldots, v_n), (R, \ldots, R)\big)$$

where $R = R(v_1, \ldots, v_n)$ is the maximal number for which the open cubes $Cu(v_i, R)$ are pairwise disjoint. (Note that R is a real valued function on $C_n(k)$.)

LEMMA 7.1.1. *The map ϕ_n is a homotopy equivalence for all n.*

PROOF. Since $\phi_n \sigma_n = id$, it will suffice to produce a homotopy from the identity map on $\tilde{C}_n(k)$ to $\sigma_n \phi_n$. We proceed as follows. For $(x, y, t) \in \mathbf{R}^3$, define

$$\begin{cases} \lambda(x, y, t) = x & \text{if } x \leqslant y, \\ \lambda(x, y, t) = (1 - t)x + ty & \text{if } x \geqslant y, \end{cases}$$

and similarly

$$\begin{cases} \mu(x, y, t) = x & \text{if } x \geqslant y, \\ \mu(x, y, t) = (1 - t)x + ty & \text{if } x \leqslant y. \end{cases}$$

The homotopy is now defined by the following formulae:

$$\begin{cases} h((v_1, \ldots, v_n), & (R_1, \ldots, R_n), t) = \\ & ((v_1, \ldots, v_n), (\lambda(R_1, R, 2t), \ldots, \lambda(R_n, R, 2t)) \\ & \text{for } 0 \leqslant t \leqslant \frac{1}{2}, \\ h((v_1, \ldots, v_n), & (R_1, \ldots, R_n), t) = \\ & ((v_1, \ldots, v_n), (\mu(R_1, R, 2t - 1), \ldots, \mu(R_n, R, 2t - 1)) \\ & \text{for } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

□

Since each ϕ_n is a homotopy equivalence we can now record the following consequence of Proposition 7.1.2.

PROPOSITION 7.1.3. *Φ induces a homotopy equivalence $\Phi[X] : \tilde{\underline{C}}(k)[X] \longrightarrow \underline{C}(k)[X]$ for all connected, based CW complexes X.*

7.2. *The May–Milgram configuration space models for $\Omega^n \Sigma^n X$*

It is on the model $\tilde{\underline{C}}(k)[X]$ that one can define a map to $\Omega^k \Sigma^k X$. The construction goes as follows. First, for any cube $Cu(v, R_1, \ldots, R_k)$, we have a canonical identification

$$\overline{Cu(v, R_1, \ldots, R_k)} \xrightarrow{\lambda(v, R_1, \ldots, R_k)} [0,1]^k,$$

which is given by

$$(x_1, \ldots, x_k) \longrightarrow \left(\frac{x_1}{2R_1} + \frac{1}{2} - \frac{v_1}{2R_1}, \frac{x_2}{2R_2} + \frac{1}{2} - \frac{v_2}{2R_2}, \ldots, \frac{x_k}{2R_k} + \frac{1}{2} - \frac{v_k}{2R_k}\right).$$

Also, we have an identification $[0,1]^k \times X / \partial([0,1]^k) \times X \cup [0,1]^k \times * \cong \Sigma^k X$. For any

$$\big(((v_1, \ldots, v_n), (R_1, \ldots, R_n)), x_1, \ldots, x_n\big) \in \tilde{C}_n(k) \times X^n,$$

we define a map θ_n,

$$\theta_n\big((v_1, \ldots, v_n), (R_1, \ldots, R_n), (x_1, \ldots, x_n)\big) : \mathbf{R}^k \longrightarrow \Sigma^k X$$

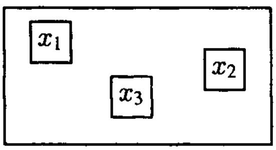
by letting

$$\theta_n \equiv * \quad \text{on } \mathbf{R}^k - \bigcup_{i=0}^{n} \overline{Cu(v_i, R_i)},$$

and on $Cu(v_i, R_i)$, we set θ_n equal to the composite

$$Cu(v_i, R_i) \xrightarrow{\lambda(v_i, R_i)} [0,1]^k \xrightarrow{id \times c_x} [0,1]^k \times X \longrightarrow \Sigma^k X$$

where c_x is the constant map with value x. This is best explained by the following picture



(7.2)

Note that since θ_n takes the value $*$ on the complement of a sufficiently large ball, θ_n extends to a map from the one point compactification of $\mathbf{R}^k$, S^k, to $\Sigma^k X$. Further, since, in this extension, ∞ is sent to $*$, we actually have an element in $\Omega^k \Sigma^k X$.

It is not hard to check that this procedure gives a map

$$\tilde{C}_n(k) \times X^n \longrightarrow \Omega^k \Sigma^k X.$$

It is also not hard to check that the θ_n's respect the equivalence relation and we obtain a map $\Theta : \tilde{\underline{\mathcal{C}}}(k)[X] \longrightarrow \Omega^k \Sigma^k X$.

THEOREM 7.2.1. *For connected X, the map Θ is a homotopy equivalence.*

PROOF. The proof of this result is too long and technical to present in its entirety here. We will, however, give a brief outline.

Sketch proof of the homotopy equivalence $\Theta : \tilde{\underline{\mathcal{C}}}(k)[X] \longrightarrow \Omega^n \Sigma^n X$
The first observation is that we have a map π of $\mathcal{O}$-spaces from $\underline{\mathcal{C}}(1) \longrightarrow \underline{\mathcal{F}}$, where $\underline{\mathcal{F}}$ is the $\mathcal{O}$-space of 7.1(B). On $C_n(1)$, it is given by the observation that an $(n+1)$-tuple of distinct points in $\mathbf{R}^1$ determines an ordering on that set of points, and hence on $\{0, \dots, n\}$. This correspondence gives a map $\pi_n : C_n(1) \longrightarrow F_n$, and it is easy to see that the π_n's give a map of $\mathcal{O}$-spaces. Further, one checks that the inverse image of the standard ordering on $\{1, \dots, n\}$ is homeomorphic to $\mathbf{R} \times (0,1)^{n-1}$, via the map $(r_1, r_2, \dots, r_n) \longrightarrow (r_1, r_1 + r_2, \dots, r_1 + \dots + r_n)$. Since this inverse image is contractible, so is the inverse image of any other ordering, and we conclude that π_n is a homotopy equivalence. It now follows from Proposition 7.1.2 that $\pi[X] : \underline{\mathcal{C}}(1)[X] \longrightarrow \underline{\mathcal{F}}[X]$ is a homotopy equivalence. Since we have already observed that $\underline{\mathcal{F}}[X]$ is homeomorphic to the James construction, we conclude that $\underline{\mathcal{C}}(1)[X]$ is homotopy equivalent to $\Omega\Sigma X$, and it isn't hard to check that the diagram

$$\begin{array}{ccc} \underline{\mathcal{C}}(1)[X] & \xrightarrow{\quad \Theta \quad} & \Omega\Sigma X \\ & {\scriptstyle\pi}\searrow \qquad \nearrow & \\ & \underline{\mathcal{F}}[X] \cong M(X,*) & \end{array}$$

commutes up to homotopy, where the right hand diagonal map is the James map. The result for $k = 1$ thus follows from James' theorem.

The idea of the rest of the proof is to use induction on k. We have the loop-path fibration from Section 2.2. Furthermore, the existence of a fibration or quasifibration with contractible total space, base space X, and fibre Y shows that $Y \simeq \Omega X$. If we apply this to the space $\Omega^k \Sigma^{k+1} X \cong \Omega^k \Sigma^k (\Sigma X)$, we obtain a fibration sequence

$$\begin{array}{ccc} \Omega^{k+1}\Sigma^{k+1}X & \longrightarrow & E(\Omega^k \Sigma^k (\Sigma X)) \\ & & \downarrow \\ & & \Omega^k \Sigma^k (\Sigma X) \end{array} \tag{7.3}$$

Suppose we have already proved the desired result for k, and all spaces X, and wish to prove it for $k+1$. If we could construct a space $\underline{\mathcal{E}}(k)[X]$, which is contractible, and so that we have a fibration sequence

$$\begin{array}{ccc} \underline{C}(k+1)[X] & \longrightarrow & \underline{\mathcal{E}}(k+1)[X] \\ & & \downarrow \\ & & \underline{C}(k)(\Sigma X) \end{array} \tag{7.4}$$

which maps to the fibration sequence (7.3), with the map on base spaces being $\theta(k)$ and the map on fibres being $\theta(k+1)$, the result would be proved for $k+1$, via the long exact sequence of a fibration, and the induction could proceed. It is not possible to construct a fibration as in (7.4), but it is possible to construct a quasifibration with the desired maps on base spaces and fibres. This suffices.

We conclude our outline by describing $\underline{\mathcal{E}}(k)[X]$. To make this definition, it is best to make a more general construction $\underline{\mathcal{E}}'(k)[X,A]$, where $A \subseteq X$ is a based subcomplex of X. $\underline{\mathcal{E}}'[X,A]$ will be defined as a subspace of $\underline{C}(k)[X]$. First, let $\pi : \mathbf{R}^k \longrightarrow \mathbf{R}^{k-1}$ denote projection on the first $k-1$ coordinates. For any point $(r_1,\dots,r_k) \in \mathbf{R}^k$, let $r^+(v)$ denote the ray $\{(r_1,\dots,r_{k-1},r_k+t) \mid t \geqslant 0\}$. For any n, let $Z_n(k) \subseteq C_n(k) \times X^n$ be the subspace of points $(v_0,\dots,v_n,x_0,\dots,x_n)$ so that, if $x_i \notin A$, then $v_j \notin r^+(v_i)$ for all $j \neq i$. $Z_n(k)$ is a closed subspace of $C_n(k) \times X^n$, and we define $\underline{\mathcal{E}}'(k)[X,A]$ to be the identification space obtained by restricting the equivalence relation defining $\underline{\mathcal{E}}(k)[X]$ to $\coprod_{n\geqslant 0} Z_n(k)$.

We will define a map

$$p : \underline{\mathcal{E}}'(k+1)[X,A] \longrightarrow \underline{C}(k)[X/A].$$

To do this, note first that $Z_n(k)$ is the union of a family of closed subsets $Z_n^S(k)$, parameterized by subsets $S \subseteq \{1,2,\dots,n\}$, where

$$Z_n^S(k) = \{(v_1,\dots,v_n),(x_1,\dots,x_n) \mid x_i \in A \text{ for } i \notin S \text{ and } v_i \notin r^+(v_j) \text{ for any } j \in S,\, i \neq j\}.$$

p is now defined as follows. For a fixed $n \geqslant 0$ and $S \subseteq \{1,2,\dots,n\}$, consider a point $(v_1,\dots,v_n,x_1,\dots,x_n)$ in $Z_n^S(k+1) \subseteq Z_n(k+1)$. We define $p|Z_n^S(k+1)$ by setting $p(v_1,\dots,v_n,x_1,\dots,x_n)$ equal to $(\pi(v_S),x_S)$ where v_S is the $\#(S)$-tuple consisting of the v_j's, $j \in S$, in increasing order, and where x_S is the $\#(S)$-tuple consisting of the x_j's, $j \in S$, also in increasing order. The fact that $\pi(v_S) \in C_{\#(S)-1}(k)$ follows from the definition of $Z_n^S(k+1)$. One now checks that the definition of P on the various $Z_n^S(k+1)$'s fit together to give a map $Z_n(k+1) \longrightarrow \underline{C}(k)[X/A]$, and that these maps respect the equivalence relation defining $\underline{\mathcal{E}}'(k+1)[X,A]$, so we obtain a map $\underline{\mathcal{E}}'(k+1)[X,A] \longrightarrow \underline{C}(k)[X/A]$. It is now possible to show that when applied to the pair (CX,X), $\underline{\mathcal{E}}(k+1)[CX,X]$ is contractible, and p is a quasifibration,

$$p : \underline{\mathcal{E}}(k+1)[cX,X] \longrightarrow \underline{C}(k)[\Sigma X].$$

Further, it also isn't hard to check that $p^{-1}(*)$ is equal to the subspace $\underline{C}(k+1)[X] \subseteq \underline{\mathcal{E}}(k+1)[CX,X]$. We set $\underline{\mathcal{E}}(k)[X] = \underline{\mathcal{E}}'(k)[CX,X]$, and obtain the desired quasifibrations. To get the map to the fibration sequence (7.3), one replaces $\underline{\mathcal{E}}(k+1)[CX,X]$

by a homotopy equivalent version $\underline{\tilde{\mathcal{E}}}(k+1)[CX, X]$, by analogy with the construction $\underline{\tilde{\mathcal{C}}}(k+1)[X]$. □

7.3. *The homology of* $\Omega^n \Sigma^n X$

We now wish to discuss how these constructions can be used to obtain homological calculations. Originally, as was remarked in (6.12), the Zilchgon model was used in [24] to show that at each level $m < n$ and for any field $\mathbf{F}$ the *Cotor*-spectral sequence with E_2-term

$$Ext_{H^*(\Omega^m \Sigma^n X;\mathbf{F})}(\mathbf{F}, \mathbf{F})$$

which converges to $H_*(\Omega^{m+1}\Sigma^n X; \mathbf{F})$ collapses for any connected CW complex X. Furthermore, it was shown there that $H_*(\Omega^{m+1}\Sigma^n X; \mathbf{F})$ is a primitively generated Hopf algebra as long as $m + 1 < n$.

This makes the computation effective since we can start with

$$H_*(\Omega\Sigma^n X; \mathbf{F}) = T(H_*(\Sigma^{n-1}X; \mathbf{F})),$$

the primitively generated Hopf algebra. Here, using the Poincaré–Birkhoff–Witt theorem, one finds that $H^*(\Omega\Sigma^n X; \mathbf{F})$ is a tensor product of exterior algebras on (explicit) odd dimensional generators and $C(\mathbf{F})$-truncated algebras on (explicit) even dimensional generators. (Here $C(\mathbf{F})$-truncated means the free polynomial algebra on even dimensional generators b_i, subject only to the relation $b_i^p = 0$ where p is the characteristic of $\mathbf{F}$.)

Then, since

$$Ext_{A\otimes B}(\mathbf{F}, \mathbf{F}) = Ext_A(\mathbf{F}, \mathbf{F}) \otimes Ext_B(\mathbf{F}, \mathbf{F})$$

we are reduced to considering Ext for an exterior algebra $E(e_{2n+1})$ – which is $\mathbf{F}[b_{2n}]$ – and for a $C(\mathbf{F})$-truncated polynomial algebra, $\mathbf{F}[b_{2n+2}]/R$ – where it is $E(e_{2n+1})$ if R is empty and $E(e_{2n+1}) \otimes \mathbf{F}[b_{2p(n+1)-2}]$ otherwise. Since these are primitively generated if $n > 1$, the dual of $\mathbf{F}[b_{2n}]$ is a tensor product of $C(\mathbf{F})$-truncated algebras and one can repeat the calculation to obtain the homology of each successive stage.

REMARK. A special case is when $\mathbf{F}$ has characteristic zero. Then, for each n there is the natural inclusion $i : \Sigma^n X \longrightarrow SP^\infty(\Sigma^n X)$. Passing to loop spaces and noting that $\Omega^n SP^\infty(\Sigma^n X) \simeq SP^\infty(X)$ by the Dold–Thom theorem, in the limit we have a map $i_\infty : Q(X) \longrightarrow SP^\infty(X)$. Then, from the discussion above it is direct to see that

$$i_{\infty *} : H^*(SP^\infty(X); \mathbf{F}) \longrightarrow H^*(Q(X); \mathbf{F})$$

is an isomorphism of rings for X a connected CW complex.

There are, of course many other paths to these results. But having the homology is not quite the same thing as understanding what it means.

To this end, initially J. Moore, then W. Browder, Araki and Kudo and finally Dyer and Lashof, [26], [10], [4], [15], constructed families of homology operations in $\Omega^n X$, $Q(X)$, and Fred Cohen showed, using the results of [24], that these operations together with loop sum, completely describe the homology of $\Omega^n \Sigma^n X$ for X a connected CW complex.

From another point of view V. Snaith proved that stably, we obtain a splitting

$$\Sigma^\infty \underline{C}(k)[X] \simeq \Sigma^\infty \bigvee_{k=1}^{\infty} C_l(k) \ltimes_{S_k} X^{\wedge n}$$

where Σ^∞ denotes "suspension spectrum"; see Section 8 for the definition of this concept. This is a direct consequence of the following result.

THEOREM 7.3.1. *There is a homotopy equivalence*

$$Q(\underline{C}(k)[X]) \simeq \prod_{k=1}^{\infty} Q(C_l(k) \ltimes_{S_k} X^{\wedge n}).$$

(See, e.g., [8] for details of a very slick proof due to F. Cohen.)

COROLLARY 7.3.1.

$$H^*(\Omega^n \Sigma^n X; \mathbf{A}) = \bigoplus_{k=1}^{\infty} H^*(C_l(k) \ltimes_{S_k} X^{\wedge n}; \mathbf{A})$$

for arbitrary untwisted coefficients $\mathbf{A}$.

Using Snaith splitting and the calculations above one can easily obtain the homology of the spaces $C_l(k) \ltimes_{S_k} X^{\wedge n}$ for any connected CW complex X and arbitrary k, l. This has had very important applications recently in many areas of mathematics. For example, in [9], it is the crucial input needed in the proof of the Atiyah–Jones conjecture.

7.4. *Barratt–Eccles simplicial model and J. Smith's unstable version*

The first two constructions we have exhibited work in the category of topological spaces. This has many advantages, for instance that the relationship of the combinatorial constructions with the iterated loop spaces is very explicit. It is also possible, as shown by Barratt and Eccles, to make the constructions for $k = \infty$ entirely inside the category of simplicial sets. Their construction has three main advantages. One is that the proofs become simpler. For instance, the analogue of the map p which could only be shown to be a quasifibration in the configuration space model is a surjective homomorphism of simplicial groups in this context, and hence automatically a Kan fibration. A second advantage is that in the Barratt–Eccles context, there is a natural extension of the result which applies to nonconnected simplicial sets. The third is that the loop sum operation arises as the multiplication operation in a simplicial group, hence is strictly associative. This is not the case for $\underline{C}[X]$.

J. Smith in his thesis, [30], constructed simplicial versions of the finite stage constructions $\underline{C}(k)[X]$. We will examine these at the end of this section.

The Barratt–Eccles model begins by constructing a simplicial version of the $\mathcal{O}$-space $\underline{C}(\infty)$.

DEFINITION 7.4.1. An $\mathcal{O}$-simplicial set is a contravariant functor from the category Γ^0 of Definition 7.1.1 to the category of simplicial sets, which takes $\varnothing$ to the one point simplicial set $*$.

An example is the $\mathcal{O}$-space $\underline{\mathcal{F}}$ of 1.7(B), in which F_n can be viewed equally well as a discrete topological space and as a discrete simplicial set. We also note that we have a functor e from the category of sets to the category of simplicial sets, given by $e(X)_n = X^{n+1}$, where $d_i : X^{n+1} \longrightarrow X^n$ deletes the i-th coordinate for $0 \leqslant i \leqslant n$, and where s_i repeats the i-th coordinate. It is readily checked that $e(X)$ is always a contractible simplicial set. We now consider the $\mathcal{O}$-simplicial set $\underline{B}$ defined as the composite

$$\Gamma^{op} \xrightarrow{\underline{\mathcal{F}}} \text{Sets} \xrightarrow{e} \text{Simplicial sets}. \tag{7.5}$$

For any $\mathcal{O}$-simplicial set $\underline{C}$, and based simplicial set, X, we define the simplicial set $\underline{C}[X]$ to be

$$\coprod_{n \geqslant 1} C_n \times X^n_{\cdot} / \cong,$$

where $\cong$ is the equivalence relation generated by relations of the following two forms

(a) $(c, x_1, \ldots, x_n) \cong (\sigma c, x_{\sigma(1)}, \ldots, x_{\sigma(n)})$, where $\sigma \in S_n$.

(b) $(c, x_1, \ldots, x_{i-1}, *, x_{i+1}, \ldots, x_n) \cong (\delta^i c, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

Of course, these relations are precisely analogous to those used in defining the topological version.

One of the advantages of the construction is made apparent by the following proposition.

PROPOSITION 7.4.1 ([6]). *For any based, simplicial set $X_{\cdot}$, $\underline{B}[X]$ is a free simplicial monoid, in such a way that the natural map $\underline{\mathcal{F}}[X] \longrightarrow \underline{B}[X]$ is a homomorphism of monoids, where $\underline{\mathcal{F}}[X_{\cdot}]$ is identified with the free monoid on X.*

PROOF. We first observe that we have maps $\underline{\mathcal{F}}_m \times \underline{\mathcal{F}}_n \longrightarrow \underline{\mathcal{F}}_{m+n}$, by assigning to a pair of orderings $(<_m, <_n)$ on $\{1, \ldots, n\}$ and $\{1, \ldots, m\}$, respectively, the ordering on $\{1, \ldots, m+n\}$ which we obtain by identifying $\{1, \ldots, m\} \cup \{1, \ldots, n\}$ with $\{1, \ldots, m+n\}$, where $\{1, \ldots, m\}$ is sent into $\{1, \ldots, m+n\}$ by adding m to each element of $\{1, \ldots, n\}$. Since e preserves products, we get maps $e(\{1, \ldots, m\}) \times e(\{1, \ldots, n\}) \longrightarrow e(\{1, 2, \ldots, m+n\})$. This gives a family of maps

$$B_m \times X^m \times B_n \times X^n \longrightarrow B_{m+n} \times X^{m+n},$$

and one checks that these maps respect the equivalence relations involved, to yield the required multiplication map. It is easy to check associativity, and the basepoint acts as an identity element. It is also easy to check freeness. □

This functor from simplicial sets to simplicial monoids is referred to by Barratt and Eccles as $\Gamma^+(X)$. They also compose Γ^+ with the group completion functor from simplicial monoids to simplicial groups, and call the result $\Gamma(X)$. $\Gamma(X)$ is a free simplicial group. Their main theorem now reads as follows.

THEOREM 7.4.1 ([6]). (a) *For any connected, based simplicial set $X.$, the natural inclusion $\Gamma^+(X) \longrightarrow \Gamma(X.)$ is a weak equivalence of simplicial sets.*

(b) *For any simplicial set $X.$, $|\Gamma(X.)|$ has the homotopy type of $Q(|X.|)$.*

PROOF. Part (a) is a standard fact about group completions of simplicial monoids. See [13] for details. It is essential here that $\Gamma^+(X)$ be a levelwise free simplicial monoid. To prove 7.4.1(b), one first proves that if $A. \hookrightarrow X.$ is an inclusion of simplicial sets, then the natural map $\Gamma(A.) \longrightarrow \mathrm{Ker}(\Gamma(X.) \longrightarrow \Gamma(X./A.)$, is a homotopy equivalence. This is proved in two steps. The first is to observe that Γ carries disjoint unions of based discrete simplicial sets to products, in the sense that the natural homomorphism $\Gamma(X \vee Y) \longrightarrow \Gamma(X) \times \Gamma(Y)$ is a weak equivalence of simplicial sets for all based sets X and Y. (Note that this is a special case of the required result, since it shows that $Ker(\Gamma(X \vee Y) \longrightarrow \Gamma(X))$ has the homotopy type of $\Gamma(Y)$). One first proves the analogous result for the monoid valued construction Γ^+, and concludes the result for Γ via a general comparison theorem for the homology of a simplicial monoid with that of its group completion. The second step is to prove that this special case suffices. Specifically, let T be any functor from the category of based sets to simplicial groups, and let T^s be the functor from simplicial sets to simplicial groups obtained by applying T levelwise and taking diagonal simplicial groups. Then Barratt and Eccles prove that if the natural map $T(X \vee Y) \longrightarrow T(X) \times T(Y)$ is a homotopy equivalence for all X and Y, then for all pairs of simplicial sets $(X., A.)$ the natural homomorphism $T^s(A.) \longrightarrow Ker(T^s(X.) \longrightarrow T^s(X./A.))$ is a weak equivalence of simplicial sets. Since Γ^+ is of the form T^s, this gives the result.

Since surjective homomorphisms of simplicial groups are Kan fibrations, applying the above discussion to the inclusion $X \longrightarrow CX$ shows that $|\Gamma(X)| \simeq \Omega|\Gamma\Sigma X|$, since $CX/X \simeq \Sigma X$. Iteratively, $|\Gamma(X.)| \simeq \Omega^k|\Gamma\Sigma^k X|$ for all k. On the other hand, if a simplicial set is l-connected, it is easy to check that the inclusion $X. \longrightarrow \Gamma^+(X.)$ is $(2l-1)$-connected, consequently, the map $\Omega^l|X.| \longrightarrow \Omega^l|\Gamma^+(X.)|$ is $(l-1)$-connected. Therefore the inclusion $|\Gamma^k\Sigma^k X| \longrightarrow \Omega^k|\Gamma\Sigma^k X|$ is $(k-1)$ connected. Thus, there is a map

$$\Omega^k|\Sigma^k X| \xrightarrow{\beta} \Omega^k|\Gamma(\Sigma^k X)| \xrightarrow{\theta} |\Gamma(X.)|,$$

where θ is a homotopy inverse to the inclusion $|\Gamma X.| \longrightarrow \Omega^k|\Gamma\Sigma^k X|$, where β is $(k-1)$ connected, hence $\theta \circ \beta$ is $(k-1)$ connected. Letting $k \longrightarrow \infty$ shows that $|\Gamma(X.)| \simeq Q(|X.|)$. □

This then gives a simplicial construction when $k = \infty$. For finite k, we have J. Smith's models [30]. Smith produces simplicial submonoids $\Gamma^{(n)+}(X.) \subseteq \Gamma^+(X.) \subseteq \Gamma(X.)$, whose realizations both give $\Omega^n\Sigma^n(|X.|)$ when $X.$ is connected and so that $|\Gamma^{(n)}(X.)) \cong \Omega^n\Sigma^n|X.|$ for arbitrary $X.$. First, we examine $\Gamma^{(n)+}(X.)$. $\Gamma^{(n)+}(X.)$ is constructed as $\underline{\mathcal{B}}^{(n)}[X.]$, where $\underline{\mathcal{B}}^{(n)}$ is a certain sub-$\mathcal{O}$-simplicial set of $\underline{\mathcal{B}}$ above, which we now describe. Let $\mathcal{B} = \{B_l\}_{l\geqslant 0}$, and consider the simplicial set B_l. Its k-simplices are $(k+1)$-tuples of orderings on $\{1,\dots,l\}$. For any pair (i,j), with $1 \leqslant i,j \leqslant l$, we have the restriction map from the set of orderings on $\{1,\dots,l\}$ to the set of orderings on $\{i,j\}$, which we identify with $\{1,\dots,2\}$ via $i \longrightarrow 1, j \longrightarrow 2$. This yields a simplicial map $\phi_{ij} : B_l \longrightarrow B_2$. Now, B_2 can be filtered by skeleta. It turns out that $|sk_nB_2| \cong S^n$, and the $\mathcal{S}_2$-action is identified with the antipodal action on S^n. Now define $B_l^{(n)}$ to be

$$\bigcap_{1\leqslant i<j\leqslant l} \phi_{ij}^{-1}(sk_nB_2).$$

$\underline{\mathcal{B}}^{(n)}$ becomes a sub $\mathcal{O}$-simplicial set, and $\underline{\mathcal{B}}^{(n)}[X]$ is a subsimplicial monoid of $\underline{\mathcal{B}}[X]$. $\Gamma^{(n)+}(X.)$ is defined to be $\underline{\mathcal{B}}^{(n)}[X.]$, and $\Gamma^{(n)}(X.)$ is defined to be its group completion.

THEOREM 7.4.2 (Smith). *If $X.$ is connected, then $|\Gamma^{(n)+}(X.)|$ and $|\Gamma^{(n)}(X.)|$ are homotopy equivalent to $\Omega^n\Sigma^n|X.|$. In general $|\Gamma^{(n)}(X.)| \cong \Omega^n\Sigma^n|X.|$.*

It is not known that the realizations of Smith's simplicial $\mathcal{O}$-sets are equivalent to the $\mathcal{O}$-spaces $\underline{\mathcal{C}}(n)$ although one suspects that they will be.

8. Spectra, infinite loop spaces, and category theoretic models

By the *homotopy category Ho* of based spaces, we mean the category whose objects are based spaces (X,x), and where the morphisms from (X,x) to (Y,y) are given by $[X,Y]_0$, the based homotopy classes of maps from X to Y. Similarly, one could define the *stable homotopy category Ho^s* as the category whose objects are based spaces (X,x), and where the morphisms from (X,x) to (Y,y) are given by

$$\{X,Y\} = \varinjlim_{n} [\Sigma^n X, \Sigma^n Y]_0.$$

It is proved in [3] that for any fixed X, the graded set $A_n^Y(X) = \{\Sigma^n X, Y\}$ for $n \geqslant 0$, and $A_n^Y(X) = \{X, \Sigma^{-n}Y\}$ for $n \leqslant 0$, is actually a graded abelian group, and yields a long exact sequence of graded abelian groups when applied to a cofibration sequence $X_1 \longrightarrow X_2 \longrightarrow X_2/X_1$. A_*^Y is referred to as a *generalized cohomology theory*, i.e. a graded abelian group valued functor which satisfies all the Eilenberg–Steenrod axioms except the dimension axiom which asserts that $A_n^X(S^0) = 0$ for $n \neq 0$, $A_0(S^0) = \mathbf{Z}$.

Generalized cohomology theories have turned out to be extremely useful in stable homotopy theory. K-theory and various bordism theories have been particularly so. These theories, and also singular cohomology theory are not, however of the form A_*^Y for any Y in the above Ho^s. One says they are not *representable* in Ho^s. It turns out, though, that by enlarging Ho^s a bit, one can make these theories representable. Moreover, by a theorem of E.H. Brown, [11], one can obtain a precise criterion for when a graded abelian group valued functor is representable.

To see how to construct this enlargement we consider the case of ordinary integral cohomology, $H^*(\ ;\mathbf{Z})$. In Ho the functor $X \mapsto H^n(X;\mathbf{Z})$ is representable. Let $K(\mathbf{Z},n)$ be an Eilenberg–MacLane space, i.e.

$$\pi_i\big(K(\mathbf{Z},n)\big) = \begin{cases} 0, & i \neq n, \\ \mathbf{Z}, & i = n. \end{cases}$$

Then there exists a class $\iota_n \in H^n(K(\mathbf{Z},n))$ so that the homomorphism $[X, K(\mathbf{Z},n)] \longrightarrow H^n(X;\mathbf{Z})$, given by $f \mapsto H^n(f)(\iota_n)$, is an isomorphism of functors.

Although H^n is defined on Ho^s, it is not the case that $\{X, K(\mathbf{Z},n)\} \cong H^n(X,\mathbf{Z})$, as one can easily check. The point is that, e.g., $H^{n+1}(\Sigma X;\mathbf{Z}) \ncong [\Sigma X, \Sigma K(\mathbf{Z},n)]$ in general, since $\Sigma K(\mathbf{Z},n) \ncong K(\mathbf{Z},n+1)$. What this suggests is that one wants to allow objects which, in a sense, contain all of the $K(\mathbf{Z},n)$'s at once. We therefore introduce the concepts of *prespectra* and *spectra*.

8.1. *Prespectra, spectra, triples, and a delooping functor*

DEFINITION 8.1.1. (a) A prespectrum $\mathbf{X}$ is a family of based spaces $\{X_i\}_{i\geqslant 0}$, together with "bonding maps" $\sigma_i : \Sigma X_i \longrightarrow X_{i+1}$.

(b) A morphism f from a prespectrum $\mathbf{X} = \{X_i\}$ to $\mathbf{Y} = \{Y_i\}$ is a family of based maps $f_i : X_i \longrightarrow Y_i$, so that $f_i\sigma_{i-1}^X = \sigma_{i-1}^Y f_{i-1}$ for all i.

(c) A prespectrum is an Ω-spectrum if, for each $i \geqslant 0$, the adjoint to σ_i,

$$Ad(\sigma_i) : X_i \longrightarrow \Omega X_{i+1}$$

is a homeomorphism.

(d) If X is any based space, the suspension prespectrum of X, $\Sigma^\infty X$, is given by $\{\Sigma^i X\}_{i\geqslant 0}$, with the evident bonding maps.

Note that given any prespectrum $\mathbf{X} = \{X_i\}_{i\geqslant 0}$, and based space Z, one can form a new prespectrum $\mathbf{X} \wedge Z = \{X_i \wedge Z\}_{i\geqslant 0}$ where the i-th bonding map is $\sigma_i \wedge id_Z$. In particular, we can let $Z = I^+$, the unit interval with a disjoint basepoint added, and declare that two maps $f, g : \mathbf{X} \longrightarrow \mathbf{Y}$ of prespectra are homotopic if there is a map $H : \mathbf{X} \wedge I^+ \longrightarrow \mathbf{Y}$ so that $H|\mathbf{X} \wedge 0^+ \longrightarrow \mathbf{Y} = f$ and $H|\mathbf{X} \wedge 1^+ \longrightarrow \mathbf{Y} = g$, where $\mathbf{X} \wedge 0^+$ and $\mathbf{X} \wedge 1^+$ are identified with $\mathbf{X}$ in the obvious way. By letting $\mathbf{X}$ be the suspension spectrum $\Sigma^\infty(S^n)$, we now obtain a definition of the homotopy groups $\pi_n(\mathbf{Y})$ for any prespectrum $\mathbf{Y}$, and of the homotopy classes of maps $[\mathbf{X}, \mathbf{Y}]$ for any pair of spectra.

EXAMPLE 8.1. Let $\mathbf{K}(\mathbf{Z},n)$ denote the prespectrum whose i-th entry is $K(\mathbf{Z}, n+i)$, and where $\sigma_i : \Sigma K(\mathbf{Z}, n+i) \longrightarrow K(\mathbf{Z}, n+i+1)$ is the map representing a generator in $H^{n+i+1}(\Sigma K(\mathbf{Z}, n+i);\mathbf{Z}) \cong \mathbf{Z}$. $\mathbf{K}(\mathbf{Z},n)$ can be taken to be an Ω-spectrum, and

$$\begin{cases} \pi_0(\mathbf{K}(\mathbf{Z},n)) = \mathbf{Z}, \\ \pi_i(\mathbf{K}(\mathbf{Z},n)) = 0 \quad \text{if } i \neq n. \end{cases}$$

More generally, $[\Sigma^\infty X, \mathbf{K}(\mathbf{Z}, n)] \cong H^n(X; \mathbf{Z})$, so, in this enlarged category $H^n(-, \mathbf{Z})$ is representable.

REMARK. The above mentioned definition of $[\mathbf{X}, \mathbf{Y}]$ does not, in fact, have good properties when $\mathbf{Y}$ is not an Ω-spectrum. The actual definition of $[\mathbf{X}, \mathbf{Y}]$ can be carried out as in [3] or by replacing $[\mathbf{X}, \mathbf{Y}]$ with $[\mathbf{X}, \omega(\mathbf{Y})]$, where ω is a functorial construction of an Ω-spectrum from $\mathbf{Y}$. We will not dwell on this point.

From the definitions, if $\mathbf{X}$ is an Ω-spectrum, it is clear that $\pi_i(\mathbf{X})$ is isomorphic to $\pi_i(X_0)$, the ordinary i-th homotopy group of the zeroth space of the spectrum $\mathbf{X}$. Consequently, the homology and other invariants of X_0 are of interest. Further, each X_i is an "i-fold delooping" of X_0 in the sense that $X_0 \simeq \Omega^i X_i$ via a composite of adjoints to the bonding maps, so X_0 is referred to as an *infinite loop space*. We also obtain maps

$$\theta_i : \Omega^i \Sigma^i(X_0) \xrightarrow{\Omega^i \sigma_i} \Omega^i X_i \simeq X_0.$$

Further, the θ_i's are compatible in the sense that $\theta_{i+1} \circ \eta_i = \theta_i$, where

$$\eta_i : \Omega^i \Sigma^i(X_0) \longrightarrow \Omega^{i+1} \Sigma^{i+1}(X_0)$$

is the inclusion, and so we obtain a map

$$Q(X_0) \xrightarrow{\nu} X_0.$$

It will turn out that this map ν will, in the case of connective Ω-spectra, determine the entire spectrum $\mathbf{X}$ up to homotopy equivalence. We will now discuss this fact.

DEFINITION 8.1.2. A triple on a category $\mathcal{C}$ is a functor $T : \mathcal{C} \longrightarrow \mathcal{C}$, together with natural transformations $\mu : T^2 \longrightarrow T$ and $\eta : Id \longrightarrow T$, so that the following diagrams commute for all $X \in \mathcal{C}$.

$$\begin{array}{ccccc} TX & \xrightarrow{T\eta(X)} & T^2X & \xleftarrow{\eta(TX)} & TX \\ & \searrow{\scriptstyle id} & \downarrow{\scriptstyle \mu(X)} & \swarrow{\scriptstyle id} & \\ & & TX & & \end{array}$$

$$\begin{array}{ccc} T^3X & \xrightarrow{\mu(TX)} & T^2X \\ \downarrow{\scriptstyle T\mu(X)} & & \downarrow{\scriptstyle \mu(X)} \\ T^2X & \xrightarrow{\mu(X)} & TX \end{array}$$

EXAMPLE 8.2. (a) Let C be the category of based sets, and let F be the functor from C to C which assigns to each based set, X, the free group on X with the basepoint set to the identity. (Note that a group is, via a forgetful functor, a set). F is a triple, since any set includes in the free group on that set as the words of length 1, and μ is obtained by evaluating a "word of words" as, simply, a word.

(b) Again, let C denote the category of based sets, and let F^{ab} denote the free abelian group functor, with basepoint set to 0. F^{ab} is also a triple on C.

(c) Let C be the category of based spaces, and let T be the functor $\Omega\Sigma$. There is the James inclusion $X \longrightarrow \Omega\Sigma X$, which is the natural transformation η. To construct μ, we first observe that there is a natural transformation $e : \Sigma\Omega \longrightarrow Id$, which is given by $e(t \wedge \phi) = \phi(t)$. $\mu(X)$ is now given by the composite

$$\Omega\Sigma\Omega\Sigma(X) \xrightarrow{\Omega(e(\Sigma X))} \Omega \circ Id \circ \Sigma(X) = \Omega\Sigma(X).$$

With this choice of μ and η, $\Omega\Sigma$ becomes a triple.

(d) Again, C will be the category of based spaces, and we let $T = \Omega^k\Sigma^k$. T becomes a triple by a construction identical to that in example (c). Even

$$Q = \varinjlim \Omega^k\Sigma^k$$

also becomes a triple on C.

DEFINITION 8.1.3. An algebra (X, ξ) over a triple T is an object $X \in C$ and a map $\xi : TX \longrightarrow X$ so that the diagrams below commute.

$$\begin{array}{ccc} X & \xrightarrow{\eta(X)} & TX \\ & {\scriptstyle id}\searrow & \downarrow{\scriptstyle \xi} \\ & & X \end{array} \qquad \begin{array}{ccc} TTX & \xrightarrow{\mu(X)} & TX \\ \downarrow{\scriptstyle T\xi} & & \downarrow{\scriptstyle \xi} \\ TX & \xrightarrow{\xi} & X \end{array}$$

Morphisms of T-algebras are defined as morphisms in C making the evident diagrams commute. Also, for any object X in C, (TX, μ) is an algebra over the triple T, to be thought of as the free T-algebra on X.

This is quite a useful notion. For instance, the reader should verify that if F is the triple in Example 8.2(a), an F-algebra structure on a based set X is the same thing as a group structure on X, where the basepoint is the identity. Similarly, if F^{ab} is as in 8.2(b), an F^{ab}-algebra structure on X is the same thing as an abelian group structure on X, where the basepoint is equal to zero. Also, any loop space has an algebra structure over the triple $\Omega\Sigma$ in 8.2(c), given by

$$\Omega\Sigma\Omega Z \xrightarrow{\Omega(e(Z))} \Omega Z,$$

and similarly, any k-fold loop space is an algebra over the triple $\Omega^k\Sigma^k$ of 8.2(d). In fact, by analogy with 8.2(a) and (b), we view $\Omega^k\Sigma^k$ as the "free k-fold loop space" functor, and it can be shown that $\Omega^k\Sigma^k$-algebra structures on a space X are the same thing (up to an obvious notion of homotopy equivalence) as k-fold deloopings Z of X, i.e. spaces, Z, together with an equivalence $X \xrightarrow{\simeq} \Omega^k Z$. This result is originally due to Beck [7]. We will indicate a proof of the $k = \infty$ version, i.e. we will show that a Q-algebra structure on a space X determines an infinite family of deloopings, with certain compatibility conditions, i.e. a spectrum with X as zeroth space.

We first discuss some generalities. Let T be a triple on a category C, and let (X, ξ) be a T-algebra. We define a simplicial object $T.(X, \xi)$ in C by setting $T_k(X, \xi) = T^{k+1}(X)$, and letting the face and degeneracies be given by the following formulae.

$$\begin{cases} d_i : T^{k+1}(X) \longrightarrow T^k(X) := T^i\mu(T^{k-i-1}X) & \text{for } 0 \leqslant i \leqslant k-1, \\ d_k : T^{k+1}(X) \longrightarrow T^k(X) := T^k(\xi), & \\ s_i : T^{k+1}(X) \longrightarrow T^{k+2}(X) := T^{i+1}(\eta(T^{k-i}X)) & \text{for } 0 \leqslant i \leqslant k. \end{cases} \tag{8.1}$$

One easily checks that $T.(X, \xi)$ is a simplicial object in the category of T-algebras. In fact $T.(X, \xi)$ should be viewed as a simplicial resolution of (X, ξ) by free T-algebras in C. Note that there is a map of simplicial objects $\alpha : T.(X, \xi) \longrightarrow X.$, where $X.$ is the constant simplicial object with value X, given in level k by the composite

$$\xi \circ T(\xi) \circ \cdots \circ T^{k-1}(\xi) \circ T^k(\xi).$$

PROPOSITION 8.1.1 ([22]). *Let C be the category of based topological spaces, and T a triple. Then the map $|T.(X, \xi)| \longrightarrow X$ induced by α is a weak equivalence.*

PROOF. This is proved in [22, Proposition 9.8, p. 90]. □

To produce deloopings we must also use the interaction of the suspension functor with the triple in question. We formalize this as follows.

DEFINITION 8.1.4. Let T be a triple on a category C. By an intertwiner Σ for T, we mean a functor $\Sigma : C \longrightarrow C$ together with a natural transformation $\zeta : \Sigma T \longrightarrow T\Sigma$, so that the following diagrams commute.

$$\begin{array}{ccccc} \Sigma T^2 X & \xrightarrow{\zeta(TX)} & T\Sigma TX & \xrightarrow{T(\zeta X)} & T^2\Sigma X \\ \Big\downarrow{\scriptstyle \Sigma\mu(X)} & & & & \Big\downarrow{\scriptstyle \mu(\Sigma X)} \\ \Sigma TX & \xrightarrow{id} & \Sigma TX & \xrightarrow{\zeta(X)} & T\Sigma X \end{array}$$

$$\begin{array}{ccc} \Sigma X & \xrightarrow{\Sigma\eta(X)} & \Sigma TX \\ \Big\downarrow{\scriptstyle \eta(\Sigma X)} & & \Big\downarrow{\scriptstyle \zeta(X)} \\ T\Sigma X & \xrightarrow{id} & T\Sigma X \end{array}$$

DEFINITION 8.1.5. Given any intertwiner Σ for T, and T-algebra (X, ξ), we construct a simplicial object $T^{\Sigma}_{.}(X, \xi)$ by setting

$$T^{\Sigma}_{k}(X, \xi) = T\Sigma T^{k}X,$$

and declaring that the faces and degeneracies are given by the following formulae:

$$\begin{cases} d_0 : T\Sigma T^k X \longrightarrow T\Sigma T^{k-1}X := \mu(T^{k-1}X) \circ T(\zeta(T^{k-1}X)) & \\ d_i : T\Sigma T^k X \longrightarrow T\Sigma T^{k-1}X := T\Sigma T^{i-1}\mu(T^{k-i-1}X) & \text{for } i > 0. \\ s_i : T\Sigma T^k X \longrightarrow T\Sigma T^{k+1}X := T\Sigma T^{i}\eta(T^{k-i}X) & \end{cases}$$

Note that $T^{\Sigma}_{.}(X, \xi)$ is a simplicial object in the category of T-algebras. We also note that there is a morphism

$$\Sigma X. \xrightarrow{\lambda} T.^{\Sigma}(X, \xi),$$

where $\Sigma X.$ is the constant simplicial object with value ΣX. There is also a map ν : $\Sigma T.(X, \xi) \longrightarrow T.^{\Sigma}(X, \xi)$, given in level k by $\zeta(T^k X)$. $T^{\Sigma}_{.}$ is a functor from the category of T-algebras to the category of simplicial T-algebras.

We now apply this to our situation, where $T = Q$ and C is the category of based CW complexes. Σ is now ordinary suspension. To define a map $\zeta(X) : \Sigma QX \longrightarrow Q\Sigma X$, we define, for $f : S^n \longrightarrow S^n \wedge X$ and $s \in S^n$, $t \in [0, 1]$,

$$\zeta(X)[t, f](s) = [t, f(s)] \tag{8.2}$$

where t is the suspension coordinate and $S^n \wedge \Sigma X$ is identified with $\Sigma(S^n \wedge X)$. It is easy to check that with this definition, the pair (Σ, ζ) forms an intertwiner for Q. Thus, for any Q-algebra (X, ξ), we obtain a simplicial Q-algebra $Q.^{\sigma}(X, \xi)$, and a map of spaces

$$\Sigma X \xrightarrow{\lambda} |Q.^{\Sigma}(X, \xi)|.$$

If we consider the adjoint $ad(\lambda) : X \longrightarrow \Omega|Q.^{\Sigma}(X, \xi)|$, then $|Q.^{\Sigma}(X, \xi)|$, is a candidate for a first delooping for X.

PROPOSITION 8.1.2. *Let X be any Q-algebra. Then $ad(\lambda)$ is a weak equivalence. Further, if X is k-connected, then $|Q.^{\Sigma}(X, \xi)|$ is $(k + 1)$-connected.*

PROOF. We first observe that λ factors as

$$\Sigma X \longrightarrow \Sigma Q.(X,\xi) \xrightarrow{\zeta(X)} Q.^{\Sigma}(X,\xi),$$

where the left arrow is $\Sigma\eta(X) : \Sigma X \longrightarrow \Sigma Q(X) = \Sigma Q_0(X,\xi)$. It is an equivalence by Theorem 8.1.1. It consequently suffices to show that the adjoint to $\nu(X)$, $ad(\nu(X)) : |Q.(X,\xi)| \longrightarrow \Omega|Q.^{\Sigma}(X,\xi)|$, is an equivalence. Secondly, it is standard in this case (where $\pi_0(Q_k^{\Sigma}(X,\xi))$ is a group for all k) that the natural map $|\Omega Q.^{\Sigma}(X,\xi)| \longrightarrow \Omega|Q.^{\Sigma}(X,\xi)|$ is an equivalence. See [12] for details. It therefore suffices to show that the adjoint to $\zeta(Q^k X) : \Sigma Q^{k+1}X \longrightarrow Q\Sigma Q^k X$ is an equivalence, and for this it clearly suffices to show that $ad(\zeta(X))$ is an equivalence for all X. But the adjoint of $\zeta(X)$ is the inclusion $QX \longrightarrow \Omega Q\Sigma X$, which is easily checked to be an equivalence. The connectivity statement is easy. □

Let Ψ denote the functor $(X,\xi) \longrightarrow Q^{\Sigma}_{.}(X,\xi)$ from Q-algebras to simplicial Q-algebras. Applying Ψ levelwise to $Q^{\Sigma}_{.}$, we obtain a functor $\Psi[2]$ from Q-algebras to bisimplicial Q-algebras, and by iteration of this procedure functors $\Psi[k]$ to k-fold simplicial Q-algebras. By applying Proposition 8.1.2 levelwise, one obtains a natural (on the category of Q-algebras) equivalence $|\Psi[k](X,\xi)| \simeq \Omega|\Psi[k+1](X,\xi)|$. In other word, we have constructed a functor S from the category of Q-algebras to the category of Ω-spectra. It is not hard to check that the functor actually takes its values in the full subcategory of connective spectra. Further, S is homotopy invariant in the sense that if $f : (X_1,\xi_1) \longrightarrow (X_2,\xi_2)$ is a morphism of Q-algebras, so that $f : X_1 \longrightarrow X_2$ is a weak equivalence of spaces, then $S(f)$ is a weak equivalence of spectra.

8.2. *The May recognition principle for Ω-spectra*

We wish to use the $\mathcal{O}$-space constructions $\underline{C}[X]$ to minimize the amount of data required to construct the deloopings. As it stands, for a general $\mathcal{O}$-space, $\underline{C}$, $X \longrightarrow \underline{C}[X]$ is not a triple on the category of based spaces. To have the triple structure requires that $\underline{C}$ actually be an "operad" in the sense of May. We now describe this notion. In order to simplify the definition a bit, we introduce some terminology. By a *graded topological space,* we mean a space C equipped with a decomposition

$$C = \coprod_{n\geqslant 0} C_n.$$

If X is a space, we will write

$$F_C(X) = \coprod_{n\geqslant 0} C_n \times X^n.$$

This is of course functorial on the category of spaces. If

$$X = \coprod_{i \geqslant 0} X_i$$

is a graded space, then $F_C(X)$ becomes a graded space, with

$$F_C(X)_n = \coprod_{l \geqslant 0} \coprod_{j_1 + \cdots + j_l = n} C_l \times X_{j_1} \times \cdots X_{j_l} \subseteq F_C(X).$$

DEFINITION 8.2.1. Let $\underline{C}$ be an $\mathcal{O}$-space.

$$C = \coprod_{n \geqslant 0} C_n$$

is viewed as a graded set. An operad structure on $\underline{C}$ is a natural transformation of functors $\mu : F_C \circ F_C \longrightarrow F_C$, satisfying the following requirements.

(a) The diagrams

$$\begin{array}{ccc} F_C \circ F_C \circ F_C(X) & \xrightarrow{F_C(\mu(X))} & F_C \circ F_C(X) \\ \Big\downarrow {\scriptstyle \mu(F_C(X))} & & \Big\downarrow {\scriptstyle \mu(X)} \\ F_C \circ F_C(X) & \xrightarrow{\mu(X)} & F_C(X) \end{array}$$

commute for all X.

(b) $\mu(*)$ gives maps $C_k \times C_{j_1} \times \cdots \times C_{j_k} \longrightarrow C_j$, where $j = j_1 + \cdots + j_k$. Since $\underline{C}$ is an $\mathcal{O}$-space, C_j is equipped with an action by the symmetric group $\mathcal{S}_j$. On the other hand, $C_k \times C_{j_1} \times \cdots \times C_{j_k}$ is equipped with an action of $\mathcal{S}_{j_1} \times \cdots \times \mathcal{S}_{j_k}$, with each symmetric group acting on its corresponding factor, and all acting trivially on C_k. Let $\rho : \mathcal{S}_{j_1} \times \cdots \times \mathcal{S}_{j_k} \longrightarrow \mathcal{S}_j$ be the homomorphism which views $\mathcal{S}_{j_1}$ as acting on $\{1, \ldots, j_1\}$, $\mathcal{S}_{j_2}$ as acting on $\{j_1 + 1, \ldots, j_2\}$, etc. We require that $\mu(*)$ restricted to $C_k \times C_{j_1} \times \cdots \times C_{j_k}$ be equivariant with respect to ρ, i.e.

$$\mu(*)(c; \sigma_1 c_1, \ldots, \sigma_k c_k) = \rho(\sigma_1, \ldots, \sigma_k)\mu(*)(c; c_1, \ldots, c_k).$$

(c) Let $j_1, \ldots, j_k$ be given, with $j_1 + \cdots + j_k = j$. Note that $\{j_1, \ldots, j_k\}$ determines a partition

$$\{1, \ldots, j_1 | j_1 + 1, \ldots, j_1 + j_2 | j_2 + 1, \ldots, j_1 + \ldots j_{k-1} | j_1 + \ldots \\ + j_{k-1} + 1, \ldots, j_1 + \ldots + j_k\}$$

of $\{1, \ldots, j\}$, with k blocks. For any $\sigma \in \mathcal{S}_k$ let $\theta = \theta(\sigma; j_1, \ldots, j_k)$ be the unique permutation of $\{1, \ldots, j\}$ which is order preserving on each block $\{j_1 + \ldots + j_{s-1} +$

$1, \ldots, j_1 + \ldots + j_s\}$, and so that $\sigma(s) > \sigma(t)$ implies that θ carries elements of $\{j_1 + \ldots + j_{s-1} + 1, \ldots, j_1 + \ldots j_s\}$ to elements which are strictly greater than all elements in $\{j_1 + \ldots + j_{t-1} + 1, \ldots, j_1 + \ldots + j_t\}$. Then

$$\mu(*)(\sigma c; c_1, \ldots c_k) = \theta(\sigma; j_1, \ldots, j_k)\mu(*)(c; c_{\sigma(1)}, \ldots, c_{\sigma(k)}).$$

(d) There exists an element $1 \in C_1$, so that $\mu(*)(c; \underbrace{1, \ldots, 1}_{k \text{ factors}}) = c$ for all $c \in C_k$.

PROPOSITION 8.2.1. *An operad structure on an $\mathcal{O}$-space gives a triple structure on the functor $X \longrightarrow \underline{\mathcal{C}}[X]$.*

(The proof is a direct but tedious verification. See [22] for details.)

Not all $\mathcal{O}$-spaces described in Section 7.1 extend to operad structures. For instance, $\underline{\mathcal{C}}(k)$ does not, nor does the $\mathcal{O}$-space $\underline{\mathcal{C}}^d(k)$ of Example 7.1(D). However, let $\underline{\mathcal{F}}$ be the $\mathcal{O}$-space of 7.1(B). $\underline{\mathcal{F}}$ extends to an operad structure as follows. Let $j_1 + \ldots + j_k = j$, and let $B_s \subseteq \{1, \ldots, j\}$ be the subset

$$\{n | j_1 + \ldots + j_{s-1} + 1 \leqslant n \leqslant j_1 + \ldots + j_s\}.$$

The structure map $\mu(*) : F_k \times F_{j_1} \times \cdots \times F_{j_k} \longrightarrow F_j$ is given by assigning to a $(k+1)$-tuple $(\leqslant, \leqslant_1, \ldots, \leqslant_k)$ of orderings the unique ordering on $\{1, \ldots, j\}$ which restricts to the ordering $\leqslant_s$ on B_s, when B_s is identified with $\{1, \ldots, j_s\}$ in an order preserving way, and so that if $m \in B_s$ and $n \in B_t$, with $s \neq t$, $m < n$ if and only if $s < t$.

Also, recall the Barratt–Eccles $\mathcal{O}$-simplicial set $\underline{B}$ from Section 7. Here, B_n was defined as $e(F^n)$, where e was a product preserving functor from sets to simplicial sets. The above defined operad structure map for $\underline{\mathcal{F}}$ now defines similar maps $e(F_k) \times e(F_{j_1}) \times \cdots \times e(F_{j_k}) \longrightarrow e(F_j)$. Applying geometric realization gives an operad structure on the $\mathcal{O}$-space $\{|B_n|\}_{n \geqslant 0}$.

With simple modifications one can modify $\underline{\mathcal{C}}(k)$ into an $\mathcal{O}$-space with operad structure. We define $\underline{\mathcal{C}}^B(k) = \{C_n^B(k)\}_{n \geqslant 0}$ by letting $C_n^B(k)$ be the space of disjoint n-tuples of open n-cubes in $[0,1]^k$. It is understood that these are cubes with sides parallel to the coordinate axes. $\underline{\mathcal{C}}^B(k)$ now admits an operad structure. For any $(k+1)$-tuple $(c; c_1, \ldots, c_l)$ with $c_x \in C_{j_s}^B(k)$, and $c \in C_l^B(k)$, say $c = (Cu_1, \ldots, Cu_l)$, we have the identification $\lambda_i : [0,1]^k \longrightarrow Cu_i$, which is an affine linear map and carries sides parallel to a coordinate axis to sides parallel to the same coordinate axis. The j_s-tuple of cubes in $[0,1]^k$ specified by c_s is identified with a new j_s-tuple of disjoint cubes $\lambda_s(c_s)$ contained in Cu_s. The j-tuple of cubes $\{\lambda_1(c_1), \ldots, \lambda_l(c_l)\}$ consists of disjoint cubes, since the Cu_i's are disjoint. This gives the operad structure. We also remark that $\underline{\mathcal{C}}^B(k)$ includes in $\underline{\mathcal{C}}(k)$ as a sub-$\mathcal{O}$-space, and that this inclusion satisfies the hypothesis of Proposition 8.1.1. It follows that $\underline{\mathcal{C}}^B[X]$ and $\underline{\mathcal{C}}[X]$ are weakly equivalent for all based CW-complexes X.

Let T^B denote the triple $X \longrightarrow \underline{\mathcal{C}}^B[X]$. We observe that there is a functor from the category of connective spectra to Q-algebras, which assigns to each connective spectrum its zeroth space. There is a natural transformation of triples $T^B \longrightarrow Q$, which means that any Q-algebra can be viewed as a T^B-algebra, and hence we obtain a composite functor $\mathcal{U}$ from the category of connective spectra to the category of T^B-algebras. In order to state our theorem, we also define a weak natural transformation of functors to the category of spaces (or spectra, or T-algebras, where T is a triple) from a functor F^0 to a functor F^1 to be a sequence of functors $\{G_0, \ldots, G_{2k}\}$, with $G_0 = F_0$ and $G_{2k} = F_1$, together with natural transformations $G_{2l+1} \longrightarrow G_{2l+2}$ and natural transformations $G_{2l+1} \longrightarrow G_{2l}$ which are weak equivalences for all objects in the domain category. A weak natural transformation is said to be a weak equivalence if in addition the natural transformations $G_{2l+1} \longrightarrow G_{2l+2}$ are weak equivalences when evaluated on any object in the domain category. Note that a morphism of T-algebras is said to be a weak equivalence if the map on spaces is a weak equivalence in the usual sense.

The May recognition principle is now stated as follows.

THEOREM 8.2.1. *There is a functor $\mathcal{S}$ from the category of T^B-algebras to the category of connective spectra, satisfying the following properties.*

(a) *If $f : (X, \xi) \longrightarrow (X', \xi')$ is a map of T^B-algebras, and the map $f : X \longrightarrow X'$ is a weak equivalence, then $\mathcal{S}(f)$ is a weak equivalence of spectra.*

(b) *There is a natural weak equivalence of functors on the category of connective spectra from $\mathcal{S} \circ \mathcal{U}$ to the identity functor.*

(c) *There is a weak natural transformation of functors on the category of T^B-algebras from the identity to $\mathcal{U} \circ \mathcal{S}$, which is a weak equivalence on T^B-algebras (X, ξ) for which $\pi_0(X)$ is a group. (Note that in general, if (X, ξ) is a T^B-algebra, we have a map*

$$C_2^B \underset{\mathcal{S}_2}{\times} X \times X \longrightarrow X,$$

and hence by choosing a point in C_2^B a map $X \times X \longrightarrow X$. Consequently, X is an H-space, and the multiplications are independent of the choice of point up to homotopy. Thus, $\pi_0(X)$ is given a well-defined monoid structure.)

We do not give a proof of this theorem, but refer to [22] or [29]. However, we do give a description of $\mathcal{S}$. We first note that the suspension functor Σ acts as an intertwiner for T^B. The map $\Sigma \underline{C}^B[X] \longrightarrow \underline{C}^B[\Sigma X]$ is induced by the evident maps

$$\Sigma\big(C_n(\infty) \underset{\mathcal{S}_n}{\times} X^n\big) \longrightarrow C_n(\infty) \times \mathcal{S}_n(\Sigma X)^n,$$

after factoring out the equivalence relation defining $\underline{C}^B[-]$. Consequently, we may construct the simplicial T^B-algebra $T^B_{\cdot}(X, \xi)$, and iterate this construction levelwise, to obtain spaces $\mathcal{S}(X, \xi)_k$, with maps

$$\Sigma \mathcal{S}(X, \xi)_k \xrightarrow{\lambda_k} \mathcal{S}(X, \xi)_{k+1}.$$

Here, $\mathcal{S}(X, \xi)_0 = X$. One is able to show that the adjoint of λ_k is an equivalence if $k \geqslant 1$, and if X is connected so is the adjoint to λ_0. In any case, we obtain a functor to connective spectra. Requirement (a) is clearly satisfied, since T^B preserves weak equivalences.

8.3. *G. Segal's construction of Ω-spectra*

There is another point of view on these ideas, due to G.B. Segal. He enlarges the category of finite ordered sets and order preserving maps to a larger category Γ, so that (roughly) given a simplicial space $X.$, i.e. a functor $\Delta^{op} \longrightarrow X.$, an extension of $X.$ to Γ gives rise to a connective spectrum with zeroth space homotopy equivalent to X.

We outline his ideas. We first define a category Γ to have objects the finite sets $\gamma_n = \{1, \ldots, n\}$ for $n > 0$, and $\gamma_0 = \varnothing$. A morphism $\varphi : \gamma_n \longrightarrow \gamma_k$ is a function $\Phi : \mathcal{P}(\gamma_n) \longrightarrow \mathcal{P}(\gamma_k)$ (where $P(X)$ denotes the power set of X), so that $\Phi(V \cup W) = \Phi(V) \cup \Phi(W)$ and $\Phi(V - W) = \Phi(V) - \Phi(W)$. The first condition shows that Φ is determined by the sets $\Phi(\{i\})$ $1 \leqslant i \leqslant n$, and the second condition shows it is equivalent to $\Phi(\{i\}) \cap \Phi(\{j\}) = \varnothing$ if $i \neq j$.

Given morphisms $\varphi : \gamma_n \longrightarrow \gamma_m$ and $\psi : P(\gamma_m) \longrightarrow P(\gamma_k)$ corresponding to maps

$$\Phi : \mathcal{P}(\gamma_n) \longrightarrow \mathcal{P}(\gamma_m)$$

and $\Psi : P(\gamma_m) \longrightarrow P(\gamma_k)$, then $\psi \circ \varphi$ corresponds to $\Theta : P(\gamma_n) \longrightarrow P(\gamma_k)$, where $\Theta(V) = \Psi \circ \Phi$.

There is an isomorphism of categories from the opposite of the category of based finite sets $\{0, 1, \ldots, n\}$ (0 is the basepoint) and based maps to Γ given by

$$\{0, 1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$$

and $(f : \{0, 1, \ldots, n\} \longrightarrow \{0, 1, \ldots, m\}) \longrightarrow \varphi_f$. Here φ_f corresponds to the map

$$\Phi_f : P(\gamma_m) \longrightarrow P(\gamma_n), \quad \text{where } \Phi_f(V) = f^{-1}(V),$$

for any $V \subset \{1, \ldots, m\}$.

There is also a functor $i : \Delta \longrightarrow \Gamma$, where Δ is the category whose objects are the sets $\{0, 1, \ldots, n\}$, equipped with their standard ordering, and whose morphisms are the order preserving maps. To define i, we first define, for $p, q \in \{0, 1, \ldots, n\}$, $[p, q] = \{r \in \{0, 1, \ldots, n\} \mid p \leqslant r \leqslant q\}$. Note that if $q < p$, $[p, q] = \varnothing$. i is now defined on objects by $i(\{0, 1, \ldots, n\}) = \{1, \ldots, n\}$, and on morphisms by $i(f : \{0, 1, \ldots, n\} \longrightarrow \{0, 1, \ldots, m\}) = \varphi_f$, where φ_f corresponds to the function $\Phi_f : P(\gamma_n) \longrightarrow P(\gamma_m)$ defined by $\Phi_f([1, \ldots, , r]) = [1, \ldots, f(r)]$. This gives, for instance,

$$\Phi_f(\{r\}) = [f(r-1) - 1, f(r)] \cap \{1, \ldots, m\}.$$

By a Γ-space, we mean a contravariant functor from Γ to topological spaces. By restriction to Δ, we obtain a simplicial topological space. In particular, we may define $|\Phi|$ for any Γ-space.

Let $\lambda_i : \{1\} \longrightarrow \{1, \ldots, n\}$ be the morphism in Γ corresponding to

$$\Lambda_i : P(\{1\}) \longrightarrow P(\{1, \ldots, n\})$$

given by $\Lambda_i(\{1\}) = \{i\}$. Then, given any Γ-space Φ we have the map

$$\Phi(\{1,\dots,n\}) \xrightarrow{\prod_{i=1}^{n}\Phi(\lambda_i)} \prod_{i=1}^{n}\Phi(\{1\}) \tag{8.3}$$

for each n.

Φ is said to be *special* if (8.3) is a weak homotopy equivalence for each n and if $\Phi(\varnothing) \simeq *$. Segal then proves the following result.

THEOREM 8.3.1. *Let X be any Γ-space. Then there is a sequence of functors B^n from the category of Γ-spaces to itself, and natural transformations*

$$|B^n\Phi| \longrightarrow \Omega|B^{n+1}\Phi|,$$

which are weak equivalences if Φ is special. In particular, the sequence

$$|\Phi|, B|\Phi|, B^2|\Phi|, \dots,$$

form an Ω-spectrum, and we obtain a functor B from special spaces to Ω-spectra. Further, there is a functor A from Ω-spectra to Γ-spaces, together with natural equivalences $BA \longrightarrow Id$ and $AB \longrightarrow Id$.

One can go a bit further. In any simplicial space $X.$ with X_0 contractible, one has a well defined homotopy class of maps from ΣX_1 to $|X.|$. Let $\mu : \{1\} \longrightarrow \{1,2\}$ be the morphism in Γ given by $\{1\} \longrightarrow \{1.2\}$. Also, let

$$\tau : \Phi(\{1\}) \times \Phi(\{1\}) \longrightarrow \Phi(\{1,2,\})$$

be the inverse to the weak equivalence occurring in the definition of the notion of a special Γ-space. Then $\Phi(\mu) \circ \tau$ gives an H-space structure on $\Phi(\{1\})$.

THEOREM 8.3.2 (Segal). *The adjoint to the inclusion $\Sigma\Phi(\{1\}) \longrightarrow |\Phi|$ is a weak equivalence if the above described H-space structure admits a homotopy inverse. In particular, this holds if $\Phi(\{1\})$ is connected.*

8.4. *The combinatorial data which build Ω-spectra*

These constructions also allow one to construct spectra from purely combinatorial data. To understand this, we recall the nerve construction, which associates to any category, $\mathcal{C}$, a simplicial set, $N.\mathcal{C}$, and hence a topological space. The k-simplices are composable k-tuples of arrows

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_k} x_k$$

in C if $k > 0$, and are simply objects in C if $k = 0$. The face maps are given by the following formulae.

$$\begin{cases} d_0(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} x_k) = (x_1 \xrightarrow{f_2} x_2 \xrightarrow{f_3} \cdots \xrightarrow{f_k} x_k), \\ d_i(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} x_k) = (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots x_{i-1} \xrightarrow{f_{i+1} \circ f_i} x_{i+1} \cdots \xrightarrow{f_k} x_k) \\ \qquad \text{for } 1 \leqslant i < k, \\ d_k(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} x_k) = (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \cdots \xrightarrow{f_{k-1}} x_{k-1}), \\ s_i(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} x_k) = (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \longrightarrow x_{i-1} \xrightarrow{f_i} x_i \xrightarrow{id} x_i \\ \qquad \xrightarrow{f_{i+1}} x_{i+1} \longrightarrow \cdots \xrightarrow{f_k} x_k). \end{cases}$$

This is often a convenient way to construct spaces and maps, since it is clear that functors induce maps of simplicial sets. Indeed, any simplicial complex is homeomorphic to the nerve of a category, hence any CW complex has the homotopy type of the nerve of a suitable category. It is reasonable to ask what additional structure on the category allows one to construct a spectrum from $N.C$ in the same way as the Q or $\underline{T}^B$-algebra structures allowed one to construct spectra out of a space X. In order to describe this structure, we need a definition.

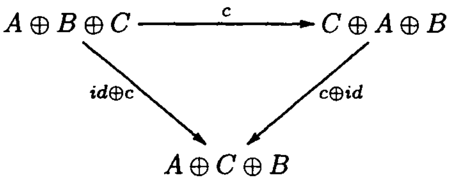
DEFINITION 8.4.1. A permutative category is a triple $(C, \oplus, c)$, where C is a category, $\oplus : C \times C \longrightarrow C$ is a functor, and c is a natural isomorphism of functors, from $\oplus$ to $\oplus \circ \tau$, where $\tau : C \times C \longrightarrow C \times C$ is the "reverse coordinates" map, subject to the following conditions.

(a) $\oplus$ is associative in the sense that

$$\oplus \circ (Id \times \oplus) = \oplus \circ (\oplus \times Id).$$

(b) $c(y, x) \circ c(x, y) = id_{(x,y)}$ for all $(x, y) \in C \times C$.

(c) The diagram



commutes.

(d) $c(A \oplus *) = Id_A$.

The nerve of a permutative category becomes a simplicial monoid. Further, its realization is a $\mathcal{BE}$-algebra, where $\mathcal{BE}$ is the triple corresponding to the Barratt–Eccles $\mathcal{O}$-space $|\mathcal{B}|$. To see this, one observes that $\underline{\mathcal{B}}[|NC|]$ can itself be described as the realization of the nerve of a category, what one might call the free permutative category on C (see [34] or or [23]). One can now use the above described space level constructions to arrive at a connective spectrum $\mathcal{S}pt(C)$.

THEOREM 8.4.1. *Spt defines a functor from the category of permutative categories to the category of connective spectra. Further, the zeroth space of* $Spt(C)$ *has the homotopy type of the group completion of the monoid* N_0C.

The last part of the statement is crucial for computations. It has as a corollary the well-known theorem of Barratt, Priddy and Quillen.

COROLLARY 8.4.1 ([28]). *Let* $\mathcal{S}_\infty$ *denote the infinite symmetric group, i.e.*

$$\varinjlim_n \mathcal{S}_n,$$

where $\mathcal{S}_n$ *is included in* $\mathcal{S}_{n+1}$ *in the evident way. Let* $B\mathcal{S}_\infty^+$ *denote Quillen's plus construction on* $BS\mathcal{S}_\infty$, *which abelianizes the fundamental group without affecting homology. Then* $Q(S^0) \cong B\mathcal{S}_\infty^+ \times \mathbf{Z}$. *In particular, if* $Q(S^0)_0$ *denotes the component consisting of maps of degree* 0, $H_*(Q(S^0)_0;\mathbf{Z}) \cong H_*(B\mathcal{S}_\infty;\mathbf{Z})$.

PROOF. The Barratt–Eccles monoid valued construction on S^0, which is the nerve of a category with two objects $*$ and p, and only identity morphisms, is isomorphic to $\coprod_{n\geqslant 0} B\mathcal{S}_n$, equipped with an associative multiplication, carrying $B\mathcal{S}_n \times B\mathcal{S}_m$ into $B\mathcal{S}_{n+m}$. It is not hard to see that the group completion is homotopy equivalent to $B\mathcal{S}_\infty \times \mathbf{Z}$. The result now follows from the above results. □

We conclude with some examples.

(A) The category of finite sets can be given the structure of a permutative category, with the sum operation corresponding to disjoint union. The resulting spectrum is the sphere spectrum.

(B) Let G be a finite group, and consider the category of finite sets with G-action. As in (A) above, we obtain a permutative category, which corresponds to Segal's G-equivariant sphere spectrum. It is a bouquet of spectra parameterized by the conjugacy classes of subgroups K of G, where the summand corresponding to the conjugacy class of K is the suspension spectrum of the classifying space of the group $N_G(K)/K$.

(C) Let A be any abelian group. View it as a category whose objects are the elements of A, and whose only morphisms are identity maps. The addition in A makes this category into a permutative category, in which c is actually an identity map for all pairs of objects in the category. The associated spectrum is the Eilenberg–MacLane spectrum $K(A,0)$.

(D) Let R be any ring, and consider the category of all finitely generated projective R-modules. This can be given the structure of a permutative category, where the sum operation corresponds to direct sum of modules. The corresponding spectrum is Quillen's algebraic K-theory spectrum for the ring R.

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