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CONTROLLED ALGEBRA AND THE NOVIKOV CONJECTURES FOR K- AND L-THEORY

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0. INTRODUCTION

IN THIS paper we combine the methods of [5] with the continuously controlled algebra of [1] and the *L*-theory of additive categories with involution [22] to split assembly maps in K- and *L*-theory. Specifically, we prove the following theorems.

Let Γ be a group with finite classifying space $B\Gamma$. Assume $E\Gamma$ admits a compactification X (meaning X compact, and $E\Gamma$ is an open dense subset) satisfying the following conditions, (denoting $X - E\Gamma$ by Y).

- (i) The Γ -action extends to X.
- (ii) X is metrizable.
- (iii) X is contractible.
- (iv) Compact subsets of $E\Gamma$ become small near Y, i.e. for every point $y \in Y$, for every compact subset $K \subset E\Gamma$ and for every neighborhood U of y in X, there exists a neighborhood V of y in X so that if $g \in \Gamma$ and $gK \cap V \neq \emptyset$ then $gK \subset U$.

Conditions of this type were first utilized by Farrell and Hsiang in [12]. Let R be a ring, and let $K^{-\infty}(R)$ denote the Gersten-Wagoner (non-connective) K-theory spectrum of R. Then we have the following theorem.

THEOREM A. The spectrum $B\Gamma_+ \wedge K^{-\infty}(R)$ is a split summand of $K^{-\infty}(R\Gamma)$.

Let R be a ring with involution, satisfying that $K_{-i}(R) = 0$ for sufficiently large *i*, and let $\mathbb{L}^{-\infty}(R)$ be the periodic L-theory spectrum of R with homotopy groups the Wall surgery obstruction groups [22]. Then we have the following theorem.

THEOREM B. The spectrum $B\Gamma_+ \wedge \mathbb{L}^{-\infty}(R)$ is a split summand in $\mathbb{L}^{-\infty}(R\Gamma)$.

If $\tilde{K}_i(R) = 0$ for all $i \le 1$, say for $R = \mathbb{Z}$ we may replace $\mathbb{L}^{-\infty}$ by \mathbb{L}^s . In this case we get the following corollary.

COROLLARY C. The spectrum $B\Gamma_+ \wedge \mathbb{L}^s(\mathbb{Z})$ is a split summand in $\mathbb{L}^s(\mathbb{Z}\Gamma)$.

Novikov conjectured the homotopy invariance of higher signatures, and proved it for free abelian groups. It is well known that the Novikov conjecture for a group Γ is equivalent

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to the rational split injectivity of the assembly map. In Sections 3 and 5, we identify the above splitting maps with the usual assembly map, so Corollary C verifies the Novikov conjecture for the class of groups considered here (actually a stronger integral version of the Novikov conjecture), see [22, Proposition 24.5] for a discussion of this.

The condition that X be metrizable is not actually necessary. It is sufficient that X is compact Hausdorff. This is proved using a generalized Čech theory in a sequel to this paper [7], where we also give conditions to ensure splitting, depending intrinsically on the group. We choose to present a proof in the metrizable case since the proof is fairly easy, and it does cover a large class of groups, in particular by [2] the above conditions are satisfied for word hyperbolic groups.

Jointly with W. Vogell and J. Roe respectively, we have extended the methods presented here to work in A-theory and topological K-theory as well [8,9].

A theorem similar to Theorem B has been announced by S. Ferry and S. Weinberger. Specifically, they replace the condition that the compactification is equivariant, by the condition that $E\Gamma$ is compactified by Y, a Z-set. The definition of a Z-set is that there is a homotopy h_t of the identity such that $h_t(Y) \subset E\Gamma$ for t > 0.

Assembly maps and related problems have been studied for a long time by many mathematicians under various assumptions on the group, and with various conclusions such as rational splitting, integral splitting, integral isomorphism, and for various functors such as K-, L-, A-, or C^* -theory, see e.g. [3-5, 10, 12-14, 17, 19, 21, 26, 28].

1. CONTINUOUSLY CONTROLLED ALGEBRA

In this section we recapitulate and extend results from [1].

Let *E* be a topological space, *R* a ring. We denote the free *R*-module generated by $E \times \mathbb{N}$ by $R[E]^{\infty}$. Here \mathbb{N} denotes the natural numbers. Notice that a set map $f: E \to F$ induces a map $R[E]^{\infty} \to R[F]^{\infty}$, in particular if $x \in E$, we have $R[x]^{\infty}$ a submodule of $R[E]^{\infty}$.

Definition 1.1. The category $\mathscr{B}(E; R)$ of finitely generated free R-modules parameterized by E has objects A, submodules of $R[E]^{\infty}$ such that denoting $A \cap R[x]^{\infty}$ by A_x

(i) $A = \bigoplus A_x$.

(ii) A_x is a finitely generated free *R*-module.

(iii) $\{x | A_x \neq 0\}$ is locally finite in E.

Morphisms are all *R*-module morphisms.

Given a morphism $\phi: A \to B$ and two points $x, y \in E$ we denote the component of ϕ from A_x to B_y by ϕ_y^x . Clearly, ϕ is determined by, and determines $\{\phi_y^x\}$. We call ϕ_y^x the components of ϕ .

Notice we have an isomorphism $R[x]^{\infty} \times R[x]^{\infty} \cong R[x]^{\infty}$ sending ((x, i), 0) to (x, 2i) and (0, (x, i)) to (x, 2i - 1).

Definition 1.2. Direct sum \oplus in the category $\mathscr{B}(E; R)$ is defined by

$$(A \oplus B)_x = \begin{cases} A_x \subset R[x]^\infty & \text{if } B_x = 0\\ B_x \subset R[x]^\infty & \text{if } A_x = 0\\ A_x \oplus B_x \subset R[x]^\infty \oplus R[x]^\infty \cong R[x]^\infty & \text{otherwise} \end{cases}$$

The special case made when A_x or B_x is zero, ensures the convenient formulae $A \oplus 0 = A = 0 \oplus A$. With this direct sum we clearly have the following proposition.

PROPOSITION 1.3. $\mathcal{B}(E; R)$ is a small additive category.

Remark 1.4. The reader may find it artificial to require the objects of $\mathscr{B}(E; R)$ to be submodules of $R[E]^{\infty}$, specially in view of the extra trouble in defining direct sum as above. The justification is that $\mathscr{B}(E; R)$ is a *small* category with convenient equivariant properties as exemplified by the following proposition.

PROPOSITION 1.5. Assume the group Γ acts freely, properly discontinuously on the space E. The induced action on $R[E]^{\infty}$ gives an action on $\mathscr{B}(E; R)$. We then have an equivalence of categories

$$\mathscr{B}(E; R)^{\Gamma} \cong \mathscr{B}(E/\Gamma; R[\Gamma]).$$

Proof. The Γ -action on E induces an $R[\Gamma]$ -module structure on $R[E]^{\infty}$. A set theoretic section $s: E/\Gamma \to E$ induces an $R[\Gamma]$ -module morphism

$$R\Gamma[E/\Gamma]^{\infty} \rightarrow R[E]^{\infty}$$

which is an isomorphism with inverse Φ given by

$$\Phi(\varepsilon)([x],n) = \sum_{g} \varepsilon(gs([x]),n)g.$$

An object A in $\mathscr{B}(E; R)$ fixed under the Γ -action is a submodule $A \subset R[E]^{\infty} \cong R\Gamma[E/\Gamma]^{\infty}$ which is setwise fixed under the Γ -action, hence an $R\Gamma$ -submodule. We have

$$A_{[x]} = A \cap R\Gamma[[x]]^{\infty} = \bigoplus_{g \in \Gamma} A_{g \cdot x}$$

hence $A_{[x]}$ is a free $R\Gamma$ -module of rank the *R*-rank of A_x , and the local finiteness of $\{x \in E | A_x \neq 0\}$ and proper discontinuity of the Γ -action implies that $\{[x] \in E/\Gamma | A_{[x]} \neq 0\}$ is locally finite in E/Γ . Obviously, *A* is generated by the $\{A_{[x]}\}$ as $R\Gamma$ -modules. Morphisms of $\mathscr{B}(E; R)$, fixed under the Γ -action satisfy

$$\phi_{gx}^{gy} = g\phi_x^y g^{-1}$$

so they are $R\Gamma$ -module morphisms, so we have completed the proof. Notice the equivalence does depend on the set theoretic splitting $E/\Gamma \rightarrow E$.

We shall consider categories with various degrees of control on the morphisms. Examples of this are bounded control as in [20] and continuous control as in [1], but also something that might be considered a mixture of the two.

In the following definitions, let X be a topological space, Y a subspace, T an open subset of X and $p: T \to K$ a map to a topological space K, which is continuous at points of $Y \cap T$. Denote X - Y by E.

Definition 1.6. Let U be a subset of X and A an object in $\mathscr{B}(E)$. We define A|U by $(A|U)_x = A_x$ if $x \in U - Y$ and $(A|U)_x = 0$ if $x \in X - U - Y$.

Definition 1.7. A morphism $\phi: A \to B$ in the category $\mathscr{B}(E; R)$ is said to be continuously controlled at a point $y \in Y$, if for every neighborhood U of y in X, there is a neighborhood V of y in X, so that

$$\phi(A|V) \subset B|U$$
 and $\phi(A|X-U) \subset B|X-V$

in other words if $a \in V - Y$ and $b \in X - U - Y$ implies $\phi_b^a = 0$ and $\phi_a^b = 0$.

This is the standard control definition in [1].

LEMMA 1.8. If X is compact Hausdorff, and $\phi: A \to B$ satisfies half the control condition at all points of Y, i.e. for every $y \in Y$ and every neighborhood U there is a neighborhood V so that $\phi(A|V) \subset B|U$, then ϕ is continuously controlled.

Proof. Let y and U be given. Find $V \subset \overline{V} \subset U$ so that $\phi(A|V) \subset B|U$. For $z \in Y - U$ we may find a neighborhood W_z so that $\phi(A|W_x) \subset B|X - \overline{V}$. Since $X - U - \bigcup W_z$ is compact we have $A|X - U - \bigcup W_z$ is a finitely generated R-module, so we can find a compact set $K \subset X - Y$ so that $\phi(A|X - U - \bigcup W_z) \subset B|K$, and we may replace V by V - K.

The next definition deals with control, but only in the direction of a certain map, reminiscent of the long thin handles in [13].

Definition 1.9. A morphism $\phi: A \to B$ in the category $\mathscr{B}(E; R)$ is said to be controlled in the *p*-direction or *p*-controlled at a point $y \in Y \cap T$, if for every neighborhood U of p(y) in K, there is a neighborhood V of p(y) in K so that

$$\phi(A|p^{-1}(V)) \subset B|p^{-1}(U)$$
 and $\phi(A|X-p^{-1}(U)) \subset B|X-p^{-1}(V)$

in other words if $a \in p^{-1}(V) - Y$ and $b \in X - p^{-1}(U) - Y$ then $\phi_b^a = 0$ and $\phi_a^b = 0$.

Definition 1.10. The category $\mathscr{B}(X, Y; R)$ has the same objects as $\mathscr{B}(E; R)$, E = X - Y, but morphisms are required to be continuously controlled at all points of Y.

These are the categories defined in [1]. The reader should think of E = X - Y as an open dense subset of X.

Definition 1.11. The category $\mathscr{B}(X, Y, p; R)$ has the same objects as $\mathscr{B}(E; R)$, E = X - Y but morphisms have to be continuously controlled at all points of Y - T, and p-controlled at all points of $T \cap Y$.

Example 1.12. The main examples we shall consider in this paper are the categories

$$\mathscr{B}(CX, CY \cup X, p_X; R), \qquad \mathscr{B}(\Sigma X, \Sigma Y, p_X; R)$$

where X is a compactification of $E\Gamma$, $Y = X - E\Gamma$ and p_X is the projection $X \times (0, 1) \rightarrow X$.

Finally, we need to introduce germs. Using notation as above let W be a (typically open) subset of Y.

Definition 1.13. The category $\mathscr{B}(X, Y, p; R)^{W}$ has the same objects as $\mathscr{B}(X, Y, p; R)$, but morphisms are identified if they agree in a neighborhood of W. Specifically $\phi, \psi: A \to B$ are identified if there is a neighborhood U of W in X, so that $\psi_{b}^{a} = \phi_{b}^{a}$ when $a \in U - Y$ or $b \in U - Y$. Similarly, the category $\mathscr{B}(X, Y; R)^{W}$ has the same objects as $\mathscr{B}(X, Y; R)$, but morphisms are identified if they agree in a neighborhood of W.

Following [1] we shall study the functoriality of $\mathscr{B}(X, Y; R)^W$, but first let us recall categorical terminology. Two functors between categories $F, G: \mathscr{A} \to \mathscr{B}$ are naturally equivalent if there is a natural transformation from F to G which is an isomorphism for each object in \mathscr{A} . The categories \mathscr{A} and \mathscr{B} are equivalent if there are functors $F: \mathscr{A} \to \mathscr{B}$ and $G: \mathscr{B} \to \mathscr{A}$ and natural equivalences from FG to $1_{\mathscr{B}}$ and from GF to $1_{\mathscr{A}}$. Two additive

categories are isomorphic when they are equivalent by functors which give a 1-1 correspondence of objects. We use the symbol = in case an isomorphism is the identity on objects. A functor between additive categories is called *lax* if it commutes with direct sum up to natural equivalence, sends 0 to 0 and induces a homomorphism of Hom-sets.

Definition 1.14. A set map $f: (X, Y) \to (X', Y')$ satisfying the following conditions is said to be eventually continuous at Y.

(i) If K is a compact subset of X' - Y', then $f^{-1}(K)$ has compact closure in X - Y.

(ii) $f(X - Y) \subset X' - Y'$.

(iii) f is continuous at points of Y.

Definition 1.15. The support at infinity $\operatorname{supp}_{\infty}(A)$ of an object A, in $\mathscr{B}(X, Y; R)$ is the set of limit points of $\{x | A_x \neq 0\}$.

Clearly, the support at infinity is a subset of Y.

There is a slight problem getting induced morphisms from eventually continuous maps. We need to consider two cases, as to whether f is monic on X - Y or not. Let (X, Y) and (X', Y') be compact Hausdorff pairs.

PROPOSITION 1.16. An eventually continuous map $f:(X, Y) \rightarrow (X', Y')$ which is monic on X - Y, induces a functor

$$\mathscr{B}(X, Y; R) \rightarrow \mathscr{B}(X', Y'; R)$$

sending $A \subset R[X - Y]^{\infty}$ to $A \subset R[X - Y]^{\infty} \xrightarrow{f_{\pi}} R[X' - Y']^{\infty}$, and the map induced by the identity on morphisms. When f is not assumed to be 1-1, we get a functor by sending $C = \{C_x\}$ to $f_{\pi}C$ with

$$(f_{\sharp}C)_{x'} = \bigoplus_{f(x)=x'} C_x$$

and choosing an embedding of $(f_{\sharp}C)_{x'} \subset R[x']^{\infty}$. The functor is induced by the identity on morphisms. In this case f_{\sharp} is only well-defined up to natural equivalence. If f sends Y - W to Y' - W' we get a functor

$$\mathscr{B}(X, Y; R)^{W} \to \mathscr{B}(X', Y'; R)^{W'}$$

Proof. The case when f is 1-1 is clear, so consider the case when f is not necessarily 1-1. Condition (i) ensures the direct sum is finite. We need to show that if ϕ is a continuously controlled map in $\mathscr{B}(X, Y; R)$, then $f_{\sharp}\phi$ is continuously controlled. Let $y' \in Y'$ and a neighborhood U' be given. Assume by contradiction that we can find a sequence $x'_i, y'_i \in X' - Y'$, so that $x'_i \notin U', y'_i \to y'$ and $f_{\sharp}\phi$ has a nontrivial component between x'_i and y'_i . We then have $x_i, y_i \in X - Y$ so that $f(x_i) = x'_i, f(y_i) = y'_i$ and ϕ has a nontrivial component between x_i and y'_i . We then have $x_i, y_i \in X - Y$ so that $f(x_i) = x'_i, f(y_i) = y'_i$ and ϕ has a nontrivial component between x_i and y_i . Since X is compact Hausdorff $\{y_i\}$ has a convergent subsequence $\{y_j\}$ with limit point y. The local finiteness condition on objects ensures that y must lie in Y. By continuity f(y) = y', and we may find a neighborhood V of y, so that $f(V) \subset V'$. Clearly $x_j \notin V$ and we have a contradiction. In the germ case note that two morphisms are identified if and only if the difference factors through an object A with $\supp_{\infty}(A) \subset Y - W$. But then $\supp_{\infty} (f_{\sharp}(A)) \subset Y' - W'$ because if $f_{\sharp}(A)_{y_i} \neq 0$ and $y'_i \to y'$, then we can find y_i such that $A_{y_i} \neq 0$ and $f(y_i) = y'_i$. The set $\{y_i\}$ must have a limit point y. This point belongs to $\supp_{\infty}(A) \subset Y - W$. By continuity y' = f(y) so $y' \in Y' - W'$. *Remark* 1.17. When composing eventually continuous maps f and g that are 1-1 in the interior, we do get $f_{\sharp}g_{\sharp} = (fg)_{\sharp}$. When they are not 1-1 in the interior we only get $f_{\sharp}g_{\sharp}$ is naturally equivalent to $(fg)_{\sharp}$.

PROPOSITION 1.18. If f_1 and f_2 are eventually continuous maps $(X, Y) \rightarrow (X', Y')$ and $f_1 | Y = f_2 | Y$, then f_{1*} and f_{2*} are naturally equivalent functors.

Proof. The natural equivalence is induced by the identity. \Box

Remark 1.19. Given any set map f we shall use the symbol f_* as given in Proposition 1.16. In any given case of course one needs to make sure that this does indeed define a functor.

Remark 1.20. In case of a locally compact Hausdorff pair (X, Y), $Y \subset X$ a proper inclusion, the obvious identification

$$\mathscr{B}(X, Y; R)^{Y} = \mathscr{B}(X_{+}, Y_{+}; R)^{Y}$$

ensures that an eventually continuous map $f: (X, Y) \rightarrow (X', Y')$ induces a functor

$$f_{\sharp}:\mathscr{B}(X, Y; R)^{Y} \to \mathscr{B}(X', Y'; R)^{Y}$$

in $f: (X_+, Y_+) \rightarrow (X'_+, Y'_+)$ is an eventually continuous map, i.e. if f satisfies an appropriate properness condition.

Let C be a (typically closed) subset of Y.

Definition 1.21. The full subcategory of $\mathscr{B}(X, Y; R)$ on objects with support at infinity contained in C, is denoted by $\mathscr{B}(X, Y; R)_c$. Similarly we have $\mathscr{B}(X, Y, p; R)_c$, $\mathscr{B}(X, Y; R)_c^W$ and $\mathscr{B}(X, Y, p; R)_c^W$ as full subcategories of $\mathscr{B}(X, Y, p; R)$, $\mathscr{B}(X, Y; R)^W$ and $\mathscr{B}(X, Y, p; R)^W$, respectively.

Let Y be a closed subset of X, C a closed subset of Y, S a closed subset of C (e.g. S = C). Then Y - C may be thought of as an open subset of Y/S. In case S = C then Y/S is just the one-point compactification of Y - C.

LEMMA 1.22. We have

$$\mathscr{B}(X, Y; R)^{Y-C} = \mathscr{B}(X/S, Y/S; R)^{Y/S-C/S}.$$

In particular, we have

$$\mathscr{B}(X, Y; R)^{Y-C} = \mathscr{B}(X/C, (Y/C - C/C)_+; R)^{Y/C - C/C}.$$

Proof. The identity induces a homeomorphism

$$E = X - Y = X/S - Y/S.$$

Hence, the objects on both sides are the objects of $\mathscr{B}(E; R)$. Any morphism has a representative which is 0 except in a neighborhood of Y - C, and for such a representative the control conditions are the same, since Y - C is thought of as an open subset of Y as well as of Y/S.

We next reduce the dependency of X.

THEOREM 1.23. Let X be a compact metrizable space, Y a closed subset, so that X - Y is dense in X, CY denotes the cone on Y. We then have eventually continuous maps $f: (X, Y) \rightarrow (CY, Y)$ and $g: (CY, Y) \rightarrow (X, Y)$ so that $f | Y = g | Y = 1_Y$.

Proof. We define the eventually continuous maps

$$f:(X, Y) \rightarrow (CY, Y)$$
 and $g:(CY, Y) \rightarrow (X, Y)$

so that $f | Y = g | Y = 1_Y$. Choose a metric on X so that $\sup_{x \in X} (d(x, Y)) = 1$. We orient the cone so that the conepoint is $Y \times 0$. Given $x \in X$ there exists $y \in Y$ so that d(x, y) = d(x, Y). Choose one such y and define f(x) = (y, 1 - d(x, Y)). If $x \in Y$, obviously f(x) = (x, 1), and if x is close to Y then f(x) will be close to (x, 1). Given $(y, t) \in CY$, we can find x so that $d(x, Y) = \sup(\{d(x, Y) | d(x, y) \le 1 - t\}, using the open denseness condition and compactness. Choose one such x and define <math>g(y, t) = x$. If t = 1 clearly g(y, t) = y and if t is close to 1, g(y, t) must be close to y.

COROLLARY 1.24. There is a natural equivalence

 $f_{\sharp}:\mathscr{B}(X, Y; R) \cong \mathscr{B}(CY, Y; R)$

natural in the sense that if h is an eventually continuous map $h:(X, Y) \rightarrow (X', Y')$, then the diagram

$$\mathscr{B}(X, Y; R) \longrightarrow \mathscr{B}(CY, Y; R)$$

$$\downarrow \qquad \qquad \downarrow$$

$$\mathscr{B}(X', Y'; R) \longrightarrow \mathscr{B}(CY', Y'; R)$$

commutes up to natural equivalence.

Proof. Let $f:(X, Y) \to (CY, Y)$ and $g:(CY, Y) \to (X, Y)$ be chosen as in Theorem 1.23. Then by Proposition 1.18, f_{\sharp} will be an equivalence of categories with inverse g_{\sharp} since f and g are the identity on Y. If $f':(X', Y') \to (CY', Y')$ is chosen as in Theorem 1.23 we have $Ch \cdot f$ and $f' \cdot h$ agree when restricted to Y, so once again it follows from Proposition 1.18 that $Ch_{\sharp} \cdot f'_{\sharp}$ and $f'_{\sharp} \cdot h_{\sharp}$ are naturally equivalent.

If W is an open subset of Y, the above technique also yields the following theorem.

THEOREM 1.25. $\mathscr{B}(X, Y; R)^{W} \cong \mathscr{B}(CY, Y; R)^{W}$.

Following Karoubi [16] we have the following definition.

Definition 1.26. An additive category is flasque if it admits an endofunctor Σ^{∞} and a natural equivalence $1 \oplus \Sigma^{\infty} \cong \Sigma^{\infty}$.

Let \mathscr{A} be a small additive category. We let $K(\mathscr{A})$ be a functorial assignment of \mathscr{A} to a spectrum whose homotopy groups are the K-groups of the symmetric monoidal category obtained from \mathscr{A} by restricting to isomorphisms. It is well known (from the additivity theorem) that the K-groups of a flasque additive category are trivial. Following [20], we define $\mathscr{C}_i(\mathscr{A})$ to be the category of \mathscr{A} -objects parameterized by \mathbb{Z}^i and bounded morphisms. The inclusion $\mathbb{Z}^i = \mathbb{Z}^i \times 0 \subset \mathbb{Z}^{i+1}$ induces a map

$$K(\mathscr{C}_i(\mathscr{A})) \to K(\mathscr{C}_{i+1}(\mathscr{A}))$$

which is naturally homotopy trivial in two ways since it factors through the categories

parameterized by $Z^i \times Z_+$ and $Z^i \times Z_-$, respectively, which have obvious natural flasque structures by Eilenberg swindle see e.g. [20]. This gives a functorial map

$$\Sigma K(\mathscr{C}_i(\mathscr{A})) \to K(\mathscr{C}_{i+1}(\mathscr{A})) \quad \text{or} \quad K(\mathscr{C}_i(\mathscr{A})) \to \Omega K(\mathscr{C}_{i+1}(\mathscr{A})).$$

We denote the homotopy colimit of

 $K(\mathscr{A}) \to \Omega K(\mathscr{C}_1(\mathscr{A})) \to \cdots \to \Omega^i(K(\mathscr{C}_i(\mathscr{A})) \to \cdots$

by $K^{-\infty}(\mathscr{A})$. It follows from [20], see [5] for a more detailed explanation, that $K^{-\infty}(\mathscr{A})$ is the nonconnective K-theory spectrum associated with the symmetric monoidal category obtained from \mathscr{A} by restricting to isomorphisms. Clearly, $K^{-\infty}$ is a functor from the category of small additive categories and lax functors to the category of spectra.

Recall the notion of an \mathscr{A} -filtered additive category \mathscr{U} [16].

Definition 1.27. Let \mathscr{A} be a full subcategory of an additive category \mathscr{U} . Denote objects of \mathscr{A} by the letters A-F and objects of \mathscr{U} by the letters U-W. We say that \mathscr{U} is \mathscr{A} -filtered, if every object U has a family of decompositions $\{U = E_{\alpha} \oplus U_{\alpha}\}$, so that

- (i) for each U, the decomposition form a filtered poset under the partial order that E_α⊕U_α ≤ E_β⊕U_β, whenever U_β ⊆ U_α and E_α ⊆ E_β;
- (ii) every map $A \to U$, factors $A \to E_{\alpha} \to E_{\alpha} \oplus U_{\alpha} = U$ for some α ;
- (iii) every map $U \to A$ factors $U = E_{\alpha} \oplus U_{\alpha} \to E_{\alpha} \to A$ for some α ;
- (iv) for each U, V the filtration on $U \oplus V$ is equivalent to the sum of filtrations $\{U = E_{\alpha} \oplus U_{\alpha}\}$ and $\{V = F_{\beta} \oplus V_{\beta}\}$ i.e. to $U \oplus V = (E_{\alpha} \oplus F_{\beta}) \oplus (U_{\alpha} \oplus V_{\beta})$.

Karoubi defines \mathscr{U}/\mathscr{A} to be the category with the same objects as \mathscr{U} , but with ϕ , $\psi: U \to V$ identified if $\phi - \psi$ factors through \mathscr{A} . We have the following theorem [20, Corollary 5.7].

THEOREM 1.28. $K^{-\infty}(\mathscr{A}) \to K^{-\infty}(\mathscr{U}) \to K^{-\infty}(\mathscr{U}/\mathscr{A})$ is a homotopy fibration.

Proof. This follows from [20, Corollary 5.7]. By taking the spectra that include the negative homotopy groups we avoid idempotent completions. \Box

Let X be a compact Hausdorff space, Y a closed subspace, so that X - Y is dense, C a closed subspace of Y, W an open subset of Y so that $C \subset W$. Put $\mathscr{U}_1 = \mathscr{B}(X, Y; R)$, $\mathscr{A}_1 = \mathscr{B}(X, Y; R)_C$, $\mathscr{U}_2 = \mathscr{B}(X, Y; R)^W$, $\mathscr{A}_2 = \mathscr{B}(X, Y; R)^W_C$. Then we have the following lemma.

LEMMA 1.29. \mathcal{U}_i is \mathcal{A}_i -filtered and

$$\mathscr{U}_1/\mathscr{A}_1 = \mathscr{B}(X, Y; R)^{Y-C}, \qquad \mathscr{U}_2/\mathscr{A}_2 = \mathscr{B}(X, Y; R)^{W-C}.$$

Proof. Direct from definitions.

Theorem 1.28 gives the following corollary.

COROLLARY 1.30. We have fibrations up to homotopy

$$K^{-\infty}(\mathscr{B}(X,Y;R)_{\mathcal{C}}) \to K^{-\infty}(\mathscr{B}(X,Y;R)) \to K^{-\infty}(\mathscr{B}(X,Y;R)^{Y-\mathcal{C}})$$

and

$$K^{-\infty}(\mathscr{B}(X,Y;R)_{\mathcal{C}}^{W}) \to K^{-\infty}(\mathscr{B}(X,Y;R)^{W}) \to K^{-\infty}(\mathscr{B}(X,Y;R)^{W-\mathcal{C}}).$$

LEMMA 1.31. Let X be a compact metrizable space, Y a closed subspace so that X - Y is dense in X, and * a point of Y. The natural map

$$\mathscr{B}(X, Y; R) \to \mathscr{B}(X, Y; R)^{Y^{-*}}$$

induces an isomorphism on K-theory (i.e. a weak homotopy equivalence of K-theory spectra).

Proof. Let $\mathscr{A} = \mathscr{B}(X, Y; R)_*$. By Corollary 1.30 we shall be done if we can prove that $K^{-\infty}(\mathscr{A})$ is weakly contractible. Choose a metric on X and a sequence of points x_0, x_1, \ldots , in X - Y converging to *, and such that $d(x_i, Y)$ is strictly decreasing. This is possible since X - Y is open dense in X. Let \mathscr{A}' be the full subcategory of $\mathscr{B}(X, Y; R)$ with $A_x = 0$ except for $x \in \{x_i\}$. Clearly \mathscr{A}' is a subcategory of \mathscr{A} , but actually \mathscr{A} and \mathscr{A}' are equivalent. One way is the inclusion. To get the other direction define a set map $f: X \to X$ by $f(x) = x_i$ if $d(x, Y) \ge d(x_i, Y)$ and $d(x, Y) < d(x_{i-1}, Y)$, when $x \in X - Y$, and f(y) = y for $y \in Y$. This map is clearly continuous at * (and discontinuous at all other points of Y), but since objects of \mathscr{A} are required to be 0 in a neighborhood of Y - *, we nevertheless get f_*A , an object of \mathscr{A}' and an equivalence of categories. To finish off the proof of the lemma notice that \mathscr{A}' is flasque i.e. it admits an endofunctor $U^{\infty}: \mathscr{A}' \to \mathscr{A}'$ so that $1 \oplus U^{\infty} \cong U^{\infty}$ (an Eilenberg swindle), and hence $K^{-\infty}(\mathscr{A}')$ is contractible. Specifically, U^{∞} is given by

$$(U^{\infty}(A))_{x_i} = \bigoplus_{j < i} A_{x_j}.$$

This completes the proof of Lemma 1.31.

LEMMA 1.32. Let X be a compact metrizable space, Y a closed nowhere dense subset, and W an open subset of Y. Then

$$\mathscr{B}(X, Y; R)^{W} = \mathscr{B}(X - (Y - W), W; R)^{W}.$$

Proof. The functor forgets the control along Y - W. The categories have the same objects, namely the objects of $\mathscr{B}(X - Y; R)$. If a morphism in $\mathscr{B}(X, Y; R)^{W}$ becomes 0 in $\mathscr{B}(X - (Y - W), W; R)^{W}$, that means that the components of the morphism are 0 in a neighborhood of W, but that means the morphism is 0 in $\mathscr{B}(X, Y; R)^{W}$. This shows the functor is monic on Hom-sets. To see it is epic on Hom-sets, consider a morphism in $\mathscr{B}(X - (Y - W), W; R)^{W}$ represented by $\phi: A \to B$. We may assume A and B are 0 except in a neighborhood of W, so ϕ is automatically controlled at interior points of Y - W. Hence, it is no loss of generality to assume that Y is the closure of W. We need to show that ϕ is equivalent to a morphism which is controlled at points of ∂W . For every point $z \in W$ we find a neighborhood $U_z \subset B(z, \frac{1}{2}d(z, \partial W))$ so that no non-zero component of ϕ reaches from U_z outside the ball $B(z, \frac{1}{2}d(z, \partial W))$. We let U be the union of all the U_z s and replace ϕ by a map whose components are equal to ϕ s when two points are in U and 0 otherwise. Rechoosing ϕ to be this new representative, we claim ϕ represents a morphism in $\mathscr{B}(X, Y; R)^{W}$ i.e. that it is controlled at points of ∂W . Let y be a point in ∂W and consider a δ -ball around y. We let $V = B(y, \delta/4)$. If $a \in V$ and ϕ_b^a or ϕ_a^b is different from 0, we must have $a \in U_z$, some $z \in W$. We have

$$d(z, y) \le d(z, a) + d(a, y) < d(z, \partial W)/2 + \delta/4 \le d(z, y)/2 + \delta/4$$

so $d(z, y) < \delta/2$. Since $a \in U_z$ and ϕ_b^a or ϕ_a^b is different from 0, we have $b \in B(z, \frac{1}{2}d(z, \partial W))$ and

$$d(b, y) \le d(b, z) + d(z, y) < d(z, \partial W)/2 + \delta/2 \le d(z, y)/2 + \delta/2 < \delta.$$

Hence a non-zero component of ϕ does not reach outside the δ -ball. Thus ϕ represents a map in $\mathscr{B}(X, Y)^{W}$ and we have shown the functor is epic on Hom-sets.

Definition 1.33. A (reduced) Steenrod homology theory [11, 15] is a functor from the category of component metrizable spaces and continuous maps, to graded abelian groups (with $h_*(pt) = 0$), satisfying the following axioms:

- (i) h is homotopy invariant.
- (ii) Given any closed subset A of X there is a natural transformation $\partial: h_*(X/A) \to h_{*-1}(A)$ fitting into a long exact sequence

$$\cdots \to h_n(A) \to h_n(X) \to h_n(X/A) \to h_{n-1}(A) \to \cdots$$

(iii) Given a compact metric space, which is the countable union of metric spaces along a single common point (like Hawaiian earrings), $\bigvee X_i$, then the projection maps $p_i: \bigvee X_i \to X_i$ induce an isomorphism

$$h_*(\bigvee X_i) \to \prod h_*(X_i).$$

These axioms are sometimes called the Kaminker-Schochet axioms.

Given any generalized homology theory, there is a unique Steenrod homology extension. Uniqueness was proved by Milnor [18], existence by Kahn-Kaminker-Schochet and Edwards-Hastings [11, 15].

Definition 1.34. A functor k from compact metrizable spaces to spectra is called a Steenrod functor if it satisfies the following conditions:

- (i) k(CX) is contractible for any cone CX.
- (ii) If $A \subset X$ is closed, then

$$k(A) \rightarrow k(X) \rightarrow k(X/A)$$

is a fibration (up to natural weak homotopy equivalence).

(iii) Given a compact metric space $\bigvee X_i$ which is the countable union along a single point of metric spaces X_i , then the projections induce a weak homotopy equivalence $k(\bigvee X_i) \simeq \prod k(X_i)$.

We have the following proposition.

PROPOSITION 1.35. Let k be a Steenrod functor. Then $\pi_*(k(X))$ is the unique Steenrod homology theory associated with the spectrum $k(S^0)$. If X is a finite CW-complex, then k(X) is weakly homotopy equivalent to $X \wedge k(S^0)$.

Proof. The maps $X \to X \times I$ sending x to (x, 0) or (x, 1) fit into $X \to X \times I \to CX$ giving homotopy invariance. The connecting homomorphism of the fibration gives the long exact sequence, and axiom (iii) gives the wedge axiom, hence $X \to \pi_*(k(X))$ is a Steenrod homology theory. We need to show that when we restrict to finite CW-complexes we get the homology theory associated to the spectrum $k(S^0)$. This however is proved in [27] (see Theorem 3.1 for the full statement). In particular it is proved in [27] that for any finite CW complex, X, there is a weak homotopy equivalence of spectra

$$X \wedge k(S^0) \cong k(X),$$

thus $\pi_*(k(X))$ restricts to the usual homology theory on the category of finite CW-complexes.

Our definition of a Steenrod functor might be called a reduced Steenrod functor, since it sends a point to a contractible spectrum. We may of course get an unreduced Steenrod functor by the usual device of adding an extra basepoint. THEOREM 1.36. Let X be a compact metrizable space, then $K^{-\infty}(\mathscr{B}(CX, X; R))$ is a reduced Steenrod functor with value $\Sigma K^{-\infty}(R)$ on S⁰. In particular if X a finite CW-complex then $\Omega K^{-\infty}(\mathscr{B}(CX, X; R))$ is weakly homotopy equivalent to $X \wedge K^{-\infty}(R)$.

Proof. We need to prove conditions (i)-(iii) for the functor sending X to $K^{-\infty}(\mathscr{B}(CX, X, R))$. (i) is proved in [1], and so is (ii), but not quite in this generality. Consider

$$K^{-\infty}(\mathscr{B}(CX,X;R)_A) \to K^{-\infty}(\mathscr{B}(CX,X;R)) \to K^{-\infty}(\mathscr{B}(CX,X)^{X-A}).$$

This is a fibration by Corollary 1.30. We first want to show that the inclusion

$$\mathscr{B}(CA, A; R) \to \mathscr{B}(CX, X; R)_A$$

is an equivalence of categories. Define a set map f from CX to CX to be the identity on X, and on CX - X a point x is sent some point a in CA such that d(x, a) realizes $\inf\{d(x, a) | a \in CA, d(a, A) \ge d(x, X)\}$. If this set is empty, send x to the cone point. Clearly f is continuous at points of A (and discontinuous at points of X - A), but since objects of $\mathscr{B}(CX, X; R)_A$ are required to be 0 in a neighborhood of X - A, we still get a functor

$$f_{\sharp}:\mathscr{B}(CX,X;R)_{A}-\mathscr{B}(CA,A;R)$$

which is an inverse to the inclusion up to natural equivalence. It follows from Lemma 1.22, Lemma 1.31 and Corollary 1.24 that $K^{-\infty}(\mathscr{B}(CX, X; R)^{X-A})$ is naturally weakly homotopy equivalent to $K^{-\infty}(\mathscr{B}(C(X/A), X/A; R))$, so we have verified condition (ii). Finally, consider $\mathscr{B}(C(\bigvee X_i), \bigvee X_i; R)$. Let * denote the wedge point. We have

$$K^{-\infty}(\mathscr{B}(C(\bigvee X_i),\bigvee X_i;R)) \simeq K^{-\infty}(\mathscr{B}(C(\bigvee X_i),\bigvee X_i;R)^{\bigvee X_i-\bullet})$$

by Lemma 1.31. The category of germs $\mathscr{B}(C(\bigvee X_i), \bigvee X_i; R)^{\bigvee X_{i-*}}$ is equivalent to the product of the categories $\mathscr{B}(CX_i, X_i; R)^{X_i-*}$ since only small neighborhoods of X_i-* matter, so there can be no interaction between the components at the germ. It is proved in [5, 6] that $K^{-\infty}$ commutes with infinite products, so

$$K^{-\infty}(\prod (\mathscr{B}(CX_i, X_i; R)^{X_i^{-*}})) \simeq \prod K^{-\infty}(\mathscr{B}(CX_i, X_i; R)^{X_i^{-*}}).$$

Moreover,

$$K^{-\infty}(\mathscr{B}(CX_i, X_i; R)^{X_i^{-*}}) \simeq K^{-\infty}(\mathscr{B}(CX_i, X_i; R))$$

by Lemma 1.31, and we are done. The final remark follows from Proposition 1.35. \Box

We define Steenrod homology of a pair by $h_n(X, A) = h_n(X/A)$. Combining Theorem 1.36 with Corollary 1.24 we have proved the following.

THEOREM 1.37. Let X be a compact metrizable space, Y a closed subspace so that X - Y is dense in X, W an open subset of Y. Then there is a natural isomorphism

$$K^{-\infty}_{\star}(\mathscr{B}(X,Y;R)^{W}) \cong h_{\star^{-1}}(Y,Y-W;K^{-\infty}R)$$

where $h_{*-1}(-; K^{-\infty} R)$ denotes the Steenrod homology theory associated to the algebraic K-theory spectrum of the ring R (non-connective version).

Remark 1.38. Notice that the proof of Theorem 1.36 identifies the map

$$K^{-\infty}(\mathscr{B}(CX,X;R)) \to K^{-\infty}(\mathscr{B}(CX,X;R)^{X-A})$$

with the Steenrod functor of $K^{-\infty}R$ applied to the collapse map $X \to X/A$.

Definition 1.39. Let E be a locally compact space such that E_+ , the one-point compactification is metrizable (e.g. E a finite dimensional CW-complex), and S a spectrum. We define

$$h^{l.f.}_{\star}(E;\mathbb{S}) = h^{st}_{\star}(E_+;\mathbb{S})$$

where h^{st} is the unique reduced Steenrod homology theory associated to the homology theory with spectrum S.

THEOREM 1.40. Let E be a locally compact space with metrizable one point compactification. Consider $E = E \times 1 \subset E \times (0, 1]$. Then

$$K_{*}^{-\infty}(\mathscr{B}(E \times (0, 1], E; R)^{E}) \cong h_{*-1}^{l.f.}(E, K^{-\infty}R)$$

Proof. By Lemma 1.32,

$$\mathscr{B}((E \times (0,1])_+, E_+; R)^E \cong \mathscr{B}(E \times (0,1], E; R)^E$$

and by Lemma 1.31 $\mathscr{B}((E \times (0, 1])_+, E_+; R)$ and $\mathscr{B}((E \times (0, 1])_+, E_+; R)^E$ have the same K-theory. Finally, Theorem 1.37 shows that the K-theory of $\mathscr{B}((E \times (0, 1])_+, E_+; R))$ is the reduced Steenrod homology of E_+ , with a shift in dimension.

2. SPLITTING THE K-THEORY ASSEMBLY MAP

Let Γ be a group. We shall consider a finite free Γ -CW complex E with a compactification X (meaning X compact, and E is an open dense subset). This compactification is supposed to satisfy conditions (i)-(iv) from Section 1.

If E is an $E\Gamma$ with a compactification satisfying these conditions, (so in particular $B\Gamma$ is a finite complex) the aim of this section is to show that the K-theory assembly map splits.

THEOREM 2.1. $B\Gamma_+ \wedge K^{-\infty}(R)$ is a split factor of $K^{-\infty}(R\Gamma)$.

Some of the lemmas needed do not require all the conditions. The proof follows the strategy of [5], but replacing $h^{l.f.}$ as well as bounded K-theory by continuously controlled K-theory, and the natural transformation by an induced map. In view of Theorem 1.40, what we are doing is of course using a continuously controlled model for $h^{l.f.}$. Given a spectrum A with Γ -action, recall the definition of the homotopy fixed set

$$A^{h\Gamma} = \operatorname{map}_{\Gamma}(E\Gamma_{+}, A_{+}).$$

The collapse map $E\Gamma_+ \to S^0$ induces a map from the fixed set $A^{\Gamma} = \text{Map}_{\Gamma}(S^0, A_+)$ to the homotopy fixed set.

Remark 2.2. There is an important special case where the map from the fixed set to the homotopy fixed set is a weak homotopy equivalence. When $A = \prod_{\Gamma} B$, and the Γ -action permutes the factors we have

$$A^{h\Gamma} = (\operatorname{Map}(\Gamma_+, B))^{h\Gamma} = \operatorname{Map}_{\Gamma}(E\Gamma_+, \operatorname{Map}(\Gamma_+, B)) = \operatorname{Map}_{\Gamma}((E\Gamma \times \Gamma)_+, B)$$
$$= \operatorname{Map}_{\Gamma}(\Gamma_+, \operatorname{Map}(E\Gamma_+, B)) = \operatorname{Map}(E\Gamma_+, B) \simeq B = A^{\Gamma}.$$

Let CX be the cone of X, with $X \times 0$ as the cone point, and identify $X \times 1$ with X. We denote X - E by Y and the projection map $X \times (0, 1) \rightarrow X$ by p_X , and consider the

 Γ -equivariant functor induced by collapsing X, from CX to ΣX :

$$\mathscr{B}(CX, CY \cup X, p_X; R) \rightarrow \mathscr{B}(\Sigma X, \Sigma Y, p_X; R).$$

Notice that the modules on both sides are parameterized by $E \times (0, 1)$, but the control conditions are quite different. Denoting $\Omega K^{-\infty}(\mathscr{B}(CX, CY \cup X, p_X; R))$ by S and $\Omega K^{-\infty}(\mathscr{B}(\Sigma X, \Sigma Y, p_X; R))$ by T, we study the diagram

$$\begin{array}{cccc} S^{\Gamma} & \longrightarrow & T^{\Gamma} \\ \downarrow & & \downarrow \\ S^{h\Gamma} & \longrightarrow & T^{h\Gamma} \end{array}$$

and prove the following statements:

(i) $S^{\Gamma} \simeq (E/\Gamma)_{+} \wedge K^{-\infty}(R)$, (ii) $T^{\Gamma} \simeq K^{-\infty}(R\Gamma)$, (iii) $S^{\Gamma} \simeq S^{h\Gamma}$, (iv) $S^{h\Gamma} \simeq T^{h\Gamma}$.

It is only (iv) that requires X to be contractible. The proof will be a sequence of lemmas. First consider T^{Γ} .

LEMMA 2.3. $\Omega K^{-\infty} (\mathscr{B}(\Sigma X, \Sigma Y, p_X; R))^{\Gamma} \simeq K^{-\infty} (R \Gamma).$

Proof. Taking fixed sets and applying $K^{-\infty}$ clearly commutes, so we need to study the category $\mathscr{B}(\Sigma X, \Sigma Y, p_X; R)^{\Gamma}$. We shall show that

$$\mathscr{B}(\Sigma(E/\Gamma), 0 \cup 1; R\Gamma) \cong \mathscr{B}(I, 0 \cup 1; R\Gamma).$$

A priori we require control at 0 and 1 and along $Y \times (0, 1)$, but only in the p_X -direction. Control along $Y \times (0, 1)$ is automatic by equivariance because of condition (iv). To see this let $\phi: A \to B$ be an equivariant morphism, $(y, t) \in Y \times (0, 1)$ a point and U a neighborhood of y in X. We need to show that we can find a neighborhood V of (y, t) in CX so that

$$\phi(A/V) \subset B|(U \times (0,1)).$$

Choose $0 < \varepsilon_1 < t < \varepsilon_2 < 1$ and a compact fundamental domain $K \subset E$ (so $E = \bigcup_{\Gamma_g} K$). By compactness there are only finitely many *R*-module generators in $K \times [\varepsilon_1, \varepsilon_2]$, so we may find $L \subset E$ compact and $0 < \delta_1 < t < \delta_2 < 1$ so that $K \subset L$ and

$$\phi(A|K\times[\varepsilon_1,\varepsilon_2])\subset B|L\times[\delta_1,\delta_2].$$

By condition (iv) we may find a neighborhood W of y in X so that if $gL \cap W \neq \emptyset$ then $gL \subset U$. We claim $V = W \times (\varepsilon_1, \varepsilon_2)$ will do: for any point $a = (p, s) \in V$, p lies in some translate gK of K, hence $gK \cap W \neq \emptyset$, but that means $gL \subset U$ and by equivariance

$$\phi(A_a) \subset \phi(A | gK \times [\varepsilon_1, \varepsilon_2]) \subset B | gL \times [\delta_1, \delta_2] \subset B | U \times (0, 1)$$

and the proof that control is automatic along $Y \times (0, 1)$ is finished by Lemma 1.8. Arguing as in Lemma 1.5 we thus get

$$\mathscr{B}(\Sigma X, \Sigma Y, p_X; R)^{\Gamma} \cong \mathscr{B}(\Sigma(E/\Gamma), 0 \cup 1; R\Gamma).$$

Since E/Γ is compact Corollary 1.24 shows that

$$\mathscr{B}(\Sigma(E/\Gamma), 0 \cup 1; R\Gamma) \cong \mathscr{B}(I, 0 \cup 1; R\Gamma)$$

and by Theorem 1.36 $\Omega K^{-\infty}(\mathscr{B}(I, 0 \cup 1; R\Gamma))$ is weakly homotopy equivalent to $K^{-\infty}(R\Gamma)$.

We now go on to study S^{Γ} . Thinking of $E = E \times 1 \subset E \times (0,1]$ we get the following lemma.

LEMMA 2.4. $\mathscr{B}(CX, CY \cup X, p_X; R)^E = \mathscr{B}(E \times (0, 1], E; R)^E$.

Proof. We have $\mathscr{B}(CX, CY \cup X, p_X; R)^E = \mathscr{B}(CX, CY \cup X; R)^E$ and $\mathscr{B}(CX, CY \cup X; R)^E = \mathscr{B}(CX, X; R)^E$ because the categories have the same objects as in $\mathscr{B}(E \times (0,1); R)$ and morphisms subquotients of the morphisms in $\mathscr{B}(E \times (0, 1); R)$. The equalities follow because any morphism has a representative which is 0 except for a neighborhood of E, so we may assume it is 0 in a neighborhood of CY - Y, and control conditions along CY - Y are thus automatically satisfied. Finally $\mathscr{B}(CX, X; R)^E = \mathscr{B}(E \times (0, 1], E; R)^E$ is a direct consequence of Lemma 1.32.

LEMMA 2.5. We have weak homotopy equivalences

$$K^{-\infty}(\mathscr{B}(CX, CY \cup X, p_X; R)) \simeq K^{-\infty}(\mathscr{B}(CX, CY \cup X, p_X; R)^E)$$

and

$$K^{-\infty}(\mathscr{B}(CX, CY \cup X, p_X; R))^{\Gamma} \simeq K^{-\infty}(\mathscr{B}(CX, CY \cup X, p_X; R)^{E})^{\Gamma}.$$

Proof. Let \mathscr{A} denote $\mathscr{B}(CX, CY \cup X, p_X; R)_{CY}$. Clearly the quotient category

$$\mathscr{B}(CX, CY \cup X, p_X; R)/\mathscr{A} = \mathscr{B}(CX, CY \cup X, p_X; R)^E$$

but \mathscr{A} has a flasque structure, by an Eilenberg swindle shifting modules to the left as follows: choose a continuous function $a: X \to [1, \infty)$ satisfying a(m) > 1 when $m \in E$ and a(Y) = 1. Define $U^{\infty}: \mathscr{A} \to \mathscr{A}$ by

$$(U^{\infty}(A))_{(m,t)} = \bigoplus A_{m,a(m)^n t}$$

where we let $A_{(m,s)} = 0$ when $s \ge 1$. This sum is clearly finite, and

$$\{(m,t)|(U^{\infty}(A))_{(m,t)}\neq 0\}$$

is locally finite because A is 0 in a neighborhood of E. To get the statement on fixed sets, since $K^{-\infty}$ commutes with taking fixed sets, we need to consider the fixed category. Defining $\mathscr{A} = (\mathscr{B}(CX, CY \cup X, p_X; R)_{CY})^{\Gamma}$, the argument goes through as above, noting that an $R\Gamma$ -module which is 0 in a neighborhood of E must be 0 in a neighborhood of X.

Remark 2.6. This is the point where we need control along the map p_X . If we require continuous control, then equivariant maps could not have a non-zero component between (p, s) and (q, t) when $s \neq t$ because such a component would be translated by the group to points close to $Y \times (0, 1)$, contradicting control. Hence, \mathscr{A} would not be flasque, because the flasque structure requires a natural transformation with a non-zero component between (p, s) and (q, t) with $s \neq t$.

LEMMA 2.7.
$$(\mathscr{B}(CX, CY \cup X, p_X; R)^E)^{\Gamma} = (\mathscr{B}(E \times (0, 1], E; R)^E)^{\Gamma}.$$

Proof. Immediate from Lemma 2.4.

LEMMA 2.8. Assume Γ acts freely properly discontinuously on a space E. Then

$$(\mathscr{B}(E\times(0,1],E;R)^{\mathsf{E}})^{\Gamma}\cong\mathscr{B}(E/\Gamma\times(0,1],E/\Gamma;R)^{E/\Gamma}.$$

Proof. The map

$$v_{*}: R[E]^{\infty} \to R[E/\Gamma]^{\infty}$$

induced by projection induces an isomorphism

$$R[E]^{\infty} \otimes_{R\Gamma} R \to R[E/\Gamma]^{\infty}.$$

An $R\Gamma$ -module A parameterized by $E \times (0, 1)$ is sent to the R-module $A \otimes_{R\Gamma} R$. Going backwards, an R-module B parameterized by E/Γ is sent to $p_{\sharp}^{-1}(B)$. Since we are taking germs at E and components of a morphism have to become small near E, there is only one choice when lifting a morphism.

To finish off showing $S^{\Gamma} = (E/\Gamma)_+ \wedge K^{-\infty}(R)$ we need the following theorem.

THEOREM 2.9. Let E be a finite, free Γ -CW-complex. Then

$$\Omega K^{-\infty}(\mathscr{B}(E\times(0,1],E;R)^E)^{\Gamma}\simeq (E/\Gamma)_+ \wedge K^{-\infty}R$$

Proof. By Lemma 2.8,

$$(\mathscr{B}(E\times(0,1],E;R)^{E})^{\Gamma}\cong \mathscr{B}(E/\Gamma\times(0,1],E/\Gamma;R)^{E/\Gamma}.$$

Lemma 1.32 shows that

$$\mathscr{B}(E/\Gamma \times (0,1], E/\Gamma; R)^{E/\Gamma} \cong \mathscr{B}(C(E/\Gamma), E/\Gamma \cup *; R)^{E/\Gamma}$$

and Lemma 1.31 shows that

$$K^{-\infty}(\mathscr{B}(C(E/\Gamma), E/\Gamma \cup *; R)^{E/\Gamma}) \simeq K^{-\infty}(\mathscr{B}(C(E/\Gamma), E/\Gamma \cup *; R)).$$

Finally, Theorem 1.36 shows that

$$\Omega K^{-\infty}(\mathscr{B}(C(E/\Gamma), E/\Gamma \cup *; R)) \simeq (E/\Gamma)_+ \wedge K^{-\infty}R.$$

COROLLARY 2.10. $S^{\Gamma} \cong (E/\Gamma)_+ \wedge K^{-\infty}(R)$.

Proof. Combine Lemmas 2.5 and 2.7 and Theorem 2.9.

THEOREM 2.11. Let E be a finite free Γ -CW complex. Then

$$K^{-\infty}(\mathscr{B}(E\times(0,1],E;R)^E)^{\Gamma} \to K^{-\infty}(\mathscr{B}(E\times(0,1],E;R)^E)^{h\Gamma}$$

is a weak homotopy equivalence.

Proof. The proof is by induction on the Γ -cells of E. To start the induction assume $E = \Gamma$. In this case the category $\mathscr{B}(E \times (0, 1], E; R)^E$ is equivalent to the product category $\prod_{g \in \Gamma} \mathscr{B}(g \times (0, 1], g; R)^g$. This follows since we take germs at E and demand control at points of E, so near E a component of a map cannot reach from one element in $E = \Gamma$ to another. We have

 $(\mathscr{B}(E \times (0, 1], E; R)^{E})^{\Gamma} = \mathscr{B}((0, 1], 1; R)^{1}$

by Lemma 2.8. The projection maps induce a map

$$K^{-\infty}\left(\prod_{g\in T}\mathscr{B}(g\times(0,1],g;R)^g\right)\to\prod_{g\in \Gamma}K^{-\infty}(\mathscr{B}(g\times(0,1],g;R)^g)$$

which is Γ -equivariant, in the action that permutes the factors, and a weak homotopy equivalence by [6]. The fixed set on the product is the diagonal, which may be identified with $K^{-\infty}(\mathscr{B}((0,1],1;R)^1))$. Under this identification the map of fixed sets

$$K^{-\infty}(\mathscr{B}(E\times(0,1],E;R)^{E})^{\Gamma} \to \left(\prod_{g\in\Gamma} K^{-\infty}(\mathscr{B}(g\times(0,1],g;R)^{g})\right)^{\Gamma}$$

is the identity. On homotopy fixed sets we get a weak homotopy equivalence since the map is equivariant, and unequivariantly it is a weak homotopy equivalence. Finally,

$$\left(\prod_{\Gamma} K^{-\infty}(\mathscr{B}((0,1],1;R)^{1})\right)^{\Gamma} \to \left(\prod_{\Gamma} K^{-\infty}(\mathscr{B}((0,1],1;R)^{1})\right)^{k_{\Gamma}}$$

is a weak homotopy equivalence, see Remark 2.2. Assume inductively that E is obtained from N by attaching one free G-cell, e^n . Consider

$$\mathscr{B}(N\times(0,1],N;R)^{N} \xrightarrow{j} \mathscr{B}(E\times(0,1],E;R)^{E} \xrightarrow{\theta} \mathscr{B}(E\times(0,1],E;R)^{E-N}.$$

Let's denote $\Omega K^{-\infty}$ applied to this sequence by $A \to B \to C$. The sequence $A^{\Gamma} \to B^{\Gamma} \to C^{\Gamma}$ is

$$(N/\Gamma)_+ \wedge K^{-\infty} R \rightarrow (E/\Gamma)_+ \wedge K^{-\infty} R \rightarrow (E/\Gamma)/(N/\Gamma) \wedge K^{-\infty} R$$

by Theorem 2.9 and Remark 1.38, hence a fibration of spectra. The composite functor gf factors through $\mathscr{B}(N \times (0, 1], N; R)^{\emptyset}$ which is equivariantly equivalent to the trivial category. Unequivariantly $A \to B \to C$ is a homotopy fibration (a Steenrod functor applied to $N_+ \to E_+ \to E_+/N_+$). Letting D denote the homotopy fibre of $B \to C$ we get an equivariant map from $A \to D$ which unequivariantly is a weak homotopy equivalence, hence a weak homotopy equivalence of homotopy fixed sets. Since homotopy fibre and homotopy fixed sets commute we have shown that

$$A^{h\Gamma} \to B^{h\Gamma} \to C^{h\Gamma}$$

is a fibration sequence. By our induction hypothesis we have $A^{\Gamma} \to A^{h\Gamma}$ is a weak homotopy equivalence, so to finish off the proof it suffices to show that $C^{\Gamma} \to C^{h\Gamma}$ is a weak homotopy equivalence. However, this is entirely similar to the start of the induction. Since $E - N = \Gamma \times e^n$ where e^n is an open *n*-cell and we are considering germ categories we have

$$\mathscr{B}(E \times (0,1], E; R)^{E-N} \cong \prod \mathscr{B}(g \cdot e^n \times (0,1], g \cdot e^n; R)^{g \cdot e^n}$$

and the Γ -action permutes the factors, so this completes the induction step.

Corollary 2.12. $S^{\Gamma} \cong S^{h\Gamma}$.

Proof. Consider the equivariant functor combining taking germs and forgetting control

$$\mathscr{B}(CX, CY \cup X, p_X; R) \to \mathscr{B}(CX, CY \cup X, p_X; R)^E \to \mathscr{B}(E \times \{0, 1\}, E; R)^E$$

We want to show that when we apply $K^{-\infty}$ and take fixed and homotopy fixed sets, respectively, we get a weak homotopy equivalence. On fixed sets this follows from Lemmas 2.5 and 2.4. On homotopy fixed sets it follows since the functor is equivariant, and unequivariantly it induces a weak homotopy equivalence by Lemma 2.5. The proof is now finished by Theorem 2.11.

So far we have not used the assumption that X is contractible. However, that is needed to show that $S^{h\Gamma} \to T^{h\Gamma}$ is a weak homotopy equivalence. The proof of Theorem 2.1 will be completed once we have the following theorem.

Тнеогем 2.13.

$$K^{-\infty}(\mathscr{B}(CX, CY \cup X, p_X; R))^{h\Gamma} \to K^{-\infty}(\mathscr{B}(\Sigma X, \Sigma Y, p_X; R))^{h\Gamma}$$

is a weak homotopy equivalence.

Proof. Consider the diagram

$$\mathscr{B}(CX, CY \cup X; R) \xrightarrow{b} \mathscr{B}(\Sigma X, \Sigma Y; R)$$

$$\downarrow^{a} \qquad \qquad \downarrow^{c}$$

$$\mathscr{B}(CX, CY \cup X, p_{X}; R) \xrightarrow{d} \mathscr{B}(\Sigma X, \Sigma Y, p_{X}; R).$$

We need to show that d induces a weak homotopy equivalence when applying $K^{-\infty}$ and taking homotopy fixed sets. Since d is equivariant it suffices to show that d induces a weak homotopy equivalence unequivariantly, we show this by showing that a, b, and c induce weak homotopy equivalences. We can conclude that a is a weak homotopy equivalence by the following three reasons. First

$$K^{-\infty}(\mathscr{B}(CX, CY \cup X; \mathbb{R})) \to K^{-\infty}(\mathscr{B}(CX, CY \cup X; \mathbb{R})^{\mathbb{E}})$$

is a Steenrod functor applied to collapsing the contractible space CY, see Remark 1.38 and Theorem 1.25. Second, we have a weak homotopy equivalence

$$K^{-\infty}\mathscr{B}(CX, CY \cup X, p_X; \mathbb{R}) \to K^{-\infty}(\mathscr{B}(CX, CY \cup X, p_X; \mathbb{R})^{\mathbb{E}}).$$

by Lemma 2.5. Third, the categories $\mathscr{B}(CX, CY \cup X; R)^{E}$ and $\mathscr{B}(CX, CY \cup X, p_{X}; R)^{E}$ are equal by Lemma 2.4. Similarly, c induces a weak homotopy equivalence because $K^{-\infty}$ applied to $\mathscr{B}(\Sigma X, \Sigma Y; R)$ is a deloop of $K^{-\infty}(\mathscr{B}(X, Y; R))$ by Theorems 1.23 and 1.36, but so is $K^{-\infty}(\mathscr{B}(\Sigma X, \Sigma Y; P_{X}; R))$ by an application of Theorem 1.28. Finally, b induces a weak homotopy equivalence because it is a Steenrod functor applied to collapsing the contractible subset X by Theorem 1.37. This completes the proof of Theorem 2.1.

3. IDENTIFYING THE ASSEMBLY MAP

In the previous section we have described a map

$$B\Gamma_+ \wedge K^{-\infty}(R) \to K^{-\infty}(R[\Gamma]).$$

We called it the assembly map. In this section we justify this by proving the map is the same as what is usually called the assembly map. This is based on results of Weiss and Williams [27]. To describe the result we need to recall some definitions from [27]. Let F be a homotopy invariant functor from finite complexes to spectra, sending the empty set to a contractible space. The functor F is called excisive if it sends a homotopy pushout of spaces to a homotopy pushout of spectra (i.e. if $\pi_*(F(B))$ is a homology theory). Weiss-Williams prove the following theorem.

THEOREM 3.1. Let G be a homotopy functor from finite CW-complexes to spectra, such that $G(\emptyset)$ is contractible. Then there is an excisive functor $G^{\%}$ and a natural transformation $G^{\%}(B) \to G(B)$, which is the identity on a point. If $F(B) \to G(B)$ is a natural transformation from an excisive functor F which is a homotopy equivalence on a point, then $F(B) \simeq G^{\%}(B)$ by a homotopy equivalence making $F(B) \simeq G^{\%}(B) \to G(B)$ the given natural transformation. Furthermore, $F(B) \simeq G^{\%}(B) \simeq B_+ \wedge F(*)$.

The natural transformation $F(B) \rightarrow G(B)$ is called an assembly map, and the theorem thus says that a homotopy invariant functor has an essentially unique assembly map.

To apply this theorem, we need to describe two functors F and G. Let

$$F(B) = \Omega K^{-\infty} (\mathscr{B}(B \times (0, 1], B; R)^B).$$

Then F on the category of finite complexes is an excisive homotopy invariant functor to the category of spectra. We define G as follows. Let B be a finite complex, E a universal covering space for B, and Γ the group of covering transformations (so Γ is isomorphic to the fundamental group of B). Give E a length metric induced from B and let q be the projection on the second factor $E \times (0, 1] \rightarrow (0, 1]$. Notice that if X is a compactification of E satisfying conditions (i) and (iv) of Theorem 2.1, then $(\mathscr{B}(E \times (0, 1], E, q; R)^E)^{\Gamma}$ and $(\mathscr{B}(CX, CY \cup Y, p_X; R)^E)^{\Gamma}$ are the same subcategories of $\mathscr{B}(E \times (0, 1))$. Hence, we may try to define G by

$$G(B) = \Omega K^{-\infty}((\mathscr{B}(E \times (0, 1], E, q; R)^E)^{\Gamma}).$$

The problem is that the assignment of E to B is not functorial without a basepoint. This however is solved by the following lemma.

LEMMA 3.2. Let $p: E \to B$ be a universal covering and $g: E \to E$ a map so that pg = p. Then the induced map on $\mathscr{B}(E \times (0, 1], E, q; R)^E$ fixes $(\mathscr{B}(E \times (0, 1], E, q; R)^E)^{\Gamma}$.

Proof. By covering space theory, g is multiplication by some element γ . Hence g is equivariant with respect to the Γ -action and the Γ -action conjugated by γ , but the fixed category with respect to the Γ -action and the conjugated Γ -action is the same.

We thus do have that G is a functor of B. Applying $\Omega K^{-\infty}$ to the equivalence

$$\mathscr{B}(B \times \{0, 1\}, B; R)^B \cong (\mathscr{B}(E \times \{0, 1\}, E; R)^E)^{\Gamma}$$

followed by the forget control map

$$(\mathscr{B}(E \times (0, 1], E; R)^E)^{\Gamma} \rightarrow (\mathscr{B}(E \times (0, 1], E, q; R)^E)^{\Gamma}$$

gives a natural transformation $F(B) \rightarrow G(B)$ which by the above mentioned results of Weiss-Williams is the assembly map, since it is the identity on a point.

Remark 3.3. In the above discussion G(B) is homotopy equivalent to $K^{-\infty}(R[\Gamma])$. This identification is of course not independent of choice of basepoint.

Remark 3.4. The category $\mathscr{B}(E \times \{0, 1\}, E, q; R)$ is easily seen to be flasque, whenever E is noncompact, say $E = E\Gamma$ with $B\Gamma$ finite. It follows that $\mathscr{B}(E \times \{0, 1\}, E, q; R)^E$ has trivial K-theory since $\mathscr{B}(E \times \{0, 1\}, E, q; R)_{\emptyset}$ is also flasque. Hence, we obtain that whenever we can find a Γ -equivariant subcategory $\mathscr{U} \subset \mathscr{B}(E \times \{0, 1\}, E, q; R)^E$ containing $(\mathscr{B}(E \times \{0, 1\}, E, q; R)^E)^\Gamma$ such that $\mathscr{B}(E \times \{0, 1\}, E; R)^E \to \mathscr{B}(E \times \{0, 1\}, E, q; R)^E$ factors through \mathscr{U} by a functor inducing isomorphism in K-theory, we get a splitting of the assembly map. In this paper we choose \mathscr{U} by continuous control in a compactification of $E\Gamma$. One may always try to let \mathscr{U} be the subcategory where morphisms also are required to be bounded. This remark recovers a result from [5], saying that if bounded K-theory with labels in $E\Gamma$ is isomorphic to $h^{l. f.}(E\Gamma; K^{-\infty}(R))$, we get a splitting of the assembly map.

4. L-THEORY OF ADDITIVE CATEGORIES

In this section we recall and expand various results due to Ranicki. Specifically, we prove the following theorem.

THEOREM 4.1. Let \mathcal{U} be an \mathscr{A} -filtered additive category with involution preserving \mathscr{A} , thus inducing an involution on \mathscr{A} . Let K be the inverse image of $K_0(\mathcal{U})$ under $K_0(\mathscr{A}^{\wedge}) \rightarrow K_0(\mathcal{U}^{\wedge})$. Then up to homotopy there is a fibration of spectra

$$\mathbb{L}^{h}(\mathscr{A}^{\wedge K}) \to \mathbb{L}^{h}(\mathscr{U}) \to \mathbb{L}^{h}(\mathscr{U}/\mathscr{A}).$$

THEOREM 4.2. Let \mathcal{U} be an \mathscr{A} -filtered additive category with involution preserving \mathscr{A} . Then up to homotopy there is a fibration of spectra

$$\mathbb{L}^{-\infty}(\mathscr{A}) \to \mathbb{L}^{-\infty}(\mathscr{U}) \to \mathbb{L}^{-\infty}(\mathscr{U}/\mathscr{A}).$$

To explain the terms in these theorems we recall Ranicki's *L*-theory of additive categories with involution. Let \mathscr{U} be an additive category. We shall consider chain complexes U_* in \mathscr{U} . If not otherwise stated, the chain complexes are always supposed to be finite, i.e. $U_i = 0$ except for finitely many *i*, but we do not require $U_i = 0$ in i < 0.

An additive category with involution \mathcal{U} , is an additive category together with a contravariant functor $*: \mathcal{U} \to \mathcal{U}$, sending U to U*, and a natural equivalence $** \cong 1$. We shall be working with small categories and in our applications ** will be equal to the identity.

Given a chain complex U_* , the *n*-dual chain complex U^{n-*} is defined by $(U^{n-*})_r = U^{n-r} = (U_{n-r})^*$, and boundary $(-1)^r$ times * applied to the boundary map. If U_* and V_* are two chain complexes, $\operatorname{Hom}(U_*, V_*)$ is the chain complex of abelian groups which in degree r is $\bigoplus_{p+q=r} \operatorname{Hom}(U_p, V_q)$.

Let U_* be a chain complex, then $\operatorname{Hom}(U^*, U_*)$ has a \mathbb{Z}_2 action given by $T(f) = (-1)^{pq} f^*, f: U^p \to U_q$ and T the nontrivial element in \mathbb{Z}_2 . Notice an *n*-cycle in $\operatorname{Hom}(U^*, U_*)$ is a chain map $\phi: U^{n-*} \to U_*$.

Let W be the standard $\mathbb{Z}[\mathbb{Z}_2]$ -module resolution of \mathbb{Z} :

$$W: \cdots \to \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2].$$

Define the $\mathbb{Z}[\mathbb{Z}_2]$ -module chain complex

$$W_{\mathcal{V}}U = W \otimes_{\mathbb{Z}[\mathbb{Z}_2]}(\operatorname{Hom}(U^*, U_*)).$$

An *n*-chain ψ is a collection $\{\psi_s \in \text{Hom}(U^*, U_*)_{n-s} | s \ge 0\}$, so with an *n*-cycle ψ comes a chain map $\psi_0: U^{n-*} \to U_*$. Ranicki defines an *n*-dimensional quadratic complex in \mathcal{U} , (U_*, ψ) to consist of a chain complex U_* in \mathcal{U} and an *n*-cycle $\psi \in W_{\%}U_*$. The quadratic complex is called *Poincaré* if the chain map $(1 + T)\psi_0: U^{n-*} \to U_*$ is a chain homotopy equivalence. Similarly, Ranicki defines quadratic pairs, and quadratic Poincaré pairs, so that bordism and gluing operations are defined as in a manifold category. The *n*-dimensional quadratic L^h -group $L_n^h(\mathcal{U})$ is defined to be the cobordism group of *n*-dimensional quadratic Poincaré complexes. Notice that the (n + 4)-dual of the double suspension of a chain complex U_* , is the double suspension of the *n*-dual of U_* , and this together with the 2-periodicity of W gives a 1-1 correspondence between *n*-dimensional and (n + 4)-dimensional Poincaré complexes. This correspondence is called double skew suspension, and it works in all dimensions (including *n* negative).

Important for our purposes is that to a quadratic *n*-dimensional complex, a pair is assigned so that the pair is Poincaré specifically [23, Proposition 13.1].

PROPOSITION 4.3. There is a natural one-one correspondence between the homotopy equivalence classes of n-dimensional Poincaré pairs in \mathcal{U} and the homotopy equivalence classes of n-dimensional quadratic complexes in \mathcal{U} .

This correspondence is given by sending the pair to the algebraic mapping cone. Roughly speaking, the algebraic mapping cone of the duality map of an *n*-dimensional quadratic complex is a null cobordant (n - 1)-dimensional Poincaré complex.

Ranicki relates this categorical approach to the usual approach in the following theorems. Let R be a ring with involution.

THEOREM 4.4. If \mathcal{U} is the category of f.g. free R-modules, and involution given by duality, then $L_n^h(\mathcal{U})$ coincides with the usual L-groups $L_n^h(R)$.

Idempotent completion enters as related to L^{p} -groups.

THEOREM 4.5. Let \mathscr{U} be as above. If K is an involution invariant subgroup of $K_0(\mathscr{U}^{\wedge})$, then $L_n^h(\mathscr{U}^{\wedge K}) = L_n^K(R)$.

The L-groups being defined as bordism groups, Ranicki [22] defines a (Kan)- Δ -set $\mathbb{L}_{i}^{h}(\mathscr{U})$ where the *n*-simplexes are *n*-ads of (n + i)-dimensional Poincaré complexes. This Δ -set is naturally based by the 0-chain complex. A Δ -set model for the circle has one 0-simplex, the basepoint, and two 1-simplexes, the basepoint and one more simplex. The Δ -set model for the loop space is determined by what that one-simplex is sent to, so

$$\Omega \mathbb{L}_i^h(\mathscr{U}) = L_{i+1}^h(\mathscr{U})$$

in the world of Δ -sets. Upon realization of the Δ -set there is a map

$$|\Omega \mathbb{L}_{i}^{h}(\mathscr{U})| \to \Omega |\mathbb{L}_{i}^{h}(\mathscr{U})|$$

which is a homotopy equivalence. The skew double suspension

$$\mathbb{L}^h_i(\mathscr{U}) \to \mathbb{L}^h_{i+4}(\mathscr{U})$$

which is the double suspension of the chain complex, and the identity on the quadratic structure gives an isomorphism of Δ -sets. In Ranicki's original treatment of algebraic Poincaré complexes an *n*-dimensional Poincaré complex was assumed to be concentrated between dimensions 0 and *n*. By giving that up, and only assuming chain complexes to be finite [22] the 4-periodicity becomes exact, and negative dimensional Poincaré complexes make perfectly good sense. Realization of the Δ -sets $\mathbb{L}_i^h(\mathscr{U})$ now gives a four periodic spectrum. We denote this spectrum, with homotopy groups, the *L*-groups of \mathscr{U} functorially assigned to \mathscr{U} by $\mathbb{L}^h(\mathscr{U})$. We wish to establish the analogue of Theorem 1.28 for this spectrum.

Definition 4.6. Let \mathscr{A} be a full subcategory of \mathscr{U} . A chain complex U_* is \mathscr{A} -dominated if there is a chain complex C_* in \mathscr{A} and chain maps $r: C_* \to U_*$ and $i: U_* \to C_*$ such that ri is chain homotopic to the identity.

Recall that the idempotent completion \mathscr{U}^{\wedge} of an additive category \mathscr{U} has objects (U, p), $p: U \to U$, $p^2 = p$ and $\operatorname{Hom}((U, p), (V, q)) \subset \operatorname{Hom}(U, V)$, the subset for which $q\phi p = \phi$. If $K \subset K_0(\mathscr{U}^{\wedge})$ is a subgroup, we denote the full subcategory of \mathscr{U}^{\wedge} with objects (A, p) so that the stable isomorphism class of (A, p) lies in K by $\mathscr{U}^{\wedge K}$.

We have an inclusion of \mathscr{U} in \mathscr{U}^{\wedge} as a full subcategory sending A to $(A, 1_A)$. Clearly this inclusion factors through $\mathscr{U}^{\wedge K_0(\mathscr{U})}$.

PROPOSITION 4.7. Let \mathcal{U} be an \mathscr{A} -filtered category. A chain complex U_* in \mathcal{U} is \mathscr{A} -dominated if and only if the induced \mathcal{U}/\mathscr{A} -chain complex is contractible.

Proof. Assume U_* is \mathscr{A} -dominated by C_* . The induced \mathscr{U}/\mathscr{A} -complex of C_* is isomorphic to the 0-chain complex, hence U_* is \mathscr{U}/\mathscr{A} -homotopy equivalent to the 0-chain complex, so contractible. If U_* is \mathscr{A}/\mathscr{U} -contractible, we have maps $\overline{\Gamma}_i: U_i \to U_{i+1}$ in \mathscr{U}/\mathscr{A} so that $d_{i+1}\overline{\Gamma}_i + \overline{\Gamma}_{i-1}d_i = 1$ in \mathscr{U}/\mathscr{A} . Choose representatives Γ_i in \mathscr{U} for $\overline{\Gamma}_i$. Assume $U_i = 0$ for i > n. The map $1 - \Gamma_{n-1}d_n: U_n \to U_n$ factors through \mathscr{A} , hence (since \mathscr{U} is \mathscr{A} -filtered) we may choose $U_n = A_n \oplus V_n$ so that $1 - \Gamma_{n-1}d_n$ factors through A_n . Next decompose $U_{n-1} = A_{n-1} \oplus V_{n-1}$ so that $d_n | A_n$ factors through A_{n-1} and $1 - d_n \Gamma_{n-1} - \Gamma_{n-2}d_{n-1}$: $U_{n-1} \to U_{n-1}$ factors through A_{n-1} . Continuing this process we get a chain complex A_* in \mathscr{A} which we claim dominates U_* . We define $r: A_* \to U_*$ to be the inclusion and $i: U_* \to A_*$ to be $1 - \Gamma d - d\Gamma$. It is easy to see that i is a chain map and $ri = 1 - \Gamma d - d\Gamma$ so Γ is a chain homotopy from ri to the identity.

This is an algebraic analogue of the connection established in [25] between a proper homotopy equivalence and a finitely dominated space.

We need the following lemma from [24].

LEMMA 4.8. Let \mathscr{A} be a full subcategory of \mathscr{U} , U_* an \mathscr{A} -dominated chain complex in \mathscr{U} . Let K be the inverse image of $K_0(\mathscr{U})$ under the induced map $K_0(\mathscr{A}^{\wedge}) \to K_0(\mathscr{U}^{\wedge})$, and let $\mathscr{U}^{\wedge K}$ be the full subcategory of \mathscr{U}^{\wedge} with objects $(A, p) \oplus U$, $[(A, p)] \in K$. Then the induced chain complex in $\mathscr{U}^{\wedge K}$, under the inclusion $\mathscr{U} \to \mathscr{U}^{\wedge K}$ is chain homotopy equivalent to a chain complex in $\mathscr{A}^{\wedge K}$.

Proof. The explicit formulas in [24] show that U_* is homotopy equivalent to an infinite chain complex of the form

$$\cdots A \xrightarrow{p} A \xrightarrow{1-p} A \xrightarrow{p} A \xrightarrow{1-p} A \xrightarrow{p} \xrightarrow{p} A \xrightarrow$$

But in \mathscr{A}^{\wedge} this is homotopy equivalent to

$$0 \rightarrow (A, p) \rightarrow (A_{n-1}, 1) \rightarrow \cdots$$

and $[(A, p)] \in K_0(\mathscr{A}^{\wedge})$ must map to an element of $K_0(\mathscr{U})$.

We may now prove the main theorem of this section. Special cases of this theorem have been proved by Ranicki.

Proof of Theorem 4.1. Let $\mathscr{U} \wedge {}^{K}$ be the full subcategory of $\mathscr{U} \wedge$ with objects $(A, p) \oplus U$, where (A, p) is an object of $\mathscr{A} \wedge {}^{K}$. Clearly $\mathscr{U} \wedge {}^{K}$ is $\mathscr{A} \wedge {}^{K}$ -filtered, and $\mathscr{U} \wedge {}^{K}/\mathscr{A} \wedge {}^{K}$ is equivalent to \mathscr{U}/\mathscr{A} . By the bordism approach to L-theory we get a fibration

$$\mathbb{L}^{h}(\mathscr{A}^{\wedge K}) \to \mathbb{L}^{h}(\mathscr{U}^{\wedge K}) \to \mathbb{L}^{h}(\mathscr{A}^{\wedge K} \to \mathscr{U}^{\wedge K}).$$

We have $\mathbb{L}^{h}(\mathcal{U}) \to \mathbb{L}^{h}(\mathcal{U} \wedge K)$ is a weak homotopy equivalence since any chain complex in $\mathcal{U} \wedge K$ is homotopy equivalent to a chain complex in \mathcal{U} . Given a Poincaré pair $((A_{*}, \psi_{A}) \to (U_{*}, \psi_{U}))$ with A_{*} in $\mathscr{A} \wedge K$ and U_{*} in $\mathscr{U} \wedge K$, the quadratic chain complex (U_{*}, ψ_{U}) in $\mathscr{U} \wedge K/\mathscr{A} \wedge K$ is Poincaré, since chain complexes in $\mathscr{A} \wedge K$ are equivalent to the 0-chain complex in $\mathscr{U} \wedge K/\mathscr{A} \wedge K$. Applying this to *n*-ads of Poincaré pairs induces a map $\mathbb{L}(\mathscr{A} \wedge K \to \mathscr{U} \wedge K) \to \mathbb{L}(\mathscr{U}/\mathscr{A})$. Given a chain complex \overline{U}_{*} in \mathscr{U}/\mathscr{A} , we want to find a chain complex U_{*} in \mathscr{U} such that the induced chain complex in \mathscr{U}/\mathscr{A} is isomorphic to \overline{U}_{*} . Choose representatives $\overline{d}_{i}: \overline{U}_{i} \to \overline{U}_{i-1}$ in \mathscr{U} for the boundary morphisms. This may not be a chain

complex because $\bar{d}_{i-1}\bar{d}_i$ is not necessarily 0. It factors through \mathscr{A} , however, so we define U_* inductively by $U_0 = \bar{U}_0$, $U_1 = \bar{U}_1$, and $d_1 = \bar{d}_1$. Write $\bar{U}_2 = A_2 \oplus U_2$ so that $\bar{d}_2 d_1$ factors $U_2 \oplus A_2 \to A_2 \to \bar{U}_0$, and let d_2 be $U_2 \to U_2 \oplus A_2 \xrightarrow{d_2} U_1$. Then $d_2 d_1 = 0$ and renaming \bar{d}_3 to be the composite $\bar{U}_3 \to \bar{U}_2 \to U_2$, we may decompose \bar{U}_3 similarly, and by a finite induction we get the chain complex U_* . Noting that the projection maps $\bar{U}_i \to U_i$ induce isomorphisms in \mathscr{U}/\mathscr{A} , it is clear that the induced chain complex in \mathscr{U}/\mathscr{A} is isomorphic to \bar{U}_* . Similarly, a quadratic Poincaré structure $\bar{\psi}$ on \bar{U}_* lifts to a quadratic structure on U_* , by choosing representatives, but not necessarily to a quadratic Poincaré structure. Since quadratic complexes are in 1–1 correspondence with Poincaré pairs (the correspondence being the algebraic mapping cone) there is a Poincaré pair $(A_* \to V_*)$ such that the algebraic mapping cone is homotopy equivalent to U_* . In \mathscr{U}/\mathscr{A} , U_* is Poincaré which is the same as saying that A_* is contractible by Proposition 4.7. That in turn by Lemma 4.8 means that A_* is homotopy equivalent to a chain complex in $\mathscr{A} \wedge K$, so we have found a Poincaré pair in $\mathscr{A} \wedge K \to \mathscr{U} \wedge K$ mapping to \bar{U}_* , hence

$$\mathbb{L}(\mathscr{A}^{\wedge K} \to \mathscr{U}^{\wedge K}) \to \mathbb{L}(\mathscr{U}/\mathscr{A})$$

is an epimorphism on homotopy groups. A relative version of this argument shows it is a monomorphism.

So far we have not discussed torsion. Following Ranicki, given an additive category \mathscr{U} with involution, and a system of stable isomorphisms $\phi_{X,Y}: X \to Y$ so that any composite which happens to be an automorphism represents $0 \in K_1(\mathscr{U})$, and given a *-invariant subgroup S of $K_1(\mathscr{U})$, one may define the groups $L_n^S(\mathscr{A})$ and the simple \mathbb{L} -spectra $\mathbb{L}_i^S(\mathscr{A})$. Slightly more generally, if we only require that any composite which happens to be an automorphism lies in a *-invariant subgroup H of $K_1(\mathscr{A})$, $\mathbb{L}_i^S(\mathscr{A})$ may be defined for any *-invariant subgroup such that $H \subset S$. By methods entirely analogous to the above one obtains the following.

THEOREM 4.9. Let \mathcal{U} be an \mathcal{A} -filtered additive category with involution. Assume \mathcal{A} and \mathcal{U} have compatible systems of stable isomorphisms. Then we have a quasifibration

$$\mathbb{L}^{s}(\mathscr{A}) \to \mathbb{L}^{s}(\mathscr{U}) \to \mathbb{L}^{K}(\mathscr{U}/\mathscr{A})$$

where $K = \text{Im}(K_1(\mathcal{U}) \to K_1(\mathcal{U}/\mathcal{A}))$, and s refers to the trivial subgroup.

Remark 4.10. If E is non-compact, it is usually quite easy to find a system of stable isomorphisms for the various subcategories of $\mathscr{B}(E; R)$, by Eilenberg swindle on the *objects*.

In the above remark it is important to note that we only have Eilenberg swindle on the objects, not a functorial Eilenberg swindle, since this would imply that the *L*-theory of the category vanishes, as seen in the next lemma.

Definition 4.11. An additive category \mathscr{A} with involution is flasque if there is a functor $\Sigma^{\infty}: \mathscr{A} \to \mathscr{A}$ and natural equivalences $\Sigma^{\infty} * \cong *\Sigma^{\infty}$ and $1 \oplus \Sigma^{\infty} \cong \Sigma^{\infty}$.

LEMMA 4.12. If \mathscr{A} is flasque then $L_i(\mathscr{A}) = 0$.

Proof. Let (A_*, v) be an element of $L_i(\mathcal{A})$. Then

$$(\Sigma^{\infty}A_{*},\Sigma^{\infty}v)\cong(\Sigma^{\infty}A_{*}\oplus A,\Sigma^{\infty}v\oplus v)$$

hence $(A_*, v) = 0$.

Let $\mathscr{C}_i(-)$ be the functor from small additive categories to itself with objects parameterized by the \mathbb{R}^i , and bounded morphisms. We recover a theorem due to Ranicki.

THEOREM 4.13.

$$L_{n+1}^{s}(\mathscr{C}_{i+1}(\mathscr{U})) \cong L_{n}^{h}(\mathscr{C}_{i}(\mathscr{U}))$$

and

$$L_{n+1}^{h}(\mathscr{C}_{i+1}(\mathscr{U})) \cong L_{n}^{h}(\mathscr{C}_{i}(\mathscr{U})^{\wedge}).$$

In particular,

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L_{n+1}^{s}(\mathscr{C}_{1}(\mathscr{U})) \cong L_{n}^{h}(\mathscr{U})
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and

 $L_{n+1}^{h}(\mathscr{C}_{1}(\mathscr{U}))\cong L_{n}^{h}(\mathscr{U}^{\wedge}).$

Proof. We prove the particular case, since the general case is entirely similar. Let \mathscr{A} be the full subcategory of $\mathscr{C}_1(\mathscr{U})$ with objects A such that there is an n with $A_i = 0$ for i > n sufficiently large. Clearly, \mathscr{A} is flasque so

$$\mathrm{Im}(K_1(\mathscr{U})) \to K_1(\mathscr{C}_1(\mathscr{U}/\mathscr{A})) = K_1(\mathscr{C}_1(\mathscr{U}/\mathscr{A}))$$

and the quasifibration

$$\mathbb{L}^{h}(\mathscr{A}) \to \mathbb{L}^{h}(\mathscr{C}_{1}(\mathscr{U})) \to \mathbb{L}^{h}(\mathscr{C}_{1}(\mathscr{U}/\mathscr{A}))$$

shows that $L_n^h(\mathscr{C}_1(\mathscr{U})) \cong L_n^h(\mathscr{C}_1(\mathscr{U})/\mathscr{A})$. Let $\mathscr{C}_+(\mathscr{U})$ be the full subcategory of $\mathscr{C}_1(\mathscr{U})$ with objects A such that there is an n with $A_i = 0$ for i < n. The category $\mathscr{C}_+(\mathscr{U})$ is flasque, and $\mathscr{C}_+(\mathscr{U})$ is \mathscr{B} -filtered, where \mathscr{B} is the full subcategory of $\mathscr{C}_1(\mathscr{U})$ with objects that are 0 both at large positive and negative values of the point in \mathbb{R} . We thus get $L_n^h(\mathscr{B}) \cong L^s(\mathscr{C}_+(\mathscr{U})/\mathscr{B})$ but \mathscr{B} is equivalent to \mathscr{U} and $\mathscr{C}_+(\mathscr{U})/\mathscr{B}$ is equivalent to $\mathscr{C}_1(\mathscr{U})/\mathscr{A}$ and we are done.

Following Ranicki we make the following definition.

Definition 4.14. Let \mathscr{A} be an additive category with involution

 $L_n^{-i}(\mathscr{A}) = L_{n+i+2}^s(\mathscr{C}_{i+2}(\mathscr{A}))$

where we write s for -i when -i = 2, h for -i when -i = 1, and p for -i when -i = 0.

The point of Ranicki's definition is that he shows it agrees with the usual definitions in case \mathcal{A} is the category of finitely generated *R*-modules, *R* a ring with involution.

LEMMA 4.15. If a functor $f: \mathcal{A} \to \mathcal{B}$ of additive categories with involution induces isomorphism on K-theory (when restricted to isomorphisms) and on L-theory with one decoration, then it induces isomorphism on L-theory with any decoration.

Proof. This follows from Ranicki-Rothenberg exact sequences.

Let \mathscr{A} be an additive category and consider the inclusion

$$\mathscr{C}_i(\mathscr{A}) \subset \mathscr{C}_{i+1}(\mathscr{A}).$$

The map on Δ -sets

$$\mathbb{L}_{i}^{h}(\mathscr{C}_{i}(\mathscr{A})) \to \mathbb{L}_{i}^{h}(\mathscr{C}_{i+1}(\mathscr{A}))$$

is homotopic to the constant map in two ways: the cellular chain complex of ([0, ∞), 0) obtained by subdividing at integral lattice points



defines an element in symmetric L-theory

$$L^1(\mathscr{C}_0(\mathbb{Z}) \to \mathscr{C}_{[0,\infty)}(\mathbb{Z}))$$

and pairing with this element gives a map

$$L_{i}^{h}(\mathscr{C}_{i}(\mathscr{A})) \to \mathbb{L}_{i+1}^{h}(\mathscr{C}_{i+1}(\mathscr{A}))$$

which is a Δ -set homotopy from the inclusion to the constant map. (The element in $L^1(\mathscr{C}_0(\mathbb{Z}) \to \mathscr{C}_{[0,\infty)}(\mathbb{Z}))$ is actually the generator under the isomorphism $L^1(\mathscr{C}_0(\mathbb{Z}) \to \mathscr{C}_{[0,\infty)})(\mathbb{Z})) \cong L^0(\mathbb{Z}) \cong \mathbb{Z}$). Similarly, pairing with the chain complex of $[0, -\infty)$ gives a null homotopy, so combined we get a map

$$\Sigma \mathbb{L}^{h}(\mathscr{C}_{i}(\mathscr{A})) \to \mathbb{L}^{h}(\mathscr{C}_{i+1}(\mathscr{A}))$$

and hence

$$\mathbb{L}^{h}(\mathscr{C}_{i}(\mathscr{A})) \to \Omega \mathbb{L}^{h}(\mathscr{C}_{i+1}(\mathscr{A}))$$

and the composite

$$\mathbb{L}^{h}(\mathscr{C}_{i}(\mathscr{A})) \to \Omega \mathbb{L}^{h}(\mathscr{C}_{i+1}(\mathscr{A})) \to \Omega \mathbb{L}^{h}(\mathscr{C}_{i+1}(\mathscr{A})^{\wedge})$$

is a weak homotopy equivalence, since on homotopy groups this is exactly the map described in Theorem 4.13.

Definition 4.16. The spectrum $\mathbb{L}^{-\infty}(\mathscr{A})$ is defined to be the homotopy colimit of

 $\mathbb{L}^{h}(\mathscr{A}) \to \Omega \mathbb{L}^{h}(\mathscr{C}_{1}(\mathscr{A})) \to \Omega^{2} \mathbb{L}^{h}(\mathscr{C}_{2}(\mathscr{A})) \to \cdots$

It is clear that $\mathbb{L}^{-\infty}$ is a functor from the category of small additive categories and lax involution preserving functors to the category of spectra.

LEMMA 4.17. Let \mathscr{B} be an additive category. If \mathscr{A} is a full subcategory, inducing isomorphism on K_0 , then

$$\mathbb{L}(\mathscr{A}) \to \mathbb{L}(\mathscr{B})$$

is a weak homotopy equivalence (any decoration on the L-theory).

Proof. We need to show the map induces isomorphism on homotopy groups, but any chain complex in \mathscr{B} is homotopy equivalent to a chain complex in \mathscr{A} , so from this it follows that $L^h(\mathscr{A}) \to L^h(\mathscr{B})$ is an isomorphism, and since the inclusion induces isomorphism on K-theory Lemma 4.15 finishes off the proof.

Finally, we are ready for the proof of Theorem 4.2.

Proof of Theorem 4.2. We get this result using Theorem 4.1. For any additive category with involution \mathscr{C} we have $\mathbb{L}^{h}(\mathscr{C}) \simeq \mathbb{L}^{h}(\mathscr{C}^{\wedge 0})$ by Lemma 4.17 above. Let

$$K_i = \ker(K_0(\mathscr{C}_i(\mathscr{U})) \to K_0(\mathscr{C}_i(\mathscr{U}/\mathscr{A}))).$$

Using Theorem 4.1, we then have fibrations

$$\mathbb{L}^{h}(\mathscr{C}_{i}(\mathscr{A})^{\wedge K_{i}}) \to \mathbb{L}^{h}(\mathscr{C}_{i}(\mathscr{U})) \to \mathbb{L}^{h}(\mathscr{C}_{i}(\mathscr{U})/\mathscr{C}_{i}(\mathscr{A})).$$

Noting that $\mathscr{C}_i(\mathscr{U})/\mathscr{C}_i(\mathscr{A})$ is equivalent to $\mathscr{C}_i(\mathscr{U}/\mathscr{A})$ by an equivalence which is the identity on objects, the proof is completed by contemplating the diagram

which shows that taking homotopy colimit of the upper row is the same as taking homotopy colimit of the lower row.

5. THE L-THEORY SPLITTING

In this section we apply the results of the last section to obtain a splitting of the L-theory assembly map. Throughout this section R will be a ring with involution satisfying $K_{-i}(R) = 0$ for sufficiently large *i*. The proofs are extremely similar to the K-theory proofs, given the requisite techniques from the preceding section. First, however, we need to worry about the involution. Given an object of one of the categories $\mathscr{B}(X; R)$, $\mathscr{B}(X, Y), \ldots$, we denote the full subcategory with objects A such that A_x has a basis of the form $\{(x, i)\}$ by $\mathscr{B}_b(X; R), \mathscr{B}_b(X, Y; R), \ldots$ All the results of Section 1 hold for these categories since they are cofinal and have the same K_0 .

If R is a ring with involution, we get an involution on $\mathscr{B}_b(X; R)$ as follows. On objects, * is the identity. On a morphism $\phi: A \to B$, we define $\phi^*: B^* \to A^*$ by $(\phi^*)_y^x: B_x^* \to A_y^*$ is the map $B_x \to A_y$ with matrix, the conjugate transpose of $\phi_y^y: A_y \to B_x$. Here elements of the ring R are conjugated by the involution on the ring.

Remark 5.1. We are identifying A_x with its dual via the basis, and dual basis, and use the matrix description of the dual map.

We shall prove L-theory results similar to the K-theory results, replacing Theorem 1.28 by Theorems 4.1 and 4.2.

Notice equivalent additive categories (with equivalence preserving the involution) have the same *L*-theory.

As in K-theory we get the following lemma.

LEMMA 5.2. Let X be a compact metrizable space, Y a closed subspace, so that X - Y is dense in X. Let C be a closed subset of Y, W an open subset of Y with $C \subset W$. We then have fibrations up to homotopy

$$\mathbb{L}^{-\infty}(\mathscr{B}(X,Y;R)_{\mathcal{C}}) \to \mathbb{L}^{-\infty}(\mathscr{B}(X,Y;R)) \to \mathbb{L}^{-\infty}(\mathscr{B}(X,Y;R)^{Y-\mathcal{C}})$$

and

$$\mathbb{L}^{-\infty}(\mathscr{B}(X,Y;R)^{W}_{\mathcal{C}}) \to \mathbb{L}^{-\infty}(\mathscr{B}(X,Y;R)^{W}) \to \mathbb{L}^{-\infty}(\mathscr{B}(X,Y;R)^{W-C}).$$

Proof. Direct from Lemma 1.29 and Theorem 4.2.

LEMMA 5.3. Let X be a compact metrizable space, Y a closed subspace, so that X - Y is dense in X. The natural map

$$\mathscr{B}_b(X, Y; R) \to \mathscr{B}_b(X, Y; R)^{Y^-*}$$

induces an isomorphism in L-theory with any decoration.

Proof. The flasque structure on $\mathscr{B}_b(X, Y; R)_*$ preserves the involution, so this follows from Lemma 4.12 and Corollary 5.2.

THEOREM 5.4. Let X be a compact metrizable space, R a ring with involution. Then $\mathbb{L}_{*}^{-\infty}(\mathscr{B}_{b}(CX, X; \mathbf{R}))$ is a Steenrod functor, associated to the 4-periodic $\mathbb{L}^{-\infty}(\mathbf{R})$ -spectrum.

Proof. We need to verify the conditions of Definition 1.34. In [1] it is proved that the K-theory $\mathscr{B}(CCX, CX; R)$ is 0 by showing one can write $\mathscr{B}(CCX, CX; R)$ as the union of flasque categories. If (C_*, v) represents an element in $L_n^h(\mathscr{B}_b(CCX, CX; R))$, the (C_*, v) will have to be in one of these flasque categories, hence it must represent zero, so $L^h(\mathscr{B}_b(CCX, CX; R)) = 0$, and hence so is $L^{-\infty}(\mathscr{B}_b(CCX, CX; R))$ by Lemma 4.15. Assume \mathscr{A}_i are additive categories with involution satisfying that there exists *j* independent of *i*, so that $K_{-j}(\mathscr{A}_i) = 0$. It is clear that $\mathbb{L}_n^h(\Pi \mathscr{A}_i) \simeq \Pi \mathbb{L}_n^h(\mathscr{A}_i)$ as Δ -sets since a quadratic Poincaré complex is represented by a length one or two chain complex. Since these Δ -sets satisfy the Kan condition, we get a weak homotopy equivalence of spectra $\mathbb{L}^h(\Pi \mathscr{A}_i) \simeq \Pi \mathbb{L}^h(\mathscr{A}_i)$ upon realization. Combining with the K-theory result and Rothenberg-Ranicki exact sequences we get

$$\mathbb{L}^{-\infty}(\Pi \mathscr{A}_i) \simeq \Pi \mathbb{L}^{-\infty}(\mathscr{A}_i).$$

It thus follows that

$$\mathbb{L}^{-\infty}(\mathscr{B}_b(C(\backslash X_{\alpha}), \backslash X_{\alpha}; R)) \simeq \Pi \mathbb{L}^{-\infty}(\mathscr{B}_b(CX_{\alpha}, X_{\alpha}; R))$$

since the K-theory proof only involved flasque subcategories and equivalences of categories. Finally, using Theorem 4.2, we get a fibration

$$\mathbb{L}^{-\infty}(\mathscr{B}_b(CX,X;R)_A) \to \mathbb{L}^{-\infty}(\mathscr{B}_b(CX,X;R)) \to \mathbb{L}^{-\infty}(\mathscr{B}_b(CX,X;R)^{X-A})$$

but then the proof is finished as in Theorem 1.36, using Lemma 5.3.

We can now formulate and prove a splitting theorem in L-theory.

THEOREM 5.5 Let Γ be a group satisfying the conditions in the introduction, and let R be a ring with involution so that $K_{-i}(R) = 0$ for sufficiently large i. Then $B\Gamma_+ \wedge \mathbb{L}^{-\infty}(R)$ is a split factor of $\mathbb{L}^{-\infty}(R\Gamma)$.

Proof. Having provided the requisite technical tools in Theorems 4.2 and 5.4, the proof proceeds formally as in the K-theory case. Apply $\Omega L^{-\infty}$ to

$$\mathscr{B}(CX, CY \cup X, p_X; R) \rightarrow \mathscr{B}(\Sigma X, \Sigma Y, p_X; R)$$

and call the resulting spectra S and T. We now study the diagram

$$\begin{array}{cccc} S^{\Gamma} & \to & T^{\Gamma} \\ \downarrow & & \downarrow \\ S^{h\Gamma} & \to & T^{h\Gamma} \end{array}$$

and prove the statements

(i) $S^{\Gamma} \simeq (E/\Gamma)_{+} \wedge \mathbb{L}^{-\infty}(R)$, (ii) $iT^{\Gamma} \simeq \mathbb{L}^{-\infty}(R\Gamma)$, (iii) $S^{\Gamma} \simeq S^{h\Gamma}$, (iv) $S^{h\Gamma} \simeq T^{h\Gamma}$.

Statements (i) and (ii) follow from Theorem 4.2, (iii) is an inductive argument over the cells as in the K-theory case using Theorem 4.2, and the above mentioned fact that L-theory commutes with products. Finally, (iv) follows because $\mathbb{L}^{-\infty}$ is a Steenrod functor, and we are collapsing a contractible subspace.

Finally, we need to identify the splitting map with the usual assembly map.

THEOREM 5.6. The map of spectra $B\Gamma_+ \wedge \mathbb{L}^{-\infty}(R) \to \mathbb{L}^{-\infty}(R[\Gamma])$ is the usual assembly map.

Proof. This follows formally arguing as in Section 3.

This is needed to see that our results imply the Novikov conjecture for the class of groups considered, see [22, Proposition 24.5]. It also has the useful consequence that a diagram of assembly maps

$$B\Gamma_{+} \wedge \mathbb{L}^{q}(R) \rightarrow \mathbb{L}^{q}(R[\Gamma])$$

$$\downarrow \qquad \qquad \downarrow$$

$$B\Gamma_{+} \wedge \mathbb{L}^{-\infty}(R) \rightarrow \mathbb{L}^{-\infty}(R[\Gamma])$$

will be commutative. Here q denotes any decoration. In particular if the decoration is chosen so that $\mathbb{L}^{q}(R) \cong \mathbb{L}^{-\infty}(R)$ we obtain the splitting with other decorations than $-\infty$. This can also be proved more directly using Theorem 4.1.

Remark 5.7. The analogue of Remark 3.4 holds in L-theory as well.

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REFERENCES

- 1. D. R. ANDERSON, F. CONNOLLY, S. C. FERRY and E. K. PEDERSEN: Algebraic K-theory with continuous control at infinity, J. Pure Appl. Algebra 94 (1994), 25-47.
- 2. M. BESTVINA and G. MESS: The boundary of negatively curved groups. J. Amer. Math. Soc. 4 (1991), 469-481.
- 3. M. G. BÖKSTEDT, W. C. HSIANG and I. MADSEN: The cyclotomic trace and algebraic K-theory of spaces, *Invent. Math.* 111 (1993), 465–540.
- 4. S. E. CAPPELL: On homotopy invariance of higher signatures, Invent. Math. 33 (1976), 171-179.
- 5. G. CARLSSON: Bounded K-theory and the assembly map in algebraic K-theory I, to appear in Proc. Oberndfach Conf. on the Novikov conjecture S.C. Ferry, J. Rosenberg and A. A. Ranicki, Eds.
- 6. G. CARLSSON: On the algebraic K-theory of infinite product categories, Stanford University preprint, (1992).
- G. CARLSSON and E. K. PEDERSEN: Čech homology and the Novikov conjecture, Stanford University preprint, (1993).
- 8. G. CARLSSON, E. K. PEDERSEN and J. ROE: In Preparation, (1994).
- 9. G. CARLSSON, E. K. PEDERSEN and W. VEGELL: Continuously controlled algebraic K-theory of spaces and the Novikov conjecture, Report No. 17 1993/1994, Institut Mittag-Leffler, (1993).
- A. CONNES and MOSCOVICI: Cyclic cohomology, the Novikov conjecture, and hyperbolic groups, *Topology* 29 (1990), 345–388.
- 11. D. A. EDWARDS and H. M. HASTINGS: Čech and Steenrod homotopy theories with applications to geometric topology, Lecture Notes in Mathmatics, Vol. 542, Springer, Berlin (1976).
- F. T. FARRELL and W.-C. HSIANG: On Novikov's conjecture for nonpositively curved manifolds, I, Ann. Math. 113 (1981), 197-209.

- 13. F. T. FARRELL and L. E. JONES: K-theory and dynamics, Ann. Math. 124 (1986), 531-569.
- 14. S. C. FERRY and S. WEINBERGER: Curvature, tangentiality, and controlled topology, *Invent. Math.* 105 (1991), 401-414.
- D. S. KAHN, J. KAMINKER and C. SCHOCHET: Generalized homology theories on compact metric spaces, Michigan Math. J. 24 (1977), 203-224.
- 16. M. KAROUBI: Foncteur dérivés et K-théorie, Lecture Notes in Mathematics, 136, Springer-Verlag, Berlin (1970).
- 17. G. G. KASPAROV: Equivariant KK-theory and the Novikov conjecture, Invent. Math. 91 (1988), 147-201.
- 18. J. MILNOR: On the Steenrod homology theory, mimeographed, Berkeley (1960).
- 19. A. S. MISHCHENKO: Infinite dimensional representations of discrete groups and higher signatures, *Izv. Akad.* SSSR, Ser. Mat. 38 (1974), 81-106.
- E. K. PEDERSEN and C. WEIBEL: K-theory homology of spaces, in Algebraic Topology, Proceedings Arcata 1986, G. Carlsson, R. L. Cohen, H. R. Miller and D. C. Ravenel, Eds, Lecture Notes in Mathematics, Vol. 1370, Springer, Berlin, (1989), pp. 346-361.
- 21. F. QUINN: Algebraic K-theory of poly-(finite or cyclic) groups, Bull. Amer. Math. Soc. 12 (1985), 221-226.
- 22. A. A. RANICKI: Algebraic L-theory and topological manifolds, Cambridge Tracts in Mathematics, Vol. 102, Cambridge University Press, Cambridge (1992).
- 23. A. A. RANICKI: Lower K- and L-theory, LMS Lecture Notes, Vol. 178, Cambridge University Press (1992).
- 24. A. A. RANICKI: The algebraic theory of finiteness obstructions, Math. Scand. 57 (1985), 105-126.
- 25. L. C. SIEBENMANN: Infinite simple homotopy types, Indag. Math. 32 (1970), 479-495.
- 26. F. WALDHAUSEN: Albegraic K-theory of generalized free products. Ann. of Math. 108 (1978), 135-256.
- 27. M. WEISS and B. WILLIAMS: Automorphisms of Manifolds and Algebraic K-theory III, Aarhus University, (preprint) (1991).
- 28. M. YAMASAKI: L-groups of crystallographic groups, Invent. Math. 88 (1987), 571-602.

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