Desuspension in the Symmetric L-groups

by

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Introduction

In view of the recent strides made in the computation of Wall groups (see e.g. [C-M], [Pardon], [Wall]), the problem of determining which of the surgery obstructions occur as the obstruction of a degree one normal map of closed manifolds has become of increased importance. One approach to this is by product formulae, i.e. formulae which express the surgery obstruction of the degree one normal map

$$M \times N \longrightarrow X \times N$$

in terms of the obstruction of $(M \to X)$ and data derived from N. Morgan has recently analyzed this problem for the case $\pi_1(N) = 0$. (see [Morgan]). The problem reduces to a bordism problem, in fact to the analysis of a bilinear pairing

$$\Omega_*(K(\pi_1(N), 1)) \otimes L_*(\pi_1(X)) \longrightarrow L_*(\pi_1(X \times N)).$$

In [Ranicki] it is shown that this pairing actually factors through a pairing

$$L^*(\pi_1(N)) \otimes L_*(\pi_1(X)) \longrightarrow L_*(\pi_1(X \times N)),$$

where the L^*-groups are symmetric versions of the Wall-groups, defined in [Miščenko] and [Ranicki]. For purposes of computing product formulae, then, computing the groups L^* becomes of great interest. L^0 turns out to be the Witt group of $\mathbb{Z}\pi$, and L^1 is quite closely tied to the surgery group $L_1(\mathbb{Z}\pi)$. Secondly, there are skew suspension maps relating the high-dimensional L^*-groups to the lower dimensional ones. The approach to calculating the L^*'s, then, is to measure the cokernel of the skew-suspension maps, thereby reducing the problem to a Witt group problem, about which much is known (see [C]).

The method for analyzing this cokernel is closely related to the method of characteristic elements, which one may use to calculate $W(\hat{\mathbb{Z}}_2)$, and a generalization of which was used in [C] to calculate $W(\hat{\mathbb{Z}}_2)$ for π a 2-group.

§I defines the groups L^*, §II defines the target groups for our invariants, §III proves that the invariants are well-defined, and §IV proves the main theorem, IV.3, which asserts that the defined invariant is the complete obstruction to desuspension.

I. Preliminaries

We recall from [Ranicki] the definition of algebraic Poincaré complexes over a ring Λ with involution and their bordism groups. Given a projective module over Λ, let P^* denote its dual module, $\mathrm{Hom}_\Lambda(P, \Lambda)$, endowed with a Λ-module structure in the usual way.

Definition 1 An n-dimensional ϵ-symmetric complex over Λ is a chain complex of projective Λ-modules, having the chain homotopy type of an n-dimensional chain complex, $\{C_*, \partial_*\}$, together with a collection of Λ-module maps $\Phi = \{\varphi_s \in \mathrm{Hom}_\Lambda(C^{n-r+s}, C_r) \mid r \in \mathbb{Z}, S \geq 0\}$, so that

$$(*)\ \partial\varphi_s + (-1)^r\varphi_s\partial^* + (-1)^{n+s-1}(\varphi_{s-1} + (-1)^{s+(n-r+s)r}\epsilon\varphi^*_{s-1}) = 0$$

$$: C^{n-r+s-1} \longrightarrow C_r \quad (s \geq 0,\ \varphi_{-1} = 0).$$

(Of course, each φ_s really stands for a collection $\varphi^r_s : C^{n-r+s} \longrightarrow C_r$, $\forall r$. We suppress the superscript for simplicity of notation) Here $C^k = C^*_k$, and ∂^* and φ^*_{s-1} denote the duals to the maps ∂ and φ_{s-1}. Note that φ_0 is thus a chain map from the complex $\{C^{n-*}, \partial^*\}$ to the complex $\{C_*, \partial_*\}$. If φ_0 is a chain equivalence, the symmetric complex is said to be Poincaré.

Definition 2 Let (C_*, Φ) be an n-dimensional Poincaré complex, and let $f : C \to D$ be a chain map. where D is an $(n+1)$ - dimensional chain complex of projective Λ-modules. Then by surgery data for f we will mean a collection $\Psi = \{\psi_s\}_{s=0}^{s=n+1}$ of Λ-module homomorphisms, $\psi_s : D^{n-r-1} \longrightarrow D_{r+s}$, so that

$$\partial\psi_s + (-1)^r \psi_s \partial^* + (-1)^{n+s} (\psi_{s-1} + (-1)^{s+(n-r-1)(r+s)} \epsilon \psi^*_{s-1})$$

$$+ (-1)^n f\varphi_s f^* = 0.$$

We say that the surgery data Ψ is connected if the map $D^* \longrightarrow MC(f)$ induced by ψ_0, where $MC(f)$ denotes the algebraic mapping cone on f, is surjective in 0-dimensional homology.

Definition 3 The Poincaré complex C'_* obtained from C_* by surgery on the map f, using connected surgery data Ψ, is defined by

$$C'_r = D^{n-r+1} \oplus C_r \oplus D_{r+1},$$

$d_{C'}$ is given by the matrix

$$\begin{bmatrix} (-1)^r d^*_D & 0 & 0 \\ (-1)^{n+1}\varphi_0 f^* & d_C & 0 \\ (-1)^r \psi_0 & (-1)^r f & d_D \end{bmatrix}$$

$$\varphi_0' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \varphi_0 & 0 \\ (-1)^{r(n-r)} & (-1)^{n-r}\epsilon f\varphi_1^* & (-1)^{n-r}\epsilon\psi_1^* \end{bmatrix}$$

$$: (C')^{n-r} = D_{r+1} \oplus C^{n-r} \oplus D^{n-r+1} \longrightarrow C_r = D^{n-r+1} \oplus C_r \oplus D_{r+1}.$$

$$\varphi_s' = \begin{bmatrix} 0 & 0 & \\ 0 & \varphi_s & \\ 0 & (-1)^{n-r}\epsilon f\varphi_{s+1}^* & (-1)^{n-r+s}\epsilon\psi_{s-1}^* \end{bmatrix}$$

$$: (C')^{n-r} = D_{r+1} \oplus C^{n-r} \oplus D^{n-r+1} \longrightarrow C_{r+s} = D^{n-r-s+1} \oplus C_{r+s} \oplus D_{r+s+1}.$$

The equivalence relation generated by all equivalences of the form $(C_*, \Phi) \sim (C_*', \Phi')$, where C_*' is obtained from C_* by surgery, and homotopy equivalence, is called algebraic cobordism. The set of equivalence classes becomes group under direct sum becomes a group under direct sum of Poincaré complexes, and is denoted $L^n(\Lambda, \epsilon)$.

Definition 4 The skew-suspension of an n-dimensional ϵ-symmetric Poincaré complex (C_*, Φ) is an $(n+2)$-dimensional Poincaré complex $(\overline{C}_*, \overline{\Phi})$, where

$$\underline{\overline{C}_k = C_{k-1},\ \overline{C}_0 = \overline{C}_{n+2} = 0.}$$

$$\underline{\varphi_s : \overline{C^k} \longrightarrow \overline{C}_{n+2-k+s}}$$

$$= \underline{\varphi_s : C^{k-1} \longrightarrow C_{n+1-k+s}.}$$

It is easily verified that this defines a homomorphism

$$\sigma : L^n(\Lambda, \epsilon) \longrightarrow L^{n+2}(\Lambda, -\epsilon)$$

Remark 1 For the surgery groups $L_n(\mathbb{Z}\pi, \epsilon)$ the analogue to the double skew-suspension $\sigma^2 : L_n(\mathbb{Z}\pi, \epsilon) \longrightarrow L_{n+4}(\mathbb{Z}\pi, \epsilon)$ may be identified with the periodicity isomorphism $L_n(\mathbb{Z}\pi, \epsilon) \xrightarrow{\times[\mathbb{C}P^2]} L_{n+4}(\mathbb{Z}\pi, \epsilon)$. In the case of ${}^{/}L^n$, however, σ fails to be an isomorphism, and it is this failure we shall analyze.

Remark 2 For complexes C and D of projective Λ-modules, define the complex $\mathrm{Hom}_\Lambda(C, D)$ by $\mathrm{Hom}_\Lambda(C, D)_n = \bigoplus_{q-p=n} \mathrm{Hom}_\Lambda(C_p, D_q)$, $d_{\mathrm{Hom}_\Lambda(C,D)}(f) = d_D f + (-1)^q f d_C$. Note that duality provides an involution on $\mathrm{Hom}_\Lambda(C^*, C_*)$ by $f \to (-1)^{pq} \epsilon f^*$; so the complex $\mathrm{Hom}_\Lambda(C^*, C_*)$ becomes a complex of $\mathbb{Z}[\mathbb{Z}/_2]$-modules. Let W_* denote the standard $\mathbb{Z}[\mathbb{Z}/_2]$-resolution of $\mathbb{Z}$, $W_n = \mathbb{Z}[\mathbb{Z}/_2]$, $\partial e_n = (1 + (-1)^n T)e_{n-1}$, $n \geq 0$, $W_n = 0$ for

$n < 0$. Let $Q^*(C, \epsilon) = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/_2]}(W_*, \mathrm{Hom}_\Lambda(C^*, C_*))$, where $\mathrm{Hom}_\Lambda(C^*, C_*)$ is acted on by $\mathbb{Z}/_2$ by $T\varphi = (-1)^{pq}\,\epsilon\varphi^*$. We may now observe that choosing an ϵ-symmetric structure on a complex C_* amounts to choosing a cycle in $Q^n(C, \epsilon)$. Note also that $Q^n(C, \epsilon)$ is a functor in C_*, since given $f : C_* \longrightarrow D_*$, we may define a map

$$Q^n(C, \epsilon) \xrightarrow{Q^n(f,\epsilon)} Q^n(D, \epsilon)$$

by letting $\hat{f} : \mathrm{Hom}_\Lambda(C^*, C_*) \longrightarrow \mathrm{Hom}_\Lambda(D^*, D_*)$ denote the map $\varphi \longrightarrow f\varphi f^*$, and noting that $\hat{f}$ is $\mathbb{Z}/_2$ - equivariant. Surgery data for the map f consists of a choice of $\psi \in Q^{n+1}(D, \epsilon)$ so that $\partial\psi = Q^n(f, \epsilon)(\varphi)$, where φ is a cycle defining the symmetric structure on C_*.

II The Groups $\mathbb{W}_n(\Lambda, \epsilon)$

As in the previous section, let Λ be a ring with involution, and let

$$Q^*(C, \epsilon) = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/_2]}(W_*, \mathrm{Hom}_\Lambda(C^*, C_*)),$$

as in remark 2, §I.

Recall that the abelian group

$$H_\epsilon(\mathbb{Z}/_2, \Lambda) = \{\lambda \in \Lambda | \lambda = \epsilon\bar{\lambda}\}/\{\lambda + \epsilon\bar{\lambda} , \lambda \in \Lambda\}$$

becomes a Λ-module by

$$\lambda\alpha = \lambda \alpha \bar{\lambda},$$

for $\lambda \in \Lambda$, $\alpha \in H(\mathbb{Z}/_2, \Lambda)$, and that if $\varphi : M^* \longrightarrow M$ is an ϵ-symmetric Λ-homomorphism (i.e. $\varphi = \epsilon\varphi^*$), we obtain a Λ-map $\hat{\varphi} : M^* \longrightarrow H_\epsilon(\mathbb{Z}/_2, \Lambda)$ by $x \rightarrow \langle x, \varphi x \rangle$, where $\langle \, , \, \rangle$ denotes the evaluation pairing $\langle \, , \, \rangle : M^* \otimes M \to \Lambda$.

Let $\mathfrak{K}_* = \mathfrak{K}_*(\Lambda, \epsilon)$ denote a Λ-projective resolution of $H_\epsilon(\mathbb{Z}/_2, \Lambda)$, and let $\mathfrak{K}_*^{(n)}$ denote its n-skeleton. We consider the two complexes $Q(\mathfrak{K}_*^{(n)}, \epsilon)$ and $Q(\mathfrak{K}_*^{(n+1)}, \epsilon)$. Recall that the n-cycles of $Q(\mathfrak{K}_*^{(n)}, \epsilon)$, $Z^n(\mathfrak{K}_*^{(n)}, \epsilon)$ consist of collections $\Phi = \{\varphi_s\}$ of Λ-homomorphisms, satisfying

$\partial\varphi_s + (-1)^q\varphi_s\partial^* + (-1)^{n+s-1}(\varphi_{s-1} + (-1)^{s+pq}\,\epsilon\,\varphi^*_{s-1}) = 0 : C^p \longrightarrow C_q$. Therefore, $\varphi_n + (-1)^{n+1}(-1)^{n^2}\,\epsilon\,\varphi^*_n = 0$, or $\varphi_n = \epsilon\varphi^*_n$. We obtain a homomorphism $\lambda(\Phi) = \varphi_n : \mathfrak{K}^*_n \to H_\epsilon(\mathbb{Z}/_2, \Lambda)$. Secondly, φ_0 provides a Λ-homomorphism $\varphi_0 : \mathfrak{K}^*_n \longrightarrow \mathfrak{K}_0$, which when composed with augmentation map $\eta : \mathfrak{K}_0 \to H_\epsilon(\mathbb{Z}/_2, \Lambda$ from the resolution gives a second homomorphism

$$\rho(\Phi) : \mathfrak{K}^*_n \longrightarrow H_\epsilon(\mathbb{Z}/_2, \Lambda)$$

These two correspondences define homomorphisms

$$\lambda, \rho : Z^n(\mathfrak{K}^{(n)}_*, \epsilon) \longrightarrow \mathrm{Hom}_\Lambda(\mathfrak{K}^*_n, H_\epsilon(\mathbb{Z}/_2, \Lambda))$$

Define $\tilde{Z}^n \subseteq Z^n(\mathfrak{K}^{(n)}_*, \epsilon)$ by

$$\tilde{Z}^n = \{x \in Z^n(\mathfrak{K}^{(n)}_*, \epsilon) \mid \rho(x) = \lambda(x)\}.$$

We now let B_{n+1} denote the subgroup of the $(n+1)$-chains of $Q(\mathfrak{K}^{(n+1)}_*, \epsilon)$ consisting of those chains whose boundary is in the image of $Q(\mathfrak{K}^{(n)}_*, \epsilon)$ in $Q(\mathfrak{K}^{(n+1)}_*, \epsilon)$ under the natural inclusion. This means that an element of B_{n+1} is a collection $\Psi = \{\psi_s\}$ of Λ-module momomorphisms such that $\psi_{n+1} = \epsilon\,\psi^*_{n+1}$, since

$$\psi_{n+1} + (-1)^{n+2}(-1)^{(n+1)^2} \epsilon \psi^*_{n+1} = 0.$$

This defines a homomorphism

$$\alpha = \psi_{n+1} : B_{n+1} \longrightarrow \mathrm{Hom}_\Lambda(\mathfrak{K}^*_{n+1}, H_\epsilon(\mathbb{Z}/_2, \Lambda))$$

A second homomorphism β is obtained by $\beta(\Psi) = \eta \circ \psi_0$, where $\eta : \mathfrak{K}_0 \longrightarrow H_\epsilon(\mathbb{Z}/_2, \Lambda)$ is the augmentation. Define

$$B_{n+1} = \{\Psi | \alpha(\Psi) = \beta(\Psi)\}$$

<u>Proposition 1</u> $\partial\widetilde{B}_{n+1} \subseteq \widetilde{Z}^n$

<u>Pf.</u> Let $\Phi = \partial\Psi$. Then

(i) $\varphi_0 = \partial\psi_0 + (-1)^q\psi_0\partial^*$

(ii) $\varphi_n = (-1)^{n+1}(\partial\psi_n + (-1)^{n+1}\psi_n\partial^* - (\psi_{n-1} + \epsilon\psi^*_{n-1}))$

and since $\Psi \in B_{n+1}$,

(iii) $0 = \partial\psi_{n+1} + (-1)^n\psi_{n+1}\partial^* + (\psi_n + (-1)^{n+1}\epsilon\psi^*_n)$

Now,

$$\partial\psi_n + (-1)^{n+1}\psi_n\partial^* = \partial\psi_n + \epsilon\psi^*_n\partial^* + (-1)^{n+1}\partial\psi_{n+1}\partial^*$$

so

$$\varphi_n = (-1)^{n+1}(\partial\psi_n + \epsilon\psi_n^*\partial^* - (\psi_{n-1} + \epsilon\psi_{n-1})) + \partial\psi_{n+1}\partial^*$$

The left hand term in the sum is of the form $\beta + \epsilon\beta^*$, so $\hat{\varphi}_n = \partial\psi_{n+1}\partial^*$.

Equation (i) asserts that $\eta\circ\varphi_0 = \eta\circ\psi_0\circ\partial^*$, since $\eta\circ\partial = 0$, and $H_\epsilon(\mathbb{Z}/_2, \Lambda)$ is a $\mathbb{Z}/_2$-vector space. The condition $\alpha(\Psi) = \beta(\Psi)$ guarantees that $\eta\circ\psi_0 = \hat{\psi}_{n+1}$, or $\eta\circ\psi_0\circ\partial^*(x) = \hat{\psi}_{n+1}(\partial^* x) = \partial\psi_{n+1}\partial^*(x) = \hat{\varphi}_n(x)$, so $\eta\circ\varphi_0 = \hat{\varphi}_n$, which implies $\Phi \in \widetilde{Z}^n$. (*)

We now define

$\mathbb{W}_n(\Lambda, \epsilon, \mathcal{R}) = \widetilde{Z}^n/\partial\widetilde{B}_{n+1}$, and conclude this section by showing that $\mathbb{W}_n(\Lambda, \epsilon, \mathcal{R})$ is independent of the choice of resolution $\mathcal{R}$.

<u>Proposition 2</u> <u>If</u> $\mathcal{R}_*, \mathcal{S}_*$ <u>are two resolutions of</u> $H_\epsilon(\mathbb{Z}/_2, \Lambda)$, <u>then</u> $\mathbb{W}_n(\Lambda, \epsilon, \mathcal{R}) \cong \mathbb{W}_n(\Lambda, \epsilon, \mathcal{S})$. <u>We then define</u> $\mathbb{W}_n(\Lambda, \epsilon) = \mathbb{W}_n(\Lambda, \epsilon, \mathcal{R}) = \mathbb{W}_n(\Lambda, \epsilon, \mathcal{S})$.

<u>Proof</u>. We may assume that there is a chain map $\mathcal{S}\to\mathcal{R}$ which is surjective in each degree, since in any event, there is a resolution $\mathcal{T}$ which maps surjectively in each degree to both $\mathcal{R}$ and $\mathcal{S}$. it is then easily seen that $\mathcal{S}$ is isomorphic to $\mathcal{E}\oplus\mathcal{R}$, where $\mathcal{E}$ is a contractible complex. Since any sum of elementary complexes

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow P \xrightarrow{id} P \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

with P projective, we may assume that $\mathcal{S}$ is obtained from $\mathcal{R}$ by addition with a single elementary complex.

The complex

$\mathrm{Hom}_\Lambda((\mathcal{E} \oplus \mathcal{R})^*, (\mathcal{E} \oplus \mathcal{R}))$ splits as

$\mathrm{Hom}_\Lambda(\mathcal{E}^*, \mathcal{E}) \oplus \mathrm{Hom}_\Lambda(\mathcal{E}^*, \mathcal{R}) \oplus \mathrm{Hom}_\Lambda(\mathcal{R}^*, \mathcal{E}) \oplus \mathrm{Hom}\ (\mathcal{R}^*, \mathcal{R})$,

and the involution preserves the first and fourth summands and permutes the middle two. Thus, $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/_2]}(W_*, \mathrm{Hom}\ ((\mathcal{E} \oplus \mathcal{R})^*, \mathcal{E} \oplus \mathcal{R}))$ splits into three summands,

$$\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/_2]}(W_*, \mathrm{Hom}_\Lambda(\mathcal{E}^*, \mathcal{E})) \oplus$$
$$\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/_2]}(W_*, \mathrm{Hom}_\Lambda(\mathcal{E}^*, \mathcal{R}) \oplus \mathrm{Hom}_\Lambda(\mathcal{R}^*, \mathcal{E})) \oplus$$
$$\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/_2]}(W_*, \mathrm{Hom}\ (\mathcal{R}^*, \mathcal{R}))$$

Furthermore, the homomorphisms ρ and β vanish identically on the first two of these, and λ and α vanish identically on the middle summand.

It is now easily verified that the middle term contributes nothing to $\mathfrak{w}_n(\Lambda, \epsilon, \mathcal{S})$, since any cycle Z in

$$\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/_2]}(W_*, \mathrm{Hom}\,(\mathcal{E}^*, \mathcal{R}_*^{(n)}) \quad \mathrm{Hom}\,((\mathcal{R}^{(n)})^*, \mathcal{E}))$$

is a boundary in

$$\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/_2]}(W_*, \mathrm{Hom}\,(\mathcal{E}^*, \mathcal{R}_*^{(n+1)}) \quad \mathrm{Hom}\,((\mathcal{R}^{(n+1)})^*, \mathcal{E})),$$

$\mathcal{E}$ being contractible, and the fact that α vanishes identically on this summand guarantees that we may choose the chain x such that $\partial x = z$ with $x \in \widetilde{B}_{n+1}$. We must therefore check that the contribution of the first summand is also zero. Let $Z^n(\mathcal{E})$ be the group of n-cycles in $Q(\mathcal{E}, \epsilon)$ and let $\widetilde{Z}^n(\mathcal{E}) = \{\Phi \in Z^n(\mathcal{E}) | \varphi_n = \gamma + \epsilon\gamma^*\}$. Also, let $B_{n+1}(\mathcal{E})$ be the group of $(n + 1)$-chains x in $Q(\mathcal{E}, \epsilon)$ so that $\partial x \in Q(\mathcal{E}^{(n)}, \epsilon)$, and let $\widetilde{B}_{n+1}(\mathcal{E}) = \{\Psi \in B_{n+1}(\mathcal{E}) | \psi_{n+1} = \gamma + \epsilon\gamma^*\}$. It is easily seen that $\partial\widetilde{B}_{n+1}(\mathcal{E}) \subseteq \widetilde{Z}^n(\mathcal{E})$, as in Proposition 1. Moreover, since ρ and β vanish identically on this summand, the contribution of this summand to $\mathfrak{W}_n(\Lambda, \epsilon, \mathcal{S})$ is isomorphic to

$$\widetilde{Z}^n(\mathcal{E})/_{\partial\widetilde{B}_{n+1}(\mathcal{E})}.$$

It is now an easy calculation with the elementary complexes that this group is zero. (*).

III Defining the Invariant

We assume from now on that all Poincaré complexes will in fact be n-dimensional compleses, i.e. that $C_* = 0$ for $* < 0$, $* > n$. This involves no loss of generality since the complexes have the homotopy type of an n-dimensional complex.

Let (C_*, Φ) be an ϵ-symmetric Poincaré complex. From the identity (*) in the definition of Poincaré complexes, we find

$\partial\varphi_{n+1} + (-1)^r\varphi_{n+1}\partial^* + (-1)(\varphi_n - \epsilon\,\varphi_n^*) = 0 \ : C^{2n-r} \to C_r$. Since C_* is n-dimensional, $C^{2n-r} = 0$ for $r < n$, $C_r = 0$ for $r > n$, so the map $\varphi_{n+1} = 0$, and we obtain $\varphi_n = \epsilon\,\varphi_n^*$. Therefore, we have the n-th "Wu class" map $\hat{\varphi}_n : C^n \longrightarrow H_\epsilon(\mathbb{Z}/_2, \Lambda)$, as in [Ranički]

Lemma 1 Let C_* be a chain complex of projective Λ-modules, bounded below ($C_* = 0$ for $* < 0$) Then any homomorphism $f : C_0 \to M$, where M is a Λ-module, may be extended to a chain map (unique up to chain homotopy) $f : C_* \to \mathcal{R}_*(M)$, where $\mathcal{R}_*(M)$ denotes a resolution of the module M.

Proof The usual argument for maps of resolutions does not use the acyclicity of C_*. (*)

The map $\hat{\varphi}_n$ defines a homotopy class of chain maps

$$W : C^* \longrightarrow \mathcal{R}_*(H_\epsilon(\mathbb{Z}/_2, \Lambda)).$$

The invariant we construct will lie in the group

$$\mathbb{W}_n(\Lambda, \epsilon)$$

Since (C_*, Φ) is a Poincaré complex, the chain map $\varphi_0 : C^* \to C_*$ is a chain equivalence. We choose $\bar{\varphi}_0$ to be a chain inverse to φ_0(the choice is unique up to chain homotopy).

Proposition 2 The element $\{w\bar{\varphi}_0\varphi_s\bar{\varphi}_0^*w^*\} \in Z^n(\mathcal{R}^{(n)}, \epsilon)$ lies in $Z^n(\mathcal{R}^{(n)}, \epsilon)$.

Pf. Let $\Phi = \{w\bar{\varphi}_0\varphi_s\bar{\varphi}_0^*w^*\}$. Then $\lambda(\Phi)(x) = \widehat{w\bar{\varphi}_0\varphi_n\bar{\varphi}_0^* w^*}(x) = \hat{\varphi}_n(\bar{\varphi}_0^*w^*x)$ Also, $\rho(\Phi)(x) = \eta w\bar{\varphi}_0\varphi_0\bar{\varphi}_0^*w^*(x)$. By the choice of w and $\bar{\varphi}_0$, $\eta w\bar{\varphi}_0\varphi_0 = \hat{\varphi}_n$, so $\rho(\Phi)(x) = \hat{\varphi}_n(\bar{\varphi}_0^*w^*x) = \lambda(\Phi)(x)$. (*)

Let $\xi(C_*, \Phi) \in \mathbb{W}_n(\Lambda, \epsilon)$ be defined by $\xi(C_*, \Phi) = \{w\bar{\varphi}_0\varphi_s\bar{\varphi}_0^*w^*\}$.

Proposition 3 $\xi(C_*, \Phi)$ is independent of the choice of w and $\bar{\varphi}_0$ within homotopy classes.

Pf. If $w \simeq w'$, $\bar{\varphi}_0 \simeq \bar{\varphi}_0'$, $w\varphi_0 \simeq w'\varphi_0'$, wo we suppose that we have a chain homotopy $h : w\varphi_0 \simeq w'\varphi_0'$

According to [Raničķi]

$$\Psi = \{w\bar{\varphi}_0\varphi_s h^* + (-1)^q h\varphi_s\bar{\varphi}_0'^* w'^* + (-1)^{q+1} h\varphi_{s-1}h^*\}$$

satisfies

$$\partial\Psi = \{w\bar{\varphi}_0\varphi_s\bar{\varphi}_0^* w^*\} - \{w'\bar{\varphi}_0'\varphi_s\bar{\varphi}_0'^* w'^*\}$$

We must show that $\Psi \in \tilde{B}_{n+1} \subseteq B_{n+1}$. To verify this, it will suffice to show $\alpha(\Psi) = \beta(\Psi)$.

$$\alpha(\Psi) = \hat{\psi}_{n+1} = (-1)^{q+1}\widehat{h\varphi_n h^*} = \widehat{h\varphi_n h^*},$$

the last equality since $H_\epsilon(\mathbb{Z}/_2, \Lambda)$ is a $\mathbb{Z}/_2$-vector space.

$\beta(\Psi) = \eta \circ \psi_0 = \eta w\bar{\varphi}_0\varphi_0 h^* + (-1)^q \eta h\varphi_0\bar{\varphi}_0'^* w'^* : \mathcal{R}_{n+1}^* \to \mathcal{R}_{n+1}$. The second summand factors through a zero group, hence is zero. By the choice of $\bar{\varphi}_0$ and w, we have

$$\eta w\bar{\varphi}_0\varphi_0 h^* = \hat{\varphi}_n \circ h^* = \widehat{h \circ \varphi_n \circ h^*}, \quad \text{so}$$

$\alpha(\Psi) = \beta(\Psi)$. (*)

<u>Cor. 4</u> $\xi(C_*, \Phi)$ <u>is independent of the homotopy type of</u> C_*.

<u>Pf.</u> Clear. (*)

<u>Cor. 5.</u> <u>Let</u> (C_*, Φ) <u>and</u> (C_*', Φ') <u>be two Poincaré complexes over</u> Λ. <u>Then</u>

$$\underline{\xi(C_* \oplus C_*', \Phi \oplus \Phi') = \xi(C_*, \Phi) \oplus \xi(C_*, \Phi')}$$

<u>Pf.</u> Clear, since the homomorphism $\varphi_n \oplus \varphi_n'$ is equal to $\hat{\varphi}_n \oplus \hat{\varphi}_n'$. (*)

IV The Homomorphism $\mathbb{W}_n : L^n(\Lambda, \epsilon) \to \mathbb{W}_n(\Lambda, \epsilon)$ and Desuspension in the L-groups

In the previous section, it was shown that there is an invariant of the homotopy type of (C_*, Φ), $\xi(C_*, \Phi)$. In this section, we show that $\xi(C_*, \Phi)$ is an invariant of the algebraic cobordism class of (C_*, Φ), and hence induces a homomorphism $\mathbb{W}_n : L^n(\Lambda, \epsilon) \longrightarrow \mathbb{W}_n(\Lambda, \epsilon)$, in view of corollary III. 5.

Proposition 1 *Let* (C_*, Φ) *be a Poincaré complex,* $f : C_* \longrightarrow D_*$ *a chain* map, and $\Psi = \{\psi_s\}$ surgery data for f. If (C', Φ') denotes the Poincaré complex obtained by surgery on f, then $\xi(C'_*, \Phi') = \xi(C_*, \Phi)$.

Pf. We note that C'_* is obtained by a double mapping cone construction on C_*. That is, we first form the algebraic mapping cone $MC(f)$, and observe that surgery data for f determines a homotopy class om maps $\tilde{f} : D^* \to MC(f)$, together with a Poincaré structure on $MC(\tilde{f})$. In particular, the underlying chain complex of C'_* is $MC(\tilde{f})$. Similarly, C'^* admits D as a subcomplex, as well as $MC(f\varphi_0^*)$. By the definition of the top Wu class of C'_*, $\varphi'_n | D_* = 0$. Therefore, we may choose the chain map w from C'^* to $\mathcal{R}_*(H_\epsilon(\mathbb{Z}/_2, \Lambda))$ so that w vanishes on $D_* \subseteq C'^*$. Therefore, there is a splitting of graded Λ-modules (not of chain complexes)

$$C'_* \cong D^* \oplus C_* \oplus D_*$$

$$C'^* \cong D_* \oplus C^* \oplus D^*$$

And the map $w : C'^* \to \mathfrak{R}_*(H_\epsilon(\mathbb{Z}/_2, \wedge))$ has "matrix" $(0, w', w'')$, where w' is a lifting of the n-th Wu class of C_* to $\mathfrak{R}_*(H_\epsilon(\mathbb{Z}/_2, \wedge))$. Consequently, the dual map w^* has matrix

$$\begin{pmatrix} 0 \\ w'^* \\ w''^* \end{pmatrix}$$

Recall from §I that the map φ_0' is given by the matrix

$$\begin{bmatrix} 0 & 0 & (-1)^{q(n-q)}\varepsilon \\ 0 & \varphi_0 & 0 \\ 1 & (-1)^{(-q)+pq} f\varphi_1^* & (-1)^{(n-q)+pq}\epsilon\psi_1 \end{bmatrix}$$

Consequently, if $\bar{\varphi}_0$ is a chain inverse to φ_0, we find that the matrix of a chain inverse to φ_0' is given by.

$$\begin{pmatrix} * & * & 1 \\ 0 & \varphi_0 & 0 \\ (-1)^{q(n-q)}\epsilon & 0 & 0 \end{pmatrix}$$

where the $*$'s represent certain maps, the values of which will not concern us. Now, $w\overline{\varphi_0} =$

$$(0\ w'\ w'')\begin{pmatrix} * & * & 1 \\ 0 & \overline{\varphi_0} & 0 \\ (-1)^{q(n-q)}\epsilon & 0 & 0 \end{pmatrix}$$

$$= ((-1)^{q(n-q)}\epsilon w'',\ w'\overline{\varphi_0},\ 0),$$ so $\overline{\varphi}_0^* w^*$ has matrix

$$\begin{pmatrix} w''^* \\ \overline{\varphi}_0^* w'^* \\ 0 \end{pmatrix}$$

Finally, $w\overline{\varphi}_0\varphi_0 = (0,\ w'\overline{\varphi}_0\varphi_0,\ w'')$, so $w\overline{\varphi}_0\varphi_0\overline{\varphi}_0^* w^* =$

$$(0,\ w'\overline{\varphi}_0\varphi_0, w'')\begin{pmatrix} (-1)^{q(n-q)}\epsilon w'' \\ w'\overline{\varphi}_0 \\ 0 \end{pmatrix}$$

$= w'\overline{\varphi}_0\varphi_0\overline{\varphi}_0^* w^*$. Similarly, $w\overline{\varphi}_0'\varphi_s'\overline{\varphi}_0' w^* = w\varphi_0\varphi_s\varphi_0^* w^*$, so the value of $\xi(C_*, \Phi) = \xi(C_*, \Phi)$. (*)

Applying the definition of the groups $L^n(\Lambda, \epsilon)$ and Corollary II.5, we have defined a homomorphism $\mathfrak{w}_n : L^n(\Lambda, \epsilon) \rightarrow \mathfrak{w}_n(\Lambda, \epsilon)$ by $\mathfrak{w}_n(C_*, \Phi) = \xi(C_*, \Phi)$.

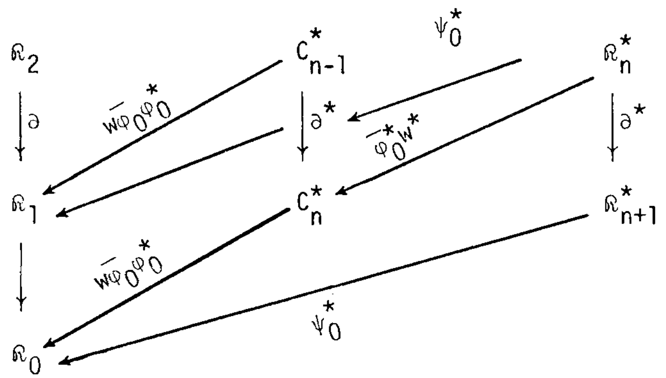
Proposition 2 $\mathfrak{w}_n$ vanishes on the image of the skew-suspension map σ.

Proof. It is immediate that the n-th Wu class map $\hat{\varphi}_n$ is trivial on a skew-suspension, since it is defined on a trivial group. Thus, the chain map w may be taken to be zero. (*)

It is shown in [Ranički] that a chain complex is in the image of the skew-suspension if its n-th Wu class vanishes. This allows us to prove the main theorem.

Theorem 3. $x \in L^n(\Lambda, \epsilon)$ is in the image of the skew-suspension if and only if $\mathfrak{w}_n(x) = 0$.

Proof. Consider a representative Poincaré complex (C_*, Φ) for x. We may suppose that the n-th Wu class map $\hat{\varphi}_n$ is onto $H_\epsilon(\mathbb{Z}/_2, \Lambda)$. If not, we may add on some null-cobordant complexes for which $\hat{\varphi}_n$ is onto. Now, if the invariant $\mathfrak{w}_n(x)$ is trivial, there exists $\Psi \in \widetilde{B}_{n+1}$ with $\partial\Psi = \Phi$, where Φ is a cycle representing $\mathfrak{w}_n(x)$. Ψ thus represents surgery data for the chain map $C_* \xrightarrow{w\bar{\varphi}_0} \mathfrak{R}_*^{(n+1)}$. Thus, we form the chain complex (C_*', Φ') by surgery on the map w_0. I claim that the n-th Wu class φ'_n is trivial on the n-dimensional cohomology of $C'_\star$. To see this, we analyze $H^n(C)$. In the relevant dimensions, the complex may be represented by



Note that since as a chain map, $\varphi_0 \simeq \varphi_0^*$, and $\bar{\varphi}_0\varphi_0 \simeq \text{id}$, the map $w\bar{\varphi}_0\varphi_0^*$ has the same effect homologically as w. Thus it is surjective in cohomology, and we find

$$H^n(MC(w\bar{\varphi}_0\varphi_0)) \cong \text{Ker}(w_n : H^n(C^*) \longrightarrow H_\epsilon(\mathbb{Z}/_2, \wedge)).$$

The remaining dimensions are unchanged from C^*, since $\Re_\star$ is acyclic above dimension 0. We conclude, then, that any cohomology class in $H^n(C'_\star)$ is represented by a pair $(x, y) \in C_n^* \oplus \Re_{n+1}^*$, so that $w\bar{\varphi}_0\varphi_0^*(x) = -\psi_0^*(y)$.

Applying the augmentation $\eta : \mathcal{R}_0 \to H_\epsilon(\mathbb{Z}/_2, \wedge)$, we find $\eta w\bar{\varphi}_0\varphi_0^*(x) = -\eta\psi^*(x)$. Now by the construction of w and the above remarks about φ_0^*, $\eta w\bar{\varphi}_0\varphi_0^*(x) = \hat{\varphi}_n(x)$. We also note that $\eta\psi_0^* = \eta\psi_0$, for $\psi_0^* = \pm(\psi_0 \pm \partial\psi_1 \pm \psi_1\partial^*) \pm \bar{\varphi}_0\varphi_1\bar{\varphi}_0^*w^*$ $\eta\partial\psi_1 = 0$ since $\eta\partial = 0$, $\psi_1\partial^* = 0$, $\psi_1\partial^*|\mathcal{R}_{n+1}^* = 0$ since $\partial^*|\mathcal{R}_{n+1}^* = 0$, and $w^*|\mathcal{R}_{n+1}^* = 0$, so $\psi_0^* = \pm\psi_0$. Again, since $H_\epsilon(\mathbb{Z}/_2, \wedge)$ is a $\mathbb{Z}/_2$-vector space, $\eta\psi_0 = \pm\eta\psi_0$, so $\eta\psi_0^* = \eta\psi_0$. Now, since $\Psi \in \widetilde{B}_{n+1}$, $\eta\psi_0 = \hat{\psi}_{n+1}$, so $-\eta\psi_0^*(y) = \hat{\psi}_{n+1}(y)$, thus for any cycle (x,y) representing a class in $H^n(C^*)$, $\hat{\varphi}_n(x) + \hat{\psi}_{n+1}(y) = 0$. But the matrix representing φ_n' is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \varphi_0 & 0 \\ 0 & * & \psi_{n+1} \end{bmatrix}, \text{ so } \hat{\varphi}_n'(x,y) = \hat{\varphi}_n(x) + \hat{\psi}_{n+1}(y) = 0.$$

This proves that $\hat{\varphi}_n$ is identically zero on $H^n(C')$. It is shown in [Ranički] that under these circumstances, one may perform a sequence of elementary surgeries to kill $H^n(C)$, leaving a complex C'' with $H^n(C'') = 0$. Such a complex is the homotopy type of a skew-suspension (again, see [Ranički]). This concludes the proof (*).

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