ON THE WITT GROUP OF A 2-ADIC GROUP RING

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Introduction

LET Λ denote the 2-adic group ring $\hat{Z}_2\pi$, where π is a 2-group, endowed with the "orientable" involution $\bar{g} = g^{-1}$. In this paper we explicitly calculate the Witt group $W(\Lambda)$ of Hermitian forms over Λ with respect to this involution. The difficult part of the calculation is the quotient $W(\Lambda)/W^{ev}(\Lambda)$, where $W^{ev}(\Lambda)$ denotes the subgroup of *even* Hermitian forms. To obtain $W(\Lambda)$, one must observe that $W^{ev}(\Lambda) \cong Z/2$ and is a direct summand of $W(\Lambda)$. This was pointed out by W. Pardon.

The calculation is a generalization of the method of characteristic elements, which one uses to compute $W(Z)/W^{ev}(Z) \cong Z/8$. (See [4].) There one associates to every non-singular bilinear form over Z a characteristic element χ in the underlying free Z-module and observes that $\langle \chi, \chi \rangle$ is an invariant of the isomorphism class of the form if reduced mod 8. Moreover, this invariant vanishes on even forms, hence the above isomorphism. In the case of $\Lambda = \hat{Z}_2\pi$, we find that we must associate a collection of characteristic elements $\chi_1, \ldots, \chi_k$, in the underlying free Λ-module, one for each conjugacy class of involutions in π. Furthermore, the elements $\langle \chi_i, \chi_j \rangle$ are all invariants in suitable quotient groups of Λ, and most of the work in this paper is the evaluation of the quotient groups. These invariants detect all but a small subgroup of $W(\Lambda)/W^{ev}(\Lambda)$, i.e., the subgroup of Hermitian forms for which all these invariants vanish is isomorphic to $\pi_{ab} \otimes Z/2$, where π_{ab} is the commutator quotient $\pi/[\pi, \pi]$.

Besides being of algebraic interest, this group is of topological interest for two reasons. First, the composite

$$W(\Lambda) \to W(\hat{Q}_2\pi) \to L_0^{+,\mathrm{tor},q}(\pi) \to L_1^{-,q}(\pi)$$

(see [1] for definitions) is a surjection from $W(\Lambda)$ to the kernel of the forgetful map $L_1^{-,q}(\pi) \to L_1^{-}(\pi)$ which neglects quadratic structure. Thus the image of $W(\Lambda)$ is the collection of surgery obstructions which are zero as "symmetric" obstructions.

Secondly, $W(\hat{Z}_2\pi) \cong L^0(\hat{Z}_2\pi)$, the 0-dimensional "algebraic cobordism group", as in [5] or [6]. One has the double skew-suspension map

$$L^n(\hat{Z}_2\pi) \to L^{n+4}(\hat{Z}_2\pi),$$

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and it seems likely that $W(\hat{Z}_2\pi)$ should play an important role in $L^{4k}(\hat{Z}_2\pi)$.

The description of $W(\wedge)$ is given by Theorem 7 and Corollary 8. As a particular consequence, we have

THEOREM. *Let* Π *be a* 2*-group, and suppose that all elements of order* 2 *are central. Then*

$$W(\hat{Z}_2\Pi) \cong Z/8 \oplus (Z/4)^r \oplus (Z/2)^{r(r-1)/2}$$
$$\oplus (\Pi_{ab} \otimes Z/2) \oplus (Z/2)$$

where $r = \#\{X \in \Pi \mid X^2 = e, X \neq e\}$ *and* $\Pi_{ab} = \Pi/[\Pi, \Pi]$, *the commutator quotient of* Π.

The paper is organized as follows—§ I discusses generalities about 2-adic group rings. § II defines the "off-diagonal" invariants, obtained by evaluating distinct characteristic elements on each other, and computes the target groups for these invariants. § III defines the "diagonal" invariants, obtained by evaluating characteristic elements on themselves, and computes the target groups for them. § IV evaluates the image of $W(\wedge)$ in the sum of the diagonal and off-diagonal target groups. § V proves that $W(\wedge)/W^{ev}(\wedge)$ is isomorphic to this, and concludes with the structure theorem V. 7 for $W(\wedge)$.

Finally, we remark that the methods in this paper extend readily to the non-orientable case; that we have not included that extension in the interest of brevity. I would like to thank W. Pardon for pointing out the structure of $W^{ev}(\wedge)$ to me.

I. Preliminaries

Let Λ be a ring with involution $-$. Given a free left Λ-module H, define

$$H^* = \mathrm{Hom}_\Lambda(H, \Lambda)$$

H^* is given a left Λ-module structure by $(\lambda \cdot \phi)(h) = \phi(h)\bar{\lambda}$, $\phi \in H^*$, $h \in H$, $\lambda \in \Lambda$.

DEFINITION 1. A Hermitian space over Λ is a pair (H, β), where
 (i) H is a free Λ-module
 (ii) β: $H \times H \to \Lambda$ is a pairing satisfying
  (a) $\beta(x, y) = \overline{\beta(y, x)}$, $x, y \in H$
  (b) $\beta(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1\beta(x_1, y) + \lambda_2\beta(x_2, y)$
  (c) $ad(\beta)$: $H \to H^*$ is an isomorphism, where $(\mathrm{ad}(\beta)(y))(x) = \beta(x, y)$.

DEFINITION 2. We say a Hermitian space (H, β) is split if there is a direct summand $K \subseteq H$, with $K = K^\perp$ ($K^\perp = \{h \in H \mid \beta(h, k) = 0\ \forall k \in K\}$). We call such a K a kernel of (H, β).

Orthogonal direct sums of Hermitian spaces, $(H_1, \beta_1) \perp (H_2, \beta_2)$, may be formed in the evident way, so we obtain the commutative monoid of isomorphism classes of Hermitian spaces over Λ, $M(\Lambda, -)$. The split spaces form a submonoid.

DEFINITION 3. We define $W(\Lambda, -)$, the Witt group of Λ, with respect to the involution $-$, to be the quotient of $M(\Lambda, -)$ by the submonoid of split spaces. When no confusion can arise, we simply write $W(\Lambda)$.

LEMMA 4. *$W(\Lambda, -)$ is a group.*

Proof. $(H, \beta) \perp (H, -\beta)$ admits a kernel, namely $\Delta = \{(h, h), h \in H\}$, so $(H, -\beta)$ is an inverse to (H, β).

Let $\tilde{\Lambda} = \{\lambda \in \Lambda \mid \lambda = \bar{\lambda}\}$.

DEFINITION 5. $I(\Lambda) = \tilde{\Lambda}/\{\lambda + \bar{\lambda}, \lambda \in \Lambda\}$.

$I(\Lambda)$ is a $Z/2$-vector space.

PROPOSITION 6. *$I(\Lambda)$ admits a left Λ-module structure by $\lambda \cdot x = \lambda x \bar{\lambda}$, for $x \in \tilde{\Lambda}$, $\lambda \in \Lambda$.*

Proof. That $1 \cdot x = x$, $\lambda \cdot (x_1 + x_2) = \lambda \cdot x_1 + \lambda \cdot x_2$, and $(\lambda_1\lambda_2) \cdot x = \lambda_1 \cdot (\lambda_2 \cdot x)$ are clear. We must only show that $(\lambda_1 + \lambda_2) \cdot x = \lambda_1 \cdot x + \lambda_2 \cdot x$. But, $(\lambda_1 + \lambda_2) \cdot x = (\lambda_1 + \lambda_2)x(\bar{\lambda}_1 + \bar{\lambda}_2) = \lambda_1 x \bar{\lambda}_1 + \lambda_2 x \bar{\lambda}_2 + \lambda_1 x \bar{\lambda}_2 + \lambda_2 x \bar{\lambda}_1$. Now $\lambda_1 x \bar{\lambda}_2 + \lambda_2 x \bar{\lambda}_1 = \lambda_1 x \bar{\lambda}_2 + \overline{(\lambda_1 x \bar{\lambda}_2)} = 0$ in $I(\Lambda)$.

We say an element in $\tilde{\Lambda}$ is even if it is of the form $\mu + \bar{\mu}$, for some $\mu \in \Lambda$. We say a Hermitian space (H, β) is *even* if $\beta(x, x)$ is even for all $x \in H$. We let $W^{ev}(\Lambda, -)$ be the subgroup of $W(\Lambda, -)$ generated by even Hermitian spaces.

LEMMA 7. *Given a Hermitian space (H, β) over Λ, the function φ: $H \to I(\Lambda)$ given by $\varphi(x) = \beta(x, x)$ is a Λ-module homomorphism.*

Proof. Clear, since $\beta(\lambda x, \lambda x) = \lambda \beta(x, x)\bar{\lambda}$ and $\beta(x_1 + x_2, x_1 + x_2) = \beta(x_1, x_1) + \beta(x_2, x_2) + \beta(x_1, x_2) + \beta(x_2, x_1) = \beta(x_1, x_1) + \beta(x_2, x_2)$ in $I(\Lambda)$.

We now specialize to the case where Λ is the 2-adic group ring $\hat{Z}_2\pi$, where π is a 2-group, and the involution is given by

$$\overline{\sum_{g \in \pi} \alpha_g g} = \sum_{g \in \pi} \alpha_g g^{-1}.$$

We state a well-known result.

PROPOSITION 8. *Let J denote the kernel of the augmentation $\Lambda \to Z/2$,*

$\sum \alpha_g g \to \sum \alpha_g$. *Then for some n,* $J^n \subseteq 2\Lambda$. Λ *is a complete local ring with maximal ideal J.*

Proof. See, e.g., [**3**].

Notation. Given a ring R and a set X, let $R(X)$ denote the free R-module on X. If X is a π-set, where π is a group, then $R(X)$ becomes an $R\pi$-module in the evident way.

Given a finite group π, let $X(\pi)$ denote the set of involutions of π, i.e.,

$$X(\pi) = \{g \in \pi \mid g^2 = e\}.$$

$X(\pi)$ becomes a π-set under the conjugation action by π. If we let $C(\pi)$ denote the set of conjugacy classes of involutions in π, we have an isomorphism of π-sets

$$X(\pi) \cong \bigcup_{\alpha \in C(\pi)} X_\alpha, \tag{$*$}$$

where X_α is the collection of all elements in α. Note that each X_α is a transitive π-set.

LEMMA 9. $I(\Lambda) \cong Z/2(X(\pi)) \cong \bigoplus_{\alpha \in C(\pi)} Z/2(X_\alpha)$, *as* Λ-*modules.*

Proof. The second equivalence follows from ($*$) above. That the first is true as an isomorphism of abelian groups is clear, since $I(\Lambda)$ will be the $Z/2$-vector space on the fixed points of the "inverse" map from π to itself, $g \to g^{-1}$, which are just the involutions of π. Moreover, if we calculate the action of a given basis element g in Λ, we find that $gx\bar{g} = gxg^{-1}$ for $x \in \tilde{\Lambda}$, which gives the isomorphism as Λ-modules.

Let i: $\tilde{\Lambda} \to I(\Lambda)$ denote the projection, and let $i_\alpha(x)$ denote the component of $i(x)$ in $Z/2(X_\alpha)$. Thus $i(x) = \bigoplus_\alpha i_\alpha(x)$.

We now have $I(\Lambda)$ split as a direct sum of cyclic modules, one for each conjugacy class of involutions in π. Given $\alpha \in C(\pi)$, if we select a particular T_α in X_α, we obtain a surjection

$$\tau_\alpha\colon \Lambda \to Z/2(X_\alpha)$$

by $g \to gT_\alpha g^{-1}$. Let J_α denote the left ideal which is the kernel of this map.

LEMMA 10. J_α *is generated as a left ideal by* 2 *and by elements of the form* $(1-s)$, *where s is in the centralizer of* T_α, *which we henceforth call* N_α.

Proof. Suppose an element $\sum_{g \in \pi} \alpha_g g \in \Lambda$ is in J_α. Then $\sum \alpha_g g T_\alpha g^{-1} = 0$ (mod 2). After modifying by multiples of 2, we see that the basis elements in this expansion must pair off and cancel. Thus, we must have a sum $\sum_i (g_i T_\alpha g_i^{-1} - g_i' T_\alpha (g_i')^{-1})$, where $g_i T_\alpha g_i^{-1} = g_i' T_\alpha (g_i')^{-1}$, so $g_i' = g_i s_i$, with $s_i \in N_\alpha$. This gives $\sum_g \alpha_g g = 2 \cdot x - \sum g_i'(1-s_i)$.

We conclude this section with

LEMMA 11. *Given any class in $W(\Lambda)$, there is a representative for that class which is diagonalizable, i.e., is isomorphic to a Hermitian space of the form*

$$\langle u_1\rangle \perp \langle u_2\rangle \perp \cdots \perp \langle u_k\rangle,$$

where $\langle u\rangle$ is the Hermitian space with $H = \Lambda \cdot e$, $\beta(e, e) = u$. Thus, u is a unit of Λ and $u = \bar{u}$.

Proof. Let (H, β) be a Hermitian space over Λ, and $x \in H$ be such that $\beta(x, x) = u$ is a unit in Λ. Then $(H, \beta) \cong \langle u\rangle \perp ((u)^\perp, \beta \mid (u)^\perp \times (u)^\perp)$. Inductively, we may show that $(H, \beta) \cong \Delta \perp (H', \beta')$, where $\Delta = \langle u_1\rangle \perp \cdots \perp \langle u_k\rangle$, and $\beta(x', x') \in J$ for all $x \in H'$. Pick $x \in H'$. By the non-singularity of β', $\exists y \in H'$ such that $\beta'(x, y) = 1$. x and y generate a summand L, and the matrix of the form restricted to this summand with respect to the basis $\{x, y\}$ is $\begin{pmatrix} \alpha_1 & 1 \\ 1 & \alpha_2 \end{pmatrix}$, with $\alpha_1, \alpha_2 \in J$. Thus, $(H', \beta') \cong (L, \beta' \mid L \times L) \perp (L^\perp, \beta' \mid L^\perp \times L^\perp)$. Thus, it suffices to prove that $L(\alpha_1, \alpha_2)$ is equivalent to a space of type $\langle u_1\rangle \perp \cdots \perp \langle u_s\rangle$, $\forall \alpha_1, \alpha_2 \in J$, where $L(\alpha_1, \alpha_2)$ has basis $\{x_1, x_2\}$, $\beta(x_i, x_i) = \alpha_i$, $\beta(x_1, x_2) = 1$. We'll prove that $\langle 1\rangle \perp L(\alpha_1, \alpha_2) \cong \langle 1\rangle \perp \langle u_2\rangle \perp \langle u_3\rangle$ for appropriate choices of u_i, $i = 1, 2, 3$, u_i a unit in Λ. Thus $L(\alpha_1, \alpha_2) \cong \langle -1\rangle \perp \langle 1\rangle \perp L(\alpha_1, \alpha_2) \cong \langle -1\rangle \perp \langle u_1\rangle \perp \langle u_2\rangle \perp \langle u_3\rangle$.

Let z be a basis for the summand $\langle 1\rangle$ in $H = \langle 1\rangle \perp L(\alpha_1, \alpha_2)$. Then $\beta(z + x_1, z + x_1) = 1 + \alpha_1$ and $\beta(z - x_2, z - x_2) = 1 + \alpha_2$ are units, and $\beta(z + x_1, z - x_2) = 1 - 1 = 0$, so $H \cong \langle 1 + \alpha_1\rangle \perp \langle 1 + \alpha_2\rangle \perp (M^\perp, \beta \mid M^\perp \times M^\perp)$, where M is generated by $z + x_1$ and $z - x_2$. Since $M^\perp$ is one-dimensional, we are done.

II. Off-diagonal invariants

As in Section I, we choose elements $T_\alpha \in X$, obtaining surjections

$$\tau_\alpha\colon \Lambda \to Z/2(X_\alpha)$$

Let (H, β) be a Hermitian space over Λ, and let $\varphi\colon H \to I(\Lambda)$ be the homomorphism of Lemma I.7. Note that φ splits as

$$\varphi = \bigoplus_\alpha \varphi_\alpha,$$

where $\varphi_\alpha(x)$ is the component of $\varphi(x)$ in the direct summand $Z/2(X_\alpha)$.

Consider $\varphi_\alpha\colon H \to Z/2(X_\alpha)$. Since H is free, there is a factorization $\tilde{\varphi}_\alpha$ so that

$$\begin{tikzcd} & \Lambda \arrow[d, "\tau_\alpha"] \\ H \arrow[ur, "\tilde{\varphi}_\alpha"] \arrow[r, "\varphi_\alpha"'] & Z/2(X_\alpha) \end{tikzcd}$$

commutes. Since the pairing β is non-singular (i.e., $ad(\beta)$ is an isomorphism), there is $\chi_\alpha \in H$, called an α-characteristic element for (H, β), so that

$$\tilde{\varphi}_\alpha(x) = \beta(x, \chi_\alpha)\ \forall x \in H.$$

The choice of characteristic element depends on the choice of factorization $\tilde{\varphi}_\alpha$. However, we have

LEMMA 1. *If $\tilde{\varphi}_\alpha$ and $\tilde{\varphi}'_\alpha$ are two factorizations of φ_α as above, and χ_α and χ'_α are the associated α-characteristic elements, then $\chi_\alpha - \chi'_\alpha \in \bar{J}_\alpha \cdot H$, where $\bar{J}_\alpha \cdot H$ is the subgroup of H generated by all elements $\bar{j}_\alpha \cdot h$, $\bar{j}_\alpha \in \bar{J}_\alpha$, $h \in H$.*

Proof. Since $\tau_\alpha \circ \tilde{\varphi}_\alpha = \tau_\alpha \circ \tilde{\varphi}'_\alpha$, we find that $\beta(x, \chi_\alpha - \chi'_\alpha) \in J_\alpha\ \forall x \in H$. Thus, $ad(\beta)(\chi_\alpha - \chi'_\alpha) \in \mathrm{Hom}_\Lambda(H, J_\alpha) \subseteq H^*$. But with H^* given the left module structure from section I, $\mathrm{Hom}_\Lambda(H, J_\alpha) = \bar{J}_\alpha \cdot H^*$. Since $ad(\beta)$ is an isomorphism,

$$\chi_\alpha - \chi'_\alpha \in \bar{J}_\alpha \cdot H.$$

If $\alpha \neq \alpha'$, this lemma shows that the value of $\beta(\chi_\alpha, \chi_{\alpha'}) \in \Lambda/J_{\alpha'} + \bar{J}_\alpha$ is independent of the choice of characteristic elements, and hence is an invariant of the isomorphism class of (H, β). Let $\Gamma_{\alpha,\alpha'} = \Lambda/J_{\alpha'} + \bar{J}_\alpha$. These elements are the off-diagonal invariants in the title of this section. We let $\sigma_{\alpha,\alpha'}$ denote the function from $M(\Lambda, -)$ to $\Gamma_{\alpha,\alpha'}$ obtained by setting $\sigma_{\alpha,\alpha'}(H, \beta) = \beta(\chi_\alpha, \chi_{\alpha'}) \in \Gamma_{\alpha,\alpha'}$, where χ_α, $\chi_{\alpha'}$ are any choices of α- and α'-characteristic elements respectively.

LEMMA 2. *$\sigma_{\alpha,\alpha'}$ induces a homomorphism $\sigma_{\alpha,\alpha'}\colon W(\Lambda)/W^{ev}(\Lambda) \to \Gamma_{\alpha,\alpha'}$.*

Proof. We first observe that $\sigma_{\alpha,\alpha'}$ is a homomorphism from the monoid $M(\Lambda, -)$ to $\Gamma_{\alpha,\alpha'}$, i.e., that $\sigma_{\alpha,\alpha'}((H, \beta) \perp (H', \beta')) = \sigma_{\alpha,\alpha'}((H, \beta)) + \sigma_{\alpha,\alpha'}((H', \beta'))$. But if χ_α and χ'_α are α-characteristic elements for H and H' respectively, then $(\chi_\alpha, \chi'_\alpha)$ is α-characteristic for $(H, \beta) \perp (H', \beta')$, so $\sigma_{\alpha,\alpha'}((H, \beta) \perp (H', \beta')) = \beta(\chi_\alpha, \chi_{\alpha'}) + \beta'(\chi'_\alpha, \chi'_{\alpha'}) = \sigma_{\alpha,\alpha'}((H, \beta)) + \sigma_{\alpha,\alpha'}((H', \beta'))$.

We note that $\sigma_{\alpha,\alpha'}$ factor through $W(\Lambda)$. To see this, we must show that $\sigma_{\alpha,\alpha'}$ vanishes on a split space (H, β). But if (H, β) is split, it admits a kernel K, and the functions φ_α all vanish on K. Hence, the characteristic elements of H may be chosen to lie in $K^\perp$. But $K=K^\perp$, hence the characteristic elements may all be taken to lie in K, so $\beta(\chi_\alpha, \chi_{\alpha'})=0$ $\forall\alpha, \alpha'$, which gives the result.

Finally, we must show that $\sigma_{\alpha,\alpha'}$ vanishes on $W^{ev}(\Lambda)$. But if $(H, \beta)\in W^{ev}(\Lambda)$, $\beta(x, x)$ is even for all $x\in H$, hence $\varphi\equiv 0$, so all the characteristic elements may be taken to be zero, which concludes the proof of the lemma.

LEMMA 3. *$\Gamma_{\alpha,\alpha'}\cong Z/2[N_\alpha\backslash\pi/N_{\alpha'}]$, where $N_\alpha\backslash\pi/N_{\alpha'}$ denotes the collection of double cosets $N_\alpha\cdot x\cdot N_{\alpha'}$.*

Proof. By Lemma I.10, $\Gamma_{\alpha,\alpha'}$ is the additive group of Λ factored by the relations $2g=0$, $g=sg$ for $s\in N_\alpha$, and $g=gt$ for $t\in N_{\alpha'}$, hence $\Gamma_{\alpha,\alpha'}\cong Z/2[N_\alpha\backslash\pi/N_{\alpha'}]$.

We now compute the image of $W(\Lambda)$ in $\bigoplus_{\alpha\neq\alpha'}\Gamma_{\alpha,\alpha'}$. The involution of Λ induces a natural isomorphism of abelian groups

$$-:\Gamma_{\alpha,\alpha'}\to\Gamma_{\alpha',\alpha},$$

which carries $\sigma_{\alpha,\alpha'}(H,\beta)$ to $\sigma_{\alpha',\alpha}(H,\beta)$. (In terms of the presentation $\Gamma_{\alpha,\alpha'}\cong Z/2[N_\alpha\backslash\pi/N_{\alpha'}]$, $\overline{N_\alpha\cdot xN_{\alpha'}}=N_{\alpha'}x^{-1}N_\alpha$). Therefore, we may order the collection of conjugacy classes $C(\pi)$ in some way, and need only consider the image of $W(\Lambda)$ in $\bigoplus_{\alpha<\alpha'}\Gamma_{\alpha,\alpha'}$. For convenience, let the conjugacy class of the identity be least in this ordering.

Each $\Gamma_{\alpha,\alpha'}$ admits a natural augmentation, $\varepsilon_{\alpha,\alpha'}$: $\Gamma_{\alpha,\alpha'}\to Z/2$, by $\varepsilon_{\alpha,\alpha'}(x)=1$ for any double coset x. We first consider the image of

$$\bigoplus_{\alpha<\alpha'}\varepsilon_{\alpha,\alpha'}\circ\sigma_{\alpha,\alpha'}\colon\ W(\Lambda)\to\bigoplus_{\alpha<\alpha'}Z/2$$

By Lemma I.11, $W(\Lambda)$ is generated by one-dimensional Hermitian spaces $\langle u\rangle$, where u is a unit in Λ and $u=\bar{u}$. Let a basis for $\bigoplus_{\alpha<\alpha'}Z/2$ be given by $e_{\alpha,\alpha'}$, and let e denote the conjugacy class of the identity.

LEMMA 4. *An element in the image of $\bigoplus_{\alpha<\alpha'}\varepsilon_{\alpha,\alpha'}\circ\sigma_{\alpha,\alpha'}$ is completely determined by its image in $\bigoplus_{\substack{\alpha<\alpha'\\ \alpha>e}}Z/2$, and $\bigoplus_{\substack{\alpha<\alpha'\\ \alpha>e}}\varepsilon_{\alpha,\alpha'}\circ\sigma_{\alpha,\alpha'}\colon\ W(\Lambda)\to\bigoplus_{\substack{\alpha<\alpha'\\ \alpha>e}}Z/2$ is a surjection, so $\operatorname{im}\left(\bigoplus_{\alpha<\alpha'}\varepsilon_{\alpha,\alpha'}\circ\sigma_{\alpha,\alpha'}\right)\cong\bigoplus_{\substack{\alpha<\alpha'\\ \alpha>e}}Z/2$.*

Proof. We have pointed out that $W(\Lambda)$ is generated by elements of the form $\langle u\rangle$. $I(\pi)$ admits an augmentation ε: $I(\pi)\to\bigoplus_{\alpha\in C(\pi)}Z/2(e_\alpha)$. Since

$u = \bar{u}$, we may consider $\varepsilon(u) \in \bigoplus_{\alpha \in C(\pi)} Z/2(e_\alpha)$, say $\varepsilon(u) = \sum_{\alpha \in C(\pi)} c_\alpha e_\alpha$, $c_\alpha \in Z/2$. Then it is easy to verify that

$$\bigoplus_{\alpha<\alpha'} \varepsilon_{\alpha,\alpha'} \circ \sigma_{\alpha,\alpha'}(\langle u \rangle) = \sum_{\alpha<\alpha'} c_\alpha c_{\alpha'} e_{\alpha,\alpha'}.$$

Since u is a unit in Λ, $\sum_\alpha c_\alpha = 1$. We now have that the image of $\langle u \rangle$ can be identified with the subset of odd order of $C(\pi)$ consisting of all those $\alpha \in C(\pi)$ with $C_\alpha = 1$. Consequently, generators of the image of $\bigoplus_{\alpha<\alpha'} \varepsilon_{\alpha,\alpha'} \circ \sigma_{\alpha,\alpha'}$ are given by g_S, where S is a subset of $C(\pi)$ of odd order, and $g_S = \sum_{\substack{\alpha,\alpha' \in S \\ \alpha<\alpha'}} e_{\alpha,\alpha'}$. We claim that the set

$$\{g_{\{e,\alpha_1,\alpha_2\}}, \alpha_1, \alpha_2 \in C(\pi), \alpha_1, \alpha_2 \neq e\}$$

generates the image of $\bigoplus_{\alpha<\alpha'} \varepsilon_{\alpha,\alpha'} \circ \sigma_{\alpha,\alpha'}$. For if $e \notin S$,

$$g_S = \sum_{\substack{\alpha_1<\alpha_2 \\ \alpha_1,\alpha_2 \in S}} g_{\{e,\alpha_1,\alpha_2\}}$$

and if $e \in S$,

$$g_S = \sum_{\substack{\alpha_1<\alpha_2 \\ \alpha_1,\alpha_2 \neq e \\ \alpha_1,\alpha_2 \in S}} g_{\{e,\alpha_1,\alpha_2\}}.$$

To see this, note that in the first case,

$$\begin{aligned}\sum_{\substack{\alpha_1,\alpha_2 \in S \\ \alpha_1<\alpha_2}} g_{\{e,\alpha_1,\alpha_2\}} &= \sum_{\substack{\alpha_1,\alpha_2 \in S \\ \alpha_1<\alpha_2}} e_{e,\alpha_1} + e_{e,\alpha_2} + e_{\alpha_1,\alpha_2} \\ &= (\#(S)-1)\Big(\sum_{\alpha \neq e} e_{e,\alpha}\Big) + \sum_{\substack{\alpha_1<\alpha_2 \\ \alpha_1,\alpha_2 \in S}} e_{\alpha_1,\alpha_2} = g_S.\end{aligned}$$

In the second case, if $e \in S$, then

$$\begin{aligned}\sum_{\substack{\alpha_1<\alpha_2 \\ \alpha_1,\alpha_2 \in S \\ \alpha_1,\alpha_2 \neq e}} g_{\{e,\alpha_1,\alpha_2\}} &= \sum_{\substack{\alpha_1<\alpha_2 \\ \alpha_1,\alpha_2 \in S \\ \alpha_1,\alpha_2 \neq e}} e_{e,\alpha_1} + e_{e,\alpha_2} + e_{\alpha_1,\alpha_2} \\ &\doteq (\#(S)-2) \sum_{\alpha>e} e_{e,\alpha} + \sum_{\substack{\alpha_1<\alpha_2 \\ \alpha_1,\alpha_2 \in S \\ \alpha_1,\alpha_2 \neq e}} e_{\alpha_1,\alpha_2} = \sum_{\alpha_1<\alpha_2} e_{\alpha_1,\alpha_2} = g_S.\end{aligned}$$

Moreover, the set $\{g_{\{e,\alpha_1,\alpha_2\}}\}$ forms a basis for $\operatorname{im}\left(\bigoplus_{\alpha<\alpha'} \varepsilon_{\alpha,\alpha'}\circ\sigma_{\alpha,\alpha'}\right)$ since they are linearly independent—note that each has only one non-zero coordinate not involving e, namely e_{α_1,α_2}, and that they are all distinct. Consequently, the image of $\bigoplus_{\alpha<\alpha'} \varepsilon_{\alpha,\alpha'}\circ\sigma_{\alpha,\alpha'}$ is isomorphic to

$$\bigoplus_{\substack{\alpha<\alpha'\\ \alpha\neq e}} Z/2(e_{\alpha,\alpha'}).$$

We denote by $I_{\alpha,\alpha'}$ the kernel of the augmentation $\varepsilon_{\alpha,\alpha'}\colon \Gamma_{\alpha,\alpha'}\to Z/2$, and remark that the augmentation filtration on Λ gives us a filtration on $\Gamma_{\alpha,\alpha'}=\Lambda/\bar{J}_\alpha+J_{\alpha'}$, by

$$\Gamma^{(s)}_{\alpha,\alpha'}=(J^{(s)}+\bar{J}_\alpha+J_{\alpha'})/\bar{J}_\alpha+J_{\alpha'},$$

where $J^{(s)}$ denotes the sth power of the augmentation ideal J. As before, $\Gamma^{(s)}_{\alpha,\alpha'}=0$ for s sufficiently large.

LEMMA 5. $\bigoplus_{\alpha<\alpha'} I_{\alpha,\alpha'}\subseteq \operatorname{im}\left(\bigoplus_{\alpha<\alpha'}\sigma_{\alpha,\alpha'}\right)$.

Proof. We will show that for any $x\in\Gamma^{(s)}_{\alpha,\alpha'}$ $(s\geq 1)$ there is a Hermitian space (H,β) so that $\sigma_{\alpha,\alpha'}((H,\beta))\equiv x \pmod{\Gamma^{(s+1)}_{\alpha,\alpha'}}$, and so that $\sigma_{\beta,\beta'}((H,\beta))=0$ if $\beta<\beta'$, $\beta\neq\alpha$, or $\beta'\neq\alpha'$. By an obvious argument, this gives the lemma. Let T_α and $T_{\alpha'}$ be the chosen fixed elements in the conjugacy classes α and α', so $\tau_\alpha(1)=T_\alpha$ and $\tau_{\alpha'}(1)=T_{\alpha'}$. Then consider the Hermitian space whose matrix is

$$\begin{bmatrix} T_\alpha & \lambda\\ \bar{\lambda} & T_{\alpha'}\end{bmatrix}\qquad \lambda\in J^{(s)}.$$

φ_β is zero unless $\beta=\alpha$ or α', hence $\chi_\beta=0$ unless $\beta=\alpha$ or α'.

We compute choices for the remaining characteristic elements, say

$$\chi_\alpha=y_1e_1+y_2e_2\quad\text{and}\quad \chi_{\alpha'}=z_1e_1+z_2e_2.$$

We must solve

$$\begin{cases}\beta(e_1,\chi_\alpha)\equiv 1(J_\alpha+J^{(s+1)})\\ \beta(e_2,\chi_\alpha)\equiv 0(J_\alpha+J^{(s+1)})\end{cases}$$

and

$$\begin{cases}\beta(e_1,\chi_{\alpha'})\equiv 0(J_{\alpha'}+J^{(s+1)})\\ \beta(e_2,\chi_{\alpha'})\equiv 1(J_{\alpha'}+J^{(s+1)}).\end{cases}$$

This gives

$$\begin{aligned} T_\alpha \bar{y}_1 + \lambda \bar{y}_2 &\equiv 1(J_\alpha + J^{(s+1)}) \\ \overline{\lambda y_1} + T_{\alpha'} \bar{y}_2 &\equiv 0(J_\alpha + J^{(s+1)}) \\ T_\alpha \bar{z}_1 + \lambda \bar{z}_2 &\equiv 0(J_\alpha + J^{(s+1)}) \\ \overline{\lambda z_1} + T_{\alpha'} \bar{z}_2 &\equiv 1(J_\alpha + J^{(s+1)}) \end{aligned}$$

so $\bar{y}_2 = T_{\alpha'} \overline{\lambda y_1}$, $\bar{z}_1 = T_\alpha \lambda \bar{z}_2$, and we obtain

$$T_\alpha \bar{y}_1 + \lambda T_{\alpha'} \overline{\lambda y_1} \equiv 1(J_\alpha + J^{(s+1)})$$

and

$$T_{\alpha'} \bar{z}_2 + \bar{\lambda} T_\alpha \lambda \bar{z}_2 \equiv 1(J_\alpha + J^{(s+1)})$$

or

$$\bar{y}_1 \equiv (T_\alpha + \lambda T_{\alpha'} \bar{\lambda})^{-1}(J_\alpha + J^{(s+1)})$$

and

$$\bar{z}_2 \equiv (T_{\alpha'} + \bar{\lambda} T_\alpha \lambda)^{-1}(J_\alpha + J^{(s+1)}).$$

But note that $(T_\alpha + \lambda T_{\alpha'} \bar{\lambda})^{-1} = \left(1 + \sum_{i=1}^{\infty} (T_\alpha \lambda T_{\alpha'} \bar{\lambda})^i T_\alpha \equiv T_\alpha (J_\alpha + J^{(s+1)}\right)$ (since $T_\alpha \lambda T_{\alpha'} \bar{\lambda} \in J^{(2s)}$), so $\bar{y}_1 \equiv T_\alpha (J_\alpha + J^{(s+1)})$. Similarly, $\bar{z}_2 \equiv T_{\alpha'}(J_\alpha + J^{(s+1)})$. Therefore,

$$\begin{gathered} \bar{y}_2 = T_\alpha, \qquad \overline{\lambda y_1} = T_{\alpha'} \bar{\lambda} T_\alpha, \\ \bar{z}_1 = T_\alpha \lambda \bar{z}_2 = T_\alpha \lambda T_{\alpha'} (J_\alpha + J^{(s+1)}). \end{gathered}$$

We now compute $\sigma_{\alpha,\alpha'}((H, \beta)) (\operatorname{mod} \Gamma_{\alpha,\alpha'}^{(s+1)})$, which is

$$\begin{aligned} \beta(\chi_\alpha, \chi_{\alpha'}) &= \beta(y_1 e_1 + y_2 e_2, z_1 e_1 + z_2 e_2) \\ &= y_1 \beta(e_1, e_1) \bar{z}_1 + y_1 \beta(e_1, e_2) \bar{z}_2 + y_2 \beta(e_2, e_1) \bar{z}_1 \\ &\quad + y_2 \beta(e_2, e_2) \bar{z}_2. \end{aligned}$$

Now $y_1 \equiv T_\alpha (\bar{J}_\alpha + J^{(s+1)})$, $y_2 \equiv T_\alpha \lambda T_{\alpha'} (\bar{J}_\alpha + J^{(s+1)})$. We get

$$\begin{aligned} &T_\alpha T_\alpha T_\alpha \lambda T_{\alpha'} + T_\alpha \lambda T_{\alpha'} + T_\alpha \lambda T_{\alpha'} \bar{\lambda} T_\alpha \lambda T_{\alpha'} + T_\alpha \lambda T_{\alpha'} T_{\alpha'} T_{\alpha'} \\ &\qquad = T_\alpha \lambda T_{\alpha'} + T_\alpha \lambda T_{\alpha'} + T_\alpha \lambda T_{\alpha'} = T_\alpha \lambda T_{\alpha'} (\bar{J}_\alpha + J_{\alpha'} + J^{(s+1)}). \end{aligned}$$

Thus, if we let $\lambda = T_\alpha x T_{\alpha'}$, we obtain $\sigma_{\alpha,\alpha'}((H, \beta)) \equiv x(\bar{J}_\alpha + J_{\alpha'} + J^{(s+1)})$, if (H, β) has matrix $\begin{bmatrix} T_\alpha & \lambda \\ \bar{\lambda} & T_{\alpha'} \end{bmatrix}$, which concludes the proof.

Now observe that if $\alpha = e$, $I_{\alpha,\alpha'} = 0$, since $\Gamma_{e,\alpha'} = \Lambda/J + J_{\alpha'} = Z/2$. Consequently, applying Lemmas 4 and 5, we find that

$$\operatorname{im}\left(\bigoplus_{\alpha < \alpha'} \sigma_{\alpha,\alpha'}\right) \cong \bigoplus_{\substack{\alpha < \alpha' \\ \alpha \neq e}} \Gamma_{\alpha,\alpha'}.$$

We now wish to study the kernel of the map $\bigoplus_{\substack{\alpha<\alpha' \\ \alpha\neq e}} \sigma_{\alpha,\alpha'}$. Let $\gamma = \bigoplus_{\substack{\alpha<\alpha' \\ \alpha\neq e}} \sigma_{\alpha,\alpha'}$.

LEMMA 6. *Given $x \in W(\Lambda)$ with $\gamma(x)=0$. Then there is an element (H, β) in the equivalence class x with*

$$(H, \beta) \cong \langle u_1 \rangle \perp \langle u_2 \rangle \perp \cdots \perp \langle u_k \rangle \perp (\bar{H}, \bar{\beta}),$$

where $u_j = T_\alpha + \lambda + \bar{\lambda}$ for some α and some $\lambda \in \Lambda$, and $(\bar{H}, \bar{\beta})$ in $W^{ev}(\Lambda)$.

Proof. Let χ_α be α-characteristic for (H, β). We may assume that $\beta(\chi_\alpha, \chi_\alpha)$ is a unit for all α. If not, we may add a copy of the split space $\langle -T_\alpha \rangle \perp \langle T_\alpha \rangle$ to (H, β) and consider the α-characteristic element $(1, \chi_\alpha) = \tilde{\chi}_\alpha$ in $\langle T_\alpha \rangle \perp (H, \beta)$. $\beta(\tilde{\chi}_\alpha, \tilde{\chi}_\alpha) = T_\alpha + \beta(\chi_\alpha, \chi_\alpha)$, which is a unit if χ_α is not. Furthermore, if $\langle T_\alpha \rangle \perp (H, \beta)$ is equivalent to a space of the desired form, then so is $\langle -T_\alpha \rangle \perp \langle T_\alpha \rangle \perp (H, \beta)$, so we may assume $\beta(\chi_\alpha, \chi_\alpha)$ to be a unit without loss of generality.

We are assuming that $\sigma_{\alpha,\alpha'}((H, \beta)) = 0$ for all $\alpha \neq \alpha'$. Consequently,

$$\beta(\chi_\alpha, \chi_{\alpha'}) \in \bar{J}_\alpha + J_{\alpha'} \qquad \forall \alpha'.$$

Therefore, we find that there are elements $h_{\alpha'} \in \bar{J}_\alpha$, $j_{\alpha'} \in J_{\alpha'}$ for all $\alpha' = \alpha$, $\alpha' \in C(\pi)$, so that

$$\beta(\chi_\alpha, \chi_{\alpha'}) = h_{\alpha'} + j_{\alpha'} \qquad \forall \alpha'.$$

Define a homomorphism $f\colon H \to \Lambda$ by splitting H as $K \oplus L$, where K is the direct summand generated by $\chi_{\alpha'}$, $\alpha' \neq \alpha$, L is any complementary summand, and defining f by $f(\chi_{\alpha'}) = -\bar{h}_{\alpha'}$ $\forall \alpha' \neq \alpha$, and $f \mid L \equiv 0$. Thus $f(h) \in J_\alpha$ $\forall h \in H$, since $h_{\alpha'} \in \bar{J}_\alpha$ $\forall \alpha'$. By the non-singularity of the form, there exists an element $h \in H$ so that

$$\beta(x, h) = f(x) \qquad \forall x \in H.$$

Again by the non-singularity of the form, we may conclude that $h \in \bar{J}_\alpha \cdot H$, since $\beta(x, h) \in J_\alpha$ $\forall x \in H$. Consequently, $\chi_\alpha + h$ is another legitimate α-characteristic element, and

$$\begin{aligned}\beta(\chi_\alpha + h, \chi_{\alpha'}) &= \beta(\chi_\alpha, \chi_{\alpha'}) + \beta(h, \chi_{\alpha'}) \\ &= h_{\alpha'} + j_{\alpha'} + \overline{\beta(\chi_{\alpha'}, h)} = h_{\alpha'} + j_{\alpha'} - h_{\alpha'} = j_{\alpha'}.\end{aligned}$$

Thus we may suppose that $\beta(\chi_\alpha, \chi_{\alpha'}) \in J_{\alpha'}$ $\forall \alpha' \neq \alpha$. Now, since $\beta(\chi_\alpha, \chi_\alpha)$ is a unit, there is an orthogonal splitting $(H, \beta) \cong \langle \beta(\chi_\alpha, \chi_\alpha) \rangle \perp (H', \beta')$, where $H' = (\chi_\alpha)^\perp$ and β' is $\beta \mid H' \times H'$. By the definition of characteristic elements, $i_{\alpha'}(\beta(\chi_\alpha, \chi_\alpha)) = \beta(\chi_\alpha, \chi_{\alpha'}) T_{\alpha'} \beta(\chi_{\alpha'}, \chi_\alpha)$ in $Z/2(X_{\alpha'}) \subseteq I(\Lambda)$. But $\beta(\chi_\alpha, \chi_{\alpha'}) \in J_{\alpha'}$ $\forall \alpha' \neq \alpha$, so $i_{\alpha'}(\beta(\chi_\alpha, \chi_\alpha)) = 0$ $\forall \alpha' \neq \alpha$.

This means that $\beta(\chi_\alpha, \chi_\alpha) \in Z/2(X_\alpha) \subseteq I(\Lambda)$, and since $\beta(\chi_\alpha, \chi_\alpha)$ is a unit, a basis change will bring it to the form $\langle T_\alpha + \lambda + \bar{\lambda}\rangle$.

(H', β') is now a Hermitian space, and since $\beta(h', \chi_\alpha) = 0\ \forall h' \in H'$, we have

$$i_\alpha(\beta(h', h')) = 0 \qquad \forall h' \in H'.$$

We now repeat the above procedure using another conjugacy class, until we have exhausted them. We are now left with $\langle u_1\rangle \perp \langle u_2\rangle \perp \cdots \perp \langle u_k\rangle \perp (\bar{H}, \bar{\beta})$, where the u_i's are of the required form and $\bar{\beta}$ satisfies $i(\bar{\beta}(\bar{h}, \bar{h})) = 0\ \forall \bar{h} \in \bar{H}$, so $\bar{\beta}(\bar{h}, \bar{h})$ is even for all $\bar{h} \in \bar{H}$, so $(\bar{H}, \bar{\beta})$ is in $W^{ev}(\Lambda)$. This proves the theorem.

III. Diagonal invariants

Let χ_α be an α-characteristic element as in Section I. Let $\Delta_\alpha = \tilde{\Lambda}/K_\alpha$, where K_α is the subgroup of $\tilde{\Lambda}$ generated by elements of the form $j_\alpha + \bar{j}_\alpha + \bar{j}'_\alpha x j'_\alpha$, where $j_\alpha, j'_\alpha \in J_\alpha$ satisfy $i_\alpha(\bar{j}_\alpha T_\alpha j_\alpha) = i_\alpha(\bar{j}'_\alpha x j'_\alpha)$.

Lemma 1. *If χ_α and χ'_α are two α-characteristic elements for (H, β), then*

$$\beta(\chi_\alpha, \chi_\alpha) \equiv \beta(\chi'_\alpha, \chi'_\alpha) \qquad (\text{mod } K_\alpha).$$

Proof. By Lemma II.1, $\chi_\alpha = \chi'_\alpha + \bar{j}_\alpha$, where $\bar{j}_\alpha \in \bar{J}_\alpha \cdot H$.

$$\begin{aligned}\beta(\chi_\alpha, \chi_\alpha) &= \beta(\chi'_\alpha + \bar{j}_\alpha, \chi'_\alpha + \bar{j}_\alpha)\\ &= \beta(\chi'_\alpha, \chi'_\alpha) + \beta(\chi'_\alpha, \bar{j}_\alpha) + \beta(\bar{j}_\alpha, \chi'_\alpha) + \beta(\bar{j}_\alpha, \bar{j}_\alpha).\end{aligned}$$

But the defining condition for an α-characteristic element guarantees that $i_\alpha(\beta(\bar{j}_\alpha, \chi'_\alpha) T_\alpha \beta(\chi'_\alpha, \bar{j}_\alpha)) = i_\alpha(\beta(\bar{j}_\alpha, \bar{j}_\alpha))$. Hence, $\beta(\chi'_\alpha, \bar{j}_\alpha) + \beta(\bar{j}_\alpha, \chi'_\alpha) + \beta(\bar{j}_\alpha, \bar{j}_\alpha) \in K_\alpha$, which proves the lemma.

Hence, the value of $\beta(\chi_\alpha, \chi_\alpha)$ in Λ/K_α is an invariant of the isomorphism class of (H, β). This gives a function $\Psi_\alpha\colon M(\Lambda, -) \to \Delta_\alpha$. The same proof as that of Lemma II.2 gives

Lemma 2. *Ψ_α induces a homomorphism $\Psi_\alpha\colon W(\Lambda)/W^{ev}(\Lambda) \to \Delta_\alpha$.*

We will now describe the group Δ_α. If G is an abelian group with involution, let

$$\tilde{G} = \{g \in G \mid g = \bar{g}\}$$

and

$$H^{\pm}(G) = \{g \in G \mid g = \pm\bar{g}\}/\{g \pm \bar{g},\ g \in G\}.$$

Let Y_α be the collection of double cosets $N_\alpha\backslash\pi/N_\alpha$. The involution $x \to x^{-1}$ acts in the natural way on Y_α, hence on $Z/2(Y_\alpha)$.

LEMMA 3. *Let* $\bar{K}_\alpha \subset \tilde{\Lambda}$ *be defined by* $\bar{K}_\alpha = \{j_\alpha + \bar{j}_\alpha, j_\alpha \in J_\alpha\}$. *Then there is an exact sequence*

$$0 \to \widetilde{Z/2(Y_\alpha)} \to \tilde{\Lambda}/\bar{K}_\alpha \to Z/2(X(\pi)) \to 0.$$

Moreover, the extension is given by $2 \cdot g = \langle g \rangle$, *where* $g \in X(\pi)$ *and* $\langle g \rangle$ *denotes the double coset containing* g.

Proof. First, we have the natural projection

$$\tilde{\Lambda}/\bar{K}_\alpha \to \tilde{\Lambda}/J_\alpha + \bar{J}_\alpha \cong \widetilde{Z/2(Y_\alpha)}.$$

The kernel of this map is

$$\widetilde{J_\alpha + \bar{J}_\alpha}/\{j_\alpha + \bar{j}_\alpha\} \cong H^+(J_\alpha + \bar{J}_\alpha).$$

But this can be evaluated by considering the long exact homology sequence associated to the short exact sequence of groups with involution

$$0 \to J_\alpha + \bar{J}_\alpha \to \Lambda \to \Lambda/J_\alpha + \bar{J}_\alpha \to 0.$$

For, we obtain

$$H^-(\Lambda) \to H^-(\Lambda/J_\alpha + \bar{J}_\alpha) \to H^+(J_\alpha + \bar{J}_\alpha) \to H^+(\Lambda) \to H^+(\Lambda/J_\alpha + \bar{J}_\alpha)$$

and $H^-(\Lambda) = 0$, $H^-(\Lambda/J_\alpha + \bar{J}_\alpha) \cong H^-(Z/2(Y_\alpha)) \cong Z/2(Z_\alpha)$, where $Z_\alpha = \{y \in Y_\alpha \mid y = \bar{y}\}$, $H^+(\Lambda) \cong Z/2(X(\pi))$, and $H^+(\Lambda/J_\alpha + \bar{J}_\alpha) \cong Z/2(Z_\alpha)$.

Here the map $Z/2(X(\pi)) \to Z/2(Z_\alpha)$ is the obvious one, taking $g \in X(\pi)$ to $\langle g \rangle$, the double coset containing g. Since $H^+(J_\alpha + \bar{J}_\alpha)$ is a $Z/2$-vector space, $H^+(J_\alpha + \bar{J}_\alpha) \cong Z/2(Z_\alpha) \oplus L$, where $L = \ker(Z/2(X(\pi)) \to Z/2(Z_\alpha))$.

So far, we have an exact sequence

$$0 \to Z/2(Z_\alpha) \oplus L \to \tilde{\Lambda}/\bar{K}_\alpha \to \widetilde{Z/2(Y_\alpha)}.$$

We now note that for $\lambda \in \tilde{\Lambda}$, $2\lambda \in \widetilde{J_\alpha + \bar{J}_\alpha}$, and furthermore it lies in

$$\ker(H^+(J_\alpha + \bar{J}_\alpha) \to H^+(\Lambda)).$$

Hence, in the exact sequence above, 2λ lies in $Z/2(Z_\alpha)$. We conclude that L is a direct summand of $\tilde{\Lambda}/\bar{K}_\alpha$, and we may rewrite our exact sequence as

$$0 \to Z/2(Z_\alpha) \to \tilde{\Lambda}/\bar{K}_\alpha \to \widetilde{Z/2(Y_\alpha)} \oplus L \to 0.$$

Also, $\widetilde{Z/2(Y_\alpha)} \cong Z/2(Z_\alpha) \oplus E$, where $E = \{x + \bar{x} \mid x \in Y_\alpha\}$. Note $2(x + \bar{x}) = 2x + 2\bar{x} \in \bar{K}_\alpha$, so E is a direct summand of $\tilde{\Lambda}/\bar{K}_\alpha$, and we obtain

$$0 \to Z/2(Z_\alpha) \oplus E \to \tilde{\Lambda}/\bar{K}_\alpha \to Z/2(Z_\alpha) \oplus L \to 0.$$

Observing that $Z/2(Z_\alpha)\oplus E\cong\widetilde{Z/2(Y_\alpha)}$ and $Z/2(Z_\alpha)\oplus L\cong Z/2(X(\pi))$, we have the promised exact sequence.

A moment's reflection shows that the map

$$\tilde{\Lambda}/\bar{K}_\alpha\to Z/2(X(\pi))$$

can be identified with the projection

$$\tilde{\Lambda}/\bar{K}_\alpha\to\tilde{\Lambda}/\{x+\bar{x},\ x\in\Lambda\}\cong Z/2(X(\pi))$$

and that the generators for $\widetilde{Z/2(Y_\alpha)}$ are all of the form $x+\bar{x}$, $x\in\Lambda$. Thus, for $\langle g\rangle\in Z/2(Z_\alpha)\subseteq\widetilde{Z/2(Y_\alpha)}$, $\langle g\rangle$ represents the element $g+\bar{g}\in\tilde{\Lambda}$. If $g\in X(\pi)$, $\langle g\rangle$ represents $2g\in\Lambda$. This gives the statement about the extension.

We continue the analysis of Δ_α. Let L_α denote the subgroup of $\tilde{\Lambda}$ generated by all elements of the form $\bar{j}_\alpha x j_\alpha$, where $x\in\tilde{\Lambda}$, $j_\alpha\in J_\alpha$. Let $C_\alpha(\pi)$ denote the set of N_α-conjugacy classes of involutions in π, i.e., the set of orbits of $X(\pi)$ under the action of N_α by conjugation. Note that each N_α-conjugacy class is contained in a unique double coset $N_\alpha\cdot x\cdot N_\alpha$.

Lemma 4. *There is an exact sequence*

$$0\to\widetilde{Z/2(Y_\alpha)}\to\tilde{\Lambda}/\bar{K}_\alpha+L_\alpha\to Z/2(C_\alpha(\pi))\to 0$$

Moreover, the extension is given by $2\cdot x=\langle x\rangle$, *where* $x\in C_\alpha(\pi)$ *and* $\langle x\rangle$ *denotes the double coset in* $\widetilde{Z/2(Y_\alpha)}$ *containing* X.

Proof. Note first that

$$\begin{aligned}(\bar{j}_\alpha+\bar{j}'_\alpha)x(j_\alpha+j'_\alpha)&=\bar{j}_\alpha x j_\alpha+\bar{j}'_\alpha x j'_\alpha+\bar{j}_\alpha x j'_\alpha+\bar{j}'_\alpha x j_\alpha\\&=\bar{j}_\alpha x j_\alpha+\bar{j}'_\alpha x j'_\alpha+\kappa,\end{aligned}$$

where $\kappa\in\bar{K}_\alpha$, so to calculate the image of L_α in $\tilde{\Lambda}/\bar{K}_\alpha$, it suffices to consider the subgroup generated by $\bar{j}_\alpha x j_\alpha$, where j_α runs over a set of generators for J_α as an abelian group. By Lemma I.10, such a set of generators is given by elements of the form $2g$ and $g(1-s)$, where $s\in N_\alpha$. Note that if $x=\lambda+\bar{\lambda}$, then $\bar{j}_\alpha x j_\alpha=\bar{j}_\alpha\lambda j_\alpha+\overline{\bar{j}_\alpha\lambda j_\alpha}\in\bar{K}_\alpha$, so we need only consider generators $(1-s)gTg^{-1}(1-s^{-1})$, where $T^2=e$ and $s\in N_\alpha$, and $4T$, where $T^2=e$. $4T\in\bar{K}_\alpha$, so only generators of the first type need be considered. But $(gTg^{-1})^2=e$ if $T^2=e$, so we need only consider generators of the form $(1-s)T(1-s^{-1})$, $s\in N_\alpha$, $T^2=e$.

Multiplying out, this generator is equal to $T+sTs^{-1}-sT-Ts^{-1}$. But in $\widetilde{Z/2(Y_\alpha)}$, $sT+Ts^{-1}$ is equal to the generator $\langle T\rangle$, i.e., the double coset $N_\alpha\cdot T\cdot N_\alpha$, since $s\in N_\alpha$, which is also represented by $2T$ in $\tilde{\Lambda}$. Thus, $T+sTs^{-1}=2T$, and we find that $T=sTs^{-1}$, hence the result. The assertion about the extension follows from Lemma 3.

We now consider a new homomorphism μ from $\tilde{\Lambda}$ to $Z/2(X(\pi))\cong I(\Lambda)$, defined by

$$\mu(x) = i(x) + i(xT_\alpha x)$$

(Recall that i: $\tilde{\Lambda} \to I(\Lambda)$ is the usual projection.) Note that $\tilde{\Lambda}$ is filtered by $\tilde{\Lambda}^{(s)} = \tilde{\Lambda} \cap J^{(s)}$, where $J^{(s)}$ denotes the sth power of the augmentation ideal J. From Proposition I.8, we know that

$$\bigcap_s J^{(s)} = 0, \quad \text{so} \quad \bigcap_s \tilde{\Lambda}^{(s)} = 0.$$

$I(\Lambda)$ inherits this filtration via the map i. We now wish to describe the image of μ in $I(\Lambda)$.

We first point out the splitting

$$I(\Lambda) = Z/2(X(\pi)) \cong Z/2(X_\alpha) \oplus \bigoplus_{\alpha' \neq \alpha} Z/2(X_{\alpha'}).$$

$Z/2(X_\alpha)$ and $\bigoplus_{\alpha' \neq \alpha} Z/2(X_{\alpha'})$ admit natural augmentations ε_1: $Z/2(X_\alpha) \to Z/2$ and

$$\varepsilon_2\colon \bigoplus_{\alpha' \neq \alpha} Z/2(X_{\alpha'}) \to Z/2,$$

by

$$\varepsilon_1(\Sigma \alpha_s S T_\alpha S^{-1}) = \Sigma \alpha_s$$

$$\varepsilon_2(\Sigma \alpha_x x) = \Sigma \alpha_x.$$

LEMMA 5. *The image of μ consists of all $(x, y) \in Z/2(X_\alpha) \oplus \bigoplus_{\alpha' \neq \alpha} Z/2(X_{\alpha'})$ such that $\varepsilon_1(x) = \varepsilon_2(y)$.*

Proof. It is clear that the image of μ lies in the subgroup described above. Also, if π' is the projection from $I(\Lambda)$ to $\bigoplus_{\alpha' \neq \alpha} Z/2(X_{\alpha'})$, $\pi'\mu(x) = \pi'(i(x))$, $x \in \tilde{\Lambda}$, since $i(xT_\alpha x) \in Z/2(x_\alpha)$. Therefore, we have shown that the composite $\pi' \circ \mu$ is surjective, and it only remains to show that any element in $\ker(\varepsilon_1)$ is in the image of μ. But $Z/2(X_\alpha)$ inherits the augmentation filtration from

$$\ker\left(\tilde{\Lambda} \xrightarrow{\pi' \circ \mu} \bigoplus_{\alpha' \neq \alpha} Z/2(X_{\alpha'})\right) \overset{(\text{defn.})}{=} \tilde{\Lambda}',$$

$\tilde{\Lambda}'^{(s)} = \tilde{\Lambda}' \cap J^{(s)}$. Now, if $x \in \tilde{\Lambda}'^{(s)}$, then $i(x) \equiv \mu(x) \pmod{Z/2(X_\alpha)^{(2s)}}$, since if $x \in \tilde{\Lambda}'^{(s)}$, then $xT_\alpha x \in \tilde{\Lambda}'^{(2s)}$. Consequently, given an element $x \in Z/2(X_\alpha)^{(s)}$, we may find an element $\tilde{x}$ in $\tilde{\Lambda}'^{(s)}$ so that $\mu(\tilde{x}) \equiv x \pmod{Z/2(X_\alpha)^{(2s)}}$. Since $\bigcap_s Z/2(X_\alpha)^{(s)} = 0$, the lemma is proved.

Observe that there is a map $\bar{\mu}$ which makes the following diagram commute:

$$\begin{tikzcd} \tilde{\Lambda} \arrow[r, "\mu"] \arrow[d] & I(\Lambda) = Z/2(X(\pi)) \arrow[d] \\ \tilde{\Lambda}/\bar{K}_\alpha + L_\alpha \arrow[r, "\bar{\mu}"] & Z/2(C_\alpha(\pi)) \end{tikzcd}$$

(The right-hand vertical arrow is the evident projection.) For, μ factors through $\tilde{\Lambda} \otimes_Z Z/2$, since $Z/2(C_\alpha(\pi))$ is a $Z/2$-vector space, and by Lemma 4,

$$\tilde{\Lambda}/\bar{K}_\alpha + L_\alpha \otimes_Z Z/2 \cong Z/2(C_\alpha(\pi)) \oplus Z/2(\Sigma'_\alpha),$$

where Σ'_α is the collection of double cosets $N_\alpha \cdot x \cdot N_\alpha$ which do not contain an involution. Thus, we have three types of basis elements.

I. $[x] \in C_\alpha(\pi)$, which is represented by the element $x \in \tilde{\Lambda}$.

II. $\langle x \rangle + \langle x^{-1} \rangle$, where $\langle x \rangle$ denotes the double coset $N_\alpha x N_\alpha$, with $N_\alpha x N_\alpha \neq N_\alpha x^{-1} N_\alpha$. This is represented by $x + x^{-1} \in \tilde{\Lambda}$.

III. $\langle x \rangle$, a double coset $N_\alpha x N_\alpha$ with $N_\alpha x N_\alpha = N_\alpha x^{-1} N_\alpha$, but so that $N_\alpha x N_\alpha$ does not contain an involution. This is represented by $x + x^{-1} \in \tilde{\Lambda}$.

To see that the map $\bar{\mu}$ exists, we must show

(i) If $x' = SxS^{-1}$, $x^2 = e$, $S \in N_\alpha$, then $\mu(x) = \mu(x')$.
(ii) If $x' = S_1 x S_2$, $S_1, S_2 \in N_\alpha$, then $\mu(x + x^{-1}) = \mu(x' + x'^{-1})$.

To see (i), note that

$$\begin{aligned} \mu(x) &= [x] + [xT_\alpha x] = [SxS^{-1}] + [SxT_\alpha xS^{-1}] \\ &= [SxS^{-1}] + [SxS^{-1}T_\alpha SxS^{-1}] = [x'] + [x'T_\alpha x'] = \mu(x'), \end{aligned}$$

where $S \in N_\alpha$.

For (ii),

$$\begin{aligned} \mu(x + x^{-1}) &= [xT_\alpha x^{-1}] + [x^{-1}T_\alpha x] = [xS_2T_\alpha S_2^{-1}x^{-1}] + [S_2^{-1}x^{-1}T_\alpha xS_2] \\ &= [S_1xS_2T_\alpha S_2^{-1}x^{-1}S_2^{-1}] + [S_2^{-1}x^{-1}S_1^{-1}T_\alpha S_1xS_2] \\ &= [x'T_\alpha x'^{-1}] + [x'^{-1}T_\alpha x'] = \mu(x' + x'^{-1}). \end{aligned}$$

Thus, $\bar{\mu}$ exists.

We now want to determine the kernel of $\bar{\mu}$. Lemma 4 can be interpreted as saying that there is a direct sum splitting

$$\bar{\Lambda}/\bar{K}_\alpha + L_\alpha \cong \widetilde{Z/2(\Sigma'_\alpha)} \oplus \Omega_\alpha$$

where $\Omega_\alpha = Z/4(C_\alpha(\pi))/R$, and R is the subgroup generated by elements of the form $2([x_1]+[x_2])$, where x_1 and x_2 belong to the same double coset in $N_\alpha\backslash\pi/N_\alpha$. We will now calculate the kernel of $\bar{\mu}$ by first calculating $\ker(\bar{\mu})\cap\Omega_\alpha$, then $\ker(\bar{\mu})/(\ker(\bar{\mu})\cap\Omega_\alpha)$, and finally the extension.

In what follows, let $\Sigma_\alpha \subseteq N_\alpha\backslash\pi/N_\alpha$ consist of those double cosets containing an involution, and let $\langle T_\alpha\rangle \in \Sigma_\alpha$ denote the double coset $N_\alpha T_\alpha N_\alpha$.

Since the range of $\bar{\mu}$ is a $Z/2$-vector space, $2\cdot\Omega_\alpha \subseteq \ker(\bar{\mu}\mid\Omega_\alpha)$. $\Omega_\alpha/2\Omega_\alpha \cong Z/2(C_\alpha(\pi))$, and we consider the induced map $\bar{\mu}\colon Z/2(C_\alpha(\pi)) \to Z/2(C_\alpha(\pi))$.

Noting as before that for $x \in Z/2(C_\alpha(\pi))^{(s)}$, $i(x) \equiv \bar{\mu}(x)$ (mod $Z/2(C_\alpha(\pi))^{(2s)}$), we find that $\bar{\mu}$ restricted to $Z/2(C_\alpha(\pi))^{(1)}$ is an isomorphism and that $[T_\alpha]$ is in the kernel of $\bar{\mu}$, hence that $\ker(\bar{\mu})\cap\Omega_\alpha = Z/4[T_\alpha] + Z/2(\Sigma_\alpha - \langle T_\alpha\rangle)$. Further, given any element $z \in \widetilde{Z/2(\Sigma'_\alpha)}$, there is by the above remark an element $y \in \Omega_\alpha$ with $\bar{\mu}(z) = \bar{\mu}(y)$ (since $\bar{\mu}(z) \in Z/2(C_\alpha(\pi))^{(1)}$).

Thus, we have an exact sequence

$$0 \to Z/4[T_\alpha] + Z/2(\Sigma_\alpha - \langle T_\alpha\rangle) \to \ker(\bar{\mu}) \to \widetilde{Z/2(\Sigma'_\alpha)} \to 0$$

and standard arguments show that the extension is entirely determined by the choice of y, which amounts to a homomorphism

$$\widetilde{Z/2(\Sigma'_\alpha)} \to Z/2(C_\alpha(\pi))^{(1)},$$

namely,

$$\widetilde{Z/2(\Sigma'_\alpha)} \xrightarrow{\bar{\mu}} Z/2(C_\alpha(\pi))^{(1)} \xrightarrow{\bar{\mu}^{-1}} Z/2(C_\alpha(\pi))^{(1)}$$

where $\bar{\mu}^{-1}$ is the inverse to $\bar{\mu}$ restricted to $Z/2(C_\alpha(\pi))^{(1)}$. The composite may be expressed as

$$\bar{\mu}^{1} \circ \bar{\mu}(\langle x\rangle + \langle x^{-1}\rangle) = \sum_{j=1}^{\infty} [\nu_j(x)] + [\nu_j(x^{-1})],$$

with $\nu_j(x)$ defined inductively by $\nu_1(x) = xT_\alpha x^{-1}$, $\nu_j(x) = \nu_{j-1}(x)T_\alpha\nu_{j-1}(x)^{-1}$. Here the sum is finite since π is a 2-group, hence is solvable. To check that the formula above is actually correct for $\bar{\mu}^{-1}\circ\mu$, we need only check that

$$\bar{\mu}\left(\sum_{j=1}^{\infty} [\nu_j(x)] + [\nu_j(x^{-1})]\right) = xT_\alpha x^{-1} + x^{-1}T_\alpha x = \nu_1(x) + \nu_1(x^{-1}).$$

But,

$$\begin{aligned}
&\bar\mu\left(\sum_{j=1}^{\infty}[\nu_j(x)]+[\nu_j(x^{-1})]\right)\\
&\quad=\sum_{j=1}^{\infty}[\nu_j(x)]+[\nu_j(x^{-1})]+\left(\sum_{j=1}^{\infty}[\nu_j(x)]+[\nu_j(x^{-1})]\right)\\
&\quad\quad\cdot(T_\alpha)\cdot\left(\sum_{j=1}^{\infty}[\nu_j(x)]+[\nu_j(x^{-1})]\right)\\
&\quad=\sum_{j=1}^{\infty}[\nu_j(x)]+[\nu_j(x^{-1})]+\sum_{j=1}^{\infty}[\nu_j(x)T_\alpha\nu_j(x)^{-1}]\\
&\quad\quad+[\nu_j(x^{-1})T_\alpha\nu_j(x^{-1})^{-1}]\\
&\quad=\sum_{j=1}^{\infty}[\nu_j(x)]+[\nu_j(x^{-1})]+\sum_{j=1}^{\infty}[\nu_{j+1}(x)]+[\nu_{j+1}(x^{-1})]\\
&\quad=\nu_1(x)+\nu_1(x^{-1}).
\end{aligned}$$

What we have shown is

PROPOSITION 6. *There is an exact sequence*

$$0\to Z/4[T_\alpha]+Z/2(\Sigma_\alpha-\langle T_\alpha\rangle)\to\ker(\bar\mu)\to\widetilde{Z/2(\Sigma'_\alpha)}\to 0$$

with extension given by $2(\langle x\rangle+\langle x^{-1}\rangle)=\sum_{j=1}^{\infty}[\nu_j(x)]+[\nu_j(x^{-1})]$, ν_j *defined as above. Alternatively,* $\ker(\bar\mu)\cong(Z/4[T_\alpha]+Z/2(\Sigma_\alpha-\langle T_\alpha\rangle)+Z/4(\Sigma'_\alpha))/R$, *where* R *is generated by elements of the form* $2(\langle x\rangle+\langle x^{-1}\rangle)-\sum_{j=1}^{\infty}[\nu_j(x)]+[\nu_j(x^{-1})]$, *and* $\langle x\rangle+\langle x^{-1}\rangle$ *is a typical generator in* $Z/2(\Sigma'_\alpha)$.

Remark. We will not alter the isomorphism class of the group by changing the extension by an automorphism of the kernel. Hence, to simplify the calculation we could rewrite the extension as $2(\langle x\rangle+\langle x^{-1}\rangle)=[xT_\alpha x^{-1}]+[x^{-1}T_\alpha x]$. Of course, this would rename the generators.

We are now prepared to compute Δ_α. For, $\Delta_\alpha\cong\bar\Lambda/K_\alpha$, and we have a commuting diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \ker(\bar\mu) \arrow[r] \arrow[d] & \ker(\bar\mu) \arrow[d] & \\
 & \dfrac{\bar K_\alpha+L_\alpha}{K_\alpha} \arrow[r] \arrow[d] & \bar\Lambda/K_\alpha \arrow[r] \arrow[d, "\bar\mu"] & \bar\Lambda/\bar K_\alpha+L_\alpha \arrow[d, "\bar\mu"] & \\
0 \arrow[r] & \Gamma \arrow[r] & Z/2(W_\alpha(\pi)) \arrow[r] & Z/2(C_\alpha(\pi)) \arrow[r] & 0
\end{tikzcd}$$

where $W_\alpha(\pi) = X_\alpha \cup \bigcup_{\alpha' \neq \alpha} (X_{\alpha'}/N_\alpha)$, $X_{\alpha'}/N_\alpha$ denoting the orbit space of $X_{\alpha'}$ by the conjugation action by N_α, and $\bar{K}_\alpha + L_\alpha/K_\alpha$ is readily seen to be isomorphic to Γ (which is defined by the exactness of the lower row) since $K_\alpha = \ker(\bar{\mu} \mid L_\alpha + \bar{K}_\alpha)$. The map

$$X_\alpha \cup \bigcup_{\alpha' \neq \alpha} (X_{\alpha'}/N_\alpha) \to C_\alpha(\pi) = \bigcup_{\alpha'} (X_{\alpha'}/N_\alpha)$$

is the identity on $X_{\alpha'}/N_\alpha$ and projection on X_α. Using the presentation for $\ker(\bar{\mu})$ provided by Proposition 6, we obtain the following description of Δ_α.

PROPOSITION 7. *$\Delta_\alpha \cong (\ker(\bar{\mu}) + Z/4(W_\alpha(\pi)))/S$, where S is generated by elements of the form $2g - \langle g \rangle$, where $g^2 = e$, $2g \in Z/4(W_\alpha(\pi))$, and $\langle g \rangle$ represents the element $N_\alpha g N_\alpha \in \Sigma_\alpha$, so $\langle g \rangle \in Z/4[T_\alpha] + Z/2(\Sigma_\alpha - \langle T_\alpha \rangle) \subseteq \ker(\bar{\mu})$.*

We have now completed the description of the groups Δ_α and $\Gamma_{\alpha,\alpha'}$.

IV. The image of $W(\Lambda)$ in $\bigoplus_{\alpha \neq e} \Delta_\alpha + \bigoplus_{\alpha \neq \alpha'} \Gamma_{\alpha,\alpha'}$

In Section II, we described the image of $W(\Lambda)$ in $\bigoplus_{\alpha \neq \alpha'} \Gamma_{\alpha,\alpha'}$ and characterized the kernel of $\bigoplus_{\alpha \neq \alpha'} \sigma_{\alpha,\alpha'}: W(\Lambda) \to \bigoplus_{\alpha \neq \alpha'} \Gamma_{\alpha,\alpha'}$. Denote this kernel by $\overline{W(\Lambda)}$. In this section, we will identify $\left(\bigoplus_{\alpha \neq e} \Psi_\alpha\right)(\bar{W}(\Lambda)) \subseteq \bigoplus_{\alpha \neq e} \Delta_\alpha$, and describe the extension in the resulting exact sequence

$$0 \to \left(\bigoplus_{\alpha \neq e} \Psi_\alpha\right)(\bar{W}(\Lambda)) \to \operatorname{im}\left(\bigoplus_{\alpha \neq e} \Psi_\alpha \oplus \bigoplus_{\alpha \neq \alpha'} \sigma_{\alpha,\alpha'}\right)$$
$$\to \operatorname{im}\left(\bigoplus_{\alpha \neq \alpha'} \sigma_{\alpha,\alpha'}\right) \to 0. \qquad (*)$$

LEMMA 1. *Let j be in the augmentation ideal J, so $T_\alpha + j$ is a unit, and suppose $j = \bar{j}$, with $i_{\alpha'}(j) = 0\ \forall \alpha' \neq \alpha$. Then $(T_\alpha + j)^{-1}$ is of the form $T_\alpha + \lambda + \bar{\lambda}$ if and only if $i_\alpha(j) = i_\alpha(jT_\alpha j)$.*

Proof. $i_\alpha(j) = i_\alpha(jT_\alpha j)$ implies that $j + jT_\alpha j$ is of the form $\lambda + \bar{\lambda}$, since it is easily verified that $(T_\alpha + j)^{-1}$ satisfies $i_{\alpha'}((T_\alpha + j)^{-1}) = 0\ \forall \alpha' \neq \alpha$. Since Λ is a complete local ring, $(T_\alpha + j)^{-1}$ may be computed as a power series, namely,

$$(T_\alpha + j)^{-1} = \sum_{i=0}^{\infty} (T_\alpha j)^i T_\alpha = T_\alpha + \sum_{i=0}^{\infty} (T_\alpha j)^i T_\alpha (j + jT_\alpha j) T_\alpha (jT_\alpha)^i$$
$$= T_\alpha + \lambda' + \bar{\lambda}',$$

since $j + jT_\alpha j$ is of the form $\lambda + \bar{\lambda}$.

Conversely, suppose $(T_\alpha+j)^{-1}$ is of the form $T_\alpha+\lambda+\bar{\lambda}$. We must show that $j=(T_\alpha+\lambda+\bar{\lambda})^{-1}-T_\alpha$ is even. Let $\nu=\lambda+\bar{\lambda}$, and note that

$$\left(\sum_{i=0}^{\infty} x_i\right)z\left(\sum_{i=0}^{\infty} \bar{x}_i\right)=\sum_{i=0}^{\infty} x_i z\bar{x}_i+\alpha+\bar{\alpha},$$

for some $\alpha\in\Lambda$, with $z=\bar{z}$, $x_i\in\Lambda$.

$$(T_\alpha+\nu)^{-1}=\sum_{i=0}^{\infty}(T_\alpha\nu)^i T_\alpha, \qquad \text{so } j=\sum_{i=1}^{\infty}(T_\alpha\nu)^i T_\alpha,$$

and

$$\begin{aligned} j+jT_\alpha j&=\sum_{i=1}^{\infty}(T_\alpha\nu)^i T_\alpha+\sum_{i=1}^{\infty}(T_\alpha\nu)^i T_\alpha(\nu T_\alpha)^i+\xi+\bar{\xi}\\ &=\sum_{i=1}^{\infty}(T_\alpha\nu)^i T_\alpha+\sum_{j=1}^{\infty}(T_\alpha\nu)^{2j}T_\alpha+\xi+\bar{\xi}=\sum_{i=1}^{\infty}(T_\alpha\nu)^{2i-1}T_\alpha+\xi'+\bar{\xi}'. \end{aligned}$$

But $(T_\alpha\nu)^{2i-1}T_\alpha=(T_\alpha\nu)^{i-1}T_\alpha\nu T_\alpha(\nu T_\alpha)^{i-1}$, which is even since ν is.

PROPOSITION 2. *The homomorphism* $\bigoplus_{\alpha\neq e}\Psi_\alpha$ *restricted to* $\overline{W(\Lambda)}$ *surjects onto the direct sum* $\bigoplus_{\alpha\neq e} M_\alpha$, *where* $M_\alpha\subseteq\Delta_\alpha$ *is the kernel of the map* $\bar{\mu}\colon \Delta_\alpha\to Z/2(W_\alpha(\pi))$.

Proof. By Lemma II.6, if $(H,\beta)\in\overline{W(\Lambda)}$, there is a Hermitian space equivalent to (H,β) of the form $\langle u_1\rangle\perp\langle u_2\rangle\perp\cdots\perp\langle u_l\rangle$ where $u_s=T_{\alpha_s}+\lambda_s+\bar{\lambda}_s$, for some $\alpha_s\in C(\pi)$ and $\lambda_s\in\Lambda$. Therefore, since $\varphi_{\alpha'}(u_s)=0$ for $\alpha'\neq\alpha_s$, we need only consider the image of Hermitian spaces of the form $\langle T_\alpha+\lambda+\bar{\lambda}\rangle$ in Δ_α, and the total image of $W(\Lambda)$ will be the direct sum of the images in the various factors.

The previous lemma shows that if x satisfies $x=T_\alpha+j$, $j=\bar{j}$, $j\in J$, and $i_{\alpha'}(j)=0$ $\forall\alpha'\neq\alpha$, then $x^{-1}\in M_\alpha$ if and only if x is of the form $T_\alpha+\lambda+\bar{\lambda}$. Consequently, we must show that

$$\Psi_\alpha(\langle T_\alpha+\lambda+\bar{\lambda}\rangle)=(T_\alpha+\lambda+\bar{\lambda})^{-1}.$$

Consider the Hermitian space $\langle T_\alpha+\lambda+\bar{\lambda}\rangle$, with basis element e, so $\beta(e,e)=T_\alpha+\lambda+\bar{\lambda}$. Then $\varphi_\alpha(e)=T_\alpha$, so a lifting $\tilde{\varphi}_\alpha$ may be chosen by $\tilde{\varphi}_\alpha(e)=1$. Therefore, the α-characteristic element may be taken to be

$$\chi_\alpha=(T_\alpha+\lambda+\bar{\lambda})^{-1}e,$$

and

$$\beta(\chi_\alpha,\chi_\alpha)=(T_\alpha+\lambda+\bar{\lambda})^{-1}(T_\alpha+\lambda+\bar{\lambda})(T_\alpha+\lambda+\bar{\lambda})^{-1}=(T_\alpha+\lambda+\bar{\lambda})^{-1},$$

concluding the proof.

To completely describe $\operatorname{im}\left(\bigoplus_{\alpha\neq e}\Psi_\alpha\oplus\bigoplus_{\alpha\neq\alpha'}\sigma_{\alpha,\alpha'}\right)$ we must analyze the extension (*). Let $P(\pi)=\bigoplus_{\alpha\neq e}M_\alpha$, and $Q(\pi)=\bigoplus_{\alpha\neq e}\Delta_\alpha$, so $P(\pi)\subseteq Q(\pi)$.

The homomorphism $\bigoplus_{\alpha\neq e}\Psi_\alpha\oplus\bigoplus_{\alpha\neq\alpha'}\sigma_{\alpha,\alpha'}$ takes values in $Q(\pi)\oplus \operatorname{im}\left(\bigoplus_{\alpha\neq\alpha'}\sigma_{\alpha,\alpha'}\right)$, and we have that the image of the kernel of $\bigoplus_{\alpha\neq\alpha'}\sigma_{\alpha,\alpha'}$ lies in $P(\pi)\oplus(0)$. Therefore, the extension (*) is completely determined by a homomorphism

$$p\colon \operatorname{im}\left(\bigoplus_{\alpha\neq\alpha'}\sigma_{\alpha,\alpha'}\right)\to Q(\pi)/P(\pi)+Q(\pi)_2,$$

where $Q(\pi)_2=\{x\in Q(\pi)\mid 2x=0\}$, defined as follows: given an element $z\in\operatorname{im}\left(\bigoplus_{\alpha\neq\alpha'}\sigma_{\alpha,\alpha'}\right)$, pick an element x of $W(\Lambda)$ with $\bigoplus_{\alpha\neq\alpha'}\sigma_{\alpha,\alpha'}(x)=z$, and compute the image of $\bigoplus_{\alpha\neq e}\Psi_\alpha(x)$ in $Q(\pi)/P(\pi)+Q(\pi)_2$.

Proposition III.7 shows that $\dfrac{Q(\pi)}{P(\pi)+Q(\pi)_2}=\bigoplus_{\alpha\neq e}D_\alpha$, where $D_\alpha\cong\dfrac{Z/2(\Sigma_\alpha)/\langle T_\alpha\rangle}{R}$, and R is generated by all elements of the form $\sum_{i=0}^{\infty}\nu^i(x)+\nu^i(x^{-1})$, $x\in\pi$, and we recall that $\nu(x)=xT_\alpha x^{-1}$, ν^i denotes the ith iterate of ν. Note that as an element in D_α, $\nu^0(x)+\nu^0(x^{-1})=x+x^{-1}$ is zero.

Before describing the homomorphism p, we must analyze D_α more carefully.

Claim. Let $\tilde R$ be the subgroup of $Z/2(\Sigma_\alpha)/\langle T_\alpha\rangle$ generated by all elements of the form $\sum_{i=0}^{\infty}\nu^i(x)$, and x is an element of the form $\lambda+\bar\lambda$. (To apply ν to an element x of Λ, set $\nu(x)=xT_\alpha\bar x$.) Then $\tilde R=R$.

To see this, we observe that $\nu\left(\sum_{i=1}^{k}x_i\right)=\sum_{i=1}^{k}\nu(x_i)+y$, where y is even, since

$$\begin{aligned}\nu\left(\sum_{i=1}^{k}x_i\right)&=\left(\sum_{i=1}^{k}x_i\right)T_\alpha\left(\sum_{i=1}^{k}\bar x_i\right)\\&=\sum_{i=1}^{k}x_iT_\alpha\bar x_i+\sum_{i>j}x_iT_\alpha\bar x_j+x_jT_\alpha\bar x_i=\sum_{i=1}^{k}\nu(x_i)+y.\end{aligned}$$

DEFINITION 3. *A collection of elements of* π, $\{S_{k,j}\}_{\substack{k=0,\dots,n\\ j=1,\dots,m(k)}}$ *is an* R-

approximation of order n to x if

$$\sum_{i=0}^{\infty} \nu^i(x) = \sum_{k=0}^{n} \sum_{l=0}^{n-k} \sum_{j=1}^{m(k)} \nu^l(S_{k,j}) + \nu^l(S_{k,j}^{-1}).$$

We say a sequence of R-approximations $\{S_{k,j}^{(t)}\}$ of order t, $t = 1, 2, \ldots$ is compatible if $S_{k,j}^{(t)} = S_{k,j}^{(w)}$ for $k \leq t, w$.

The remark preceding the definition allows us to construct a compatible sequence of R-approximations to x, if x is even. For, suppose we are given an R-approximation

$$\{S_{k,j}^{(n)}\}_{\substack{k=0,\ldots,n \\ j=1,\ldots,m(k)}}$$

to x of order n. Then

$$\nu^n(x) = \sum_{k=0}^{n} \sum_{j=1}^{m(k)} \nu^{n-k}(S_{k,j}^{(n)}) + \nu^{n-k}(S_{k,j}^{(n)-1}).$$

Therefore

$$\begin{aligned} \nu^{n+1}(x) &= \nu^n(x) T_\alpha \overline{\nu^n(x)} \\ &= \sum_{k=0}^{n} \sum_{j=1}^{m(k)} \nu^{n+1-k}(S_{k,j}^{(n)}) + \nu^{n+1-k}(S_{k,j}^{(n)-1}) + y \end{aligned}$$

where y is even, say $y = \sum_{s=1}^{p} S_s + S_s^{-1}$. Then if we set $S_{k,j}^{(n+1)} = S_{k,j}^{(n)}$ for k $N+1$, and $S_{n+1,j}^{(n+1)} = S_j$, so $m(n+1) = p$, we have constructed the required R-approximation.

Now, if we are given a compatible sequence of R-approximations $\{S_{k,j}^{(t)}\}$, $t = 1, 2, \ldots$, then x is in R, since

$$\sum_{i=0}^{\infty} \nu^i(x) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{j=1}^{m(k)} \nu^l(S_{k,j}) + \nu^l(S_{k,j}^{-1}),$$

where we may omit the superscript, since $S_{k,j}^{(t)} = S_{k,j}^{(w)}$ whenever both exist. The above sequence is a finite sum of elements in R, since $\sum_{l=0}^{\infty} \nu^l(S_{k,j}) + \nu^l(S_{k,j}^{-1}) \in R$, and since $\sum_{i=0}^{\infty} \nu^i(x)$ is finite. This proves the claim.

LEMMA 4. *Let $x, y \in \bar{\Lambda}$. Then $\sum_{i=0}^{\infty} \nu^i(x+y) = \sum_{i=0}^{\infty} \nu^i(x) + \nu^i(y)$ in D_α. In particular, if $y = \lambda + \bar{\lambda}$, $\sum_{i=0}^{\infty} \nu^i(y) \in R$, so $\sum_{i=0}^{\infty} \nu^i(x+y) = \sum_{i=0}^{\infty} \nu^i(x)$.*

Proof. Define e_n by

$$e_1 = \nu(x+y) - \nu(x) - \nu(y)$$

$$e_n = \nu^n(x+y) - \nu^n(x) - \nu^n(y) - \sum_{j=1}^{n-1} \nu^{n-j}(e_j).$$

e_n is even, for

$$\nu^{n-1}(x+y) = \nu^{n-1}(x) + \nu^{n-1}(y) + \sum_{j=1}^{n-1} \nu^{n-1-j}(e_j).$$

Applying ν to this, we obtain

$$\nu^n(x+y) = \nu(\nu^{n-1}(x+y)) = \nu(\nu^{n-1}(x) + \nu^{n-1}(y) + \sum_{j=1}^{n-1} \nu^{n-1-j}(e_j))$$
$$= \nu^n(x) + \nu^n(y) + \sum_{j=1}^{n-1} \nu^{n-j}(e_j) + \lambda,$$

where λ is even. But by definition, $e_n = \lambda$, so e_n is even. Now

$$\sum_{i=0}^{\infty} \nu^i(x+y) = \sum_{i=0}^{\infty} \nu^i(x) + \nu^i(y) + \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \nu^i(e_j).$$

The latter term is in R, so we have equality in D_α.

In order to determine p, we perform the following calculation.

LEMMA 5. *Let (H, β) be the Hermitian space over Λ represented by the symmetric matrix* $\begin{bmatrix} T_\alpha & \lambda \\ \bar{\lambda} & T_{\alpha'} \end{bmatrix}$, *with basis e_1, e_2.*

Then representatives for $\Psi_\alpha((H, \beta))$, $\Psi_{\alpha'}((H, \beta))$, and $\sigma_{\alpha,\alpha'}((H, \beta))$ may be taken to be, respectively, $(T_\alpha - \lambda T_{\alpha'}\bar{\lambda})^{-1}$, $(T_{\alpha'} - \bar{\lambda} T_\alpha \lambda)^{-1}$, and $(T_\alpha - \lambda T_{\alpha'}\bar{\lambda})^{-1}(-\lambda + \lambda T_{\alpha'}\bar{\lambda} T_\alpha \lambda)(T_{\alpha'} - \bar{\lambda} T_\alpha \lambda)^{-1}$.

Proof. We compute choices for characteristic elements. Say

$$\chi_\alpha = x_1 e_1 + x_2 e_2, \qquad \chi_{\alpha'} = y_1 e_1 + y_2 e_2$$

Then $\beta(e_1, \chi_\alpha) = 1$, $\beta(e_2, \chi_\alpha) = 0$, $\beta(e_1, \chi_{\alpha'}) = 0$, and $\beta(e_2, \chi_{\alpha'}) = 1$, yielding the equations

$$\begin{cases} T_\alpha \bar{x}_1 + \lambda \bar{x}_2 = 1 \\ \overline{\lambda x_1} + T_{\alpha'} \bar{x}_2 = 0 \end{cases} \quad \text{and} \quad \begin{cases} T_\alpha \bar{y}_1 + \lambda \bar{y}_2 = 0 \\ \overline{\lambda y_1} + T_{\alpha'} \bar{y}_2 = 1 \end{cases}$$

which we solve and obtain

$$\chi_\alpha = (T_\alpha - \lambda T_{\alpha'} \bar{\lambda})^{-1}(e_1 - \lambda T_{\alpha'} e_2)$$
$$\chi_{\alpha'} = (T_{\alpha'} - \bar{\lambda} T_\alpha \lambda)^{-1}(e_2 - \bar{\lambda} T_\alpha e_1)$$

$$\begin{aligned}\Psi_\alpha((H,\beta)) &= \beta(\chi_\alpha, \chi_\alpha)\\ &= (T_\alpha - \lambda T_{\alpha'}\bar{\lambda})^{-1}\beta(e_1 - \lambda T_{\alpha'}e_2, e_1 - \lambda T_{\alpha'}e_2)(T_\alpha + \lambda T_{\alpha'}\bar{\lambda})^{-1}\\ &= (T_\alpha - \lambda T_{\alpha'}\bar{\lambda})^{-1}(T_\alpha - \lambda T_{\alpha'}\bar{\lambda})(T_\alpha - \lambda T_{\alpha'}\bar{\lambda})^{-1}\\ &= (T_\alpha - \lambda T_{\alpha'}\bar{\lambda})^{-1}.\end{aligned}$$

The other two follow similarly.

Notice also that $(-\lambda + \lambda T_{\alpha'}\bar{\lambda}T_\alpha\lambda) = (T_\alpha - \lambda T_{\alpha'}\bar{\lambda})(-T_\alpha\lambda) = (-\lambda T_{\alpha'}) \times (T_{\alpha'} - \bar{\lambda}T_\alpha\lambda)$. Hence the representative for $\sigma_{\alpha,\alpha'}(H,\beta)$ may be taken to be either $(-T_\alpha\lambda)(T_{\alpha'} - \bar{\lambda}T_\alpha\lambda)^{-1}$ or $(T_\alpha - \lambda T_{\alpha'}\bar{\lambda})^{-1}(-\lambda T_{\alpha'})$.

We now expand $(T_\alpha - \lambda T_{\alpha'}\bar{\lambda})^{-1}$ in a power series.

$$(T_\alpha - \lambda T_{\alpha'}\bar{\lambda})^{-1} = (1 - T_\alpha\lambda T_{\alpha'}\bar{\lambda})^{-1}T_\alpha = \sum_{i=0}^{\infty}(T_\alpha\lambda T_{\alpha'}\bar{\lambda})^i T_\alpha.$$

A straightforward manipulation of power series shows that this is equal to

$$T_\alpha + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\nu^i((T_\alpha\lambda T_{\alpha'}\bar{\lambda})^j T_\alpha\lambda T_{\alpha'}\bar{\lambda}T_\alpha(\lambda T_{\alpha'}\bar{\lambda}T_\alpha)^j) \tag{i}$$

where $\nu(x) = xT_\alpha\bar{x}$ as before. Lemma 4 showed that the map $x \to \sum_{i=0}^{\infty}\nu^i(x)$ induced a homomorphism $\omega_\alpha\colon I(\Lambda) \to D_\alpha$. Letting $\lambda_{\alpha'}\colon \Lambda \to I(\Lambda)$ be given by $\lambda_{\alpha'}(x) = xT_\alpha\bar{x}$, we obtain a composite homomorphism $\omega_\alpha \circ \lambda_{\alpha'}\colon \Lambda \to D_\alpha$. We claim that $\omega_\alpha \circ \lambda_{\alpha'}$ factors as

$$\begin{tikzcd} \Lambda \arrow[r, "\omega_\alpha \circ \lambda_{\alpha'}"] \arrow[d] & D_\alpha \\ \Gamma_{\alpha,\alpha'} = \Lambda/\bar{J}_\alpha + J_{\alpha'} \arrow[ur, "\eta_\alpha"'] & \end{tikzcd}$$

For, if $x \in J_{\alpha'}$, then $\lambda_{\alpha'}(x) = 0$, so $\omega_\alpha \circ \lambda_{\alpha'}$ vanishes on $J_{\alpha'}$. On the other hand, if $x \in \bar{J}_\alpha$, we have $\omega_\alpha \circ \lambda_{\alpha'}(x) = \omega_\alpha(xT_{\alpha'}\bar{x}) = \sum_{i=0}^{\infty}\nu^i(xT_{\alpha'}\bar{x})$. But these terms are all zero in D_α since $x \in \bar{J}_\alpha$, and D_α is a quotient of $\Lambda/\bar{J}_\alpha + J_\alpha = Z/2(N_\alpha\backslash\pi/N_\alpha)$.

Lemma 5 showed that one representative for $\sigma_{\alpha,\alpha'}(H,\beta)$ is

$$-\sum_{j=0}^{\infty}(T_\alpha\lambda T_{\alpha'}\bar{\lambda})^j T_\alpha\lambda T_{\alpha'} \overset{(\text{defn})}{=} z,$$

and the calculation (i) showed that

$$\Psi_\alpha(H,\beta) = \omega_\alpha(zT_{\alpha'}\bar{z}) = \omega_\alpha \circ \lambda_{\alpha'}(z).$$

By the definition of p, this says that $p = \omega_\alpha \circ \lambda_{\alpha'}$ on $\Gamma_{\alpha,\alpha'}$, at least on the augmentation ideal. A similar argument using the Hermitian space $\langle 1+T_\alpha+T_{\alpha'}\rangle$ shows that $\omega_\alpha\circ\lambda_{\alpha'}$ determines p on all of $\Lambda/\bar{J}_\alpha+J_\alpha=\Gamma_{\alpha,\alpha'}$.

We have shown:

PROPOSITION 6. *The extension is determined by the homomorphism* $p\colon \operatorname{im}\left(\bigoplus_{\alpha\neq\alpha'}\sigma_{\alpha,\alpha'}\right)\to\bigoplus_{\alpha\neq e}D_\alpha$, *defined by* $p=\omega_\alpha\circ\lambda_{\alpha'}$ *on* $\Gamma_{\alpha,\alpha'}$.

V. The kernel of $\bigoplus_{\alpha\neq e}\Psi_\alpha+\bigoplus_{\alpha\neq\alpha'}\sigma_{\alpha,\alpha'}$

In this section, we characterize the set of Hermitian spaces (H,β) for which the invariants of the previous section vanish. We will show that such a space is equivalent to a space in which the maps $\varphi_\alpha\colon H\to Z/2(X_\alpha)$ are all zero except for φ_e. We must first produce a convenient generating set for K_α.

LEMMA 1. *K_α is generated by elements of the form $\bar{j}_\alpha Sj_\alpha$, where $\bar{S}$ is an involution not conjugate to T_α and $j_\alpha\in J_\alpha$, and of the form $\bar{j}_\alpha+j_\alpha+\bar{j}_\alpha T_\alpha j_\alpha$, $j_\alpha\in J_\alpha$.*

Proof. Recall that K_α is the subgroup of $\tilde{\Lambda}$ generated by elements of the form $\bar{j}_\alpha+j_\alpha+\bar{j}'_\alpha xj'_\alpha$, where $x\in\tilde{\Lambda}$, and $j_\alpha, j'_\alpha\in J_\alpha$ satisfy

$$i_\alpha(\bar{j}_\alpha T_\alpha j_\alpha)=i_\alpha(\bar{j}'_\alpha xj'_\alpha)$$

If we let $L_\alpha+\bar{K}_\alpha$ be the group generated by all elements of the form $\bar{j}_\alpha+j_\alpha$ and $\bar{j}_\alpha xj_\alpha$, $x\in\tilde{\Lambda}$, then K_α is the kernel of the homomorphism

$$\mu\colon \bar{K}_\alpha+L_\alpha\to Z/2(X_\alpha),\qquad \mu(x)=x+xT_\alpha x.$$

We consider the quotient K_α/Σ, where Σ is generated by $\bar{j}_\alpha Sj_\alpha$, $i_\alpha(S)=0$, and $\bar{j}_\alpha+j_\alpha+\bar{j}_\alpha T_\alpha j_\alpha$, and show it is zero. We first claim that any element in K_α/Σ is equivalent to one of the form $\sum_l j_l+\bar{j}_l$, $j_l\in J_\alpha\ \forall i$. For, if $x\in L_\alpha+\bar{K}_\alpha$, $x=\sum_l j_l+\bar{j}_l+\sum_s \bar{j}_s x_s j_s$, $j_l, j_s\in J_\alpha$, $\sum_l j_l+\bar{j}_l$ is already of the required form, and if $i_\alpha(x_s)=0$, $\bar{j}_s x_s=0$ in K_α/Σ. We must therefore show that $\bar{j}gT_\alpha g^{-1}j$ is equivalent to $\bar{j}_\alpha+j_\alpha$ for some $j_\alpha\in J_\alpha$, but clearly $\bar{j}gT_\alpha g^{-1}g=-(\bar{j}g+g^{-1}j)$ in K_α/Σ, which is of the required form. We thus assume from now on that x is of the form $\sum_l j_l+\bar{j}_l$, $j_l\in J_\alpha$.

Next we show that $4x\in\Sigma$ for any $x\in\tilde{\Lambda}$. This is clear if $i_\alpha(x)=0$, for then $4x=2\cdot x\cdot 2$ which is a generator of Σ. Now let x be an arbitrary generator for $\tilde{\Lambda}$. Then $4x=2x+2x=-4xT_\alpha x$, so if $i_\alpha(x)\in Z/2(X_\alpha)^{(1)}$, iteration of this step will eventually show that $4x=4x'$, $i_\alpha(x')=0$. Finally, $2+2+4T_\alpha$ is a generator for Σ, so $4T_\alpha=-4$, which shows $4x=0$ in K_α/Σ.

Also, $2(x+\bar{x})=2x+2\bar{x}=-4xT_\alpha\bar{x}=0$, so $2(x+\bar{x})=0$ in K_α/Σ. Now we suppose we have $x=\sum_l j_l+\bar{j}_l$ with $i_\alpha(xT_\alpha x)=0$. Since $2(x+\bar{x})=0$ in K_α/Σ, we may suppose that $x=\sum_l (1-s_l)g_l^{-1}+g_l^{-1}(1-s_l^{-1})$, with $s_l\in N_\alpha$. The condition $i_\alpha(xT_\alpha x)=0$ says that the expression $\sum_l g_lT_\alpha g_l^{-1}+s_lg_lT_\alpha g_ls_l^{-1}=0$ in $Z/2(X_\alpha)$. Using the relations in K_α/Σ, we have

$$\begin{aligned}
&(1-s_l)g_l+g_l^{-1}(1-s_l^{-1})\\
&\quad=-(1-s_l)g_lT_\alpha g_l^{-1}(1-s_l^{-1})\\
&\quad=-g_lT_\alpha g_l^{-1}-s_lg_lT_\alpha g_l^{-1}s_l^{-1}+s_lg_lT_\alpha g_l^{-1}+g_lT_\alpha g_l^{-1}s_l^{-1}\\
&\quad=-g_lT_\alpha g_l^{-1}-s_lg_lT_\alpha g_l^{-1}s_l^{-1}+2g_lT_\alpha g_l^{-1}-(1-s_l)g_lT_\alpha g_l^{-1}-g_lT_\alpha g_l^{-1}(1-s_l^{-1})\\
&\quad=g_lT_\alpha g_l^{-1}-s_lg_lT_\alpha g_l^{-1}s_l^{-1}-(1-s_l)g_lT_\alpha g_l^{-1}-g_lT_\alpha g_l^{-1}(1-s_l^{-1}),
\end{aligned}$$

so

$$\begin{aligned}
&\sum_l (1-s_l)g_l+g_l^{-1}(1-s_l^{-1})\\
&\qquad=\sum_l g_lT_\alpha g_l^{-1}-s_lg_lT_\alpha g_l^{-1}s_l^{-1}-\Big(\sum_l (1-s_l)g_lT_\alpha g_l^{-1}+g_lT_\alpha g_l^{-1}(1-s_l^{-1})\Big).
\end{aligned}$$

But since $\sum_l g_lT_\alpha g_l^{-1}+s_lg_lT_\alpha g_l^{-1}s_l^{-1}=0$ in $Z/2(X_\alpha)$, the basis elements in $\sum_l g_lT_\alpha g_l^{-1}-s_lg_lT_\alpha g_l^{-1}s_l^{-1}$ can be made to cancel in pairs, provided we change the signs of appropriate terms in $\sum_l j_l+\bar{j}_l$, which is permissible since $2(j_l+\bar{j}_l)=0$.

Thus, we have set the original sum equal to $\sum_l (1-s_l)g_lT_\alpha g_l^{-1}+g_lT_\alpha g_l^{-1}(1-s_l^{-1})$, which is a sum of the same type as we started with, except with g_l replaced by $g_lT_\alpha g_l^{-1}$. The solvability of the group π guarantees that the sequence $\{x_j\}$ of elements of π defined by $x_1=g_l$, $x_{j+1}=x_jT_\alpha x_j^{-1}$ eventually stabilizes at T_α, i.e., $x_j=T_\alpha$ for large j. Therefore, the original element x is equivalent to one of the form

$$\sum_l (1-s_l)T_\alpha+T_\alpha(1-s_l^{-1}),\qquad s_l\in N_\alpha$$

But since $T_\alpha\in N_\alpha$, this is equivalent to an expression

$$\sum_l (1-s_l)+(1-s_l^{-1})=\sum_l 2-s_l-s_l^{-1}.$$

$2-s_l-s_l^{-1}=(1-s_l)(1-s_l^{-1})\in\Sigma$, completing the proof.

COROLLARY 2. *If* $x \in \Sigma$, *and* $i_{\alpha'}(x) = 0$ $\forall x' \in S$, *where* $S \subseteq C(\pi)$ *satisfies* $e, \alpha \notin S$, *then* x *may be written as a sum*

$$\sum_s \bar{j}_s + j_s + \bar{j}_s T_\alpha j_s + \sum_l \bar{j}_l x_l j_l, \qquad x_l \in \tilde{\Lambda},$$

where $i_{\alpha'}(x_l) = 0$ $\forall \alpha' \in S$.

Proof. We have seen in Lemma 1 that x may be written as

$$x = \sum_s \bar{j}_s + j_s + \bar{j}_s T_\alpha j_s + \sum_l \bar{j}_l x_l j_l, \qquad x_l \in \tilde{\Lambda}.$$

x_l may be written as a sum

$$\sum_k \lambda_k + \bar{\lambda}_k + \sum_{\alpha' \in C(\pi)} \left[\sum_{i=1}^{n(\alpha')} g_i^{\alpha'} \right],$$

where $g_i^{\alpha'}$ is an involution in the conjugacy class α'.

$$i_{\alpha'}\left(\sum_k \lambda_k + \bar{\lambda}_k \right) = 0 \qquad \forall \alpha,$$

and

$$i_{\alpha''}\left(\left[\sum_{i=1}^{n(\alpha')} g_i^{\alpha'} \right] \right) = 0 \quad \text{if} \quad \alpha'' \neq \alpha'.$$

Therefore, it will suffice to show that if

$$i_{\alpha'}\left(\sum_l \bar{j}_l x_l j_l \right) = 0,$$

and each x_l is an element of order 2 in the conjugacy class α', then we may rewrite $\sum_l \bar{j}_l x_l j_l$ as a sum of the form

$$\sum_s \bar{j}_s + j_s + \bar{j}_s T_\alpha j_s + \sum_t \bar{j}_t x_t j_t,$$

with $i_{\alpha''}(x_t) = 0$ $\forall \alpha'' \neq e, \alpha$. A further reduction as in the proof of Lemma 1 allows us to consider only sums of the form

$$x = \sum_l (1 - s_l) g_l T_{\alpha'} g_l^{-1} (1 - s_l^{-1}), \qquad s_l \in N_\alpha.$$

We assume that $i_{\alpha'}(x) = 0$, so

$$\sum g_l T_{\alpha'} g_l^{-1} + s_l g_l T_\alpha g_l^{-1} s_l^{-1} = 0 \quad \text{in} \quad Z/2(X_{\alpha'}).$$

$$(1 - s_l) g_l T_\alpha g_l^{-1} (1 - s_l^{-1}) = g_l T_{\alpha'} g_l^{-1} + s_l g_l T_{\alpha'} g_l^{-1} s_l^{-1} - s_l g_l T_{\alpha'} g_l^{-1} - g_l T_{\alpha'} g_l^{-1} s_l^{-1}.$$

Now

$$g_lT_\alpha g_l^{-1}-s_lg_lT_{\alpha'}g_l^{-1}s_l^{-1}+g_lT_{\alpha'}g_l^{-1}-g_lT_{\alpha'}g_l^{-1}s_l^{-1}$$
$$+(g_lT_{\alpha'}g_l^{-1}-s_lg_lT_{\alpha'}g_l^{-1})T_\alpha(g_lT_{\alpha'}g_l^{-1}-g_lT_{\alpha'}g_l^{-1}s_l^{-1})$$

is of the form $j+\bar{j}+\bar{j}T_\alpha j$, hence we may subtract it and obtain

$$s_lg_lT_{\alpha'}g_l^{-1}s_l^{-1}-g_lT_{\alpha'}g_l^{-1}-\bar{j}T_\alpha j,$$

where $j\in J_\alpha$ is the element $g_lT_{\alpha'}g_l^{-1}-g_lT_{\alpha'}g_l^{-1}s_l^{-1}$. Since $i_{\alpha'}(x)=0$, we are free to change the signs of some of the summands $(1-s_l)g_lT_\alpha g_l^{-1}(1-s_l^{-1})$ and make the terms conjugate to $T_{\alpha'}$ cancel, completing the proof of the corollary.

LEMMA 3. *Suppose $x\in W(\Lambda)$, $\bigoplus_{\alpha\neq\alpha'}\sigma_{\alpha,\alpha'}(x)=0$ and $\bigoplus_{\alpha\neq e}\Psi_\alpha(x)=0$. Then x is equivalent in $W(\Lambda)/W^{ev}(\Lambda)$ to an element of the form $\langle u_1\rangle\perp\langle u_2\rangle\perp\cdots\perp\langle u_k\rangle$, where $u_j=1+\lambda_j+\bar{\lambda}_j$ for some $\lambda_j\in\Lambda$.*

Proof. Let us order $C(\pi)-\{e\}$, say as $\alpha_1,\alpha_2,\ldots,\alpha_q$. Then Lemma III.6 asserts that a representative for x may be taken to be $\coprod_{j=1}^{q}x_j$, where

$$x_j=\langle u_1^j\rangle\perp\cdots\perp\langle u_{n(j)}^j\rangle,\qquad u_i^j=T_{\alpha_j}+\lambda_i^j+\bar{\lambda}_i^j.$$

The proof will be by induction, and the inductive step will be

(A) Suppose (H,β) is equivalent to (H',β') where $i_{\alpha_s}(\beta'(x,x))=0$ $\forall x\in H'$, $s<k$, and $\Psi_{\alpha_k}(H,\beta)=0$. Then (H,β) is equivalent to (H'',β''), where $i_{\alpha_s}(\beta''(x,x))=0$ $\forall x\in H''$, $s<k+1$. This will imply the lemma, since we assume $\Psi_{\alpha_k}((H,\beta))=0$ $\forall k$. We now prove the inductive step. Suppose we are given (H',β') with $i_{\alpha_s}(\beta'(x,x))=0$ $\forall x\in H'$, $s<k$. If we form the space

$$\langle -T_{\alpha_k}\rangle\perp\langle T_{\alpha_k}\rangle\perp(H',\beta'),$$

equivalent to (H',β'), it will also satisfy the inductive hypothesis. Let $\chi_k\in H'$ be α_k-characteristic, and consider the element $(1,\chi_k)$ in $\langle T_{\alpha_k}\rangle\perp(H',\beta')$.

$T_{\alpha_k}+\beta(\chi_k,\chi_k)$ is a unit, since $\Psi_{\alpha_k}(H',\beta')=0$, so this element generates a summand of the space $\langle T_{\alpha_k}\rangle\perp(H',\beta')$. Moreover, by Corollary 2, $T_{\alpha_k}+\beta(\chi_k,\chi_k)$ may be written as a sum

$$\sum_{l=1}^{r}\bar{j}_l(\lambda_l+\bar{\lambda}_l)j_l+\sum_{s=1}^{k-1}\sum_{t=1}^{m(s)}\bar{j}_tT_{\alpha_s}j_t+\sum_{w=1}^{p}j_w+\bar{j}_w+\bar{j}_wT_{\alpha_k}j_w.$$

Form the Hermitian space

$$(\bar{H},\bar{\beta})=\langle T_{\alpha_k}\rangle\perp(H',\beta')\perp\coprod_{l=1}^{r}L_l\perp\coprod_{s=1}^{k-1}\coprod_{t=1}^{m(s)}M_{s,t}\perp\coprod_{w=1}^{p}N_w,$$

where

(i) d is a generator for the $\langle T_{\alpha_k}\rangle$ summand

(ii) L_l is the space with basis e_1, e_2 and matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(iii) $M_{s,t}$ is the space $\langle T_{\alpha_s}\rangle \perp \langle -T_{\alpha_s}\rangle$ with basis $f_1^{s,t}$, $f_2^{s,t}$

(iv) N_w is the space $\langle T_{\alpha_k}\rangle \perp \langle -T_{\alpha_k}\rangle$ with basis g_1^w, g_2^w.

$(\tilde{H}, \tilde{\beta})$ is equivalent to $\langle T_{\alpha_k}\rangle \perp (H', \beta')$ since each of the summands we added was split. To prove the lemma, we observe that the element

$$y = d + \chi_k + \sum_{l=1}^{r} \bar{j}_l\lambda_l e_1^l + j_l e_2^l + \sum_{s=1}^{k-1}\sum_{t=1}^{m(s)} \bar{j}_t f_1^{s,t}$$
$$+ \sum_{w=1}^{p} (1+\bar{j}_w) g_1^w + g_2^w$$

is α_k-characteristic for $(\tilde{H}, \tilde{\beta})$ and satisfies $\tilde{\beta}(y, y) = T_{\alpha_k}$.

$(\tilde{H}, \tilde{\beta})$ therefore breaks up as a sum $(\tilde{H}, \tilde{\beta}) = \langle T_{\alpha_k}\rangle \perp (H'', \beta'')$, where $H'' = (y)^\perp$, and β'' is $\tilde{\beta} \mid H'' \times H''$. Since y was α_k-characteristic, $i_{\alpha_s}(\beta''(x, x)) = 0\ \forall x \in H''$, $s < k+1$, and we have that (H', β') is equivalent to $\langle -T_{\alpha_k}\rangle \perp \langle T_{\alpha_k}\rangle \perp (H'', \beta'')$, from which the lemma follows since $\langle -T_{\alpha_k}\rangle \perp \langle T_{\alpha_k}\rangle$ is split.

We now complete the description of $W(\Lambda)/W^{ev}(\Lambda)$ by analyzing the subgroup of $W(\Lambda)/W^{ev}(\Lambda)$ generated by spaces of the form $\langle 1+\lambda+\bar{\lambda}\rangle$.

LEMMA 4. *Let μ_1 and μ_2 be even. Then in $W(\Lambda)/W^{ev}(\Lambda)$,*

$$\langle -1\rangle \perp \langle 1+\mu_1\rangle \perp \langle 1+\mu_2\rangle = \langle 1+\mu_1+\mu_2\rangle$$

Proof. Let a basis for the space $\langle -1\rangle \perp \langle 1+\mu_1\rangle \perp \langle 1+\mu_2\rangle$ be given by e_1, e_2, e_3. Then by considering the basis

$$e_1+e_2+e_3, \qquad (1+\mu_1)e_1+e_2, \qquad (1+\mu_2)e_1+e_3,$$

we find that the given space is isomorphic to $\langle 1+\mu_1+\mu_2\rangle \perp Y$, where Y has matrix

$$\begin{bmatrix} (1+\mu_1)^2-(1+\mu_1) & -(1+\mu_1)(1+\mu_2) \\ -(1+\mu_2)(1+\mu_1) & (1+\mu_2)^2-(1+\mu_2) \end{bmatrix}$$

One now sees that it is sufficient to show that μ_i^2 is even. To show this, one need only check on generators $s+s^{-1}$. But

$$(s+s^{-1})^2 = 2+s+s^{-1} = (1+s)+(1+s^{-1}).$$

LEMMA 5. *There is a surjection from $Z/8 \oplus \pi_{ab} \otimes Z/2$ to a subgroup of*

$W(\Lambda)/W^{ev}(\Lambda)$ generated by elements $\langle 1+\lambda+\bar{\lambda}\rangle$ (π_{ab} denotes the commutator quotient $\pi/[\pi, \pi]$).

Proof. We define the map by requiring that the generator of the $Z/8$ summand go to the space $\langle 1\rangle$, and if $x \in \pi_{ab} \otimes Z/2$, then

$$x \to \langle -1\rangle \perp \langle -1\rangle \perp \langle -1\rangle \perp \langle 1+\tilde{x}+\tilde{x}^{-1}\rangle$$

where $\tilde{x}$ is an element of π mapping to x. All that requires proof is that the map restricted to $\pi_{ab} \otimes Z/2$ is well-defined.

To do this, we consider the map $\hat{\mathbf{Z}}_2\pi \to \pi_{ab} \otimes Z/2$, defined by $g \to (g)$, where (g) is the image of g in π_{ab}. The function $g \to \langle -1\rangle \perp \langle -1\rangle \perp \langle -1\rangle \perp \langle 1+g+g^{-1}\rangle$ defines a homomorphism δ: $\hat{\mathbf{Z}}_2\pi \to W(\Lambda)/W^{ev}(\Lambda)$. Since

$$(1+g_1+g_2^{-1})(1+g_1^{-1}+g_2) = 3+g_1+g_1^{-1}+g_2+g_2^{-1}+g_1g_2+(g_1g_2)^{-1},$$
$$\langle 1\rangle = \langle 3+g_1+g_1^{-1}+g_2+g_2^{-1}+g_1g_2+(g_1g_2)^{-1}\rangle$$

in $W(\Lambda)/W^{ev}(\Lambda)$, and we have by Lemma 4 that

$$\begin{aligned}
&\delta(1)+\delta(g_1)+\delta(g_2)+\delta(g_1g_2)\\
&\quad = \langle -1\rangle^{(12)} \perp \langle 1+2\rangle \perp \langle 1+g_1+g_1^{-1}\rangle \perp \langle 1+g_2+g_2^{-1}\rangle \perp \langle 1+g_1g_2+(g_1g_2)^{-1}\rangle\\
&\quad = \langle -1\rangle^{(4)} \perp \langle 1+2\rangle \perp \langle 1+g_1+g_1^{-1}\rangle \perp \langle 1+g_2+g_2^{-1}\rangle \perp \langle 1+g_1g_2+(g_1g_2)^{-1}\rangle\\
&\quad = \langle -1\rangle \perp \langle 3+g_1+g_1^{-1}+g_2+g_2^{-1}+g_1g_2+(g_1g_2)^{-1}\rangle = 0.
\end{aligned}$$

This means that we may factor δ through $\hat{\mathbf{Z}}_2\pi/(1+g_1+g_2+g_1g_2)$. Secondly, $\delta(1) = \langle -1\rangle \perp \langle -1\rangle \perp \langle -1\rangle \perp \langle 3\rangle = 0$, so we further factor δ through $\hat{\mathbf{Z}}_2\pi/(1, (1+g_1+g_2+g_1g_2))$. It is a standard calculation that this group is isomorphic to $J/J^2 \cong \pi_{ab}$. Finally, the relation $\delta(g) = \delta(g^{-1})$ guarantees that the map factors through $\pi_{ab} \otimes Z/2$.

PROPOSITION 6. *The group generated by elements $\langle 1+\lambda+\bar{\lambda}\rangle$ is isomorphic to $Z/8 \oplus \pi_{ab} \otimes Z/2$. It is a direct summand of $W(\Lambda)/W^{ev}(\Lambda)$.*

Proof. We must produce a homomorphism from $W(\Lambda)/W^{ev}(\Lambda) \to Z/8 \oplus \pi_{ab} \otimes Z/2$. To obtain the map to $Z/8$, simply take $W(\Lambda)/W^{ev}(\Lambda) \to W(\hat{\mathbf{Z}}_2)/W^{ev}(\hat{\mathbf{Z}}_2) \cong Z/8$. As for the map to $\pi_{ab} \otimes Z/2$, let $\Lambda' = Z/4(\pi_{ab} \otimes Z/2)$, and consider $W(\Lambda)/W^{ev}(\Lambda) \to W(\Lambda')/W^{ev}(\Lambda')$. There is a determinant homomorphism

$$\det\colon\ W(\Lambda)/W^{ev}(\Lambda) \to U(\Lambda')/N^{+}(\Lambda'),$$

where $U(\Lambda')$ denotes the group of units of Λ' and $N^{+}(\Lambda')$ denotes the subgroup of $U(\Lambda')$ generated by units of the form $u\bar{u}$, and 3.

Since the involution on Λ' is trivial, this group is isomorphic to $U(\Lambda') \otimes Z/2$. Now it is a straightforward check that the elements of the form $1+2\lambda' \in U(\Lambda')$, $\lambda' \in \Lambda'$, generate a direct summand isomorphic to

$\pi_{ab} \otimes Z/2$, and that the composite

$$\pi_{ab} \otimes Z/2 \to W(\Lambda)/W^{ev}(\Lambda) \to \pi_{ab} \otimes Z/2$$

is the identity. This completes the proof.

Finally, we recapitulate. Let the image of

$$W(\Lambda)/W^{ev}(\Lambda) \quad \text{in} \quad \bigoplus_{\alpha \neq e} \Delta_\alpha \oplus \bigoplus_{\alpha \neq \alpha'} \Gamma_{\alpha,\alpha'}$$

be denoted by $\mathscr{A}$. $\mathscr{A}$ was completely described by Lemmas II.4 and II.5 together with Propositions III.7 and IV.6.

THEOREM 7. $W(\Lambda)/W^{ev}(\Lambda) \cong \mathscr{A} \oplus Z/8 \oplus \pi_{ab} \otimes Z/2$.

COROLLARY 8. $W(\Lambda) \cong Z/2 \oplus \mathscr{A} \oplus Z/8 \oplus \pi_{ab} \otimes Z/2$.

Proof. As remarked in the introduction $W^{ev}(\Lambda) \cong Z/2$, and it is seen to be a direct summand, since it is in $W(\hat{\mathbf{Z}}_2)$.

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