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Wu Invariants of Hermitian Forms

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INTRODUCTION

In this paper, we consider the following question: given a Hermitian ϵ-symmetric form (H, β) over a ring with involution Λ, when is (H, β) Witt equivalent to an *even* form, i.e. one where $\beta(x, x)$ is always of the form $\lambda + \epsilon\bar{\lambda}$? The answer is given by constructing a group $Q^\epsilon(\Lambda)$, functorial on the category of rings with involution, and a homomorphism w: $W^\epsilon(\Lambda) \to Q^\epsilon(\Lambda)$ so that (H, β) is equivalent to an even form if and only if $w(H, \beta) = 0$. The group $Q^\epsilon(\Lambda)$ is readily computable—it is the homology of a certain two-term chain complex defined over Λ. Furthermore, the homomorphism w is easily evaluated on any particular form. For example, in the case $\Lambda = \mathbb{Z}$, with trivial involution, $Q^\epsilon(\Lambda) = \mathbb{Z}/8$ and $w(H, \beta)$ is the reduction of the signature mod 8. This work is a generalization of the results of [1], in which $Q^1(\Lambda)$ was implicitly computed for $\Lambda = \hat{\mathbb{Z}}_2\pi$, where π is a 2-group. $W^1(\Lambda)$ was then obtained directly. The definition of $Q^1(\Lambda)$ given here would have simplified many of the computations in the specific case [1], and would have simplified immensely the proof of the analogue to our Theorem IV.2.

This reformulation of the characteristic elements of [1] owes much to the work of Andrew Ranicki [6]. The argument in the proof of Theorem IV.2 makes use of the zero dimensional version of Ranicki's algebraic surgery on the Wu map ψ_β. Our invariant $w(H, \beta)$ is also the zero-dimensional version of the desuspension invariant introduced in [2].

In the final section, we apply these results to rederive the calculation of $W(\mathcal{O}_K)$, where K is an unramified extension of $\hat{\mathbb{Q}}_2$. We also produce invariants of $W(K)$, where K is any field of characteristic 2 so that $[K : K^2] < \infty$, which detect the Witt classes, and compute the Witt group $W(\mathbb{Z}/2\pi)$, $\pi = (\mathbb{Z}/2)^i$, with trivial involution.

Section I contains preliminary material on rings with involution. Section II defines $Q^\epsilon(\Lambda)$, Section III prove the functoriality of Q^ϵ on the category of rings with involution, Section IV defines the natural transformation w: $W^\epsilon \to Q^\epsilon$,

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and Section V proves the main theorem, Theorem V.2, which asserts that (H, β) is Witt equivalent to an even form if and only if $w(H, \beta) = 0$. Section VI contains the above-mentioned examples, namely $\Lambda = \mathbb{Z}$, $\Lambda = \mathcal{O}_K$, $\Lambda = K$, and $\Lambda = \mathbb{Z}/2\pi$. The invariants for K a field of characteristic 2, $[K : K^2]$ finite, are described in Theorem VI.3.

We remark that work has been done in the case of Dedekind rings on this problem (see [7] and [8]), and thank the referee for a number of helpful remarks.

I. Preliminaries

Let Λ be a ring with involution —. Given a left Λ-module H, define

$$H^* = \mathrm{Hom}_\Lambda(H, \Lambda).$$

H^* is given a left Λ-module structure by $(\lambda \cdot \phi)(h) = \phi(h)\bar{\lambda}$, $\phi \in H^*$, $h \in H$, $\lambda \in \Lambda$. If H is finitely generated projective, then so is H^*, and there is a natural isomorphism $i\colon H \to (H^*)^*$ of left Λ-modules, defined by $(i(x))(\phi) = \overline{\phi(x)}$. Given a homomorphism $f\colon H_1 \to H_2$, the dual map $f^*\colon H_2^* \to H_1^*$ is defined by $f^*(\phi)(x) = \phi(f(x))$. If $f\colon H \to H^*$, then $f^*\colon (H^*)^* \to H^*$. Composing with i, we get $f^* \circ i\colon H \to H^*$.

Proposition 1. $(f^* \circ i(x))(y) = \overline{f(y)(x)}$.

Proof. Immediate from the definition (*).

We often suppress mention of i, implicitly identifying $(H^*)^*$ and H.

Definition 2. $H^\epsilon(\Lambda) = \{\lambda \in \Lambda \mid \lambda = \epsilon\bar{\lambda}\}/\{\lambda + \epsilon\bar{\lambda}, \lambda \in \Lambda\}$, $\epsilon = \pm 1$.

$H^\epsilon(\Lambda)$ is a $\mathbb{Z}/2$-vector space. The following is also well known.

Proposition 3. *$H^\epsilon(\Lambda)$ becomes a left Λ-module under the action $\lambda \cdot \alpha = \lambda\alpha\bar{\lambda}$, $\lambda \in \Lambda$, $\alpha \in H^\epsilon(\Lambda)$.*

At this point, we assume that $H^\epsilon(\Lambda)$ is finitely generated as a Λ-module and that Λ is a Noetherian ring. This hypothesis will apply throughout the paper.

If $\varphi\colon H \to H^*$ satisfies $\varphi = \epsilon\varphi^*$ (suppressing i), then the function $x \to \varphi(x)(x)$ is a Λ-homomorphism from H to $H^\epsilon(\Lambda)$.

Definition 4. Given $\varphi\colon H \to H^*$, $\varphi = \epsilon\varphi^*$ we define $\hat{\varphi}\colon H \to H^\epsilon(\Lambda)$ by

$$\hat{\varphi}(x) = \varphi(x)(x).$$

We refer the reader to [3] or [5] for the definition of Hermitian spaces and Witt groups. For us, $W^\epsilon(\Lambda)$ will denote the Witt group of ϵ-symmetric Hermitian forms over Λ, where the underlying modules and subkernels are required to be finitely generated and projective. We conclude with

PROPOSITION 5. *If* $\varphi = \Psi + \epsilon\Psi^*$, $\hat{\varphi} \equiv 0$.

Proof. $(\Psi + \epsilon\Psi^*)(x)(x) = \Psi(x)(x) + \epsilon\Psi^*(x)(x) = \Psi(x)(x) + \epsilon\, \overline{\Psi(x)(x)}$ (by Prop. 1). This element is zero in $H^\epsilon(\Lambda)$.

II. THE GROUP $Q^\epsilon(\wedge)$

Let $\mathscr{R} = \mathscr{P}_2 \to^{\partial} \mathscr{P}_1 \to^{\rho} H^\epsilon(\Lambda)$ be the first two stages of a projective resolution of $H^\epsilon(\Lambda)$, where $\mathscr{P}_i$ is finitely generated and projective. Such a resolution exists since we assumed $H^\epsilon(\Lambda)$ to be finitely generated and Λ to be Noetherian. Define $A_1^\epsilon(\mathscr{R}) \subseteq \mathrm{Hom}_\Lambda(\mathscr{P}_1^*, \mathscr{P}_1)$ by

$$A_1^\epsilon(\mathscr{R}) = \{\Psi \in \mathrm{Hom}_\Lambda(\mathscr{P}_1^*, \mathscr{P}_1) \mid \Psi = \epsilon\Psi^*\}$$

Consider the group $\Gamma = \mathrm{Hom}_\Lambda(\mathscr{P}_2^*, \mathscr{P}_2) \oplus \mathrm{Hom}_\Lambda(\mathscr{P}_2^*, \mathscr{P}_1) \oplus \mathrm{Hom}_\Lambda(\mathscr{P}_1^*, \mathscr{P}_2)$. The duality involution acts on this by $(\phi_1, \phi_2, \phi_3) \to (\phi_1^*, \phi_3^*, \phi_2^*)$. Define $A_2^\epsilon(\mathscr{R}) \subseteq \Gamma$ by

$$A_2^\epsilon(\mathscr{R}) = \{(\phi_1, \phi_2, \phi_3) \mid (\phi_1, \phi_2, \phi_3) = \epsilon(\phi_1^*, \phi_3^*, \phi_2^*)\}$$

We define homomorphisms α_i and β_i from $A_i^\epsilon(\mathscr{R})$ to $\mathrm{Hom}_\Lambda(\mathscr{P}_1^*, H^\epsilon(\Lambda))$ by

$$\begin{aligned} \alpha_1(\psi) &= \rho \circ \psi & \beta_1(\psi) &= \hat{\psi} \\ \alpha_2(\phi_1, \phi_2, \phi_3) &= \rho \circ \phi_2 & \beta_2(\phi_1, \phi_2, \phi_3) &= \hat{\phi}_1 . \end{aligned}$$

Now define $B_i^\epsilon(\mathscr{R}) \subseteq A_i^\epsilon(\mathscr{R})$

$$B_i^\epsilon(\mathscr{R}) = \{x \in A_i^\epsilon(\mathscr{R}) \mid \alpha_i(x) = \beta_i(x)\}.$$

There is a "boundary map" δ: $B_2^\epsilon(\mathscr{R}) \to A_1^\epsilon(\mathscr{R})$, defined by $\delta(\phi_1, \phi_2, \phi_3) = \partial\phi_1\partial^* + \phi_2\partial^* + \partial\phi_3$. Since $\phi_3 = \epsilon\phi_2^*$, and $\phi_1 = \epsilon\phi_1^*$, $\delta(\phi_1, \phi_2, \phi_3) \in A_1^\epsilon(\mathscr{R})$.

PROPOSITION 1. $\delta(B_2^\epsilon(\mathscr{R})) \subseteq B_1^\epsilon(\mathscr{R})$.

Proof. We must show that the element $\partial\phi_1\partial^* + \phi_2\partial^* + \epsilon\partial\phi_2^* \in B_1^\epsilon(\mathscr{R})$, provided that $\rho \circ \phi_2 = \hat{\phi}_1$.

$$\begin{aligned} \alpha_1(\partial\phi_1\partial^* + \phi_2\partial^* + \epsilon\partial\phi_2^*)(x) &= \alpha_1(\phi_2\partial^*)(x) \\ &= \rho \circ \phi_2 \circ \partial^*(x) = \alpha_2(\phi_1, \phi_2, \phi_3) \circ \partial^*(x) \\ &= \beta_2(\phi_1, \phi_2, \phi_3) \circ \partial^*(x) = \hat{\phi}_1 \circ \partial^*(x) \\ &= \phi_1(\partial^*(x))(\partial^*(x)) = (\partial \circ \phi_1 \circ \partial^*)(x)(x) \\ &= \widehat{\partial \circ \phi_1 \circ \partial^*}(x) = \beta_1(\partial \circ \phi_1 \circ \partial^*)(x). \end{aligned}$$

It only remains to show that $\beta_1(\phi_2\partial^* + \epsilon\partial\phi_2^*) = 0$. But $\partial\phi_2^* = (\phi_2\partial^*)^*$, so this follows from Prop. I.5. ∎

DEFINITION 2. $Q^\epsilon(\Lambda) = B_1^\epsilon(\mathscr{R})/\delta(B_2^\epsilon(\mathscr{R}))$.

We must show that this definition of $Q^\epsilon(\Lambda)$ depends only on ϵ and Λ, not on the choice of resolution

$$\mathscr{P}_2 \xrightarrow{\partial} \mathscr{P}_1 \xrightarrow{\rho} H^\epsilon(\Lambda)$$

We define $C_*^\epsilon(\mathscr{R})$, where $\mathscr{R}$ is a two-step resolution $\mathscr{P}_2 \to^\partial \mathscr{P}_1 \to^\rho H^\epsilon(\Lambda) \to 0$, to be the chain complex

$$0 \to B_2^\epsilon(\mathscr{R}) \xrightarrow{\delta} B_1^\epsilon(\mathscr{R}) \to 0$$

Thus, $H_1(C_*^\epsilon(\mathscr{R})) = Q^\epsilon(\Lambda)$.

PROPOSITION 3. *Let $f\colon \mathscr{R} \to \bar{\mathscr{R}}$ be any chain map covering the identity, where $\mathscr{R}$ and $\bar{\mathscr{R}}$ are two-step resolutions of $H^\epsilon(\Lambda)$. Then f induces a chain map $C_*(f)$: $C_*(\mathscr{R}) \to C_*(\bar{\mathscr{R}})$.*

Proof. Let $f_i\colon \mathscr{P}_i \to \bar{\mathscr{P}}_i$, $i = 1, 2$, be the components of f in dimensions 1 and 2. Then we define $C_*(f)$ by

$$C_1(f)\colon B_1^\epsilon(\mathscr{R}) \to B_1^\epsilon(\bar{\mathscr{R}}),$$
$$C_1(f)(\varphi) = f_1 \circ \varphi \circ f_1^*$$
$$C_2(f)\colon B_2^\epsilon(\mathscr{R}) \to B_2^\epsilon(\bar{\mathscr{R}}),$$
$$C_2(f)(\varphi_1, \varphi_2, \varphi_3) = (f_2 \circ \varphi_1 \circ f_2^*, f_1 \circ \varphi_2 \circ f_2^*, f_2 \circ \varphi_3 \circ f_1^*)$$

It is immediate that $C_*^\epsilon(f)$ takes $A_i^\epsilon(\mathscr{R})$ to $A_i^\epsilon(\bar{\mathscr{R}})$, and that $C_*^\epsilon(f)$ is a chain map; we must show that $C_i^\epsilon(f)(B_i^\epsilon(\mathscr{R})) \subseteq B_i^\epsilon(\bar{\mathscr{R}})$. We observe that it suffices, by the definition of B_i^ϵ, to check the commutativity of the following diagrams:

$$\text{(I.)}\quad \begin{array}{ccc} A_i^\epsilon(\mathscr{R}) & \xrightarrow{\alpha_i} & \mathrm{Hom}_\Lambda(\mathscr{P}_i^*, H^\epsilon(\Lambda)) \\ \Big\downarrow{\scriptstyle C_i^\epsilon(f)} & & \Big\downarrow{\scriptstyle \xi_i} \\ A^{i\epsilon}(\bar{\mathscr{R}}) & \xrightarrow{\bar{\alpha}_i} & \mathrm{Hom}_\Lambda(\bar{\mathscr{P}}_i^*, H^\epsilon(\Lambda)) \end{array}$$

$$\text{(II.)}\quad \begin{array}{ccc} A_i^\epsilon(\mathscr{R}) & \xrightarrow{\beta_i} & \mathrm{Hom}_\Lambda(\mathscr{P}_i^*, H^\epsilon(\Lambda)) \\ \Big\downarrow{\scriptstyle C_i^\epsilon(f)} & & \Big\downarrow{\scriptstyle \xi_i} \\ A_i^\epsilon(\bar{\mathscr{R}}) & \xrightarrow{\beta_i} & \mathrm{Hom}_\Lambda(\bar{\mathscr{P}}_i\,; H^\epsilon(\Lambda)) \end{array}$$

where $\xi_i(g) = g \circ f_i^*$.

$$\text{(I)} \quad (i = 1)\ \bar{\alpha}_1(C_1^\epsilon(f)(\varphi)) = \bar{\rho} \circ f_1 \circ \varphi \circ f_1^* = \rho \circ \varphi \circ f_1^* = \alpha_1(\varphi) \circ f_1^* = \xi_1(\alpha_1(\varphi)).$$

$$\begin{aligned} \text{(I)} \quad (i = 2)\ &\bar{\alpha}_2(C_2^\epsilon(f)(\varphi_1, \varphi_2, \varphi_3)) \\ &= \rho \circ f_1 \circ \varphi_2 \circ f_2^* = \rho \circ \varphi_2 \circ f_2^* \\ &= \alpha_2(\varphi_1, \varphi_2, \varphi_3) \circ f_2^* = \xi_2(\alpha_2(\varphi_1, \varphi_2, \varphi_3)). \end{aligned}$$

$$\begin{aligned} \text{(II)} \quad (i = 1)\ &\bar{\beta}_1(C_1^\epsilon(f)(\varphi))(x) \\ &= \bar{\beta}_1(f_1 \circ \varphi \circ f_1^*)(x) = \widehat{f_1 \circ \varphi \circ f_1^*}(x) \\ &= \langle x, f_1 \circ \varphi \circ f_1^*(x)\rangle = \langle f_1^*(x), \varphi \circ f_1^*(x)\rangle = \hat{\varphi}(f_1^*(x)) \\ &= (\beta_1(\varphi) \circ f_1^*)(x) = \xi_1(\beta_1(\varphi))(x). \end{aligned}$$

Here $\langle\ ,\ \rangle\colon \mathscr{P}_1^* \times \mathscr{P}_i \to \Lambda$ is the evaluation pairing.

$$\begin{aligned} \text{(II)} \quad (i = 2)\ &\bar{\beta}_2(C_2^\epsilon(f)(\varphi_1, \varphi_2, \varphi_3))(x) \\ &= \bar{\beta}_2(f_2 \circ \varphi_1 \circ f_2^*, f_1 \circ \varphi_2 \circ f_2^*, f_2 \circ \varphi_3 \circ f_1^*)(x) \\ &= \widehat{(f_2 \circ \varphi_1 \circ f_2^*)}(x) = \langle x, f_2 \circ \varphi_1 \circ f_2^*(x)\rangle \\ &= \langle f_2^*(x), \varphi_1 \circ f_2^*(x)\rangle = \hat{\varphi}_1(f_2^*(x)) \\ &= (\beta_2(\varphi_1, \varphi_2, \varphi_3) \circ f_2^*)(x) = \xi_2(\beta_2(\varphi_1, \varphi_2, \varphi_3))(x). \end{aligned}$$

Thus, $C_*^\epsilon(f)$ induces a map $D^\epsilon(f)\colon Q^\epsilon(\Lambda) \to \bar{Q}^\epsilon(\Lambda)$, where $\bar{Q}^\epsilon(\Lambda)$ is computed using $\bar{\mathscr{R}}$.

Proposition 4. *Let $\mathscr{R}, \bar{\mathscr{R}}$ be as in Prop. 3, and let $f, g\colon \mathscr{R} \to \bar{\mathscr{R}}$ be two coverings of the identity map of $H^\epsilon(\Lambda)$. Then $D^\epsilon(f) = D^\epsilon(g)$.*

Proof. It will suffice to show that $f_1 \circ \varphi \circ f_1^* - g_1 \circ \varphi \circ g_1^* \in \bar{\delta}(B_2(\bar{\mathscr{R}}))$, for all $\varphi \in B_1^\epsilon(\mathscr{R})$. Since f and g are both coverings of the identity, $g_1 = f_1 + \bar{\partial} \circ \lambda$, where $\lambda\colon \mathscr{P}_1 \to \bar{\mathscr{P}}_2$ is a Λ-map. We compute: $g_1 \circ \varphi \circ g_1 = (f_1 + \bar{\partial} \circ \lambda) \circ \varphi \circ (f_1^* + \lambda^* \circ \bar{\partial}^*) = f_1 \circ \varphi \circ f_1^* + \bar{\partial} \circ \lambda \circ \varphi \circ f_1^* + f_1 \circ \varphi \circ \lambda^* \circ \bar{\partial}^* + \bar{\partial} \circ \lambda \circ \varphi \circ \lambda^* \circ \bar{\partial}^*$. We must show that $\gamma \in \bar{\delta}(B_2^\epsilon(\bar{\mathscr{R}}))$, where $\gamma = \bar{\partial} \circ \lambda \circ \varphi \circ f_1^* + f_1 \circ \varphi \circ \lambda^* \circ \bar{\partial}^* + \bar{\partial} \circ \lambda \circ \varphi \circ \lambda^* \circ \bar{\partial}^*$. But, $(\lambda \circ \varphi \circ \lambda^*, f_1 \circ \varphi \circ \lambda^*, \lambda \circ \varphi \circ f_1^*) \in B_2^\epsilon(\bar{\mathscr{R}})$, since

$$(f_1 \circ \varphi \circ \lambda^*)^* = \lambda \circ \varphi^* \circ f_1^* = \epsilon\lambda \circ \varphi \circ f_1^*,$$

and

$$\begin{aligned} \widehat{(\lambda \circ \varphi \circ \lambda^*)}(x) &= \langle x, \lambda \circ \varphi \circ \lambda^*(x)\rangle = \langle \lambda^*(x), \varphi \circ \lambda^*(x)\rangle = \hat{\varphi}(\lambda^*(x)) \\ &= \rho \circ \varphi(\lambda^*(x)) = (\bar{\rho} \circ f_1 \circ \varphi \circ \lambda^*)(x). \end{aligned}$$

The second to last equality follows since $\varphi \in B_1^\epsilon(\mathscr{R})$. Moreover $\bar{\delta}(\lambda \circ \varphi \circ \lambda^*, f_1 \circ \varphi \circ \lambda^*, \lambda \circ \varphi \circ f_1^*) = \gamma$, which concludes the proof. ▌

COROLLARY 5. *$Q^\epsilon(\Lambda)$ is independent of the two-step resolution used to compute it; furthermore, if $Q^\epsilon(\Lambda)$ and $\bar{Q}^\epsilon(\Lambda)$ are computed using $\mathscr{R}$ and $\bar{\mathscr{R}}$ respectively, then there is a distinguished isomorphism from $Q^\epsilon(\Lambda)$ to $\bar{Q}^\epsilon(\Lambda)$, characterized by the requirement that it be induced by a chain map f: $\mathscr{R} \to \bar{\mathscr{R}}$ covering the identity map of $H^\epsilon(\Lambda)$.*

III. FUNCTORIALITY OF THE Q-GROUPS

Let f: $\Lambda \to \Lambda'$ be a map of rings with involution.

PROPOSITION 1. *Let $\mathscr{P}$ be finitely generated and projective over Λ. Then the natural map*

$$\Lambda' \otimes_\Lambda \mathscr{P}^* \xrightarrow{i} (\Lambda' \otimes_\Lambda \mathscr{P})^*$$

defined by $i(\lambda_1 \otimes \varphi)(\lambda_2 \otimes x) = \lambda_2 f(\varphi(x))\,\bar{\lambda}_1$ is an isomorphism of Λ'-modules.

Proof. (Well-known). ▌

Let $\mathscr{P}$ be a finitely generated Λ-module, $\mathscr{M}$ a finitely generated Λ'-module, and g: $\mathscr{P} \to \mathscr{M}$ a Λ-map, where $\mathscr{M}$ is a Λ-module by restriction of scalars. Then there is a unique factorization

$$\begin{tikzcd} & \Lambda' \otimes_\Lambda \mathscr{P} \arrow[dr, "\tilde{g}"] & \\ \mathscr{P} \arrow[ur, "1\otimes\mathrm{id}"] \arrow[rr, "g"'] & & \mathscr{M} \end{tikzcd}$$

with $\tilde{g}$ a Λ-module map.

We point out that there is a canonical map of abelian groups

$$\begin{aligned} \alpha(\mathscr{P}_1, \mathscr{P}_2)\colon \mathrm{Hom}_\Lambda(\mathscr{P}_1^*, \mathscr{P}_2) &\to \mathrm{Hom}_{\Lambda'}(\Lambda' \otimes_\Lambda \mathscr{P}_1^*, \Lambda' \otimes \mathscr{P}_2) \\ &\cong \mathrm{Hom}_{\Lambda'}((\Lambda' \otimes_\Lambda \mathscr{P}_1)^*, \Lambda' \otimes_\Lambda \mathscr{P}_2) \end{aligned}$$

where the second isomorphism is that given by Prop. 1. Moreover, this map respects duality in that the diagram

$$\begin{tikzcd} \mathrm{Hom}_\Lambda(\mathscr{P}_1^*, \mathscr{P}_2) \arrow[r, "\alpha(\mathscr{P}_1, \mathscr{P}_2)"] \arrow[d] & \mathrm{Hom}_\Lambda((\Lambda' \otimes_\Lambda \mathscr{P}_1)^*, \Lambda' \otimes_\Lambda \mathscr{P}_2) \arrow[d] \\ \mathrm{Hom}_\Lambda(\mathscr{P}_2^*, \mathscr{P}_1) \arrow[r, "\alpha(\mathscr{P}_2, \mathscr{P}_1)"] & \mathrm{Hom}_{\Lambda'}((\Lambda' \otimes_\Lambda \mathscr{P}_2)^*, \Lambda' \otimes_\Lambda \mathscr{P}_1) \end{tikzcd}$$

commutes.

Let $C_*^\epsilon(\mathscr{R})$ and $C_\epsilon^*(\mathscr{S})$ be defined using resolutions $\mathscr{R}$: $\mathscr{P}_2 \to^\partial \mathscr{P}_1 \to^\rho H^\epsilon(\Lambda)$ and $\mathscr{S}$: $\mathscr{M}_2 \to^\partial \mathscr{M}_1 \to^\rho H^\epsilon(\Lambda')$, of Λ and Λ'-modules respectively. Note that there is a natural map of Λ-modules η: $H^\epsilon(\Lambda) \to H^\epsilon(\Lambda')$. Consequently, lifting η through the two-step resolutions, we obtain

$$\begin{tikzcd} \mathscr{P}_2 \arrow[r, "g_2"] \arrow[d, "\partial"'] & \mathscr{M}_2 \arrow[d, "\partial"'] \\ \mathscr{P}_1 \arrow[r, "g_1"] & \mathscr{M}_1 \end{tikzcd}$$

where the g_i's are Λ-module maps. Let $\tilde{g}_i$ be the factorization defined above. Define β_{ij}: $\mathrm{Hom}_\Lambda(\mathscr{P}_i^*, \mathscr{P}_j) \to \mathrm{Hom}_{\Lambda'}(\mathscr{M}_i^*, \mathscr{M}_j)$ by $\beta_{ij}(\varphi) = \tilde{g}_j \circ \alpha(\mathscr{P}_i, \mathscr{P}_j) \circ \tilde{g}_i^*$. β_{ij} commutes with duality in the same sense as α does. Form the complexes $C_\epsilon^*(\mathscr{R})$, $C_\epsilon^*(\mathscr{S})$ as in Section II.

PROPOSITION 2. *There is a chain map $C_\epsilon^*(\mathscr{R}) \to^\gamma C_\epsilon^*(\mathscr{S})$, defined by $\gamma_1(\varphi) = \beta_{11}(\varphi)$, $\gamma_2(\varphi_1, \varphi_2, \varphi_3) = (\beta_{22}(\varphi_1), \beta_{21}(\varphi_2), \beta_{12}(\varphi_3))$.*

Proof. We first observe that $\gamma_i(A_i^\epsilon(\mathscr{R})) \subseteq \gamma_i(A_i^\epsilon(\mathscr{S}))$, since β_{ij} respects duality. We must show that $\gamma_i(B_i^\epsilon(\mathscr{R})) \subseteq \gamma_i(B_i^\epsilon(\mathscr{S}))$.

To prove this, it suffices to observe that the diagrams

$$\begin{tikzcd} A_i^\epsilon(\mathscr{R}) \arrow[r, "\gamma_i"] \arrow[d, "\alpha_i, \beta_i"] & A_i^\epsilon(\mathscr{S}) \arrow[d, "\alpha_i, \beta_i"] \\ \mathrm{Hom}_\Lambda(\mathscr{P}_i^*, H^\epsilon(\Lambda)) \arrow[r, "\sigma_i"] & \mathrm{Hom}_{\Lambda'}(\mathscr{M}_i^*, H^\epsilon(\Lambda')) \end{tikzcd}$$

commute, where σ_i is the composite

$$\begin{aligned} \mathrm{Hom}_\Lambda(\mathscr{P}_i^*, H^\epsilon(\Lambda)) &\to \mathrm{Hom}_{\Lambda'}((\Lambda' \otimes_\Lambda \mathscr{P}_i)^*, \Lambda' \otimes_\Lambda H^\epsilon(\Lambda)) \\ &\to \mathrm{Hom}_{\Lambda'}(\mathscr{M}_i^*, H^\epsilon(\Lambda')) \end{aligned}$$

the second arrow being defined by $h \to \tilde{\eta} \circ h \circ \tilde{g}_i^*$. The commutativity of the diagram is elementary, as is the fact that γ_* is a chain map. ∎

Consequently, γ_* induces a map $Q^\epsilon(\gamma)$: $Q^\epsilon(\Lambda) \to Q^\epsilon(\Lambda')$.

PROPOSITION 3. *Let γ_*, γ_*' be chain maps $C_*^\epsilon(\mathscr{R}) \to C^\epsilon(\mathscr{S})$ associated with maps g, g', with g, g' inducing η. Then $Q^\epsilon(\gamma_*) = Q^\epsilon(\gamma_*')$.*

Proof. Since g, g' both induce η, we have that $g_1' = g_1 + \partial \circ \lambda$, where $\lambda \colon \mathscr{P}_1 \to \mathscr{M}_2$ is a Λ-map. Consequently,

$$\begin{aligned}
\gamma_1'(\varphi) &= \tilde{g}_1 \circ \alpha(\mathscr{P}_1, \mathscr{P}_1)(\varphi) \circ \tilde{g}_1^* \\
&= (\tilde{g}_1 + \partial \circ \tilde{\lambda}) \circ \alpha(\mathscr{P}_1, \mathscr{P}_1)(\varphi) \circ (\tilde{g}_1^* + \tilde{\lambda}^* \circ \partial^*) \\
&= Q^\epsilon(\gamma_*)(\varphi) + \partial \circ \tilde{\lambda} \circ \alpha(\mathscr{P}_1, \mathscr{P}_1)(\varphi) \circ \tilde{g}_1^* + \tilde{g}_1 \circ \alpha(\mathscr{P}_1, \mathscr{P}_1)(\varphi) \circ \tilde{\lambda}^* \circ \partial^* \\
&\quad + \partial \circ \tilde{\lambda} \circ \alpha(\mathscr{P}_1, \mathscr{P}_1)(\varphi) \circ \tilde{\lambda}^* \circ \partial^*.
\end{aligned}$$

We must show that the last three terms lie in $\delta(B_2^\epsilon(\mathscr{S}))$. Now set

$$x = (\tilde{\lambda} \circ \alpha(\mathscr{P}_1, \mathscr{P}_1)(\varphi) \circ \tilde{\lambda}^*, \tilde{g}_1 \circ \alpha(\mathscr{P}_1, \mathscr{P}_1)(\varphi) \circ \tilde{\lambda}^*, \tilde{\lambda} \circ \alpha(\mathscr{P}_1, \mathscr{P}_1)(\varphi) \circ \tilde{g}_1^*) \in A_2^\epsilon(\mathscr{S}).$$

It claim $x \in B_2^\epsilon(\mathscr{S})$. For,

$$\begin{aligned}
\beta_2(x) &= \widehat{\lambda \circ \alpha(\mathscr{P}_1, \mathscr{P}_1)(\varphi) \circ \tilde{\lambda}^*} = \widehat{\alpha(\mathscr{P}_1, \mathscr{P}_1)(\varphi)} \circ \tilde{\lambda}^* = \sigma_1(\hat{\varphi}) \circ \tilde{\lambda}^* \\
&= \sigma_1(\rho \circ \varphi) \circ \tilde{\lambda}^* = \rho \circ \tilde{g}_1 \circ \alpha(\mathscr{P}_1, \mathscr{P}_1)(\varphi) \circ \tilde{\lambda}^* = \alpha_2(x).
\end{aligned}$$

$\hat{\varphi} = \rho \circ \varphi$ since $\varphi \in A_1^\epsilon(\mathscr{R})$. Now,

$$\begin{aligned}
\delta(x) &= \partial \circ \tilde{\lambda} \circ \alpha(\mathscr{P}_1, \mathscr{P}_1)(\varphi) \circ \tilde{g}_1^* + \tilde{g}_1 \circ \alpha(\mathscr{P}_1, \mathscr{P}_1)(\varphi) \circ \tilde{\lambda}^* \circ \partial^* \\
&\quad + \partial \circ \tilde{\lambda} \circ \alpha(\mathscr{P}_1, \mathscr{P}_1)(\varphi) \circ \tilde{\lambda}^* \circ \partial^*. \quad \blacksquare
\end{aligned}$$

COROLLARY 4. *There is a canonical change of rings map $Q^\epsilon(\Lambda) \xrightarrow{Q^\epsilon(f)} Q^\epsilon(\Lambda')$, characterized by the requirement that it be induced by a Λ-module chain map from a Λ-resolution of $H^\epsilon(\Lambda)$ to a Λ'-resolution of $H^\epsilon(\Lambda')$, covering the Λ-map $\eta \colon H^\epsilon(\Lambda) \to H^\epsilon(\Lambda')$.*

IV. THE MAP $w : W^\epsilon(\Lambda) \to Q^\epsilon(\Lambda)$

Let (H, β) denote an ϵ-symmetric Hermitian space over Λ. We will define an element $w(H, \beta) \in Q^\epsilon(\Lambda)$ as follows: let $\mathrm{ad}(\beta) \colon H \to H^*$ denote the adjoint map to the pairing $\beta \colon H \times H \to \Lambda$. It is an isomorphism, and $\mathrm{ad}(\beta)^* = \epsilon\, \mathrm{ad}(\beta)$. From Definition I.4, we have the map

$$\widehat{\mathrm{ad}(\beta)} \colon H \to H^\epsilon(\Lambda)$$

Let $\mathscr{R} = \mathscr{P}_2 \xrightarrow{\partial} \mathscr{P}_1 \xrightarrow{\rho} H^\epsilon(\Lambda)$ be the first two stages of a resolution of $H^\epsilon(\Lambda)$, $\mathscr{P}_i$ f.g. projective, as in Section II. Then since H is projective, the map $\widehat{\mathrm{ad}(\beta)}$ lifts to $\mathscr{P}_1$. Let the lifting be denoted by $\psi_\beta \colon H \to \mathscr{P}_1$. We define $\xi_\beta \in \mathrm{Hom}_\Lambda(\mathscr{P}_1^*, \mathscr{P}_1)$ by

$$\xi_\beta = \psi_\beta \circ \mathrm{ad}(\beta)^{-1} \circ \psi_\beta^*$$

PROPOSITION 1. $\xi_\beta \in B_1^\epsilon(\mathscr{R})$.

Proof. Since $\operatorname{ad}(\beta) = \epsilon \operatorname{ad}(\beta)^*$, $\operatorname{ad}(\beta)^{-1} = \epsilon(\operatorname{ad}(\beta)^{-1})^*$.
We compute:

$$
\begin{aligned}
\alpha_1(\xi_\beta)(x) &= \alpha_1(\psi_\beta \circ (\operatorname{ad}(\beta)^{-1}) \circ \psi_\beta^*)(x) \\
&= (\rho \circ \psi_\beta \circ \operatorname{ad}(\beta)^{-1} \circ \psi_\beta^*)(x) = (\widehat{\operatorname{ad}(\beta)} \circ \operatorname{ad}(\beta)^{-1} \circ \psi_\beta^*)(x) \\
&= (\operatorname{ad}(\beta)(\operatorname{ad}(\beta)^{-1} \circ \psi_\beta^*(x)))(\operatorname{ad}(\beta)^{-1} \circ \psi_\beta^*(x)) \\
&= (\psi_\beta^*(x))(\operatorname{ad}(\beta)^{-1} \circ \psi_\beta^*(x)) \\
&= x(\psi_\beta \circ \operatorname{ad}(\beta)^{-1} \circ \psi_\beta^*(x)) = \epsilon(\psi_\beta \circ \operatorname{ad}(\beta)^{-1} \circ \psi_\beta^*(x))(x) \\
&= \widehat{\psi_\beta \circ \operatorname{ad}(\beta)^{-1} \circ \psi_\beta^*(x)} = \beta_1(\xi_\beta)(x), \qquad \text{so} \quad \xi_\beta \in B_1^\epsilon(\mathscr{R}). \qquad (*)
\end{aligned}
$$

PROPOSITION 2. *The residue class of ξ_β in $Q^\epsilon(\Lambda)$ is independent of the choice of ψ_β.* ■

Proof. Given ψ_β and ψ_β', liftings of $\widehat{\operatorname{ad}(\beta)}$, we must show that the associated elements ξ_β and ξ_β' differ by an element in $\delta(B_2^\epsilon(\mathscr{R}))$. Since ψ_β and ψ_β' are both liftings of $\operatorname{ad}(\beta)$, $\psi_\beta' = \psi_\beta + \partial \circ \chi$, where $\chi\colon H \to \mathscr{P}_2$. Thus,

$$
\begin{aligned}
\xi_\beta' - \xi_\beta &= \partial \circ \chi \circ \operatorname{ad}(\beta)^{-1} \circ \psi_\beta^* + \psi_\beta \circ \operatorname{ad}(\beta)^{-1} \circ \chi^* \circ \partial^* \\
&\quad + \partial \circ \chi \circ \operatorname{ad}(\beta)^{-1} \circ \chi^* \circ \partial^*
\end{aligned}
$$

Now, consider the element

$$\mu = (\chi \circ \operatorname{ad}(\beta)^{-1} \circ \chi^*, \psi_\beta \circ \operatorname{ad}(\beta)^{-1} \circ \chi^*, \chi \circ \operatorname{ad}(\beta)^{-1} \circ \psi_\beta^*)$$

in $A_2^\epsilon(\mathscr{R})$. We claim that $\mu \in B_2^\epsilon(\mathscr{R})$. To show this, we must have

$$\rho \circ \psi_\beta \circ \operatorname{ad}(\beta)^{-1} \circ \chi^* = \widehat{\chi \circ \operatorname{ad}(\beta)^{-1} \circ \chi^*}.$$

Now,

$$
\begin{aligned}
\rho \circ \psi_\beta \circ \operatorname{ad}(\beta)^{-1} \circ \chi^*(x) &= \widehat{\operatorname{ad}(\beta)} \circ \operatorname{ad}(\beta)^{-1} \circ \chi^*(x) \\
&= \operatorname{ad}(\beta)(\operatorname{ad}(\beta)^{-1} \circ \chi^*(x))(\operatorname{ad}(\beta)^{-1} \circ \chi^*(x)) \\
&= \chi^*(x)(\operatorname{ad}(\beta)^{-1} \circ \chi^*(x)) = x(\chi \circ \operatorname{ad}(\beta)^{-1} \circ \chi^*(x)) \\
&= \epsilon(\chi \circ \operatorname{ad}(\beta)^{-1} \circ \chi^*(x))(x) = \widehat{\chi \circ \operatorname{ad}(\beta)^{-1} \circ \chi^*(x)}
\end{aligned}
$$

$\delta(\mu) = \xi_\beta' - \xi_\beta$, proving the result. ∎

We have constructed an invariant in $Q^\epsilon(\Lambda)$ depending only on the isomorphism class of the space (H, β), which we will call $w(H, \beta) \in Q^\epsilon(\Lambda)$. We will proceed to show that this invariant induces a homomorphism $w\colon W^\epsilon(\Lambda) \to Q^\epsilon(\Lambda)$.

PPROOSITION 3. *If (H, β) is split, then $w(H, \beta) = 0$.*

Proof. Let $K \subseteq H$ be a projective summand, with $K = K^\perp$ $(=\{x \in H \mid \beta(x, K) = 0\})$, which exists since (H, β) is split. Since $\beta(k, k) = 0$ for $k \in K$, $\widehat{\mathrm{ad}(\beta)}(k) = 0$ for $k \in K$, so we may choose the map ψ_β so that $\psi_\beta \mid K \equiv 0$, K being a direct summand of H.

Claim. $\mathrm{ad}(\beta)^{-1} \circ \psi_\beta^*(x) \in K$, if ψ_β is chosen so that $\psi_\beta \mid K \equiv 0$.

We show that $\beta(\mathrm{ad}(\beta)^{-1} \circ \psi_\beta^*(x), k) = 0$ $\forall k \in K$. This will suffice, since $K = K^\perp$.

$$\beta(\mathrm{ad}(\beta)^{-1} \circ \psi_\beta^*(x), k) = (\mathrm{ad}(\beta) \circ \mathrm{ad}(\beta)^{-1} \circ \psi_\beta^*(x))(k) = \psi_\beta^*(x)(k) = x(\psi_\beta(k)) = 0.$$

Now, $\psi_\beta \circ \mathrm{ad}(\beta)^{-1} \circ \psi_\beta^*(x) = 0$ since $\psi_\beta \mid K \equiv 0$. ∎

LEMMA 4. $w((H_1, \beta_1) \perp (H_2, \beta_2)) = w(H_1, \beta_1) + w(H_2, \beta_2)$.

Proof. $\widehat{\mathrm{ad}(\beta_1 \perp \beta_2)} = \widehat{\mathrm{ad}(\beta_1)} \perp \widehat{\mathrm{ad}(\beta_2)}$, so we may take $\psi_{\beta_1 \perp \beta_2} = \psi_{\beta_1} \perp \psi_{\beta_2}$. ∎

Proposition 3 and Lemma 4 taken together prove that w induces a homomorphism

$$w\colon W^\epsilon(\Lambda) \to Q^\epsilon(\Lambda)$$

$w(H, \beta)$ will be called the *Wu element* of (H, β).

PROPOSITION 5. *$w\colon W^\epsilon(\Lambda) \to Q^\epsilon(\Lambda)$ is canonically defined; i.e., if $Q^\epsilon(\Lambda)$ and $\bar{Q}^\epsilon(\Lambda)$ are computed using resolutions $\mathscr{R}$ and $\bar{\mathscr{R}}$, and w and $\bar{w}$ are the associated Wu maps, then the diagram*

$$\begin{tikzcd} W^\epsilon(\Lambda) \arrow[r, "w"] \arrow[dr, "\bar{w}"'] & Q^\epsilon(\Lambda) \arrow[d, "i"] \\ & \bar{Q}^\epsilon(\Lambda) \end{tikzcd}$$

commutes, where i is the distinguished isomorphism of Cor. II.5.

Proof. Let ψ_β be a lift of $\widehat{\mathrm{ad}(\beta)}$ to $\mathscr{P}_1$, and $\nu\colon \mathscr{R} \to \bar{\mathscr{R}}$ a chain map. Then $\nu_1 \circ \psi_\beta$ is a lift of $\widehat{\mathrm{ad}(\beta)}$ to $\bar{\mathscr{P}}_1$, so $\bar{w}$ may be computed using $\nu_1 \circ \psi_\beta$. This readily gives the result. ∎

PROPOSITION 6. *w: $W^\epsilon \to Q^\epsilon$ is a natural transformation; i.e. if $\Lambda \to \Lambda'$ is a morphism of rings with involution, then the diagram*

$$\begin{array}{ccc} W^\epsilon(\Lambda) & \xrightarrow{W^\epsilon(f)} & W^\epsilon(\Lambda') \\ \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w} \\ Q^\epsilon(\Lambda) & \xrightarrow{Q^\epsilon(f)} & W^\epsilon(\Lambda') \end{array}$$

commutes, where the lower horizontal map is the change of rings map defined in Section II.

Proof. Let (H, β) be a Hermitian space over Λ, ψ_β: $H \to \mathscr{P}_1$ a lift of $\widehat{\mathrm{ad}(\beta)}$, where $\mathscr{R}$: $\mathscr{P}_2 \to \mathscr{P}_1 \to H^\epsilon(\Lambda) \to 0$ is a two-step resolution. Let h be a Λ-module chain map from $\mathscr{R}$ to $\mathscr{S}$ covering η: $H^\epsilon(\Lambda) \to H^\epsilon(\Lambda')$, where $\mathscr{S}$: $\mathscr{M}_2 \to \mathscr{M}_1 \to H^\epsilon(\Lambda') \to 0$ is a two step resolution of $H^\epsilon(\Lambda')$ over Λ'. It is easily verified that a choice for ψ'_β, the lifting of $\widehat{\mathrm{ad}(\beta')}$, where $(H', \beta') = W^\epsilon(f)(H, \beta)$, is the composite $\Lambda' \otimes_\Lambda H \to^{1\otimes\psi_\beta} \Lambda' \otimes_\Lambda \mathscr{P}_1 \to^{h_1} \mathscr{M}_1$. Using the fact that h is a chain map $\mathscr{R} \to \mathscr{S}$, one may now verify the commutativity of the diagram, using this specific choice of ψ'_β to compute $W(H', \beta')$. ∎

V. QUADRATIC FORMS

Recall from [3] or [6] the definition of the Wall group $L_0^{\epsilon,q}(\Lambda)$. This is the Grothendieck group of *quadratic* forms over Λ, factored by split quadratic forms. There is a natural map

$$L_0^{\epsilon,q}(\Lambda) \to W^\epsilon(\Lambda)$$

which forgets the quadratic structure and regards the quadratic space (H, β, μ) as the Hermitian space (H, β). The image of $L_0^{\epsilon,q}(\Lambda)$ is the subgroup $W_{\mathrm{ev}}^\epsilon(\Lambda) \subseteq W^\epsilon(\Lambda)$ generated by even forms, i.e. by Hermitian spaces (H, β) so that $\widehat{\mathrm{ad}(\beta)}(x) = 0$ in $H^\epsilon(\Lambda)$ for all $x \in H$.

LEMMA 1. *If $x \in W_{\mathrm{ev}}^\epsilon(\Lambda)$, $w(x) = 0$.*

Proof. It suffices to show $w(H, \beta) = 0$ for (H, β) even. But if (H, β) is even, $\widehat{\mathrm{ad}(\beta)}(x) = 0$ $\forall x \in H$, so ψ_β may be taken to be the zero map, hence $\xi_\beta = \psi_\beta \circ \mathrm{ad}(\beta)^{-1} \circ \psi_\beta^* \equiv 0$. ∎

It is the object of this section to characterize the kernel of w, i.e. to prove the converse of Lemma 1.

THEOREM 2. $w(x) = 0 \Rightarrow x \in W^{\epsilon}_{\mathrm{ev}}(\Lambda)$.

Proof. We recall from [3] or [5] that if (H, β) is a Hermitian space, and $K \subseteq H$ is a self-annihilating summand, (i.e. $\beta(x, y) = 0 \; \forall x, y \in K$), then $K^{\perp}/K$ becomes a Hermitian space, by restricting the form β to $K^{\perp} \times K^{\perp}$. Also, an equivalent formulation for K to be self-annihilating is that the composite

$$K \to H \xrightarrow{\mathrm{ad}(\beta)} H^* \to K^*$$

be zero.

We suppose we are given a Hermitian space (H, β) with $w(H, \beta) = 0$ in $Q^{\epsilon}(\Lambda)$. We may suppose that the map $\psi_\beta \colon H \to \mathscr{P}_1$ is surjective, for if it is not, add on a split space (H', β') so that $\psi_{\beta'}$ is surjective. Since $w(H, \beta) = 0$, we have

$$\psi_\beta \circ \mathrm{ad}(\beta)^{-1} \circ \psi_\beta^* = \partial \circ \chi + \epsilon\chi^* \circ \partial^* + \partial \circ \Sigma \circ \partial^*$$

where

$$\chi \colon \mathscr{P}_1^* \to \mathscr{P}_2, \; \Sigma \colon \mathscr{P}_2^* \to \mathscr{P}_2, \qquad \Sigma = \epsilon\Sigma^*, \qquad \text{and} \qquad \rho \circ \chi^* = \hat{\Sigma}$$

Form the Hermitian space $(H \oplus \mathscr{P}_2^* \oplus \mathscr{P}_2, \bar{\beta})$, where $\mathrm{ad}(\bar{\beta}) \colon H \oplus \mathscr{P}_2^* \oplus \mathscr{P}_2 \to H^* \oplus \mathscr{P}_2 \oplus \mathscr{P}_2^*$ is given by the matrix

$$\begin{bmatrix} \mathrm{id}_H & 0 & 0 \\ 0 & -\Sigma & \mathrm{id}_{\mathscr{P}_2} \\ 0 & \epsilon\, \mathrm{id}_{\mathscr{P}_2^*} & 0 \end{bmatrix}$$

$(H \oplus \mathscr{P}_2^* \oplus \mathscr{P}_2, \bar{\beta})$ is clearly Witt equivalent to (H, β). Define an inclusion $i \colon \mathscr{P}_1^* \to H \oplus \mathscr{P}_2^* \oplus \mathscr{P}_2$ by $i(x) = (\mathrm{ad}(\beta)^{-1} \circ \psi_\beta^*(x), \partial^*(x), -\epsilon\chi(x))$.

Since $\psi_\beta \colon H \to \mathscr{P}_1$ is surjective, we find that the map i is an inclusion on a direct summand. Also, the summand $i(\mathscr{P}_1^*)$ is self-annihilating, for composite $i^* \circ \mathrm{ad}(\bar{\beta}) \circ i$ can be computed to be

$$\epsilon(\psi_\beta \circ \mathrm{ad}(\beta)^{-1} \circ \psi_\beta^* - \partial\, \Sigma\, \partial^* - \partial\chi - \epsilon\chi^*\partial^*) = 0$$

Thus, by the above, (H, β) is Witt equivalent to

$$(i(\mathscr{P}_1^*)^{\perp}/i(\mathscr{P}_1^*), \beta \mid i(\mathscr{P}_1^*)^{\perp} \times i(\mathscr{P}_1^*)^{\perp})$$

If we can prove that $\beta \mid i(\mathscr{P}_1^*)^{\perp} \times i(\mathscr{P}_1^*)^{\perp}$ is even, we will have proved the theorem. By the definition of $\bar{\beta}$,

$$\widehat{\mathrm{ad}(\bar{\beta})}(x, y, z) = \widehat{\mathrm{ad}(\beta)}(x) + \hat{\Sigma}(y),$$

for $(x, y, z) \in H \oplus \mathscr{P}_2^* \oplus \mathscr{P}_2$. By the restrictions on ψ_β and χ, however, we have

$$\widehat{\mathrm{ad}(\beta)}(x) = \rho \circ \psi_\beta(x), \qquad \rho \circ \chi^*(y) = \hat{\Sigma}(y)$$

It is clear that $i(\mathscr{P}_1^*)^\perp = \ker(\mathrm{ad}(\bar{\beta}) \circ i^*)$. $(\mathrm{ad}(\bar{\beta}) \circ i^*)(x, y, z) = \psi_\beta(x) - \partial \mathrm{E}(y) - \epsilon\chi^*(y) + \partial(z)$. Applying ρ, we get $\rho \circ \mathrm{ad}(\bar{\beta}) \circ i^* = \rho \circ \psi_\beta(x) + \rho \circ \chi^*(y)$. Thus, for $(x, y, z) \in i(\mathscr{P}_1^*(x))^\perp$, $\rho\psi_\beta(x) + \rho \circ \chi^*(y) = 0$, so $\widehat{\mathrm{ad}(\beta)}(x) + \hat{\Sigma}(y) = 0$, and the form is even. ▌

VI. Computing $Q^\epsilon(\Lambda)$

In this section, w compute $Q^\epsilon(\Lambda)$ in some examples, thereby obtaining new proofs for the description of the Witt groups involved in the classical examples, and invariants which detect the Witt group in some non-classical examples.

(a) $\Lambda = \mathbb{Z}$, $\epsilon = 1$, trivial involution.

In this case $H^1(\Lambda) = \mathbb{Z}/2$, so a resolution is given by $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$. Since the involution acts trivially, $A_1{}^1 = \mathscr{P}_1 = \mathbb{Z}$. $\Gamma = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, where the involution $*$ permutes the last two factors and is trivial on the first

$$\alpha_1 \colon \mathbb{Z} \to \mathbb{Z}/2 \text{ is reduction mod } 2$$
$$\beta_1 \colon \mathbb{Z} \to \mathbb{Z}/2 \text{ is reduction mod } 2.$$

By the above computation of Γ, $A_2{}^1 = \{(x, y, z) \mid y = z\}$

$$\beta_2(x, y, z) = x \bmod 2(\mathrm{Hom}(\mathscr{P}_2, \mathbb{Z}/2) = \mathbb{Z}/2)$$
$$\alpha_2(x, y, z) = y \bmod 2.$$

Thus, $B_1{}^1 = A_1{}^1$, and $B_2{}^1$ is generated by the elements $(1, 1, 1)$ and $(2, 0, 0)$ in $A_2{}^1$. Tracing through the definition of δ, one obtains

$$\delta(1, 1, 1) = 4 + 2 + 2 = 8, \qquad \delta(2, 0, 0) = 8,$$

which gives

$$Q^1(\mathbb{Z}) = \mathbb{Z}/8\mathbb{Z}.$$

By evaluating on the space $\langle 1 \rangle$, we observe that our invariant is the signature mod 8.

(b) Let $\mathcal{O}_K$ denote the ring of integers in a finite unramified extension K of $\hat{\mathbb{Q}}_2$, $\epsilon = 1$, with trivial involution. Since the extension $\hat{\mathbb{Q}}_2 \subseteq K$ is unramified,

$2 \cdot \mathcal{O}_K$ is a prime ideal in $\mathcal{O}_K$. Moreover, $H^1(\mathcal{O}_K) = \mathcal{O}_K/2\mathcal{O}_K$ is a finite field of characteristic 2, which we denote by $\mathbb{F}$. Note, however, that the module structure of $H^1(\mathcal{O}_K)$ is not that obtained by regarding $\mathbb{F}$ as a quotient of $\mathcal{O}_K$; rather, the module structure is given by the map $\rho: \mathcal{O}_K \to \mathbb{F}$

$$\rho(x) = x^2.$$

The two module structures are isomorphic, under a non-trivial automorphism of $\mathbb{F}$ as an abelian group, namely the inverse to the Frobenius. As in the case $A = \mathbb{Z}$, a resolution is given by

$$0 \to \mathcal{O}_K \xrightarrow{2} \mathcal{O}_K \xrightarrow{\rho} \mathbb{F} \to 0.$$

As above, $A_1{}^1 = \mathscr{P}_1 = \mathcal{O}_K$. $\Gamma = \mathcal{O}_K \oplus \mathcal{O}_K \oplus \mathcal{O}_K$, $(x, y, z)^* = (x, z, y)$, giving $A_2{}^1 = \mathcal{O}_K \oplus \mathcal{O}_K$.

For $x \in A_1{}^1$

$$\beta_1(x) = x^2 \bmod 2, \qquad \alpha_1(x) = x (\bmod 2).$$

For $(x, y) \in A_2{}^1 = \mathcal{O}_K \oplus \mathcal{O}_K$,

$$\left.\begin{aligned} \alpha_2(x, y) &= x \\ \beta_2(x, y) &= y^2 \end{aligned}\right\} \text{ in } \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_K, \mathbb{F}) = \mathbb{F}.$$

This gives $B_1{}^1 = (1) + 2\mathcal{O}_K$, where (1) denotes the *additive group* generated by 1. To see this, note that if $x \in B_1{}^1$, $x \equiv x^2 (\bmod 2)$. Consequently, the reduction of x must lie in the fixed field of the Frobenius automorphism, which is $\mathbb{Z}/2 \subseteq \mathbb{F}$. Similarly, $B_2{}^1$ consists of $(x, y) \in \mathcal{O}_K \oplus \mathcal{O}_K$ such that $x \equiv y^2 \bmod 2$. Therefore, $B_2{}^1$ is generated by elements of the form (y^2, y) and $2 \cdot (\mathcal{O}_K \oplus \mathcal{O}_K)$. It remains to evaluate δ. It is immediate that $\delta(y^2, y) = 4y^2 + 4y$, $\delta(2x, 2y) = 8x + 8y$. We have, then, that $8\mathcal{O}_K \subseteq \delta(B_2{}^1)$. The quotient group

$$(1) + 2\mathcal{O}_K/8\mathcal{O}_K$$

is $\mathbb{Z}/8 + (\mathbb{Z}/4)^{d-1}$, where $d = [K : \hat{\mathbb{Q}}_2]$. We must compute the quotient by the subgroup generated by elements of the form $4(y^2 + y)$.

$$4\mathcal{O}_K/8\mathcal{O}_K = \mathcal{O}_K/2\mathcal{O}_K = \mathbb{F}$$

so to obtain the quotient of $4\mathcal{O}_K/8\mathcal{O}_K$ by the subgroup generated by $4y^2 + 4y$, it suffices to compute the quotient of $\mathbb{F}$ by elements of the form $y + y^2$. This is readily seen to be $\mathbb{Z}/2$, so we have an exact sequence

$$0 \to \mathbb{Z}/2 \to Q^1(\mathcal{O}_K) \to \frac{(1) + 2\mathcal{O}_K}{4\mathcal{O}_K} = \mathbb{Z}/4 \oplus (\mathbb{Z}/2)^{d-1} \to 0$$

In computing the extension, two cases appear:

(i) $\mathbb{F}$ is an odd-degree extension of $\mathbb{Z}/2$. In this case, $1 = x^2 + x$ for any $x \in \mathbb{F}$, hence $4 \in 4\mathcal{O}_K$ represents a non-trivial element. In this case,

$$Q^1(\mathcal{O}_K) = \mathbb{Z}/8 + (\mathbb{Z}/2)^{d-1}.$$

(ii) $\mathbb{F}$ is an even degree extension of $\mathbb{Z}/2$. 4 is now trivial in $4\mathcal{O}_K/8\mathcal{O}_K + \{4(y + y^2) \mid y \in \mathcal{O}_K\}$. On the other hand, if $x \in \mathcal{O}_K$ is such that its mod 2 reduction represents the non-trivial element in $\mathbb{F}/\{y + y^2, y \in \mathbb{F}\}$, then $4x$ is non-trivial in $4\mathcal{O}_K/8\mathcal{O}_K + \{4(y + y^2) \mid y \in \mathcal{O}_K\}$. We obtain

$$Q^1(\mathcal{O}_K) = \mathbb{Z}/4 + \mathbb{Z}/4 + (\mathbb{Z}/2)^{d-2}.$$

Combining these calculations with the well-known facts that $W^1_{\mathrm{ev}}(\mathcal{O}_K) = \mathbb{Z}/2$, that $W^1_{\mathrm{ev}}(\mathcal{O}_K)$ is a summand in $W^1(\mathcal{O}_K)$, and that $W^1(K) = W^1(\mathcal{O}_K) + \mathbb{Z}/2$, we obtain, as in [4],

$$W(K) = \mathbb{Z}/8 + (\mathbb{Z}/2)^{d+1} \text{ in case (i)}$$
$$W(K) = \mathbb{Z}/4 + \mathbb{Z}/4 + (\mathbb{Z}/2)^{d} \text{ in case (ii).}$$

Remark. The restriction to unramified extensions is made only in the interest of simplicity. In the ramified case, the module $H^\epsilon(\mathcal{O}_K)$ is no longer cyclic but is a direct sum of two cyclic modules, which makes the calculations more complicated, although not in principle more difficult. If one performs the calculations, one recovers the structure of $W^1(K)$ for any finite extension K of $\hat{\mathbb{Q}}_2$.

(c) Let $\Lambda = F$ be a field of characteristic two, so that $[F : F^2] = n < \infty$, where F^2 is the subfield of all squares in F. n is a power of two. We will see that the invariant constructed in this paper detects the Witt classes of symmetric bilinear forms over F. Let F be endowed with the trivial involution.

LEMMA 1. $W^1_{\mathrm{ev}}(F) = 0$.

Proof. If $f \in F$, $f + \bar{f} = 2f = 0$. Thus, if $(H, \beta) \in W^{+1}_{\mathrm{ev}}(F)$, $\beta(x, x) = 0$ for all $x \in H$, so (H, β) is split. ∎

This shows that the map w: $W^{+1}(F) \to Q^{+1}(F)$ is injective, by Theorem IV.2. Since $[F : F^2] = n$, we obtain that

$$H^{+1}(F) \cong F^n$$

since the module structure on $\mathscr{P}_1 = F$ is given by

$$x \cdot f = x^2 f.$$

A resolution of $H^{+1}(F)$ is given by

$$0 \to F^n \to H^{+1}(F) \to 0$$

with $\mathscr{P}_2 = 0$. Accordingly, $B_1{}^1 \cong Q^{+1}(F)$. We find $\mathrm{Hom}_F(\mathscr{P}_1^*, \mathscr{P}_1) = \bigoplus_{1 \leqslant i,j \leqslant n} F_{ij}$, duality acts by permuting F_{ij} and F_{ji}, and acts trivially on F_{ii}. Thus,

$$A_1{}^1 \cong \bigoplus_{1 \leqslant i \leqslant j \leqslant n} F_{ij}$$

A direct calculation now gives

LEMMA 2. *$B_1{}^1 \subseteq A_1{}^1$ consists of those vectors $(\alpha_{ij}) \in A_1{}^1$, $1 \leqslant i \leqslant j \leqslant n$, such that*

$$\alpha_{ii} = \sum_{j=1}^{n} \alpha_{ij}^2 b_j$$

if $\{b_j\}_{j=1}^n$ is a basis for F over F^2.

We now describe the map w: $W^{+1}(F) \to Q^{+1}(F)$. As above, let $\{b_j\}_{j=1}^n$ be a basis for F over F^2. Let (H, β) be a symmetric bilinear form over F. Then

$$x \to \beta(x, x)$$

defines a homomorphism φ: $H \to H^+(F) = \bigoplus_{i=1}^n F_i$. This gives an n-tuple of linear maps φ_i: $H \to F_i$. Since the form β is non-singular, there exist unique elements $\chi_i \in H$ so that $\beta(x, \chi_i) = \varphi_i(x)$. Equivalently, there exists a unique collection of elements $\{\chi_1, ..., \chi_n\}$ so that

$$\beta(x, x) = \sum_{i=1}^{n} \beta(x, \chi_k)^2 \, b_i$$

for all $x \in H$. The elements $w_{ij}(H, \beta) = \beta(\chi_i, \chi_j) \in F$ are now invariants of the isomorphism type of (H, β). In fact, they depend only on the Witt class, and may be identified with the (i, j)-th coordinate of $w(H, \beta)$, if $A_1{}^1$ is identified with $\bigoplus_{1 \leqslant i \leqslant j \leqslant n} F_{ij}$, as above. The conclusion is, then,

THEOREM 3. *The invariants $w_{ij}(H, \beta) \in F$, $1 \leqslant i \leqslant j \leqslant n$, detect the Witt classes of symmetric bilinear forms over F, i.e. (H, β) is stably split if and only if $w_{ij}(H, \beta) = 0$ for all i, j.*

(d) We conclude the paper by computing Q^1 in the case $\Lambda = \mathbb{Z}/2\pi$, $\pi = (\mathbb{Z}/2)^i$, endowed with the trivial involution. Recall that $\Lambda = E(x_1, ..., x_i)$, the exterior algebra on i generators. Note that the square of every element y in Λ

is 0 or 1, depending on whether y is in the maximal ideal $m = (x_1, \ldots, x_n)$ or not. From this remark, it is immediate that

$$H'(\Lambda) \cong \bigoplus_{\mu} k_\mu$$

where μ runs over all non-zero monomials in the x_i's (including 1), and k_μ denotes a copy of the cyclic module Λ/m. Thus, $\mathscr{P}_1 \cong \oplus_\mu \Lambda_\mu$, and

$$\mathrm{Hom}_\Lambda(\mathscr{P}_1^*, \mathscr{P}_1) \cong \bigoplus_{\lambda,\mu} \Lambda_{\lambda\mu}$$

A straightforward computation identifies $B_1{}^1$ as the subgroup of $\oplus_{\lambda,\mu} \Lambda_{\lambda\mu}$ consisting of all vectors $(\alpha_{\lambda\mu})$ such that

$$\alpha_{\lambda\lambda} + \alpha_{\lambda\lambda}^2\lambda = \sum_{\mu\neq\lambda} \alpha_{\lambda\mu}^2\mu \ \forall\lambda$$

and

$$\alpha_{\lambda\mu} = \alpha_{\mu\lambda} \ \forall\lambda, \mu$$

For $\lambda \neq 1$, the map $x \to x + x^2\lambda$ is an isomorphism from Λ to itself, so for $\lambda \neq 1$, $\alpha_{\lambda\lambda}$ is determined by $\{\alpha_{\lambda\mu}\}_{\lambda\neq\mu}$. If $\lambda = 1$, the map $x \to x + x^2$ has kernel $\mathbb{Z}/2$, so α_{11} is determined up to an element of the subgroup $\{0, 1\}$ by $\{\alpha_{\lambda\mu}\}_{\lambda\neq\mu}$. Another direct computation shows that $(p \circ \delta)(B_2{}^1) = m$. $\Gamma \subseteq \Gamma$, where $\Gamma \subseteq \oplus_{\lambda,\mu} \Lambda_{\lambda\mu}$, $\Gamma = \{(x_{\lambda\mu}) \mid x_{\lambda\mu} = x_{\mu\lambda}\}$, and p: $B_1{}^1 \to \Gamma$ is the projection. This gives

$$Q^1(\Lambda) \cong \mathbb{Z}/2^{2^i(2^i-1)+1}$$

Using the well-known fact that $W^1(\Lambda)$ is generated by one-dimensional spaces, one finds that on the image of w in $Q^1(\Lambda)$, the coordinates $\alpha_{1\lambda}$ are determined by $\{\alpha_{\lambda\mu}\}_{\lambda\neq\mu/\lambda,\mu\neq1}$. Thus, we obtain

$$\frac{W^1(\Lambda)}{W_{\mathrm{ev}}^1(\Lambda)} \cong \mathbb{Z}/2^{(2^i-1)(2^i-2)+1}$$

As in case (c) above, $W_{\mathrm{ev}}^1(\Lambda) = 0$ for trivial reasons, so we obtain

Proposition 4. $W^1(\Lambda) \cong \mathbb{Z}/2^{(2^i-1)(2^i-2)+1}$. (*This result was obtained by ad hoc methods in* [1].)

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