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LOCALIZATION IN LOWER ALGEBRAIC K-THEORY

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The functors K_1 and K_0 and more recently the lower K-theoretic functors K_{-i} of Bass have become useful in several contexts in algebraic topology. Typically one has a topological space with fundamental group G and some property of this space is measured by an invariant which takes values in a K-group (or a quotient of a K-group) of the integral group ring $\mathbb{Z}G$ of G. The first examples of this are J. H. C. Whitehead's work on simple homotopy types and C. T. C. Wall's results on finiteness of CW-complexes. References for these and other applications may be found in Milnor [6], page x. The first topological application of the lower K-groups occurs in Anderson and Hsiang's work on pseudo-isotopies of PL-manifolds [1].

Clearly calculation of the groups $K_*(\mathbb{Z}G)$ is of interest not only for its own sake (e.g., when G is finite, as a branch of generalized algebraic number theory), but because immediate topological applications exist.

This paper concerns computational techniques for the groups K_{-i}, and is an exposition of some of the ideas in the author's thesis [3]. Specifically, we derive the expected localization sequence for the lower K-theory, (which for some reason seems not to have surfaced previously in the literature), and indicate how this sequence may be applied in the calculation of lower K-groups for group rings and orders. The complete story of localization is as follows:

Theorem 1: Let R be a commutative ring and let S be a multiplicative set of elements of R. Let Λ be an R-algebra on which multiplication by any s in S is injective. Let $\underline{\underline{H}}_S(\Lambda)$ denote the category of S-torsion left Λ-modules which admit a finite resolution by finitely generated projective left Λ-modules. Then the localization $\psi:\Lambda \to S^{-1}\Lambda$ gives rise to a long exact sequence

$$K_{n+1}(\Lambda) \overset{\psi_*}{\to} K_{n+1}(S^{-1}\Lambda) \to K_n(\underline{\underline{H}}_S(\Lambda)) \to K_n(\Lambda) \overset{\psi_*}{\to} K_n(S^{-1}\Lambda)$$

(for all integers n).

The relative groups $K_{-i}(\underline{\underline{H}}_S(\Lambda))$ will be defined in section 1.

For the application of theorem 1 to group rings, it will be convenient to introduce some notation. Suppose R is a (commutative) integral domain with fraction field K and suppose Λ is an R-algebra on which multiplication by any non-zero element of R is injective. If A denotes $K \otimes_R \Lambda$, we shall write

$$\tilde{K}_n(\Lambda) = \ker(K_n(\Lambda) \to K_n(A))$$

$$\bar{K}_n(\Lambda) = K_n(A)/\text{im } K_n(\Lambda)$$

for the kernel and cokernel of the inclusion-induced map. With these notations we have

Theorem 2: Let R be a Dedekind ring with fraction field K, let Λ be a noetherian R-algebra which is R-torsion-free, and for each maximal ideal P of R let R_P, K_P denote the corresponding P-adic completions and $\Lambda_P = R_P \otimes_R \Lambda$. Then for all integers n there is an exact sequence

$$0 \to \bar{K}_{n+1}(\Lambda) \to \bigoplus_P \bar{K}_{n+1}(\Lambda_P) \to \tilde{K}_n(\Lambda) \to \bigoplus_P \tilde{K}_n(\Lambda_P) \to 0$$

where P ranges over all maximal ideals of R.

When R is the ring of integers in a number field and $\Lambda = RG$ is the integral group ring of a finite group, the above result may be refined to read

Theorem 3: Let R be the ring of integers in an algebraic number field K and let $\Lambda = RG$ denote the integral group ring of a finite group G of order n. Let A denote the group algebra KG and for P any maximal ideal of R let R_P, K_P, Λ_P, A_P denote the corresponding P-adic completions. Then there is an exact sequence

$$0 \to K_0(\mathbb{Z}) \to \bigoplus_{P|nR} K_0(\Lambda_P) \oplus K_0(A) \to \bigoplus_{P|nR} K_0(A_P) \to K_{-1}(\Lambda) \to 0$$

which is a free $\mathbb{Z}$-resolution of $K_{-1}(\Lambda)$. Furthermore, $K_{-i}(\Lambda) = 0$ for all $i > 1$.

Corollary: Keeping the notations of theorem 3, let k (resp. k_P, r_P) denote the number of isomorphism classes of irreducible K (resp. K_P, R/P) representations of G. Then $K_{-1}(\Lambda)$ is a finitely generated abelian group and

$$\operatorname{rank} K_{-1}(\Lambda) = 1 - k + \sum_{P|nR} (k_P - r_P).$$

Remark: In particular, it follows that the expression appearing on the right hand side of the above equation is greater than or equal to zero.

To determine whether torsion is present in $K_{-1}(RG)$ it is generally necessary to analyze in detail the map

$$\bigoplus_{P|nR} K_0(\Lambda_P) \oplus K_0(A) \to \bigoplus_{P|nR} K_0(A_P)$$

of theorem 3. We shall not present this analysis here, the details of which appear in [3], but content ourselves to remark that the localization maps $K_0(\Lambda_P) \to K_0(A_P)$ are closely related to the "Brauer decomposition maps" $G_0(K_PG) \to G_0((R/P)G)$ which arise in the modular representation theory of finite groups (for a particularly good discussion the reader is referred to Serre [8]), and that the P-adic completion maps $K_0(A) \to K_0(A_P)$ depend on considerations of Schur indices and the theory of splitting of simple algebras.

Using such information one can make an effective calculation of $K_{-1}(\mathbb{Z}G)$ for G finite whenever adequate information about the (ordinary and modular) representation theory of G is available. These calculations have been done by the author for the symmetric groups S_3, S_4, and S_5, and for all p-groups. We note in passing that torsion can occur in $K_{-1}(\mathbb{Z}G)$: this happens for example if G is the generalized quaternion group of order 16, [3]; it does not

occur if G is finite abelian ([2], p. 695 , theorem 10.6). We hope to be able to deal with questions of torsion in $K_{-1}(\mathbb{Z}G)$ in a future paper; at present, questions of this sort are rather poorly understood except in somewhat isolated cases.

§1. The Localization Sequence

In this section we shall set up the machinery necessary to discuss localization theory for lower K-theory and then give a proof of "the unknown half" of theorem 1.

Suppose R is a commutative ring and S is a multiplicative set of elements of R. It will be convenient to denote by $\underline{\underline{A}}(R,S)$ the category whose objects are R-algebras on which multiplication by any s in S is injective and whose morphisms are exactly those R-algebra homomorphisms $\Lambda \to \Gamma$ for which Γ is flat as a right Λ-module. We shall be concerned with various functors from $\underline{\underline{A}}(R,S)$ to abelian groups. Examples of such functors are of course the Grothendieck groups K_0 and G_0, the Whitehead groups K_1 and G_1, and three related functors K_0S^{-1}, K_1S^{-1}, K_0H_S which we define as follows:

$$K_iS^{-1}(\Lambda) = K_i(S^{-1}\Lambda) \quad , \; i = 0,\ 1$$

$$K_0H_S(\Lambda) = K_0(\underline{\underline{H}}_S(\Lambda))$$

Of course any functor from rings to abelian groups may be viewed as a functor from $\underline{\underline{A}}(R,S)$. On the other hand, notice that K_0H_S is a functor on $\underline{\underline{A}}(R,S)$ exactly because $\underline{\underline{A}}(R,S)$ contains only those arrows

$\Lambda \to \Gamma$ for which the operation $\Gamma \otimes_\Lambda -$ is guaranteed to take left Λ-modules of type FP to left Γ-modules of type FP.

Let F be a functor from $\underline{\underline{A}}(R,S)$ to abelian groups. Then we may define another such functor, denoted LF, by

$$LF(\Lambda) = \operatorname{coker}\,(F(\Lambda[t]) \oplus F(\Lambda[t^{-1}]) \to F(\Lambda[t,t^{-1}]))$$

where the indicated map is induced by the obvious inclusions (localizations). Mapping $F(\Lambda)$ into $F(\Lambda[t]) \oplus F(\Lambda[t^{-1}])$ via $x \to (i_*(x), -j_*(x))$, where i (resp. j) denotes the usual inclusion $\Lambda \to \Lambda[t]$ (resp. $\Lambda \to \Lambda[t^{-1}]$), we get a chain complex

$$0 \to F(\Lambda) \to F(\Lambda[t]) \oplus F(\Lambda[t^{-1}]) \to F(\Lambda[t,t^{-1}]) \to LF(\Lambda) \to 0 \qquad (\varepsilon)$$

and we say that F is <u>acyclic</u> if the complex (ε) is exact for all objects Λ of $\underline{\underline{A}}(R,S)$, and <u>contracted</u> if also the projection $F(\Lambda[t,t^{-1}]) \to LF(\Lambda)$ has a natural splitting. The following important result is known as the Fundamental Theorem of Algebraic K-theory :

<u>Theorem</u> (Bass [2], p. 663, thm. 7.4) K_0 and K_1 are contracted functors. Moreover, there is a natural isomorphism $LK_1 \cong K_0$.

Of course it follows from this that the functors K_0S^{-1} and K_1S^{-1} are also contracted, and that $LK_1S^{-1} \cong K_0S^{-1}$. As a consequence of these isomorphisms, one is led to introduce the notation

$$K_{-i} = L^iK_0 \quad , \quad K_{-i}S^{-1} = L^iK_0S^{-1} \qquad (i > 0).$$

If F and G are contracted functors, one says that a natural transformation $\alpha : F \to G$ is a <u>morphism of contracted functors</u> if α respects the natural splittings, i.e. if for all Λ the square

$$\begin{array}{ccc} LF(\Lambda) & \to & F(\Lambda[t,t^{-1}]) \\ L\alpha \downarrow & & \downarrow \alpha \\ LG(\Lambda) & \to & G(\Lambda[t,t^{-1}]) \end{array}$$

commutes. The following result is standard :

<u>Lemma 1.1</u> (Bass, [2], p. 661, prop. 7.2) Let $\alpha: F \to G$ be a morphism of contracted functors from R-algebras to abelian groups. Then $\ker(\alpha)$ and $\operatorname{coker}(\alpha)$ are contracted functors on the category of R-algebras. In particular, LF is a contracted functor.

<u>Remark</u>: The proof makes use of the augmentation arrow $\Lambda[t] \to \Lambda$, ($t \to 1$), whence the requirement that F and G be functors on the category of R-algebras rather than just on $\underline{\underline{A}}(R,S)$.

The key to our proof of the lower localization sequence is given by the following result:

<u>Lemma 1.2</u> Let $0 \to F \to G \to H \to 0$ be a short exact sequence of functors from $\underline{\underline{A}}(R,S)$ to abelian groups, and suppose that F

and H are contracted functors on the category of R-algebras. Then there is a short exact sequence $0 \to LF \to LG \to LH \to 0$ (with LF and LH contracted functors on the category of R-algebras) and G is acyclic.

<u>proof</u> Apply the snake lemma to the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & F(\Lambda) & & \ker\psi & & H(\Lambda) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F(\Lambda[t])\oplus F(\Lambda[t^{-1}]) & \to & G(\Lambda[t])\oplus G(\Lambda[t^{-1}]) & \to & H(\Lambda[t])\oplus H(\Lambda[t^{-1}]) & \to & 0\\
 & & \downarrow & & \downarrow\psi & & \downarrow & & \\
0 & \to & F(\Lambda[t,t^{-1}]) & \to & G(\Lambda[t,t^{-1}]) & \to & H(\Lambda[t,t^{-1}]) & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & LF(\Lambda) & & LG(\Lambda) & & LH(\Lambda) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where ψ is induced by the usual inclusions, and observe that since $G(\Lambda)$ injects into $\ker\psi$ (the five lemma), and since $0 \to F(\Lambda) \to G(\Lambda) \to H(\Lambda) \to 0$ is exact, it follows that $G(\Lambda) \cong \ker\psi$ (so G is acyclic) and that the connecting homomorphism $H(\Lambda) \to LF(\Lambda)$ is zero. That LF and LH are contracted has already been asserted by lemma 1.1.

<u>Definition 1.3</u>: $K_{-i}(\underline{\underline{H}}_S(\Lambda)) = L^i K_0 H_S(\Lambda)$.

With this machinery in hand, we turn to the localization sequence (theorem 1). Some historical remarks are in order: the $K_1 - K_0$ localization sequence (the case n = 0 of theorem 1) is due to Bass (circa. 1960); his result appears in [2], p. 494, theorem 6.3. Quillen's generalization to the higher K-theoretic groups (the cases n > 0) was announced in [7], page 86. Proofs were subsequently furnished by Gersten [4], and (somewhat later) by Grayson [5], p. 233. We shall now complete the proof of theorem 1 (the cases n < 0), assuming the $K_1 - K_0$ result of Bass.

<u>proof of lower localization sequence</u> As observed above, K_0, K_1, K_0S^{-1}, and K_1S^{-1} are known to be contracted functors on the category of R-algebras. The $K_1 - K_0$ localization sequence now asserts that there is an exact sequence:

$$0 \to \ker\alpha \xrightarrow{i} K_1 \xrightarrow{\alpha} K_1S^{-1} \xrightarrow{\beta} K_0H_S \xrightarrow{\gamma} K_0 \xrightarrow{\delta} K_0S^{-1} \xrightarrow{p} \operatorname{coker}\delta \to 0.$$

We may decompose this into short exact sequences as follows:

$$0 \to \ker\alpha \xrightarrow{i} K_1 \to \operatorname{coker} i \to 0$$

$$0 \to \operatorname{coker} i \xrightarrow{\alpha} K_1S^{-1} \to \operatorname{coker}\alpha \to 0$$

$$0 \to \operatorname{coker}\alpha \xrightarrow{\beta} K_0H_S \xrightarrow{\gamma} \ker\delta \to 0$$

$$0 \to \ker\delta \to K_0 \xrightarrow{\delta} \ker p \to 0$$

$$0 \to \ker p \to K_0S^{-1} \xrightarrow{p} \operatorname{coker}\delta \to 0$$

Now by lemma 1.1, since K_1, K_1S^{-1}, K_0, K_0S^{-1} are contracted, so are $\ker\alpha$, $\operatorname{coker}\alpha$, $\ker\delta$, and $\operatorname{coker}\delta$, and thus so are coker i and ker p. The lower localization sequence now follows by iterating lemma 1.2 and splicing the resulting sets of five short exact sequences back together. This completes our discussion of theorem 1.

§2. Eliminating the Relative Groups

The aim of this section is to prove theorem 2 (above), which gives a local-global approach to the calculation of K_n for noetherian algebras over Dedekind rings. The key trick will be to make use of P-adic completions to get rid of the relative groups $K_n(\underline{\underline{H}}_S(\Lambda))$ which occur in the localization sequence. This beautiful technique is due to Wilson [10]. Observe that our theorem 2 generalizes Wilson's calculations in two directions: first, we extend his calculations of K_0 to K_n for all integers n; second, we consider noetherian (rather than finite) algebras over Dedekind rings.

<u>Lemma 2.1</u>: Let R be a commutative noetherian ring, and for P a prime ideal of R let R_P denote the P-adic completion. Let Λ be any R-algebra, and suppose that M, N are left Λ-modules with M finitely presented. Let Λ_P, M_P, N_P denote $R_P \otimes_R \Lambda$, $R_P \otimes_R M$, $R_P \otimes_R N$ respectively. Then there is a natural isomorphism

$$\mathrm{Hom}_\Lambda(M,N) \otimes_R R_P \to \mathrm{Hom}_{\Lambda_P}(M_P,N_P).$$

<u>proof</u> Functoriality of $\_ \otimes_R R_P$ provides the map $\mathrm{Hom}_\Lambda(M,N) \otimes_R R_P \to \mathrm{Hom}_{\Lambda_P}(M_P,N_P)$. We must check that this map is an isomorphism when M is finitely presented.

Let $F_1 \to F_0 \to M \to 0$ be a free presentation of M with F_1, F_0 finitely generated. Then we have the commutative exact diagram

$$\begin{array}{ccccccc}
0 \to \mathrm{Hom}_\Lambda(M,N) \otimes_R R_P & \to & \mathrm{Hom}_\Lambda(F_0,N) \otimes_R R_P & \to & \mathrm{Hom}_\Lambda(F_1,N) \otimes_R R_P \\
\downarrow & & \downarrow & & \downarrow \\
0 \to \mathrm{Hom}_{\Lambda_P}(M_P,N_P) & \to & \mathrm{Hom}_{\Lambda_P}(F_{0P},N_P) & \to & \mathrm{Hom}_{\Lambda_P}(F_{1P},N_P)
\end{array}$$

from which one sees immediately that it suffices to prove the lemma for M a finitely generated free left Λ-module. But since Hom commutes with finite direct sums, it suffices to prove the lemma for $M = \Lambda$, and this case is obvious.

<u>Lemma 2.2</u>: Let Λ be a noetherian R-algebra, R a Dedekind ring, and let F denote a finitely generated left Λ-module. Then F is Λ-projective if and only if for all maximal ideals P of R $F_P = R_P \otimes_R F$ is Λ_P-projective.

<u>proof</u> If F is Λ-projective then trivially F_P is Λ_P-projective for all P. To prove the converse, we show that $\mathrm{Hom}_\Lambda(F,\_)$ is exact. Let $0 \to N' \to N \to N'' \to 0$ be a short exact sequence of left Λ-modules. Applying $\mathrm{Hom}_\Lambda(F,\_)$ we have an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(F,N') \to \mathrm{Hom}_\Lambda(F,N) \to \mathrm{Hom}_\Lambda(F,N'') \to C \to 0$$

where C denotes the cokernel. Applying $\_\otimes_R R_P$ to this sequence and invoking lemma 2.1 we have the exact sequence

$$0 \to \mathrm{Hom}_{\Lambda_P}(F_P,N'_P) \to \mathrm{Hom}_{\Lambda_P}(F_P,N_P) \to \mathrm{Hom}_{\Lambda_P}(F_P,N''_P) \to C_P \to 0$$

But $C_P = 0$ for all P since F_P is Λ_P-projective. Thus it follows that $C = 0$, whence $\mathrm{Hom}_\Lambda(F,\_)$ is exact.

Lemma 2.3: Let Λ be a noetherian R-algebra, R a Dedekind ring. Let S denote the multiplicative set of nonzero elements of R, and for P any maximal ideal of R let R_P denote the P-adic completion and $\Lambda_P = R_P \otimes_R \Lambda$. Then the map $\underline{\underline{H}}_S(\Lambda) \to \coprod_P \underline{\underline{H}}_S(\Lambda_P)$ induced by the P-adic completions is an equivalence of categories.

proof Let M be an element of $\underline{\underline{H}}_S(\Lambda)$. Then M is the direct sum of its P-primary components

$$M = \bigoplus_P M_P$$

$$M_P = \{ m \in M \mid \mathrm{Ann}_R(m) = P^{e_m} \}$$

(observe that M_P is in fact a sub-Λ-module), and clearly $R_P \otimes_R M = M_P$. This shows that $\underline{\underline{H}}_S(\Lambda)$ is a subcategory of $\coprod_P \underline{\underline{H}}_S(\Lambda_P)$.

Let N be an element of $\underline{\underline{H}}_S(\Lambda_P)$. If $N \neq 0$ it follows that $\mathrm{Ann}_R(N) = P^e$ for some $e > 0$ since N is finitely generated. But therefore any set of Λ_P-generators for N is also a set of Λ-generators, whence N is a finitely generated S-torsion left Λ-module.

Suppose that $\mathrm{hd}_{\Lambda_P}(N) = n$. Since Λ is noetherian, we may choose finitely generated free left Λ-modules $F_0, \cdots, F_{n-1}$ such that there is an exact sequence of left Λ-modules

$$0 \to F \to F_{n-1} \to F_{n-2} \to \cdots \to F_1 \to F_0 \to N \to 0$$

(here F is the kernel of $F_{n-1} \to F_{n-2}$; it is finitely generated).

Let Q be any maximal ideal of R, and apply $R_Q \otimes_R -$ to the above exact sequence. If $Q \neq P$, then since $N_Q = 0$, it follows that F_Q is Λ_Q- projective. If $Q = P$, F_Q is also Λ_Q-projective, since $N = N_P$ and by assumption $\mathrm{hd}_{\Lambda_P}(N) = n$. By lemma 2.2, it follows that F is Λ-projective, i.e. that $\mathrm{hd}_\Lambda(N) \leq n$. Thus N is in $\underline{\underline{H}}_S(\Lambda)$, which completes the proof.

With the aid of lemma 2.3 the proof of theorem 2 will be quite easy. Let R be a Dedekind ring with fraction field K, let Λ be a noetherian R-algebra which is R-torsion-free, and let S denote the multiplicative set of nonzero elements of R. If A denotes $K \otimes_R \Lambda$, we have the long localization sequence

$$K_{n+1}(\Lambda) \to K_{n+1}(A) \to K_n(\underline{\underline{H}}_S(\Lambda)) \to K_n(\Lambda) \to K_n(A)$$

which, using the notation introduced before the statement of theorem 2, may be written as the collection of short exact sequences

$$0 \to \bar{K}_{n+1}(\Lambda) \to K_n(\underline{\underline{H}}_S(\Lambda)) \to \tilde{K}_n(\Lambda) \to 0.$$

We map this sequence into the product of the analogous sequences for the P-adic completions Λ_P.

But by lemma 2.3 we may conclude that the P-adic completions give rise to isomorphisms:

$$K_n(\underline{\underline{H}}_S(\Lambda)) \to \bigoplus_P K_n(\underline{\underline{H}}_S(\Lambda_P))$$

(we use the Q-construction of Quillen or the L-construction above, depending on whether $n \geq 0$ or $n < 0$). But now since the center terms of our "global" short exact sequences are mapped into the coproduct of the center terms in our "local" sequences, then so also are the outer terms; i.e., we may conclude that there is a commutative exact diagram

$$\begin{array}{ccccccccc} 0 \to & \bar{K}_{n+1}(\Lambda) & \to & K_n(\underline{\underline{H}}_S(\Lambda)) & \to & \tilde{K}_n(\Lambda) & \to 0 \\ & \downarrow & & \| & & \downarrow & \\ 0 \to & \bigoplus_P \bar{K}_{n+1}(\Lambda_P) & \to & \bigoplus_P K_n(\underline{\underline{H}}_S(\Lambda_P)) & \to & \bigoplus_P \tilde{K}_n(\Lambda_P) & \to 0. \end{array}$$

Applying the snake lemma to this diagram, the desired exact sequence of theorem 2 is immediate.

§3. Integral Group Rings of Finite Groups

In this section we show how theorem 2 can be used to compute the lower K-theory for integral group rings of finite groups. The main result, given by theorem 3 above, will be seen to emerge as a consequence of the following fact: in a certain sense, the integral group ring of a finite group is close to being a regular ring.

Recall that a ring Λ is called quasi-regular if it has a nilpotent two-sided ideal N such that Λ/N is (left) regular. The following result is an easy consequence of Serre's calculation of $K_0(\Lambda[t,t^{-1}])$ for Λ a regular ring:

Lemma 3.1: (Bass. [2], p. 685, prop. 10.1) If a ring Λ is quasi-regular then $K_{-i}(\Lambda) = 0$ for all $i > 0$.

Notice that (left) Artin rings and (left) hereditary rings are clearly quasi-regular.

Let R be a Dedekind ring with fraction field K and let Λ (resp. A) denote the group algebra RG (resp. KG) of a finite group G. For P any maximal ideal of R, let Λ_P, A_P denote the corresponding P-adic completions. Since A and A_P are Artin rings, the conclusions of lemma 3.1 apply, and by theorem 2 we may conclude there are exact sequences

$$0 \to \bar{K}_0(\Lambda) \to \bigoplus_P \bar{K}_0(\Lambda_P) \to K_{-1}(\Lambda) \to \bigoplus_P K_{-1}(\Lambda_P) \to 0$$

(3.2)

$$0 \to K_{-i}(\Lambda) \to \bigoplus_P K_{-i}(\Lambda_P) \to 0 \qquad (i > 1).$$

When possible we wish to dispense with the groups $K_{-i}(\Lambda_P)$. We shall do this by an appeal to the theory of maximal orders. The first result we shall require is well-known (see, for example, [9], p. 63, prop. 4.9).

Lemma 3.3: Let R be a Dedekind ring with fraction field K of characteristic zero, and let Λ denote the integral group ring RG of a finite group G of order n. If Γ is any R-order in A = KG which contains Λ then $n\Gamma$ is contained in Λ. Furthermore, Λ is a maximal R-order if and only if n is invertible in R.

As a consequence of this result, for Γ any R-order containing Λ we have a cartesian square

$$\begin{array}{ccc} \Lambda & \to & \Gamma \\ \downarrow & & \downarrow \\ \Lambda/n\Gamma & \to & \Gamma/n\Gamma \end{array}$$

which gives rise to a long K-theory Mayer-Vietoris sequence (for $n < 2$)

$$K_n(\Lambda) \to K_n(\Lambda/n\Gamma) \oplus K_n(\Gamma) \to K_n(\Gamma/n\Gamma) \to K_{n-1}(\Lambda) \to \cdots$$

Now $\Lambda/n\Gamma$ and $\Gamma/n\Gamma$ are clearly Artinian rings. If also Γ is a hereditary (e.g. maximal) order containing Λ, then by lemma 3.1 we may conclude that this sequence reduces to the exact sequence

$$\cdots \to K_0(\Lambda) \to K_0(\Lambda/n\Gamma) \oplus K_0(\Gamma) \to K_0(\Gamma/n\Gamma) \to K_{-1}(\Lambda) \to 0$$

(3.4)

$$\text{and} \qquad K_{-i}(\Lambda) = 0 \quad , \quad i > 1.$$

Thus this calculation gives already the vanishing of $K_{-i}(RG)$ for $i > 1$ whenever R is a Dedekind ring in a field of characteristic zero and G is a finite group. If we now impose the extra condition that R be a complete discrete valuation ring, then either Γ is n-adically complete or $\Gamma/n\Gamma = 0$. In either case the map $K_0(\Gamma) \to K_0(\Gamma/n\Gamma)$ is surjective, whence $K_{-1}(\Lambda) = 0$.

Applying these considerations to (3.2), we are now able to assert that if R is a Dedekind ring with fraction field K of characteristic zero and Λ (resp. A) denotes the group algebra RG (resp. KG) of a finite group G, then there is an exact sequence

$$0 \to \bar{K}_0(\Lambda) \to \bigoplus_P \bar{K}_0(\Lambda_P) \to K_{-1}(\Lambda) \to 0$$

$$\text{and} \qquad K_{-i}(\Lambda) = 0 \quad , \quad i > 1.$$

At this point there is not too much left to do to finish the proof of theorem 3. By lemma 3.3, if G has order n and a maximal ideal P of R does not lie over nR, then n is invertible in R_P so $\Lambda_P = R_P G$ is a maximal R_P-order, hence regular. But it is well-known that localization of a regular ring induces surjection on K_0. Hence, if $P \nmid nR$ we have $\bar{K}_0(\Lambda_P) = 0$, and thus the exact

$$0 \to \bar{K}_0(\Lambda) \to \bigoplus_{P|nR} \bar{K}_0(\Lambda_P) \to K_{-1}(\Lambda) \to 0$$

Rearranging this sequence slightly, we get the exact sequence

$$K_0(\Lambda) \to \bigoplus_{P|nR} K_0(\Lambda_P) \oplus K_0(A) \to \bigoplus_{P|nR} K_0(A_P) \to K_{-1}(\Lambda) \to 0$$

which is valid for R any Dedekind ring with fraction field K of characteristic zero.

If finally we impose the additional requirement that no prime factor of n be invertible in R, which holds for example if R is the ring of integers in an algebraic number field, then by Swan's theorem ([9], p. 57, thm. 4.2; also p. 36, thm. 2.21) it follows that all finitely generated projective left Λ-modules are locally free. In particular, the image of $K_0(\Lambda)$ in $\bigoplus_P K_0(\Lambda_P) \oplus K_0(A)$ is generated by the images of free Λ-modules. Since these images are clearly all distinct, and since any free Λ-module is induced from a free $\mathbb{Z}$-module, this yields the desired exact sequence

$$0 \to K_0(\mathbb{Z}) \to \bigoplus_{P|nR} K_0(\Lambda_P) \oplus K_0(A) \to \bigoplus_{P|nR} K_0(A_P) \to K_{-1}(\Lambda) \to 0$$

This completes the proof of theorem 3 modulo the observation that the above exact sequence is a free $\mathbb{Z}$-resolution. But this is a

consequence of A and A_P being semisimple Artinian and Λ_P (being P-adically complete) having the same K_0 as the finite ring $\Lambda/P\Lambda$.

proof of corollary: Since A (A_P) is semisimple, $K_0(A)$ ($K_0(A_P)$) is free abelian on isomorphism classes of simple left A-modules (A_P-modules). Further, $K_0(\Lambda_P) \cong K_0(\Lambda/P\Lambda)$, which is free abelian on isomorphism classes of indecomposable left $\Lambda/P\Lambda$-modules. But a simple argument based on the lifting of idempotents shows that indecomposable $\Lambda/P\Lambda$-modules may be put in one-to-one correspondence with simple $\Lambda/P\Lambda$-modules by reducing mod the radical of the finite ring $\Lambda/P\Lambda$.

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REMARKS ON PRIMITIVE RINGS WITH NON-ZERO SOCLES

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I.N. Herstein and Lance W. Small in conversation have asked whether for a finite set $A_1, A_2, \ldots, A_n$ of non-zero m×m matrices ($m \geq 2$) over an infinite field there exists an idempotent matrix E different from the identity matrix I such that $EA_iE \neq 0$ for $i = 1, \ldots, n$. In this note we answer this question affirmatively and generalize the result to the class of primitive rings with non-zero socle.

It is clear that without some other restriction the set of matrices must have entries from an infinite field. To see this consider the finite set $\{A_j\}$ of all non-zero idempotent m×m matrices over a finite field where for any idempotent m×m matrix $E \neq I$ we have that $I - E$ is one of the A_j.