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ON SLICE KNOTS IN DIMENSION THREE

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1. Introduction. Under the equivalence relation of concordance (sometimes called cobordism), smooth knots in the 3-sphere S^3 form an abelian group with respect to connected sum [4]. The knots K representing the zero class are precisely those which are *slice*, that is, satisfy $(S^3, K) = \partial(B^4, D)$ for some smooth 2-disc D in the 4-ball B^4. Now associated with any knot K and a Seifert surface V spanning K, is a bilinear Seifert pairing $\theta_V: H_1(V) \times H_1(V) \to \boldsymbol{Z}$ [12], [6]. We say that K is *algebraically slice* if θ_V vanishes on a subgroup of $H_1(V)$ whose rank is $\frac{1}{2}$ rank $H_1(V)$ (this condition is independent of the choice of V). It is known that a necessary condition for K to be slice is that it be algebraically slice. Moreover, in higher (odd) dimensions analogous definitions may be made, and there the conditions are equivalent [6]. We shall show that this is not the case in dimension 3.

The Seifert pairing (up to appropriate equivalence) and a fortiori the 'algebraic concordance' class of K, is determined by the Blanchfield linking pairing on $H_1(\tilde{X})$, where $\tilde{X}$ is the universal abelian cover of the complement X of K [14]. Our 'second order' obstructions may be regarded as arising from certain cyclic covers of $\tilde{X}$, or (as in the present paper), from certain metacyclic branched covers of (S^3, K). In particular, our method provides potentially nontrivial obstructions to null-concordance for any knot with Alexander polynomial $\Delta(t) \neq 1$. (Whether or not there exist knots with $\Delta(t) = 1$ which are not slice is an interesting open question.)

The present paper and the earlier account [2] are related as follows. First, a fairly simple method was found for showing that certain (algebraically slice) knots K were not ribbon knots, using signatures associated with certain cyclic covers of (say) the 2-fold branched cover of (S^3, K) (see [2]). This was extended to give a necessary condition for K to be slice, in terms of the behavior as $n \to \infty$ of the corresponding invariants associated with the 2^n-fold branched cyclic cover of (S^3, K) (see §4). Calculations for certain specific examples, however, disclosed a multiplicativity in

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the invariants which showed that this behaviour was determined by the 2-fold branched cover (see §5). The search for an explanation of this phenomenon led to the approach in [2]. Thus the purpose of the present paper is to fill the existing historical gap, to motivate [2], and to provide variety. It should also make clear the relationship to earlier work of Massey [8], Hsiang-Szczarba [5], and Rohlin [10].

Organization is as follows. In §2 we use the Atiyah-Singer G-signature theorem [1] to associate with a 3-manifold M and an epimorphism $\varphi: H_1(M) \to Z_m$, certain rational numbers $\sigma_r(M, \varphi)$, $0 < r < m$. In §3 we show that if M is obtained by surgery on a link L in S^3 (and if φ is appropriately related to L), then $\sigma_r(M, \varphi)$ may be expressed in terms of standard invariants of L, in particular, signatures of the type introduced by Tristram [13]. In §4 we establish a necessary condition, in terms of certain $\sigma_r(M, \varphi)$, for a knot K to be slice. More precisely, we consider, for some fixed prime q, the q^n-fold branched cyclic cover M_n of (S^3, K), and show that if K is a slice knot, then, for suitable $\varphi_n: H_1(M_n) \to Z_m$, $\sigma_r(M_n, \varphi_n)$ must remain bounded as $n \to \infty$. Finally, in §5, we study the class of knots consisting of the various doubles of the unknot, and use the result of §4 to show that although there are infinitely many algebraically slice knots in this class, only two are slice. The calculation of the relevant invariants $\sigma_r(M_n, \varphi_n)$ is based on §3.

We work throughout in the smooth category. In the absence of evidence to the contrary, manifolds are to be assumed compact and oriented, and homology to be with integer coefficients.

2. An invariant. Let $\tilde{N} \to N$ be an m-fold cyclic branched covering of closed 4-manifolds, branched over a surface $F \subset N$ with inverse image $\tilde{F} \subset \tilde{N}$. The (symmetric) intersection form on $H_2(\tilde{N})$ extends naturally to a nonsingular Hermitian form $\cdot$ on $H = H_2(\tilde{N}) \otimes C$. Let $\tau: H \to H$ be the automorphism induced by the covering translation of $\tilde{N}$ which rotates each fibre of the normal bundle of $\tilde{F}$ through $2\pi/m$. Note that τ is an isometry of $(H, \cdot)$, and that $\tau^m = \text{id}$. Write $\omega = e^{2\pi i/m}$, and let E_r be the ω^r-eigenspace of τ, $0 \leqq r < m$. Then $(H, \cdot)$ decomposes as an orthogonal direct sum $E_0 \oplus E_1 \oplus \cdots \oplus E_{m-1}$. Let $\varepsilon_r(\tilde{N})$ be the signature of the restriction of $\cdot$ to E_r.

The following identity is proved by Rohlin in [10]. For the convenience of the reader we include a proof, which follows closely that of Rohlin.

LEMMA 2.1. $\varepsilon_r(\tilde{N}) = \text{sign}\, N - 2[F]^2 r(m-r)/m^2$.

PROOF. We can write $E_r = E_r^+ \oplus E_r^-$, where $\cdot$ is $\pm$ definite on $E_r^\pm$. Then $H = H^+ \oplus H^-$, where $H^\pm = E_0^\pm \oplus E_1^\pm \oplus \cdots \oplus E_{m-1}^\pm$, and for $0 \leqq s < m$ we have the τ^s-signatures

$$\begin{aligned}\text{sign}(\tau^s, \tilde{N}) &= \text{trace}(\tau^s|H^+) - \text{trace}(\tau^s|H^-)\\ &= \sum_{r=0}^{m-1} \omega^{rs}\, \varepsilon_r(\tilde{N}).\end{aligned}$$

A standard transfer argument gives $\varepsilon_0(\tilde{N}) = \text{sign}\, N$. Thus

$$\text{sign}(\tau^s, \tilde{N}) - \text{sign}\, N = \sum_{r=1}^{m-1} \omega^{rs}\, \varepsilon_r(\tilde{N}).$$

Inverting, we obtain, for $0 < r < m$,

$$\varepsilon_r(\tilde{N}) = \frac{1}{m}\sum_{s=1}^{m-1}(\omega^{-rs} - 1)(\operatorname{sign}(\tau^s, \tilde{N}) - \operatorname{sign} N)$$
$$= \operatorname{sign} N + \frac{1}{m}\sum_{s=1}^{m-1}(\omega^{-rs} - 1)\operatorname{sign}(\tau^s, \tilde{N}).$$

By the G-signature theorem [1, Proposition 6.18], $\operatorname{sign}(\tau^s, \tilde{N}) = [\tilde{F}]^2 \operatorname{cosec}^2(\pi s/m)$, $0 < s < m$. We see geometrically that the self-intersection number $[\tilde{F}^2]$ is equal to $[F]^2/m$. Therefore

$$\varepsilon_r(\tilde{N}) = \operatorname{sign} N + \frac{[F]^2}{m^2}\sum_{s=1}^{m-1}(\omega^{-rs} - 1)\operatorname{cosec}^2\frac{\pi s}{m}.$$

Now

$$\sum_{s=1}^{m-1}(\omega^{-rs} - 1)\operatorname{cosec}^2\frac{\pi s}{m} = -2\sum_{s=1}^{m-1}\sin^2\frac{\pi rs}{m}\operatorname{cosec}^2\frac{\pi s}{m}$$
$$- i\sum_{s=1}^{m-1}\sin\frac{2\pi rs}{m}\operatorname{cosec}^2\frac{\pi s}{m}.$$

The second sum must vanish, and one may easily verify that it does. To evaluate the first sum, let $\xi = e^{\pi i/m}$. Then

$$\sum_{s=1}^{m-1}\sin^2\frac{\pi rs}{m}\operatorname{cosec}^2\frac{\pi s}{m} = \sum_{s=1}^{m-1}\left(\frac{\xi^{rs} - \xi^{-rs}}{\xi^s - \xi^{-s}}\right)^2$$
$$= \sum_{s=1}^{m-1}(\xi^{s(r-1)} + \xi^{s(r-3)} + \cdots + \xi^{-s(r-1)})^2$$
$$= \sum_{s=1}^{m-1} P(\xi^s), \quad \text{say.}$$

Now $P(z) = P(z^{-1})$, and $\xi^{2m} = 1$. Therefore

$$\sum_{s=1}^{m-1} P(\xi^s) = \frac{1}{2}\sum_{s=0}^{2m-1} P(\xi^s) - \frac{1}{2}(P(1) + P(-1))$$
$$= \frac{1}{2}\left(2m\sum_t \text{coefficient of } z^{2mt} \text{ in } P(z)\right) - r^2$$
$$= r(m - r)$$

as the only contribution to the sum of coefficients comes from $t = 0$, and is r. Hence $\sum_{s=1}^{m-1}(\omega^{-rs} - 1)\operatorname{cosec}^2(\pi s/m) = -2r(m - r)$, and the proof is complete.

Now let M be a closed 3-manifold, and $\varphi\colon H_1(M) \to Z_m$ an epimorphism. φ induces an m-fold cyclic covering $\tilde{M} \to M$, with a canonical generator, corresponding to $1 \in Z_m$, for the group of covering translations.

Suppose that for some positive integer n, there is an mn-fold cyclic branched covering of 4-manifolds $\tilde{W} \to W$, branched over a surface $F \subset \operatorname{int} W$, such that $\partial(\tilde{W} \to W) = n(\tilde{M} \to M)$, and such that the covering translation of $\tilde{W}$ which induces rotation through $2\pi/m$ on the fibres of the normal bundle of $\tilde{F}$ restricts on each component of $\partial\tilde{W}$ to the canonical covering translation of $\tilde{M}$ determined by φ. Let this covering translation induce τ on $H = H_2(\tilde{W}) \otimes C$. As in the closed case, $(H, \cdot)$ is an orthogonal direct sum of eigenspaces of τ, and again we have the eigenspace signatures $\varepsilon_r(\tilde{W})$, the only difference being that the form $\cdot$ will not now in general be nonsingular. Define, for $0 < r < m$, the rational number

$$\sigma_r(M, \varphi) = \frac{1}{n}\left(\operatorname{sign} W - \varepsilon_r(\tilde{W}) - \frac{2[F]^2 r(m-r)}{m^2}\right).$$

It follows readily from Lemma 2.1, and Novikov additivity of sign W and $\varepsilon_r(\tilde{W})$ (valid for the latter because they are linear combinations of τ^s-signatures) that $\sigma_r(M, \varphi)$ depends only on (M, φ) and r.

As we shall see in Lemma 2.2 below, it is always possible to take $n = 1$, but the extra generality in the definition will be useful in §4. We shall, however, always be in a situation where either $n = 1$ or $F = \emptyset$.

The following lemma shows that $\sigma_r(M, \varphi)$ is always defined.

LEMMA 2.2. *Given (M, φ) as above, suppose $M = \partial W$ with $H_1(W; Z_m) = 0$. Then $\tilde{M} \to M$ extends to an m-fold cyclic branched covering $\tilde{W} \to W$ over a surface $F \subset$ int W, such that the canonical covering translation of $\tilde{M}$ corresponds to rotation through $2\pi/m$ on each fibre of the normal bundle of $\tilde{F} \subset$ int $\tilde{W}$.*

PROOF. $\varphi \in \operatorname{Hom}(H_1(M), Z_m) \cong H^1(M; Z_m)$. Since $H_1(W; Z_m) = 0$, there is a surface $F \subset$ int W such that the image in $H_2(W; Z_m)$ of $[F] \in H_2(W)$ is the Lefschetz dual of $\delta\varphi \in H^2(W, M; Z_m) \cong \operatorname{Hom}(H_2(W, M), Z_m)$. Thus, in terms of intersections,

$$[F] \cdot x \pmod m = \delta\varphi(x) = \varphi(\partial x) \quad \text{for all } x \in H_2(W, M).$$

Let $\rho \in H^2(W, W - F; Z_m) \cong \operatorname{Hom}(H_2(W, W - F), Z_m)$ be dual to the fundamental class in $H_2(F; Z_m)$. Comparing the cohomology exact sequences of the pairs (W, M), $(W, W - F)$, with Z_m coefficients, we see that $\rho = \delta\psi$ for some $\psi \in H^1(W - F; Z_m) \cong \operatorname{Hom}(H_1(W - F), Z_m)$ which extends φ. Note also that since

$$[F] \cdot y \pmod m = \delta\psi(y) = \psi(\partial y) \quad \text{for all } y \in H_2(W, W - F),$$

ψ evaluates to $1 \in Z_m$ on a meridian of F. Then ψ determines the desired branched covering $\tilde{W} \to W$.

3. Surgery descriptions. We now describe a method for computing $\sigma_r(M, \varphi)$. A framed oriented link L, with components $L_1, \cdots, L_n$, in S^3, is a *surgery description* of (M, φ) if

(i) M is obtained by surgery on L (according to its framing), and

(ii) if $\bar{\mu}_i \in H_1(M)$ is the image of the class of a meridian μ_i of L_i, then $\varphi(\bar{\mu}_i) = 1 \in Z_m$ for each $i = 1, \cdots, n$.

(Note that the orientation of L is irrelevant to (i), but not to (ii).)

Surgery descriptions in this sense always exist, for it is known that, given M, there exists a link L satisfying (i) [16], [7], which may now be modified by moves corresponding to handle additions and handle slides until (ii) is also satisfied.

The invariants $\sigma_r(M, \varphi)$ can be expressed in terms of certain invariants of L, as follows. Let $A = (a_{ij})$ be the matrix of linking numbers of L, that is, $a_{ij} = lk(L_i, L_j)$, $i \neq j$, and a_{ii} is the framing integer associated with L_i. Choose a Seifert surface V spanning L, let S be the corresponding Seifert matrix, and let S^T denote the transpose of S. Recall that $\omega = e^{2\pi i/m}$.

LEMMA 3.1. *Let the framed oriented link L be a surgery description of (M, φ). Then, for $0 < r < m$,*

$$\sigma_r(M, \varphi) = \operatorname{sign} A - \operatorname{sign}((1 - \omega^{-r})S + (1 - \omega^r)S^T) - \frac{2(\sum_{i,j} a_{ij}) r(m-r)}{m^2}.$$

PROOF. Let W be the 4-manifold obtained by attaching n 2-handles to the 4-ball B^4 along disjoint tubular neighbourhoods of the components of L, according to the framing of L. Then W is 1-connected and $\partial W = M$. Recalling the proof of Lemma 2.2, we seek $F \subset \text{int } W$ such that

$$[F]\cdot x \pmod m = \delta\varphi(x) = \varphi(\partial x) \quad \text{for all } x \in H_2(W, M).$$

For $i = 1, \cdots, n$, let $c_i \in H_2(W)$ be the class represented by the core of the ith 2-handle together with (say) the cone (in B^4) on L_i. Then $H_2(W)$ is free abelian on $c_1, \cdots, c_n$. Also, $H_2(W, M)$ is free abelian on $c_1^*, \cdots, c_n^*$, where c_i^* is the class of the co-core of the ith 2-handle. Let $f = \sum_{i=1}^n c_i$. Then $f\cdot c_j^* = 1, j = 1, \cdots, n$, and by hypothesis $\varphi(\partial c_j^*) = \varphi(\tilde{\mu}_j) = 1 \in Z_m$. Hence $f\cdot x \pmod m = \varphi(\partial x)$ for all $x \in H_2(W, M)$. Let V' be obtained by pushing the interior of V, the Seifert surface for L, into the interior of B^4, in the obvious way, using a collar of S^3 in B^4. The union of V' with the cores of all the 2-handles is then a surface $F \subset \text{int } W$ representing f. By the proof of Lemma 2.2, we then have an m-fold cyclic branched cover $\tilde{W}$ of (W, F) such that $\partial(\tilde{W} \to W)$ is the covering $\tilde{M} \to M$ determined by φ.

The intersection form on $H_2(W)$ is given, with respect to the basis $c_1, \cdots, c_n$, by the matrix A of linking numbers of L; hence sign $W =$ sign A. Also, $[F]^2 = (\sum_{i=1}^n c_i)^2 = \sum_{i,j} a_{ij}$. This accounts for the first and last terms on the right-hand side of the assertion of the lemma; it remains to identify the middle term as $\varepsilon_r(\tilde{W})$.

Now $\tilde{W} = \tilde{B} \cup \tilde{H}$, where $\tilde{B}$ is the m-fold cyclic branched cover of (B^4, V'), and $\tilde{H}$ is the m-fold cyclic branched cover of ($\bigcup$ 2-handles, $\bigcup$ cores). Thus $\partial\tilde{B}$ is the m-fold cyclic branched cover of (S^3, L), and $\tilde{H}$ is a disjoint union of n 2-handles, attached to $\tilde{B}$ along a tubular neighbourhood of $\tilde{L} \subset \partial\tilde{B}$, where $\tilde{L}$ is the inverse image of L. Since $H_2(\tilde{L}) = 0 = H_2(\tilde{H})$, there is a Mayer-Vietoris exact sequence $0 \to H_2(\tilde{B}) \to H_2(\tilde{W}) \to H_1(\tilde{L})$ which is equivariant with respect to the action of the group of covering translations. Now tensor with C, and observe that the resulting exact sequence induces a corresponding exact sequence of eigenspaces. In particular, since the covering translations act trivially on $\tilde{L}$, we have $\varepsilon_r(\tilde{B}) = \varepsilon_r(\tilde{W})$ for $0 < r < m$.

For calculating intersections, it turns out to be more convenient to use, instead of $\tilde{B}$, the corresponding unbranched cover. So consider a tubular neighbourhood $V' \times D^2$ of V' in B^4, and let $\hat{B}$ be the m-fold cyclic cover of $B^4 - V' \times \text{int } D^2$. Then $\tilde{B} \cong \hat{B} \cup V' \times D^2$, and, since $H_2(V') = 0$, we have an equivariant Mayer-Vietoris exact sequence

$$H_2(V' \times S^1) \longrightarrow H_2(\hat{B}) \longrightarrow H_2(\tilde{B}) \longrightarrow H_1(V' \times S^1).$$

Since the covering translations induce the identity on $H_*(V' \times S^1)$, an elementary argument shows that inclusion induces an isomorphism of eigenspaces $E_r(\hat{B}) \cong E_r(\tilde{B})$ for $0 < r < m$, and hence $\varepsilon_r(\hat{B}) = \varepsilon_r(\tilde{B}), 0 < r < m$.

$\hat{B}$ may be described as follows. First let C be obtained by cutting $B^4 - V' \times \text{int } D^2$ along the trace T of the isotopy which pushed the interior of V into the interior of B^4. Observe that $T \cong V \times I$, that $C \cong B^4$, and that C contains two copies $T^\pm$ of T in its boundary. Now take m copies C_s of C, $s \in Z_m$, and identify T_s^+ with T_{s+1}^- for each s. The result is $\hat{B}$. Let $z_1, \cdots, z_k$ be cycles in V representing a basis for $H_1(V)$. These determine cycles $z_1^+, \cdots, z_k^+$, say, in T_0^+ and $z_1^-, \cdots, z_k^-$ in T_1^-, and for each $i = 1, \cdots, k$, the union of the cone on z_i^+ in C_0 and the cone on z_i^- in C_1 determines a class $x_i \in H_2(\hat{B})$. A Mayer-Vietoris argument shows that $x_1, \cdots, x_k$ is a $Z[Z_m]$-basis for $H_2(\hat{B})$. Letting τ as usual denote the automorphism of $H_2(\hat{B})$ induced by the canonical covering translation, which takes each C_s to C_{s+1}, a basis for $H_2(\hat{B})$ over Z is $\{\tau^s x_i : 0 \leqq r < m, 1 \leqq i \leqq k\}$. The intersection form on $H_2(\hat{B})$ with respect to this basis can be readily described. Recall that S is the Seifert matrix of L with respect to V; write $S = (v_{ij})$. Then (with an appropriate modification if $m = 2$)

$$\tau^s x_i \cdot \tau^t x_j = \begin{cases} v_{ij} + v_{ji}, & s = t, \\ -v_{ij}, & s = t + 1, \\ -v_{ji}, & s = t - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Now pass to $H_2(\hat{B}) \otimes C$, but continue to write τ, x_i instead of $\tau \otimes \mathrm{id}$, $x_i \otimes 1$. Let $y_{i,r} = \sum_{s=0}^{m-1} \omega^{-rs} \tau^s x_i$, $0 \leqq r < m$, $1 \leqq i \leqq k$. Then $\{y_{i,r} : 1 \leqq i \leqq k\}$ is a linearly independent set of elements of E_r, $0 \leqq r < m$. Since $E_0 \oplus E_1 \oplus \cdots \oplus E_{m-1} = H_2(\hat{B}) \otimes C$ has dimension mk, it follows that in fact it is a basis for E_r. With respect to this basis, the Hermitianized intersection form on E_r is given by

$$\begin{aligned} y_{i,r} \cdot y_{j,r} &= \sum_{t=0}^{m-1} \sum_{s=0}^{m-1} (\omega^{-rs} \tau^s x_i) \cdot (\omega^{-rt} \tau^t x_j) \\ &= \sum_{t=0}^{m-1} \sum_{s=0}^{m-1} \omega^{-r(s-t)} (\tau^s x_i \cdot \tau^t x_j) \\ &= \sum_{s=0}^{m-1} (v_{ij} + v_{ji} - \omega^{-r} v_{ij} - \omega^r v_{ji}) \\ &= m((1 - \omega^{-r}) v_{ij} + (1 - \omega^r) v_{ji}). \end{aligned}$$

Hence, for $0 < r < m$, $\varepsilon_r(\tilde{W}) = \varepsilon_r(\hat{B}) = \mathrm{sign}((1 - \omega^{-r})S + (1 - \omega^r)S^T)$, and the proof is complete.

REMARK. The signatures $\mathrm{sign}((1 - \omega^{-r})S + (1 - \omega^r)S^T)$, for m prime and $r = [m/2]$, were used by Tristram in [13]. (Compare also [9] and [6].) The above interpretation of them as eigenspace signatures associated with an m-fold branched cyclic cover has also been given, in somewhat greater generality, by Viro [15].

4. Slice knots. Let K be a knot in S^3. Fix a prime q, and let M_n denote the q^n-fold branched cyclic cover of (S^3, K), $n = 1, 2, \cdots$. (By an argument analogous to the proof of Lemma 4.2 below, $H_*(M_n; Q) \cong H_*(S^3; Q)$.) Suppose we have an epimorphism $\varphi: H_1(M_1) \to Z_m$. It is not hard to show that the branched covering projection $M_n \to M_1$ induces a surjection on π_1, and hence on H_1. Composition with φ then defines epimorphisms $\varphi_n: H_1(M_n) \to Z_m$ for all n.

THEOREM 4.1. *Suppose K is a slice knot. Then there is a constant c, and a subgroup G of $H_1(M_1)$ with $|G|^2 = |H_1(M_1)|$, such that if m is a prime power and $\varphi: H_1(M_1) \to Z_m$ is an epimorphism satisfying $\varphi(G) = 0$, then $|\sigma_r(M_n, \varphi_n)| < c$ for all n.*

We remark that the proof of Theorem 4.1 will apply without essential change to any knot K in a homology 3-sphere M such that $(M, K) = \partial(W, D)$ for some 2-disc D in a homology 4-ball W.

We require some preliminary lemmas.

LEMMA 4.2. *Let D be a 2-disc in B^4, and let V_n be the q^n-fold branched cyclic cover of (B^4, D), q prime. Then $\tilde{H}_*(V_n: Q) = 0$.*

PROOF. Let $\tilde{X}$ be the infinite cyclic cover of $B^4 - D$. We then have the exact sequence (see [9])

$$\cdots \longrightarrow \tilde{H}_i(\tilde{X}; Z_q) \xrightarrow{t^{q^n}-1} \tilde{H}_i(\tilde{X}; Z_q) \longrightarrow \tilde{H}_i(V_n; Z_q) \longrightarrow \tilde{H}_{i-1}(\tilde{X}; Z_q) \longrightarrow \cdots$$

where t is the automorphism induced by the canonical covering translation of $\tilde{X}$. Since $V_0 = B^4$, $t - 1$ is an isomorphism. Hence, with Z_q coefficients, $t^{q^n} - 1 = (t - 1)^{q^n}$ is also an isomorphism, giving $\tilde{H}_*(V_n; Z_q) = 0$. Since V_n is compact, the result follows.

LEMMA 4.3. *Let V be a Q-homology 4-ball. If the image of $H_1(\partial V) \to H_1(V)$ has order l, then $H_1(\partial V)$ has order l^2.*

PROOF. Since $H_2(\partial V) = 0$, we have an exact sequence

$$0 \to H_2(V) \to H_2(V, \partial V) \to H_1(\partial V) \to H_1(V) \to H_1(V, \partial V) \to 0.$$

By duality and universal coefficient theorems, $|H_2(V)| = |H_1(V, \partial V)|$ and $|H_2(V, \partial V)| = |H_1(V)|$; hence the result.

A slight extension of [5, Lemma 4.1] yields

LEMMA 4.4. *Let X be a connected complex with $\pi_1(X)$ finitely generated, $H_1(X)$ finite, and $H_1(X; Z_p)$ cyclic for some prime p. Let $\tilde{X} \to X$ be a regular p^r-fold cyclic covering. Then $H_1(\tilde{X}; Q) = 0$.*

For a proof of the following, see [9].

LEMMA 4.5. *Let X be a finite connected complex and $\tilde{X} \to X$ a regular infinite cyclic covering. Let F be a field. If $H_1(X; F) \cong F$, then* dim *$H_1(\tilde{X}; F)$ is finite.*

PROOF OF THEOREM 4.1. By hypothesis, $(S^3, K) = \partial(B^4, D)$ for some 2-disc $D \subset B^4$. Let V_n be the q^n-fold branched cyclic cover of (B^4, D); thus $\partial V_n = M_n$. By Lemma 4.2, $\tilde{H}_*(V_n; Q) = 0$. Let $i_{n*}: H_1(M_n) \to H_1(V_n)$ be induced by inclusion, and let $G = \ker i_{1*}$. By Lemma 4.3, $|G|^2 = |H_1(M_1)|$.

Suppose $m = p^a$, p prime. Since $\varphi(G) = 0$, there is an epimorphism $\psi: H_1(V_1) \to Z_{p^b}$ for some b making the diagram

$$\begin{array}{ccc} H_1(M_1) & \xrightarrow{i_{1*}} & H_1(V_1) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} \\ Z_{p^a} & \longrightarrow & Z_{p^b} \end{array}$$

commute, where $Z_{p^a} \to Z_{p^b}$ is multiplication by p^{b-a}. Composing ψ with the epimorphism $H_1(V_n) \to H_1(V_1)$ induced by the branched covering projection gives a commutative diagrm

$$\begin{array}{ccc} H_1(M_n) & \xrightarrow{i_{n*}} & H_1(V_n) \\ \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \psi_n} \\ Z_{p^a} & \longrightarrow & Z_{p^b} \end{array}$$

for all n.

Let $d_n = \dim H_1(V_n; Z_p)$. By doing surgery on $d_n - 1$ circles in int V_n we may obtain W_n with $H_1(W_n; Z_p)$ cyclic and a commutative diagram

$$\begin{array}{ccc} H_1(M_n) & \xrightarrow{i'_{n*}} & H_1(W_n) \\ \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \psi'_n} \\ Z_{p^a} & \longrightarrow & Z_{p^b} \end{array}$$

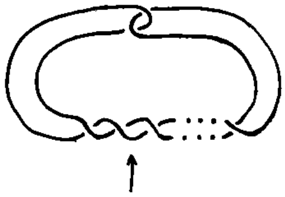
where i'_n is inclusion and ψ'_n is surjective. Let $\tilde{W}_n \to W_n$ be the p^b-fold cyclic covering induced by ψ'_n; then $\partial(\tilde{W}_n \to W_n)$ consists of p^{b-a} copies of the p^a-fold cyclic covering $\tilde{M}_n \to M_n$ induced by φ_n.

Since $\tilde{H}_*(V_n; Q) = 0$, the euler characteristic $\chi(V_n) = 1$. Hence $\chi(W_n) = \chi(V_n) + 2(d_n - 1) = 2d_n - 1$, giving $\chi(\tilde{W}_n) = p^b(2d_n - 1)$. By Lemma 4.4, $H_1(\tilde{W}_n; Q) = 0$. Therefore $H_3(\tilde{W}_n; Q)$, which is isomorphic to $H_1(\tilde{W}_n, \partial\tilde{W}_n; Q)$ by duality, has dimension $p^{b-a} - 1$. It follows that $\dim H_2(\tilde{W}_n; Q) = p^b(2d_n - 1) + p^{b-a} - 2$. Note also that since signature is unaffected by surgery, sign W_n = sign $V_n = 0$. Hence

$$\begin{aligned} |\sigma_r(M_n, \varphi_n)| &\leq \frac{1}{p^{b-a}}(p^b(2d_n - 1) + p^{b-a} - 2) \\ &< p^a(2d_n - 1) + 1. \end{aligned}$$

Finally, let $\tilde{X}$ denote the infinite cyclic cover of $X = B^4 - D$, and let t: $H_1(\tilde{X}; Z_p) \to H_1(\tilde{X}; Z_p)$ be the automorphism induced by the canonical covering translation. Then (see the proof of Lemma 4.2) $H_1(V_n; Z_p) \cong \operatorname{coker}(t^{q^n} - 1)$. In particular, $d_n \leq d = \dim H_1(\tilde{X}; Z_p)$, which is finite by Lemma 4.5. We may now set $c = |G|(2d - 1) + 1$, and the proof is complete.

5. Some calculations. Let us consider the knots K_k ($k \in Z$) illustrated in Figure 1.



k full positive twists

FIGURE 1

Thus K_k may be described as the k-twisted double of the unknot, or alternatively as the rational (2-bridge) knot corresponding to the rational number $(4k + 1)/2$ **[11]**, **[3]**. Its 2-fold branched cover is the lens space $L(4k + 1, 2)$.

K_k has a Seifert surface of genus 1 with corresponding Seifert matrix

$$\begin{pmatrix} -1 & 1 \\ 0 & k \end{pmatrix}.$$

It follows easily that K_k is algebraically slice precisely when $4k + 1 = l^2$ for some integer l. The first two such values of k, namely 0 and 2, give the unknot and the stevedore's knot respectively, both of which are slice (indeed ribbon) knots. However,

THEOREM 5.1. *K_k is slice only if $k = 0, 2$.*

In fact, the proof shows that if $k \neq 0, 2$, then K_k does not bound a disc in any homology 4-ball (see remark after Theorem 4.1).

PROOF. If K_k is slice, then it is certainly algebraically slice. So for some fixed k such that $4k + 1 = l^2$, let M_n be the 2^n-fold branched cyclic cover of (S^3, K_k). For any divisor m of l we have epimorphisms $\varphi\colon H_1(M_1) \cong Z_{l^2} \to Z_m$, which necessarily satisfy $\varphi(G) = 0$ where $G \subset H_1(M_1)$ has order l. We compute $\sigma_r(M_n, \varphi_n)$ (for suitable φ) using the following surgery description. In Figure 2, surgery with framing $+1$ on the unknotted curve J indicated yields S^3 in such a

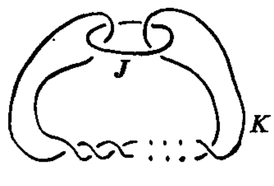


FIGURE 2

way that the other unknotted curve, K, becomes K_k. By an isotopy of S^3, Figure 2 may be transformed to Figure 3. Then M_n, the 2^n-fold branched cyclic cover of (S^3, K_k), is obtained by surgery on the link L consisting of the 2^n lifts of J in the

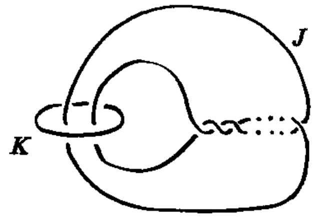


FIGURE 3

2^n-fold branched cyclic cover of (S^3, K). The latter is just S^3, and L is illustrated in Figure 4. To determine the appropriate framing x of a component L_i of L, choose (temporarily) an equivariant orientation of L. Consider a 2-chain C_i whose boundary is a slightly pushed-off copy of L_i determined by the framing of L_i. This projects to a 2-chain C whose boundary is a similarly defined push-off of J. Consideration of the intersections of $\bigcup_i C_i$ and C with L and J respectively gives, for each i,

$$1 = \text{framing of } J = x + \sum_{j \neq i} lk(L_i, L_j) = x - 2k;$$

hence $x = 2k + 1$.

We must now consider $\varphi_n\colon H_1(M_n) \to Z_m$. M_1 is obtained by surgery on the framed link shown in Figure 5, to which, for reasons soon to become apparent,

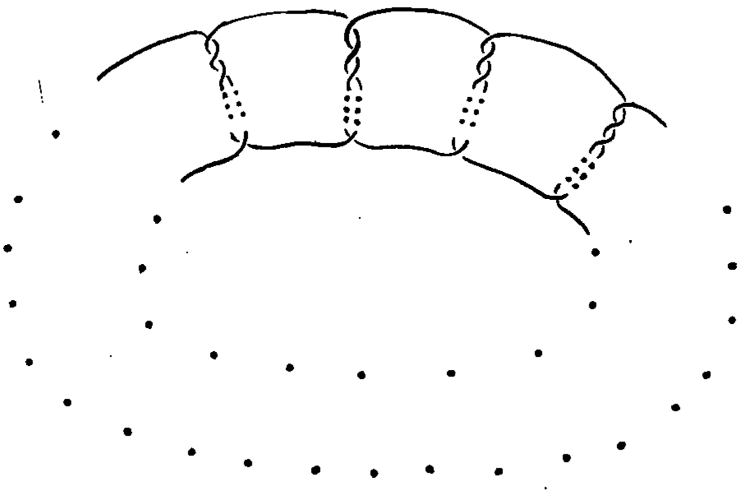


FIGURE 4

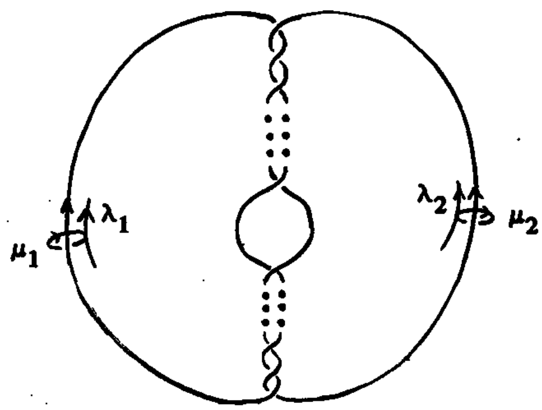


FIGURE 5

we have assigned a nonequivariant orientation. Let λ_1, μ_1 and λ_2, μ_2 be oriented longitude-meridian pairs for the two components, where λ_i is chosen to be null-homologous in the complement of the ith component. The first homology of the complement of the link is free abelian on μ_1, μ_2, and we have $\lambda_1 = 2k\mu_2$, $\lambda_2 = 2k\mu_1$. Surgery has the effect of adding the relations

$$\lambda_1 + (2k + 1)\mu_1 = 0, \qquad \lambda_2 + (2k + 1)\mu_2 = 0.$$

Thus if $\bar{\mu}_i$ denotes the image of μ_i in $H_1(M_1)$, we see that $H_1(M_1)$ is cyclic of order $4k + 1 = l^2$, generated by $\bar{\mu}_1 = \bar{\mu}_2$. Hence $\varphi\colon H_1(M_1) \to Z_m$ can be chosen to satisfy $\varphi(\bar{\mu}_1) = \varphi(\bar{\mu}_2) = 1$.

More generally, we give the link L which yields M_n the alternating orientation shown in Figure 6. Recalling that φ_n is defined to be the composition $H_1(M_n) \to H_1(M_1) \xrightarrow{\varphi} Z_m$, it follows that we then have $\varphi(\bar{\mu}_i) = 1$ for each i, where $\bar{\mu}_i \in H_1(M_n)$ corresponds to the meridian of the ith component L_i of L. Thus L is a surgery presentation for (M_n, φ_n) in the sense of §3.

The linking matrix A of L is the $2^n \times 2^n$ matrix

$$\begin{pmatrix} 2k+1 & k & 0 & \cdots & 0 & k \\ k & 2k+1 & k & & & 0 \\ 0 & k & 2k+1 & & & \vdots \\ \vdots & & & & \ddots & 0 \\ 0 & & & & & k \\ k & 0 & & \cdots & 0 \quad k & 2k+1 \end{pmatrix}.$$

Note that $A = kB + I$, where B is

$$\begin{pmatrix} 2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 2 & 1 & & & 0 \\ 0 & 1 & 2 & & & \vdots \\ \vdots & & & \ddots & & 0 \\ 0 & & & & 2 & 1 \\ 1 & 0 & \cdots & 0 & 1 & 2 \end{pmatrix}.$$

It is easy to verify (for example by inductively computing the principal minors) that B is positive definite. Hence A is also positive definite, that is, $\operatorname{sign} A = 2^n$. Also, $\sum_{i,j} a_{ij} = 2^n(4k+1) = 2^n l^2$.

Let V be the Seifert surface for L illustrated in Figure 6.

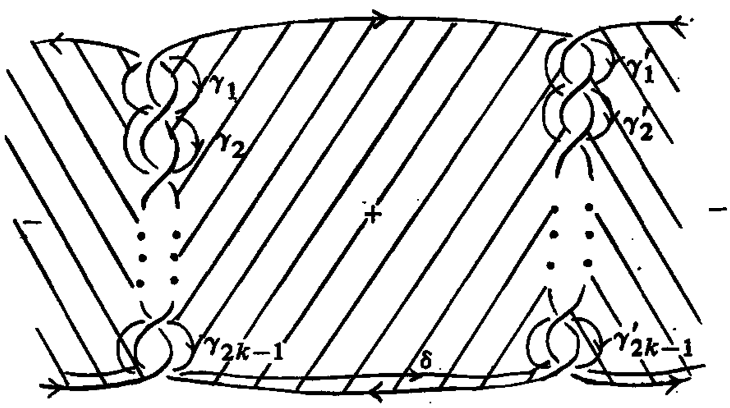


FIGURE 6

The 2^n $(2k-1)$-element sets of which $\{\gamma_1, \cdots, \gamma_{2k-1}\}$, $\{\gamma'_1, \cdots, \gamma'_{2k-1}\}$ is a typical pair, together with δ, determine a basis for $H_1(V)$. In the corresponding Seifert matrix, S_n, say, $\gamma_1, \cdots, \gamma_{2k-1}$ contribute the $(2k-1) \times (2k-1)$ block

$$C = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & & \vdots \\ \cdot & & -1 & & \\ \cdot & & & & 0 \\ \cdot & & & \ddots & 1 \\ 0 & \cdots & & 0 & -1 \end{pmatrix}$$

and $\gamma_1', \cdots, \gamma_{2k-1}'$ contribute the transpose C^T of C. Thus S_n is the $(2^n(2k-1)+1) \times (2^n(2k-1)+1)$ matrix

$$\left(\begin{array}{ccccccc|c} C & & & & & & & 1 \\ & C^T & & & & & & \\ & & C & & & & & 1 \\ & & & C^T & & & & \\ & & & & \ddots & & & \\ & & & & & C & & 1 \\ & & & & & & C^T & \\ \hline & 1 & & 1 & & & 1 & -2^{n-1} \end{array}\right).$$

Let $\omega = e^{2\pi i/m}$. Write $S_{n,r}$ for the Hermitian matrix $(1-\omega^{-r})S_n + (1-\omega^r)S_n^T$, and similarly for C. Then $S_{n,r}$ is

$$\left(\begin{array}{ccccccc|c} C_r & & & & & & & \bar{\alpha} \\ & C_r^T & & & & & & \alpha \\ & & C_r & & & & & \bar{\alpha} \\ & & & C_r^T & & & & \alpha \\ & & & & \ddots & & & \\ & & & & & C_r & & \bar{\alpha} \\ & & & & & & C_r^T & \alpha \\ \hline \alpha & \bar{\alpha} & \alpha & \bar{\alpha} & & \alpha & \bar{\alpha} & x \end{array}\right)$$

where $\alpha = 1 - \omega^r$ and $x = -2^{n-1}\alpha\bar{\alpha}$. Now choose P such that $PC_rP^* = D$ is diagonal, and let Q be the $(2^n(2k-1)+1) \times (2^n(2k-1)+1)$ matrix

$$\left(\begin{array}{ccccc|c} P & & & & & \\ & \bar{P} & & & & \\ & & \ddots & & & 0 \\ & & & P & & \\ & & & & \bar{P} & \\ \hline & & 0 & & & 1 \end{array}\right).$$

Then $QS_{n,r}Q^*$ is

$$\left(\begin{array}{ccccc|c} D & & & & & * \\ & D & & & & * \\ & & \ddots & & & \vdots \\ & & & D & & * \\ & & & & D & * \\ \hline * & * & \cdots & * & * & x \end{array}\right)$$

where the entries in the last row (and last column) are periodic with period $2(2k-1)$. We shall see later (in the proof of Lemma 5.2) that C_r is nonsingular. Assuming this, use the diagonal entries of the D's in $QS_{n,r}Q^*$ to clear all the entries (except the last) from the last row and column. Because of the periodicity noted above, this process changes the entry x to $x + 2^{n-1}y = 2^{n-1}(y - \alpha\bar{\alpha})$ for some y independent of n. Hence sign $S_{n,r} = 2^n$ sign $C_r + \eta_r$, where $|\eta_r| \leqq 1$ and η_r is independent of n. By Lemma 3.1, for $0 < r < m$,

$$\sigma_r(M_n, \varphi_n) = 2^n - 2^n \operatorname{sign} C_r - \eta_r - 2^{n+1}r(m - r)(l/m)^2.$$

In particular, note the multiplicative relation

$$\sigma_r(M_n, \varphi_n) + \eta_r = 2^n(\sigma_r(M_1, \varphi) + \eta_r).$$

It follows from Theorem 4.1 that for K_k to be slice we must have $\sigma_r(M_1, \varphi) + \eta_r = 0$, or, equivalently,

$$\operatorname{sign} C_r = 1 - 2r(m - r)(l/m)^2,$$

for every prime power divisor m of l, and every r, $0 < r < m$. (We may remark that the present examples are somewhat deceptive in that the multiplicativity noted above does not hold in general. Nevertheless, a good deal of information can be extracted from a single branched cover; see [2, Theorems 2 and 3].)

Since $C_{m-r} = \bar{C}_r$, and m is odd, there is no loss of generality in restricting to $0 < r \leqq (m - 1)/2$. Theorem 5.1 will follow easily from

LEMMA 5.2. *Suppose m is odd and $0 < r \leqq (m - 1)/2$. Then*

$$\operatorname{sign} C_r = -2[2kr/m] - 1.$$

PROOF. Let D_n be the $n \times n$ principal minor of C_r, $n = 1, \cdots, 2k - 1$. Then, writing $\alpha = 1 - \omega^r$ as before, and expanding D_n by (say) the first row, we obtain the difference equation

$$D_n = -(\alpha + \bar{\alpha})D_{n-1} - \alpha\bar{\alpha}D_{n-2}, \qquad n = 2, \cdots, 2k - 1.$$

Since the roots of the corresponding characteristic equation $x^2 + (\alpha + \bar{\alpha})x + \alpha\bar{\alpha} = 0$ are $-\alpha$, $-\bar{\alpha}$, the general solution of this difference equation is $(-1)^n(A\alpha^n + B\bar{\alpha}^n)$, where A and B are arbitrary constants. Our initial values $D_0 = 1$, $D_1 = -(\alpha + \bar{\alpha})$ give $A = \alpha/(\alpha - \bar{\alpha})$, $B = -\bar{\alpha}/(\alpha - \bar{\alpha})$; hence

$$D_n = (-1)^n\left(\frac{\alpha^{n+1} - \bar{\alpha}^{n+1}}{\alpha - \bar{\alpha}}\right).$$

Write $\alpha = \rho e^{i\theta}$, $\rho > 0$. Then $D_n = (-\rho)^n \sin(n + 1)\theta/\sin\theta$. Also, since $\tan\theta = (-\sin 2\pi r/m)/(1 - \cos 2\pi r/m) = -\cot \pi r/m$, and since $-\pi/2 < \theta < 0$, we have $\theta = \pi r/m - \pi/2$. In particular, we see that $D_{2k-1} = \det C_r \neq 0$, a fact which we used earlier. We also see that there are no two consecutive zeros among the D_n, so sign C_r = (number of permanences of sign) − (number of changes of sign) in the sequence $D_0, \cdots, D_{2k-1}$ (where 0's may be assigned either sign). Thus sign $C_r = 2c - (2k - 1)$, where c = number of changes of sign of $\sin n\theta = \sin n(\pi r/m - \pi/2)$, $n = 1, \cdots, 2k$. Write $r = (m - s)/2$, $1 \leqq s \leqq m - 2$, s odd. Then $\sin n(\pi r/m - \pi/2) = -\sin \pi ns/2m$. Hence c = number of changes of sign of $\sin \pi ns/2m$, $n = 1, \cdots, 2k$, $= [2ks/2m] = [k - 2kr/m] = k - 1 - [2kr/m]$. Therefore,

$$\operatorname{sign} C_r = 2\left(k - 1 - \left[\frac{2kr}{m}\right]\right) - (2k - 1) = -2\left[\frac{2kr}{m}\right] - 1$$

as stated.

Returning to the proof of Theorem 5.1, recall that K_k slice implies

$$2r(m - r)(l/m)^2 - 1 + \text{sign } C_r = 0$$

for every prime power divisor m of $l = 4k + 1$, and every r, $0 < r < m$. By Lemma 5.2, this is equivalent to the condition that for every r, $0 < r \leqq (m - 1)/2$,

$$r(m - r)(l/m)^2 - \left[\frac{2kr}{m}\right] - 1 = 0.$$

Replacing $[2kr/m]$ by $(l^2 - 1)r/2m = 2kr/m > [2kr/m]$, it follows that we must have

$$r(m - r)(l/m)^2 - \frac{(l^2 - 1)r}{2m} - 1 < 0.$$

Multiplying by $2/r$, we obtain $(m - 2r)\,(l/m)^2 + 1/m - 2/r < 0$, and hence, since $m|l$, $m + 1/m < 2(r + 1/r)$. But putting $r = (m - 1)/2$, the value which maximizes $r + 1/r$, gives $m^2 - 4m - 1 < 0$, which is clearly violated by (odd) $m > 3$. Moreover, if $l > 3$, then l has a prime power divisor $m > 3$. Hence K_k can be slice only if $l = 1, 3$, that is, $k = 0, 2$. Indeed we have shown that this fact is detected by the invariants $\sigma_r(M_n, \varphi_n)$ for any r, $0 < r < m$.

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