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FIBRATIONS OVER SPHERES

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IN THIS paper we consider the following problem; if V is a closed smooth manifold, under what conditions is a map $f: V \to S^m$ homotopic to a smooth fibration? Browder and Levine [1] have given some very general conditions for the case $m = 1$, so we assume that $m \geqq 2$.

Let us factorise f through a homotopy equivalence and a Hurewicz fibre map with fibre L. A *necessary* condition for our problem to be soluble is that L have the homotopy type of a finite dimensional CW complex. At present we also need to assume that L is 1-connected. In §1 we define a *pre-fibration* to be a map with a regular value b such that $f^{-1}(b)$ has the homotopy properties of a fibre. Theorem 1 shows that the above conditions on L are almost sufficient to ensure that f is homotopic to a pre-fibration.

In §4 we classify pre-fibrations over S^m with given fibre F; we find that they are in (1–1) correspondence with the elements of a group $D_{m-1}(F)$. A recent theorem of Cerf [2] shows that every pre-fibration over S^2 is equivalent to a smooth fibration, provided F is 2-connected and has dimension at least nine. This enables us to give a solution to our problem when $m = 2$ (Theorem 2).

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§1. PRE-FIBRATIONS

If V and W are topological spaces and $f: V \to W$ is a continuous map, define $X = \{(v, p) : v \in V, p : I \to W \text{ such that } p(0) = f(v)\}$, with the compact open topology. Let $i : V \to X$, $\phi : X \to W$ be defined by $i(v) = (v, \text{constant})$ and $\phi(v, p) = p(1)$. Then $f = \phi i$, i is a homotopy equivalence and ϕ is a Hurewicz fibre map; the fibre of ϕ will also be called the *fibre* of f.

Let V^{m+d} be a smooth manifold and let $f: V \to S^m$ be a map with simply connected fibre L. Suppose the base-point b of S^m is a regular value of f, $i|f^{-1}(b) \to L$ is a weak homotopy equivalence and $V - f^{-1}(b)$ is simply connected. Then (V, f) is a *pre-fibration* over S^m; the smooth manifold $f^{-1}(b)$ is called the *fibre* of (V, f).

Following Levine [5, §2], we define the group P_d to be $\mathbf{Z}$, 0, $\mathbf{Z}_2$ or 0 according as $d \equiv 0$, 1, 2, or 3 (mod 4).

THEOREM 1. *Let V^{m+d} be a closed smooth manifold, and let $f: V \to S^m$ be a map with simply connected fibre L. If $m \geqq 2$, $d \geqq 5$ and $H_n(L) = 0$ for all sufficiently large n, then there is a well-defined obstruction $\gamma(V, f) \in P_d$; f is homotopic to a pre-fibration if and only if $\gamma(V, f) = 0$.*

§2. HOMOTOPY THEORY

LEMMA 1. *Let X be a topological space and let $\phi : X \to S^m$ be a fibre map with fibre L. Suppose $m \geqq 2$ and X satisfies Poincaré duality in dimension $m + d$ (with integer coefficients). If $H_n(L) = 0$ for all sufficiently large n, then L satisfies Poincaré duality in dimension d.*

PROPOSITION. *Suppose $X = Y \cup Z$, where Y, Z are open sets, and let $x \in H_n(X)$ (integer coefficients). Let $p_* : H_n(X) \to H_n(Y, Y \cap Z)$, $q_* : H_n(X) \to H_n(Z, Y \cap Z)$ be the natural maps (via excisions). Then there is a skew-commutative exact diagram*

$$\begin{array}{ccccccc}
H^{n-r}(Y, Y\cap Z) & \xrightarrow{p^*} & H^{n-r}(X) & \longrightarrow & H^{n-r}(Z) & \xrightarrow{\delta} & H^{n-r+1}(Y, Y\cap Z) \\
\downarrow{\scriptstyle p_*x\cap} & & \downarrow{\scriptstyle x\cap} & & \downarrow{\scriptstyle q_*x\cap} & & \downarrow{\scriptstyle p_*x\cap} \\
H_r(Y) & \longrightarrow & H_r(X) & \xrightarrow{q_*} & H_r(Z, Y\cap Z) & \xrightarrow{\partial} & H_{r-1}(Y)
\end{array}$$

Proof. The first two squares commute by naturality of the cap product. The following squares also commute.

$$\begin{array}{ccc}
H^{n-r}(Z) & \longrightarrow & H^{n-r}(Y\cap Z) \\
\downarrow{\scriptstyle q_*x\cap} & & \downarrow{\scriptstyle \partial q_*x\cap} \\
H_r(Z, Y\cap Z) & \xrightarrow{\delta} & H_{r-1}(Y\cap Z)
\end{array}
\qquad
\begin{array}{ccc}
H^{n-r}(Y\cap Z) & \xrightarrow{\delta} & H^{n-r+1}(Y, Y\cap Z) \\
\downarrow{\scriptstyle \partial p_*x\cap} & & \downarrow{\scriptstyle p_*x\cap} \\
H_{r-1}(Y\cap Z) & \longrightarrow & H_{r-1}(Y)
\end{array}$$

Since $\partial p_* x + \partial q_* x = 0$, the result follows.

Proof of Lemma 1. Let a, b be distinct points of S^m, and let $Y = \phi^{-1}(S^m - a)$, $Z = \phi^{-1}(S^m - b)$. Let $r : S^m \to S^m$ be an orientation-preserving homeomorphism interchanging a and b; then r is covered by a fibre homotopy equivalence $\rho : X \to X$. Observe that ρ induces a homotopy equivalence of pairs $\rho : Y,\ Y \cap Z \to Z,\ Y \cap Z$ such that, if $x \in H_{m+d}(X)$, then $\rho_* p_* x = q_* x$. There is a commutative diagram

$$\begin{array}{ccc}
H^{m+d-r}(Y) & \xleftarrow{\rho^*} & H^{m+d-r}(Z) \\
\downarrow{\scriptstyle p_*x\cap} & & \downarrow{\scriptstyle q_*x\cap} \\
H_r(Y, Y\cap Z) & \xrightarrow{\rho_*} & H_r(Z, Y\cap Z)
\end{array} \tag{1}$$

in which ρ^* and ρ_* are isomorphisms.

Let $\mathbf{R}^m_* = \mathbf{R}^m - 0$, and let $S : \mathbf{R}^m \to S^m - a$ be a homeomorphism taking 0 to b. Then s is covered by a fibre homotopy equivalence $\sigma : L \times \mathbf{R}^m, L \times \mathbf{R}^m_* \to Y, Y \cap Z$. Let $\theta : H_r(L) \cong H_{r+m}(L \times \mathbf{R}^m, L \times \mathbf{R}^m_*)$ be the Thom isomorphism; then there is a unique element $\lambda \in H_d(L)$ such that $\sigma_* \theta(\lambda) = p_*(x)$. There are commutative diagrams

(2)
$$\begin{array}{ccccc} H^{d-r}(L) \cong & H^{m+d-r}(L \times \mathbf{R}^m, L \times \mathbf{R}^m_*) & \xleftarrow{\sigma^*} & H^{m+d-r}(Y, Y \cap Z) \\ \downarrow{\scriptstyle \lambda\cap} & \downarrow{\scriptstyle \theta\lambda\cap} & & \downarrow{\scriptstyle p_*\times\cap} \\ H_r(L) \cong & H_r(L \times \mathbf{R}^m) & \xrightarrow{\sigma_*} & H_r(Y) \end{array}$$

(3)
$$\begin{array}{ccccc} H^{m+d-r}(L) \cong & H^{m+d-r}(L \times \mathbf{R}^m) & \xrightarrow{\sigma^*} & H^{m+d-r}(Y) \\ \downarrow{\scriptstyle \lambda\cap} & \downarrow{\scriptstyle \theta\lambda\cap} & & \downarrow{\scriptstyle p_*\times\cap} \\ H_{r-m}(L) \cong & H_r(L \times \mathbf{R}^m, L \times \mathbf{R}^m_*) & \xleftarrow{\sigma_*} & H_r(Y, Y \cap Z) \end{array}$$

in which σ^* and σ_* are isomorphisms.

Now use diagrams (1), (2), (3) and the Proposition to obtain the following skew-commutative exact diagram

$$\begin{array}{ccccccc} H^{d-r}(L) & \longrightarrow & H^{m+d-r}(X) & \longrightarrow & H^{m+d-r}(L) & \longrightarrow & H^{d-r+1}(L) \\ \downarrow{\scriptstyle \lambda\cap} & & \downarrow{\scriptstyle \times\cap} & & \downarrow{\scriptstyle \lambda\cap} & & \downarrow{\scriptstyle \lambda\cap} \\ H_r(L) & \longrightarrow & H_r(X) & \longrightarrow & H_{r-m}(L) & \longrightarrow & H_{r-1}(L) \end{array}$$

Let x generate $H_{m+d}(X)$; since X satisfies Poincaré duality, $x \cap : H^{m+d-r}(X) \cong H_r(X)$. Let λ be defined as above, and suppose inductively that $\lambda \cap : H^{d-r}(L) \cong H_r(L)$ for all $r \leqq k$. This is true if k is sufficiently negative, since $H^n(L)$ is zero for all sufficiently large n. By the 5-Lemma, it follows that $\lambda \cap : H^{d-r}(L) \cong H_r(L)$ for $r \leqq k + m - 1$. Since $m \geqq 2$, this proves that $\lambda \cap : H^{d-r}(L) \cong H_r(L)$ for all r, as required.

LEMMA 2. *Let V^{m+d} be a closed smooth manifold and let $f: V \to S^m$ be a map with fibre L. If $m \geqq 2$, $H_n(L) = 0$ for all sufficiently large n and b is a regular value of f, then $i \mid f^{-1}(b) \to L$ has degree* 1.

Proof. Let B be a disc neighbourhood of b in S^m containing no critical values of f. Let $f = \phi i$ as in §1, let $T = f^{-1}(B)$, $K = \phi^{-1}(B)$, $\partial K = \phi^{-1}(\partial B)$, $F = f^{-1}(b)$ and note that $L = \phi^{-1}(b)$.

Now $\phi \mid K \to B$ is fibre homotopy equivalent to a trivial bundle over B. It follows that $\phi^* : H^m(B, \partial B) \cong H^m(K, \partial K)$, and similarly $f^* : H^m(B, \partial B) \cong H^m(T, \partial T)$. If κ generates $H^m(K, \partial K)$, then $i^*\kappa$ generates $H^m(T, \partial T)$, and there is a commutative diagram.

(4)
$$\begin{array}{ccccc} H_r(V, \overline{V - T}) & \cong H_r(T, \partial T) & \xrightarrow{\cap i^*\kappa} & H_{r-m}(T) \cong & H_{r-m}(F) \\ \downarrow{\scriptstyle i_*} & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & \downarrow{\scriptstyle i_*} \\ H_r(X, \overline{X - K}) & \cong H_r(K, \partial K) & \xrightarrow{\cap\kappa} & H_{r-m}(K) \cong & H_{r-m}(L) \end{array}$$

in which the horizontal maps are isomorphisms.

By Lemma 1, L satisfies Poincaré duality in dimension d. Since $\overline{X - K} \simeq L$, $H_{m+d-1}(\overline{X - K}) = 0$, so the inclusion induces an epimorphism $H_{m+d}(X) \to H_{m+d}(X, \overline{X - K})$.

Since $i_* : H_{m+d}(V) \cong H_{m+d}(X)$, it follows easily that $i_* : H_{m+d}(V, \overline{V-T}) \to H_{m+d}(X, \overline{X-K})$ is an epimorphism. Putting $r = m + d$ in diagram (4) shows that $i_* : H_d(F) \to H_d(L)$ is an epimorphism, so $i : F \to L$ has degree 1.

We refer the reader to [8] for some consequences of Lemma 2. In particular, there is a monomorphism $\iota : H_r(L) \to H_r(F)$ such that $i_* \iota = 1 : H_r(L) \to H_r(L)$, so

$$H_r(F) \cong \partial H_{r+1}(L, F) \oplus \iota H_r(L).$$

If $d = 2k$, then the intersection of any $\alpha \in \partial H_{k+1}(L, F)$ with any $\beta \in \iota H_k(L)$ vanishes.

§3. SURGERY

We shall attempt to simplify F by surgery in V; the following lemma describes the form of surgery required. We assume that V^{m+d} is a compact smooth manifold and $f : V \to S^m$ is a map with no critical values in a small disc $A \subset S^m$. Choose distinct points, $a, b \in \text{Int } A$ and let $F = f^{-1}(b)$, $G = f^{-1}(a)$. As usual, we assume that $m \geqq 2$.

LEMMA 3. *If $\alpha \in \pi_{k+1}(V - G, F)$ and $2k \leqq d - 1$, then there is a map $f' \simeq f : V \to S^m$ with no critical values in A and such that $F' = f'^{-1}(b)$ is obtained from F by surgery, killing $\partial\alpha$. If $2k \leqq d - 2$ and $G' = f'^{-1}(a)$, then there is a commutative exact diagram*

$$\begin{array}{ccccc} \pi_{k+1}(V - G) & \xrightarrow{j_*} & \pi_{k+1}(V - G, F) & \xrightarrow{\partial} & \pi_k(F) \\ \downarrow & & \downarrow \theta & & \downarrow \\ \pi_{k+1}(V - G') & \xrightarrow{j_*} & \pi_{k+1}(V - G', F') & \xrightarrow{\partial} & \pi_k(F') \end{array}$$

in which the vertical maps are epimorphisms, and $\theta(\alpha) = 0$. If $2k = d \geqq 6$ and $V - G$, F are both simply connected, then there is an obstruction $\xi \in \pi_k(V_{m+k-1, m-1})$ (where $V_{r,s}$ is the Stiefel manifold of s-frames in r-space). If $\xi = 0$, then f' can be found as above.

Proof. Let $\mathscr{E} = (\mathbf{e}_1, \ldots \mathbf{e}_m)$ be an orthonormal frame at b such that $\mathbf{e}_1$ points along the great circle $a\,b$ (away from a). Let $\mathscr{F} = (\mathbf{f}_1, \ldots \mathbf{f}_m)$ be the normal framing of F in V induced by f from $\mathscr{E}$. Let the great circle $a\,b$ (produced in the direction of $\mathbf{e}_1$) meet ∂A in c, and let $l : I \to [b, c]$ be the linear map sending 0 to b and 1 to c. Since f has no critical values in A, $f^{-1}(A)$ is a smooth fibration, equivalent to a trivial bundle with fibre F. Therefore there is an imbedding $h : F \times A \to V$ such that $fh = p_2 : F \times A \to A$.

If $2k \leqq d - 1$, then $\partial\alpha$ can be represented by an imbedding $u : S^k \to F$. The same is true if $2k = d \geqq 6$ and F is simply connected, by Lemma 6 of [6]. Let $v = h(u \times l) : S^k \times I \to V$; then v imbeds $S^k \times I$ in $h(F \times A)$ so that $v(S^k \times I) \cap F = u(S^k)$ and $\mathbf{f}_1 \mid u(S^k)$ points along generators of $v(S^k \times I)$. Extend $v \mid S^k \times 1 \to h(F \times \partial A)$ to a map $w : D^{k+1} \times 1 \to V - G$ such that $w \cup v : (D^{k+1} \times 1) \cup (S^k \times I), (S^k \times 0) \to V - G$, F represents α.

Since w avoids G, it avoids a neighbourhood N of G in $h(F \times A)$. Choose a homeomorphism $H : h(F \times A) \to h(F \times A)$ keeping $G \cup h(F \times \partial A)$ fixed and taking N onto a neighbourhood of F. We can extend H to a homeomorphism of V by defining $H \mid V - h(F \times A)$ to be the identity. Then $Hw : D^{k+1} \times 1 \to V - G$ is an extension of $v \mid S^k \times 1$, avoiding F and such that α is represented by $Hw \cup v : (D^{k+1} \times 1) \cup (S^k \times I), (S^k \times 0) \to V - G$, F.

If $2k \leqq d-1$ or $m \geqq 3$, then Hw can be approximated (rel $S^k \times 1$) by an imbedding $w' : D^{k+1} \times 1 \to V - F - G$ such that $w' \cup v$ is a smooth imbedding. If $2k = d \geqq 4$ and $m = 2$ then (by hypothesis) $V - G$ is simply connected. It follows by methods of Whitney that Hw is homotopic (rel $S^k \times 1$) to an imbedding w' as above. In either case α is represented by a smoothly imbedded disc $D \subset V - G$ with boundary $S = D \cap F$, and $\mathbf{f}_1 \mid S$ points radially into D.

There is an obstruction $\xi \in \pi_k(V_{m+d-k-1,\, m-1})$ to extending $(f_2, \ldots f_m) \mid S$ to a partial normal framing of D in V. If $2k \leqq d-1$, then $\pi_k(V_{m+d-k-1,\, m-1}) = 0$, so $\xi = 0$. Given such an extension, Haefliger [3, §3] shows how to perform a framed spherical modification on $(F, \mathscr{F})$ which kills $\partial\alpha$; let the resulting framed submanifold be $(F', \mathscr{F}')$. By the tubular neighbourhood theorem, $\mathscr{F}'$ defines an imbedding $h' : F' \times A \to V$, unique up to isotopy, with $h'(x, b) = x$ for $x \in F'$. To define $f' : V \to S^m$, put $f' \mid h'(F' \times A) = p_2 h'^{-1}$ and extend $f' \mid h'(F' \times \partial A) \to \overline{S^m - A}$ arbitrarily over $\overline{V - h'(F' \times A)}$. Then f' has no critical values in A, $F' = f'^{-1}(b)$ and f' induces $\mathscr{F}'$ from $\mathscr{E}$. Using the fact that $(F, \mathscr{F})$ and $(F', \mathscr{F}')$ are framed cobordant in $V \times I$, we see that $f' \cong f$.

To complete the proof of Lemma 3, it only remains to produce the commutative diagram when $2k \leqq d-2$. Observe that F', G' are obtained from F, G by surgeries (in V) of type $(k+1, d-k)$ [6, §1]. Since $k+1 < m-k$, there is an epimorphism $\pi_k(F) \to \pi_k(F')$.

There is an imbedding $\psi : D^{k+1} \times D^{d-k} \to V$ such that $\psi(D^{k+1} \times D^{d-k}) \cap G = \psi(S^k \times D^{d-k})$ and G' is topologically isotopic to $G'' = \overline{G - \psi(S^k \times D^{d-k})} \cup \psi(D^{k+1} \times S^{d-k-1})$. Write G_0 for $G \cup \psi(D^{k+1} \times D^{d-k}) = G'' \cup \psi(D^{k+1} \times D^{d-k})$. Since $k+2 < (m+d) - (k+1)$, the inclusion $V - G_0 \to V - G$ induces an isomorphism $\pi_{k+1}(V - G_0) \cong \pi_{k+1}(V - G)$. Since $k+1 < (m+d) - (d-k)$, the inclusion $V - G_0 \to V - G''$ induces an epimorphism $\pi_{k+1}(V - G_0) \to \pi_{k+1}(V - G'')$. Since $\pi_{k+1}(V - G'') \cong \pi_{k+1}(V - G')$, we have defined an epimorphism $\pi_{k+1}(V - G) \to \pi_{k+1}(V - G')$.

The epimorphism $\theta : \pi_{k+1}(V - G, F) \to \pi_{k+1}(V - G', F')$ is defined similarly; it is clear that $\theta(\alpha) = 0$ and the diagram commutes. This completes the proof of Lemma 3. We shall need to investigate the dependence of ξ on α; this will be postponed until Lemma 5.

LEMMA 4. *Let V^{m+d} be a compact smooth manifold and let $f : V \to S^m$ be a 2-connected map. If $m \geqq 2$ and $d \geqq 4$, then there is a map $g \simeq f$ with regular value $b \in S^m$ such that $g^{-1}(b)$ and $V - g^{-1}(b)$ are simply connected.*

Proof. We use the notation of Lemma 3, assuming (by a preliminary homotopy) that f has no critical values in the disc A. Since $\pi_0(V - G) = 0$, the boundary map $\partial : \pi_1(V - G, F) \to \pi_0(F)$ is an epimorphism. F is compact (since V is), so $\pi_0(F)$ is finite; therefore we can choose elements $\alpha_1, \ldots \alpha_n \in \pi_1(V - G, F)$ such that $\pi_0(F) = \{\partial\alpha_1, \ldots \partial\alpha_n\}$. By Lemma 3, we can do surgery to kill $\partial\alpha_1, \ldots \partial\alpha_n$, thus killing $\pi_0(F)$.

$V - G$ has the homotopy type of a finite CW complex, so $\pi_1(V - G)$ is generated by a finite set $\{\beta_1, \ldots \beta_n\}$. Let $\alpha_i = j_* \beta_i \in \pi_1(V - G, F)$, and use Lemma 3 to do surgery on $\alpha_1, \ldots \alpha_n$ in turn. If F_1, G_1 are the resulting submanifolds of V, then there is a commutative exact diagram

$$\begin{array}{ccccc} \pi_1(V-G) & \xrightarrow{j_*} & \pi_1(V-G,F) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \theta} & & \\ \pi_1(V-G_1) & \xrightarrow{j_{1*}} & \pi_1(V-G_1,F_1) & \longrightarrow & 0 \end{array}$$

in which η, θ are epimorphisms and $\theta(\alpha_i) = 0$ for $i = 1, \ldots n$. Therefore $j_{1*}\,\eta(\beta_i) = \theta j_*(\beta_i) = \theta(\alpha_i) = 0$. Since η is an epimorphism, $\pi_1(V - G_1)$ is generated by $\eta(\beta_1), \ldots \eta(\beta_n)$. It follows from the homotopy sequence of $(V - G_1, F_1)$ that $\ker j_{1*}$ is a subgroup of $\pi_1(V-G_1)$. Therefore $j_{1*} = 0$, so we have killed $\pi_1(V - G, F)$.

The following result is given in the Princeton thesis of L. C. Siebenmann.

PROPOSITION. *Let π, π' be finitely presented groups and let $\sigma : \pi \to \pi'$ be an epimorphism. Then* $\ker \sigma$ *is the normal closure of a finite subset of π.*

Proof. Let $\pi = \{x; r(x) = 1\}$, $\pi' = \{x'; r'(x') = 1\}$ be finite presentations of π, π'. There is a sequence of words s such that $\sigma x = s(x')$. Since σ is an epimorphism, there is a sequence of words t such that $x' = \sigma t(x) = t(\sigma x)$.

Observe that, in π', $r(s(x')) = 1$ and $x' = t(s(x'))$. Therefore

$$\begin{aligned} \pi' &\cong \{x'; r'(x') = 1, r(s(x')) = 1, x' = t(s(x'))\} \\ &\cong \{x', x''; r'(x') = 1, x'' = s(x'), r(x'') = 1, x' = t(x'')\} \\ &\cong \{x''; r(x'') = 1, x'' = s(t(x'')), r'(t(x'')) = 1\} \end{aligned}$$

Now $\sigma x = s(x') = x''$, so $\ker \sigma$ is the normal closure of the finitely many elements in $x^{-1}.\ s(t(x))$ and $r'(t(x))$, as required.

Since F and $V - G$ have the homotopy type of finite CW complexes, $\pi_1(F)$ and $\pi_1(V - G)$ are finitely presented. The inclusion now induces an epimorphism $\sigma : \pi_1(F) \to \pi_1(V - G)$, so $\ker \sigma$ is the normal closure of a finite set $\{\beta_1, \ldots \beta_n\}$. Choose $\alpha_i \in \pi_2(V - G, F)$ such that $\beta_i = \partial\alpha_i$; by Lemma 3 we can do surgery to kill $\partial\alpha_1, \ldots \partial\alpha_n$, thus killing $\ker \sigma$. Then the inclusion $F \to V - G$ induces an isomorphism of fundamental groups, and the same is true of the inclusion $G \to V - F$.

If $m \geqq 3$, $V - G$ is already simply connected, so the lemma is proved in this case (indeed, we could have used a much simpler argument). Let B be a disc neighbourhood of b in Int A with $a \in \partial B$. Let $T = f^{-1}(B)$, and suppose $m = 2$. Observe that there are natural inclusions $\pi_1(G) \subset \pi_1(\partial T)$ and $\pi_1(S^1) \subset \pi_1(\partial T)$. If z generates $\pi_1(S^1)$, then $\pi_1(G) \cup \{z\}$ generates $\pi_1(\partial T)$ and z is in the centre of $\pi_1(\partial T)$. Let $\lambda : \pi_1(\partial T) \to \pi_1(T)$ and $\mu : \pi_1(\partial T) \to \pi_1(\overline{V - T})$ be induced by inclusions.

By Van Kampen's theorem, $\pi_1(V)$ is obtained from $\pi_1(T) *_{\pi_1(\partial T)} \pi_1(\overline{V - T})$ by adding relations $\lambda(\alpha) = \mu(\alpha)$ (for all $\alpha \in \pi_1(G)$) and $\lambda(z) = \mu(z)$. Since $\lambda \mid \pi_1(G) \to \pi_1(T)$ and $\mu \mid \pi_1(G) \to \pi_1(\overline{V - T})$ are isomorphisms, and $\lambda(z) = 1$, $\pi_1(V)$ is obtained from $\pi_1(\overline{V - T})$ by adding the relation $\mu(z) = 1$. But $\pi_1(V) = 1$, so $\pi_1(\overline{V - T})$ is generated by conjugates of $\mu(z)$. Since μ is an epimorphism and z is the centre of $\pi_1(\partial T)$, $\mu(z)$ is in the centre

of $\pi_1(\overline{V-T})$. Therefore $\pi_1(F) \cong \pi_1(V-F)$ is cyclic, and both are isomorphic to $H_1(V-F) \cong H_1(\overline{V-T})$.

Since f is 2-connected, $f_*: H_2(V) \to H_2(S^2)$ is an epimorphism. Clearly $f_*: H_2(T, \partial T) \cong H_2(B, \partial B)$, so by excision $f_*: H_2(V, \overline{V-T}) \cong H_2(S^2, \overline{S^2-B})$. Therefore there is a commutative exact diagram

$$
\begin{array}{ccccccc}
H_2(V) & \xrightarrow{q_*} & H_2(V, \overline{V-T}) & \longrightarrow & H_1(\overline{V-T}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \cong\, f_*} & & & & \\
H_2(S^2) & \xrightarrow{\cong} & H_2(S^2, \overline{S^2-B}) & & & &
\end{array} \tag{5}
$$

It follows that q_* is an epimorphism, so $H_1(\overline{V-T}) = 0$. This proves Lemma 4.

Proof of Theorem 1. By Lemmas 1 and 2, L satisfies Poincaré duality in dimension d and $i: F \to L$ has degree 1. Since L is simply connected, f is 2-connected, so by Lemma 4 we may assume that F and $V-F$ are simply connected. It follows from the remarks after Lemma 2 that $i: F \to L$ is 2-connected. Suppose inductively that i has been made k-connected ($4 \leqq 2k \leqq d$) and $V-F$ is simply connected.

Diagram (4) shows that $i_*: H_r(V, \overline{V-T}) \cong H_r(X, \overline{X-K})$ for $r \leqq k+m-1$, and i_* is an epimorphism for $r = k+m$. Applying the 5-Lemma to the exact homology sequences of $(V, \overline{V-T})$ and $(X, \overline{X-K})$, we see that $(\overline{X-K}, \overline{V-T})$ is homologically $(k+m-1)$-connected. Since $\overline{V-T}$ and $\overline{X-K}$ are simply connected, $(\overline{X-K}, \overline{V-T})$ is $(k+m-1)$-connected. Write $M = \phi^{-1}(a)$; then $(X-M, V-G)$ is $(k+m-1)$-connected. By the 5-Lemma, $i_*: \pi_{k+1}(V-G, F) \to \pi_{k+1}(X-M, F)$ is an epimorphism.

Let $\beta \in H_{k+1}(L, F)$; then β is the Hurewicz image of some $\gamma \in \pi_{k+1}(L, F)$. Let η be the image of γ in $\pi_{k+1}(X-M, F)$. There is an element $\alpha \in \pi_{k+1}(V-G, F)$ such that $\eta = i_*\alpha$, since i_* is an epimorphism. By Lemma 3, we can do surgery to kill $\partial\alpha$ if $2k \leqq d-1$. The Hurewicz image of $\partial\alpha$ is $\partial\beta$; since $\partial: H_{k+1}(L, F) \to H_k(F)$ is a monomorphism, surgery on α kills β. Note that $V-F$ is still simply connected.

If $2k \leqq d-2$, we can use this construction to kill $H_{k+1}(L, F)$ (which is finitely generated since it is isomorphic to a subgroup of $H_k(F)$). Thus we have completed our inductive step; it remains to consider the cases $2k = d-1$ and $2k = d$.

If $2k = d-1$ and k is even, we can use the methods of Kervaire and Milnor [4, §5] to kill $H_{k+1}(L, F)$. If k is odd, the argument of [4, §6] applies, provided we show that the framing used to construct a spherical modification can be altered by any element in the kernel of the map $\pi_k(SO_{k+1}) \to \pi_k(SO)$. But this is proved in [8, §4], so we can kill $H_{k+1}(L, F)$ whenever $2k = d-1 \geqq 4$.

If $2k = d$ and $\beta, \beta' \in H_{k+1}(L, F)$, we write $\beta.\beta'$ for the intersection number $\partial\beta.\partial\beta'$. This defines a non-singular bilinear form on $H_{k+1}(L, F)$ (which is a free Abelian group). Given $\beta \in H_{k+1}(L, F)$, we can construct γ, η, α as above. When we try to apply Lemma 3 to

kill $\partial\alpha$, we meet an obstruction $\xi \in \pi_k(V_{m+k-1,\,m-1})$. If k is even, $\pi_k(V_{m+k-1,\,m-1}) \cong \mathbf{Z}$. If k is odd, then $\pi_k(V_{m+k-1,\,m-1}) \cong \mathbf{Z}_2$ unless $m = 2$, when $\pi_k(V_{k+1,\,1}) \cong \mathbf{Z}$. We write ξ_2 for the element of $\mathbf{Z}_2$ which corresponds to the reduction of ξ mod 2.

LEMMA 5. *If k is even, then ξ is zero when $\beta.\beta$ vanishes. If k is odd, then ξ_2 depends only on β, and the map $\beta \to \xi_2$ defines a quadratic function on $H_{k+1}(L, F)$ with values in $\mathbf{Z}_2$. A primitive element β can be killed if $\xi_2 = 0$.*

Proof. The result for k even is given in [5, 4.10]. Suppose k is odd, and represent α by an imbedded disc D as in the proof of Lemma 3. Then $\tau(V) \mid S$ has a canonical framing $\mathscr{G}$ which extends over $\tau(V) \mid D$. Let $(\mathbf{g}_1, \ldots \mathbf{g}_{k+1})$ be the canonical framing of $\tau(D) \mid S$, so $\mathscr{G}_0 = (\mathbf{f}_2, \ldots \mathbf{f}_m, \mathbf{g}_1, \ldots \mathbf{g}_{k+1})$ is a partial framing of $\tau(V) \mid S$. Then $\mathscr{G}, \mathscr{G}_0$ define an element ζ of $\pi_k(V_{m+2k,\,m+k}) \cong \mathbf{Z}_2$. It is easily checked that ζ is the image of ξ under the epimorphism $\pi_k(V_{m+k-1,\,m-1}) \to \pi_k(V_{m+2k,\,m+k})$, so ζ can be identified with ξ_2.

Clearly ζ depends only on the regular homotopy class of S in F and the homotopy class of (D, S) in (V, F). But two immersed k-spheres in F which are homotopic are necessarily regularly homotopic, so ζ (and hence ξ_2) is determined by the homotopy class of (D, S) in (V, F).

Now γ, η are uniquely determined by β; let α, $\alpha' \in \pi_{k+1}(V - G, F)$ be such that $i_*\alpha = i_*\alpha' = \eta$. Let δ be the image of $\alpha - \alpha'$ in $\pi_{k+1}(V, F)$. Then $i_*\delta$ is the zero element of $\pi_{k+1}(X, F)$; but i_*, $\pi_{k+1}(V, F) \to \pi_{k+1}(X, F)$ is an isomorphism, so $\delta = 0$. This proves that ξ_2 depends only on β. It follows by the argument of [5, 4.2] that, for β, $\beta' \in H_{k+1}(L, F)$, $\xi_2(\beta + \beta') = \xi_2(\beta) + \xi_2(\beta') + (\beta.\beta')_2$.

Finally, suppose β is a primitive element of $H_{k+1}(L, F)$ and $\xi_2(\beta) = 0$. If $m \geqq 3$, it follows that $\xi = 0$, so β can be killed using Lemma 3. Now suppose $m = 2$, and let $\xi(D)$ be $2a$ times a generator of $\pi_k(V_{k+1,1})$ (where a is an integer). Following Levine [5, 4.11], we construct a new disc D_2 with the same boundary S as D but with $\xi(D_2) = 0$.

Let S_0 be a $(k+1)$-sphere immersed in a small $(m+d)$-disc in $V - (F \cup D)$ with self-intersection number $-a$. Join D and S_0 by a tube to obtain an immersion of $D_0 = D \# S_0$ with self-intersection number $-a$. Then the given framing $(\mathbf{f}_2, \ldots \mathbf{f}_m)$ on $S = \partial D_0$ can be extended over D_0 (see [5, 4.11]).

Since β is a primitive element, there is an element $\theta \in H_{k+1}(L, F)$ such that $\theta.\beta = a$. Diagram (4) defines an isomorphism $H_k(F) \cong H_{k+2}(V, \overline{V - T})$; let $\psi \in H_{k+2}(V, \overline{V - T})$ be the image of $\partial\theta$. Since $i_*(\partial\theta) = 0$, it follows from diagram (4) that $i_*(\partial\psi) = 0$. The pair $(\overline{X - K}, \overline{V - T})$ is $(k+1)$-connected, so $\partial\psi$ is spherical in $\overline{V - T}$. By Lemma 6 of [6], $\partial\psi$ can be represented by an imbedded sphere S_1 in $\overline{V - T}$, meeting D_0 transversely. It can be checked that the intersection number D_0 and S_1 is a. Join D_0 and S_1 by a tube to obtain an immersion $D_1 = D_0 \# S_1$ with zero self-intersection number.

Since $\partial\psi$ is a boundary in V, the intersection number $\partial\psi.\partial\psi$ is zero. It follows from Lemma 7 of [6] that S_1 has trivial normal bundle in V. Therefore the given framing on $S = \partial D_1$ can be extended over D_1. Since $V - F$ is simply connected, D_1 is regularly homotopic to an imbedded disc D_2 such that $\partial D_2 = S = D_2 \cap F$. Now we can do surgery on D_2, killing β as required, and Lemma 5 is established.

If $2k = d \geqq 6$ and k is even, we have shown that any element $\beta \in H_{k+1}(L, F)$ with $\beta.\beta = 0$ can be killed. It follows from arguments of Milnor [6, §6] that $H_{k+1}(L, F)$ can be killed provided it has zero signature. It also follows that the intersection pairing on $H_{k+1}(L, F)$ has determinant ± 1 and is even, so its signature σ is divisible by 8. Define $\gamma(V, f) = \sigma/8 = (1/8)$ (signature of F − signature of L).

By the Hirzebruch signature theorem, the signature is equal to $\langle L_j(V) \cup f^*[S^m], [V]\rangle$, where $k = 2j$, $L_j(V)$ is the Hirzebruch class in $H^d(V; \mathbf{Q})$ and $[S^m]$, $[V]$ are the fundamental classes in $H^m(S^m)$, $H_{m+d}(V)$. Therefore $\gamma(V, f)$ is equal to $(1/8)(\langle L_j(V) \cup f^*[S^m], [V]\rangle -$ signature of L), which depends only on the homotopy class of f and the manifold V.

Now suppose $2k = d$ and k is odd. Let $\gamma(V, f)$ be the Arf invariant of the quadratic function ξ_2 on $H_{k+1}(L, F)$. The methods of [6, §8] show that $H_{k+1}(L, F)$ can be killed provided $\gamma(V, f) = 0$.

In all cases, therefore, $H_{k+1}(L, F)$ can be killed if $\gamma(V, f) = 0$; note that $V - F$ remains simply connected. It follows by Poincaré duality that f is homotopic to a pre-fibration provided $\gamma(V, f) = 0$. To complete the proof of Theorem 1, we have to show that $\gamma(V, f)$ is invariant under a homotopy of f when $d = 2k$ and k is odd. It is enough to show that, if f is homotopic to a pre-fibration, then $\gamma(V, f) = 0$. But this follows from an argument similar to that of [5, 4.4.].

§4. CLASSIFICATION OF PRE-FIBRATIONS

Two pre-fibrations (V, f) and (V', f') over S^m are *equivalent* if there is a diffeomorphism $e : V \to V'$ such that $f = f'e$ on a neighbourhood of $f^{-1}(b)$. It follows that $f \simeq f'e : V \to S^m$.

We shall take the north pole of S^n as base-point, and shall identify D^n with the northern hemisphere. Let $D'_n(F)$ be the group of diffeomorphisms of $F \times S^n$ keeping $F \times D^n$ fixed. Let $D''_n(F)$ be the subset of $D'_n(F)$ consisting of diffeomorphisms which are concordant to the identity keeping $F \times D^n$ fixed. Then $D''_n(F)$ is a normal subgroup of $D'_n(F)$; define $D_n(F) = D'_n(F)/D''_n(F)$.

LEMMA 6. *Let F^d be a closed simply connected smooth manifold. If $m \geqq 2$, $m + d \geqq 6$, then equivalence classes of pre-fibrations over S^m with fibre F are in* (1–1) *correspondence with elements of $D_{m-1}(F)$.*

Proof. Let $P_m(F)$ be the set of equivalence classes of pre-fibrations over S^m with fibre F. If $h : F \times S^{m-1} \to F \times S^{m-1}$ represents an element of $D_{m-1}(F)$, define $V = (F \times D^m) \cup_h (F \times D^m)$. Let $f_2 : F \times D^m \to D^m$ be a continuous extension of $p_2 h^{-1}$, and define $f = p_2 \cup f_2 : V \to D^m \cup_1 D^m = S^m$. Let $j : F \times D^m \to V$ be the inclusion of the first summand; we shall identify F with $j(F \times 0)$.

By Van Kampen's theorem, V is simply connected. It follows from diagram (5) that f is 2-connected. Suppose inductively that $i : F \to L$ is k-connected; it follows from diagram (4) that $i : V - F \to X - L$ is $(k + m - 1)$-connected. Since h leaves $F \times D^{m-1}$ fixed, we deduce that $i : F \to L$ is $(k + m - 1)$-connected, completing the induction. Therefore (V, f) is a pre-fibration over S^m with fibre F. It is easy to see that the equivalence class of (V, f) depends only on the concordance class of h, so we have defined a map $\theta : D_{m-1}(F) \to P_m(F)$.

Suppose h_1, h_2 represent elements of $D_{m-1}(F)$ such that $\theta(h_1) = \theta(h_2)$. Define V_r, f_r, j_r from h_r as above ($r = 1, 2$); there is an equivalence $e : V_1 \to V_2$. By the tubular neighbourhood theorem, we may choose e so that $ej_1 = j_2$. Thus $h_2 h_1^{-1} : F \times S^{m-1} \to F \times S^{m-1}$ can be extended to a diffeomorphism $F \times D^m \to F \times D^m$. Since $h_2 h_1^{-1}$ keeps $F \times D^{m-1}$ fixed, it follows easily that it represents the identity element of $D_{m-1}(F)$. Therefore θ is a monomorphism.

We now show that any element $\alpha \in P_m(F)$ is in the image of θ. Recall that D^m is the northern hemisphere of S^m; we shall identify $2D^m$ with the region to the north of the Antarctic circle. Now α is represented by a pre-fibration (V, f) such that f has no critical values in $2D^m$. Choose an imbedding $j : F \times 2D^m \to V$ such that $fj = p_2 : F \times 2D^m \to 2D^m$. We shall show that the closure of the complement of $j(F \times D^m)$ is diffeomorphic to $F \times D^m$, and is attached to $j(F \times D^m)$ by an element of $D_{m-1}(F)$.

Let B be an m-disc in $2D^m - D^m$. Let $b : S^{m-1} \to \partial B$ be an orientation-reversing diffeomorphism which extends to a diffeomorphism $D^m \to B$. Let $a : D^{m-1} \times I \to \overline{2D^m - D^m - B}$ be an imbedding such that $a_1 : D^{m-1} \to S^{m-1}$ is the inclusion and $a_0 = ba_1$. Let $A = D^m \cup B \cup a(D^{m-1} \times I)$ (so A is topologically an m-disc), $\partial_0 A = \partial B - a_0(D^{m-1})$, $\partial_1 A = \overline{S^{m-1} - D^{m-1}}$ and $\partial_c A = a(S^{m-2} \times I)$. Define $W = \overline{V - j(F \times A)}$, $\partial_0 W = j(F \times \partial_0 A)$ $\partial_1 W = j(F \times \partial_1 A)$ and $\partial_c W = j(F \times \partial_c A)$. Then W is a smooth manifold with corners, and $\partial W = \partial_0 W \cup \partial_1 W \cup \partial_c W$.

Recall that $W \simeq V - F$ is simply connected, by definition of pre-fibration. Clearly $\partial_0 W$ and $\partial_1 W$ are simply connected. It follows from diagram (4) that the inclusion $j(F \times B) \subset \overline{V - j(F \times D^m)}$ is a homotopy equivalence, so $\partial_0 W \subset W$ and $\partial_1 W \subset W$ are homotopy equivalences. Thus W is a (relative) h-cobordism between $\partial_0 W$ and $\partial_1 W$. By Smale's theorem [7], we may extend $j(1 \times a) : F \times S^{m-2} \times I \to \partial_c W$ to a diffeomorphism $l : F \times D^{m-1} \times I \to W$ with $l_0 = j(1 \times b) \mid F \times D^{m-1} \to \partial_0 W$. It follows that there is a diffeomorphism $k : F \times S^{m-1} \times I \to \overline{V - j(F \times D^m) - j(F \times B)}$ such that $j(1 \times a) = k \mid F \times D^{m-1} \times I$ and $k_0 = j(1 \times b)$. Thus $h = k_1^{-1} j \mid F \times S^{m-1} \to F \times S^{m-1}$ represents an element of $D_{m-1}(F)$. It is easy to see that (V, f) represents $\theta(h)$, so we have proved that θ is an epimorphism, and therefore an isomorphism.

If $m = 2$, $d \geqq 9$ and F is 2-connected, we can strengthen our result by using the following theorem of Cerf [2].

THEOREM. *Let F^d be a 2-connected closed smooth manifold, and let $h : F \times I \to F \times I$ be a diffeomorphism with $h \mid F \times 0 = 1$. If $d \geqq 9$, then there is an isotopy of $F \times I$ keeping $F \times 0$ fixed and taking h to the identity.*

This implies the following result about $D_1(F)$.

COROLLARY. *If F^d is a 2-connected closed smooth manifold and $d \geqq 9$, then every element of $D_1(F)$ can be represented by a diffeomorphism $h : F \times S^1 \to F \times S^1$ with $p_2 h = p_2 : F \times S^1 \to S^1$.*

Suppose h satisfies the conditions of this corollary. Define V as in the proof of Lemma 6. In the definition of $f : V \to S^2$, $f_2 : F \times D^2 \to D^2$ is any continuous extension of

$p_2 h^{-1} = p_2$. If we take $f_2 = p_2$, then $f = p_2 \cup p_2 : V \to S^2$ is a fibration. The differential structure on V is defined by means of collar neighbourhoods of $F \times S^1$ in $F \times D^2$. If these are chosen suitably, then f is a smooth fibration representing $\theta(h)$. Therefore, under the conditions of the above corollary, every pre-fibration over S^2 with fibre F is equivalent to a smooth fibration. Combining this with Theorem 1, we obtain the following result.

THEOREM 2. *Let V^{d+2} be a closed smooth manifold and let $f: V \to S^2$ be a map with 2-connected fibre L. If $d \geqq 9$ and $H_n(L) = 0$ for all sufficiently large n, then there is a well-defined obstruction $\gamma(V, f) \in P_d$; f is homotopic to a smooth fibration if and only if $\gamma(V, f) = 0$.*

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