SIMPLE HOMOTOPY THEORY
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1. Projective modulus and automorphisms

We construct $\mathbb{Z}[\pi]$ the integral group ring of the group π. This consists of formal linear combinations $\sum_{g\in\pi} n_g g$ ($n_g \in \mathbb{Z}$, $n_g = 0$ for all but finitely many g).

$$\sum n_g g + \sum m_g g = \sum (n_g + m_g) g$$
$$\left(\sum m_g g\right)\left(\sum n_g g\right) = \sum_g \left(\sum_{hk=g} m_h n_k\right) g$$

Ring R (associative, has 1, not nec. comm.)

(Left) R-module is Abelian group A with ra defined in A for all $r \in R$, $a \in A$ such that

$$r(a+b) = ra + rb$$
$$(r+s)a = ra + sa$$
$$(rs)a = r(sa)$$
$$1a = a$$

R-homomorphism $f : A \to B$ is group homotopic with $f(ra) = rf(a)$ for all $r \in R$, $a \in A$.

a_i R-module ($i \in I$) $\oplus_{i\in I} A_i$ consists of formal sums $\sum_{i\in I} a_i$ with $a_i \in A_i$ and $a_i = 0$ for almost all i.

A is *free* if it is isomorphic to a direct sum of copies of R; equivalently, A has a *basis* $\{a_i\}_{i\in I}$ such that for all $a \in A$ there exists unique $r_i \in R$ such that $a = \sum r_i a_i$ ($r_i = 0$ for almost all i).

A is *projective* if, given R-modules B, C and R-homotopic $\phi : B \to C$, $f : A \to C$ with ϕ onto, there exists $g : A \to B$ such that $\phi g = f$.

Lemma 1.1. *A is projective iff it is a direct summand of a free module.*

A is finitely generated if there exists finite subset $\{a_1, \dots, a_n\}$ of A which spans A.

Corollary 1.2. *A f.g. projective module is a direct summand of a f.g. free module.*

R any ring. Define $K_0(R)$ to be Abelian group with one generator $[A]$ for each isomorphism class of f.g. projective R-modules, subject to relations

$$[A] + [B] = [A \oplus B] \ .$$

Define $\tilde{K}_0(R) = K_0(R)/\text{subgroup generated by } [R])$ projective class group of R.

Examples. 1) $R = \mathbb{Z}$ f.g. proj. $\mathbb{Z}$-modules all free

$$\tilde{K}_0(\mathbb{Z}) = 0 \ , \quad K_0(\mathbb{Z}) \cong \mathbb{Z}$$

2) $R =$ field f.g. proj. R-modules are f.d. vector spaces.

$$\tilde{K}_0(R) = 0 \ , \quad K_0(R) \cong \mathbb{Z} \ .$$

3) p, q distinct primes, $K_0(\mathbb{Z}_{pq}) \cong \mathbb{Z} \oplus \mathbb{Z}$, $\tilde{K}_0(\mathbb{Z}_{pq}) \cong \mathbb{Z}$.

4) $R =$ ring of algebraic integers in some algebraic number field,
$K_0(R) = \mathbb{Z} \oplus$ (ideal class group of R), $\tilde{K}_0(R) \cong$ ideal class group.

Lemma 1.3. *Any element of $K_0(R)$ can be expressed as $[A] - [B]$, where A, B are f.g. proj. modules; $[A] - [B] = [C] - [D]$ iff $\exists$ f.g. proj. X such that $A \oplus D \oplus X \cong B \oplus C \oplus X$.*

Proof. Consider ordered pairs of f.g. proj. modules (A, B) . Define $(A, B) \sim (C, D)$ if $A \oplus D \oplus X \cong B \oplus C \oplus X$ for some X, let G be set of equivalence classes.

Addition in G: $(A, B) + (C, D)$ represented by $(A \oplus C, B \oplus D)$. G is a group.

Define $\phi : K_0(R) \to G$, $\psi : G \to K_0(R)$ by $\phi[A] = (A, 0)$, $\psi(A, B) = [A] - [B]$. □

Corollary 1.4. *Any element of $\tilde{K}_0(R)$ can be expressed as $[A]$; $[A] = [B]$ iff $A \oplus F \cong B \oplus G$ for some f.g. free $F.G$.*

Proof. Any element of $K_0(R)$ can be expressed as $[A] - [B]$. Any B f.g. proj $\Rightarrow \exists X$ such that $B \oplus X$ is f.g. free.

$\therefore$ Any element of $K_0(R)$ is of the form $[A \oplus X] - [B \oplus X]$.

$\therefore$ Any element of $\tilde{K}_0(R)$ is of the form $[A \oplus X]$.

Suppose $[A] = [B]$ in $\tilde{K}_0(R)$. So $[A] - [B]$ in $K_0(R) \in$ subgroup generated by $[R]$.

$\therefore$ $[A] - [B] = [F] - [G]$; F, G f.g. free, so $A \oplus G \oplus X \cong B \oplus F \oplus X$ some f.g. proj. X.

$X \oplus Y$ is f.g. free some Y

$\therefore$ $A \oplus (G \oplus X \oplus Y) \cong B \oplus (F \oplus X \oplus Y)$

$$\begin{aligned} A \oplus F \cong B \oplus G &\Longrightarrow [A] - [B] = [G] - [F] \text{ in } K_0 \\ &\Longrightarrow [A] = [B] \text{ in } \tilde{K}_0 \ . \end{aligned}$$

□

Tensor products.

Let A be a right R-module, B a left R-module. $A \otimes_R B$ is the universal Abelian group of bilinear maps $\phi : A \times B \to G$ such that $\phi(ar, b) = \phi(a, rb)$.

If A is an (S, R)-bimodule [i.e., left S-module, right R-module such that $(sa)r = s(ar)$], then $A \otimes_R B$ inherits structure of left S-module

$$\left[\begin{array}{l} s \in S \text{ induced by } A \times B \longrightarrow A \otimes_R B \\ \qquad\qquad\qquad\qquad (a, b) \longmapsto Sa \otimes b \end{array}\right]$$

If A is an (S, R)-bimodule and B is an (R, T)-bimodule, then $A \otimes_R B$ is an (S, T)-bimodule.

$R \xrightarrow{f} S$ ring homomorphism preserving 1.

Construct $f_* : K_0(R) \to K_0(S)$.

Regard S as (S, R)-bimodule; S acts on S by left multiplication, R acts on S on right by $s.r = sf(r)$.

A is left R-module $\Longrightarrow S \otimes_R A$ is a left S-module.

Lemma 1.5. *$S \otimes_R (A \oplus B) \cong (S \otimes_R A) \oplus (S \otimes_R B)$ and if A is f.g. projective R-module then $S \otimes_R A$ is f.g. projective S-module.*

Proof. The first part is obvious. Note that $S \otimes_R R \cong S$. Therefore $S \otimes_R$ (f.g. free module) is f.g. free.

If A is f.g. projective R-module, then $A \oplus X$ is f.g. free for some X. Therefore $(S \otimes_R A) \oplus (S \otimes_R X)$ is f.g. free.

Therefore $S \otimes_R A$ is f.g. projective S-module. $\square$

Theorem 1.6. *K_0 and $\tilde{K}_0$ are covariant functors from the category of rings and ring homomorphisms (preserving 1) to the category of Abelian groups and homomorphisms.*

Examples. 1) Suppose there exists homomorphism $R \to K$, K a field. Then $\mathbb{Z} \to R \to K$ induce homomorphisms

$$\begin{array}{ccc} K_0(\mathbb{Z}) & \to K_0(R) \to & K_0(K) \\ \| & & \| \\ \mathbb{Z} & \xrightarrow{\quad\cong\quad} & \mathbb{Z} \end{array}$$

Therefore $K_0(R) \cong \mathbb{Z} \oplus \tilde{K}_0(R)$.

In particular, this holds for commutative rings, and integral group rings $(\sum n_g g \to \sum n_g)$.

2) $K_0(M_n(R)) \cong K_0(R)$

R^n can be regarded as an $(R, M_n(R))$-bimodule

$$r(x_1, \dots, x_n) = (rx_1, \dots rx_n)$$
$$(x_1, \dots, x_n)a_{ij} = (\sum x_i a_{i1}, \dots, \sum x_i a_{in})$$

or an $(M_n(R), R)$-bimodule.

$$R^n \otimes_R R^n \cong M_n(R0 \text{ as an } M_n(R)\text{-bimomodule}$$
$$R^n \otimes_{M_n(R)} R^n \cong R \text{ as an } R\text{-bimodule.}$$

If A is left $M_n(R)$-module, $A^* = R^n \otimes_{M_n(R)} A$ left R-module; B.... R-module, $B_* = R^n \otimes_R B$ left $M_n(R)$-module.

${}^*, {}_*$ preserve $\oplus$ to f.g. projectives; $(A^*)_* \cong A$ and $B_*)^* \cong B$.

$\therefore$ defines inverse isomorphisms $K_0(M_n(R)) \cong K_0(R)$.

In general $\tilde{K}_0(M_n(R)) \ncong \tilde{K}_0(R)$, e.g., $\tilde{K}_0(M_n(\mathbb{Z})) \cong \mathbb{Z}_n$.

Any ring R; $GL(n, R)$ = group of invertible $n \times n$ matrices $/R$.

Regard $GL(n, R)$ as a subgroup of $GL(n+1, R)$.

$M \in GL(n, R)$ identified with $\left(\begin{smallmatrix} M & 0 \\ 0 & 1 \end{smallmatrix}\right) \in GL(n+1, R)$

$$GL(1, R) \subset GL(2, R) \subset \cdots \subset GL(n, R) \subset GL(n+1, R) \subset \cdots$$

Define $GL(R) = \bigcup_{n=1}^{\infty} GL(n, R)$.

A liter [??] as $\infty \times \infty$ matrices,, $a_{ij} = \delta_{ij}$ for all but finitely many i, j.

Let e_{ij} be the matrix with 1 in (i, j)th place, zero elsewhere.

If $i \neq j$ and $r \in R$, then $1 + re_{ij} \in GL(R)$, inverse $1 - re_{ij}$.

Let $E(R)$ be the group generated by these elementary matrices.

Lemma 1.7 (J.H.C. Whitehead)**.** *$E(R)$ is the commutator subgroup of $GL(R)$.*

Proof. Suppose i, j, k distinct. Then

$$\begin{aligned}&(1 + re_{ij})(1 + se_{jk})(1 - re_{ij})(1 - se_{jk}) \\ &\quad = (1 + re_{ij} + se_{jk} + rse_{ik})(1 - re_{ij} - se_{jk} + rse_{ik}) = 1 + rse_{ik}\end{aligned}$$

Therefore all elementary matrices are commutators.

Let $X, Y \in GL(n, R)$; then in $GL(R)$ we have

$$XYX^{-1}Y^{-1} = \begin{pmatrix} XYX^{-1}Y^{-1} & 0 \\ 0 & 1_n \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & X^{-1} \end{pmatrix} \begin{pmatrix} Y & 0 \\ 0 & Y^{-1} \end{pmatrix} \begin{pmatrix} (YX)^{-1} & 0 \\ 0 & YX \end{pmatrix}$$
$$\begin{pmatrix} Z & 0 \\ 0 & Z^{-1} \end{pmatrix} \begin{pmatrix} 1 & Z^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -Z & 1 \end{pmatrix} \begin{pmatrix} 1 & Z^{-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\therefore \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$\begin{pmatrix} 1 & Z^{-1} \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ -Z & 1 \end{pmatrix}$, are products of elementary matrices

$$\begin{pmatrix} 1 & 1 \\ -Z & 1 \end{pmatrix} = \prod_{\substack{n+1\leq i\leq 2n \\ 1\leq j\leq n}} (1 + z_{ij}e_{ij})$$

Therefore $E(R) \cong GL(R)'$ $\square$

Define $K_1(R) = GL(R)/E(R)$; this is Abelian, usually written additively.

Let A be f.g. projective, and let $\alpha : A \to A$ be an automorphism of A. Define $\tau(\alpha) \in K_1(R)$ (the Whitehead determinant of α) as follows.

If A is free, pick basis and represent α by invertible matrix M.

Then $\tau(\alpha) =$ image of M in $K_1(R)$; independent of basis as in $\operatorname{im} M = \operatorname{im} S^{-1}MS$.

If A is f.g. projective, pick X such that $A \oplus X$ is f.g. free. Define $\tau(\alpha) = \tau(\alpha \oplus 1_X)$ (already defined).

Examples. Independent of X.

1) $\tau(\alpha\beta) = \tau(\alpha) + \tau(\beta)$ if α, β onto of A
2) $\tau(\alpha \oplus \beta) = \tau(\alpha) + \tau(\beta)$ if α onto of A, β onto of B.

In fact, τ is universal with respect to 1) and 2).

Let π be any group $g \in \pi \Rightarrow \underset{|X| \text{ matrices}}{[\pm g]} \in GL(1, \mathbb{Z}[\pi]) \subset GL(\mathbb{Z}[\pi])$.

Definition. $\mathrm{Wh}[\pi] = K_1(\mathbb{Z}[\pi])/\{\tau(\pm g) : g \in \pi\}$ the *Whitehead group* of π.

$f : R \to S$ induces homomorphism $f_* : GL(R) \to GL(S)$.

By Abelianism, get $f_* : K_1(R) \to K_1(S)$.

Theorem 1.8. *K_1 is a covariant functor from the category of rings and ring homomorphisms to the category of Abelian groups and homomorphisms. Analogous result for* Wh.

Examples. 1) If R is commutative, $\det : GL(R) \to U(R) =$ group of units of R.

$$U(R) \underrightarrow{= GL(1,R) \subset GL(R) \to} K_1(R) \xrightarrow{\det} U(R)$$

$$u \longmapsto \begin{pmatrix} u & & & & \\ & 1 & & & 0 \\ & & 1 & & \\ 0 & & & 1 & \\ & & & & \ddots \end{pmatrix} \mapsto u$$

Therefore $K_1(R) \cong U(R) \oplus SK_1(R)$ for commutative R.

2) $\mathrm{Wh}(C_S) \neq 0$. Enough to find a unit in $\mathbb{Z}[C_S]$ not of form $\pm g$ $(g \in C_S)$ $\mathrm{Wh}(\pi) \cong \frac{U(\mathbb{Z}[\pi])}{\pm\pi} \oplus SK_1$

   t generates C_S.

   $1 - t - t^4$ is a unit in $\mathbb{Z}[C_S]$ inverse $1 - t^2 - t^3$

   In fact $\mathrm{Wh}(C_S) \cong \mathbb{Z}$ generated by $1 - t - t^4$ (hard to prove).

3) $K_1(\mathbb{Z}) \cong \mathbb{Z})2 \cong U(\mathbb{Z})$, $SK_1(\mathbb{Z}) = 0$

   Implies that Wh (trivial group) $= 0$

   $A \in GL(n, \mathbb{Z})$ with $\det A = 1$

   RTP that A is a product of elementary matrices.

$$\begin{bmatrix} a_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

   Simplify $(a_{11} \cdots a_{1n})$ by Euclidean algorithm. Suppose a_{1r} has maximal modulus in top row. Suppose $a_{1s} \neq 0$ for some $s \neq r$. Pick $\lambda \in \mathbb{Z}$ such that $|a_{1r} - \lambda a_{1s}| < |a_{1s}|$. $A(1 - \lambda e_{sr})$ has same top row as A except that a_{1r} is replaced by $a_{1r} - \lambda a_{1s}$. Repeat until the top row has only one non-zero element – must be ± 1. If $n \geq 2$, can make top row $(1, 0, \ldots, 0)$.

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ \Big| & & A' & \\ & & & \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{bmatrix}$$

   Premultiply by elementary matrices to kill the first column. Therefore $A \equiv$ some element of $GL(n01, \mathbb{Z})$ (mod $E(\mathbb{Z})$). Continue until

$$A \equiv \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \pm 1 \end{pmatrix}$$

   But $\det A = 1$, so $A \equiv I \bmod E(\mathbb{Z})$

4) If R is a field then $K_1(R) \cong R^* = U(R)$. Similar to above, but easier.

5) $K_1(M_n(R)) \cong K_1(R)$

   $GL(k, M_n(R)) \cong GL(nk, R)$ (portioned matrices)

   $\therefore GL(M_n(R)) \cong GL(R)$

   Abelianize $\Rightarrow K_1(M_n(R)) \cong K_1(R)$

Lemma 1.9 (π group). *If $\gamma : \pi \to \pi$ is conjugation by some $g \in \pi$, then $\gamma_* : K_i(\mathbb{Z}[\pi]) \to K_i(\mathbb{Z}[\pi])$ is the identity* ($i = 0, 1$).

Proof. If A is f.g. projective over $\mathbb{Z}[\pi]$, then $\gamma_*[A]$ represented by $C \otimes_{\mathbb{Z}[\pi]} A$ where $C = \mathbb{Z}[\pi]$ as left $\mathbb{Z}[\pi]$-module with right $\mathbb{Z}[\pi]$-action given by $c \cdot r = cgrg^{-1}$ ($c \in C$, $r \in \mathbb{Z}[\pi]$, $\cdot$ denotes right action on C).

Define $\phi : C \to \mathbb{Z}[\pi]$ by $\phi(c) = cg$. Left $\mathbb{Z}[\pi]$-module isomorphism, and

$$\phi(c \cdot r) = \phi(cgrg^{-1}) = cgr$$

$$\phi(c)r = cgr$$

Therefore ϕ is a bimodule isomorphism,, so $C \otimes_{\mathbb{Z}[\pi]} A \cong A$.

Therefore $\gamma_* : K_0(\mathbb{Z}[\pi]) \to K_0(\mathbb{Z}[\pi])$ is identity.

If $M \in GL(n, \mathbb{Z}[\pi])$, then $\gamma_* M = (gI_n)M(gI_n)^{-1}$.

Therefore $\gamma_* M \equiv M \bmod E(\mathbb{Z}[\pi])$, so $\gamma_* : K_1 \to K_1$ is identity. $\square$

Wh(π) is f.g. if π is finite (Bass).

$\tilde{K}_0(\mathbb{Z}[C_\infty \times C_{p^2}])$ not f.g.

$\tilde{K}_0(\mathbb{Z}[\pi])$ is summand of Wh($\pi \times C_\infty$).

Wh(π) $= \tilde{K}_0(\pi) = 0$ if π free or free Abelian.

2. Chain Complexes

Consider chain complexes of left R-modules.

$$C_* : \cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

∂ is an R-homomorphism such that $\partial^2 = 0$.

$H_n(C_*)$ is a left R-module.

$$C_* \text{ is free/proj/f.g.} \iff C_n \text{ is free/proj } \forall\, n$$

$$C_* \text{ is f.g.} \iff \bigoplus_{n=0}^{\infty} C_n \text{ is f.g.}$$

Examples. X a (simplicial) complex, fundamental group π, and universal cover $\tilde{X}$ triangulated canonically. Chain complex $C_*(\tilde{X})$ (finite simplicial chains). π acts n $\tilde{X}$, so $C_*(\tilde{X})$ is chain complex of $\mathbb{Z}[\pi]$-modules. Free: one basis element for each simplex of X.

If X dominated by finite complex $X \xrightarrow{f} K \xrightarrow{g} X - zf \simeq 1$.

$C_*(\tilde{X}) \to \underset{\text{f.g. free}}{C_*(\tilde{K})} \to C_*(\tilde{X})$ with $g_* f_* \simeq 1$.

Lemma 2.1. *If C_* is projective and acyclic, then there exists R-homomorphisms $\Gamma_i : C_i \to C_{i+1}$ such that $\partial\Gamma + \Gamma\partial = 1$.*

Proof. $C_1 \xrightarrow{\partial} C_0$ onto, C_0 projective, so there exists $\Gamma_0 : C_0 \to C_1$ with $\partial\Gamma_0 = 1$.

Suppose inductively that $\Gamma_0, \dots, \Gamma_{n-1}$ defined. $x \in C_n$; $\partial x = (\partial\Gamma_{n-1} + \Gamma_{n-2}\partial)\partial x = \partial\Gamma\partial x$

$\therefore \quad (1 - \Gamma_{n-1}\partial)x \in Z_n = \ker\partial : C_n \to C_{n-1}$

$Z_n = \mathrm{im}\ : \partial : C_{n+1} \to C_n = B + n.$

C_n projective $\Rightarrow \exists\, \Gamma_n : C_n \to C_{n+1}$ s.t. $\partial\Gamma_n = 1 - \partial\Gamma_{n-1}$, i.e. $\partial\Gamma_n + \Gamma_{n-1}\partial = 1$ completes induction step.

$f : C_* \to D_*$ chain map. □

Algebraic mapping cylinder. M_* of f has $M_n = C_n \oplus C_{n-1} \oplus D_n$ with $\partial : M_n \to M_{n-1}$ defined by $\partial(x, y, z) = (\partial x - y, -\partial y, \partial z + fy)$. Check $\partial^2 = 0$.

Chain maps

$$\lambda : C_* \to M_*, \qquad \mu : M_* \to D_*$$
$$x \mapsto (x, 0, 0), \quad (x, y, z) \mapsto z + fx$$

$\mu\lambda = f$ and μ is a chain equivalence.

Inverse $\bar{\mu} : D_* \to M_*$; $z \mapsto (0, 0, z)$.

$\mu\bar{\mu} = 1$. homotopy $\bar{\mu}\mu \simeq 1$ given by $\Delta_n : M_n \to M_{n+1}$

$$(x, y, z) \mapsto (0, x, 0)$$
$$\begin{aligned}(\partial\Delta + \Delta\partial)(x, y, z) &= (-x, -\partial x, fx) + (0, \partial x - y, 0)\\ &= (-x, -y, fx)\\ &= (\bar{\mu}\mu - 1)(x, y, z)\end{aligned}$$

algebraic mapping cone. $Q_* = M_* / \operatorname{im}\lambda$

$\therefore\ Q_n = C_{n-1} \oplus D_n$, $\partial(y, z) = (-\partial y, \partial z + fy)$

$$\begin{tikzcd} 0 \arrow[r] & C_* \arrow[r, "\lambda"] \arrow[dr, "f"'] & M_* \arrow[r, "\pi"] \arrow[d, "\mu"] & Q_* \arrow[r] & 0 \\ & & D_* & & \end{tikzcd}$$

Commutes, top row exact.

Define $H_n(f) = H_n(Q_*)$; get exact homology sequence of f.

$$H_n(C_{(}) \xrightarrow{f_*} H_n(D_*) \longrightarrow H_n(f) \longrightarrow H_{n-1}(C_*) \xrightarrow{f_*} a$$

Lemma 2.2. *If $f : C_* \to D_*$ induces homology group isomorphisms, and C_*, D_* projective, then f is a chain equivalence.*

Proof. M_*, Q_* mapping cylinder and cone of f. It is enough to show $\lambda : C_* \to M_*$ is an equivalence. Q_* is acyclic and projective, therefore by 2.1 there exists the contraction Γ_*.

Put $M_n = C_n \oplus Q_n$ in obvious way.

Put $\Delta_n = 0 \oplus \Gamma_n : M_n \to M_{n+1}$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_i & \xrightarrow{\lambda} & M_i & \xrightarrow{\pi} & Q_i & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \Delta_i} & & \downarrow{\scriptstyle \Gamma_i} & & \\
0 & \longrightarrow & C_{i+1} & \xrightarrow{\lambda} & M_{i+1} & \xrightarrow{\pi} & Q_{i+1} & \longrightarrow & 0
\end{array}$$

commutes $\pi(1 - \partial\Delta - \Delta\partial) = (1 - \partial\Gamma - \Gamma\partial)\pi = 0$,

therefore there exists $\bar\lambda : M_* \to C_*$ such that $\lambda\bar\lambda = 1 - \partial\Delta - \Delta\partial$

$\lambda\bar\lambda \simeq 1$

$\lambda\bar\lambda\lambda(x) = (1 - \partial\Delta - \Delta\partial)\lambda(x) = \lambda(x)$

λ mono $\Rightarrow \bar\lambda\lambda = 1$

So $\bar\lambda$ chain inverse to λ as required. $\square$

C_* *dominated* by D_* if there exists $f : C_* \to D_*$, $g : D_* \to C_*$, $gf \sim 1$. *Dimension* of C is $\dim(C_*) = \sup\{n : C_n \neq 0\}$.

Theorem 2.3 (C.T.C. Wall). *If C_*, D_* is projective, D_* dominates C_*, and D_* is f.g., then C_* is equivalent to a f.g. projective complex of dimension $\leq \dim(D_*)$.*

Definition. C_* is of finite type if C_n is f.g. for all n.

Lemma 2.4. *If C_*, D_* is projective, D_* dominates C_*, and D_* is of finite type, then $C_* \simeq$ some complex of finite type.*

Proof. $f : C_* \to D_*$, $g : D_* \to C_*$, $gf \simeq 1$. Suppose inductively that $H_i(f) = 0$ for $i < n$ (start with $n = 0$).

First step: $H_n(f)$ is f.g.

Homology sequence of f:

$$0 \longrightarrow H_i(C_*) \underset{g_*}{\overset{f_*}{\rightleftarrows}} H_i(D_*) \longrightarrow H_i(f) \longrightarrow 0 \tag{1}$$

Let $r = fg : D_* \to D_*$.

f, g, r induces homology isomorphisms in dimensions $< n$. Exact sequence of r:

$$H_n(D_*) \xrightarrow{r_*} H_n(D_*) \longrightarrow H_n(r) \longrightarrow 0$$
$$r_* = f_*g_*\ , \quad f_* = r_*f_* \Rightarrow \operatorname{im} r_* = \operatorname{im} f_*$$
$$\Rightarrow_n (f) = H_n(r)$$

Let Q_* be mapping cone of r. $H_i(Q_*) = 0$ for $i < n$.

Exact sequence. $0 \to Z_n(Q_*) \underset{\subset}{\longrightarrow} Q_n \xrightarrow{\partial} Q_{n-1} \xrightarrow{\partial}\to Q_0 \to 0$. Q_i is projective so argument of 2.1 $\Rightarrow \exists$ contraction ? (don't use Z_n projective).

$\Gamma_n|Z_n(Q_*) = 1$, so Z_n is direct summand of Q_n.

$\therefore$ Z_n is f.g., $\therefore$ $H_n(f) \cong H_n(Q_*)$ is f.g.

From $(*)$, $H_n(f) \cong \ker g_* : H_n(D_*) \to H_n(C_*)$

Pick f.g. projective E and epimorphism $e : E \to \ker g_*$

$\exists$ d such that

$$\begin{tikzcd} E \arrow[r, "d"] \arrow[d, "e"'] & Z_n(D_*) \arrow[d, "\text{proj.}"] \\ \ker g_* \arrow[r, "\text{inc.}"] & H_n(D_*) \end{tikzcd} \quad \text{commutes.}$$

(2)
$$\begin{tikzcd} {} \arrow[r, "\partial"] & C_{n+2} \arrow[r, "\partial"] \arrow[d, shift right, "f"'] & C_{n+1} \arrow[r, "\partial"] \arrow[d, shift right, "f\oplus 0"'] & C_n \arrow[r, "\partial"] \arrow[d, shift right, "f"'] & {} \\ {} \arrow[r, "\partial"] & D_{n+2} \arrow[r, "\partial\oplus 0"] \arrow[u, shift right, "g"'] & D_{n+1}\oplus E \arrow[r, "\partial\oplus d"] \arrow[u, shift right, "g\oplus C"'] & D_n \arrow[r, "\partial"] \arrow[u, shift right, "g"'] & {} \end{tikzcd}$$

To choose c, note that $gd(C_n) \subset B_n(C_*)$ since $e(E) \subset \ker g_*$.

E projective, so $\exists$ $c : E \to C_{n+1}$ such that $\partial c = gd$.

Replace D_* by bottom row of (2): chain complex of finite type. Haven't changed gf, so D_* still dominates C_*.

g induces homology isomorphisms in dimensions sn.

Therefore f does too. Therefore $H_i(f) = 0$ for $i \leq n$.

Only changed D_{n+1}.

Iterate infinitely, obtain complex D'_* and map $f' : C_* \to D'_*$ inducing homology isomorphisms in all dimensions. Therefore by 2.2, $C_* \simeq D'_*$, which is of finite type. $\square$

Proof of Theorem 2.3. By Lemma 2.4, replace C_* by an equivalent complex of finite type. $f : C_* \to D_*$, $g : D_* \to C_*$ such that $gf \simeq 1$, say $1 - gf = \partial\Delta + \Delta\partial$ where $\Delta_i : C_i \to C_{i+1}$.

Let $n = \dim D_*$. Then $gf : Cn + 1 \to C_{n+1}$ is zero.

$\therefore \partial\Delta_{n+1} + \Delta_n\partial = 1_{C_{n+1}} \Rightarrow \partial\Delta_n\partial = \partial$

$\therefore$ we have the map $\partial\Delta_n : C_n \to B_n$ such that $\partial\Delta_n|B_n = 1$.

$\therefore B_n$ is a direct summand of C_n.

$\therefore C_n/B_n$ is f.g. projective.

Let E_* be complex

$$0 \longrightarrow C_n/B_n \xrightarrow{\partial} C_{n_1} \xrightarrow{\partial} C_{n_2} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \longrightarrow 0$$

Projection: $C_* \to E_*$ induces homology isomorphisms (clear for dimensions $\leq n$, and $H_i(D_*) = 0$ for $i > n$, from $H_i(C_*) \leftrightarrows H_i(D_*)$).

Therefore $C_* \simeq E_*$ by 2.2; and E_* is f.g. proj., $\dim E_* = \dim D_*$. $\square$

Let C_* be f.g. projective. Define *Wall invariant* $\sigma(C_*)$ to be $\sum_i (-1)^i [C_i] \in \tilde{K}_0(R)$.

Lemma 2.5. *If $C_* \simeq D_*$, then $\sigma(C_*) = \sigma(D_*)$ (where C_*, D_* are f.g. projective).*

Proof. Let Q_* be mapping cone of a chain equivalence $C_* \to D_*$. Then Q_* is acyclic, so $\exists$ contraction Γ_*.

$$\begin{aligned}
\therefore \quad & 0 \to B_n \xrightarrow{\subset} Q_n \xrightarrow{\partial} B_{n-1} \to 0 \ \text{ splits} \\
\therefore \quad & B_n \oplus B_{n-1} \cong Q_n \cong C_{n-1} \oplus D_n \\
\therefore \quad & \sigma(C_*) - \sigma(D_*) = \sum_n (-1)^{n-1} \{[C_{n-1} + [D_n]\} \\
& \qquad = \sum (-1)^{n-1} \{[B_n] + [B_{n-1}]\} \\
& \qquad = 0\ .
\end{aligned}$$

$\square$

C an generalize definition of $\sigma(C_*)$ to case when C_* is projective and dominated by a f.g. proj. complex F or such a $C_* \simeq$ f.g. proj. complex E_* (by 2.3) and define $\sigma(C_*)$ to be $\sigma(E_*)$; well defined by Lemma 2.5.

Theorem 2.6. *A f.g. projective complex C_* is equivalent to a f.g. free complex of dimension at most* $\dim C_*$ *iff* $\sigma(C_*) = 0$.

Proof. "Only if" is clear.

"If" : Suppose $\sigma(C_*) = 0$. Suppose inductively that C_i free for $i < n$. C_n f.g. proj. $\Rightarrow$ $\exists$ R-module E, f.g. proj., such that $C_n \oplus E$ is free.

Replace C_* by complex

$$\xrightarrow{\partial} C_{n+2} \xrightarrow{\partial \oplus 0} C_{n+1} \oplus E \xrightarrow{\partial \oplus 1} C_n \oplus E \xrightarrow{\partial \oplus 0} C_{n-1} \xrightarrow{\partial} C_{n-2}$$

which is equivalent to C_* by Lemma 2.2.

This completes the induction; only had to alter C_n and C_{n+1}.

Let $m = \dim C_*$: continue this process until C_i is free, $i < m$ (doesn't increase $\dim C_*$). $\sigma(C_*) = 0$ but $\sigma(C_*) = (-1)^m [C_m]$.

$\therefore$ $\exists$ f.g. free F, G such that $c_m \oplus F \cong G$.

Replace C_* by complex

$$0 \to C_m \oplus F \xrightarrow{\partial \oplus 1} C_{m-1} \oplus F \xrightarrow{\partial \oplus 0} C_{m-2} \xrightarrow{\partial}$$

which is $\simeq C_*$ by 2.2; and it is f.g. free of $\dim m$. $\square$

Whitehead Torsion.

Hypothesis (for rest of Section 2): R is such that free modules R^m, R^n are isomorphic iff $m = n$.

Examples 1) if R any ring: R^∞ = free left R-module on countably many generators. $S = \mathrm{End}_R(R^\infty)$. If A is any left R-module, $\mathrm{Hom}_R(A, R^\infty)$ is a left S-module. But, as left S-modules

$$\begin{aligned} S = \mathrm{Hom}_R(R^\infty, R^\infty) &\cong \mathrm{Hom}_R(R^\infty \oplus R^\infty, R^\infty) \\ &\cong S \oplus S \end{aligned}$$

so hypothesis doesn't hold for S.

2) Hypothesis does hold if R can be mapped homomorphically into a field, e.g., commutative rings, $\mathbb{Z}[\pi]$. □

Let A be a f.g. free R-module, and let $b = (b_1, \dots, b_m)$, $c = (c_1, \dots, c_n)$ be bases for A. Then $m = n$, so $\exists$ unique square matrix $[a_{ij}] \in GL(n, R)$ such that $c_i = \sum a_{ij} b_j$. Write $[c/b]$ for $\tau[a_{ij}] \in K_1(R)$.

A *based* chain complex is a f.g. free chain complex C_* together with a basis $c_n = (c_n^{(1)}, \dots, c_n^{(d_n)})$ of C_n, $\forall\, n$.

Let C_* be based and acyclic. By 2.1 $\exists$ contraction Γ_*.

Exact sequence.

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & B_n & \xrightarrow{\subset} & C_n & \xrightarrow{\partial} & B_{n-1} & \longrightarrow & 0 & \text{splits} \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \partial\oplus\partial\Gamma_n} & & \downarrow{\scriptstyle 1} & & & \\ 0 & \longrightarrow & B_n & \xrightarrow{\subset} & B_{n-1} \oplus B_n & \xrightarrow{p_1} & B_{n-1} & \longrightarrow & 0 & \end{array}$$

commutative diagram. Five lemma $\Rightarrow \partial \oplus \partial\Gamma_n$ isomorphism $= \gamma_n$

$$\gamma_n : C_n \longrightarrow B_{n-1} \oplus B_n$$

Let $\gamma = (\oplus\gamma_{2i})^{-1}(\oplus\gamma_{2i+1}) : \oplus C_{2i+1} \to \oplus C_{2i}$.

Bases $\oplus c_{2i}$, $\gamma(\oplus c_{2i+1})$ for $\oplus C_{2i}$

Define $\tau(C_*)$ to be $[\gamma(\oplus c_{2i+1}) / \oplus c_{2i}]$.

Re-ordering bases: $\tau\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) = \tau(-1)$, so that re-ordering bases adds $\tau(\pm 1)$ to $\tau(C_*)$.

Define $\overline{K}_1(R) = K_1(R)/\{\tau(\pm 1)\} = \mathrm{coker}(K_1(\mathbb{Z}) \to K_1(R))$.

Torsions of chain complexes will be regarded as elements of $\overline{K}_1(R)$.

Lemma 2.7. *The torsion $\tau(C_*)$ depends only on C_* and bases c_*.*

Proof. Let Γ'_* be another contraction giving isomorphisms $\gamma'_n : C_N \to B_{n-1} \oplus B_n$.

Let $\beta_n = \gamma'_n \gamma_n^{-1} : B_{n-1} \oplus B_n \to B_{n-1} \oplus B_n$.

It is enough to prove $\tau(\beta) = 0$.

Commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_n & \longrightarrow & B_{n-1} \oplus B_n & \longrightarrow & B_{n-1} & \longrightarrow & 0 \\ & & {\scriptstyle 1}\big\downarrow & & \big\downarrow{\scriptstyle \beta_n} & & \big\downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & B_n & \longrightarrow & B_{n-1} \oplus B_n & \longrightarrow & B_{n-1} & \longrightarrow & 0 \end{array}$$

B_{n-1}, B_n are f.g. projective: $\exists X_{n-1}, X_n$ such that $X_{n-1} \oplus B_{n-1}$, $B_n \oplus X_n$ f.g. free. Let $F_n = B_n \oplus X_n$.

$$\phi_n = 1 \oplus \beta_n \oplus 1 : F_{n-1} \oplus F_n \longrightarrow F_{n-1} \oplus F_n$$

$$\tau(\phi_n) = \tau(\beta_n)$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_n & \longrightarrow & F_{n-1} \oplus F_n & \longrightarrow & F_{n-1} & \longrightarrow & 0 \\ & & \circlearrowleft 1 & & \circlearrowleft \phi_n & & \circlearrowleft 1 & & \end{array}$$

w.r.t bases for F_{n-1}, F_n, ϕ_n has matrix $\begin{pmatrix} 1 & M \\ 0 & 1 \end{pmatrix}$ which is a product of elementary matrices.

Therefore $\tau(\beta_n) = \tau(\phi_n) = 0$ as required. □

C_*, D_* based $f : C_* \to D_*$ chain map, mapping cone $Q_* : Q_n = C_{n-1} \oplus D_n$: basis $q_n = c_{n-1} \oplus d_n$.

Q_* is based and acyclic if f is a chain equivalence

Define $\tau(f) = \tau(Q_*)$.

Call f a *simple* equivalence if $\tau(f) = 0$.

Theorem 2.8. *If $f : C_* \to D_*$ is a chain equivalence of based chain complexes, and $g \simeq f$, then $\tau(g) = \tau(f)$.*

Proof. $f - g = \partial\Delta + \Delta\partial$

Let Q^f_*, Q^g_* be the mapping cones of f, g.

$$Q^f_n = Q^g_n = C_{n-1} \oplus D_n \ , \qquad q^f_n = q^g_n = c_{n-1} \oplus d_n$$

$$\partial^f(y, z) = (-\partial y, \partial z + fy)$$

$$\partial^g(y, z) = (-\partial y, \partial z + gy)$$

Define $\phi : Q^f_* \to Q^g_*$ by $\phi(y, z) = (y, z + \Delta y)$.

$$\text{Chain map:} \quad \phi\partial^f(y, z) = (-\partial y, \partial z + fy - \Delta\partial y)$$

$$\partial^g\phi(y, z) = (-\partial y, \partial z + \partial\Delta y + gy)$$

ϕ is an isomorphism of chain complexes. In fact, $\phi_n : C_{n-1} \oplus D_n \to C_{n-1} \oplus D_n$ is a product of elementary automorphisms, so $[\phi(q_n)/q_n] = 0$. Therefore $\tau(Q^f_n) = \tau(Q^g_*)$ as required. □

Lemma 2.9. *Let $0 \to C'_* \xrightarrow{i} C_* \xrightarrow{j} C''_* \to 0$ be a s.a.s. of based acyclic complexes. Suppose i, j preserve bases, in the sense that $i(c'_n) \subset c_n$ and $j(c_n - i(c'_n)) = c''_n$. Then $\tau(c_*) = \tau(c'_*) + \tau(c''_*)$.*

Proof. Claim $\exists$ contractions $\Gamma_*, \Gamma'_*, \Gamma''_*$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C'_n & \xrightarrow{i} & C_n & \xrightarrow{j} & C''_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Gamma'_n} & & \downarrow{\scriptstyle \Gamma_n} & & \downarrow{\scriptstyle \Gamma''_n} & & \\ 0 & \longrightarrow & C'_{n+1} & \xrightarrow{i} & C_{n+1} & \xrightarrow{j} & C''_{n+1} & \longrightarrow & 0 \end{array}$$

commutes.

Let Γ''_* be any contraction of C''_*.

C_n free $\Rightarrow$ $\exists\ \Delta_n : C_n \to C_{n+1}$ such that $j\Delta_n = \Gamma''_n j$. Therefore $j(1 - \partial\Delta - \partial\Delta) = (1 - \partial\Gamma'' - \Gamma''\partial)j$.

$\exists$ unique $k : C_* \to C'_*$ such that $ik = 1 - \partial\Delta - \Delta\partial : C_* \to C_*$.

C'_* contractible, so $k \simeq 0$, say $k = \partial\Delta' + \Delta'\partial$, $\Delta'_n : C_n \to C'_{n+1}$.

Put $\Gamma_n = \Delta_n + i\Delta'_n$, then $\partial\Gamma + \Gamma\partial = 1$; Γ_* contraction, $j\Gamma_n = j\Delta_n = \Gamma''_n j$.

Diagram chasing $\Rightarrow 0 \to B'_n \xrightarrow{i} B_n \xrightarrow{j} B''_n \to 0$ exact.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C'_n & \xrightarrow{i} & C_n & \xrightarrow{j} & C''_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma'_n} & & \downarrow{\scriptstyle \gamma_n} & & \downarrow{\scriptstyle \gamma''_n} & & \\ 0 & \longrightarrow & B'_{n-1} \oplus B'_n & \longrightarrow & B_{n-1} \oplus B_n & \longrightarrow & B''_{n-1} \oplus B''_n & \longrightarrow & 0 \end{array}$$

$$\partial + \partial\Gamma = \gamma_n : C_n \longrightarrow B_{n-1} \oplus B_n$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus C'_{2r+1} & \xrightarrow{i} & \bigoplus C_{2r+1} & \xrightarrow{j} & \bigoplus C''_{2r+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma'} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma''} & & \\ 0 & \longrightarrow & \bigoplus C'_{2r} & \longrightarrow & \bigoplus C_{2r} & \longrightarrow & \bigoplus C''_{2r} & \longrightarrow & 0 \end{array}$$

both commute.

Let M, M', M'' be matrices of $\gamma, \gamma', \gamma''$ w.r.t. given bases.

i, j preserve bases. Re-order bases c_n of C_n to bring M into form $\begin{pmatrix} M' & x \\ 0 & M'' \end{pmatrix} = \begin{pmatrix} M' & 0 \\ 0 & M'' \end{pmatrix} \begin{pmatrix} 1 & (M')^{-1}x \\ 0 & 1 \end{pmatrix}$.

$$\therefore \quad \tau(M) \equiv \tau(M') + \tau(M'') \bmod \tau(\pm 1)$$

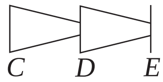
$$\therefore \quad \tau(C_*) = \tau(C'_*) + \tau(C''_*) \in \bar{K}_1(R)$$

$\square$

Theorem 2.10. *If $f : C_* \to D_*$, $g : D_* \to E_*$ are chain equivalences of based complexes, then $\tau(gf) = \tau(g) + \tau(f)$.*

Proof. Let Q_*^f, Q_*^g, Q_*^{gf} be mapping cones. Define S_* by

$$\begin{array}{ccccccccc} S_n = & C_{n-1} & \oplus & D_n & \oplus & D_{n-1} & \oplus & E_n \\ & y & & z & & v & & w \end{array}$$



squash C

bases $s_n = c_{n-1} \oplus d_n \oplus d_{n-1} \oplus e_n$

$$\partial(y, z, v, w) = (-\partial y, \partial z + fy - v, -\partial v, \partial w + gv) \ .$$

Based exact sequence

$$\begin{array}{c} 0 \longrightarrow Q_*^f \longrightarrow S_* \longrightarrow Q_*^g \longrightarrow 0 \\ (y,z) \quad \longmapsto \quad (y,z,0,0) \qquad\qquad \\ \qquad\qquad (y,z,v,w) \quad \longmapsto \quad (v,w) \\ \tau(S_*) = \tau(f) + \tau(g) \ \text{ by 2.9.} \end{array}$$

Define $i : Q_*^{gf} \to S_*$ by $i(y, w) = (y, 0, fy, w)$ chain map.

Define complex T_* by $T_n = D_n \oplus D_{n-1}$ basis $t_n = d_n \oplus d_{n-1}$

$$\partial(z, v) = (\partial z - v, -\partial v)$$

$$0 \longrightarrow Q_x^{gf} \xrightarrow{i} S_* \xrightarrow{j} T_* \longrightarrow 0$$

$$(y, z, v, w) \longrightarrow (z, v - fy)$$

This is not based.

Now basis ?? for $S_n : s_n' = i(c_{n-1} \oplus e_n) \cup d_n \oplus d_{n-1}$. In fact, $[s_n'/s_n] = 0 \in \bar{K}_1(R)$ related to s_n by transformation $(y, z, v, w) \mapsto (y, z, v + fy, w)$. By Lemma 2.9, $\tau(gf) + \tau(T_*) = \tau(S_*) = \tau(f) + \tau(g)$

$$T_n = D_n \oplus D_{n-1} \qquad \partial(z, v) = (\partial z - v, -\partial v) \ , \quad t_n = d_n \oplus d_{n-1}$$

Define T_*' by $T_n' = T_n$, $t_n' = t_n$, $\partial'(z, v) = (-v, 0)$. Define $\phi : T_* \to T_*'$ by $\phi(z, v) = (z, v - \partial z)$ chain map.

ϕ is elementary automorphism of T_n.

$[\phi t_n / t_n] = 0$

$\therefore \tau(T_*) = \tau(T_*')$

To calculate $\tau(T_*')$, use contraction Γ_*', with $\Gamma'(z, v) = (0, -z)$.

Matrix of $\gamma : \oplus T_{2i+1}' \to \oplus T_{2i}'$ has integer coefficients $\bar{K}_1(\mathbb{Z}) = 0$, so γ has zero torsion. Therefore $\tau(T_*') = 0$. $\square$

Corollary 2.11. *Let $0 \to C_*' \xrightarrow{i} C_* \xrightarrow{j} C_*'' \to 0$ be an exact sequence of based complexes. Suppose i is a chain equivalence, and i, j preserve bases. Then $\tau(i) = \tau(C_*'')$.*

Proof. Let Q_* be the mapping cone of i; let Q'_* be the mapping cone of $1_{C'_*}$. Then $\tau(Q_*) = \tau(i)$ and $\tau(Q'_*) = 0$ by (2.10).

Define $u : Q'_* \to Q_*$ by $u(y,z) = (y, i(z))$.

Define $v : Q_* \to C''_*$ by $v(y,z) = j(z)$ preserve bases.

Exact sequence $0 \to Q'_* \to Q_* \to C''_* \to 0$.

By Lemma 2.9, $\tau(i) = \tau(Q_*) = \tau(C''_*)$. □

$f : C_* \to D_*$ any chain map of based complexes.

$M_* =$ mapping cylinder : $M_n = C_n \oplus C_{n-1} \oplus D_n$, basis $m_n = c_n \oplus c_{n-1} \oplus d_n$

$$\partial(x,y,z) = (\partial x - y, -\partial y, \partial z + fy) \ .$$

Chain equivalence $\mu : M_* \to D_*$

Corollary 2.12. *μ is a simple equivalence, i.e. $\tau(\mu) = 0$.*

Proof. Recall from 2.2 that a chain inverse of μ is given by $\bar{\mu}(z) = (0,0,z)$. Define T_* by $T_n = C_n \oplus C_{n-1}$, basis $t_n = c_n \oplus c_{n-1}$

$$\partial(x,y) = (\partial x - y, -\partial y) \ .$$

Based exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D_* & \xrightarrow{\bar{\mu}} & M_* & \longrightarrow & T_* & \longrightarrow & 0 \\ & & & & (x,y,z) & \longmapsto & (x,y) & & \end{array}$$

$$\therefore \quad \tau(\bar{\mu}) = \tau(T_*) = 0 \ \text{ as in proof of 2.10.}$$

$$\therefore \quad \mu\bar{\mu} = 1 \ , \quad \text{so} \quad \tau(\mu) = 0 \ \text{ by 2.10.}$$

□

An *elementary* based chain complex of dimension n is one of form

$$0 \to \cdots \to 0 \to E_n \to E_{n-1} \to 0 \to \cdots$$

with $E_i = 0$ if $i \neq n, n-1$.

$$E_n = E_{n-1} = R \ , \quad e_n = e_{n-1} = 0 \ .$$

$$\partial : E_n \to E_{n-1} \ \text{ is } \pm \text{ identity.}$$

Example. K, L (finite) simplicial complexes. Suppose $K \searrow L$ by elementary simplicial collapse. $\widetilde{K}, \widetilde{L}$ universal covers.

Exact sequences

$$0 \longrightarrow C_*(\widetilde{L}) \xrightarrow{\subset_*} C_*(\widetilde{K}) \longrightarrow E_* \longrightarrow 0$$

where E_* is elementary, of same dimension as collapse.

Suppose C_*, D_* are based, and there is a based exact sequence

$$0 \longrightarrow C_* \xrightarrow{i} D_* \longrightarrow E_* \longrightarrow 0$$

with E_* elementary.

Then i is called an *elementary expansion.*

By 2.2, i is a homotopy equivalence.

Any chain inverse is called an *elementary collapse.*

Theorem 2.13. *A chain map $f : C_* \to D_*$ is a simple equivalence iff it can be factored into finitely many elementary expansions and collapses.*

Proof. The torsion of an elementary complex is 0; by Lemma 2.11, an elementary expansion or collapse has torsion zero.

Lemma 2.14. *A based acyclic complex with zero torsion can be reduced to 0 by finitely many elementary expansions and collapses.*

Proof. C_* based acyclic, $n = \dim C_*$.

First we show how to alter basis $c_{n-1} = (c', \dots, c^d)$ of C_{n-1} by an elementary matrix $1 + \lambda eij$.

$$\begin{tikzcd}
 & 0 \arrow[r] & C_n \arrow[r, "\partial"] \arrow[d, "i_1"] & C_{n-1} \arrow[r, "\partial"] \arrow[d] & C_{n-2} \arrow[r, "\partial"] \arrow[d, "1"] & {} \\
 & 0 \arrow[r] \arrow[d] & C_n \oplus R \arrow[r, "\partial^2"] \arrow[d, "i_{2,3}"] & C_{n-1} \oplus R \arrow[r, "\partial^1"] \arrow[d, "1"] & C_{n-2} \arrow[r, "\partial"] \arrow[d, "1"] & {} \\
0 \arrow[r] & C_n \arrow[r] & C_n \oplus C_n \oplus R \arrow[r, "\partial^3"] & C_{n-1} \oplus R \arrow[r, "\partial^1"] & C_{n-2} \arrow[r, "\partial"] & {} \\
 & 0 \arrow[r] \arrow[u] & C_n \oplus R \arrow[r, "\partial^4"] \arrow[u, "i_{1,3}"] & C_{n-1} \oplus R \arrow[r, "\partial^1"] \arrow[u, "1"] & C_{n-2} \arrow[r, "\partial"] \arrow[u, "1"] & {} \\
 & 0 \arrow[r] & C_n \arrow[r, "\partial"] \arrow[u, "i_1"] & C_{n-1} \arrow[r, "\partial"] \arrow[u, "\phi"] & C_{n-2} \arrow[r, "\partial"] \arrow[u, "1"] & {}
\end{tikzcd}$$

where

$$\begin{aligned}
\partial^1(z, r) &= \partial z + r(c^j + \lambda c^i) \\
\partial^2(y, r) &= (\partial y - r(c^j + \lambda c^i), r) \\
\partial^3(x, y, r) &= \partial^2(x + y, r + (\partial x)j) \qquad \partial x = \sum (\partial x)_r c^r \\
\partial^4 &= \partial^3(x, 0, r) \\
\partial^5 &= (x, -x, -(\partial x)j) \\
\phi(z) &= (z - (z); (c^j + \lambda c^i), (z)j)
\end{aligned}$$

Vertical maps define elementary expansions; except that ϕ isn't based. To make it based, we have to replace $(c^1, \dots, c^d)$ by $(c^1, \dots, c^{j-1}, c^j + \lambda c^i, c^{j+1}, \dots, c^d)$. But this is the change we wanted to produce.

If $n \geq 2$, make expansion

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_n & \xrightarrow{\partial} & C_{n-1} & \xrightarrow{\partial} & C_{n-2} & \xrightarrow{\partial} \\ & & \Big\downarrow 1 & & \Big\downarrow i_1 & & \Big\downarrow i_1 & \\ 0 & \longrightarrow & C_n & \xrightarrow{\partial\oplus 0} & C_{n-1} \oplus C_n & \xrightarrow{\partial\oplus 1} & C_{n-2} \oplus C_n & \xrightarrow{\partial\oplus 0} \end{array}$$

This makes B_{n-2} (rottom row) $\cong B_{n-2}$ (top row) $\oplus\, C_n$

$$0 \longrightarrow C_n \longrightarrow C_{n-1} \longrightarrow B_{n-2} \longrightarrow 0 \text{ splits}$$

i.e. it makes B_{n-2} free.

Bases c_n, c_{n-1} for C_n, C_{n-1}.

From the exact sequence $0 \longrightarrow C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} B_{n-2} \longrightarrow 0$ and freeness of B_{n-2}, we can extend ∂c_n to a basis $\overline{\partial c_n}$ of C_{n-1}.

$\exists$ matrix $M \in GL(k, R)$ ($k =$ rank of C_{n-1}) such that $\overline{\partial c_n} = M c_{n-1}$. Make another expansion

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_n & \xrightarrow{\partial} & C_{n-1} & \xrightarrow{\partial} & C_{n-2} & \xrightarrow{\partial} \\ & & \Big\downarrow 1 & & \Big\downarrow i_1 & & \Big\downarrow i_1 & \\ 0 & \longrightarrow & C_n & \xrightarrow{\partial\oplus 0} & C_{n-1} \oplus R^k & \xrightarrow{\partial\oplus 1} & C_{n-2} \oplus R^k & \xrightarrow{\partial\oplus 0} & . \end{array}$$

Extend c_{n-1} to bases of $C_{n-1} \oplus R^k$ by adjoining standard basis $(e^1, \dots, e^k)$ of R^k.

Extend $\overline{\partial c_n}$ to basis of $C_{n-1} \oplus R^k$ by adjoining $(M^{-1}e^1, \dots, M^{-1}e^k)$.

Now $\overline{\partial c_n} = \left(\begin{smallmatrix} M & 0 \\ 0 & M^{-1} \end{smallmatrix}\right) c_{n-1}$ and $\left(\begin{smallmatrix} M & 0 \\ 0 & M^{-1} \end{smallmatrix}\right)$ is product of elementary matrices. So we can change c_{n-1} into $\overline{\partial c_n}$ by elementary expansions and collapses. Then $\partial : C_n \to C_{n-1}$ is based injection, so we can collapse C_* onto $0 \to \frac{C_{n-1}}{\partial c_n} \xrightarrow{\partial} C_{n-2} \xrightarrow{\partial} C_{n-3} \xrightarrow{\partial}$. This reduces $\dim C_*$.

Continue until $\dim C_* = 1$

$$0 \to C_1 \xrightarrow[\cong]{\partial} C_? \longrightarrow$$

Since $\tau(C_*) = 0$, ∂ is given (wrt bases ??) by matrix M with $\tau(M) = 0$. Expand until M is a product of elementary matrices.

Change basis of C_0 to make ∂ based (by expansions and collapses as above). Now C_* can be collapsed to 0. This proves the lemma. $\square$

Proof of Theorem 2.13. $f : C_* \longrightarrow D_*$ is simple again, C_*, D_* based. M_* = mapping cylinder of f.

$$\bar{\mu} : D_* \longrightarrow M_*$$
$$D_n \ni z \longmapsto (0,0,z) \in C_n \oplus C_{n-1} \oplus D_n$$

$\square$

Exercise. $\bar{\mu} : D_* \to M_*$ is a product of elementary expansions

$$\begin{array}{ccccccccc}
\xrightarrow{\partial} & D_2 & \xrightarrow{\partial} & D_1 & \xrightarrow{\partial} & D_0 & \longrightarrow & 0 \\
 & \Big\downarrow & & \Big\downarrow{\scriptstyle i_2} & & \Big\downarrow{\scriptstyle i_2} & & \\
\xrightarrow{\partial} & D_2 & \longrightarrow & C_0 \oplus D_1 & \longrightarrow & C_0 \oplus D_0 & \longrightarrow & 0 \\
 & \Bigg| & & (y,z) & \longmapsto & (y, fy + \partial z) & & \\
 & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\
\longrightarrow & C_1 \oplus D_2 & \longrightarrow & C_1 \oplus C_0 \oplus D_1 & \longrightarrow & C_0 \oplus D_0 & \longrightarrow & 0
\end{array}$$

Replace $f : C_* \to D_*$ by a based injection.

Exact sequence $0 \to C_* \xrightarrow{f} D_* \xrightarrow{\pi} A_* \to 0$ based $\tau(A_*) = \tau(f) = 0$. A_* cyclic.

Therefore can reduce A_* to 0 by Lemma 2.14. We show how to "cover" expansions and collapses of A_* by corresponding expansions and collapses of D_*. If $A_* \longrightarrow A'_*$ is an elementary collapse then $D_* \to D'_* = \pi^{-1}(A'_*)$. Let $A_* \to A'_*$ be an elementary expansion. Let $h : A'_* \to A_*$ be a collapse. Then $h|A_*$ is chain homotopic to 1. Extend homotopy to get collapse $g : A'_* \to A_*$ with $g|A_* = 1$. (A_n direct summand of A'_n).

Define $D'_* = \{(x,y) \in D_* \oplus A'_* : \pi(x) = g(y)\}$

$$\partial(x,y) = (\partial x, \partial y)$$

$x \mapsto (x, \pi(x))$ is a based injection $D_* \to D'_*$.

Extend basis of D_* to basis of D'_* suitably; then $D_* \to D'_*$ is elementary expansion.

Still have exact sequence $0 \to C_* \xrightarrow{f'} D'_* \xrightarrow{\pi'} A'_* \to 0$ (based).

This finishes the proof of Theorem 2.13.

$\square$

Exercise. 1) Can get from D_* to C_* by expansions and collapses of dimension at most $\max(\dim C_* + 1, \dim D_* + 1)$.

2) In Lemma 2.14, we can get from C_* to 0 by expansions and collapses of dimension ≥ 2.

3. CW complexes

e^n closed n-cell.

C.W. complex is Hausdorff space X with maps $\phi_\alpha : e^n \to X$ $(\alpha \in A_n)$

i) If $X^n = \bigcup_{r \leq n} \bigcup_{\alpha \in A_r} \phi_\alpha(e^r)$, then $X = \cup X^n$ and $\phi_\alpha(\partial e^n) \subset X^{n-1}$.

ii) $\phi_\alpha(\text{int } e^n)_n \phi_\beta(\text{int } e^m) = \phi$ unless $\alpha = \beta$ and $n = m$, i.e. $\phi_\alpha|\text{int } e^n$ is 1–1

iii) $\forall$ $alpha$, $\phi_\alpha(e^n) =$ finite union of interiors of cells

iv) $C \subset X$ closed $\Leftrightarrow$ $\phi_\alpha^{-1}(C)$ closed in e^n for all α

Lemma 3.1. *Any CW complex has the homotopy type of a simplicial complex.*

Proof. Suppose $\simeq$ equiv $f : X^{n-1} \to K^{n-1} =$ simplicial complex.

A_n discrete topology.

$\phi : A_n \times \partial e^n \to X^{n-1}$ given by $\phi(\alpha, x) = \phi_x(x)$.

Let ψ be simplicial approximation of $f\phi$

$f\phi : A_n \times \partial e^n \to K^{n-1}$.

By homotopy theory,

$$\begin{aligned} X^n &= X^{n-1} \cup_\phi (A_n \times e^n) \\ &\simeq K^{n-1} \cup_\psi (A_n \times e^n) \\ &= K^n \end{aligned}$$

which can be triangulated. □

Corollary 3.2. *Any CW complex is locally path connected and weakly locally simply connected (i.e. $\forall$ $x \in X$ $\exists$ neighborhood U of x such that any loop in U is null homotopic in X).*

Let X be a connected CW complex, $x_0 \in X$, $G \subset \pi_1(X, x_0)$. Then there exists covering space, $p : \tilde{X} \to X$, with $\tilde{x}_0 \in \tilde{X}$ such that $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = G$; $\tilde{X}$ connected.

A *covering translation* of $p : \tilde{X} \to X$ is ??

$h : \tilde{X} \to \tilde{X}$ with $ph = p$.

Example. $\mathbb{R}^n$ is a cover of n-fold torus T^n.

$$(x_1, \ldots, x_n) \longmapsto (e^{2\pi i x_1}, \ldots, e^{2\pi i x_n})$$

Group of covering translations is $\mathbb{Z}^n$.

Lemma 3.3. *If G is normal in $\pi_1(X, x_0)$ [*regular *cover] then the group of covering translations is $\cong \pi = \frac{\pi_1(X,x_0)}{G}$.*

Proof. Suppose covering translation $h : \tilde{X} \to \tilde{X}$

$\exists$ path $f : I \to \tilde{X}$ with $f(0) = \tilde{x} - 0$, $f(1) = h(\tilde{x}_0)$.

$pf : I \to X$ is a loop in X, representing $\eta(h) \in \pi$. Well defined, homomorphism.

Injective: suppose $\eta(h) = 1$. pf represents element of $G = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

$\therefore pf \simeq p\ell$, ℓ some loop in $\tilde{X}$, rel ends.

Lift this homotopy to $\tilde{X}$ to prove $f(0) = f(1)$, so $h(\tilde{x}_0) = \tilde{x}_0$.

$\therefore h = 1$

Surjective: take loop $\ell : I \to X$. Lift to path $\tilde{\ell} : I \to \tilde{X}$, $\tilde{\ell}(0) = \tilde{x}_0$.

$p_*(\pi_1(\tilde{X}, \tilde{\ell}(1))) = G$ (since G normal)

$$\left[\begin{array}{lll} A \xrightarrow{\tilde{u}} & \tilde{X} & \exists \text{ unique } \tilde{u} : A \to \tilde{X} \text{ with } p\tilde{u} = u \\ \quad\;\; \searrow^{u} & \downarrow^{p} & \qquad\qquad\qquad\quad \text{and } \; \tilde{u}(\alpha_0) = \tilde{x} \\ & X & \text{provided } u_*\pi_1(A, u_0) \subset p_*(\pi_1(\tilde{X}, \tilde{x})). \end{array}\right]$$

By covering space theory, $\exists\, \tilde{h} : \tilde{X} \to \tilde{X}$ with $\tilde{h}(\tilde{x}_0) = \tilde{\ell}(1)$. Clearly $\eta(h)$ rep by ℓ $\square$

Lemma 3.4. *If X is a connected CW complex, then any covering $\tilde{X}$ of X has the structure of a CW complex.*

Proof. Any map $\phi : e^n \to X$ has a lift (non-unique) $\tilde{\phi} : e^n \to \tilde{X}$ with $p\tilde{\phi} = \phi$.

Two lifts $\tilde{\phi}_1, \tilde{\phi}_2$ with $\tilde{\phi}_1(x) = \tilde{\phi}_2(x)$ for some $x \in e^n$ are equal everywhere.

Take for n-cells of $\tilde{X}$ all lifts of all $\phi_x : e^n \to X$ $(\alpha \in A_n)$. Easy to check that this is CW complex. $\square$

Example.

$$P^2 = S^1 \cup_2 e^2 = e^0 \cup e^1 \cup_2 e^2$$

$$\begin{aligned} S^2 &= \text{universal cover of } P^2 \\ &= (e^0 \cup e^0) \cup (e^1 \cup e^1) \cup (e^2 \cup e^2) \end{aligned}$$

If $\tilde{X} \xrightarrow{P} X$ is a *regular* cover of CW complex X, with π = group of translations, then π permutes cells of $\tilde{X}$ freely ($g \in \pi$, e^n_α cell of $\tilde{X}$, $ge^n_\alpha = e^n_\alpha \Rightarrow g = 1$). π permutes n-cells of p^{-1}(n-cell of X) transitively.

Cellular homology

$H_*(X, Y)$ = singular homology

CW complex X; define $C_n(X) = H_n(X^n, x^{n-1})$

$\partial_n : C_n(X) \to C_{n-1}(X)$ defined as composite

$$H_n(X^n, X^{n-1}) \underset{(X^n, X^{n-1})}{\xrightarrow{\partial}} H_{n-1}(X^{n-1}) \underset{(X^{n-1}, X^{n-2})}{\xrightarrow{j_*}} H_{n-1}(X^{n-1}, X^{n-2})$$

$\partial^2 = 0$; chain complex $C_*(X)$.

Lemma 3.5. *$C_*(X)$ is free Abelian with one generator for each n-cell of X. $C_*(X)$ is chain equivalent to the singular chain complex $S_*(X)$.*

Proof. By ?? and homotopy properties of singular homology

$$\begin{aligned} H_m(X^n, X^{n-1}) &\cong H_m(A_n \times e^n, A_n \times \partial e^n) \\ &\cong 0 \quad \text{for} \quad m \neq n \end{aligned}$$

$\therefore$ $C_n(X) \cong$ free Abelian with one generator for each n-cell.

It follows that

$$H_m(X^{m-1}) \cong H_m(X^{m-2}) \cong H_m(X^{m-3}) \cong \cdots \cong H_m(X^0) = 0$$

and

$$H_m(X^{m+1}) \cong H_m(X^{m+2}) \cong H_m(X^{m+3}) \cong \cdots \cong H_m(S)$$

$$\begin{aligned} Z_m(C_*(X)) &= \ker(j_*\partial : H_m(X^m, X^{m-1}) \to H_{m-1}(X^{m-1}, X^{m-2})) \\ &= \ker(\partial : H_m(X^m, X^{m-1}) \to H_{m-1}(X^{m-1})) \text{ as } j_* \\ &= \text{im } j_* \\ Z_m/B_m &\cong H_m(X^m)/j_*^{-1}(B_m) \\ &\cong H_m(X^m)/j_*^{-1}(\text{im } j_*\partial) \\ &= H_m(X^m)/\text{im } \partial \end{aligned}$$

Exact sequence $H_m(X^{m+1}, X^m) \xrightarrow{\partial} H_m(X^m) \to H_m(X^{m+1}) \to 0$ gives $H_m(X^m)/\text{im } \partial \cong H_m(X^{m+1}) \cong H_m(X)$. Cycle $z \in C_*(X) = H_m(X^m, X^{m-1})$. Put $z = j_*y$, $y \in H_m(X^m)$. Now image of y in $H_m(X)$ is image of homology class of z in $H_m(X)$. $e^n_\alpha =$ basis element of $C_n(X)$ corresponding to n-cell $\phi_\alpha : e^n \to X$. Seek map $\theta : C_*(X) \to S_*(X)$ such that $\theta\partial = \partial\theta$, $\theta(C_*(X^n)) \subset S_*(X^n)$, $\theta(e^n_\alpha)$ represents $e^n_\alpha \in H_n(X^n, X^{n-1})$. Define inductively; for $n = 0$, define $\theta(e^0_\alpha) =$ 0-simplex at e^0_α. Suppose $\theta : C_*(X^{n-1}) \to S_*(X^{n-1})$ defined. If e^n_α is a basis element of $C_n(X)$, $\theta(\partial e^n_\alpha)$ already defined, represents ∂e^n_α in $H_{n-1}(X^{n-1}, X^{n-2})$. Also, $\partial\theta(\partial e^n_\alpha) = 0$ as chain. Pick chain $c^n_\alpha \in S_n(X^n)$ representing e^n_α in $H_n(X^n, X^{n-1})$ [so $\partial c^n_\alpha \in S_{n-1}(X^{n-1})$]. Now $\partial c^n_\alpha - \theta(\partial e^n_\alpha)$ represents 0 in $H_{n-1}(X^{n-1}, X^{n-2})$. But $j_* : H_{n-1}(X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$ is mono so $\partial c^n_\alpha - \theta(\partial e^n_\alpha)$ represents 0 in $H_{n-1}(X^{n-1})$.

$\therefore$ $\exists$ $d^n_\alpha \in S_n(X^{n-1})$ such that $\partial c^n_\alpha - \theta(\partial e^n_\alpha) = \partial d^n_\alpha$. Put $\theta(e^n_\alpha) = c^n_\alpha - d^n_\alpha$. Then $\partial\theta(e^n_\alpha) = \theta\partial(e^n_\alpha)$. $\theta(e^n_\alpha)$ represents e^n_α in $H_n(X^n, X^{n-1})$, because $d^n_\alpha \in S_n(X^{n-1})$. This completes the induction.

It follows that θ induces homology isomorphisms given above

$$\begin{array}{cccc} \underset{\text{cycle}}{z} \in C_n(X) = H_n(X^n, X^{n-1}) & \xleftarrow{j_*} & H_n(X^n) & \theta(z) \\ & & \downarrow & j_*[\theta(z)] = z \\ & & H_n(x) & \end{array}$$

□

Theorems from homotopy theory:

Whitehead Theorem. *Let X, Y be connected CW complexes and let $f : X \to Y$ be a map inducing* homology *isomorphisms in all dimensions; then f is a homotopy equivalence.*

Hurewicz Theorem. *Let X, Y be connected, simply connected CW complexes and let $f : X \to Y$ be a map. If $H_r(f) = 0$ for all $r < n$, then $\pi_r(f) = 0$ for all $r < n$, and the natural map $\pi_n(f) \to H_n(f)$ is an isomorphism*

$$\Phi \Uparrow \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow & & \uparrow \phi \\ S^{n-1} & \subset & D^n \end{array} \qquad \begin{array}{rcl} \pi_n(f) & = & \textit{homotopy classes of } \Phi \\ & \cong & \pi_n(M_f, X) \end{array}$$

Connected CW complex X; $\tilde{X} \to X$ regular covering, group π. $C_*(\tilde{X})$ is a complex of free $\mathbb{Z}[\pi]$-modules

$$\sum n_g g \in \mathbb{Z}[\pi] \ , \quad f^n_\alpha \text{ is a cell of } \tilde{X}$$

Define $(\sum n_g g)(f^n_\alpha) = \sum n_g(g.f^n_\alpha) \in C_n(\tilde{X})$. ∂ is a $\mathbb{Z}[\pi]$-homomorphism.

For each cell e^n_α of X, pick lift $\tilde{e}^n_\alpha$ in $\tilde{X}$. Then $\{\tilde{e}^n_\alpha\}$ is a basis for $C_n(\tilde{X})$ over $\mathbb{Z}[\pi]$. (Any n-cell in $\tilde{X}$ can be expressed uniquely as $g\tilde{e}^n_\alpha$.)

Similarly, $S_*(\tilde{X})$ is a free chain complex over $\mathbb{Z}[\pi]$. Slight modification of 3.5 shows that $C_*(\tilde{X}) \cong S_*(\tilde{X})$ over $\mathbb{Z}[\pi]$. (Actually get canonical homotopy class of equivalences $C_* \simeq S_*$.)

CW complexes X, Y (connected) $f : X \to Y$, f induces π_1 surjection. Let $G = \ker f_* : \pi_1(X) \to \pi_1(Y)$, let $\tilde{X}$ be covering of X com. to G, let $\tilde{Y}$ be universal cover of Y. Then $\exists$ lift $\tilde{f} : \tilde{X} \to \tilde{Y}$ of f. If $\tilde{f}'$ is another lift, then $\tilde{f}' = g\tilde{f}$ for some covering translation g of $\tilde{Y}$. If h is a covering translation of $\tilde{X}$, then $\tilde{f}h = \bar{h}\tilde{f}$ for some unique translation $\bar{h}$ of $\tilde{Y}$. $h \mapsto \tilde{h}$ defines isomorphism, translation group of $\tilde{X} \to$ group of $\tilde{Y} \cong \pi_1(Y)$. Use this isomorphism to identify the groups.

Now $\tilde{f}_* : S_*(\tilde{X}) \to S_*(\tilde{Y})$ is a chain map over $\mathbb{Z}[\pi_1(Y)]$ whence

$$C_*(\tilde{X}) \xrightarrow{\simeq} S_*(\tilde{X}) \xrightarrow{\tilde{f}_*} X_*(\tilde{Y}) \xleftarrow{\simeq} C_*(\tilde{Y})$$

So we obtain $\tilde{f}_* : C_*(\tilde{X}) \to C_* * (\tilde{Y})$, but defined only up to chain homotopy.

A cellular map $f : X \to Y$ is one with $f(X^n) \subset Y^n \ \forall\ n$. Then we obtain a unique $\tilde{f}_* : C_*(\tilde{X}) \to C_*(\tilde{Y})$ (unique up to covering translations).

Lemma 3.6. *If connected CW complex X is dominated by a finite CW complex K, then $C_*(\tilde{X})$, $S_*(\tilde{X})$ are dominated by a finitely generated free $\mathbb{Z}[\pi_1(X)]$-complex ($\tilde{X}$ = universal cover).*

Proof. $X \xrightarrow{f} K \xrightarrow{g} X$, $gf \simeq 1_x$. Wlg K connected. Let $G = \ker g_* : \pi_1 K \to \pi_1 X$. Let $\tilde{K}$ be covering of K ?? to G; let $\tilde{X}$ be universal cover of X. Lift f, g to $\tilde{f} : \tilde{X} \to \tilde{K}$, $\tilde{g} : \tilde{K} \to \tilde{X}$.

Lift $gf \simeq 1$ to get $\tilde{g}\tilde{f} \simeq$ covering translation of $\tilde{X}$, choose $\tilde{g}$ to make $\tilde{g}\tilde{f} \simeq 1_{\tilde{x}}$.

$\tilde{g}_* : C_*(\tilde{K}) \to C_*(\tilde{X})$; also $\tilde{f}_* : C_*(\tilde{X}) \to C_*(\tilde{K})$ and $\tilde{g}_*\tilde{f}_* \simeq 1_{C_*(\tilde{X})}$; so $C_*(\tilde{K})$ dominates $C_*(\tilde{X})$; hence also $S_*(\tilde{X})$. $C_*(\tilde{K})$ $f.g.$ free.

By Theorem 2.3, $C_*(\tilde{X}) \simeq f.g.$proj $\mathbb{Z}[\pi_1 X]$-complex E_*. Define wall invariant $\sigma(X) \subseteq \tilde{K}_0(\mathbb{Z}[\pi_1, X])$ to be $\sigma(E_*)$. By Theorem 2.5, $\sigma(X)$ depends only on homotopy type of X. By Theorem 1.9, $\sigma(X)$ doesn't depend on base point of X. $\square$

Theorem 3.7. *Let X be a connected CW complex, A_* a free $\mathbb{Z}[\pi_1 X]$-complex, and let $\varphi : A_* \to C_*(\tilde{X})$ be a chain equivalence, such that $\varphi_i : A_i \to C_i(\tilde{X})$ is bijective for $i \le 2$. Then $\exists$ a CW complex Z, a cellular homotopy equivalence $Z \xrightarrow{f} X$ and chain equivalence $\alpha : C_*(\tilde{Z}) \to A_*$ such that $\tilde{f}_* = \varphi\alpha$ and $\alpha : C_i(\tilde{Z}) \to A_i$ is bijective for all i.*

Proof. Suppose inductively that Z^{n-1}, $f|Z^{n-1} \to X$, $\alpha|C_*(\tilde{Z}^{n-1}) \to A_*$ already constructed, with f cellular, $\alpha : C_i(\tilde{Z}^{n-1}) \to A_i$ bijective for $i < n$ and $\tilde{f}_* = \varphi\alpha$.

Induction starts with $n = 3$, $Z^2 = X^2$, $f = \text{incl} : Z^2 \to X$; $\alpha = \varphi^{-1} : C_i(\tilde{Z}) \to A_i$ ($i \le 2$). Note that $\pi_1(X^2) \cong \pi_1(X)$, so that all complexes are over $\mathbb{Z}[\pi_1 X]$. f induces map $g : Z^{n-1} \to X^n$, α induces $\beta : C_*(\tilde{Z}^{n-1}) \to A_*^n$ the "n-skeleton" of A_*.

$$\begin{array}{ccccc}
C_*(\tilde{Z}^{n-1}) & \xrightarrow{1} & C_*(\tilde{Z}^{n-1}) & \xrightarrow{\varphi|A_*^{n-1}} & C_*(\tilde{X}^{n-1}) \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\tilde{g}_*} & & \\
A_*^n & \xrightarrow{\varphi|A_*^n} & C_*(\tilde{X}^n) & \longrightarrow & C_*(\tilde{X}^n)
\end{array}$$

Induces maps $(\varphi|A^n)_* : H_i(\beta) \to H_i(\tilde{g}_*)$, isomorphisms for $i < n$ (because $\varphi : A_* \to C_*(\tilde{X})$ was chain equivalent). But

$$H_i(\beta) = 0 \ \text{ for } \ i < n, \quad H_n(\beta) = A_n$$

$$\therefore H_i(\tilde{g}_*) = 0 \ \text{ for } \ i < n, \ \text{ get map } \ \theta : A_n \to H_n(\tilde{g}_*)$$

[Note that composition $A_n \xrightarrow{\theta} H_n(\tilde{g}_*) \to H_n(\tilde{X}^n, \tilde{X}^{n-1}) = C_n(\tilde{X})$ is just φ.]

By Hurewicz theorem applied to $\tilde{g} : \tilde{Z}^{n-1} \to \tilde{X}^{n-1}$

$$H_n(\tilde{g}_*) \cong \pi_n(\tilde{g}) \cong \pi_n(g) \ .$$

Pick basis $\{a_t\}_{t\in T}$ for A_n; we can represent $\theta(a_t) \in H_n(\tilde{g}_*)$ by the diagram

$$\begin{array}{ccc} Z^{n-1} & \longrightarrow & X^n \\ \uparrow{\scriptstyle v_t} & & \uparrow{\scriptstyle u_t} \\ \partial e^n & \longrightarrow & e^n \end{array}$$

Given T discrete topology, define $v : T \times \partial e^n \to Z^{n-1}$ by $v(t,x) = v_t(x)$. Let $Z^n = Z^{n-1} \cup_v (T \times e^n)$, define $f|T \times e^n \to X^n$ by $f(t,x) = u_t(x)$ extends g to a map $f : Z^n \to X^n$. Define $\alpha : C_n(\tilde{Z}^n) \to A_n$ by $\alpha(\tilde{e}^n_t) = a_t$, where $\tilde{e}^n_t$ is a lift of cell $t \times e^n$ in $\tilde{Z}^n$. (Choose lift to make this a chain map.) But $\tilde{f}_*(\tilde{e}^n_t)$ is represented by

$$\begin{array}{ccc} \tilde{X}^{n-1} & \xrightarrow{\text{inc}} & \tilde{X}^n \\ \uparrow{\scriptstyle \tilde{f}\tilde{v}_t} & & \uparrow{\scriptstyle \tilde{u}^n_t} \\ \partial e^n & \longrightarrow & e^n \end{array}$$

But this is $\tilde{f}_*(a_t) = \varphi(a_t) = \varphi\alpha(\tilde{e}^n_t)$ therefore $\tilde{f}_* = \varphi\alpha$. ☐

A group π is finitely presented if it is defined by a finite set of generators and relations $\{g_1, \dots g_k : f_1(\mathbf{g}) == r_\ell(\mathbf{g})\}$. Group H is a *retract* of G if $\exists$ homomorphisms $\varphi : H \to G$, $\psi : G \to H$ with $\psi\varphi = 1_H$.

Lemma 3.8. *A retract of a finitely presented group is finitely presented.*

Proof. G finitely presented ad $\{g_i : r_j(\mathbf{g}) = 1\}$.

$\varphi : H \to G$, $\psi : G \to H$ such that $\psi\varphi = 1_H$.

$\varphi\psi(g_i) = w_i(\mathbf{g})$ for some word w_i.

Let $L = \{g_i : r_j(\mathbf{g}) = 1,\ w_i(\mathbf{g}) = g_i\}$.

$\exists$ homomorphism $\pi : G \to L$, $\pi(g_i) = g_i$.

$\exists$ homomorphism $\theta : L \to L$, $\theta(g_i) = \psi(g_i)$.

[well defined since

$$\theta(w_i(\mathbf{g})) = \psi(w_i(\mathbf{g})) = \psi\varphi\psi(g_i) = \psi(g_i) = \theta(g_i) = w_i(\theta(\mathbf{g})) = w_i(\psi(\mathbf{g}))\ ]$$

θ is isomorphism with inverse $\pi\varphi : H \to L$

$$\pi\varphi\theta(g_i) = \pi\varphi\psi(g_i) = \pi w_i(\mathbf{g}) = w_i(\pi\mathbf{g}) = w_i(\mathbf{g}) \in L = g_i \in L$$

$\{\psi(g_i)\}$ is set of generators for H.

$$\theta\pi\varphi(\psi g_i) = \theta\pi w_i(\mathbf{g}) = \theta w_i(\pi\mathbf{g}) = \theta w_i(\mathbf{g}) = \theta(g_i) = \psi(g_i)$$

$\therefore\ \pi\varphi\theta = 1$, $\theta\pi\varphi = 1$, so $H \cong L$ which is infinitely presented. ☐

Lemma 3.9. *If connected CW complex X is dominated by a finite complex, then $X \simeq$ CW complex Y with Y^2 finite.*

Proof. Let $f : X \to K$, $g : K \to X$ be such that K is finite, $gf \simeq 1_X$. $f_* : \pi_1(X) \to \pi_1(K)$, $g_* : \pi_1(K) \to \pi_1(X)$ with $g_* f_* = 1$.

$\exists\, r_1, \ldots, r_\ell \in \ker g_* : \pi_1(K) \to \pi_1(X)$ such that

$$\pi_1(K)/\{r_1, \ldots, r_\ell\} \cong \pi_1(X)$$

Let $v_j : \partial e^2 \to K^1$ represent $r_j \in \pi_1(K)$. Let $u_j : e^2 \to X$ be a null-homotopy of gv_j. Define $Y^2 = K^2 \cup_{v_1} e_1^2 \cup \cdots \cup_{v_\ell} e_\ell^2$. Define $g|e_j^2 :\to X$ to be u_j. Then we have $g : Y^2 \to X$ induces bijection $g_* : \pi_1 Y^2 \to \pi_1 X$, and

$$\begin{tikzcd}
g_* : \pi_2 Y^2 \arrow[r] & \pi_2 X & \text{is onto} \\
\pi_2(K^2) \arrow[u] \arrow[r, "\text{onto}"'] & \pi_2(K) \arrow[u, "\text{onto}"', "g_*"] &
\end{tikzcd}$$

so $\pi_i(g) = 0$ for $i \leq 2$.

Suppose we have $Y^{n-1} \supset Y^2$ so 2-skeleton, and $g : Y^{n-1} \to X$ with $\pi_i(y) = 0$ for $i < n$. Let $\{\xi_t\}_{t \in T}$ be a (not ? finite) set of generators of $\pi_n(y)$. Represent ξ_t by

$$\begin{tikzcd}
Y^{n-1} \arrow[r, "z"] & X \\
\partial e^n \arrow[u, "v_t"'] \arrow[r] & e^n \arrow[u, "u_t"']
\end{tikzcd}$$

use v_t to attach n-cells to Y^{n-1}, giving Y'', u_t to extend g to $g : Y^n \to X$, so that $\pi_i(g) = 0$, $i \leq n$. Construct $Y^2 \subset Y^3 \subset Y^4 \subset \cdots$ with union Y, map $g : Y \to X$ with $\pi_*(g) = 0$. Therefore g is a homotopy equivalent. $\square$

A gap in the proof of Theorem 3.7

$$\alpha : C_*(\tilde{Z}^n) \longrightarrow A_* \ , \quad \text{chain map in} \quad \dim < n \ .$$

1. Commutes if lift $\tilde{e}_t^n$ of $t \times e^n$ is carefully chosen.

$$\begin{tikzcd}
A_n \arrow[rr, "\partial"] \arrow[d, "\alpha^{-1}"] \arrow[dr, "1.\ \theta"'] & & A_{n-1} \arrow[d, "\alpha^{-1}"] \arrow[from=1-1, to=2-3, phantom, "2."] \\
C_n(\tilde{Z}^n) \arrow[r] \arrow[d, equal] & H_n(\tilde{g}) \arrow[r] \arrow[d, "\partial"] & C_{n-1}(\tilde{Z}^n) \arrow[d, equal] \\
H_n(\tilde{Z}^n, \tilde{Z}^{n-1}) \arrow[r, "\partial"] \arrow[ur] & H_{n-1}(\tilde{Z}^{n-1}) \arrow[r] & H_{n-1}(\tilde{Z}^{n-1}, \tilde{Z}^{n-2})
\end{tikzcd}$$

2. commutes

$$\begin{array}{ccccc}
A_*^{n-2} & \subset & A_*^{n-1} & \subset & A_*^n \\
\downarrow{\scriptstyle \alpha^{-1}} & & \downarrow{\scriptstyle \alpha^{-1}} & & \downarrow{\scriptstyle \phi} \\
C_*(\tilde{Z}^{n-2}) & \subset & C_*(\tilde{Z}^{n-1}) & \xrightarrow{\tilde{g}} & C_*(\tilde{X}^n)
\end{array}$$

Homology sequence of triples

$$\begin{array}{ccc}
A_n & \xrightarrow{\partial} & A_{n-1} \\
\downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \alpha^{-1}} \\
H_n(\tilde{g}) & \longrightarrow & C_{n-1}(\tilde{Z}^n)
\end{array}$$

Theorem 3.10. *If the connected CW complex X is dominated by a finite complex K, and $\sigma(X) = 0$, then $X \simeq$ finite complex of dimension $\leq \max(4, \dim K)$.*

Remark. 4 can be replaced by 3. [CTC Wall; Finiteness conditions I]

Proof. By 3.9, we can assume X^2 finite.

By 3.6, 2.3, 2.6, $C_*(\tilde{X})$ is equivalent to a f.g. free complex E_*, by maps $f : C_*(\tilde{X}) \to E_*$, $g : E_* \to C_*(\tilde{X})$, inverse equivalences. Define complex A_* suitable for 3.7 as follows

$$A_*^2 = C_*(\tilde{X}^2) \qquad \text{– f.g. free}$$

$$A_n = E_n \ , \quad n \geq 4 \quad \text{– f.g. free.}$$

$$\begin{array}{cccccccccccccc}
\longrightarrow & A_5 & \longrightarrow & A_4 & \xrightarrow{\partial_4} & A_3 & \xrightarrow{\partial_3} & A_2 & \longrightarrow & A_1 & \longrightarrow & A_0 & \longrightarrow & 0 \\
& \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \\
\longrightarrow & E_5 & \longrightarrow & E_4 & \longrightarrow & E_3 & \longrightarrow & E_2 & \longrightarrow & E_1 & \longrightarrow & E_0 & \longrightarrow & 0
\end{array}$$

Let Q_* be mapping cone of $f|A_*^2 \to E_*$. This has $H_i(Q_*) = 0$ for $i \leq 2$

$$0 \longrightarrow Z_3(Q_*) \longrightarrow Q_3 \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow 0 \ \text{ exact.}$$

Define $A_3 = Z_3(Q_*)$ – f.g. proj.

$$A_3 = \{(y, z) \in A_2 \oplus E_3 : \partial y = 0 \ , \ \partial z = -fg\} \ .$$

Define

$$\partial_4(x) = (0, \partial x)$$

$$f_3(y, z) = z$$

$$\partial_3(y, z) = -y$$

A_* is a chain complex, and vertical maps induce homology isomorphisms. So

$f : A_* \longrightarrow E_*$ is chain equivalence.

$gf : A_* \longrightarrow C_*(\tilde{x})$ chain equivalence.

$gf|A_*^2 \simeq$ inclusion.

$\therefore \ gf \simeq \phi : A_* \longrightarrow C_*(\tilde{X})$ with $\phi|A_*^2$ bijection.

A_* is f.g. projective, free except in dim 3, $\sigma(A_*) = 0$. Enlarge A_3, A_4 to replace A_* by $\simeq$ equivalent free complex.

By 3.7, $X \simeq Y$ with $C_*(\tilde{Y}) \cong A_*$.

In particular, Y finite, $\dim Y = \max(\dim E_*, 4)$.

By 2.3, we can choose E_* such that $\dim E_* = \dim K$. □

Exercise. Use Theorem 3.7 and methods of 3.9, 3.10, to show: (Milnor): If X is simply connected CW complex, and $H_n(X; \mathbb{Z})$ has rank β_n and has τ_n "torsion coefficients", then $X \simeq CW$ complex with $\beta_n + \tau_{n-1} + \tau_n$ n-cells for each n.

Theorem 3.11. *Given ????*

`[NOTE: PAGE 39 AND 40 ARE IMPOSSIBLE TO READ. TOO FADED.]`

4. Torsion for CW complexes

π any group. A f.g. from $\mathbb{Z}[\pi]$-modules, $(a_1, \dots, a_k)$ basis. $(a'_1, \dots, a'_k)$ is *equivalent* to $(a_1, \dots, a_k)$ if $a'_i = \pm g_i a_i$ where $g_i \in \pi$ (so $\pm g_i \in \mathbb{Z}[\pi]$).

Chain complexes C_*, D_* (based), $f : C_* \to D_*$ chain equivalent. Then image of $\tau(f)$ in $Wh(\pi)$ depends only on equiv. classes of bases of C_*, D_*.

K finite CW complex. Equivalence class of basis of $C_n(\tilde{K})$ $(\tilde{e}^n_1, \dots, \tilde{e}^n_k)$ depends only on cell structure of K, not on choice of lifts $\tilde{e}^n_k$ or on orientation of cells.

$f : K \to L$ homotopy equivalence of finite CW complexes define $\tau(f) =$ image of $\tau(\tilde{f}_* : C_*(\tilde{K}) \to C_*(\tilde{L}))$ in $Wh(\tau)$.

This depends only on cell structures of K, L and homotopy class of f (by 2.8).

Theorem 4.1. *If $f : K \to L$, $g : L \to M$ use homotopy equivalences of finite CW complexes, then $\tau(gf) = \tau(g) + \tau(f) \in Wh(\pi_1 K = \pi_1 L = \pi_1 M = \pi)$.*

Problem. Is $\tau(f)$ a topological invariant of K, L, f? Yes, if K, L are complex manifolds.

X any CW complex. Complex X' is a subdivision of X if $|X'| = |X|$ and the interior of each cell in X' is contained in the interior of some cell in X.

Identity map $\chi : X \to X'$ is cellular.

Theorem 4.2. *$\chi : X \to X'$ is a simple homotopy equivalence, i.e. $\tau(\chi) = 0$.*

Proof. X finite CW complex. $\exists$ subcomplexes.

$$\phi = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_k = X$$

such that $X_i - X_{i-1}$ consists of just one cell. Let X_i' be subdivision of X_i induced by X'. Let $Y_i = X_i' \cup (\text{cells of } X - X_i)$.

Maps $X = Y_{-1} \xrightarrow{X} Y_0 \xrightarrow{X} Y_1 \xrightarrow{X} \cdots \xrightarrow{X} Y_k = X'$.

Enough to prove $X : Y_{i-1} \to Y_i$ is s.h.e. i.e. $\tau(X) = 0$. Choose lift $\tilde{e}$ for each cell e of X. If e' is a cell in X', int $e' \subset$ int e for some unique cell e in X. Choose lift $\tilde{e}'$ of e' so that int $\tilde{e}' \subset$ int $\tilde{e}$.

Exact sequence

$$0 \longrightarrow C_*(\tilde{Y}_{i-1}) \xrightarrow{\tilde{X}_*} C_*(\tilde{Y}_i) \longrightarrow D_* \longrightarrow 0 \quad (\text{defines } D_*) \tag{3}$$

Let $X_i - X_{i-1} = e_i^n$.

Then $\tilde{X}_*$ maps each cell $\tilde{e}$ of $\tilde{Y}_{i-1}$ to a cell of $\tilde{Y}_i$, except that $\tilde{X}_*(\tilde{e}_i^n) = \tilde{f}_1^n + \cdots + \tilde{f}_r^n$ where $\tilde{f}_1^n, \ldots, \tilde{f}_r^n$ are the n-cells of $\tilde{Y}_i$ with int $\tilde{f}_j^n \subset$ int $\tilde{e}_i^n$. Change basis of $C_n(\tilde{Y}_i)$ by replacing $\tilde{f}_1^n$ by $\tilde{f}_1^n + \cdots + \tilde{f}_r^n$ (leave other basis elements alone). This is an elementary operation, so it doesn't affect the torsion of $\tilde{X}_*$.

But now (3) is a broad exact sequence, so $\tau(\tilde{X}_*) = \tau(D_*)$.

Boundary maps of D_* have matrices with integer coefficients (by the choice of lifts ?? need to translate by an element of π).

$\therefore$ Torsion of D_* is in image of $\bar{K}(\mathbb{Z}) = 0$.

$\therefore$ $\tau(\chi) = 0$, as required. □

Corollary 4.3. *If $f : X \to Y$ is a homotopy equivalence of compact polyhedra, then $\tau(f)$ is well defined (i.e. independent of PL triangulations chosen for X, Y).*

Theorem 4.4. *Given finite CW complex K with fundamental group π, and element $\tau \in Wh(\pi)$, $\exists$ finite CW complex L and homotopy equivalence $f : K \to L$ with $\tau(f) = \tau$.*

Proof. Represent τ by a matrix $M \in GL(k, \mathbb{Z}[\pi])$. Let $Y = K \vee \bigvee_{i=1}^k x_i^n$, where ????. $p : Y \to K$ sends s_i^n to base point.

As in 3.11, $\pi_{n+1}(p) \cong \bigoplus_{i=1}^k \mathbb{Z}[\pi]$, one ?? for each s_i^n; let ξ_i be i^{th}.

Let $\phi : \pi_{n+1}(p) \to \pi_{n+1}(p)$ have matrix M. Represent image of $\phi(\xi_i)$ in $\pi_n(Y)$ by map $v_i : \partial e_i^{n+1} \to Y$. Use the v_i's to attach $e_1^{n+1}, \ldots, e_r^{n+1}$ to Y, giving complex $L \supset K$.

Then $C_*(\tau)$ has form

$$0 \to \pi_{n+1}(p) \xrightarrow{\phi} \pi_{n+1}(p) \xrightarrow{0} C_{n-1}(\tilde{K}) \xrightarrow{\partial} \cdots$$

By 2.3 and the Whitehead theorem, inclusion $K \subset L$ is homotopy equivalence,

$$0 \to C_0(\tilde{K}) \to C_*(\tilde{\tau}) \to (?????)$$

By 2.11, $\tau(f) - \tau(\phi) = \tau$. □

Let Δ^n be an n-simplex, but Δ_n be an ????. Let $\Lambda = \overline{\partial\Delta - \Delta_n}$. K finite CW complex, $f : \Lambda, \partial\Lambda \to K^{n-1}, K^{n-2}$.

Let $L = K \cup_f \Delta$; this is CW complex with cells of K and Δ_0^{n-1}; Δ^n.

Then $K \subset L$ is called an elementary expansion of dimension n, and a homotopy image is an elementary collapse.

Both are homotopy equivalences, and have zero torsion.

Example. There exist finite complex K, L, which are homeomorphic but don't have isomorphic subdivisions. Thus $\exists$ compact polyhedra $|K|$, $|L|$, which homeomorphic but not PL homeomorphic. (Hauptvermutung is false.)

Proof. Group π with $Wh(\pi) \neq 0$, e.g. C_5, π finitely presented.

$\exists$ finite simplicial complex X, with $\pi_1(X_1) \cong \pi$. By method of 4.4, $\exists$ finite simplicial complex $X_2 \supset X_1$ such that inclusion $X_1 \subset X_2$has torsion $\tau \neq 0$.

$\exists$ finite simplicial complex $X_3 \supset X_2$ such that $X_3 \searrow X_1$ (e.g. take k large, and extend $X_1 \to X)1 \times \Delta^k$ to an embedding $X_2 \to X_1 \times \Delta^k$ by general position). $\exists$ finite simplicial complex $X_4 \supset X_3$ such that $X_4 \searrow X_2$. Embed X_4 in some $\mathbb{R}^n$.

Let W_4 be a regular neighborhood of X_4 in $\mathbb{R}^n$.

Let W_i be a regular neighborhood of X_i in W_{i+1}, $i = 3, 2, 1$.

W_4 is a regular neighborhood of X_4, $X_4 \searrow X_2$. $\therefore$ W_4 is a regular neighborhood of X_2.

W_2 is a regular neighborhood of X_2, $W_2 \subset \operatorname{int} W_4$.

$\therefore \overline{W_4 = W_2} \cong \partial W_2 \times I$.

Similarly,

$$\overline{W_3 - W_1} \cong \partial W_1 \times I \ . \tag{4}$$

Let $V = \overline{W_2 - W_1}$, $V' = \overline{W_3 - W_2}$.

V is a cobordism from $M = \partial W_1$ to $N = \partial W_2$.

V' is a cobordism from N to $\partial W_3 \cong \partial W_1$ by (4). Now

$$V \cup V' \cong M \times I$$

$$\begin{aligned} V &\cong V \cup (V' \cup \overline{W_4 - W_3}\,) \\ &\cong (V \cup F') \cup (\,\overline{W_4 - W_3}\,) \\ &\cong \overline{W_4 - W_3} \end{aligned}$$

$$\therefore\ V' \cup V \cong V' \cup \overline{W_4 - W_3} \cong N \times I$$

V is an *invertible* cobordism. $M \hookrightarrow V$ has torsion τ. $\square$

Theorem 4.5. *If V is an invertible cobordism from M to N, then $V - N \cong M \times [0, \infty)$.*

Proof. Let V' = inverse of V. Let $U = V \cup_N V' \cup_M V \cup_N V' \cup_M \cdots$.

$$\begin{aligned} U &\cong (V \cup V') \cup (V \cup V') \cup \cdots \\ &\cong (M \times I) \cup (M \times I) \cup \cdots \\ &\cong M \times [0, \infty)\ . \end{aligned}$$

But

$$\begin{aligned} U &\cong V \cup (V' \cup V) \cup (V' \cup V) \cup \cdots \\ &\cong V \cup (N \times I) \cup (N \times I) \cup \cdots \\ &\cong V \cup_N (N \times [0, \infty))\ . \end{aligned}$$

$\exists$ collar neighborhood C of N in V, so $U \cong V - N$. $\square$

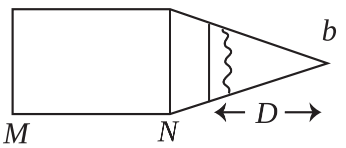
Take

$$\begin{aligned} K &= (M \times I \cup (\text{cone on } M \times I) \\ L &= V \cup (\text{cone on } N) \end{aligned}$$

Topologically,

$$\begin{aligned} K &= \text{1-pt compactification of } M \times [0, \infty) \\ L &= \text{1-pt compactification of } V \cup (N \times [0, \infty)) \\ \therefore\ K &\equiv L \text{ topologically.} \end{aligned}$$

Suppose K', L' are isomorphic subdivisions of K, L. Let a, b be vertices of the cones in K, L. Let $P = \overline{K' - st(a, K')}$, $Q = \overline{L' - st(b, L')}$. Then $M \times 0 \subset M \times I \subset P$, $M \subset V \subset Q$; and $(P, M \times 0) \cong (M \times I, M)$, $(Q, M) \cong (V, M)$. For



CN = cone on $N = \{tx + (1-t)b : x \in N,\ t \in I\}$.
Let $D = \{tx + (1-t)b : t \geq 1/2\}$.
$\exists$ pseudo radial homeomorphism $CN \to CN$ fixing N, b and taking $st(b, K')$ onto D. Extends to a PL homeomorphism $L \to L$ fixing M and taking Q onto $\overline{L - D} \cong V \cup (N \times I) \cong V$.

Similarly $(P, M \times 0) \cong (M \times I, M \times 0)$. Isomorphism $K' \xrightarrow{h} L'$ must take a onto b (for these are the only points with non-simply connected links).

$(n \geq 2 + \dim X_4)$.

$\therefore$ must take P onto Q by PL homeomorphism.

Now

$$\begin{tikzcd} M \arrow[r, phantom, "\subset"] \arrow[d, "h"'] & P \arrow[d, "h"] \\ M \arrow[r, phantom, "\subset"] & Q \end{tikzcd}$$

Vertical maps are PL homeomorphisms. Therefore they have zero torsion. $M \subset P$ has torsion zero and $M \subset Q$ has torsion $\tau \neq 0$. Contradiction Theorem 4.2.

Remark. Every invertible cobordism V from M to N is an h-cobordism.

Stallings proved that any h-cobordism V of dimension ≥ 5 is invertible. Therefore $V - N \cong M \times [0, \infty)$. s-cobordism is an h-cobordism in which $M < V$, $N \subset V$ are simple homotopy equivalences. Smale, Barden Mazur Stallings.

Theorem 4.6. *If V^n is an s-cobordism and $n \geq 6$, then $V \cong M \times I$.*

Exercise. M^n closed PL manifold, $n \geq 4$. Then, if $\tau \in Wh(\pi, M)$, $\exists$ an h-cobordism W o M with torsion $\tau(W, M) = \tau$ [e.g., take W = regular neighborhood of $M \cup$ suitable 2-complex in $M \times I$].

Theorem 4.7. *A cellular homotopy equivalence $f : K \to L$ between finite CW complexes has $\tau(f) = 0$ iff f can be factored into finitely many elementary expansions and collapses.*

Proof. From now on, "elementary collapse" means retraction $L \to K$ where $K \subset L$ is an elementary expansion.

Elementary expansions and collapses have zero torsion.

Converse. First note that $L \subset M_f$ is a composite of expansions. Put M_f^i = mapping cylinder of $f|K^i \to L$

$$L \subset M_f^0 \subset M_f^1 \subset \cdots \subset M_f^k = M_f \qquad (k = \dim K)\ .$$

$M_f^{i-1} \subset M_f^i$ is composite of elementary expansions of dimension $i+1$, one for each i-cell of K. So we can replace L by M_f, and $f : K \to L$ by an inclusion. Assume from now on that f is an inclusion. □

Lemma 4.8. *If $f : K \to L$ is a composite of elementary expansions and collapses, and $\phi : \partial e^n \to K^{n-1}$ is a map, then*

$$f \cup 1 : K \cup_\phi e^n \to L \cup_{f\phi} e^n \text{ is a composite of expansions and collapses.}$$

Proof. Enough to consider case when f is an elementary expansion or collapse. Expansion case is trivial, so suppose $f : K \to L$ is a collapse. $\exists$ cellular homotopy $H : 1 \simeq f$, rel 1.

Let $h = H\phi \cup 1 : (\partial e^n \times I) \cup (e^n \times 1) \to K \cup_\phi e^n$. Let $J = (K \cup_\phi e^n) \cup_h (e^n \times I)$, regard as CW complex with cells of K, $e^n \times 1$, $e^n \times 0$, $e^n \times I$.

$$K \cup_\phi e^n \subset J \text{ is elementary expansion (add cells } e^n \times 0,\ e^n \times I)$$

$$K \cup_{f\phi} e^n \subset J \text{ is elementary expansion (add cells } e^n \times 1,\ e^n \times I)$$

Now $L \cup_{f\phi} e^n \subset K \cup_{f\phi} e^n$ is elementary expansion. Hence result. □

Proof of Theorem 4.7. $f : K \to L$ inclusion, $\tau(f) = 0$. Assume inductively that $L - K$ has no cells of dimension $< r$. We modify L keeping K fixed so that $L - K$ has no cells of dimension $\leq r$.

Let e^r be an r-cells of $L - K$

$$\pi_r(L^{r+1}, K) \cong \pi_r(L, K) = 0 \ .$$

$\exists$ cellular homotopy $H : e^r \times I \to L$ such that $H_0 =$ inclusion, $H_1(e^r) \subset K$, $H_t|\partial e^r$ independent of t. Let $e^{r+2} = e^r \times I \times I$, $e^{r+1} = \overline{\partial(e^r \times I \times I) - (e^r \times I \times 0)}$, $h : e^r \times I \times 0 \to L$ induced by H. Let $M = L \cup_h e^{r+2}$: CW complex with cells of L together with e^{r+1}, e^{r+2}.

Now $K \cup e^r \cup e^{r+1}$ is a subcomplex of M, collapsing onto K. By repeated use of Lemma 4.8 (once for each cell of $M - (K \cup e^r \cup e^{r+1})$) we obtain a complex $L' \supset K$, obtained from L by elementary expansions and collapses, such that $L' \to K$ has fewer r-cells than $L - K$ (we have removed e^r, but introduced e^{r+2}). Repeat until $L - K$ has no r-cells, completing induction. Continue until $L - K$ has n-cells and $(n-1)$ cells only, with $n > \dim K$.

We show how to alter basis of $C_n(\tilde{L})$ by elementary matrix $1 + ae_{ij}$ ($a \in \mathbb{Z}[\pi, K]$). Let $\tilde{e}_i^n$, $\tilde{e}_j^n$ be n-cells of $\tilde{L}$. By Hurewicz theorem, $H_n(\tilde{L}^n, \tilde{L}^{n-1}) \cong \pi_n(\tilde{L}^n, \tilde{L}^{n-1})$. Therefore $\exists$ map $\varphi : e^n, \partial e^n \to L^n, L^{n-1}$, reps. class $\tilde{e}_j^n + a\tilde{e}_i^n$. $\exists$ homotopy $G : \partial e^n \times I \to L^n$ with

$$G_0 = \text{ attaching map of } e_j^n$$

$$G_1 = \varphi \partial e^n \to L \ .$$

Define $\psi = 1 \cup G : (e^n \times 0) \cup (\partial e^n \times I) \to L^n$. Let $M = L \cup_\psi (e^n \times I)$ with cells of L and $e^n \times 1$, $e^n \times I$. Then L expands to $L \cup_\psi (e^n \times I)$ which collapses onto $(L - e_j^n) \cup_\phi e^n$. This performs desired change of basis. Since $\tau(L \subset K) = 0$, we may expand (to increase chain groups) and then reduce matrix of $\partial : C_n(\tilde{L}, \tilde{K}) \to C_{n-1}(\tilde{L}, \tilde{K})$ to 1. (May also have to change lifts and orientation.) Let $\tilde{e}_i^n$ be an n-cell of L, so $\partial \tilde{e}_i^n = \tilde{e}_i^{n-1}$ in $H_{n-1}(\tilde{L}^{n-1}, \tilde{L}^{n-2} \cup \tilde{K})$. Let $\varphi : \partial e^n \to L$ be attaching map. Claim that φ is homotopic to map $\psi : \partial e^n \to L$ such that $\psi(\partial e^n) \cap (L - K) = e_i^{n-1}$ and $\psi|\psi^{-1}(\text{int } e_i^{n-1})_{1-1}$ and $\overline{\psi^{-1}(\text{int } e_i^{n-1})} \cong n-1$ cell. For let $\theta : \partial e_i^{n-1} \to K$ be attaching map of e_i^{n-1}. $\pi_{n-1}(L, K) = 0$, so $\exists$ homotopy $H : e_i^{n-1} \to K$

such that $H|\partial e_i^{n-1} = \theta$. Then

$$1 \cup H : \underbrace{e_i^{n-1} \cup_\partial e_i^{n-1}}_{\cong S^{n-1}} \to L$$

represents same element of $H_{n-1}(\tilde{L}^{n-1}, \tilde{k})$ as φ.

Therefore $1 \cup H$ represents same element of $\pi_{n-1}(\tilde{L}^{n-1}, \tilde{K})$ as φ.

$$\pi_{n-1}(\tilde{K}) \to \pi_{n-1}(\tilde{L}^{n-1}) \to \pi_{n-1}(\tilde{L}^{n-1}, \tilde{k})$$

Therefore φ represents same element of $\pi_{n-1}(\tilde{L}^{n-1})$ as $\psi = (1 \cup H) +$ (some element of $\pi_{n-1}(\tilde{K})$) and ψ has required properties.

By the trick used above for elementary change of basis, $L^{n-1} \cup_\psi e_i^n$ is obtained from $L^{n-1} \cup_\varphi e_i^n$ by elementary expansion and collapse. Also, $K \cup e_i^{n-1} \cup_+ e_i^n$ collapses to K, so we can reduce number of cells in $L - K$.

Continue until $L - K$ has no cells; then we have obtained K from L by elementary moves. □

5. Open Manifolds

X any Hausdorff space. An *end* of X is a collection $\mathcal{E}$ of non-empty open sets in X, such that

i) $U \in \mathcal{E} \Rightarrow U$ connected and $Fr(U)$ compact.
ii) $U, V \in \mathcal{E} \Rightarrow \exists W \in \mathcal{E}$ with $W \subset U \cap V$.
iii) $\cap\{\bar{U}; U \in \mathcal{E}\} = \emptyset$.
iv) $\mathcal{E}$ maximal w.r.t. i)–iii).

Example. $\mathbb{R}$ has just two ends, namely

$$\{(a,\infty); a \in \mathbb{R}\} \quad \text{and} \quad \{(-\infty, b); b \in \mathbb{R}\}\ .$$

Lemma 5.1. *Suppose $\mathcal{E}'$ satisfies i)–iii), and $A \subset X$ has compact frontier. Then $\exists\ U \in \mathcal{E}'$ such that either $\bar{U} \cap A = \emptyset$ or $\bar{U} \subset A$.*

Proof. Since $Fr(A)$ is compact, and $\bigcap_{U \in \mathcal{E}'}(\bar{U} \cap Rf(A)) = \emptyset$, $\exists\ U_1, \dots, U_k \in \mathcal{E}'$ such that $\bar{U}_1 \cap \cdots \cap \bar{U}_k \cap Fr(A) = \emptyset$. By ii), $\exists\ U \in \mathcal{E}'$ such that $U \subset U_1 \cap \cdots \cap U_k$, so $\bar{U} \subset \bar{U}_1 \cap \cdots \cap \bar{U}_k$, so $\bar{U} \subset X - Fr(A)$. Since U connected, $\bar{U}$ connected, so $\bar{U} \subset A$ or $X - A$. □

Corollary 5.2. *If $\mathcal{E}'$ satisfies i)–iii), then $\mathcal{E}'$ is contained in a unique end of X.*

Proof. Let $\mathcal{E}$ be the collection of all non-empty connected open sets V such that $V \supset U$ for some $U \in \mathcal{E}'$ and $Fr(V)$ compact. Then $\mathcal{E}$ satisfies i)–iii).

Suppose $\mathcal{E}'' \supset \mathcal{E}'$ also satisfies i)–iii). Then if $V \in \mathcal{E}''$, $\exists\ U \in \mathcal{E}'$ such that $\bar{U} \cap V = \emptyset$ or $\bar{U} \subset V$. $\bar{U} \cap V$ impossible by i,ii) so $\bar{U} \subset V$, so $V \in \mathcal{E}$. So $\mathcal{E}'' \subset \mathcal{E}$, so $\mathcal{E}$ is unique and containing $\mathcal{E}'$.

A *neighborhood* of $\mathcal{E}$ is a set-N containing some $U \in \mathcal{E}$. □

Corollary 5.3. *Distinct ends of X have disjoint neighborhoods.*

Proof. Suppose $\mathcal{E}, \mathcal{E}'$ are ends without disjoint neighborhoods. Choose $U \in \mathcal{E}$, $\exists\ V \in \mathcal{E}'$ such that $\bar{V} \subset U$ (by 5.1). By maximality of $\mathcal{E}'$, $U \in \mathcal{E}'$ so $\mathcal{E} \subset \mathcal{E}'$. Similarly $\mathcal{E}' \subset \mathcal{E}$. Therefore $\mathcal{E} = \mathcal{E}'$. □

Definition. A space X is σ-compact if it is the union of countable many compact subspaces.

Theorem 5.4. *Let X be locally connected, locally compact, connected, σ-compact, Hausdorff. Then X has an end iff X is not compact.*

Proof. A compact space has no ends, by ii)–iii) for ends.

Conversely, $X = \cup C_i$ where C_i is compact, $C_1 \subset C_2 \subset \cdots$, X non-compact.

X locally compact, so C_i has compact neighborhood D_i in X. Every component V of $X - C_i$ is open (X is locally connected), and meets D_i (therefore X connected if $V \cap D_i = \emptyset$, then $\bar{V} - V \subset C_i$ and $\bar{V} \subset X - C_i$; therefore $\bar{V} - V = \emptyset$, so V open and closed in X, contradiction).

$Fr(D_i)$ compact, so covered by finitely many components $V_i^1, \ldots, V_i^k$ of $X - C_i$;

$X = D_i \cup V_i^1 \cup \cdots \cup V_i^k$

$\therefore$ some V_i^j has non-compact closure.

Choose inductively $U_1, U_2, \ldots$ such that U_i is a component of $X - C_i$, $\bar{U}_i$ non-compact, $U_i \subset U_{i-1}$. $Fr(U_i) \subset C_i$ because X connected, therefore $\{U_1, U_2, \ldots, \} = \mathcal{E}'$ satisfies i)–iii), so contained in an end of X. □

Examples. i) $\mathbb{R}^n$, $n \geq 2$, has just one end.

$B^n_\lambda =$ closed ball radius λ.

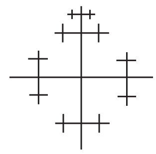
$\{\mathbb{R}^n - B^n_\lambda : \lambda \in \mathbb{R}\}$ defines an end $\mathcal{E}$ of $\mathbb{R}^n$.

If $\mathcal{E}'$ is another end, $\exists$ disjoint neighborhoods $U \in \mathcal{E}$, $V \in \mathcal{E}'$.

$Fr(U) \cup Fr(V)$ is compact, $\mathbb{R}^n - (Fr(U) \cup Fr(V))$ has at least two unbounded components U, V, which is impossible.

An end $\mathcal{E}$ is *isolated* if it has a neighborhood U which is not a neighborhood of any other end. It follows that $\bar{U}$ has just one end.

Example. The universal cover of $S^1 \vee S^1$ has infinitely many ends, none of which is isolated.



An *open* manifold is a non-compact manifold without boundary.

If W is an open manifold, a *completion* of W is a homeomorphism (PL) from W onto $\bar{W} - \partial\bar{W}$ ($=$ int $\bar{W}$) where $\bar{W}$ is a compact PL manifold.

Theorem 5.5. *An open PL manifold has a completion iff it has finitely many ends, each of which has a collar.*

A collar *of an end* $\mathcal{E}$ *of* W *is a submanifold* V *of* W *such that* int $V \in \mathcal{E}$, $V \cong \partial V \times [0, \infty)$.

Proof. Suppose W homeomorphic to int $\bar{W}$, where $\bar{W}$ is compact.

Let $M_1, \dots, M_k$ be components of $\partial\bar{W}$. Let $\gamma_i : M_i \times I \to \bar{W}$ be a collar neighborhood of M_i in $\bar{W}$ such that $\operatorname{im}\gamma_i \cap \operatorname{im}\gamma_j = \emptyset$ if $i \neq j$.

Then $\{\gamma_i(M_i \times (a, 1)) : a \in (0, 1)\}$ defines an end $\mathcal{E}_i$ of W. $\mathcal{E}_i, \dots, \mathcal{E}_k$ are the only ends of W. If $\mathcal{E}$ were another, with neighborhood $U \in \mathcal{E}$ disjoint from $\gamma_i(M_i \times (a_i, 1))$, so $\bar{U} =$ closure of U in int $\bar{W} \subset$ int $\bar{W} - \cup\gamma_i(M_i \times (a_i, 1))$ which is compact.

Converse by similar argument. $\square$

A 0-*neighborhood* of an end $\mathcal{E}$ of W is a submanifold V of W such that int $V \in \mathcal{E}$, V has just one end, V is closed in W, and ∂V is connected.

Theorem 5.6. *Any isolated end of an open manifold* W *has a* 0*-neighborhood.*

Proof. $\exists$ neighborhood $U \in \mathcal{E}$ which isn't a neighborhood of any other end. $Fr(U)$ is compact.

$\exists$ compact polyhedron $K \overset{\text{PL}}{\hookrightarrow} W$ which is a neighborhood of $Fr(U)$. Now let N be a regular neighborhood of K in W. Now $\overline{U - N}$ has a non-compact component V [because $\overline{U - N}$ has an end]. V is connected, $Fr(V) \subset N$ (because U is connected), so $Fr(V)$ compact. V is a PL submanifold with $\partial V = Fr(V)$, and it is a neighborhood of $\mathcal{E}$.

Let M_1, M_2 be two components of ∂V. $\exists$ PL arc (embedded) $A \subset V$ with ends in M_1, M_2. [Possible for $\dim W \geq 3$ by general position for $\dim W \leq 2$, easy.]

Now let H be a regular neighborhood of A in V.

Replace V by $\overline{V - H}$, which is still a neighborhood of $\mathcal{E}$, contained in U, PL manifold, connected; but with fewer boundary components than V.

Repeat the process until we get a 0-neighborhood of $\mathcal{E}$. $\square$

Remark. This process gives a 0-neighborhood in U, so it gives arbitrarily small 0-neighborhoods of $\mathcal{E}$.

Inverse sequence of groups $\cdots \xrightarrow{f_4} G_3 \xrightarrow{f_3} G_2 \xrightarrow{f_2} G_1$ is *stable* if $\exists$ a subsequence $\cdots \xrightarrow{G_{n_3}} g_{n_2} \xrightarrow{g_{n_2}} G_{n_1}$ such that g_{n_i} induces an isomorphism $\operatorname{im} g_{n_{i+1}} \to \operatorname{im} g_{n_i}$, $\forall i$.

Then $\varprojlim G_n$ has $\operatorname{im} g_{n_i}$ for inverse limit. Note: $g_{n_i} = f_{n_{i-1}+1} f_{n_{i-1}+2} \cdots f_{n_i}$.

Let $\mathcal{E}$ be an end of X. π_1 is *stable at* $\mathcal{E}$ if $\exists$ path-connected neighborhoods $U_1 \supset U_2 \supset U_3 \supset \cdots$ of $\mathcal{E}$ with $\bigcap \bar{U}_i = \emptyset$, with base points $u_i \in U_i$, paths p_i from u_i to u_i (in U_i) such that

$$\cdots \longrightarrow \pi_1(U_3, u_3) \longrightarrow \pi_1(U_2, u_2) \longrightarrow \pi_1(U_1, u_1)$$

is stable.

Theorem 5.7. *If π_1 is stable at $\mathcal{E}$, and $V_1 \supset V_2 \supset \cdots$ is sequence of path-?? neighborhoods of $\mathcal{E}$ with $\bigcap \bar{V}_i = \emptyset$, (and with base points and paths), then*

$$\longrightarrow \pi_1(V_3) \longrightarrow \pi_1(V_2) \longrightarrow \pi_1(V_1)$$

is stable, with inverse limit equal to $\varprojlim \pi_1(U_i)$.

Proof. Suppose wlog that $\longrightarrow \pi_1(U_3) \xrightarrow{f_3} \pi_1(U_2) \xrightarrow{f_2} \pi_1(U_1)$ has f_n inducing an isomorphism $\operatorname{im} f_{n+1} \cong \operatorname{im} f_n$. Choose $V_{n_1} \subset U_1$, $U_{r_1} \subset V_{n_1}$, $V_{n_2} \subset U_{r_1}$, $U_{r_2} \subset V_{n_2}$, etc..

Choose paths joining the base points. Have diagram

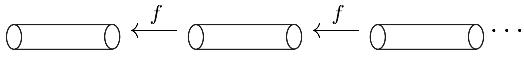
$$\begin{tikzcd}
\pi_1(U_{r_4}) \arrow[r, "h_4"] \arrow[d] & \pi_1(U_{r_3}) \arrow[r, "h_3"] \arrow[d] & \pi_1(U_{r_2}) \arrow[r, "h_2"] \arrow[d] & \pi_1(U_{r_1}) \arrow[d] \\
\pi_1(V_{n_4}) \arrow[r, "g_4"'] \arrow[ur] & \pi_1(V_{n_3}) \arrow[r, "g_3"'] \arrow[ur] & \pi_1(V_{n_2}) \arrow[r, "g_2"'] \arrow[ur] & \pi_1(V_{n_1})
\end{tikzcd}$$

Then $\operatorname{im} h_3 h_4 \longrightarrow \operatorname{im} g_3 g_4 \longrightarrow \operatorname{im} h_2$ whose composite is an isomorphism. But $\operatorname{im} h g_3 g_4 \to \operatorname{im} h_2$ is 1–1. Therefore $\operatorname{im} h_3 h_4 \longrightarrow \operatorname{im} g_3 g_4$ is an isomorphism. So $\longrightarrow \pi_1(V_{n_6} 0 \xrightarrow{g_5 g_6} \pi_1(V_{n_4}) \xrightarrow{g_3 g_4} \pi_1(V_{n_2})$ has same inverse limit as $\longrightarrow \pi_1(U_{r_6}) \xrightarrow{h_5 h_6} \pi_1(U_{r_4}) \xrightarrow{h_3 h_4} \pi_1(U_{r_2})$ so $\underleftarrow{\ }\lim\ \pi_1(U_i) = \underset{\leftarrow}{l\,im}\ \pi_1(V_i)$. $\square$

An end $\mathcal{E}$ of X is *tame* if π_1 is stable at $\mathcal{E}$, and $\mathcal{E}$ has arbitrarily small open neighborhoods dominated by finite CW complexes, and $\mathcal{E}$ is isolated.

Examples. 1) Let $f : S^1 \to S^1$ be squaring map

$$X = S^1 \times I \underset{f}{\cup} S^1 \times I \underset{f}{\cup} S^1 \times I \cup \cdots$$

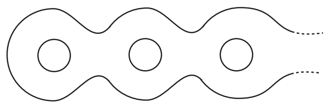


just one end. Let $U_i = X-$ union of first i $S^1 \times I$'s. $\simeq X \simeq S^1$ is dominated by a finite complex.

But π_1 isn't stable at $\mathcal{E}$.

$$\cdots \pi_1(U_3) \longrightarrow \pi_1(U_2) \longrightarrow \pi_i(U_1) \text{ is the same as } \cdots \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$$

2) $X = X^2 \times S^2 \# S^2 \times S^2 \# \cdots$



Just one end, with arbitrarily small simply connected neighborhoods, therefore π_1 stable.

No neighborhood of end is dominated by a finite CW complex.

3) If W is an open manifold with a completion, then all ends of W are tame

$$V = \partial V \times [0, \infty) \,. \qquad \text{Look at } \partial V \times [n, \infty) \,.$$

If $\mathcal{E}$ is an end of X at which π_1 is stable. A 1-*neighborhood* of $\mathcal{E}$ is a 0-neighborhood V with extra properties:

1) $\pi_1(\partial V) \cong \pi_1(V)$ (induced by inclusion)
2) The natural map $\pi_1(\mathcal{E}) \to \pi_1(V)$ is an isomorphism.

Theorem 5.8. *Suppose $\mathcal{E}$ is an isolated tame end of an open manifold W. If* $\dim W \geq 5$, *then $\mathcal{E}$ has a* 1*-neighborhood.*

Proof. First show that $\pi_1\mathcal{E}$ is finitely presented. Choose 0-neighborhoods $V_1 \supset V_2 \supset \cdots$ of $\mathcal{E}$ with $\cap \bar{V}_n = \emptyset$ and such that $g_n : \pi_1(V_n) \to \pi_1(V_{n-1})$ induces iso $\operatorname{im}(g_{n+1}) \to \operatorname{im}(g_n)$.

$\exists$ neighborhood U of $\mathcal{E}$, $U \subset V_1$, and U dominated by finite complex K.

$\exists$ n such that $V_n \subset U$: we have

$$\operatorname{im} g_{n+1} \longrightarrow \pi_1(U) \longrightarrow \operatorname{im} g_2 \subset \pi_1(V_1)$$

Composite is an isomorphism, so $\pi_1(\mathcal{E}) \cong \operatorname{im}(g_2)$ is a retract of $\pi_1(U)$, which is a retract of finitely presented group $\pi_1(K)$. By Lemma 3.8, $\pi_1(\mathcal{E})$ is finitely presented.

Let E_n be the image of map $\pi_1(\mathcal{E}) \to \pi_1(V_n)$.

Seek $V^1 \subset \operatorname{Int} V_3$ such that $\pi_1(\partial V^1) \to E_2$ is onto.

E_2 is finitely generated: represent finite set of generators by arcs $A_1, \ldots, A_k$ embedded in V_3 with ends in ∂V_4. By general case pos^n, $A_i \cap \partial V_4$ is finite set of points.

Subdivide $A_1, \ldots, A_k$ into arcs $B'_1, \ldots, B'_\ell$ such that $B'_j \cap \partial V_4 = \partial B'_j$; say $B'_1, \ldots, B'_p$ in V_4 and $B'_{p+1}, \ldots, B'_\ell \subset \overline{V_3 - V_4}$.

Adjust B'_j slightly to obtain disjoint arcs $B_1, \dots, B_\ell$. Let $H_1, \dots, H_p$ be regular neighborhoods of $B_1, \dots, B_p$ in V_4. Let $H_{p+1}, \dots, H_\ell$ be regular neighborhoods of $B_{p+1}, \dots, B_\ell$ in $\overline{V_3 - V_4}$. $\square$

Replace V_4 by $V' = \overline{V_4 - H_1 \cup \cdots \cup H_p} \cup H_{p+1} \cup \cdots \cup H_\ell$. This has the desired effect: $\pi_1(\partial V') \to \pi_1(V_3) \to E_2$ is onto.

Now we modify V' further to make $\pi_1(\partial V') \xrightarrow{\varphi} E_2$ an isomorphism. [It will then be a 1-neighborhood.]

Lemma 5.9. *Let π, E be finitely presented groups and let $\varphi : \pi \to E$ be an epimorphism. Then* $\ker \varphi$ *is the normal closure of a finite subset of π.*

Proof. Let $\{g_i; r_j(\mathbf{g}) = 1\}$, $\{h_i : s_j(\mathbf{h}) = 1\}$ be finite presentation of π, E. $\exists$ words $w_i \to v$, $\varphi(g_i) = w_i(\mathbf{h})$. Since φ is onto, $\exists$ words v_i such that $h_i = \varphi(v_i(\mathbf{g})) = v_i(\varphi(\mathbf{g}))$. Now

$$\begin{aligned} E &\cong \{h_i : s_j(\mathbf{h}) = 1,\ r_j(\mathbf{w}(\mathbf{h})) = 1,\ h_i = v_i(\mathbf{w}(\mathbf{h}))\} \\ &\cong \{h_i, g'_i : s_j(\mathbf{h}) = 1,\ g'_i = w_i(\mathbf{h}),\ r_j(\mathbf{g}') = 1,\ h_i = v_i(\mathbf{g}')\} \\ E &\cong \{g'_i : s_j(\mathbf{v}(\mathbf{g}')) = 1,\ r_j(\mathbf{g}') = 1,\ g'_i = w_i(\mathbf{v}(\mathbf{g}'))\} \end{aligned}$$

$\varphi : \pi \to E$ has $\varphi(g_i) = w_i(\mathbf{h}) = g'_i$, so $\ker \varphi$ is normal closure of $\{s_j(v(\mathbf{g}'))\} \cup \{g_i^{-1} w_i(\mathbf{v}(\mathbf{g}))\}$ as required. $\square$

So $\varphi : \pi_1(\partial V') \to E_2$ onto, $\ker \varphi =$ normal closure of finite set $\{z_1, \dots, z_k\}$. Represent z_i by embedded circle s_i in $\partial V'$. Since $\varphi(z_1) = 0$, S bounds a disc D in V_2. By general position ($\dim W \geq 5$), we can suppose D embedded, and int $D \cap \partial V' =$ finite union of circles.

Suppose first that int $D \cap \partial V' = \emptyset$, so $D \subset V'$ or $D \subset \overline{V_2 - V'}$. Let H be a regular neighborhood of D in V' or $\overline{V_2 - V'}$. Replace V' by $V'_1 = \overline{V' - H}$ or $V' \cup H$

$$\begin{tikzcd} \pi_1(\partial V') \arrow[r] \arrow[dr] & \pi_1(((\partial V') \cup H) - D) \arrow[d] & \pi_1(\partial V'_1) \arrow[l, "j_*"'] \arrow[dl] \\ & \pi_1(V_2) & \end{tikzcd}$$

Now j_* is composite

$$\begin{tikzcd} \pi_1(\partial V'_1) \arrow[r, "\simeq"] & \pi_1(((\partial V') \cup H) - D) \arrow[d, "\wr"] \\ & \pi_1((\partial V') \cup H) \end{tikzcd}$$

isomorphism since $\dim H = \dim W \geq 5$, $\dim D =$?. So j_* is an isomorphism. So $\pi_1(\partial V'_1) \cong \pi_1(\partial V')$ (normal closure of ?) so we have killed z_1. Describe this process as swapping the disc D across ∂V. In general, (int $D) \cap \partial V'$ = finite union of circles S_1 and S_ℓ. S_i bounds a disc D_i in int D. Label S_i so that $D_i \subset D_j \Rightarrow i \leq j$.

Swap D_1 across $\partial V'$; this reduces the number of interior components of (int $D) \cap \partial V'$. Repeat the process until (int $D) \cap \partial V' = \emptyset$; now swap D across $\partial V'$, killing z_1. Repeat to kill $z_2, \ldots, z_k$; then $\varphi : \pi_1(\partial V') \to E_2$, is isomorphism. Therefore $\varphi : \pi_1(\partial V') \to E_1$ is also isomorphism ($\pi_1(V_2) \to \pi_1(V_1)$ induces iso $E_2 \to E_1$). Therefore $\pi_1(V') \to \pi_1(V_2) \to \pi_1(V_1)$ maps onto E_1. Suppose $z \in$ kernel of $\psi : \pi_1(V') \to \pi_1(V_1)$. Represent z by a circle S in V'. S bounds a disk D in V_1 : by general position, embedded with $D \cap \partial V' = S_1 \cup \cdots \cup S_\ell$ (circles).

Let S_1 be innermost circle, bounding disc $D_1 \subset D$. $S_1 \subset \partial V'$ is null-homotopic in V_1. Since $\pi_1(\partial V') \to \pi_1(V_1)$ is 1–1, S_1 is null homotopic in $\partial V'$. Let D_1' be a small disc neighborhood of D_1 in D, not meeting $S_2, \ldots, S_k$. $\partial D_1' \subset V'$ or $\overline{V_1 - V'}$; use the null-homotopy of S_1 in $\partial V'$ to span $\partial D_1'$ by a disc D_1'' in V' or $\overline{V_1 - V'}$ by general position D_1'' is embedded and disjoint from $\partial V'$. Replace D by $\overline{D - D_1'} \cup D_1''$, which meets $\partial V'$ in fewer components than D. Repeat until $D \cap \partial V'$ is empty; then S bounds disc D in V'. Therefore S is null-homotopic in V', so $z = 0$. So $\psi : \pi_1(V') \to \pi_1(V_1)$ is 1–1. Therefore $\pi_1(V') \to E_1$ is isomorphism. But $\pi_1(\mathcal{E}) \to \pi_1(V') \to E_1$ is an isomorphism. Therefore $\pi_1(\mathcal{E}) \to \pi_1(V')$ is isomorphism, and $\pi_1(\partial V') \to \pi_1(V')$ is isomorphism. Therefore v' is 1-neighborhood of $\mathcal{E}$; in fact $\exists$?? small 1-neighborhoods.

$\mathcal{E}$ tame end of W. $\pi = \pi_1(\mathcal{E}) \cong \pi_1(\partial V) \cong \pi_1(V)$ for any 1-neighborhood of $\mathcal{E}$. $\tilde{V}, \partial\tilde{V}$ will be universal coverings, $C_* =$?? chain group.

Lemma 5.10. *If V is a sufficiently small 1-neighborhood of $\mathcal{E}$, then $C_*(\tilde{V}, \partial\tilde{V})$ is homotopy euqivalent to a f.g. projective complex over $\mathbb{Z}[\pi]$.*

Proof. $\mathcal{E}$ tame, so $\exists$ open path-? neighborhood U of $\mathcal{E}$ which is dominated by a finite complex. Let V be any 1-neighborhood with $\bar{V} \subset U$. Let $X = \overline{U - V}$ in U, so $U = X \cup V$, $X \cap V = \partial V$ (all CW complexes)

$$C_*(\tilde{U}, \partial\tilde{V}) \simeq C_*(\tilde{V}, \partial\tilde{V}) \oplus C_*(\tilde{X}, \partial\tilde{V}) \quad \text{(by ?? and homotopy)}$$

Therefore $C_*(\tilde{V}, \partial\tilde{V})$ is dominated by $C_*(\tilde{U}, \partial\tilde{V})$.

U is dominated by finite complex, so by 3.6, $C_*(\tilde{U})$ is equivalent to a f.g. projective complex, say

$$f : C_*(\tilde{U}) \xrightarrow{\simeq} D_* \,.$$

∂V is a finite complex, so $C_*(\partial\tilde{V})$ is equivalent to a f.g. free complex, say

$$g : C_*(\partial\tilde{V}) \xrightarrow{\simeq} E_* \ .$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_*(\partial\tilde{V}) & \longrightarrow & C_*(\tilde{U}) & \longrightarrow & C_*(\tilde{U},\partial\tilde{V}) & \longrightarrow & 0 \\ & & {\scriptstyle g}\big\downarrow & & {\scriptstyle f}\big\downarrow & & & & \\ & & E_* & \underset{\varphi}{\longrightarrow} & D_* & & & & \end{array}$$

commutes up to homotopy for suitable φ. It follows that $C_*(\tilde{U},\partial\tilde{V}) \simeq$ mapping ?? of $\varphi(E?)$ which is f.g. projective.

$C_*(\tilde{V},\partial\tilde{V})$ is dominated by $C_*(\tilde{U},\partial\tilde{V})$, hence by f.g. projective complex. Therefore by Theorem 2.3, $C_*(\tilde{V},\partial\tilde{V})$ is equivalent to a f.g. projective complex. □

Definition. A *k-neighborhood* of end $\mathcal{E}$ of open manifold W is a 1-neighborhood V such that $H_i(\tilde{V},\partial\tilde{V}) = 0$ for $i \le k \ge 2$.

Theorem 5.11. *A tame end $\mathcal{E}$ of a manifold W of dimension $n \ge 5$ has arbitrarily small $(n-3)$-neighborhoods.*

Proof. Suppose inductively that $\mathcal{E}$ has arbitrarily small $(k-1)$-neighborhoods. Start with $k = 2$; suppose $k \le n-3$. Let V be a $(k-1)$-neighborhood.

$C_*(\tilde{V},\partial\tilde{V})$ is equivalent to a f.g. projective complex, say E_*. Since $H_i(E_*) = 0$, $i < k$, $\exists$ exact sequence

$$0 \longrightarrow Z_k(E_*) \longrightarrow E_k \longrightarrow E_{k-1} \longrightarrow \cdots \longrightarrow E_0 \to 0 \ .$$

Therefore $Z_k(E_*)$ is f.g. projective (as in 2.3). Therefore $H_i(E_*) \cong H_k(\tilde{V},\partial\tilde{V})$ is f.g. Let $\{x_1,\dots,x_m\}$ be finite set of generators.

Lemma 5.12. *Let V be a $(k-1)$-neighborhood of end $\mathcal{E}$, and suppose $\mathcal{E}$ has arbitrarily small $(k-1)$-neighborhoods. Then any element of $H_k(\tilde{V},\partial\tilde{V})$ can be represented by a PL embedded disc $(D^k,\partial D^k) \subset (V,\partial V)$, provided $k \le n-3$.*

Completion of proof of 5.11. Represent x_1 by an embedded disc $(D^k,\partial D^k) \subset (V,\partial V)$. Let H be a regular neighborhood of D^k in V, and repalce V by $V' = \overline{V - H}$.

V' is still a 1-neighborhood, for $\pi_1(V') \cong \pi_1(V - D) \underset{n-k\ge 3}{\cong} \pi_1(V)$

$$\pi_1(\partial V') \cong \pi_1((\partial V \cup H) - D) \underset{n-k\ge 3}{\cong} \pi_1(\partial V) \cong \pi_1(V) \cong \pi_1(V')$$

Homology exact sequence of $(\tilde{V}, \partial\widetilde{(V)\cup H}, \partial\tilde{V})$ gives

$$H_i(\widetilde{(\partial V)\cup H},\partial\tilde{V}) \longrightarrow H_i(\tilde{V},\partial\tilde{V}) \longrightarrow H_i(\tilde{V}',\partial\tilde{V}') \longrightarrow H_{i-1}(\widetilde{\partial V)\cup H},\partial\tilde{V})$$

Therefore $H_i(\tilde{V}', \partial\tilde{V}') \simeq 0$ for $k < k$, so V' is $(k-1)$-neighborhood. $H_k(\tilde{V}, \partial\tilde{V}) \to H_k(\tilde{V}', \partial\tilde{V}')$ is onto, and ??. Repeat process to kill off $x_2, \dots, x_r$; we finish with a k-neighborhood of $\mathcal{E}$. Continue to get an $(n-3)$-neighborhood. □

Proof of Lemma. Represent $x \in H_k(\tilde{V}, \partial\tilde{V})$ by a map $\varphi : D^k, \partial D^k \longrightarrow V, \partial V$ by the Hurewicz Theorem. Image of φ is compact, so $\exists$ small $(k-1)$-neighborhood $V' \subset V$ so that $\operatorname{im}\varphi \subset V - V'$. Then $x \in$ image of $\psi : H_k(\widetilde{V - V'}, \partial\tilde{V}) \longrightarrow H_k(\tilde{V}, \partial\tilde{V})$, say $x = \psi(y)$. Let $U = \overline{V - V'}$, $y \in H_k(\tilde{U}, \partial\tilde{V})$. $\partial\tilde{V} \subset \tilde{V}$, $\tilde{U} \subset \tilde{V}$ induce isomorphisms of homology up to dimension $k-2$ (sinc V, V' are $(k-1)$-neighborhoods). Therefore $\partial\tilde{V} \subset \tilde{U}$ induces homology ?? in dimensions $\leq k-2$. Therefore $H_i(\tilde{U}, \partial\tilde{V}) = 0$ for $i \leq k-2$. Take handle decomposition of U based on ∂V. We can remove 1-handles, and cancel handles of dimension $\leq k-2$, so there are no handles of dim $\leq \max(1, k-2)$.

Let X = regular neighborhood of union of $(k-1)$-handles in U. Let $Y = \overline{U - X}$, let $Z = X \cap Y$. Let $\bar{y}$ = image of y in $H_k(\tilde{U}, \tilde{X}) \cong H_k(\tilde{Y}, \tilde{Z})$. Let $h_1, \dots, h_r$ be the k-handles in Y. Let $\eta_1, \dots, \eta_r$ be the hommology classes in $H_k(\tilde{Y}, \tilde{Z})$ represented by $h_1, \dots, h_r$. Wlog $\eta_r = 0$ (otherwise introduce irrelevant k and $k+1$ which cancel; then irrelevant k-handle represents 0 in $H_k(\tilde{Y}, \tilde{Z})$. Then $\eta_1, \dots, \eta_r$ generate $H_k(\tilde{Y}, \tilde{Z})$ as $\mathbb{Z}[\pi]$-module.

Let $\bar{y} = \sum_{i=1}^{r} \rho_i \eta_i$ ($p \in \mathbb{Z}[\pi]$), wlog $\rho_r = 1$. Start with $(D^k, \partial D^k) \subset (Y, Z)$ as the ?? of h_r. Apply handle addition theorem to add on translates of $h_1, \dots, h_{r-1}$ to obtain disc $(D^k, \partial D^k) \subset (Y, Z)$ representing $\bar{y}$.

Suppose $k = 2$; since there are no 1-handles, so as $(k-1)$-handles, so X is a collar neighborhood of ∂V in V, so we are home. Suppose now $k \geq 3$, so $n \geq k+3 \geq 6$. X is a collar neighborhood of $\partial V cup (k-1)$-handles. Let $X' = \partial V \cup (k-1)$-hanldes. Let h' be a $(k-1)$-handles. X is a collar neighborhood of $\partial X'$ in V, so we replace D^k by a disc $\bar{D}$ with $\partial\bar{D} \subset \partial X'$, $h' \cong D^{k-1} \times D^{n-k+1}$. Let S' = image of $0 \times S^{n-k}$, i.e. cocone body.

By general position, $\partial D \cap S'$ is a finite union of points, $P_1, \dots, P_j$; each intersection transverse.

Choose path p_i from P_1 to P_i in ∂D.

Choose path p'_i from P_1 to P_i in S'.

Let g_i = element of $\pi_1(Z) \cong \pi$ represented by $p_i \circ \bar{p}'_i$ (out along p_i, back along p'_i). Let ε_i be sign of intersection at P_i (depends on orientation of spheres S', ∂D). Now $\sum \varepsilon_i g_i \in \mathbb{Z}[\pi]$ is coefficient of h' in $\partial\bar{y}$, which is 0. Therefore we can pair off $P_1, \dots, P_j$ so that, if P_s, P_t are paired, then $g_s = g_t$ and $\varepsilon_s = -\varepsilon_t$. Now choose a path p from P_s to P_t in ∂D, path p' from P_s to P_t in s'. Then loop $p \circ \bar{p}'$ is null-homotopic in Z. So we can apply whitney argument to remove intersections at P_s, P_t (need $n \geq 6$). this reduces the number of intersections of $S', \partial D$; repeat until $S, \cap \partial D = \emptyset$.

Now deform ∂D until it doesn't meet h', by an isotopy. Do this for all $(k-1)$-handles h'; then $(D, \partial D) \subset (V, \partial V)$, and represents the right homotopy class x. □

Lemma 5.13. *Let $\mathcal{E}$ be a tame and of manifold W, $\lim W \geq 5$. If V, V' are 1-neighborhoods of $\mathcal{E}$, then the W all variants $\sigma(C_*(\tilde{V}, \partial\tilde{V}))$, $\sigma(C_*(\tilde{V}', \partial\tilde{V}'))$ are equal. If V is an $(n-3)$-neighborhood, then*

$$H_i(\tilde{V}, \partial\tilde{V}) = 0 \quad \textit{for} \quad i \neq n-2\ , \quad \textit{and}$$

$$H_{n-2}(\tilde{V}, \partial\tilde{V}) \quad \textit{is a f.g. projective module,}$$

representing $(-1)^n\sigma(C_*(\tilde{V}, \partial\tilde{V}))$ *in* $\tilde{K}_0(\mathbb{Z}[\pi])$.

Proof. By Theorem 5.8, it is enough to consider case $V' \subset \operatorname{int} V$. Let $U = \overline{V - V'}$. Exact sequence

$$0 \longrightarrow C_*(\tilde{U}, \partial\tilde{V}) \longrightarrow C_*(\tilde{V}, \partial\tilde{V}) \longrightarrow C_*(\tilde{V}, \tilde{U}) \longrightarrow 0$$

By excision, $C_*(\tilde{V}', \partial\tilde{V}') \cong C_*(\tilde{V}, \tilde{U})$. $\exists$ chain equivalences

$$f : C_*(\tilde{V}, \partial\tilde{V}) \longrightarrow D_*$$

$$g : C_*(\tilde{U}, \partial\tilde{V}) \longrightarrow E_*$$

with E_* f.g. free, D_* f.g. projective.

$\varphi : E_* \to D_*$ making diagram below commute, up to chain homotopy

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_*(\tilde{U}, \partial\tilde{V}) & \longrightarrow & C_*(\tilde{V}, \partial\tilde{V} & \longrightarrow & C_*(\tilde{V}, \tilde{U}) & \longrightarrow & 0 \\ & & \big\downarrow g & & \big\downarrow f & & & & \\ & & E_* & \xrightarrow{\varphi} & D_* & & & & \end{array}$$

Now $C_*(\tilde{V}, \tilde{U})$ is chain equivalent to the mapping cone of φ, say Q_*

$$\sigma(C_*(\tilde{V}', \partial\tilde{V}')) = \sigma(C_*(\tilde{V}, \tilde{U})) = \sigma(Q_*) = \sigma(D_*) = \sigma(C_*(\tilde{V}, \partial\tilde{V}))$$

since E_* is f.g. free.

Define the *Siebenmann invariant* $\sigma(\mathcal{E})$ to be $\sigma(C_*(\tilde{V}, \partial\tilde{V}))$ for any 1-neighborhood of E. Now let V be an $(n-3)$-neighborhood of $\mathcal{E}$. By Theorem 5.8, there exists 1-neighborhoods V_n of $\mathcal{E}$ with $V_0 = V$, $\cap V_n = \emptyset$, and $V_{n+1} \subset \operatorname{int} V_n$. Let $U_n = \overline{V_n - V_{n+1}}$; so $\partial U_n = \partial V_n \cup \partial V_{n+1}$. $\partial V_n \subset U_n$, $\partial V_{n+1} \subset U_n$ induces fundamental group isomorphisms (because V_i

is 1-neighborhood)

$$\begin{tikzcd} & \pi_1(V_{n+1}) \arrow[dr] & \\ \pi_1(\partial V_{n+1}) \arrow[ur] \arrow[dr] & & \pi_1(V_n) \\ & \pi_1(U_n) \arrow[ur] & \end{tikzcd}$$

Van Kampen's Theorem $\Rightarrow$ this is a pushout diagram. Therefore all isos, we know that $\pi_1(\partial V_{n+1} \cong \pi_1(V_n)$ and since V_n, V_{n+1} are 1-neighborhoods, $\pi_1(V_{n+1}) \to \pi_1(V_n)$ iso. Therefore since diagram is a pushout, $\pi_1(U_n) \to \pi_1(V_n)$ and $\pi_1(\partial V_{n+1}) \to \pi_1(U_n)$ are isos.

Similarly $\partial V_n \subset U_n$ induces π_1 iso. There exists handle decomposition of U_i on ∂V_i without handles of index $0, 1, n-1, n$. Therefore V can be obtained from ∂V by attaching handles of index $\leq n-2$.

Therefore $V \simeq CW$ complex K with ∂V as a subcomplex and with all cells of $K - \partial V$ of dimension $\leq n-2$. [Attach handles of $V - \partial V$ one at a time, giving $\partial V = X_0 \subset X_1 \subset \cdots$ with $\cup X_i = V$ and X_i obtained from X_{i-1} by attaching r-handle, $r \leq n-2$. Suppose inductively X_{i-1} complex K_{i-1} of required form. Then

$$\begin{aligned} X_i &\simeq X_{i-1} \cup \ r\text{-handle} \\ &\simeq X_{i-1} \cup \ r\text{-cell} \\ &\simeq K_{i-1} \cup e^r \end{aligned}$$

Replace attaching map of e^r by a homotopic cellular map. $K_i = K_{i-1} \cup e^r$ and $x_i \simeq K_i$. But $K = \cup K_i$; then $V \simeq K$.]

$C_*(\tilde{V}, \partial\tilde{V})$ is equivalent to a (not rec. f.g.) free complex of dim $\leq n-2$. But $C_*(\tilde{V}, \partial\tilde{V})$ is equivalent to a f.g. projective complex.

Them 2.3 (second half of proof) shows $C_*(\tilde{V}, \partial\tilde{V}) \simeq$ f.g. projective complex E_* of dimension $\leq n-2$. We have exact sequence

$$0 \longrightarrow H_{n-2}(E_*) \longrightarrow E_{n-2} \longrightarrow E_{n-3} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow 0$$

(since V is an $(n-3)$-neighborhood). Therefore $H_{n-2}(E_*)$ is f.g. projective. Moreover, $H_{n-2}(E_*)$ represents $(-1)^n\sigma(E_*)$ and $= (-1)^n\sigma(C_*(\tilde{V}, \partial\tilde{V}))$. $H_i(\tilde{V}, \partial\tilde{V}) = 0$ if $i > n-2$. $\square$

Corollary 5.14. *Let $\mathcal{E}$ be an end of manifold W, dimension ≥ 6. Then $\mathcal{E}$ has a collar iff $\mathcal{E}$ is arbitrarily small $(n-2)$-neighborhoods.*

Proof. Necessity clear. Let V be an $(n-2)$-neighborhood of $\mathcal{E}$, let v' be another $(n-2)$-neighborhood, $V' \subset \text{int } V$. $U = \overline{V - V'}$ is an h-cobordism from ∂V to $\partial V'$.

$$\begin{array}{ccccc} H_r(\tilde{V}, \tilde{U}) & \longrightarrow & H_{r-1}(\tilde{U}, \partial\tilde{V}) & \longrightarrow & H_{r-1}(\tilde{V}, \partial\tilde{V}) \\ \| & & & & \| \\ H_r(\tilde{V}', \partial\tilde{V}') & & & & 0 \\ \| & & & & \\ 0 & & & & \end{array}$$

Let $\partial V \subset U$ have torsion τ. Let $U' = U \cup$ (1-handles) $\cup$ (2-handles) where the 1-handles and 2-handles are contained in V', and are chosen so that $U \to U'$ has torsion $-\tau$. Let $V'' = \overline{V - U'}$; V'' is a neighborhood of $\mathcal{E}$ contained in V'. U' is an h-cobordism with torsion 0; i.e. an s-cobordism, dim ≥ 6. Therefore $U" \cong \partial V \times I$. Therefore there exists arbitrarily small neighborhoods V'' of $\mathcal{E}$, such that $V'' \subset \text{int } V$ and $\overline{V - V''} \cong \partial V \times I$.

Now it is easy to show that $V \cong \partial V \times [0, \infty)$, so V is a collar. □

Theorem 5.15. *Let $\mathcal{E}$ be an end of a manifold W, of dimension $n \geq 6$. Then $\mathcal{E}$ has a collar iff $\mathcal{E}$ is tame and $\sigma(\mathcal{E}) = 0$.* $[\sigma(\mathcal{E}) \in \tilde{K}_0(\mathbb{Z}[\pi]).]$

Proof. Necessity clear (take collar ? $(n-3)$-neighborhood to calculate $\sigma(\mathcal{E})$).

Conversely: let V be an $(n-3)$-neighborhood of $\mathcal{E}$, and $H_{n-2}(\tilde{V}, \partial\tilde{V})$ is stably free (since $\sigma(\mathcal{E}) = 0$) (i.e. $H_{n-2}(\tilde{V}, \partial\tilde{V}) \oplus F \cong G$ for f.g. free F, G). Wlog assume $H_{n-2}(\tilde{V}, \partial\tilde{V})$ is actually free: (for we can add $\mathbb{Z}[\pi]$ to $H_{n-2}(\tilde{V}, \partial\tilde{V})$ by swapping a trivial $(n-3)$-disc across ∂V).

Since $H_{n-2}(\tilde{V}, \partial\tilde{V})$ is f.g., there exists $(n-3)$-neighborhood $V' \subset \text{int } V$ such that if $U = \overline{V - V'}$, then $H_{n-2}(\tilde{U}, \partial\tilde{V}) \to H_{n-2}(\tilde{V}, \partial\tilde{V})$ is onto. Exact sequence of $(\tilde{V}, \tilde{U}, \partial\tilde{V})$

$$0 \longrightarrow H_{n-2}(\tilde{U}, \partial\tilde{V}) \longrightarrow H_{n-2}(\tilde{V}, \partial\tilde{V}) \xrightarrow{0} H_{n-2}(\tilde{V}', \partial\tilde{V}') \xrightarrow[\cong]{} H_{n-3}(\tilde{U}, \partial\tilde{V}) \longrightarrow 0$$

So

$$H_{n-2}(\tilde{U}, \partial\tilde{V}) \cong H_{n-2}(\tilde{V}, \partial\tilde{V})$$
$$H_{n-2}(\tilde{V}', \partial\tilde{V}') \cong H_{n-3}(\tilde{U}, \partial\tilde{V})$$

V' is a $(n-3)$-neighborhood, wlog $H_{n-2}(\tilde{V}', \partial\tilde{V}')$ is f.g. free. Let $x_1, \ldots, x_k$ be free basis for $h_{n-3}(\tilde{U}, \partial\tilde{V})$. By Lemma 5.12, we can represent $x_1, \ldots, x_k$ by disjoint embedded D^{n-3}'s. (Embed discs one at a time; embed D_i^{n-3} in complement of regular neighborhood of $D_1^{n-3} \cup \cdots \cup D_{i-1}^{n-3}$.) Swap these discs across ∂V, givingV^*, U^*. Then $H_{n-2}(\tilde{U}^*, \partial\tilde{V}^*) \to H_{n-2}(\tilde{V}^*, \partial\tilde{V}^*)$ is still onto. It is enough to check that $H_{n-2}(\tilde{V}, \partial\tilde{V}) \twoheadrightarrow H_{n-2}(\tilde{V}^*, \partial\tilde{V}^*)$. Exact sequence

$$H_{n-2}(\tilde{V}, \partial\tilde{V}) \twoheadrightarrow H_{n-2}(\tilde{V}, \tilde{H}) \xrightarrow{0} H_{n-3}(\tilde{H}, \partial\tilde{H}) \longmapsto H_{n-3}(\tilde{V}, \partial\tilde{V})$$

Replace V by V^*, U by U^*; now $H_{n-2}(\tilde{U}, \partial\tilde{V}) = 0$. $\partial V \subset U$ induces π_1 isomorphism, $\partial\tilde{V} \subset \tilde{U}$ induces homology isomorphisms in dimension $\leq n-4$. Therefore $H_i(\tilde{U}, \partial\tilde{V}) = 0$ for $i \neq n-3$, $n-2$. U has a handle decomposition on ∂V, handles of dimension $n-3$, $n-2$ only. Let $X =$ regular neighborhood of $\partial V \cup (n-3)$-handles, $Y = \overline{U - X}$, $Z = X \cap Y$.

Let $C_{n-2} = H_{n-2}(\tilde{Y}, \tilde{Z})$, $C_{n-3} = H_{n-3}(\tilde{X}, \partial\tilde{V})$, bases C_{n-2}, C_{n-3} given by handles. Chain complex $0 \to C_{n-2} \xrightarrow{\partial} C_{n-3} \longrightarrow 0$ with homology groups $H_{n-2}(\tilde{U}, \partial\tilde{V})$, $H_{n-3}(\tilde{U}, \partial\tilde{V})$ from exact sequence of $(\tilde{U}, \tilde{X}, \partial\tilde{V})$.

Let B_{n-3} be the boundary group $\partial(C_{n-2})$. If we put in extra $(n-3)$-handle into X, and complementary $(n-2)$-handle into Y, then we add $\mathbb{Z}[\pi]$ to C_{n-2}, C_{n-3}, B_{n-3}, and do not affect the homology groups. B_{n-3} is stably free $(0 \to B_{n-3} \to C_{n-3} \to H_{n-3}(C_*) \to 0)$. By adding enough complementary pairs of handles, we can make B_{n-3} free.

Choose basis of $H_{n-2}(C_*)$, and extend to a basis of C_{n-2}, say c'_{n-2}, using exact sequence $0 \to H_{n-2}(C_*) \to C_{n-2} \to B_{n-2} \to 0$. Let $M \in GL(k, \mathbb{Z}[\pi])$ ($k =$ dimension of C_{n-2}) be such that $c'_{n-2}Mc_{n-2}$. Let $D =$ free module $(\mathbb{Z}[\pi])^k$, standard basis d. Put in extra handles as above to replace C_{n-2} by $C_{n-2} \oplus D$, and c_{n-2} by $c_{n-2} \oplus d$. Replace c'_{n-2} by $c_{n-2} \oplus M^{-1}e$; then $c'_{n-2} = Lc_{n-2}$ where $L \in GL(2k, \mathbb{Z}[\pi])$ is a product of elementary matrices.

By the handle addition theorem, we can change $(n-2)$-handles so that they give the basis c'_{n-2}. Then $H_{n-2}(C_*)$ is generated by handles $h_1^{n-2}, \ldots, h_r^{n-2}$, which form a free basis of $H_{n-2}(C_*)$. Since ∂h_i^{n-2} presents 0 in $C_{n-3} = H_{n-3}(\tilde{X}, \partial\tilde{V})$, we can apply the Whitney process to isotop h_i^{n-2} off the $(n-3)$-handles in X (as in 5.12, we need $n \geq 6$). We finish with embedded discs $D_1^{n-2}, \ldots, D_r^{n-2}$ with $\partial D_i^{n-2} \subset \partial V$ representing a basis of $H_{n-2}(\tilde{U}, \partial\tilde{V}) \cong H_{n-2}(\tilde{V}, \partial\tilde{V})$. Swap $D_1^{n-2}, \ldots, D_r^{n-2}$ across ∂V, obtaining a neighborhood V_1 of $\mathcal{E}$. Claim this is an $(n-2)$-neighborhood.

1-neighborhood: Let $U_1 = U \cap V_1$: U_1 has a handle decomposition on ∂V_1 with $(n-3)$ and $(n-2)$-handles only. $n-3 \geq 3$, so $\pi_1(\partial V_1) \to \pi_1(U_1)$ is iso.

U_1 has handle decomposition on $\partial V'$, with 2-handles and 3-handles only.

Therefore $\pi_1(\partial V') \to \pi_1(U_1)$ is onto. But we have $\pi_1(\partial V') \to \pi_1(U_1) \to \pi_1(U)$ an isomorphism; so $\pi_1(\partial V') \xrightarrow{\simeq} \pi_1(U_1)$.

Van Kampen for $\pi_1(V_1)$, $(V_1 = U_1 \cup V')$. Therefore

$$\pi_1(V_1) \cong \pi_1(U_1) \cong \pi_1(\partial V') \cong \pi_1(V') \cong \pi_1(\mathcal{E})$$

and

$$\pi_1(\partial V_1) \cong \pi_1(U_1) \cong \pi_1(V_1)$$

Therefore V_1 is a 1-neighborhood.

Let $H = \overline{U - U}$ = union of handles swapped. Exact sequence of $(\tilde{V}, \widetilde{\partial V \cup H}, \partial\tilde{V})$ gives

$$\to H_{n-2}(\widetilde{\partial V \cup H}, \partial\tilde{V}) \underset{i_*}{\longmapsto} H_{n-2}(\tilde{V}, \partial\tilde{V}) \to H_{n-2}(\tilde{V}_1, \partial\tilde{V}_1) \to 0$$

i_* is mono because $D_1^{n-2} \cup \cdots \cup D_r^{n-2}$ is free basis for $H_{n-2}(\tilde{V}, \partial\tilde{V})$.

In any case, $H_{n-2}(\tilde{V}_1, \partial\tilde{V}_1) = 0$, similarly. $H_i(\tilde{V}_1, \partial\tilde{V}_1) =$ for $i < n-2$, so V_1 is $(n-2)$-neighborhood of $\mathcal{E}$. Therefore by Corollary 5.14, $\mathcal{E}$ has a collar, as required. □

Remarks. i) There exist ends $\mathcal{E}$ which are tame but $\sigma(\mathcal{E}) \neq 0$.

ii) X finite CW complex such that $X \times S' \simeq$ closed man?? M. $\tilde{M}$ = covering corr to $\pi_1(X) \subset \pi_1(X \times S')$. Then $\tilde{M}$ has just two ends $\mathcal{E}, \mathcal{E}_2$, both tame, and both have neighborhoods, $\tilde{M}$, which is $\simeq$ finite complex X. Can happen that $\sigma(\mathcal{E}_i) \neq 0$.