

Amenable L^2 -theoretic methods and knot concordance

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Abstract. We introduce new obstructions to topological knot concordance. These are obtained from amenable groups in Strebel’s class, possibly with torsion, using a recently suggested L^2 -theoretic method due to Orr and the author. Concerning (h) -solvable knots which are defined in terms of certain Whitney towers of height h in bounding 4-manifolds, we use the obstructions to reveal new structure in the knot concordance group not detected by prior known invariants: for any $n > 1$ there are (n) -solvable knots which are not $(n.5)$ -solvable (and therefore not slice) but have vanishing Cochran-Orr-Teichner L^2 -signature obstructions as well as Levine algebraic obstructions and Casson-Gordon invariants.

1. Introduction

In this paper we introduce new obstructions to knots being topologically slice and concordant, and study the structure of the knot concordance group using these. Recall that two knots in S^3 are (topologically) *concordant* if there is a locally flat proper embedding of an annulus in $S^3 \times [0, 1]$ which restricts to the given knots on the boundary components. A knot is called (topologically) *slice* if it is concordant to the trivial knot, or equivalently, if it bounds a locally flat 2-disk properly embedded in the 4-ball. The concordance classes of knots in S^3 form an abelian group \mathcal{C} under connected sum, which is called the *knot concordance group*. Slice knots represent the identity in \mathcal{C} . Recall that obstructions to being topologically slice or concordant, particularly those we discuss in this paper, are also obstructions in the smooth category.

Since the beginning due to Fox and Milnor in the 50’s, significant progress has been made in the study of the knot concordance group. After the landmarks of Levine [Lev69b, Lev69a] and Casson-Gordon [CG86, CG78], the latest breakthrough which opened up a new direction was made in the work of Cochran, Orr, and Teichner [COT03, COT04]. They developed theory of obstructions to being slice, which detects non-slice examples for which prior invariants of Levine and Casson-Gordon vanish. Their obstructions are L^2 -signature defects of bounding 4-manifolds, or equivalently, the von Neumann-Cheeger-Gromov ρ -invariants of 3-manifolds, associated to certain homomorphisms of the fundamental group into *poly-torsion-free-abelian* (PTFA) groups. We recall that a group G is PTFA if it admits a subnormal series $G = G_0 \triangleright \cdots \triangleright G_r = \{e\}$ with each G_i/G_{i+1} torsion-free abelian. Subsequent to their work [COT03, COT04], many interesting new results on concordance, homology cobordism and related topics have been obtained using the PTFA L^2 -signatures by several authors.

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As the main result of this paper, we obtain new obstructions which detect many elements in the knot concordance group that are not distinguished by any previously known obstructions including the PTFA L^2 -signatures and the invariants of Levine and Casson-Gordon.

To give a more precise description of our results, we recall a framework of recent systematic study of the knot concordance group under which known obstructions are best understood. In [COT03], Cochran, Orr, and Teichner defined a geometrically defined filtration

$$0 \subset \cdots \subset \mathcal{F}_{n.5} \subset \mathcal{F}_n \subset \cdots \subset \mathcal{F}_1 \subset \mathcal{F}_{0.5} \subset \mathcal{F}_0 \subset \mathcal{C}$$

of the knot concordance group \mathcal{C} indexed by half integers, which is closely related to the theory of topological 4-manifolds via Whitney towers and gropes. (For definitions and related discussions, readers are referred to [COT03, Sections 7, 8].) Loosely speaking, \mathcal{F}_n is the subgroup of the classes of (n) -solvable knots, where a knot K is defined to be (n) -solvable if its zero-surgery manifold $M(K)$ bounds a spin 4-manifold W which has $H_1(W) \cong \mathbb{Z}$ generated by a meridian and admits a Whitney tower of height n for a spherical lagrangian of the intersection form of W in the sense of [COT03]. Such W is called an (n) -solution for K . An $(n.5)$ -solvable knot and its $(n.5)$ -solution are defined similarly as refinements between level n and $n + 1$. It is well known that a slice knot is (h) -solvable for any h , with the slice disk exterior as an (h) -solution. The obstructions from Levine's algebraic invariants and Casson-Gordon invariants vanish if a knot is (0.5) - and (1.5) -solvable, respectively.

In [COT03], Cochran-Orr-Teichner introduced PTFA L^2 -signature obstructions to being $(n.5)$ -solvable and initiated the study of concordance of highly solvable knots for which Levine and Casson-Gordon invariants vanish:

Theorem 1.1 ([COT03, Theorem 4.2]). *Suppose K is an $(n.5)$ -solvable knot. Suppose G is a PTFA group, $G^{(n+1)} = \{e\}$, and $\phi: \pi_1 M(K) \rightarrow G$ is a homomorphism extending to an $(n.5)$ -solution W . Then $\rho^{(2)}(M(K), \phi) = 0$.*

Here $G^{(n+1)}$ is the derived subgroup of G defined inductively by $G^{(0)} = G$, $G^{(n+1)} = [G^{(n)}, G^{(n)}]$, and $\rho^{(2)}(M, \phi)$ denotes the von Neumann-Cheeger-Gromov ρ -invariant, which is equal to the L^2 -signature defect of a bounding 4-manifold. (For its definition, see, e.g., Section 2.2.)

Since a PTFA group G satisfies $G^{(n)} = \{e\}$ for some n and a slice disk exterior of a knot K is an (h) -solution for any h , one has the following consequence of Theorem 1.1: *For any slice knot K , if G is PTFA and $\phi: \pi_1 M(K) \rightarrow G$ extends to a slice disk exterior, then $\rho^{(2)}(M, \phi) = 0$.*

We remark that for *link* concordance, there are other recent techniques revealing deep information that is invisible via L^2 -signatures. For example, the L -group valued Hirzebruch-type invariants from iterated p -covers [Cha10, Cha09], covering link calculus techniques [CLR08, CK08a, Cot], and twisted torsion invariants [CF] are known to detect interesting highly solvable non-slice links for which L^2 -signatures are not effective.

However, for *knots*, the PTFA L^2 -signatures have been known as the only useful tool to distinguish highly solvable knots up to concordance, particularly for those with vanishing Casson-Gordon and Levine obstructions. Roughly speaking, this interesting subtlety peculiar to knots is related to the “size” of the fundamental group—for knots and 3-manifolds with the first \mathbb{Z}_p -Betti number $b_1(M; \mathbb{Z}_p) \leq 1$ for any prime p , the only previously known nonabelian coverings from which one can extract information on concordance and homology cobordism are PTFA covers and certain metabelian covers considered by Casson-Gordon.

Indeed all known results on knot concordance beyond Levine and Casson-Gordon invariants essentially depend on Theorem 1.1. (See also the remarkable works of Cochran-Teichner [CT07] and Cochran-Harvey-Leidy [CHL09, CHLb, CHLc, CHLa].)

Recently, in [CO], Orr and the author have presented a new L^2 -theoretic method relating homological properties and L^2 -invariants, using a result of Lück. Their approach differs fundamentally from the prior PTFA techniques that are mostly algebraic. The approach in [CO] extends the use of L^2 -signatures to a significantly larger class of groups, namely, the class of amenable groups lying in Strebel's class $D(R)$. (Here R is a commutative ring; see Section 2.1 for definitions and related discussions.) This class contains several interesting infinite/finite non-torsion-free groups, and subsumes PTFA groups. For example, see Lemma 2.1. In [CO] they proved the homology cobordism invariance of L^2 -signature defects and L^2 -Betti numbers associated to amenable groups in $D(R)$ and gave several examples and applications.

In this paper, we further develop the L^2 -theoretic method initiated in [CO] to study the structure of the knot concordance group beyond the information from PTFA L^2 -signatures and invariants of Casson-Gordon and Levine. First we give new obstructions to a knot being slice and to being $(n.5)$ -solvable.

Theorem 1.2. *Suppose K is a slice knot and G is an amenable group lying in Strebel's class $D(R)$ for some R . If $\phi: \pi_1 M(K) \rightarrow G$ is a homomorphism extending to a slice disk exterior for K , then $\rho^{(2)}(M, \phi) = 0$.*

Theorem 1.3. *Suppose K is an $(n.5)$ -solvable knot. Suppose G is an amenable group lying in Strebel's class $D(R)$ where R is \mathbb{Q} or \mathbb{Z}_p , $G^{(n+1)} = \{e\}$, and $\phi: \pi_1 M(K) \rightarrow G$ is a homomorphism which extends to an $(n.5)$ -solution for K and sends a meridian to an infinite order element in G . Then $\rho^{(2)}(M(K), \phi) = 0$.*

We remark that the meridian condition in Theorem 1.3 is not a severe restriction since it is satisfied in most cases. It can be seen that Cochran-Orr-Teichner's Theorem 1.1 is a consequence of our Theorem 1.3.

An important aspect of the new slice obstruction (Theorem 1.2) is that it is *not* a consequence of the obstruction to being $(n.5)$ -solvable (Theorem 1.3), contrary to the PTFA case. The group G in Theorem 1.2 may be *non-solvable*. This offers a very interesting potential to detect non-slice knots which look like slice knots on invariants from solvable groups. This will be addressed in a subsequent paper.

As an application of Theorem 1.3, for each n , we produce a large family of (n) -solvable knots which are not $(n.5)$ -solvable but not detected by the PTFA L^2 -signatures:

Theorem 1.4. *For any n , there are infinitely many (n) -solvable knots J^i ($i = 1, 2, \dots$) satisfying the following: any linear combination $\sum_i a_i J^i$ under connected sum is an (n) -solvable knot with vanishing PTFA L^2 -signature obstructions, but whenever $a_i \neq 0$ for some i , $\sum_i a_i J^i$ is not $(n.5)$ -solvable. Consequently the J^i generate an infinite rank subgroup in $\mathcal{F}_n / \mathcal{F}_{n.5}$ which is invisible via PTFA L^2 -signature obstructions.*

Here we say that J is an (n) -solvable knot J with vanishing PTFA L^2 -signature obstructions if there is an (n) -solution W for J such that for any PTFA group G and for any $\phi: \pi_1 M(J) \rightarrow G$ extending to W , $\rho^{(2)}(M(J), \phi) = 0$. (We do not require $G^{(n+1)} = \{e\}$.) It turns out that such knots J form a subgroup \mathcal{V}_n of \mathcal{F}_n (see Definition 4.7 and Proposition 4.8). Obviously $\mathcal{F}_{n.5} \subset \mathcal{V}_n \subset \mathcal{F}_n$. Theorem 1.4 says that $\mathcal{V}_n / \mathcal{F}_{n.5}$ has infinite rank.

For knots in \mathcal{V}_n , the conclusion of the Cochran-Orr-Teichner theorem obstructing being $(n.5)$ -solvable (Theorem 1.1) holds even for some (n) -solution W which is not necessarily an $(n.5)$ -solution. Consequently, all the prior techniques using the PTFA obstructions (e.g., [COT03, COT04, CT07, CK08b, CHLb, CHL09, CHLc, CHLa]) fail to distinguish any knots in \mathcal{V}_n , particularly our examples in Theorem 1.4, from $(n.5)$ -solvable knots up to concordance. Obviously the invariants of Levine and Casson-Gordon also vanish for knots in \mathcal{V}_n for $n \geq 2$.

We remark that our coefficient system used in the proof of Theorem 1.4 can be viewed as a *higher-order generalization* of the Casson-Gordon metabelian setup. Recall that Casson-Gordon [CG86, CG78] extract invariants from a p -torsion abelian cover of the infinite cyclic cover of the zero-surgery manifold $M(K)$. Our coefficient system corresponds to a tower of covers

$$M_{n+1} \xrightarrow{p_n} M_n \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_1} M_1 \xrightarrow{p_0} M_0 = \text{zero-surgery manifold } M(K)$$

where p_0 is the infinite cyclic cover, p_1, \dots, p_{n-1} are torsion-free abelian covers, and p_n is a p -torsion cover. This iterated covering for knots may also be compared with the iterated p -cover construction in [Cha10, Cha09] which is useful for links.

To construct such coefficient systems that factors through a given solution, we need additional new ingredients in the proof of Theorem 1.4. First we define a *mixed-coefficient commutator series* of a group, which generalizes the ordinary and rational derived series (see Section 4.1). Also, we introduce a *modulo p version of the noncommutative higher-order Blanchfield pairing* (see Section 5). We combine known ideas of PTFA techniques of Cochran-Harvey-Leidy [CHL09] with these new tools in order to construct and analyze our coefficient systems which is into certain infinite amenable groups lying in Strebel's class with torsion.

The paper is organized as follows. In Section 2, we give a proof of our slice obstruction (Theorem 1.2), using results of [CO]. In Section 3, we develop further L^2 -theoretic techniques for amenable groups in Strebel's class and generalize some results in [CO], and using these we give a proof of our obstruction to being $(n.5)$ -solvable (Theorem 1.3). In Section 4, we construct examples and prove Theorem 1.4. In Section 5, we introduce the modulo p noncommutative Blanchfield pairing and discuss its properties.

In this paper, manifolds are assumed to be topological and oriented, and submanifolds are assumed to be locally flat.

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2. Slice knots and L^2 -signatures over amenable groups in Strebel's class

In this section we prove our new obstruction to being a slice knot, namely Theorem 1.2.

2.1. L^2 -homology and amenable groups in Strebel's class

We start by recalling necessary results in [CO] on amenable groups lying in Strebel's class $D(R)$. Recall that a group G is *amenable* if G admits a finitely-additive measure which is invariant under the left multiplication. For many other equivalent definitions and further

discussions, the reader is referred to, e.g., [Pat88]. Following [Str74], for a commutative ring R with unity, a group G is said to be in $D(R)$ if a homomorphism $\alpha: P \rightarrow Q$ between projective RG -modules is injective whenever $1_R \otimes_{RG} \alpha: R \otimes_{RG} P \rightarrow R \otimes_{RG} Q$ is injective. Here R is viewed as an (R, RG) -bimodule with trivial G -action. For more discussions on amenable groups lying in $D(R)$, the reader is referred to, e.g., [CO, Section 6].

For the purpose of this paper, it suffices to use the following description of a useful class of amenable groups in $D(\mathbb{Z}_p)$ which are *not torsion-free* in general:

Lemma 2.1 ([CO, Lemma 6.8]). *Suppose p is a fixed prime and Γ is a group with a subnormal series*

$$\{e\} = \Gamma_n \subset \cdots \subset \Gamma_1 \subset \Gamma_0 = \Gamma$$

such that for any i , Γ_i/Γ_{i+1} is abelian and has no torsion coprime to p , that is, the order of an element in Γ_i/Γ_{i+1} is either a power of p or infinite. Then Γ is amenable and in $D(\mathbb{Z}_p)$.

The fact that PTFA groups are amenable and in $D(\mathbb{Z}_p)$ can be derived as a consequence of Lemma 2.1. In fact, a PTFA group is in $D(R)$ for any R [Str74].

For a discrete countable group G , we denote by $\mathcal{N}G$ the *group von Neumann algebra* of G (e.g., see [Lüc02, Chapter 1]). $\mathbb{C}G$ is a subalgebra of $\mathcal{N}G$, and $\mathcal{N}G$ is endowed with an involution $a \rightarrow a^*$ which is induced by the inversion $g \rightarrow g^{-1}$ in G . For any module A over $\mathcal{N}G$, the L^2 -dimension $\dim^{(2)} A \in [0, \infty]$ is defined as in, e.g., [Lüc02, Chapter 6]. (For more discussions on the L^2 -dimension, see Section 3.1.)

The following result due to Cha-Orr is a key ingredient which controls the L^2 -dimension of L^2 -coefficient homology:

Theorem 2.2 ([CO, Theorem 6.6]). *Suppose G is an amenable group in $D(R)$, and C_* is a finitely generated free chain complex over $\mathbb{Z}G$. If $H_i(R \otimes_{\mathbb{Z}G} C_*) = 0$ for $i \leq n$, then $\dim^{(2)} H_*(\mathcal{N}G \otimes_{\mathbb{Z}G} C_*) = 0$ for $i \leq n$.*

2.2. Proof of Theorem 1.2

We briefly recall the definition of the von Neumann-Cheeger-Gromov ρ -invariant of a closed 3-manifold M , viewing it as an L^2 -signature defect. Given $\phi: \pi_1(M) \rightarrow G$, by replacing G with a larger group into which G injects, we may assume that there is a compact 4-manifold W with $\partial W = M$ over G , by an argument of Weinberger. Let

$$\lambda: H_2(W; \mathcal{N}G) \times H_2(W; \mathcal{N}G) \longrightarrow \mathcal{N}G$$

be the $\mathcal{N}G$ -coefficient intersection form of W . This is a hermitian form over $\mathcal{N}G$, so that the L^2 -signature $\text{sign}^{(2)} \lambda \in \mathbb{R}$ is defined. Now the L^2 -signature of W is defined by $\text{sign}_G^{(2)}(W) = \text{sign}^{(2)} \lambda$. For more detailed discussions on the L^2 -signature, see Section 3.1. (See also, e.g., [COT03, LS03, CW03, Har08, Cha08, CO].)

The von Neumann-Cheeger-Gromov ρ -invariant of (M, ϕ) is defined to be the signature defect

$$\rho^{(2)}(M, \phi) = \text{sign}_G^{(2)}(W) - \text{sign}(W)$$

where $\text{sign}(W)$ is the ordinary signature of W . $\rho^{(2)}(M, \phi)$ is known to be well-defined.

We are now ready to prove Theorem 1.2: *Suppose K is a slice knot and G is an amenable group lying in Strebel's class $D(R)$ for some R . If $\phi: \pi_1 M(K) \rightarrow G$ is a homomorphism extending to a slice disk exterior for K , then $\rho^{(2)}(M, \phi) = 0$.*

Proof of Theorem 1.2. Let W be a slice disk exterior for K , and suppose the given homomorphism $\phi: \pi_1 M(K) \rightarrow G$ factors through $\pi_1(W)$. Then, we have $\rho^{(2)}(M(K), \phi) = \text{sign}_G^{(2)}(W) - \text{sign}(W)$. By Alexander duality, $H_2(W; R) = 0$ and $H_1(W; R) \cong H_1(M(K); R)$ and thus $H_i(W, M(K); R) = 0$ for any R and $i \leq 2$. By Theorem 2.2, it follows that $\dim^{(2)} H_2(W, M(K); \mathcal{N}G) = 0$. Since the non-singular part of the $\mathcal{N}G$ -valued intersection form of W is supported by the image of $H_2(W; \mathcal{N}G) \rightarrow H_2(W, M(K); \mathcal{N}G)$, it follows that $\text{sign}_G^{(2)}(W) = 0$. Since $H_2(W, M(M)) = 0$, the ordinary signature $\text{sign}(W)$ vanishes. Therefore $\rho^{(2)}(M, \phi) = 0$. \square

We remark that the above proof actually shows that the ρ -invariant gives us an obstruction to bounding a slice disk in a homology 4-ball.

3. Obstructions to being $(n.5)$ -solvable

In this section we prove our obstruction to being an $(n.5)$ -solvable knot.

Definition 3.1. Suppose M is a closed 3-manifold and W is a compact 4-manifold with boundary M . Let $\pi = \pi_1(W)$.

- (1) W is called an *integral (n) -solution* for M if the inclusion induces $H_1(M) \cong H_1(W)$ and there exist elements $x_1, \dots, x_r, y_1, \dots, y_r \in H_2(W; \mathbb{Z}[\pi/\pi^{(n)}])$ such that $2r = \text{rank}_{\mathbb{Z}} H_2(W; \mathbb{Z})$ and the intersection form

$$\lambda_n: H_2(W; \mathbb{Z}[\pi/\pi^{(n)}]) \times H_2(W; \mathbb{Z}[\pi/\pi^{(n)}]) \longrightarrow \mathbb{Z}[\pi/\pi^{(n)}]$$

satisfies $\lambda_n(x_i, x_j) = 0$ and $\lambda_n(x_i, y_j) = \delta_{ij}$ (Kronecker symbol) for any i, j .

- (2) W is an *integral $(n.5)$ -solution* of M if (1) is satisfied for some x_i and y_j , and there exist lifts $\tilde{x}_i \in H_2(W; \mathbb{Z}[\pi/\pi^{(n+1)}])$ of the x_i such that $\lambda_{n+1}(\tilde{x}_i, \tilde{x}_j) = 0$ for any i, j .

We call the collections $\{x_i\}$ and $\{\tilde{x}_i\}$ an (n) -*lagrangian* and $(n+1)$ -*lagrangian*, respectively, and call $\{y_i\}$ their (n) -*dual*, respectively. If the zero-surgery manifold $M(K)$ of a knot K has an integral (h) -solution W , we say that K is *integrally (h) -solvable*.

We remark that if a knot K is (h) -solvable in the sense of [COT03], then K is integrally (h) -solvable as in Definition 3.1.

The main result of this section is the following:

Theorem 3.2. *Suppose a knot K is integrally $(n.5)$ -solvable, G is an amenable group lying in $D(R)$ for $R = \mathbb{Z}_p$ or \mathbb{Q} , $G^{(n+1)} = \{e\}$, and $\phi: \pi_1 M(K) \rightarrow G$ is a homomorphism which extends to an integral $(n.5)$ -solution for $M(K)$ and sends a meridian to an infinite order element in G . Then $\rho^{(2)}(M(K), \phi) = 0$.*

We note that the meridian condition holds, for example, when the abelianization map of $\pi_1(M_K)$ factors through ϕ . Theorem 1.3 in the introduction is an immediate consequence of Theorem 3.2.

Before giving a proof of Theorem 3.2, we discuss some basic preliminaries and establish further results on the L^2 -homology for amenable groups in $D(R)$, in Sections 3.1 and 3.2.

3.1. L^2 -dimension, homology, and hermitian forms

In this subsection we discuss rapidly necessary facts on L^2 -theory, under the theme that the group von Neumann algebra $\mathcal{N}G$ behaves as a semisimple ring to the eyes of the L^2 -dimension. Particularly, several standard properties of the ordinary complex dimension of vector spaces, dual spaces, homology are shared by the L^2 -dimension.

We start by reviewing the definition of the group von Neumann algebra $\mathcal{N}G$ and L^2 -dimension of modules over $\mathcal{N}G$.

Definitions. For a discrete countable group G , Let $\ell^2 G$ be the Hilbert space of formal expressions $\sum_{g \in G} z_g \cdot g$ ($z_g \in \mathbb{C}$) which are square summable, i.e., $\sum_{g \in G} |z_g|^2 < \infty$. The inner product on $\ell^2 G$ is given by $\langle \sum z_g \cdot g, \sum w_g \cdot g \rangle = \sum \overline{z_g} w_g$. Let $\mathcal{B}(\ell^2 G)$ be the algebra of bounded operators on $\ell^2 G$. (As a convention, operators act on the left.) We can view $\mathbb{C}G$ as a subalgebra of $\mathcal{B}(\ell^2 G)$ via multiplication on the left. The *group von Neumann algebra* $\mathcal{N}G$ of G is defined to be the pointwise closure of $\mathbb{C}G$ in $\mathcal{B}(\ell^2 G)$. The *von Neumann trace* of $a \in \mathcal{N}G$ is defined by $\text{tr}^{(2)} a = \langle a(e), e \rangle$ where $e \in G$ is the identity. This extends to $n \times n$ matrices $B = (b_{ij})$ over $\mathcal{N}G$ by $\text{tr}^{(2)} B = \sum_i \text{tr}^{(2)} b_{ii}$.

Now we can describe the L^2 -dimension function

$$\dim^{(2)} : \{\text{all } \mathcal{N}G\text{-modules}\} \longrightarrow [0, \infty]$$

as follows. For a finitely generated projective module P over $\mathcal{N}G$, there is a projection $(\mathcal{N}G)^n \rightarrow (\mathcal{N}G)^n$ whose image is isomorphic to P , that is, the associated $n \times n$ matrix, say $B = (b_{ij})$, has row space P . Then $\dim^{(2)} P$ is defined to be $\text{tr}^{(2)} B$. It is known that $\dim^{(2)} P$ is well-defined and that one can choose a hermitian (i.e., $B^* = (b_{ji}^*)$ equals B) projection matrix B with row space P . (See [Lüc02, Chapter 6].) For an arbitrary module A over $\mathcal{N}G$, define $\dim^{(2)} A = \sup_P \dim^{(2)} P$ where P runs over finitely generated projective submodules in A .

We will frequently use the following basic properties of the L^2 -dimension function. For proofs, see [Lüc02, Chapter 6].

- (1) $\dim^{(2)} \mathcal{N}G = 1$ and $\dim^{(2)} 0 = 0$.
- (2) If $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ is exact, then $\dim^{(2)} A = \dim^{(2)} A' + \dim^{(2)} A''$. In particular, if A' is a homomorphic image of a submodule of A , then $\dim^{(2)} A' \leq \dim^{(2)} A$.

Here we adopt the usual convention $\infty + d = \infty$ for any $d \in [0, \infty]$. A consequence is that $\dim^{(2)} M < \infty$ if M is a quotient of a submodule of a finitely generated $\mathcal{N}G$ -module.

The next lemma is a key result due to Lück, which says that a finitely generated module over $\mathcal{N}G$ can be “approximated” by a projective summand with the same L^2 -dimension.

Lemma 3.3 ([Lüc02, p. 239]). *For a finitely generated $\mathcal{N}G$ -module A , let*

$$T(A) = \{x \in A \mid f(x) = 0 \text{ for any } f: M \rightarrow \mathcal{N}G\}$$

and $P(A) = A/T(A)$. Then $\dim^{(2)} T(A) = 0$ and $P(A)$ is finitely generated and projective over $\mathcal{N}G$. Consequently, $A \cong T(A) \oplus P(A)$ and $\dim^{(2)} A = \dim^{(2)} P(A)$.

We remark that $T(-)$ and $P(-)$ are functorial. From Lemma 3.3, it follows easily that a finitely generated $\mathcal{N}G$ -module A is projective if and only if $T(A) = 0$.

L^2 -dimension of dual modules. For an $\mathcal{N}G$ -module A , its *dual* $A^* = \text{Hom}_{\mathcal{N}G}(A, \mathcal{N}G)$ is an $\mathcal{N}G$ -module under the scalar multiplication $(r \cdot f)(x) = f(x)r^*$ for $r \in \mathcal{N}G$, $f \in A^*$, $x \in A$.

Lemma 3.4. *For any finitely generated $\mathcal{N}G$ -module A , $T(A)^* = 0 = T(A^*)$, $P(A)^* = A^*$, and $\dim^{(2)} A^* = \dim^{(2)} A$.*

Proof. First we show $T(A)^* = 0$. Since $T(A)$ is a summand, any $f: T(A) \rightarrow \mathcal{N}G$ extends to A . Therefore, $f(x) = 0$ for any $x \in T(A)$ by definition.

Write the given module as $A = T(A) \oplus P(A)$. We have $A^* = T(A)^* \oplus P(A)^* = P(A)^*$. Since $P(A)^*$ is projective, $T(A^*) = 0$. Now it suffices to show that $\dim^{(2)} P^* = \dim^{(2)} P$ for a finitely generated projective module P . Choose an $n \times n$ hermitian projection matrix B with row space P . Applying $\text{Hom}(-, \mathcal{N}G)$ to the associated map $(\mathcal{N}G)^n \rightarrow (\mathcal{N}G)^n$ and considering the dual basis, it is easily seen that the row space of B^* is isomorphic to P^* . Therefore $\dim^{(2)} P^* = \text{tr}^{(2)} B^* = \text{tr}^{(2)} B = \dim^{(2)} P$. \square

L^2 -dimension of homology and cohomology. Recall that $\text{Ext}_{\mathcal{N}G}^i(-, \mathcal{N}G)$ and $\text{Tor}_i^{\mathcal{N}G}(\mathcal{N}G, -)$ are (left) $\mathcal{N}G$ -modules viewing $\mathcal{N}G$ as a bimodule.

Lemma 3.5. *For any finitely generated module A over $\mathcal{N}G$, $\dim^{(2)} \text{Ext}_{\mathcal{N}G}^1(A, \mathcal{N}G) = 0 = \dim^{(2)} \text{Tor}_1^{\mathcal{N}G}(\mathcal{N}G, A)$.*

Proof. Choose $0 \rightarrow M \rightarrow P \rightarrow A \rightarrow 0$ with P finitely generated projective. Since

$$0 \longrightarrow A^* \longrightarrow P^* \longrightarrow M^* \longrightarrow \text{Ext}_{\mathcal{N}G}^1(A, \mathcal{N}G) \longrightarrow 0$$

we have, by Lemma 3.4,

$$\begin{aligned} \dim^{(2)} \text{Ext}_{\mathcal{N}G}^1(A, \mathcal{N}G) &= \dim^{(2)} M^* - \dim^{(2)} P^* + \dim^{(2)} A^* \\ &= \dim^{(2)} M - \dim^{(2)} P + \dim^{(2)} A = 0. \end{aligned}$$

Similar argument works for $\text{Tor}_1^{\mathcal{N}G}$. \square

Following [CO], we say that a homomorphism $f: A \rightarrow B$ between $\mathcal{N}G$ -modules is an *L^2 -equivalence* if $\dim^{(2)} \text{Ker } f = 0 = \dim^{(2)} \text{Coker } f$.

Proposition 3.6. *Suppose C_* is a chain complex with each C_i finitely generated and projective over $\mathcal{N}G$. Then for any n , the Kronecker evaluation map*

$$H^n(\text{Hom}(C_*, \mathcal{N}G)) \longrightarrow \text{Hom}(H_n(C_*), \mathcal{N}G)$$

is an L^2 -equivalence. Consequently, $\dim^{(2)} H_n(C_) = \dim^{(2)} H^n(\text{Hom}(C_*, \mathcal{N}G))$.*

Proof. By [Lüc02, p. 239], $\mathcal{N}G$ is semihereditary, namely, any finitely generated submodule of a projective module is projective. It follows that $H_i(C_*)$ has projective dimension at most 1, i.e., $\text{Ext}_{\mathcal{N}G}^p(H_i(C_*), M) = 0$ for any M and $p \geq 2$. (For, it is seen easily that both the boundary and cycle submodules in C_i are finitely generated and projective.)

Consider the universal coefficient spectral sequence

$$E_2^{p,q} = \text{Ext}^p(H_q(C_*), \mathcal{N}G) \Longrightarrow H^n(\text{Hom}(C_*, \mathcal{N}G)).$$

By the above observation and Lemma 3.5, all the E_2 -terms have L^2 -dimension zero except (possibly) the bottom row $E_2^{0,q} = \text{Hom}(H_q(C_*), \mathcal{N}G)$. That is, the spectral sequence “ L^2 -collapses” at the E_2 terms. It follows that the Kronecker evaluation map is an L^2 -equivalence. By Lemma 3.4, the conclusion on the L^2 -dimension follows. \square

L^2 -signatures of hermitian forms over $\mathcal{N}G$. Here we discuss a formulation of L^2 -signatures using only $\mathcal{N}G$ as a ring, without making any explicit use of the Hilbert space $\ell^2 G$ (particularly the traditional L^2 -homology $H_*^{(2)}(-; \ell^2 G)$). While this setup seems to be known, the author could not find a treatment in the literature. Since it is the right setup for our later arguments, we give a quick outline here. (An elementary thorough treatment with full details will be given in a separate introductory article of the author.)

We call an $\mathcal{N}G$ -module homomorphism $\lambda: A \rightarrow A^*$ with A finite generated a *sesquilinear form* over $\mathcal{N}G$, viewing it often as $\lambda: A \times A \rightarrow \mathcal{N}G$. We say that λ is *hermitian* if A is finitely generated and $\lambda(x)(y) = (\lambda(y)(x))^*$ for $x, y \in A$. We say that λ is L^2 -*nonsingular* if λ is an L^2 -equivalence as a homomorphism $A \rightarrow A^*$.

Suppose $\lambda: A \rightarrow A^*$ is a hermitian form over $\mathcal{N}G$. While A is not necessarily projective, $T(A)$ always lies in the kernel of λ by the functoriality of $T(-)$ and by Lemma 3.4, and thus λ gives rise to a hermitian form $P(\lambda)$ on the projective module $P(A)$. Taking Q satisfying $P(A) \oplus Q = (\mathcal{N}G)^n$ ($n < \infty$), since $P(\lambda) \oplus 0$ on $P(A) \oplus Q$ is a hermitian form over the free module $(\mathcal{N}G)^n$, we can apply functional calculus to obtain an orthogonal decomposition

$$(\mathcal{N}G)^n = A_+ \oplus A_- \oplus A'_0$$

such that $P(\lambda) \oplus 0$ is positive definite, negative definite, and zero on the submodules A_+ , A_- , and A'_0 , respectively. Here we say that λ is positive (resp. negative) definite on A if for any nonzero $x \in A$, $\lambda(x, x)$ is of the form a^*a (resp. $-a^*a$) for some nonzero $a \in \mathcal{N}G$. In fact, since $Q \subset A'_0$ is a summand, for $A_0 = A'_0 \cap P(A)$, we also have a decomposition

$$P(A) = A_+ \oplus A_- \oplus A_0.$$

Now the L^2 -signature of λ is defined by

$$\text{sign}^{(2)} \lambda = \dim^{(2)} A_+ - \dim^{(2)} A_- \in \mathbb{R}.$$

$\text{sign}^{(2)}$ is well-defined, e.g., by carrying out a standard proof for Sylvester's law of inertia, using $\dim^{(2)}$ in place of the ordinary complex dimension. (A similar argument is given in the proof of Proposition 3.7 below.)

It is known that $\text{sign}^{(2)}$ induces a real-valued homomorphism of the L -group $L^0(\mathcal{N}G)$. We need a slightly stronger fact, namely, the following L^2 -analogue of “topologist's signature vanishing criterion”:

Proposition 3.7. *If $\lambda: A \rightarrow A^*$ is an L^2 -nonsingular hermitian form over $\mathcal{N}G$ and there is a submodule $H \subset A$ satisfying $\lambda(H)(H) = 0$ and $\dim^{(2)} H \geq \frac{1}{2} \dim^{(2)} A$, then $\text{sign}^{(2)} \lambda = 0$ and $\dim^{(2)} H = \frac{1}{2} \dim^{(2)} A$.*

Proof. A standard argument for the case of finite dimension complex vector spaces works, using $\dim^{(2)}$ in place of the complex dimension. For concreteness we describe details below: let $r = \frac{1}{2} \dim^{(2)} A$. Choose an orthogonal decomposition $A = A_+ \oplus A_- \oplus A_0$ such that λ is positive (resp. negative) definite on A_+ (resp. A_-) and $A_0 = \text{Ker } \lambda$. Suppose $\dim^{(2)} A_+ > r$. Since $\dim^{(2)} H \geq r$, $\dim^{(2)} H \cap A_+ > 0$. But since λ is zero on H and positive definite on A_+ , $H \cap A_+ = 0$. From this contradiction it follows that $\dim^{(2)} A_+ \leq r$, and similarly $\dim^{(2)} A_- \leq r$. By the L^2 -nonsingularity, $\dim^{(2)} A_0 = 0$ and $\dim^{(2)} A_+ + \dim^{(2)} A_- = 2r$. Therefore $\dim^{(2)} A_+ = r = \dim^{(2)} A_-$. If $\dim^{(2)} H > r$, then $\dim^{(2)} H \cap A_+ > 0$, which is a contradiction. \square

3.2. More about L^2 -homology and amenable groups in Strebel's class

In this subsection we generalize Theorem 2.2. We begin with an observation that $\mathcal{N}G$ looks like a flat $\mathbb{C}G$ -module to the eyes of L^2 -dimension, based on Lück's result on the vanishing of the L^2 -dimension of Tor . Here we use the amenability crucially.

Lemma 3.8. *Suppose G is amenable. Then for a $\mathbb{C}G$ -homomorphism $f: A \rightarrow B$, the natural $\mathcal{N}G$ -module homomorphisms $\mathcal{N}G \otimes \text{Ker } f \rightarrow \text{Ker}\{1_{\mathcal{N}G} \otimes f\}$ is an L^2 -equivalence. The analogues for $\text{Im}\{-\}$ and $\text{Coker}\{-\}$ hold as well.*

Note that the cokernel part is obvious since tensoring is right exact.

Proof. Tensoring with $\mathcal{N}G$ over $\mathbb{C}G$ gives us the following commutative diagram with exact row and column:

$$\begin{array}{ccccccc}
 & & \text{Tor}_1^{\mathbb{C}G}(\mathcal{N}G, \text{Im } f) & & & & \\
 & & \downarrow & & & & \\
 & & \mathcal{N}G \otimes \text{Ker } f & & & & \\
 & & \downarrow & & & & \\
 & & \mathcal{N}G \otimes A & & & & \\
 & & \downarrow \searrow^{1_{\mathcal{N}G} \otimes f} & & & & \\
 \text{Tor}_1^{\mathbb{C}G}(\mathcal{N}G, \text{Coker } f) & \longrightarrow & \mathcal{N}G \otimes \text{Im } f & \longrightarrow & \mathcal{N}G \otimes B & \longrightarrow & \mathcal{N}G \otimes \text{Coker } f \longrightarrow 0 \\
 & & \downarrow & & & & \\
 & & 0 & & & &
 \end{array}$$

Due to Lück [Lüc02, p. 259], $\dim^{(2)} \text{Tor}_1^{\mathbb{C}G}(\mathcal{N}G, -) = 0$ since G is amenable. The conclusions follow by straightforward arguments. \square

Now we introduce a lemma which makes crucial use of Strebel's class. Let R be a commutative ring with unity. Recall that, a group G is said to be in $D(R)$ if a homomorphism $f: M \rightarrow N$ between projective RG -modules is injective whenever $1_R \otimes f: R \otimes M \rightarrow R \otimes N$ is injective [Str74]. Reformulating this definition slightly, we have: a group G is in $D(R)$ if and only if for any homomorphism $f: M \rightarrow N$ between RG -modules with N projective, the restriction of f on a projective submodule $P \subset M$ is injective whenever the induced map $R \otimes P \rightarrow R \otimes N$ is injective.

Obviously a homomorphism of an R -free module, say R^I , into $R \otimes M$ lifts to a homomorphism of the RG -free module $(RG)^I \rightarrow M$. Thus, we have the following consequence:

Lemma 3.9. *Suppose G is in $D(R)$. Then for any homomorphism $f: M \rightarrow N$ between RG -modules with N projective, the maximal rank of RG -free submodules in $\text{Im } f$ is greater than or equal to the maximal rank of R -free submodules in $\text{Im}\{1_R \otimes f: R \otimes M \rightarrow R \otimes N\}$.*

Combining the above results obtained from amenability and Strebel's condition, we obtain the following L^2 -dimension estimates. (cf. [CH05, Lemma 4.4] for algebraic analogues for PTFA groups.)

Lemma 3.10. *Suppose G is amenable and in $D(R)$, $R = \mathbb{Q}$ or \mathbb{Z}_p , and $\phi: M \rightarrow N$ is a homomorphism between $\mathbb{Z}G$ -modules. Denote the induced maps on $R \otimes_{\mathbb{Z}G} -$ and $\mathcal{N}G \otimes_{\mathbb{Z}G} -$ by $1_R \otimes \phi$ and $1_{\mathcal{N}G} \otimes \phi$.*

- (1) *If N is projective, then $\dim_R \text{Im}\{1_R \otimes \phi\} \leq \dim_G^{(2)} \text{Im}\{1_{\mathcal{N}G} \otimes \phi\}$.*
- (2) *If, in addition, M is finitely generated and projective, then $\dim_R \text{Ker}\{1_R \otimes \phi\} \geq \dim_G^{(2)} \text{Ker}\{1_{\mathcal{N}G} \otimes \phi\}$.*

Proof. (1) Let $d = \dim_R \operatorname{Im}\{1_R \otimes \phi\}$. (d may be any cardinal.) Applying Lemma 3.9 to $f = 1_{RG} \otimes \phi: RG \otimes M \rightarrow RG \otimes N$, we obtain an injection $(RG)^d \rightarrow \operatorname{Im}\{1_{RG} \otimes \phi\} \subset RG \otimes N$. We may assume that it is induced by a homomorphism $i: (\mathbb{Z}G)^d \rightarrow \operatorname{Im}\phi \subset N$ (by multiplying it by an integer if necessary). Note that for any two index sets, say I and J , of arbitrary cardinality, a homomorphism $\mathbb{Z}^I \rightarrow \mathbb{Z}^J$ is injective whenever the induced map $R^I \rightarrow R^J$ is injective. Especially, in our case, i is injective. Now,

$$(\mathcal{N}G)^d = \mathcal{N}G \otimes_{\mathbb{Z}G} (\mathbb{Z}G)^d \cong \mathcal{N}G \otimes_{\mathbb{Z}G} (\operatorname{Im} i) \longrightarrow \operatorname{Im}\{1_{\mathcal{N}G} \otimes i\}$$

is an L^2 -equivalence, by Lemma 3.8. Since $\operatorname{Im}\{1_{\mathcal{N}G} \otimes i\} \subset \operatorname{Im}\{1_{\mathcal{N}G} \otimes \phi\}$, we have

$$d = \dim^{(2)} \operatorname{Im}\{1_{\mathcal{N}G} \otimes i\} \leq \dim^{(2)} \operatorname{Im}\{1_{\mathcal{N}G} \otimes \phi\}.$$

(2) We may assume that M is free by taking a projective module P such that $M \oplus P$ is finitely generated and free, and replacing ϕ with $\phi \oplus 1_P: M \oplus P \rightarrow N \oplus P$. Now, combine (1) with the equality

$$\begin{aligned} \dim^{(2)} \operatorname{Ker}\{1_{\mathcal{N}G} \otimes \phi\} + \dim^{(2)} \operatorname{Im}\{1_{\mathcal{N}G} \otimes \phi\} &= \dim^{(2)} \mathcal{N}G \otimes M \\ &= \dim_R R \otimes M = \dim_R \operatorname{Ker}\{1_R \otimes \phi\} + \dim_R \operatorname{Im}\{1_R \otimes \phi\}. \quad \square \end{aligned}$$

Using the above results, we give bounds for the L^2 -dimension of $\mathcal{N}G$ -coefficient homology in terms of the ordinary dimension of R -coefficient homology.

Theorem 3.11.

- (1) Suppose G is amenable and in $D(R)$ with $R = \mathbb{Q}$ or \mathbb{Z}_p , and C_* is a projective chain complex over $\mathbb{Z}G$ with C_n finitely generated. Then we have

$$\dim^{(2)} H_n(\mathcal{N}G \otimes_{\mathbb{Z}G} C_*) \leq \dim_R H_n(R \otimes_{\mathbb{Z}G} C_*).$$

- (2) In addition, if $\{x_i\}_{i \in I}$ is a collection of n -cycles in C_n , then for the submodules $H \subset H_n(\mathcal{N}G \otimes C_*)$ and $\overline{H} \subset H_n(R \otimes C_*)$ generated by $\{[1_{\mathcal{N}G} \otimes x_i]\}_{i \in I}$ and $\{[1_R \otimes x_i]\}_{i \in I}$, respectively, we have

$$\dim^{(2)} H_n(\mathcal{N}G \otimes C_*) - \dim^{(2)} H \leq \dim_R H_n(R \otimes C_*) - \dim_R \overline{H}.$$

Proof. First we prove (2). Let $\partial_n: C_n \rightarrow C_{n-1}$ be the boundary map, and define $f: (\mathbb{Z}G)^I \oplus C_{n+1} \rightarrow C_n$ by $f(e_i, v) \mapsto x_i + \partial_{n+1}(v)$ where $\{e_i\}_{i \in I}$ is the standard basis of $(\mathbb{Z}G)^I$. Then we have

$$\begin{aligned} H_n(\mathcal{N}G \otimes C_*)/H &= \operatorname{Ker}\{1_{\mathcal{N}G} \otimes \partial_n\} / \operatorname{Im}\{1_{\mathcal{N}G} \otimes f\} \\ H_n(R \otimes C_*)/\overline{H} &= \operatorname{Ker}\{1_R \otimes \partial_n\} / \operatorname{Im}\{1_R \otimes f\} \end{aligned}$$

Applying Lemma 3.10, we have

$$\begin{aligned} \dim^{(2)} H_n(\mathcal{N}G \otimes C_*)/H &= \dim^{(2)} \operatorname{Ker}\{1_{\mathcal{N}G} \otimes \partial_n\} - \dim^{(2)} \operatorname{Im}\{1_{\mathcal{N}G} \otimes f\} \\ &\leq \dim_R \operatorname{Ker}\{1_R \otimes \partial_n\} - \dim_R \operatorname{Im}\{1_R \otimes f\} = \dim_R H_n(R \otimes C_*)/\overline{H}. \end{aligned}$$

This proves (2). Considering the special case of $H = 0 = \overline{H}$, (1) follows. \square

Now we apply the above result on the L^2 -dimension to L^2 -homology of spaces. For a space X and a field R , we denote the i -th Betti number by $b_i(X; R) = \dim_R H_i(X; R)$. For $R = \mathbb{Q}$, we simply write $b_i(X) = b_i(X; \mathbb{Q})$. When X is endowed with $\pi_1(X) \rightarrow G$, we define the L^2 -Betti number of X by $b_i^{(2)}(X; \mathcal{N}G) = \dim^{(2)} H_i(X; \mathcal{N}G)$. When the choice of G is obvious, we write $b_i^{(2)}(X) = b_i^{(2)}(X; \mathcal{N}G)$.

We remark that for a finite CW complex X , it easily follows, from the standard properties of the L^2 -dimension we mentioned above, that the L^2 Euler characteristic $\sum_i (-1)^i \cdot b_i^{(2)}(X)$ is equal to the ordinary Euler characteristic $\chi(X)$.

Lemma 3.12. *Suppose X is a connected finite CW complex and $\phi: \pi_1(X) \rightarrow G$ is a homomorphism with G amenable and in $D(R)$. If there is a loop α in X such that $\phi([\alpha]) \in G$ has infinite order and the homology class of α generates $H_1(X; R)$, then $b_0^{(2)}(X) = 0 = b_1^{(2)}(X)$.*

Proof. Choose $S^1 \rightarrow X$ representing α . Then $H_i(X, S^1; R) = 0$ for $i = 0, 1$. By Theorem 2.2 (or Theorem 3.11), it follows that $\dim^{(2)} H_1(X, S^1; \mathcal{N}G) = 0$. Therefore $b_i^{(2)}(X) \leq b_i^{(2)}(S^1)$ for $i = 0, 1$. Since ϕ induces an injection $\pi_1(S^1) \rightarrow G$, the G -cover of S^1 is a disjoint union of lines. Therefore $H_i(S^1; \mathcal{N}G) = 0$ and $b_i^{(2)}(S^1) = 0$ for all $i \geq 1$. Since $\chi(S^1) = 0$, $b_0^{(2)}(S^1) = 0$. \square

Lemma 3.13. *Suppose $(X, \partial X)$ is a 4-dimensional Poincaré duality pair (e.g., a compact 4-manifold with boundary) with X and ∂X connected, which is endowed with $\pi_1(X) \rightarrow G$ where G is amenable and in $D(R)$. Suppose $b_0^{(2)}(\partial X)$, $b_0^{(2)}(X)$, and $b_1^{(2)}(X)$ vanish, $b_1(X; R) = 1$, and the inclusion-induced map $H_1(\partial X; R) \rightarrow H_1(X; R)$ is nontrivial. Then $b_2^{(2)}(X) = b_2(X; R)$.*

We remark that if there is a loop α in ∂X satisfies the hypotheses of Lemma 3.12 for both X and ∂X , then Lemma 3.13 applies. For example, it applies to an (h) -solution for a knot.

Proof. From the hypothesis and the long exact sequence for $(X, \partial X)$, we have $b_1^{(2)}(X, \partial X) = 0$. By duality and Proposition 3.6, we have $b_3^{(2)}(X) = b_1^{(2)}(X, \partial X) = 0$, and $b_i^{(2)}(X) = 0$ for $i \geq 4$. Similarly $b_3(X; R) = b_1(X, \partial X) = 0$, $b_i(X; R) = 0$ for $i \geq 4$. Since the L^2 Euler characteristic is equal to the ordinary Euler characteristic, it follows that $b_2^{(2)}(X) = b_2(X; R)$. \square

3.3. Proof of Theorem 3.2

We start with an observation on the integral and modulo p Betti numbers:

Lemma 3.14. *If W is a 4-manifold such that both $H_1(W)$ and $\text{Coker}\{H_1(\partial W) \rightarrow H_1(W)\}$ has no p -torsion, then $b_i(W) = b_i(W; \mathbb{Z}_p)$ for any i .*

As a special case, if W is an integral (h) -solution of a knot, then the conclusion of Lemma 3.14 holds.

Proof. The conclusion is obvious for $i = 1$. Let $H = \text{Coker}\{H_1(\partial W) \rightarrow H_1(W)\}$ and $r = \text{rank}_{\mathbb{Z}} \text{Ker}\{H_0(\partial W) \rightarrow H_0(W)\}$. Then $b_3(W) = b_1(W, \partial W) = r + \text{rank}_{\mathbb{Z}} H$ and $b_3(W; \mathbb{Z}_p) = r + \text{rank}_{\mathbb{Z}_p}(H \otimes_{\mathbb{Z}} \mathbb{Z}_p)$. From the hypothesis, the conclusion for $i = 3$ follows. By the universal coefficient theorem $H_3(W; \mathbb{Z}_p) = (H_3(W) \otimes \mathbb{Z}_p) \oplus \text{Tor}_1^{\mathbb{Z}}(H_2(W), \mathbb{Z}_p)$, it follows that both $H_3(W)$ and $H_2(W)$ have no p -torsion. By the universal coefficient theorem again, the conclusion for $i = 4$ and $i = 2$ follows. \square

Proof of Theorem 3.2. First we consider the case of $R = \mathbb{Z}_p$. Suppose K is a knot with meridian μ and zero-surgery manifold $M(K)$, and W is an integral $(n, 5)$ -solution for $M(K)$. Let $\{\tilde{x}_1, \dots, \tilde{x}_r\}$ and $\{y_1, \dots, y_r\}$ be an $(n+1)$ -lagrangian and (n) -dual as in Definition 3.1

where $2r = \text{rank}_{\mathbb{Z}} H_2(W)$. Suppose $\phi: \pi_1(X) \rightarrow G$ is a homomorphism, G is amenable and in $D(\mathbb{Z}_p)$, $G^{(n+1)} = \{e\}$, and $\phi([\mu])$ has infinite order in G .

From the meridian condition above, it follows that $b_0^{(2)}(M(K)) = 0 = b_1^{(2)}(M(K))$ by Lemma 3.12. Also, since $H_1(M(K)) \cong H_1(W)$, $b_2^{(2)}(W) = b_2(W; \mathbb{Z}_p) = b_2(W) = 2r$ by Lemma 3.13 and Lemma 3.14.

We will compute the von Neumann ρ -invariant of (M, ϕ) using W . First, since the images of the \tilde{x}_i and y_i form a lagrangian and its dual in $H_1(W; \mathbb{Q})$, it follows that the ordinary signature of W vanish. To compute the L^2 -signature of W , consider the intersection form

$$\lambda: H_2(W; \mathcal{N}G) \times H_2(W; \mathcal{N}G) \longrightarrow \mathcal{N}G.$$

Since $b_2^{(2)}(M(K)) = 0$ and $b_1^{(2)}(M(K)) = 0$, $H_2(W; \mathcal{N}G) \rightarrow H_2(W, M(K); \mathcal{N}G)$ is an L^2 -equivalence. Therefore λ is L^2 -nonsingular. Since $G^{(n+1)} = \{e\}$, $\pi_1(W) \rightarrow G$ factors through $\pi_1(W)/\pi_1(W)^{(n+1)}$ and there is an induced map

$$H_2(W; \mathbb{Z}[\pi_1(W)/\pi_1(W)^{(n+1)}]) \longrightarrow H_2(W; \mathcal{N}G).$$

Let $H \subset H_2(W; \mathcal{N}G)$ and $\overline{H} \subset H_2(W; \mathbb{Z}_p)$ be the submodules generated by the images of the \tilde{x}_i . By the existence of the (n) -dual, the images of the \tilde{x}_i in $H_2(W; \mathbb{Z}_p)$ are linearly independent over the field \mathbb{Z}_p . Therefore, \overline{H} has dimension r over \mathbb{Z}_p . From this, by Theorem 3.11, it follows that

$$\dim^{(2)} H \geq b_2^{(2)}(W) - b_2(W; \mathbb{Z}_p) + \dim_{\mathbb{Z}_p} \overline{H} = r.$$

By Proposition 3.7, it follows that $\text{sign}_G^{(2)}(W) = 0$.

The same argument also applies to $R = \mathbb{Q}$; in this case we do not need to appeal to Lemma 3.14 to deal with the \mathbb{Z}_p -coefficient Betti number, since we can use \mathbb{Q} when we apply Theorem 3.11. \square

4. Examples and computation

In this section, we give examples of knots which illustrate the usefulness of the new obstructions. The knots have very subtle behavior in the knot concordance group—these are (n) -solvable, and not distinguished from $(n.5)$ -solvable knots by any PTFA L^2 -signature obstructions, but not $(n.5)$ -solvable. This is detected using Theorem 3.2.

In order to construct non-PTFA amenable coefficient systems lying in Strebel's class to which we apply Theorem 3.2, we need some refined ingredients: mixed-coefficient commutator series and modulo p noncommutative Blanchfield form. We begin by introducing the former, and then proceed to the construction and analysis of the examples.

4.1. Mixed-coefficient commutator series

Suppose G is a group and $\mathcal{P} = (R_0, R_1, \dots)$ is a sequence of commutative rings with unity.

Definition 4.1. The \mathcal{P} -mixed-coefficient commutator series $\{\mathcal{P}^k G\}$ of G is defined by $\mathcal{P}^0 G = G$,

$$\mathcal{P}^{k+1} G = \text{Ker} \left\{ \mathcal{P}^k G \rightarrow \frac{\mathcal{P}^k G}{[\mathcal{P}^k G, \mathcal{P}^k G]} \rightarrow \frac{\mathcal{P}^k G}{[\mathcal{P}^k G, \mathcal{P}^k G]} \otimes_{\mathbb{Z}} R_k \right\}.$$

We remark that $\mathcal{P}^k G/[\mathcal{P}^k G, \mathcal{P}^k G]$ and $(\mathcal{P}^k G/[\mathcal{P}^k G, \mathcal{P}^k G]) \otimes R_k$ can be identified with $H_1(G; \mathbb{Z}[G/\mathcal{P}^k G])$ and $H_1(G; R_k[G/\mathcal{P}^k G])$, respectively. It can be verified that $\mathcal{P}^k G$ is a characteristic normal subgroup of G for any \mathcal{P} and k .

Example 4.2.

- (1) If $\mathcal{P} = (\mathbb{Z}, \mathbb{Z}, \dots)$, then $\{\mathcal{P}^k G\}$ is the standard derived series $\{G^{(k)}\}$.
- (2) If $\mathcal{P} = (\mathbb{Q}, \mathbb{Q}, \dots)$, then $\{\mathcal{P}^k G\}$ is the rational derived series of G .

Lemma 4.3. *Suppose $\mathcal{P} = (R_0, R_1, \dots)$, p is a fixed prime, and R_k has the property that any integer relatively prime to p is invertible in R_k for $k < n$. Then for any group G and $k \leq n$, $G/\mathcal{P}^k G$ is amenable and lies in $D(\mathbb{Z}_p)$.*

Proof. $H_1(G; R_k[G/\mathcal{P}^k G])$ is a module over R_k and consequently it has no torsion coprime to p as an abelian group. Since the $\mathcal{P}^k G/\mathcal{P}^{k+1} G$ is a subgroup of $H_1(G; R_k[G/\mathcal{P}^k G])$, the conclusion follows from Lemma 2.1. \square

For example, if R_k is either \mathbb{Z}_p or \mathbb{Q} for each k , then $G/\mathcal{P}^n G$ is amenable and lies in $D(\mathbb{Z}_p)$ by Lemma 4.3.

4.2. Knots with vanishing PTFA L^2 -signature obstructions

We begin by recalling a standard satellite construction, which has been used in several papers on knot concordance. It is often called “infection” or “generalized doubling”.

For a knot or link K in S^3 , we denote its exterior by $X(K) = S^3 - \text{int } N(K)$ where $N(-)$ is a tubular neighborhood. Let K be a knot in S^3 , and η be a simple closed curve in $X(K)$ which is unknotted in S^3 . Choose $N(\eta)$ disjoint from K , remove $\text{int } N(\eta)$ from S^3 , and then fill in it with the exterior $X(J)$ of another knot J along an orientation reversing homeomorphism on the boundary which identifies the meridian (resp. 0-linking longitude) of J with the 0-linking longitude (resp. meridian) of η . The result is homeomorphic to S^3 again, but K becomes a new knot in the resulting S^3 , which we denote by $K(\eta, J)$.

It is easily seen that if J and J' are concordant, then $K(\eta, J)$ and $K(\eta, J')$ are concordant.

Proposition 4.4. *Suppose K is slice, $[\eta] \in (\pi_1 X(K))^{(n)}$, and $\text{Arf}(J) = 0$. Then there exists an (n) -solution W for $K(\eta, J)$ satisfying the following: for any homomorphism $\phi: \pi_1 M(K(\eta, J)) \rightarrow G$ extending to W , if G is amenable and in $D(R)$ for some R , then we have $\rho^{(2)}(M(K), \phi) = \rho^{(2)}(M(J), \psi)$ where $\psi: \pi_1 M(J) \rightarrow \mathbb{Z}_d$ is the surjection sending a meridian to $1 \in \mathbb{Z}_d$ and d is the order of $\phi([\eta])$ in G . (If $d = \infty$, \mathbb{Z}_d is understood as \mathbb{Z} .)*

We remark that in Proposition 4.4 we do not assume $G^{(n+1)} = \{e\}$. Before proving Proposition 4.4, we give a consequence, which produces a large family of knots for which PTFA L^2 -signature obstructions vanish. For this purpose we need the following formula for abelian ρ -invariants of knots. We denote the Levine-Tristram signature function of a knot J by $\sigma_J: S^1 \rightarrow \mathbb{Z}$. Namely, choosing a Seifert matrix A of J , it is defined by $\sigma_J(w) = \text{sign}((1-w)A + (1-\bar{w})A^T)$.

Lemma 4.5 ([COT04, Proposition 5.1], [Fri05, Corollary 4.3], [CO, Lemma 8.7]). *Suppose J is a knot with meridian μ and $\alpha: \pi_1 M(J) \rightarrow G$ is a homomorphism whose image contained in an abelian subgroup of G . Then*

$$\rho^{(2)}(M(J), \alpha) = \begin{cases} \int_{S^1} \sigma_{J_0}(w) dw & \text{if } \alpha([\mu]) \in G \text{ has infinite order,} \\ \sum_{r=0}^{d-1} \sigma_{J_0}(e^{2\pi i r \sqrt{-1}/d}) & \text{if } \alpha([\mu]) \in G \text{ has order } d < \infty. \end{cases}$$

Corollary 4.6. *Suppose K is slice, $[\eta] \in \pi_1 X(K)^{(n)}$, $\text{Arf}(J) = 0$, and $\int_{S^1} \sigma_J(w) dw = 0$. Then there is an (n) -solution W for $J' = K(\eta, J)$ satisfying the following: for any homomorphism $\phi: \pi_1 M(J') \rightarrow G$ into a torsion-free amenable group G lying in $D(R)$ that extends to W , we have $\rho^{(2)}(M(J'), \phi) = 0$.*

Proof. Observing that the order of any element in a torsion-free group G is either 0 or ∞ , the conclusion follows immediately from Proposition 4.4 and Lemma 4.5. \square

Since PTFA groups are torsion-free amenable and in $D(R)$, the conclusion of Corollary 4.6 says that the PTFA L^2 -signature obstruction due to Cochran-Orr-Teichner [COT03, Theorem 4.2] does not distinguish our $J' = K(\eta, J)$ from $(n.5)$ -solvable knots, and consequently any prior methods (e.g., [CT07, CK08b, CHLb, CHL09, CHLc, CHLa]) do not neither, since all these essentially depend on [COT03, Theorem 4.2]. The following definition formalizes the property:

Definition 4.7. If K admits an (n) -solution W such that $\rho^{(2)}(M(K), \phi) = 0$ for any $\phi: \pi_1 M(K) \rightarrow G$ into a PTFA group G which extends to W , then we say that K is an (n) -solvable knot with vanishing PTFA L^2 -signature obstructions.

Obviously the knot J' obtained by the satellite construction in Corollary 4.6 is an (n) -solvable knot with vanishing PTFA L^2 -signature obstructions.

Let \mathcal{V}_n be the set of concordance classes of (n) -solvable knots with vanishing PTFA L^2 -signature obstructions.

Proposition 4.8. For any n , \mathcal{V}_n is a subgroup of the n th term \mathcal{F}_n of the solvable filtration.

Proof. Suppose W_1 and W_2 are (n) -solutions of two knots K_1 and K_2 satisfying Definition 4.7, respectively. Let $K = K_1 \# K_2$. It is known that there is a standard cobordism C between $M(K)$ and $M(K_1) \cup M(K_2)$ such that $\text{sign}_G^{(2)}(C) = 0 = \text{sign}(C)$ for any $\pi_1(C) \rightarrow G$ (e.g. see [COT04, p. 113], [CHL09, p. 1429]). Glueing C with $W_1 \cup W_2$ along $M(K_1) \cup M(K_2)$, we obtain an (n) -solution W for K . If $\psi: \pi_1(W) \rightarrow G$ is a homomorphism into a PTFA group G , then for its restrictions $\phi: \pi_1 M(K) \rightarrow G$ and $\phi_i: \pi_1 M(K_i) \rightarrow G$, we have $\rho^{(2)}(M(K), \phi) = \rho^{(2)}(M(K_1), \phi_1) + \rho^{(2)}(M(K_2), \phi_2)$ since C has vanishing (L^2 -) signatures. Each $\rho^{(2)}(M(K_i), \phi_i)$ vanishes by the hypothesis. This shows $[K] \in \mathcal{V}_n$. \square

Proof of Proposition 4.4. In [COT04, Lemma 5.4, Remark 5.7], a (0) -solution W_J for J is constructed as follows, assuming $\text{Arf}(J)$ vanishes. (See also [Cha07, Theorem 5.10].) Pushing a Seifert surface of J , we obtain a properly embedded framed surface Σ in the 4-ball B^4 . Let g be the genus of Σ . Then the desired W_J is obtained by removing a tubular neighborhood $N(\Sigma)$ from B^4 and attaching $(g \text{ 2-handles}) \times S^1$ along a collection of half basis curves on Σ .

We claim that $\pi_1(W_J) = \mathbb{Z}$, generated by a meridian. To see this, first consider the manifold N obtained by cutting B^4 along the trace of pushing from the Seifert surface to Σ . The group $\pi_1(B^4 - \Sigma)$ is an HNN extension of $\pi_1(N)$. Since N is homeomorphic to B^4 itself, it follows that $\pi_1(B^4 - \Sigma) = \mathbb{Z}$. Since any curve on Σ is null-homotopic in N , the attachment of $(g \text{ 2-handles}) \times S^1$ has no effect on the fundamental group.

In [COT04, Proposition 3.1], an (n) -solution W with $\partial W = K(\eta, J)$ is constructed by glueing W_J with a slice disk exterior X of K , along a solid torus on the boundary. It is known that $\rho^{(2)}(M(K(\eta, J)), \phi) = \rho^{(2)}(M(K), \psi) + \rho^{(2)}(M(J), \psi_J)$, where $\psi: \pi_1(M(K)) \rightarrow G$ and $\psi_J: \pi_1(M(J)) \rightarrow G$ are restrictions of an extension $\pi_1(W) \rightarrow G$ of the given ϕ (e.g., see [COT04, Proposition 3.2], [CHL09, Lemma 2.3]). By Theorem 1.2, $\rho^{(2)}(M(K), \psi) = 0$. Since ψ_J factors through $\pi_1(W_J) = \mathbb{Z}$, the image of ψ_J is exactly the cyclic subgroup in G generated by the image of a meridian of J . By the induction property for $\rho^{(2)}$, it follows that $\rho^{(2)}(M(J), \psi_J)$ is given as desired. \square

Our main examples are obtained by iterating this satellite construction as follows:

Input. A knot J_0 , a sequence of knots K_0, K_1, \dots, K_{n-1} , and simple closed curves η_k in $X(K_k)$ which are unknotted in S^3 .

Output. A sequence of knots J_0, J_1, \dots, J_n defined by $J_{k+1} = K_k(\eta_k, J_k)$.

Observe that $X(J_n) = X(J_0) \cup X(K_0 \cup \eta_0) \cup \dots \cup X(K_{n-1} \cup \eta_{n-1})$. From this we obtain another way to describe J_n : let $K'_0 = K_0$, and let $K'_{k+1} = K_k(\eta_k, K'_k)$. That is, K'_k is the knot obtained by using the unknot as J_0 in the above construction. Then, regarding $\eta_0 \subset X(K_0) \cup X(K_1 \cup \eta_1) \cup \dots \cup X(K_{n-1} \cup \eta_{n-1}) = X(K'_n)$, we can write $J_n = K'_n(\eta_0, J_0)$.

Lemma 4.9. *If $\text{lk}(K_k, \eta_k) = 0$ for each k , then $[\eta_0] \in \pi_1 X(K'_n)^{(n)}$.*

Proof. By the assumption, $[\eta_k]$ is a product of conjugates of a meridian of K_k in $\pi_1 X(K_k)$. Since a meridian of K_k is identified with a parallel of η_{k+1} in the satellite construction, an induction shows that $[\eta_0] \in \pi_1(X(K'_n))^{(n)}$. \square

From this observation, we immediately obtain the following fact by applying Proposition 4.4 and Corollary 4.6:

Proposition 4.10. *Suppose K_k is slice and $\text{lk}(K_k, \eta_k) = 0$ for each k , and $\text{Arf}(J_0) = 0$. Then there is an (n) -solution W for J_n such that if $\phi: \pi_1 M(J_n) \rightarrow G$ is a homomorphism extending to W and G is amenable and in $D(R)$ for some R , then $\rho^{(2)}(M(J_n), \phi) = \rho^{(2)}(M(J_0), \psi)$ where $\psi: \pi_1 M(J_0) \rightarrow \mathbb{Z}_d$ is a surjection and d is the order of an element in $G^{(n)}$. In addition, if $\int_{S_1} \sigma_{J_0}(w) dw = 0$, then via the (n) -solution W , J_n has vanishing PTFA L^2 -signature obstruction.*

4.3. New non-solvable knots

We apply the iterated satellite construction starting with an infinite family of initial knots J_0^i ($i = 1, 2, \dots$). Fix n and fix a family of knots $\{K_k \mid k = 0, \dots, n-1\}$ together with simple closed curves η_k in $X(K_k)$, and let $J_{k+1}^i = K_k(\eta_k, J_k^i)$. (One may use the same (K_k, η_k) for all k .)

Theorem 4.11. *Suppose $n > 0$, K_k is slice, $\text{lk}(K_k, \eta_k) = 0$, and the Alexander module $H_1(M(K_k); \mathbb{Z}[t^{\pm 1}])$ is nontrivial and generated by $[\eta_k]$ for each k . Then there is an infinite collection $\{J_0^i \mid i = 1, 2, \dots\}$ for which the knots J_n^i ($i = 1, 2, \dots$) obtained as above are (n) -solvable and have vanishing PTFA L^2 -signature obstructions, but any nontrivial linear combination $\sum_i a_i J_n^i$ of the J_n^i is not integrally $(n.5)$ -solvable (and therefore not slice.)*

Theorem 1.4 in the introduction immediately follows from Theorem 4.11. The case of $n = 0$ is classical, e.g., by Levine [Lev69b] (and Corollary 4.6).

In order to prove Theorem 4.11, we first describe how to choose $\{J_0^i\}$. Choose an infinite sequence of p_1, p_2, \dots of distinct increasing primes which are greater than the top coefficient of the Alexander polynomial $\Delta_{K_0}(t)$. Denote $w_i = e^{2\pi\sqrt{-1}/p_i}$. For the given $\{K_k\}$, let L be the maximum of the Cheeger-Gromov bound for the ρ -invariants of the $M(K_k)$ [CG85] (see also [CT07]). Namely, $|\rho^{(2)}(M(K_k), \alpha)| < L$ for any $k = 0, \dots, n-1$ and for any homomorphism α of $\pi_1 M(K_k)$.

Proposition 4.12. *There is a family of knots $\{J_0^i\}$ satisfying the following:*

- (C1) $\sigma_{J_0^i}(w_i) = \sigma_{J_0^i}(w_i^{-1}) > nL$ and $\sigma_{J_0^i}(w_i^r) = 0$ for $r \not\equiv \pm 1 \pmod{p_i}$.
- (C2) $\sigma_{J_0^i}(w_j^r) = 0$ for any r , whenever $i > j$.
- (C3) $\text{Arf}(J_0^i) = 0$ and $\int_{S_1} \sigma_{J_0^i}(w) dw = 0$.

Our construction of the $\{J_0^i\}$ is similar to that of [Cha09, Proof of Lemma 5.2], but provides a stronger conclusion. For this purpose we need the following facts:

Lemma 4.13 (Reparametrization Formula [Lit79, CO93, CK02, Cha07]). *For a knot K , let K' be the $(r, 1)$ -cable of K . Then $\sigma_{K'}(w) = \sigma_K(w^r)$ for all $w \in S^1$ but finitely many points, and $\Delta_{K'}(t) = \Delta_K(t^r)$ up to multiplication by at^n , $a \neq 0$.*

Addendum: If w is a primitive p^a th root of unity for some prime p , then $\sigma_{K'}(w) = \sigma_K(w^r)$.

The addendum is a consequence of the main statement, by the following observation: if w is a primitive p^a th root of unity, then for any knot K , σ_K is locally constant near w since w is not a zero of the Alexander polynomial $\Delta_K(t)$. Therefore we can perturb w to avoid the exceptional points, without changing the value of σ_K .

Proof of Proposition 4.12. For each i , choose small $\epsilon_i > 0$ such that the ϵ_i -neighborhood $N_{\epsilon_i}(w_i)$ is disjoint from $T_i = \{w_j^r \mid j \leq i, r \in \mathbb{Z}\} - \{w_i\}$. Appealing to [CL04], [Cha09, Lemma 5.6], there is a knot K_i with a bump signature function supported by $N_{\epsilon_i}(w_i) \cup N_{\epsilon_i}(w_i^{-1})$, that is, $\sigma_{K_i}(w_i) > 0$ and σ_{K_i} vanishes outside $N_{\epsilon_i}(w_i) \cup N_{\epsilon_i}(w_i^{-1})$. (Recall the symmetry $\sigma_K(w) = \sigma_K(w^{-1})$.)

Let K'_i be the $(p_i, 1)$ -cable of K_i , and let $J_i = K_i \# (-K'_i)$. By Reparametrization Formula, we have $\sigma_{J_i}(w_j^r) = \sigma_{K_i}(w_j^r) - \sigma_{K'_i}(w_j^{rp_i})$. Therefore, for $j = i$, we have $\sigma_{J_i}(w_i^r) = \sigma_{K_i}(w_i^r) - \sigma_{K_i}(1) = \sigma_{K_i}(w_i^r)$. Thus $\sigma_{J_i}(w_i^r) \neq 0$ if and only if $r \equiv \pm 1 \pmod{p_i}$. Also, for $j < i$, we have $\sigma_{J_i}(w_j^r) = 0$ since $\sigma_{K_i}(w_j^r)$ vanishes for any r .

By the Reparametrization Formula, we have $\int_{S^1} \sigma_{K_i}(w) dw = \int_{S^1} \sigma_{K'_i}(w) dw$. It follows that $\int_{S^1} \sigma_{J_i}(w) dw = 0$. Now the desired knot J_0^i is obtained by taking the connected sum of sufficiently large even number of copies of J_i . \square

By Proposition 4.10 and Proposition 4.12 (C3) above, we obtain immediately the conclusion that the associated J_n^i are (n) -solvable knots which have vanishing PTFA L^2 -signature obstructions. (Note that we do not need (C1) and (C2) for this purpose.)

The remaining part of this section is devoted to the proof that any nontrivial linear combination $J = \#_i a_i J_n^i$ is not integrally $(n.5)$ -solvable. This consists of two parts. In the first part, assuming that J is integrally $(n.5)$ -solvable, we construct a 4-manifold W_0 from an integral $(n.5)$ -solution of J , which has $M(J_0^i)$ as a boundary component for some fixed i for which $a_i \neq 0$. In the second part, using a non-PTFA mixed-coefficient commutator quotient of $\pi_1(W_n)$ as a coefficient system, we apply our new obstruction (Theorem 3.2) in order to derive a contradiction. The first part is similar to arguments in [CHL09] but we make several technical simplifications by taking advantages of our new obstruction. (See also Remark 4.15.)

Proof of non- $(n.5)$ -solvability, Part I: Construction of bounding 4-manifolds. Suppose a (finite) linear combination $J = \#_i a_i J_n^i$ is integrally $(n.5)$ -solvable. Dropping terms with zero coefficient, we may assume $a_i \neq 0$ for each i . Furthermore we may assume that $a_1 > 0$ by taking concordance inverses if necessary. We use the following building blocks:

- (1) Let V be an integral $(n.5)$ -solution for $M(J)$.
- (2) Let V_i be the (n) -solution of $M(J_n^i)$ given in Proposition 4.10.
- (3) Let E_k be the standard cobordism from $M(J_k^1) \cup M(K_k)$ to $M(J_{k+1}^1)$, viewing J_{k+1}^1 as $K_k(\eta_k, J_k^1)$. Namely, $E_k = (M(K_k) \times [0, 1]) \cup (M(J_k^1) \times [0, 1]) / \sim$ where $N(\eta_k)$ in $M(K_k) = M(K_k) \times 0$ is identified with the solid torus $M(J_k^1) - X(J_k^1)$ in $M(J_k^1) = M(J_k^1) \times 0$ in such a way that a zero-linking longitude of η_k is identified with a meridian of J_k^1 . For more discussions on E_k , see [CHL09, p. 1429].

- (4) Let C be the standard cobordism from $\bigcup_i a_i M(J_n^i)$ to $M(J)$ obtained by glueing copies of $M(J_n^i) \times [0, 1]$ similarly to (3), viewing the connected sum as a satellite knot formed along meridian curves. (Or, for an alternative description, see [COT04, p. 113].)

Let $\epsilon_i = a_i/|a_i|$, $b_1 = a_1 - 1$ (≥ 0) and $b_i = |a_i|$ for $i \geq 2$. For $r = 1, \dots, b_i$, let $V_{i,r}$ be a copy of $-\epsilon_i V_i$. Define

$$W_n = \left(\bigcup_i \bigcup_{r=1}^{b_i} V_{i,r} \right) \cup_i \coprod_{b_i M(J_n^i)} C \coprod_{M(J)} V.$$

Note that $\partial W_n = M(J_n^1)$. For $k = n-1, n-2, \dots, 0$, define W_k by

$$\begin{aligned} W_k &= E_k \coprod_{M(J_{k+1}^1)} E_{k+1} \coprod_{M(J_{k+2}^1)} \cdots \coprod_{M(J_{n-1}^1)} E_{n-1} \coprod_{M(J_n^1)} W_n \\ &= E_k \coprod_{M(J_{k+1}^1)} W_{k+1}. \end{aligned}$$

We have $\partial W_k = M(J_k^1) \cup M(K_k) \cup \cdots \cup M(K_{n-1})$ for $k < n$. See Figure 1.

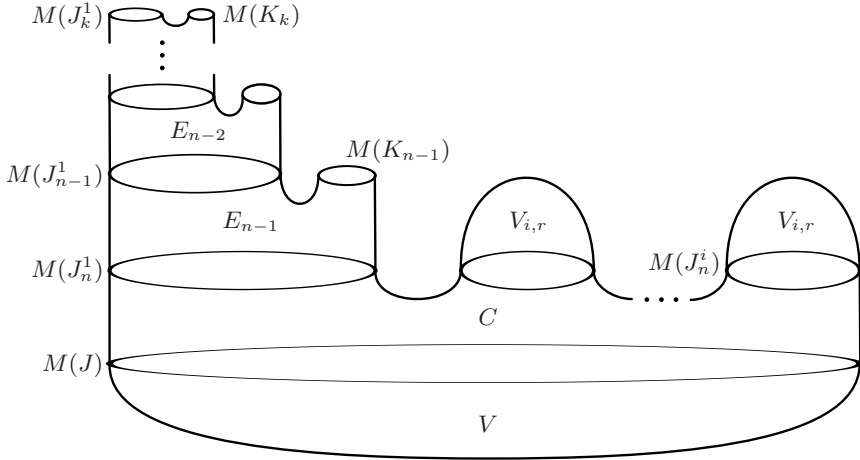


FIGURE 1. The cobordism W_k

From now on, we denote by \mathcal{P} the sequence (R_0, R_1, \dots, R_n) where $R_k = \mathbb{Q}$ for $k < n$, $R_n = \mathbb{Z}_{p_1}$. Then the mixed-coefficient commutator series $\mathcal{P}^k G$ of a group G is defined for $k = 0, 1, \dots, n+1$ as in Definition 4.1. We have $G^{(k)} \subset \mathcal{P}^k G$.

Theorem 4.14. *For $k = 0, 1, \dots, n$, the homomorphism*

$$\phi_k: \pi_1(W_k) \longrightarrow \pi_1(W_k)/\mathcal{P}^{n-k+1}\pi_1(W_k).$$

sends a meridian of J_k^1 into the abelian subgroup $\mathcal{P}^{n-k}\pi_1(W_k)/\mathcal{P}^{n-k+1}\pi_1(W_k)$. Furthermore, the image of a meridian of J_k^1 under ϕ_k has order p_1 if $k = 0$, and has order ∞ if $k > 0$.

An immediate consequence of the first statement is that ϕ_k restricted on $\pi_1 M(J_k^1)$ is into $\mathcal{P}^{n-k}\pi_1(W_k)/\mathcal{P}^{n-k+1}\pi_1(W_k)$. Roughly speaking, the first statement is essentially on the same lines with the arguments of Lemma 4.9 that $[\eta_k]$ (which is identified with the meridian of J_k) is in the $(n-k)$ th term of the derived series. The rational derived

series analogue of the second statement (for a similar but more complicated 4-manifold) was shown in [CHL09, Section 8]. Our argument is similar to their proof, but in order to handle the mixed-coefficient commutator series which gives our non-PTFA coefficient systems, we need to use a new ingredient, namely modulo p higher-order Blanchfield pairings. We postpone related discussions and the proof of Theorem 4.14 to the next section.

Proof of non-(n.5)-solvability, Part II: Computation of L^2 -signature. Now we consider the coefficient system $\phi_0: \pi_1(W_0) \rightarrow G = \pi_1(W_0)/\mathcal{P}^{n+1}\pi_1(W_0)$. As an abuse of notation, we denote by ϕ_0 various restrictions of ϕ_0 , particularly the restriction on $\pi_1 M(J_0^1)$. Note that $G^{(n+1)} = \{e\}$ obviously, and G is amenable and lies in $D(\mathbb{Z}_{p_1})$ by Lemma 4.3.

For convenience, for a 4-manifold X over G , we temporarily denote the L^2 -signature defect by $S_G(X) = \text{sign}_G^{(2)}(X) - \text{sign}(X)$. Since ∂W_0 is the disjoint union of $M(J_0^1)$, $M(K_0), \dots, M(K_{n-1})$, we have

$$S_G(W_0) = \rho^{(2)}(M(J_0^1), \phi_0) + \sum_{k=0}^{n-1} \rho^{(2)}(M(K_k), \phi_0)$$

By Novikov additivity, we have

$$S_G(W_0) = S_G(V) + S_G(C) + \sum_i \sum_{r=1}^{b_i} S_G(V_{i,r}) + \sum_{k=0}^{n-1} S_G(E_k)$$

Now we investigate the terms in right hand sides of the above two equations:

- (1) $\rho^{(2)}(M(J_0^1), \phi_0) = \sum_{r=0}^{p_1-1} \sigma_{J_0^1}(w_1^r)$. To verify this, note that the image of a meridian of J_0^1 under ϕ_0 has order p_1 by Theorem 4.14. By Lemma 4.5, the claim follows.
- (2) $S_G(V) = 0$, $S_G(C) = 0$, $S_G(E_k) = 0$. First, since V is an integral (n.5)-solution for $M(J)$, $S_G(V) = \rho^{(2)}(M(J), \phi_0)$ vanishes by Theorem 3.2. For the latter two claims, appeal to [CHL09, Lemma 2.4] (see also [COT04, Lemma 4.2]).
- (3) $S_G(V_{1,r})$ is either 0 or $-\sum_{r=0}^{p_1-1} \sigma_{J_0^1}(w_1^r)$, and $S_G(V_{i,r}) = 0$ for $i \geq 2$. To show this, first apply Proposition 4.10 to obtain

$$S_G(V_{i,r}) = -\epsilon_i \rho^{(2)}(M(J_n^i), \phi_0) = -\epsilon_i \rho^{(2)}(M(J_0^i), \alpha_{i,r})$$

for some $\alpha_{i,r}: \pi_1 M(J_0^i) \rightarrow \mathbb{Z}_d$ where d is the order of an element in $G^{(n)}$. Recall that $G^{(n)}$ is a subgroup of $\mathcal{P}^n \pi_1(W_0)/\mathcal{P}^{n+1} \pi_1(W_0)$ which is a vector space over \mathbb{Z}_{p_1} . It follows that d is either 1 or p_1 . Therefore $\rho^{(2)}(M(J_0^i), \alpha_{i,r})$ is equal to either 0 or $\sum_{r=0}^{p_1-1} \sigma_{J_0^i}(w_1^r)$ by Lemma 4.5. For $i = 1$, since $\epsilon_1 = 1$, we obtain the conclusion. For $i \geq 2$, the sum is zero since each summand vanishes by the condition (C2).

From the above facts, it follows that

$$C \cdot \sum_{r=0}^{p_1-1} \sigma_{J_0^1}(w_1^r) = - \sum_{k=0}^{n-1} \rho^{(2)}(M(K_k), \phi_0)$$

where $C \geq 1$. But it contradicts the condition (C1). This shows that the linear combination $J = \#_i a_i J_n^i$ is not integrally (n.5)-solvable.

Remark 4.15. In our construction above, the only requirement for the slice knots K_k is that their classical Alexander modules are nontrivial and generated by η_k . Interestingly, while our examples are subtler in the knot concordance group than prior examples of [CHL09] which are constructed in a similar way but detected by PTFA L^2 -signatures, the construction in [CHL09] need an additional significant restriction that certain ρ -invariants

of the $M(K_k)$ must vanish; see Step 1 of [CHL09, Theorem 8.1]. This is essentially because our L^2 -signatures are sharp enough to separate the contributions of the knots J_0^i ($i > 1$) from those of J_0^1 ; see the above analysis of the terms $S_G(V_{i,r})$. In case of PTFA L^2 -signatures, it is unable to separate these contributions, and hence [CHL09] constructs more complicated 4-manifolds (using more pieces called “ \mathcal{R} -caps”) and carries out a clever technical analysis using the additional assumption on the vanishing of certain ρ -invariants of $M(K_k)$. Our L^2 -signature obstruction enables us not only to produce subtler examples but also to simplify such complications.

5. Modulo p noncommutative Blanchfield pairing

In this section we introduce the modulo p higher-order noncommutative Blanchfield pairing on a 3-manifold M (e.g., the zero-surgery manifold of a knot). We remark that a noncommutative Blanchfield pairing first appeared in Duval’s work [Duv86] on boundary links, as a linking form over the group ring of a free group. Cochran-Orr-Teichner first introduced a noncommutative Blanchfield pairing of knots, over the group ring $\mathbb{Z}G$ (and its localizations) for PTFA groups G [COT03]. For further development of the latter, see also work of Cochran [Coc04] and Leidy [Lei06].

In the first subsection, we define the modulo p noncommutative Blanchfield pairing and discuss modulo p generalizations of various known properties of the prior noncommutative Blanchfield pairings. In the second subsection, we discuss how the modulo p version is combined with techniques of Cochran-Harvey-Leidy [CHL09] to show the nontriviality of certain non-PTFA coefficient systems obtained as mixed-coefficient commutator series quotients, as asserted in Theorem 4.14.

5.1. Definition and properties

Suppose R is a fixed commutative ring with unity and M is a closed 3-manifold endowed with a homomorphism $\phi: \pi_1(M) \rightarrow G$ satisfying the following:

- (BL1) RG is an Ore domain, namely, the quotient skew-field $\mathcal{K} = RG(RG - \{0\})^{-1}$ is well-defined.
- (BL2) $H_1(M; \mathcal{K})$ vanishes.

Assuming these conditions, the noncommutative Blanchfield pairing can be defined, following [COT03]. We outline the definition: from the short exact sequence $0 \rightarrow RG \rightarrow \mathcal{K} \rightarrow \mathcal{K}/RG \rightarrow 0$, we obtain a Bockstein homomorphism

$$H^1(M; \mathcal{K}/RG) \xrightarrow{\cong} H^2(M; RG)$$

which is an isomorphism since $H^2(M; \mathcal{K}) = H_1(M; \mathcal{K}) = 0$ by duality and (BL2) and since $H^1(M; \mathcal{K}) = \text{Hom}_{\mathcal{K}}(H_1(M; \mathcal{K}), \mathcal{K}) = 0$ by the universal coefficient theorem over the skew field \mathcal{K} . Composing the inverse of the Bockstein homomorphism with the Poincaré duality isomorphism and the Kronecker evaluation map, we obtain the following:

Proposition 5.1 (Essentially due to [COT03]). *Under the assumptions (BL1), (BL2), there is a linking form*

$$Bl: \overline{H_1(M; RG)} \rightarrow H^2(M; RG) \rightarrow H^1(M; \mathcal{K}/RG) \rightarrow \text{Hom}_{RG}(H_1(M; RG), \mathcal{K}/RG).$$

We often denote $Bl(x)(y)$ by $Bl(x, y)$, viewing it as a sesquilinear pairing. It can be shown that Bl is hermitian, i.e., $Bl(x, y) = \overline{Bl(y, x)}$ [COT03].

It is known that (BL1) is satisfied for $R = \mathbb{Q}$ (or \mathbb{Z}) and any PTFA group G . This case has been used in several recent work, most notably in [CHL09]. The following observation enables us to define $B\ell$ when $R = \mathbb{Z}_p$ (p prime) as well:

Lemma 5.2. *If G is PTFA, then $\mathbb{Z}_p G$ is an Ore domain. Consequently, there exists the Ore localization $\mathcal{K} = \mathbb{Z}_p G(\mathbb{Z}_p G - \{0\})^{-1}$, which is a skew-field containing $\mathbb{Z}_p G$.*

Proof. Since G is PTFA and \mathbb{Z}_p is a field, $\mathbb{Z}_p G$ has no zero divisor by Bovdi's theorem (see [Pas77, p. 592]). Since G is solvable, it follows that $\mathbb{Z}_p G$ is an Ore domain, by Lewin's result [Lew72] (see also [Pas77, p. 611]). \square

The main example to keep in mind is: suppose M is the zero-surgery manifold of a knot K in S^3 endowed with a nontrivial PTFA coefficient system $\phi: \pi_1(M) \rightarrow G$. Then, it is known that (BL1) and (BL2) are satisfied for $R = \mathbb{Q}$ [COT03]. For $R = \mathbb{Z}_p$, (BL1) is satisfied by Lemma 5.2, and (BL2) is satisfied by appealing to the following mod p analogue of [COT03, Proposition 2.11]. Its proof is identical with that of [COT03, Proposition 2.11] and therefore omitted.

Lemma 5.3. *Suppose G is PTFA, $R = \mathbb{Z}_p$ or \mathbb{Q} , \mathcal{K} is the quotient skew-field of RG , X is a finite CW complex, and $\phi: \pi_1(X) \rightarrow G$ is a nontrivial homomorphism. Then $\dim_{\mathcal{K}} H_0(X; \mathcal{K}) = 0$ and $\dim_{\mathcal{K}} H_1(X; \mathcal{K}) \leq \dim_R H_1(X; R) - 1$.*

If RG is a PID, then \mathcal{K}/RG is divisible and so injective over RG . Thus the evaluation map in the definition of $B\ell$ is an isomorphism. It follows that $B\ell$ is nonsingular. For example, this applies to the case of $G = \mathbb{Z}$ and $R = \mathbb{Q}$ or \mathbb{Z}_p .

Many prior known results on the noncommutative Blanchfield pairing over \mathbb{Q} also hold for our new setup, namely for $R = \mathbb{Z}_p$. In what follows we discuss some of such results (Theorems 5.4 and 5.6) that we need. For these results, since the proofs for $R = \mathbb{Q}$ in the literature can be carried out for $R = \mathbb{Z}_p$ as well (in many cases we need Lemma 5.3 instead of [COT03, Proposition 2.11]), we will give references for the $R = \mathbb{Q}$ case, without proofs for the $R = \mathbb{Z}_p$ case.

First, the following result concerns the effect of coefficient change.

Theorem 5.4 ([Lei06, Theorem 4.7], [Cha07, Theorem 5.16], [CHL09, Lemma 6.5, Theorem 6.6] for $R = \mathbb{Q}$). *Suppose $R = \mathbb{Q}$ or \mathbb{Z}_p , $\phi: \pi_1(M) \rightarrow G$ is a homomorphism and $\iota: G \rightarrow \Gamma$ is an injection such that both (M, ϕ) and $(M, \iota\phi)$ satisfy (BL1, BL2). Let \mathcal{K} and \mathcal{K}' be the quotient skew-fields of RG and $R\Gamma$, and $B\ell$ and $B\ell'$ be the Blanchfield forms on $H_1(M; RG)$ and $H_1(M; R\Gamma)$, respectively. If RG is a PID, then*

- (1) *The map ι induces an injection $\iota_*: \mathcal{K}/RG \rightarrow \mathcal{K}'/R\Gamma$.*
- (2) *$H_1(M; R\Gamma) = R\Gamma \otimes_{RG} H_1(M; RG)$ and $B\ell'(1 \otimes x, 1 \otimes y) = \iota_* B\ell(x, y)$.*

Next we introduce an R -coefficient homology version of Cochran-Harvey-Leidy's notion of a rational (n) -bordism [CHL09, Definition 5.2].

Definition 5.5. For $R = \mathbb{Q}$ or \mathbb{Z}_p , a compact 4-manifold W with $\pi = \pi_1(W)$ is called an R -coefficient (n) -bordism if there are elements $x_1, \dots, x_r, y_1, \dots, y_r \in H_2(W; R[\pi/\pi^{(n)}])$ such that $2r = \dim_R \text{Coker}\{H_2(\partial W; R) \rightarrow H_2(W; R)\}$ and the $R[\pi/\pi^{(n)}]$ -coefficient intersection form λ_n on $H_2(W; R[\pi/\pi^{(n)}])$ satisfies $\lambda_n(x_i, x_j) = 0$ and $\lambda_n(x_i, y_j) = \delta_{ij}$. We call $\{x_i\}$ and $\{y_j\}$ an (n) -langrangian and its (n) -dual, respectively.

Theorem 5.6 ([CHL09, Section 5, 6, Theorem 6.3] for $R = \mathbb{Q}$). *Suppose $R = \mathbb{Q}$ or \mathbb{Z}_p , W is an R -coefficient (n) -bordism and each boundary component $\partial_i W$ of W satisfies $b_1(\partial_i W; R) = 1$. Suppose M is a boundary component of W endowed with a non-trivial homomorphism $\phi: \pi_1(M) \rightarrow G$ into a PTFA group G , so that the Blanchfield*

pairing $B\ell$ on $H_1(M; RG)$ is defined. If ϕ extends to $\pi_1(W)$ and $G^{(n)} = \{e\}$, then $P = \text{Ker}\{H_1(M; RG) \rightarrow H_1(W; RG)\}$ satisfies $B\ell(P, P) = 0$.

Obviously an integral (n) -solution is an integral $(\mathbb{Z}$ -coefficient) (n) -bordism. We remark that an integral (n) -bordism is not necessarily a \mathbb{Z}_p -coefficient (n) -bordism, since the \mathbb{Z}_p -rank of $H_*(-; \mathbb{Z}_p)$ may be greater than the \mathbb{Z} -rank of $H_*(-; \mathbb{Z})$. However, the following observation says that, for example, an integral (n) -solution for a knot in S^3 is a \mathbb{Z}_p -coefficient (n) -bordism for any p .

Lemma 5.7. *Suppose W is an integral (n) -solution for a 3-manifold M and $H_1(M)$ has no p -torsion. Then*

$$\dim_{\mathbb{Z}_p} \text{Coker}\{H_2(M; \mathbb{Z}_p) \rightarrow H_2(W; \mathbb{Z}_p)\} = b_2(W; \mathbb{Z}_p) = b_2(W)$$

for any p . Consequently, W is a \mathbb{Z}_p -coefficient (n) -bordism.

Proof. $H_2(M; \mathbb{Z}_p) \rightarrow H_2(W; \mathbb{Z}_p)$ is equal to $H^1(M; \mathbb{Z}_p) \rightarrow H^2(W, M; \mathbb{Z}_p)$ by duality. The latter is a zero map by universal coefficient, since $H_1(M) \rightarrow H_1(W)$ is an isomorphism. From this the first equality follows. The second equality follows from Lemma 3.14. \square

5.2. Analysis of mixed-coefficient commutator quotient coefficient systems

In this subsection we give a proof of the following assertion used in Section 4.3. We use the notations in Section 4.3, and assume the hypothesis of Theorem 4.11. For simplicity, we denote J_k^1 by J_k and denote p_1 by p .

Theorem 4.14. *For $k = 0, 1, \dots, n$, the homomorphism*

$$\phi_k: \pi_1(W_k) \longrightarrow \pi_1(W_k) / \mathcal{P}^{n-k+1} \pi_1(W_k).$$

sends a meridian of J_k into the abelian subgroup $\mathcal{P}^{n-k} \pi_1(W_k) / \mathcal{P}^{n-k+1} \pi_1(W_k)$. Furthermore, the image of a meridian of J_k under ϕ_k has order p if $k = 0$, and has order ∞ if $k > 0$.

We denote the meridian of J_k and K_k in $M(J_k)$ and $M(K_k)$ by μ_k and ν_k , respectively. Recall that E_k is the standard cobordism between $M(J_k) \cup M(K_k)$ and $M(J_{k+1})$ (see Section 4.3, Proof of non- $(n.5)$ -solvability). Then $\nu_k \subset M(K_k) \subset E_k$ and $\mu_{k+1} \subset M(J_{k+1}) \subset E_k$ are isotopic in E_k , and $\mu_k \subset M(J_k) \subset E_k$ and $\eta_k \subset M(K_k) \subset E_k$ are isotopic in E_k .

Assertion 1. *Suppose η_k has linking number zero with K_k for any k . Then the inclusion $W_k \rightarrow W_{k+1}$ gives rise to an isomorphism*

$$\pi_1(W_k) / \pi_1(W_k)^{(n-k+1)} \longrightarrow \pi_1(W_{k+1}) / \pi_1(W_{k+1})^{(n-k+1)}.$$

for any $k < n$. Consequently, the induced map

$$\mathcal{P}^{n-k} \pi_1(W_k) / \mathcal{P}^{n-k+1} \pi_1(W_k) \longrightarrow \mathcal{P}^{n-k} \pi_1(W_{k+1}) / \mathcal{P}^{n-k+1} \pi_1(W_{k+1})$$

is an isomorphism.

To prove this, let X be a copy of $X(J_k)$ and view it as a subspace of

$$X \amalg_{\partial X} (M(K_k) - N(\eta_k)) = M(J_{k+1}) \subset W_{k+1}.$$

Denote a meridian and longitude of J_k on ∂X by μ and λ , respectively. From the construction of E_k , it follows that $(M(J_{k+1}) \cup (S^1 \times D^2)) / \partial X \sim S^1 \times S^1$ is a deformation retract of E_k , where μ and λ are identified with $S^1 \times *$ and $* \times S^1$, respectively. Therefore

$W_k = E_k \amalg_{M(J_{k+1})} W_{k+1}$ is homotopy equivalent to $(W_{k+1} \cup (S^1 \times D^2)) / \partial X \sim S^1 \times S^1$. It follows that

$$\pi_1(W_k)/\pi_1(W_k)^{(n-k+1)} \cong \pi_1(W_{k+1})/\langle \pi_1(W_{k+1})^{(n-k+1)}, \lambda \rangle$$

where $\langle - \rangle$ denotes the normal subgroup in $\pi_1(W_{k+1})$ generated by $-$. Since $[\lambda]$ is in $\pi_1(X)^{(2)} \subset \langle \mu \rangle^{(2)}$, it suffices to show $[\mu] \in \pi_1(W_{k+1})^{(n-k-1)}$. If $k = n - 1$, it holds obviously since $\pi_1(W_{k+1})^{(n-k-1)} = \pi_1(W_{k+1})$. Suppose $k \leq n - 2$. Since μ lies in $M(J_{k+1}) \subset W_{k+1}$, $[\mu] \in \langle \mu_{k+1} \rangle$. Since η_{k+1} has linking number zero with K_{k+1} , we have, in $\pi_1(W_{k+1})$, $\langle \mu_{k+1} \rangle = \langle \eta_{k+1} \rangle = \langle \nu_{k+1} \rangle^{(1)} = \langle \mu_{k+2} \rangle^{(1)}$. Applying it repeatedly, it follows that $[\mu] \in \langle \mu_{k+1} \rangle \subset \langle \mu_n \rangle^{(n-k-1)}$. This proves the first conclusion of Assertion 1. The second conclusion of Assertion 1 follows since $\mathcal{P}^i G$ is a characteristic subgroup of G containing $G^{(i)}$.

Also, in the above argument, we have observed that the meridian μ_k in $M(J_k) \subset W_k$ lies in $\pi_1(W_k)^{(n-k)} \subset \mathcal{P}^{n-k} \pi_1(W_k)$. From this the first conclusion of Theorem 4.14 follows.

Assertion 2. For $R = \mathbb{Q}$ or \mathbb{Z}_p and for any k , W_k is an R -coefficient (n) -bordism.

By straightforward Mayer-Vietoris arguments applied to our construction of W_k , one can show $\text{Coker}\{H_2(\partial W_k; R) \rightarrow H_2(W; R)\} \cong H_2(W_n; R) \oplus \bigoplus_{i,j} H_2(V_{i,r}; R)$. (See, for a similar argument for a more complicated 4-manifold, [CHL09, Proof of Proposition 8.2].) Since W_n and $V_{i,r}$ are integral (n) -solutions of knots, W_n and the $V_{i,r}$ are R -coefficient (n) -bordisms by Lemma 5.7. By the above $H_2(-; R)$ computation, it follows (the images of) the (n) -lagrangians and (n) -duals of W_n and the $V_{i,r}$ form an (n) -lagrangian and (n) -dual for W_k . This proves Assertion 2.

Now we use an induction on $k = n, n - 1, \dots, 0$ to prove that the order of the image of a meridian of J_k under

$$\phi_k: \pi_1(W_k) \longrightarrow \pi_1(W_k)/\mathcal{P}^{n-k+1} \pi_1(W_k)$$

is p if $k = 0$, ∞ otherwise. For $k = n$, since W_n is a solution, we have $H_1(M(J_n)) \cong H_1(W_n) = H_1(W_n)/\text{torsion} = \pi_1(W_n)/\mathcal{P}^1 \pi_1(W_n)$. The conclusion follows from this.

Now, assuming that it holds for $k + 1$, we will show the conclusion for k . To simplify notations, in this proof we temporarily denote $\pi = \pi_1(W_{k+1})$, $G = \pi/\mathcal{P}^{n-k} \pi$.

By the first part of Theorem 4.14, the coefficient system $\pi_1(M_{J_{k+1}}) \rightarrow G$ factors through the abelian group $\mathcal{P}^{n-k-1} \pi/\mathcal{P}^{n-k} \pi$, and so it factors through $H_1(M_{J_{k+1}})$. By the induction hypothesis, the meridian $[\mu_{k+1}]$ has infinite order in G . Thus, $H_1(M(J_{k+1})) \cong \langle t | \cdot \rangle$ actually injects into G . Therefore, by Theorem 5.4,

$$H_1(M(J_{k+1}); RG) = RG \otimes_{R[t^{\pm 1}]} H_1(M(J_{k+1}); R[t^{\pm 1}]) = RG \otimes_{R[t^{\pm 1}]} H_1(M(K_k); R[t^{\pm 1}])$$

Consider the following commutative diagram, where $R = R_{n-k}$:

$$\begin{array}{ccccc} H_1(M(J_{k+1}); \mathbb{Z}G) & \longrightarrow & H_1(W_{k+1}; \mathbb{Z}G) & \xrightarrow{\frac{\mathcal{P}^{n-k} \pi}{\mathcal{P}^{n-k+1} \pi}} & \\ \downarrow & & \downarrow & \swarrow & \\ H_1(M(J_{k+1}); RG) & \longrightarrow & H_1(W_{k+1}; RG) & & \end{array}$$

We claim that for $[\eta_k] \in H_1(M(K_k); \mathbb{Z}[t^{\pm 1}])$, the image of $1 \otimes [\eta_k] \in H_1(M(J_{k+1}); \mathbb{Z}G)$ in $H_1(W_{k+1}; RG)$ is nontrivial. Suppose not. Then, by Theorem 5.6, the noncommutative Blanchfield form on $H_1(M(J_{k+1}); RG)$ vanishes on $(1 \otimes [\eta_k], 1 \otimes [\eta_k])$. By Theorem 5.4, it follows that the classical Blanchfield form on $H_1(M(J_{k+1}); R[t^{\pm 1}]) = H_1(M(K_k); R[t^{\pm 1}])$

vanishes on $([\eta_k], [\eta_k])$. But this is a contradiction since $[\eta_k]$ generates the nonzero module $H_1(M(K_k); R[t^{\pm 1}])$ and the classical Blanchfield form of a knot is always nonsingular (for both $R = \mathbb{Z}_p$ and \mathbb{Q}). Recall that, for $k = 0$, we have $R_0 = \mathbb{Z}_p$ and $H_1(M(K_0); \mathbb{Z}_p[t^{\pm 1}]) \neq 0$ since the Alexander polynomial $\Delta_{K_0}(t)$ is not a unit even in $\mathbb{Z}_p[t^{\pm 1}]$, by our choice of the primes p_i . This proves the claim.

From the claim, it follows that the image of $[\eta_k]$ in $\mathcal{P}^{n-k}\pi/\mathcal{P}^{n-k+1}\pi$ is nontrivial. By Assertion 1, it follows that the image of $[\mu_k] = [\eta_k]$ in

$$\frac{\mathcal{P}^{n-k}\pi_1(W_k)}{\mathcal{P}^{n-k+1}\pi_1(W_k)} \cong \frac{\mathcal{P}^{n-k}\pi}{\mathcal{P}^{n-k+1}\pi}$$

is nontrivial. Since this is a torsion-free abelian group for $k \neq 0$ and a vector space over \mathbb{Z}_p for $k = 0$ by our choice of \mathcal{P} , the order of the image of $[\mu_k]$ is p if $k = 0$, and ∞ otherwise. This completes the proof of Theorem 4.14.

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