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Signature invariants of links from irregular covers and non-abelian covers

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Abstract

Signature invariants of odd dimensional links from irregular covers and non-abelian covers of complements are obtained by using the technique of Casson and Gordon. We show that the invariants vanish for slice links and can be considered as invariants under F_m-link concordance. We illustrate examples of links that are not slice but behave as slice links for any invariants from abelian covers.

1. *Introduction*

In the theory of link concordance, various covers of complements of links give rise to invariants. It is sometimes natural to think of extra structures on links [**1**, **11**, **14**, **15**]. An extra structure on a link is usually given by a homomorphism from the link group to another group and this defines a cover of the complement. Invariants under concordance with an extra structure are derived from the cover induced by the extra structure, but it is hard to see whether they are invariants under the ordinary link concordance. In the case of higher odd dimensional boundary or F_m-links, it was shown that invariants derived from the free cover completely determine the boundary or F_m-link concordance classes [**1**, **5**, **11**, **12**, **17**]. However, it is not known whether the information from the free cover is invariant under the ordinary link concordance.

Abelian covers can always be obtained without imposing any extra structure on links and they have been used to obtain link concordance invariants. (We say a cover is *abelian* if it is associated to a homomorphism that factors through the abelianization.) For example, the link signature is extracted from finite cyclic covers and, in [**1**], a link concordance invariant is obtained by abelianizing the information from the free cover. Abelian covers of link complements always give concordance invariants, but they are not sharp enough to distinguish some of interesting link concordance classes [**12**].

In this paper, signature invariants of arbitrary odd dimensional links are extracted from irregular covers and non-abelian covers of complements and are shown to be sharper than invariants from abelian constructions. For irregular or non-abelian covers, it is not easy to show directly that the information from the covers is invariant under ordinary link concordance due to the difficulty of algebraic setup. To show

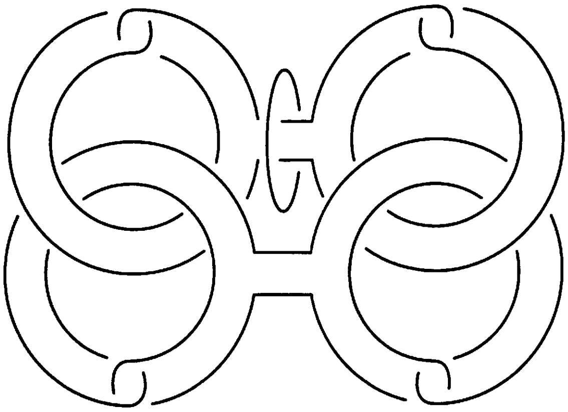

Fig. 1

the invariance, we decompose irregular or non-abelian covers into compositions of two abelian covers and apply the technique of Casson and Gordon [**2**, **3**], that was originally used to show that the Seifert form of a given 1-dimensional knot does not determine its concordance class. In Section 2, we describe the irregular covers and non-abelian covers of a bouquet that are used to obtain our invariants and how to decompose it into two abelian covers and illustrate examples of a dihedral cover and an irregular cover. Irregular or non-abelian covers of complements of F_m-links that are pull-backs of these covers of bouquet will be used to extract invariants of links.

The Casson–Gordon invariant of a knot depends on the choice of characters on the first homology of a branched cover along the knot. In general, the group of characters is not invariant under link concordance. This is a difficulty in showing that information from irregular or non-abelian covers for arbitrary links are invariants under link concordance. In Section 3, however, we show the following main result: let p be a prime. For two concordant links L_0 and L_1, there is a 1–1 correspondence between the order p characters of $\mathbf{Z}_{p^{a_1}} \oplus \cdots \oplus \mathbf{Z}_{p^{a_m}}$-branched covers of L_0 and L_1 and the Casson–Gordon invariants associated to the corresponding characters which factor through $\mathbf{Z}$ are equal (see Theorems 1 and 2). We note that the correspondence of characters depends on the choice of link concordance; nevertheless the finite set of all Casson–Gordon invariants for these characters is a link concordance invariant.

In the case of F_m-links, the situation can be understood more systematically. In Section 4, we show that there is a canonical correspondence between order p characters determined by the F_m-structure of F_m-concordant F_m-links, and the Casson-Cordon invariants associated to the corresponding characters become F_m-link concordance invariants and the characters are not required to factor through $\mathbf{Z}$ (see Theorems 3, 4 and 5).

In Section 5, we show that our invariants are computed from Seifert matrices of F_m-links and show that they vanish for all slice links (see Theorem 7). We give computational examples of non-slice links detected by our invariant but not detected by the invariants from abelian covers. One of them is the boundary link in Fig. 1 and we show that it is not a slice link as conjectured in [**11**] by using invariants from irregular covers. These examples show that irregular or non-abelian covers contain useful information on link concordance.

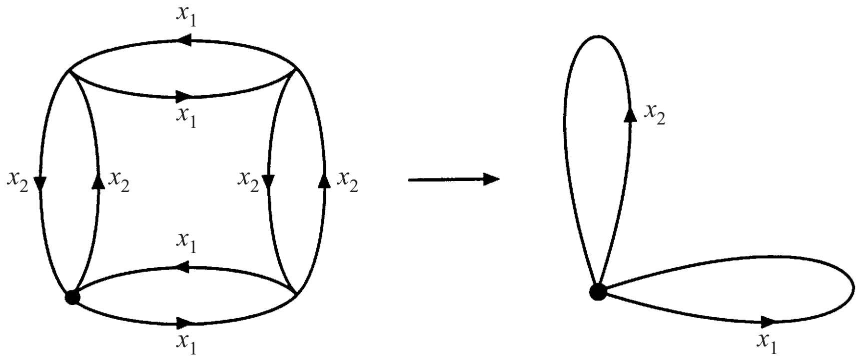

Fig. 2

2. *Covers of bouquets*

Let $B_m = \bigvee^m S^1$ be the bouquet with m circles. In this section we describe a family of covers of B_m which contains irregular covers and abelian covers and illustrate examples. The covers of this section are extended to a branched cover of the 2-complex obtained by attaching m disks to B_m, where the boundary of the ith disk is identified with the ith circle of B_m. By pulling back this branched cover along F_m-structures, covers of ambient spheres branched along F_m-links are obtained. In Section 5 we will illustrate examples of concordant invariants which are extracted from branched covers of link corresponding to the examples of this section.

We identify $\pi_1(B_m)$ with the free group F_m over m generators $x_1, \ldots, x_m$ such that x_i represents the ith circle. For m positive integers $a_1, \ldots, a_m$, let $\tilde{B}_m \to B_m$ be a covering map associated to the projection map $F_m \to \mathbf{Z}_{a_1} \oplus \cdots \oplus \mathbf{Z}_{a_m}$ which sends x_i to the ith standard generator $(0, \ldots, 1 + a_i\mathbf{Z}, \ldots, 0)$. Then $\pi_1(\tilde{B}_m)$ is isomorphic to a subgroup of F_m which is generated by conjugates of $x_i^{a_i}$ and commutators. Let α be a homomorphism from $\pi_1(\tilde{B}_m)$ onto $\mathbf{Z}_p$ that kills conjugates of $x_i^{a_i}$. This condition is needed to extend covers to branched covers as mentioned in the beginning of this section. Let $\bar{B}_m$ be the cover of $\tilde{B}_m$ determined by α. Then $\bar{B}_m \to \tilde{B}_m$ and $\tilde{B}_m \to B_m$ are regular abelian covers, but the composite $\bar{B}_m \to B_m$ of the two coverings is not necessarily regular nor abelian, as shown in the following examples.

Example 1. *Regular non-abelian case*

Consider the case of $m = 2$, $a_1 = a_2 = 2$, which is the simplest nontrivial one. The cover $\tilde{B}_2$ of B_2 is shown in Fig. 2, and $\pi_1(\tilde{B}_2)$ is isomorphic to the subgroup generated by $x_1x_2x_1^{-1}x_2^{-1}$ and conjugates of x_1^2, x_2^2. Let $\alpha\colon \pi_1(\tilde{B}_2) \to \mathbf{Z}_2$ be the homomorphism defined by $\alpha(x_1x_2x_1^{-1}x_2^{-1}) = 1 + 2\mathbf{Z}$. The cover $\bar{B}_2$ of $\tilde{B}_2$ associated to α is illustrated in Fig. 3.

The composite covering $\bar{B}_2 \to B_2$ is regular but non-abelian. The covering transformation group $F_2/\pi_1(\bar{B}_2)$ is presented by

$$\langle x_1, x_2 \mid x_1^2 = x_2^2 = (x_1x_2x_1x_2)^2 = 1 \rangle.$$

By substituting $a = x_1$, $b = x_1x_2$, it is transformed into

$$\langle a, b \mid a^2 = b^4 = 1, \quad aba^{-1} = b^{-1} \rangle$$

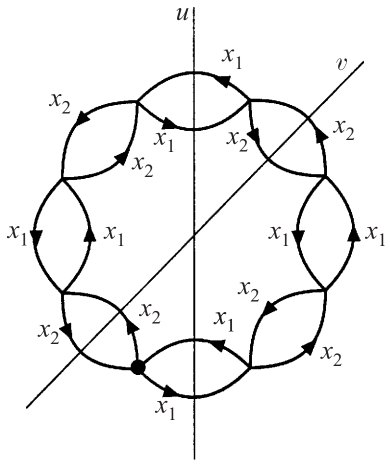

Fig. 3

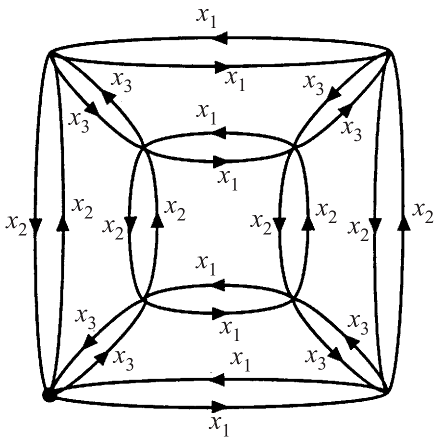

Fig. 4

which is the dihedral group D_4. Indeed, the cosets of a, x_2 and b correspond to the reflections along the lines u, v and the $(\pi/2)$-rotation, respectively.

Example 2. *Irregular case*

Let $m = 3$, $a_1 = a_2 = a_3 = 2$. In this case, the cover $\tilde{B}_3$ is as in Fig. 4 and $\pi_1(\tilde{B}_3)$ is isomorphic to the subgroup of F_3 generated by $x_1x_2x_1^{-1}x_2^{-1}$, $x_1x_3x_1^{-1}x_3^{-1}$, $x_2x_1x_3x_1^{-1}x_3^{-1}x_2^{-1}$, $x_2x_3x_2^{-1}x_3^{-1}$, $x_1x_2x_3x_2^{-1}x_3^{-1}x_1^{-1}$ and conjugates of x_i^2. Let α be the homomorphism from $\pi_1(\tilde{B}_3)$ onto $\mathbf{Z}_2$ which sends $x_2x_3x_2^{-1}x_3^{-1}$ to $1 + 2\mathbf{Z}$ and all other generators to zero. Fig. 5 illustrates the cover $\bar{B}_3$ associated to α. Even though both coverings $\bar{B}_3 \to \tilde{B}_3$ and $\tilde{B}_3 \to B_3$ are regular, the composite $\bar{B}_3 \to B_3$ is not regular. In fact, the lift based at p of the loop represented by $x_2x_3x_2x_3$ is a loop and the lift based at q is not a loop.

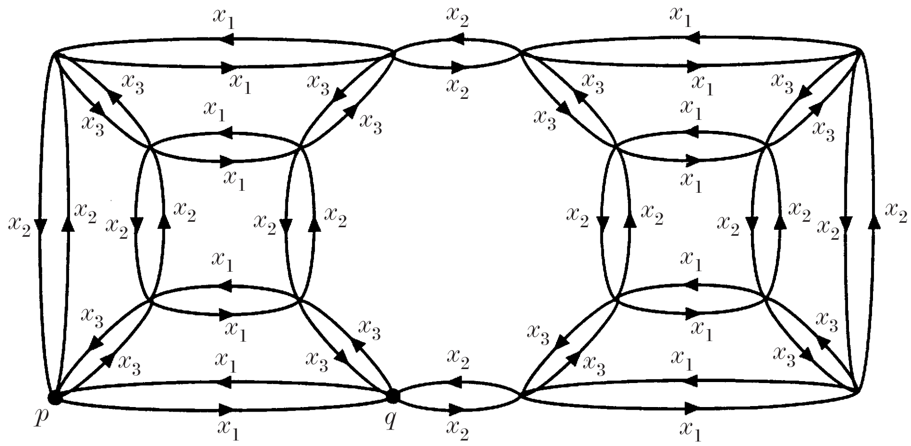

Fig. 5

3. *Invariants*

We define Casson–Gordon invariants following [**6**]. For a connected manifold M and a group G, a homomorphism $\phi\colon \pi_1(M) \to G$ is called a *G-structure* on M. In this case we call (M, ϕ) a *G-manifold*. We will denote a G-manifold (M, ϕ) by M when its G-structure ϕ is obvious. Two G-structures are equivalent if they are equal up to inner automorphisms of G. A disconnected manifold is called a G-manifold if its components have G-structures. A submanifold of a G-manifold has an induced G-structure.

Let M be a closed $(2q+1)$-manifold and let d be a positive integer. A character ϕ in $\operatorname{Hom}(H_1(M), \mathbf{Z}_d)$ determines a $\mathbf{Z}_d$-structure on M. Since the bordism group $\boldsymbol{\Omega}_{2q+1}(\mathbf{Z}_d)$ is finite, there exists a $(2q+2)$-dimensional $\mathbf{Z}_d$-manifold W and a positive integer r such that $\partial W = rM$ as $\mathbf{Z}_d$-manifolds. We can obtain $\mathbf{Z}_d$-covers $\tilde{W} \to W$ and $\tilde{M} \to M$ such that $\partial(\tilde{W} \to W) = r(\tilde{M} \to M)$. The covering transformation on $\tilde{W}$ corresponding to $1 \in \mathbf{Z}_d$ induces an automorphism t of order d on $H_{q+1}(\tilde{W}; \mathbf{C})$, which is an isometry with respect to the intersection form. Let $s(W)$ be the signature of the restriction of the intersection form to the $e^{2\pi i/d}$-eigenspace of t. (We adapt the convention that the signature of a skew-hermitian form A is the signature of iA.) Let $s_0(W)$ be the signature of the intersection form on $H_{q+1}(W; \mathbf{C})$. Define $\sigma(M, \phi) = (1/r)(s(W) - s_0(W))$; this is well defined, by bordism arguments (see [**6**, **7**] for details).

We need the following application of Milnor's exact sequence [**16**]. Let (X, A) be a pair and $p\colon \tilde{X} \to X$ be an infinite cyclic cover induced by a homomorphism $\pi_1(X) \to \mathbf{Z}$. Let t be the covering transformation corresponding to $1 \in \mathbf{Z}$. Let X_d be the $\mathbf{Z}_d$-cover of X corresponding the composition of $\pi_1(X) \to \mathbf{Z}$ and the canonical projection $\mathbf{Z} \to \mathbf{Z}_d$. Let $\tilde{A} \subset \tilde{X}$ and $A_d \subset X_d$ be the pre-images of A in $\tilde{X}$ and X_d, respectively.

LEMMA 1. *Suppose that $d = p^r$ for some prime p and let i be given.*

(1) *If $H_i(X, A; \mathbf{Z}_p) = 0$ then $H_i(X_d, A_d; \mathbf{Z}_p) = 0$.*
(2) *If $H^i(X, A; \mathbf{Z}_p) = 0$ then $H^i(X_d, A_d; \mathbf{Z}_p) = 0$.*

Proof. By Milnor [**16**], there is an exact sequence

$$\cdots \to H_i(\tilde{X},\tilde{A};\mathbf{Z}_p) \xrightarrow{t-1} H_i(\tilde{X},\tilde{A};\mathbf{Z}_p) \xrightarrow{p} H_i(X,A;\mathbf{Z}_p) \to H_{i-1}(\tilde{X},\tilde{A};\mathbf{Z}_p) \to \cdots$$

If $H_i(X,A;\mathbf{Z}_p)=0$, $t-1$ on $H_i(\tilde{X},\tilde{A};\mathbf{Z}_p)$ is surjective and $t-1$ on $H_{i-1}(\tilde{X},\tilde{A};\mathbf{Z}_p)$ is injective. Since $t^d-1=(t-1)^d$ over $\mathbf{Z}_p$, t^d-1 on $H_i(\tilde{X},\tilde{A};\mathbf{Z}_p)$ is surjective and t^d-1 on $H_{i-1}(\tilde{X},\tilde{A};\mathbf{Z}_p)$ is injective. From Milnor's sequence for $(\tilde{X},\tilde{A})\to(X_d,A_d)$, $H_i(X_d,A_d;\mathbf{Z}_p)=0$. The proof of (2) is similar. □

An *m-component n-link* is an oriented submanifold of S^{n+2} diffeomorphic to the disjoint union of m copies of S^n. We will consider odd dimensional links only. Let $n=2q-1$. We will assume that components of links are ordered and when $n=1$ we will assume that the linking number of any two distinct components is zero.

Let $E(L)=S^{n+2}-$(open tubular neighbourhood of L) be the exterior of L, so that $H_1(E(L))$ is isomorphic to $\mathbf{Z}^m$ and generated by meridians. For m positive integers $a_1,\dots,a_m$, consider a canonical projection $H_1(E(L))\to\mathbf{Z}_{a_1}\oplus\cdots\oplus\mathbf{Z}_{a_m}$ which carries the ith meridian to the ith standard generator $(0,\dots,1,\dots,0)$. This map induces the $\mathbf{Z}_{a_1}\oplus\cdots\oplus\mathbf{Z}_{a_m}$-cover $X(L)$ of $E(L)$, and the $\mathbf{Z}_{a_1}\oplus\cdots\oplus\mathbf{Z}_{a_m}$-branched cover $M(L)$ of S^{n+2} along L is constructed from $X(L)$ by filling in with $(\prod a_i.\sum a_i^{-1})$ copies of $S^n\times D^2$. We will study Casson–Gordon invariants of $M(L)$. For $\phi\in\operatorname{Hom}(H_1(M(L)),\mathbf{Z}_d)$, define $\sigma(L,\phi)=\sigma(M(L),\phi)$.

Two m-component n-links L_0 and L_1 are *concordant* if there is an oriented proper submanifold $\mathbf{L}$ in $S^{n+2}\times I$ such that $\mathbf{L}$ is diffeomorphic to the union of m copies of $S^n\times I$ and the boundary of the ith component of $\mathbf{L}$ is the union of the ith component of $-L_0\subset S^{n+2}\times 0$ and the ith component of $L_1\subset S^{n+2}\times 1$. In order to obtain a link concordance invariant from the invariant σ, we will investigate the group $\operatorname{Hom}(H_1(M(L)),\mathbf{Z}_d)$ which is identified with $H^1(M(L);\mathbf{Z}_d)$ by the the universal coefficient theorem. From now on, we fix a prime p and assume that links have m-components and that $a_1,\dots,a_m$ are powers of p.

THEOREM 1. *Let $\mathbf{L}$ be a concordance between two links L_0 and L_1. Then there is an isomorphism*

$$\Phi(\mathbf{L})\colon H^1(M(L_0);\mathbf{Z}_p)\to H^1(M(L_1);\mathbf{Z}_p)$$

which is determined by $\mathbf{L}$.

$\Phi(\mathbf{L})$ will be denoted by Φ if the choice of $\mathbf{L}$ is clear.

Proof. Denote the exterior of the concordance $\mathbf{L}$ by $E(\mathbf{L})$, the $\mathbf{Z}_{a_1}\oplus\cdots\oplus\mathbf{Z}_{a_m}$-cover of $E(\mathbf{L})$ by $X(\mathbf{L})$ and the $\mathbf{Z}_{a_1}\oplus\cdots\oplus\mathbf{Z}_{a_m}$-branched cover of $S^{n+2}\times I$ along $\mathbf{L}$ by $M(\mathbf{L})$. Then subsets of the boundaries of $E(\mathbf{L})$, $X(\mathbf{L})$ and $M(\mathbf{L})$ are identified with $-E(L_0)\cup E(L_1)$, $-X(L_0)\cup X(L_1)$ and $-M(L_0)\cup M(L_1)$, respectively. We need the following lemma.

LEMMA 2. $H_j(M(\mathbf{L}),M(L_i);\mathbf{Z}_p)=H^j(M(\mathbf{L}),M(L_i);\mathbf{Z}_p)=0$ *for all j and $i=0,1$.*

Proof. It is easy to see that $X(\mathbf{L})$ is obtained from $E(\mathbf{L})$ by taking $\mathbf{Z}_{a_n}$-covers repeatedly for $n=1,\dots,m$. Each $\mathbf{Z}_{a_n}$-cover corresponds to the composition $H_1(E(\mathbf{L}))\cong\mathbf{Z}^m\to\mathbf{Z}\to\mathbf{Z}_{a_n}$, killing all but the nth meridian. Repeated applications of Lemma 1 with the initial condition $H_j(E(\mathbf{L}),E(L_i);\mathbf{Z}_p)=0$ show that $H_j(X(\mathbf{L}),X(L_i);\mathbf{Z}_p)=0$ for all j. By Mayer–Vietoris arguments for

$$(M(\mathbf{L}),M(L_i))=(X(\mathbf{L}),X(L_i))\cup k(S^n\times I\times D^2,S^n\times\{i\}\times D^2),$$

where $k = \prod a_i \,.\, \sum a_i^{-1}$ and the union is taken along $k(S^n \times I \times S^1, S^n \times \{i\} \times S^1)$, we have $H_j(M(\mathbf{L}), M(L_i); \mathbf{Z}_p) = 0$. Similar argument proves the second conclusion.

Proof of Theorem 1 (*continued*). By Lemma 2, the maps from $H^1(M(\mathbf{L}); \mathbf{Z}_p)$ to $H^1(M(L_i); \mathbf{Z}_p)$ are isomorphisms for $i = 0, 1$. Therefore the composition

$$\Phi \colon H^1(M(L_0); \mathbf{Z}_p) \to H^1(M(\mathbf{L}); \mathbf{Z}_p) \to H^1(M(L_1); \mathbf{Z}_p)$$

is also an isomorphism.

THEOREM 2. *Suppose that L_0 and L_1 are concordant links and ϕ is a character in $H^1(M(L_0); \mathbf{Z}_p)$ which factors through the canonical projection $\mathbf{Z} \to \mathbf{Z}_p$. Then $\Phi(\phi)$ also factors through $\mathbf{Z} \to \mathbf{Z}_p$ and $\sigma(L_0, \phi) = \sigma(L_1, \Phi(\phi))$, where Φ is the isomorphism of Theorem* 1.

Proof. We use notations of the proof of Theorem 1. Consider the commutative diagram

$$\begin{array}{ccccccc} H^1(M(\mathbf{L}), M(L_0)) \to & H^1(M(\mathbf{L})) & \xrightarrow{\alpha} & H^1(M(L_0)) & \to H^2(M(\mathbf{L}), M(L_0)) \\ & f \downarrow & & \downarrow g & \\ & H^1(M(\mathbf{L}); \mathbf{Z}_p) & \xrightarrow{\beta} & H^1(M(L_0); \mathbf{Z}_p) & \end{array}$$

where f, g are induced by $\mathbf{Z} \to \mathbf{Z}_p$ and the top row is exact. By Lemma 2,

$$H_1(M(\mathbf{L}), M(L_0)) \otimes \mathbf{Z}_p = 0$$

and

$$H^1(M(\mathbf{L}), M(L_0)) = \operatorname{Hom}(H_1(M(\mathbf{L}), M(L_0)), \mathbf{Z}) = 0.$$

Since the top row is exact, we have an exact sequence

$$\operatorname{Coker} \alpha \xrightarrow{\Delta} \operatorname{Coker} f \to \operatorname{Coker} g \to 0$$

by the snake lemma. Since $pH^1(M(\mathbf{L}); \mathbf{Z}_p)$ is trivial, $p\operatorname{Coker} f$ is also trivial. By Lemma 2, $H^2(M(\mathbf{L}), M(L_0))$ is a finite abelian group without p-torsion elements and so is $\operatorname{Coker} \alpha$. Thus $\operatorname{Coker} \alpha$ is a direct sum of finite cyclic groups whose orders are coprime to p and $p\operatorname{Coker} \alpha = \operatorname{Coker} \alpha$. Therefore Δ is a trivial map. This shows that $\operatorname{Im} f = \operatorname{Im} \beta^{-1} g$. Hence $\beta^{-1}(\phi) \colon H_1(M(\mathbf{L})) \to \mathbf{Z}_p$ factors through $\mathbf{Z} \to \mathbf{Z}_p$. Applying this argument to $(M(\mathbf{L}), M(L_1))$, it is proved that $\Phi(\phi)$ also factors through $\mathbf{Z} \to \mathbf{Z}_p$.

Suppose W_0 is a $(2q + 2)$-dimensional $\mathbf{Z}_d$-manifold whose boundary is the disjoint union of r copies of $(M(L_0), \phi)$. Then $W_1 = W_0 \cup_{rM(L_0)} rM(\mathbf{L})$ has an induced $\mathbf{Z}_d$-structure and its boundary is the disjoint union of r copies of $(M(L_1), \Phi(\phi))$. To prove $\sigma(L_0, \phi) = \sigma(L_1, \Phi(\phi))$, it suffices to show that $s(M(\mathbf{L})) = s_0(M(\mathbf{L}))$ by additivity of signatures. In fact we will show that both $s(M(\mathbf{L}))$ and $s_0(M(\mathbf{L}))$ vanish. By Lemma 2, $H_{q+1}(M(\mathbf{L}), M(L_0); \mathbf{Z}_p)$ vanishes and so $H_{q+1}(M(\mathbf{L}), M(L_0); \mathbf{Q})$ also vanishes. Therefore $s_0(M(\mathbf{L}))$ vanishes.

Let $\tilde{M} \to M(\mathbf{L})$ be the $\mathbf{Z}_p$-cover and $\partial_0 \tilde{M}$ be the pre-image of $M(L_0)$. Then Lemma 1 can be applied to show that $H_{q+1}(\tilde{M}, \partial_0 \tilde{M}; \mathbf{Z}_p) = 0$ since $\beta^{-1}(\phi)$ factors through $\mathbf{Z} \to \mathbf{Z}_p$. Therefore the intersection form of $\tilde{M}$ vanishes and $s(M(\mathbf{L}))$ vanishes. $\square$

Because the isomorphism $\Phi(\mathbf{L})$ between the character groups of two concordant links L_0 and L_1 depends on the choice of a link concordance $\mathbf{L}$, we cannot see what character of L_1 is the correspondent of a given character of L_0 if we do not know $\mathbf{L}$. But the finite set of values of $\sigma(L, \phi)$ for all $\phi \in H^1(M(L); \mathbf{Z}_p)$ is a link concordance invariant. In the next section, we will show that we need not to consider the set of all values in the case of F_m-concordance, because there is a canonical correspondence between characters. Moreover, in Section 5, we will show that if L is a slice link then $\sigma(L, \phi) = 0$ for all ϕ.

We finish this section with a remark that the arguments of this section do not work for characters of order p^a. It is not clear whether the results of this section hold.

4. F_m-*links*

An m-component n-link L is called a *boundary link* if there exist m disjoint oriented $(n+1)$-manifolds $N_1, \ldots, N_m$ in S^{n+2} such that ∂N_i is the ith component of L, equivalently if there exists a group homomorphism $\theta\colon \pi_1(E(L)) \to F_m$ such that $\theta(\mu_i) = x_i$ for some choice of meridians $\mu_1, \ldots, \mu_m \in \pi_1(E(L))$ [**11**]. Note that θ induces an isomorphism on the first homology. The union $\bigcup N_i$ is called a *Seifert surface* for L. We consider a boundary link as a pair (L, θ), sometimes called an F_m-*link* and θ is called an F_m-*structure of* L [**1**]. In this section we will study $\sigma(L, \phi)$ of F_m-links. Firstly, we will observe that the covers constructed in Section 3 are pull-backs of the covers of B_m along the F_m-structure. Recall that B_m is the bouquet of m circles.

θ can be considered as a map $E(L) \to B_m$ on spaces, since B_m is the Eilenberg–MacLane space $K(F_m, 1)$. Let $\tilde{B}_m$ be the $\mathbf{Z}_{a_1} \oplus \cdots \oplus \mathbf{Z}_{a_m}$-cover of B_m described in Section 2. Then

$$\begin{array}{ccc} X(L) & \xrightarrow{\tilde{\theta}} & \tilde{B}_m \\ \downarrow & & \downarrow \\ X(L) & \xrightarrow{\theta} & B_m \end{array}$$

is a pull-back diagram where $\tilde{\theta}$ is the lift of θ. Since $\tilde{B}_m$ is a 1-complex, $\pi_1(\tilde{B}_m)$ is a free group and has the same rank as $H_1(\tilde{B}_m)$. By computing the Euler characteristic, $\operatorname{rank} H_1(\tilde{B}_m) = 1 + (m-1)\prod a_i$.

Let $\hat{B}_m$ be the 2-complex obtained from $\tilde{B}_m$ by attaching $(\prod a_i \,.\, \sum a_i^{-1})$ 2-cells along each lift of $x_i^{a_i}$ $(1 \leqslant i \leqslant m)$. Since $\pi_1(M(L))$ is the quotient group of $\pi_1(X(L))$ by a normal subgroup generated by lifts of $\mu_i^{a_i}$ and $\hat{B}_m$ has the homotopy type of a 1-complex, $\tilde{\theta}$ is extended to $\hat{\theta}\colon M(L) \to \hat{B}_m$.

Now we will show that all characters ϕ of $M(L)$ that are used to define $\sigma(L, \phi)$ are pull-backs of characters of $\hat{B}_m$ and therefore factor through $\mathbf{Z}$.

Lemma 3. *$\hat{\theta}$ induces an isomorphism between $H_1(M(L))/($torsion coprime to $p)$ and $H_1(\hat{B}_m)$.*

Proof. We apply the argument in the proof of Theorem 1 to a mapping cylinder. Let M be the mapping cylinder of θ and consider $E(L)$ as a subset of M. Let $\tilde{M}$ denote the $\mathbf{Z}_{a_1} \oplus \cdots \oplus \mathbf{Z}_{a_m}$-cover of M. Repeated applications of Lemma 2 with the initial condition $H_1(M, E(L); \mathbf{Z}_p) = H_2(M, E(L); \mathbf{Z}_p) = 0$ show that both $H_1(\tilde{M}, X(L); \mathbf{Z}_p)$ and $H_2(\tilde{M}, X(L); \mathbf{Z}_p)$ are also trivial. Since $\tilde{M}$ is the mapping cylinder of $\tilde{\theta}$, the map $H_1(X(L);\mathbf{Z}_p) \to H_1(\tilde{B}_m;\mathbf{Z}_p)$ induced by $\tilde{\theta}$ is an isomorphism. Since θ induces a surjective map on fundamental groups, so does $\tilde{\theta}$ on fundamental groups and thus on the first homologies. Thus $H_1(X(L))$ modulo torsion coprime to p is isomorphic to $H_1(\tilde{B}_m)$ under $\tilde{\theta}$. By the remark on $\pi_1(M(L))$ and $\pi_1(X(L))$ mentioned before, the conclusion follows. $\square$

From Lemma 3, it is easy to see that $H_1(M(L))$ has no p-torsion elements.

Theorem 3. *For an F_m-link (L,θ), $\hat{\theta}$ induces an isomorphism between $H^1(\hat{B}_m;\mathbf{Z}_p)$ and $H^1(M(L);\mathbf{Z}_p)$. In addition, each character in $H^1(M(L);\mathbf{Z}_p)$ factors through $\mathbf{Z} \to \mathbf{Z}_p$.*

Proof. Immediate consequences of Lemma 3. $\square$

Note that $H_1(\hat{B}_m)$ is a free abelian group of rank $1+(m-1-\sum a_i^{-1})\prod a_i$. Therefore there are nontrivial characters in $H^1(M(L);\mathbf{Z}_p)$ unless $a_1 = \cdots = a_m = 1$.

Theorem 4. *Suppose that L_0 and L_1 are two concordant links and L_0 or L_1 is concordant to a boundary link. Then $\sigma(L_0,\phi) = \sigma(L_1,\Phi(\phi))$ for any ϕ in $H^1(M(L_0);\mathbf{Z}_p)$ where Φ is the isomorphism of Theorem 1.*

Proof. An immediate consequence of Theorems 2 and 3. $\square$

Two F_m-links (L_0,θ_0) and (L_1,θ_1) are *F_m-concordant* if there is a concordance $\mathbf{L} \subset S^{n+2}\times I$ between L_0 and L_1 and a map $\mathbf{\Theta}\colon \pi_1(S^{n+1}\times I-\mathbf{L}) \to F_m$ such that the composition with $\pi_1(S^{n+2}\times\{i\}-L_i) \to \pi_1(S^{n+1}\times I-\mathbf{L})$ is the map θ_i up to inner automorphisms for $i = 0,1$. Sometimes such a concordance is called an *F_m-concordance*. Under F_m-concordance, the invariant σ behaves more systematically since the isomorphism Φ is well-understood as follows.

Theorem 5. *Let (L_0,θ_0) and (L_1,θ_1) be two F_m-concordant links. Then for any character ϕ in $H^1(\hat{B}_m;\mathbf{Z}_p)$, $\sigma(L_0,\phi\hat{\theta}_0) = \sigma(L_1,\phi\hat{\theta}_1)$.*

Proof. We have the commutative diagram

$$\begin{tikzcd} M(L_0) \arrow[d] \arrow[dr, "\hat{\theta}_0"] & \\ M(\mathbf{L}) \arrow[r] & \hat{B}_m \\ M(L_1) \arrow[u] \arrow[ur, "\hat{\theta}_1"'] & \end{tikzcd}$$

Therefore $\Phi(\phi\hat{\theta}_0) = \phi\hat{\theta}_1$ where Φ is the isomorphism in Theorem 1. By Theorem 4, the conclusion is proved. $\square$

Theorem 5 shows that $\sigma(L,\phi\hat{\theta})$ is an F_m-link concordance invariant for any character ϕ in $H^1(\hat{B}_m;\mathbf{Z}_p)$ and therefore we need not to consider the set of all values of $\sigma(L,\phi)$ as in the case of ordinary link concordance.

Most results of this section hold for homology boundary links. A *homology boundary link* is a link together with a surjection $\theta\colon \pi_1(E(L)) \to F_m$. In this case, the m-tuple $(r_1,\ldots,r_m)$ of elements of F_m is called a *pattern* of L if for some choice of meridians $\mu_1,\ldots,\mu_m \in \pi_1(E(L))$, $\theta(\mu_i) = r_i$ [**4**]. To work with homology boundary links, we need to modify the construction of the complex $\hat{B}_m$ so that 2-disks are attached along lifts of powers of elements of a pattern instead of generators x_i. Then similar arguments work. Note that boundary links are homology boundary links that admit the pattern $(x_1, \dots, x_m)$.

5. *Calculation*

Let (L, θ) denote an F_m-link. Let $N = N_1 \cup \cdots \cup N_m$ be a Seifert surface for (L, θ). We assume that θ is obtained by a Thom–Pontryagin construction for $N \subset E(L)$. The *Seifert pairing* $V: H_q(F) \times H_q(F) \to \mathbf{Z}$ is defined by $V(x, y) = \operatorname{lk}(i_+ x, y)$ where $i_+: H_*(F) \to H_*(S^{n+2} - F)$ is the map induced by a slight translation along the positive normal direction of F in S^{n+2} and lk denotes the linking number. By choosing a basis of $H_q(N_i)/\text{torsion}$ for each i, the Seifert pairing on $(H_q(N_i)/\text{torsion}) \times (H_q(N_j)/\text{torsion})$ is represented by a matrix A_{ij}. The matrix $A = (A_{ij})$ consisting of submatrices A_{ij} is called a *Seifert matrix* for (L, θ) (see [**11**]). We have $A_{ij} = (-1)^{q+1} A_{ji}^T$ for $i \neq j$.

We can construct $M(L)$ and an $(n+3)$ manifold W bounded by $M(L)$ over $\hat{B}_m$ as follows. Let N_i' be the proper submanifold of D^{n+3} obtained by pushing the interior of $N_i \subset S^{n+2}$ into D^{n+3} slightly. Let $Y = D^{n+3} -$ (open tubular neighbourhood of N_i') and let $M_i \subset Y$ denote the trace of pushing. Let $Y(j_1, \dots, j_m)$ denote a copy of $Y -$ (open tubular neighbourhood of $\bigcup M_i$) for $1 \leqslant j_i \leqslant a_i$; this is homeomorphic to D^{n+3}. Let

$$c_i^+(j_1, \dots, j_m), c_i^-(j_1, \dots, j_m): M_i \to \partial Y(j_1, \dots, j_m)$$

be the inclusions corresponding to the positive and negative normal directions, respectively. Let

$$A_i = \{(j_1, \dots, j_m) \mid 1 \leqslant j_i \leqslant a_i - 1, \quad 1 \leqslant j_l \leqslant a_l \quad \text{for } l \neq i\}$$

for $1 \leqslant i \leqslant m$. For $\lambda = (j_1, \dots, j_i, \dots, j_m) \in A_i$, let λ_+ denote $(j_1, \dots, j_i + 1, \dots, j_m)$. Let W be the manifold constructed from building blocks $Y(j_1, \dots, j_m)$ pasted as follows.

$$W = \bigcup_{(j_1, \dots, j_m)} Y(j_1, \dots, j_m) / \sim$$

where $c_i^+(\lambda)(z) \sim c_i^-(\lambda_+)(z)$ for $1 \leqslant i \leqslant m, \lambda \in A_i, z \in M_i$. Then $\partial W = M(L)$.

We consider the graph G consisting of $\prod a_i$ vertices $\{v(j_1, \dots, j_m) \mid 1 \leqslant j_i \leqslant a_i\}$ and $(\prod a_i)(m - \sum a_i^{-1})$ edges $\{e_i(\lambda) \mid 1 \leqslant i \leqslant m, \lambda \in A_i\}$. The edge $e_i(\lambda)$ joins two vertices $v(\lambda)$ and $v(\lambda_+)$. By choosing a basepoint from the 0-skeleton of $\tilde{B}_m$, we obtain a bijection from the set of vertices of G onto the 0-skeleton of $\tilde{B}_m$ carrying $v(j_1, \dots, j_m)$ to the image of the basepoint under the covering transformation corresponding to $(j_1, \dots, j_m) \in \mathbf{Z}_{a_1} \oplus \cdots \oplus \mathbf{Z}_{a_m}$. It is extended to a homotopy equivalence between G and $\hat{B}_m$. Orient edges of G such that $\partial e_i(\lambda) = \epsilon(\lambda) . (v(\lambda) - v(\lambda_+))$ for $1 \leqslant i \leqslant m$, $\lambda \in A_i$, where $\epsilon(j_1, \dots, j_m) = 1$ if $\sum_{i=1}^m (j_i - 1)$ is even, -1 otherwise. The construction of W is the 'pull-back' of the construction of G, that is, $Y(j_1, \dots, j_m)$ corresponds to $v(j_1, \dots, j_m)$ and identifications along M_is in the construction of W correspond to edges of G. From this we can see that the map $\hat{\theta}: \partial W \cong M(L) \to \hat{B}_m$ extends over W.

Let $\tilde{M}$ denote the union of the images of M_i in W under $c_i^+(\lambda)$, $1 \leqslant i \leqslant m$, $\lambda \in A_i$. Then a Mayer–Vietoris argument for $W = (W - M) \cup$ (open bicollar of $\tilde{M}$) shows that $H_{q+1}(W) \cong H_q(\tilde{M})$. Since M_i and N_i are homotopy equivalent, we have

$$H_{q+1}(W) \cong \bigoplus_{i=1}^{m} \Big(\bigoplus_{\lambda \in A_i} H_q(N_i(\lambda)) \Big),$$

where $N_i(\lambda)$ is a copy of N_i. An element $x \in H_q(N_i(\lambda))$ is represented by the $(q+1)$-chain $z'_x \cup z''_x$ where z'_x, z''_x are $(q+1)$-chains in $Y(\lambda)$, $Y(\lambda_+)$, respectively, such that $\partial z'_x = \epsilon(\lambda) . c_i^-(\lambda)(x)$ and $\partial z''_x = \epsilon(\lambda_+) . c_i^+(\lambda_+)(x)$. Therefore the intersection number of two elements of $H_{q+1}(W)$ represented by $x \in H_1(N_i(\lambda))$ and $y \in H_1(N_j(\mu))$, where $\lambda \in A_i$, $\mu \in A_j$, is given by

$$x . y = \begin{cases} V(x,y) + \epsilon V(y,x), & i = j \quad \text{and} \quad \lambda = \mu \\ V(x,y), & i = j \quad \text{and} \quad \lambda = \mu_+ \\ \epsilon V(y,x), & i = j \quad \text{and} \quad \lambda_+ = \mu \\ V(x,y), & i \neq j \quad \text{and} \quad \text{either} \\ & \quad \lambda = \mu, \lambda_+ = \mu, \lambda = \mu_+ \quad \text{or} \quad \lambda_+ = \mu_+ \\ 0, & \text{otherwise.} \end{cases}$$

where $\epsilon = (-1)^{q+1}$.

A function from the set of edges of G to $\mathbf{Z}_p$ is called a *voltage assignment* (see [**8**]). A voltage assignment α induces a map $\bar{\alpha}\colon \pi_1(G) \to \mathbf{Z}_p$; conversely, any map $\pi_1(G) \to \mathbf{Z}_p$ is realized by a voltage assignment. Hence it suffices to consider characters of the form $\bar{\alpha}\hat{\theta}\colon H_1(M(L)) \to \mathbf{Z}_p$ for some voltage assignment α, by Theorem 3.

For a given voltage assignment α for G, the $\mathbf{Z}_p$-cover G^α of G induced by α is constructed as follows. G^α has $p \prod a_i$ vertices labelled by $v(s)$ and $p(\prod a_i)(m - \sum a_i^{-1})$ edges labelled by $e(s)$, where v is a vertex of G, e is an edge of G and $s \in \mathbf{Z}_p$. The directed edge $e_i(\lambda)(s)$ $(1 \leqslant i \leqslant m, \lambda \in A_i)$ runs from $v(\lambda)(s)$ to $v(\lambda_+)(s - \epsilon(\lambda)\alpha(e_i(\lambda)))$. The covering projection $G^\alpha \to G$ sends $v(s)$ to v, $e(s)$ to e. The preferred generator t of the covering transformation group acts on G^α by $t(v(s)) = v(s+1), t(e(s)) = e(s+1)$ (indices are modulo p).

We can construct the $\mathbf{Z}_p$-cover $\tilde{W}$ of W induced by $\bar{\alpha}\hat{\theta}$ via the 'pull-back' of the construction of G^α. Let $Y(j_1, \ldots, j_m)(s)$ denote a copy of $Y(j_1, \ldots, j_m)$ for $1 \leqslant j_i \leqslant a_i$, $s \in \mathbf{Z}_p$. Let $c_i^+(j_1, \ldots, j_m)(s)$ and $c_i^-(j_1, \ldots, j_m)(s)$ be the inclusions $M_i \to Y(j_1, \ldots, j_m)(s)$. Then

$$\tilde{W} = \bigcup_{\substack{s \in \mathbf{Z}_p \\ (j_1, \ldots, j_m)}} Y(j_1, \ldots, j_m)(s) / \sim$$

where $c_i^-(\lambda)(s)(z) \sim c_i^+(\lambda_+)(s - \epsilon(\lambda)\alpha(e_i(\lambda)))(z)$ for $1 \leqslant i \leqslant m, \lambda \in A_i, s \in \mathbf{Z}_p, z \in M_i$.

The intersection form of $\tilde{W}$ can also be described. A Mayer–Vietoris argument shows that

$$H_{q+1}(\tilde{W}) \cong \bigoplus_{i=1}^{m} \Big(\bigoplus_{\lambda \in A_i, s \in \mathbf{Z}_p} H_q(N_i(\lambda)(s)) \Big),$$

where $N_i(\lambda)(s)$ is a copy of N_i. An element $x \in H_q(N_i(\lambda)(s))$ is represented by the $(q+1)$-chain $z'_x \cup z''_x$ where z'_x is a $(q+1)$-chain in $Y(\lambda)(s)$ and z''_x is one in $Y(\lambda_+)(s - \epsilon(\lambda)\alpha(e_i(\lambda)))$, such that $\partial z'_x$ equals $\epsilon(\lambda) . c_i^-(\lambda)(s)(x)$ and $\partial z''_x$ equals $\epsilon(\lambda_+) . c_i^+(\lambda_+)(s - \epsilon(\lambda)\alpha(e_i(\lambda)))(x)$.

In our case, all eigenspaces can be described as in [**9**]. For $x \in H_{q+1}(\tilde{W}; \mathbf{C})$, $u_\omega(x) = \sum_{k=0}^{p-1} \omega^{-k} t^k(x)$ is a ω-eigenvector of t if ω is a pth root of unity. By dimension counting, we have $H_{q+1}(\tilde{W};\mathbf{C}) = V_0 \oplus \cdots \oplus V_{p-1}$ where V_k is the $e^{2\pi ki/p}$-eigenspace $\{u_{e^{2\pi ki/p}}(x) \mid x \in H_q(N_i(\lambda)(0)), \lambda \in A_i\}$. Let $\omega = e^{2\pi i/p}$. The intersection number of $u_\omega(x)$ and $u_\omega(y)$ is given by

$$u_\omega(x)u_\omega(y) = \begin{cases} p(V(x,y) + \epsilon V(y,x)), & i = j,\ \lambda = \mu \\ p\omega^{\epsilon(\mu)\alpha(e_j(\mu))}V(x,y), & i = j,\ \lambda = \mu_+ \\ p\omega^{-\epsilon(\lambda)\alpha(e_i(\lambda))}\epsilon V(y,x), & i = j,\ \lambda_+ = \mu \\ pV(x,y), & i \neq j,\ \lambda = \mu \\ p\omega^{\epsilon(\mu)\alpha(e_j(\mu))}V(x,y), & i \neq j,\ \lambda = \mu_+ \\ p\omega^{-\epsilon(\lambda)\alpha(e_i(\lambda))}V(x,y), & i \neq j,\ \lambda_+ = \mu \\ p\omega^{-\epsilon(\lambda)\alpha(e_i(\lambda))+\epsilon(\mu)\alpha(e_j(\mu))}V(x,y), & i \neq j,\ \lambda_+ = \mu_+ \\ 0, & \text{otherwise.} \end{cases}$$

where $x \in H_q(N_i(\lambda)(0))$, $y \in H_q(N_j(\mu)(0))$, $\lambda \in A_i$, $\mu \in A_j$.

We summarize these results.

THEOREM 6. *Let (L,θ) be an F_m-link. Let $A = (A_{ij})$ be a Seifert matrix. Let α be a voltage assignment for G with voltage group $\mathbf{Z}_p$. Then*

$$\sigma(L, \bar{\alpha}\hat{\theta}) = \operatorname{sign}(\tilde{S}) - \operatorname{sign}(S)$$

where $S = (S_{(i,\lambda),(j,\mu)})_{1\leqslant i,j\leqslant m, \lambda\in A_i, \mu\in A_j}$ and $\tilde{S} = (\tilde{S}_{(i,\lambda),(j,\mu)})_{1\leqslant i,j\leqslant m, \lambda\in A_i, \mu\in A_j}$ are the matrices given by

$$S_{(i,\lambda),(j,\mu)} = \begin{cases} A_{ii} + \epsilon A_{ii}^T, & i = j \quad \textit{and} \quad \lambda = \mu \\ A_{ii}, & i = j \quad \textit{and} \quad \lambda = \mu_+ \\ \epsilon A_{ii}^T, & i = j \quad \textit{and} \quad \lambda_+ = \mu \\ A_{ij}, & i \neq j \quad \textit{and} \quad \textit{either} \quad \lambda = \mu, \\ & \quad \lambda_+ = \mu, \quad \lambda = \mu_+ \quad \textit{or} \quad \lambda_+ = \mu_+ \\ 0, & \textit{otherwise}, \end{cases}$$

$$\tilde{S}_{(i,\lambda),(j,\mu)} = \begin{cases} A_{ii} + \epsilon A_{ii}^T, & i = j \quad \textit{and} \quad \lambda = \mu \\ \omega^{\epsilon(\mu)\alpha(e_j(\mu))}A_{ii}, & i = j \quad \textit{and} \quad \lambda = \mu_+ \\ \omega^{-\epsilon(\lambda)\alpha(e_i(\lambda))}\epsilon A_{ii}^T, & i = j \quad \textit{and} \quad \lambda_+ = \mu \\ A_{ij}, & i \neq j \quad \textit{and} \quad \lambda = \mu \\ \omega^{\epsilon(\mu)\alpha(e_j(\mu))}A_{ij}, & i \neq j \quad \textit{and} \quad \lambda = \mu_+ \\ \omega^{-\epsilon(\lambda)\alpha(e_i(\lambda))}A_{ij}, & i \neq j \quad \textit{and} \quad \lambda_+ = \mu \\ \omega^{-\epsilon(\lambda)(\alpha(e_i(\lambda))+\alpha(e_j(\mu)))}A_{ij}, & i \neq j \quad \textit{and} \quad \lambda_+ = \mu_+ \\ 0, & \textit{otherwise.} \end{cases}$$

Note that if L is (concordant to) a boundary link, then $\sigma(L,\phi)$ is an integer for any character ϕ.

A link is said to be a *slice link* if it is concordant to a trivial link. As a consequence, we prove that our invariant vanishes for slice links as mentioned in the introduction.

THEOREM 7. *If L is a slice link, $\sigma(L,\phi) = 0$ for all character ϕ.*

Proof. It suffices to show for the trivial link. Since the null-matrix is a Seifert matrix, σ vanishes by Theorem 6. □

We remark that Theorem 7 can directly be proved by using the argument in the proof of Theorem 2.

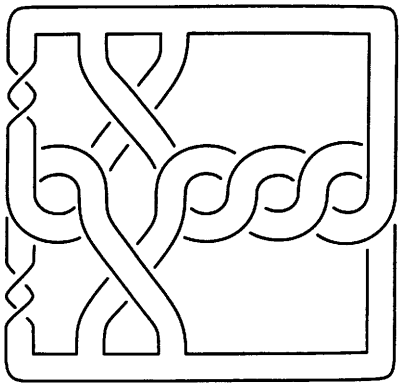

Fig. 6

Finally we illustrate some examples. In the simplest case of $m = 2$, $p = a_1 = a_2 = 2$, we have

$$S = \begin{bmatrix} A_{11} + \epsilon A_{11}^T & 0 & A_{12} & A_{12} \\ 0 & A_{11} + \epsilon A_{11}^T & A_{12} & A_{12} \\ A_{21} & A_{21} & A_{22} + \epsilon A_{22}^T & 0 \\ A_{21} & A_{21} & 0 & A_{22} + \epsilon A_{22}^T \end{bmatrix}.$$

For the voltage assignment α given by

$$\alpha(e) = \begin{cases} 1, & \text{if } e = e_1(1,1) \\ 0, & \text{otherwise} \end{cases}$$

we have

$$\tilde{S} = \begin{bmatrix} A_{11} + \epsilon A_{11}^T & 0 & A_{12} & -A_{12} \\ 0 & A_{11} + \epsilon A_{11}^T & A_{12} & A_{12} \\ A_{21} & A_{21} & A_{22} + \epsilon A_{22}^T & 0 \\ -A_{21} & A_{21} & 0 & A_{22} + \epsilon A_{22}^T \end{bmatrix}.$$

We note that this case corresponds to the first example of Section 2.

Let L be a 2-component n-link with a Seifert matrix

$$A = \left[\begin{array}{cc|cc} 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & -1 \\ \hline 2 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{array}\right].$$

where n is odd. By realization theorems (e.g. see [**11**]), such a link L exists in any odd dimension. Fig. 2 illustrates L in the classical dimension. Note that the classical link signatures [**10**, **13**] of L vanish.

A calculation aided by a computer shows that $\operatorname{sign}(S) = 0$ and $\operatorname{sign}(\tilde{S}) = 4$ if $n \equiv 1 \pmod 4$. Thus by Theorem 6, $\sigma(L, \bar{\alpha}\hat{\theta}) = 4$, and by Theorem 7, L is not a slice link.

In the case of $m = 3$, $p = a_1 = a_2 = a_3 = 2$ and α given by

$$\alpha(e) = \begin{cases} 1 & \text{if } e = e_1(1,1,1) \\ 0 & \text{otherwise} \end{cases}$$

we have

$$S = \left[\begin{array}{cccc|cccc|cccc}
B_{11} & & & & A_{12} & A_{12} & & & A_{13} & A_{13} & & \\
& B_{11} & & & A_{12} & A_{12} & & & & & A_{13} & A_{13} \\
& & B_{11} & & & & A_{12} & A_{12} & A_{13} & A_{13} & & \\
& & & B_{11} & & & A_{12} & A_{12} & & & A_{13} & A_{13} \\
\hline
A_{21} & A_{21} & & & B_{22} & & & & A_{23} & & A_{23} & \\
A_{21} & A_{21} & & & & B_{22} & & & & A_{23} & & A_{23} \\
& & A_{21} & A_{21} & & & B_{22} & & A_{23} & & A_{23} & \\
& & A_{21} & A_{21} & & & & B_{22} & & A_{23} & & A_{23} \\
\hline
A_{31} & & A_{31} & & A_{23} & & A_{23} & & B_{33} & & & \\
A_{31} & & A_{31} & & & A_{23} & & A_{23} & & B_{33} & & \\
& A_{31} & & A_{31} & A_{23} & & A_{23} & & & & B_{33} & \\
& A_{31} & & A_{31} & & A_{23} & & A_{23} & & & & B_{33}
\end{array}\right],$$

$$\tilde{S} = \left[\begin{array}{cccc|cccc|cccc}
B_{11} & & & & A_{12} & -A_{12} & & & A_{13} & -A_{13} & & \\
& B_{11} & & & A_{12} & A_{12} & & & & & A_{13} & A_{13} \\
& & B_{11} & & & & A_{12} & A_{12} & A_{13} & A_{13} & & \\
& & & B_{11} & & & A_{12} & A_{12} & & & A_{13} & A_{13} \\
\hline
A_{21} & A_{21} & & & B_{22} & & & & A_{23} & & A_{23} & \\
-A_{21} & A_{21} & & & & B_{22} & & & & A_{23} & & A_{23} \\
& & A_{21} & A_{21} & & & B_{22} & & A_{23} & & A_{23} & \\
& & A_{21} & A_{21} & & & & B_{22} & & A_{23} & & A_{23} \\
\hline
A_{31} & & A_{31} & & A_{23} & & A_{23} & & B_{33} & & & \\
-A_{31} & & A_{31} & & & A_{23} & & A_{23} & & B_{33} & & \\
& A_{31} & & A_{31} & A_{23} & & A_{23} & & & & B_{33} & \\
& A_{31} & & A_{31} & & A_{23} & & A_{23} & & & & B_{33}
\end{array}\right],$$

where $B_{ii} = A_{ii} + \epsilon A_{ii}^T$ for $1 \leqslant i \leqslant 3$. We note that this case corresponds to the second example of Section 2.

Let L be a 3-component n-link with a Seifert matrix

$$A = \left[\begin{array}{cc|cccc|cccc}
0 & 1 & 1 & & & & & & & \\
0 & 0 & & & & & 1 & & & \\
\hline
1 & & 0 & 1 & & & 1 & 0 & & \\
& & 0 & -1 & & & 0 & 0 & & \\
& & & & 0 & -1 & & & -1 & 0 \\
& & & & 0 & 1 & & & 0 & 0 \\
\hline
& 1 & 1 & 0 & & & 0 & 1 & & \\
& & 0 & 0 & & & 0 & -1 & & \\
& & & & -1 & 0 & & & 0 & -1 \\
& & & & 0 & 0 & & & 0 & 1
\end{array}\right],$$

where n is odd. By realization theorems, such a link L also exists in any odd dimension. The link L is suggested in [**11**]. Fig. 1 illustrates L in the classical dimension.

Again a calculation aided by a computer shows that sign $(S) = 0$ and sign $(\tilde{S}) = -4$ if $n \equiv 1 \pmod 4$. Therefore by Theorem 6, $\sigma(L, \bar{\alpha}\hat{\theta}) = -4$, and by Theorem 7, L is not a slice link.

In [**11**], it was proved that L is not boundary or F_m-concordant to the trivial link. Our computation extends that result. We note that the necessary condition from the classical link signatures [**10**] fails to detect that L is not a slice link. Moreover, the necessary condition obtained by abelianizing the Γ-group obstruction from the free cover [**1**] also fails to detect it, as mentioned in [**11**, **12**]. In [**18**], it was mentioned without details that L can be shown not to be a slice link by using a G-signature invariant.

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