

SIGNATURES OF LINKS IN RATIONAL HOMOLOGY SPHERES

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ABSTRACT. A theory of signatures for odd-dimensional links in rational homology spheres is studied via their generalized Seifert surfaces. The jump functions of signatures are shown invariant under appropriately generalized concordance and a special care is given to accommodate 1-dimensional links with mutual linking. Furthermore our concordance theory of links in rational homology spheres remains highly nontrivial after factoring out the contribution from links in integral homology spheres.

1. INTRODUCTION AND MAIN RESULTS

Signature invariants of knots and links have been studied in many articles, and turned out to be effective in various problems in knot theory, especially in studying concordance. Signatures were first defined as numerical invariants from Seifert matrices of classical knots in the three space by Trotter [25]. Murasugi formulated a definition of signatures for links [22]. Tristram introduced link signature functions [24]. Kauffman and Taylor found a geometric approach to signatures of links using branched covers [10]. Milnor defined a signature function from a duality in the infinite cyclic cover of a knot complement [21]. It is known that all the approaches are equivalent [5, 10, 20]. Signature invariants are known to be concordance invariants of codimension two knots and links in odd dimensional spheres. In the remarkable works of Levine [16, 17], signature invariants are used to classify concordance classes of knots. Levine showed that the knot concordance groups are isomorphic to the cobordism groups of Seifert matrices in higher odd dimensions, and signatures are complete invariants for the matrix cobordism groups modulo torsion elements.

In this paper, we develop signature invariants of arbitrary links of codimension two in odd dimensional rational homology spheres, motivated from the work of Cochran and Orr [3, 4]. Our signatures are invariant under the following equivalence of links in rational homology spheres. An n -link L in a rational homology $(n+2)$ -sphere Σ is a submanifold diffeomorphic to an ordered union of disjoint standard n -spheres. Such a link is sometimes denoted by a pair (Σ, L) . Two n -links (Σ_0, L_0) and (Σ_1, L_1) are *rational homology concordant* (or **Q**-concordant) if there is an $(n+3)$ -manifold \mathbf{S} whose boundary is the disjoint union of $-\Sigma_0$ and Σ_1 such that the triad $(\mathbf{S}, \Sigma_0, \Sigma_1)$ has the rational homology of $(S^{n+2} \times [0, 1], S^{n+2} \times 0, S^{n+2} \times 1)$, and there is a proper submanifold \mathbf{L} in \mathbf{S} diffeomorphic to $L_0 \times [0, 1]$ such that the boundary of the i -th component of \mathbf{L} is the union of the i -th components of $-L_0$ and L_1 . Concordant links are clearly rational

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homology concordant as well, and so the signatures studied in this paper are invariant under ordinary concordance as well.

As done in [25, 22, 24, 10] for links in S^{2q+1} , we extract invariants of links from the signature function

$$\sigma_A^q(\phi) = \text{sign}((1 - e^{i\phi})A + \epsilon(1 - e^{-i\phi})\overline{A}^T)$$

and its jump function

$$\delta_A^q(\theta) = \begin{cases} \lim_{\phi \rightarrow \theta^+} \sigma_A^q(\phi) - \lim_{\phi \rightarrow \theta^-} \sigma_A^q(\phi), & \text{if } \theta \text{ is not a multiple of } 2\pi \\ 0, & \text{otherwise} \end{cases}$$

where A is a rational Seifert matrix, ϕ and θ are reals, and $\epsilon = (-1)^{q+1}$. We follow the convention that the signature of an anti-hermitian matrix P is the signature of the hermitian matrix $\sqrt{-1} \cdot P$. Section 2 is devoted to an algebraic study of δ_A^q and σ_A^q . Especially, we prove a *reparametrization formula* for δ_A^q and σ_A^q of a Seifert matrix of a union of parallel copies of a Seifert surface (see Lemma 2.2, Theorem 2.1).

A Seifert surface of a link L is defined to be an oriented submanifold bounded by L in the ambient space. L has a Seifert surface if and only if there exists a homomorphism of the first integral homology of the exterior of L onto the infinite cyclic group sending each meridian of L to a fixed generator. In general, such a homomorphism does not exist and links bound no Seifert surfaces, unless the ambient space is an integral homology sphere. In the case of rational homology spheres, we show that a certain union of parallel copies of a link L bounds a Seifert surface even though L itself does not. More precisely, for an m -component link L and $\tau = (\tau_i) \in \mathbf{Z}^m$ where τ_i are relative prime, we call an oriented submanifold F a *generalized Seifert surface of type τ for L* if for some nonzero integer c , F is bounded by the union of $|c\tau_i|$ parallel copies (or negatively oriented copies if $c\tau_i < 0$) of the i -th component of L for all i . If L is a 1-link, the parallel copies are taken with respect to some framing f on L and the surface F and the framing f must induce the same framing on the parallel copies. We call c the *complexity of F* . A higher dimensional link admits a generalized Seifert surface of any type τ . But a 1-link admits a generalized Seifert surface of the types satisfying a certain condition and a framing is uniquely determined by a given type. Furthermore \mathbf{Q} -concordant links admit generalized Seifert surfaces of the same type. (See Theorem 3.2 and Theorem 3.4.)

We define a Seifert matrix to be a matrix associated to the rational-valued Seifert pairing on the middle dimensional rational homology of a generalized Seifert surface, and derive invariants of a link from the signature jump function of a Seifert matrix as follows. Let $\text{sgn } a = a/|a|$ for $a \neq 0$. The *signature jump function of type τ* for a $(2q-1)$ -link L is defined by

$$\delta_L^\tau(\theta) = \text{sgn } c \cdot \delta_A^q(\theta/c)$$

where A is a Seifert matrix of a generalized Seifert surface F of type τ for L and c is the complexity of F . Our main result is that the signature jump function δ_L^τ is a well-defined invariant under \mathbf{Q} -concordance which is independent of the choice of generalized Seifert surfaces.

Theorem 1.1. *If L_0 and L_1 are \mathbf{Q} -concordant, $\delta_{L_0}^\tau(\theta) = \delta_{L_1}^\tau(\theta)$ for all θ .*

For $\tau = (1 \cdots 1)$, we denote δ_L^τ by δ_L . If L is a link in an integral homology sphere, L bounds a Seifert surface F and $\delta_L(\theta)$ can be computed from a Seifert matrix of F . Hence our signature invariant is a generalization of the well-known signature invariant of links in the spheres.

For 1-links that are null-homologous modulo d in a rational homology 3-sphere, Gilmer defined the d -signature that depends on the choice of a specific element in the first homology of the ambient space [6]. We remark that it is equal to the evaluation of the signature function associated to a generalized Seifert matrix of type $(1 \cdots 1)$ at the d -th roots of unity, and so it can be derived from our signature jump function.

A link is called a *slice link* if its components bound properly embedded disjoint disks in a rational homology ball bounded by the ambient space. For a link L and an n -tuple $\alpha = (s_1, \dots, s_n)$ with $s_i = \pm 1$, let $i_\alpha L$ be the union of n parallel copies of L where the i -th copy is oriented according to the sign of s_i , and let n_α be the sum of s_i . The *complexity* $c^\tau(L)$ of type τ for L is defined to be the minimum of the absolute values of complexities of generalized Seifert surfaces of type τ .

For $i = 1, 2$, let (Σ_i, L_i) be an n -dimensional link with m_i -components and B_i be an $(n+2)$ -ball in Σ_i such that for some $k \geq 1$, B_i intersects k components of L_i at a trivial k -component disk link in B_i . By pasting $(\Sigma_1 - \text{int } B_1, L_1 - \text{int } B_1 \cap L_1)$ and $(\Sigma_2 - \text{int } B_2, L_2 - \text{int } B_2 \cap L_2)$ together along an orientation reversing homeomorphism on boundaries, a $(m_1 + m_2 - k)$ -component link is obtained and is called a *connected sum* of L_1 and L_2 . A connected sum depends on the choice of B_i if $m \geq 2$. Some useful properties of our signature invariant are listed in the following theorem.

Theorem 1.2.

1. If L is slice, $\delta_L^\tau(\theta) = 0$.
2. If $n_\alpha \neq 0$, $\delta_{i_\alpha L}^\tau(\theta) = \text{sgn } n_\alpha \cdot \delta_L^\tau(n_\alpha \theta)$.
3. $\delta_L^\tau(\theta)$ is a periodic function of period $2\pi \cdot c^\tau(L)$.
4. Let L be a connected sum of $(2q-1)$ -links L_1 and L_2 . If either $q > 1$ or L_1, L_2 are knots, then $\delta_L^\tau(\theta) = \delta_{L_1}^\tau(\theta) + \delta_{L_2}^\tau(\theta)$.

In Section 4, we prove the above results by combining the techniques of signatures of links in S^{2q+1} with the reparametrization formula in Section 2 that plays a key role in resolving the difficulty that links do not bound Seifert surfaces.

We give examples of links in rational homology spheres whose signature jump functions have arbitrary rational multiples of 2π as periods, and show that the theory of \mathbf{Q} -concordance of links in rational homology spheres is not reduced to the concordance theory of links in spheres. In fact we construct a family of infinitely many \mathbf{Q} -concordance classes of knots that are independent modulo knots in the spheres in the following sense. Let C_n and $C_n^{\mathbf{Q}}$ be the set of concordance classes of n -knots in S^{n+2} and the set of \mathbf{Q} -concordance classes of n -knots in rational homology spheres, respectively. They are abelian groups under connected sum. We have a natural homomorphism $C_n \rightarrow C_n^{\mathbf{Q}}$ since concordant links in a sphere are \mathbf{Q} -concordant.

Theorem 1.3. *There exist infinitely many $(2q-1)$ -knots in rational homology spheres that are linearly independent in the cokernel of $C_{2q-1} \rightarrow C_{2q-1}^{\mathbf{Q}}$.*

We remark that knots constructed by Cochran and Orr using the covering link construction [4] are nontrivial in the cokernel since they are not concordant to knots of

complexity one. Our result shows that the cokernel is large enough to contain a subgroup isomorphic to \mathbf{Z}^∞ .

On the other hand, we illustrate that our theory is also useful for links in spheres by offering alternative proofs of a result of Litherland [19] about signatures of satellite knots and a result of Kawauchi [11] about concordance of split links. In a separated paper [2], we will compute signature jump functions of covering links and study concordance of boundary links and homology boundary links. In particular, results of Cochran and Orr [3, 4], Gilmer and Livingston [7], and Levine [18] will be generalized.

2. SIGNATURES OF MATRICES

In this section, we study algebraic properties of the signature function σ_A^q and its jump function δ_A^q that will play a key role in our link signature theory. It is known that if A is a complex square matrix such that $A \pm \bar{A}^T$ is nonsingular, the function $e^{i\phi} \mapsto \sigma_A^q(\phi)$ is constant except at finitely many jumps where $e^{i\phi}$ is a zero of $\det(tA - \epsilon \bar{A}^T)$, and so the jump function δ_A^q is well-defined. In general, δ_A^q is well-defined for any complex square matrix A . We accompany a proof since we did not find one in the literature. Let $-$ denote the usual involution $(\sum \alpha_i t^i)^- = \sum \bar{\alpha}_i t^{-i}$ on the Laurent polynomial ring $\mathbf{C}[t, t^{-1}]$.

Lemma 2.1. *Let $Q(t)$ be a square matrix over $\mathbf{C}[t, t^{-1}]$ that satisfies $Q(t) = (Q(t)^T)^-$. Then for any $\phi_0 \in \mathbf{R}$, the signature of $Q(e^{i\phi})$ is locally constant on a deleted neighborhood of ϕ_0 , i.e., it is constant on each of $(\phi_0 - r, \phi_0)$ and $(\phi_0, \phi_0 + r)$ for some $r > 0$.*

Proof. First we show that the rank of $Q(e^{i\phi})$ is constant on a deleted neighborhood of ϕ_0 . Let R be a square matrix obtained by deleting some rows and columns from $Q(t)$ and denote its determinant by $d_R(t)$. Then either $d_R(t) \equiv 0$ or there is a deleted neighborhood U_R of ϕ_0 where $d_R(e^{i\phi}) \neq 0$ since $d_R(t)$ is a Laurent polynomial over \mathbf{C} . Since there are finitely many R , the intersection of all U_R is a deleted neighborhood of ϕ_0 where $\text{rank } Q(e^{i\phi})$ is constant.

Now it suffices to show that $\text{sign } Q(e^{i\phi})$ is locally constant on any open interval J where $\text{rank } Q(e^{i\phi})$ is constant. Near a fixed ϕ_1 in J , $Q(e^{i\phi})$ is congruent to a diagonal matrix whose diagonals $\lambda_i(\phi)$ are real-valued continuous functions on ϕ . We can choose a neighborhood U of ϕ_1 in J such that if $\lambda_j(\phi_1) \neq 0$, then $\lambda_j(\phi) \neq 0$ and its sign is not changed in U . Since $\text{rank } Q(e^{i\phi})$ is constant, $\lambda_j(\phi) = 0$ on U if $\lambda_j(\phi_1) = 0$. Therefore $\text{sign } Q(e^{i\phi})$ is constant on U . \square

From the lemma, it is immediate that for an arbitrary A , the function $e^{i\phi} \mapsto \sigma_A^q(\phi)$ is locally constant except at finitely many jumps on the unit circle and so δ_A^q is well-defined.

As mentioned in the introduction, signature invariants of links are defined from the signature functions of Seifert matrices of generalized Seifert surfaces. Note that a union of parallel copies of a generalized Seifert surface is again a generalized Seifert surface. Since both of them can be used to define signature invariants of links, we are naturally led to investigate a relation of the signature functions of them. Explicitly, for an r -tuple $\alpha = (s_1, \dots, s_r)$ with $s_n = \pm 1$, consider the union of r parallel copies of a Seifert surface F of a $(2q - 1)$ -link, where the n -th copy is oriented according to the sign of s_n . If A is a

Seifert matrix of F , the Seifert matrix of the union of parallel copies is given as follows.

$$i_\alpha^q A = \begin{bmatrix} A_1 & A & A & \cdots & A \\ \epsilon A^T & A_2 & A & \cdots & A \\ \epsilon A^T & \epsilon A^T & A_3 & \cdots & A \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \epsilon A^T & \epsilon A^T & \epsilon A^T & \cdots & A_r \end{bmatrix}$$

where

$$A_n = \begin{cases} A & \text{if } s_n = 1, \\ \epsilon A^T & \text{if } s_n = -1. \end{cases}$$

The following result shows that the signature jump functions of $i_\alpha^q A$ and A are essentially equivalent up to a change of parameters. To simplify statements, we define $\text{sgn } 0$ to be zero.

Theorem 2.1 (Reparametrization formula). *If $n_\alpha \neq 0$ or $A - \epsilon A^T$ is nonsingular, then $\delta_{i_\alpha^q A}^q(\theta) = \text{sgn } n_\alpha \cdot \delta_A^q(n_\alpha \theta)$.*

The rest of this section is devoted to the proof of Theorem 2.1.

For later use, we introduce some notations. For a nonzero integer r , let $i_r^q A = i_{\alpha^r}^q A$ where α^r is an $|r|$ -tuple whose entries are $r/|r|$. We denote i_α^1, i_α^2 by i_α^+, i_α^- , respectively, and use similar notations for σ and δ . Since Seifert pairing of links in rational homology spheres is rational-valued (see Section 3), we assume that A is a square matrix over \mathbf{Q} . For a positive integer a , let $\eta_a: \mathbf{R} \rightarrow \mathbf{Z}$ be the step function of period 2π given by

$$\eta_a(\phi) = \begin{cases} a + 1 - 2k, & \text{if } 2\pi(k-1)/a < \phi < 2\pi k/a \quad (k = 1, \dots, a) \\ a - 2k, & \text{if } \phi = 2\pi k/a \quad (k = 1, \dots, a-1) \\ 0 & \text{if } \phi = 0. \end{cases}$$

For negative a , let $\eta_a(\phi) = \eta_{-a}(-\phi)$ and let $\eta_0(\phi) = 0$.

Lemma 2.2. *Suppose that $e^{ik\phi} \neq 1$ for $k = 1, \dots, r$. If $n_\alpha \neq 0$ or $A - \epsilon A^T$ is nonsingular, we have*

1. $\sigma_{i_\alpha^+ A}^+(\phi) = \sigma_A^+(n_\alpha \phi)$.
2. $\sigma_{i_\alpha^- A}^-(\phi) = \eta_{n_\alpha}(\phi) \cdot \text{sign}(A + A^T) + \sigma_A^-(n_\alpha \phi)$.

Proof. Let $w = e^{i\phi}$. First we note that if $w \neq 1$,

$$\sigma_A^q(\phi) = \text{sign} \left(\frac{wA - \epsilon \bar{A}^T}{w - 1} \right).$$

We will prove some matrix congruence relations. Consider the matrix

$$\begin{bmatrix} \frac{w^k A - \epsilon A^T}{w^k - 1} & A & \bar{X}^T \\ \epsilon A^T & \frac{wB - \epsilon B^T}{w - 1} & \bar{X}^T \\ X & X & Y \end{bmatrix}$$

where $B = A$ or ϵA^T . Subtracting the second row from the first row and performing the corresponding column operation, a new matrix is obtained. Subsequently we perform the following row operations and corresponding column operations: If $k \neq -1$ and $B = A$, we add the first row times $w(w^k - 1)/(w^{k+1} - 1)$ to the second row. If $k = -1$ and $B = A$,

we assume $A - \epsilon A^T$ is nonsingular and add the top row times $(1 - w)X(A - \epsilon A^T)^{-1}$ to the bottom row. Then we obtain

$$\begin{bmatrix} \frac{(w^{k+1}-1)(A-\epsilon A^T)}{(w-1)(w^k-1)} & 0 & 0 \\ 0 & \frac{w^{k+1}A-\epsilon A^T}{w^{k+1}-1} & \bar{X}^T \\ 0 & X & Y \end{bmatrix} \text{ and } \begin{bmatrix} 0 & \frac{\epsilon A^T-A}{w-1} & 0 \\ \frac{w(\epsilon A^T-A)}{w-1} & \frac{wA-\epsilon A^T}{w-1} & 0 \\ 0 & 0 & Y \end{bmatrix}$$

respectively. If $B = \epsilon A^T$, replace w by w^{-1} and k by $-k$, respectively. Then we obtain

$$\begin{bmatrix} \frac{(w^{k-1}-1)(A-\epsilon A^T)}{(w^{-1}-1)(w^k-1)} & 0 & 0 \\ 0 & \frac{w^{k-1}A-\epsilon A^T}{w^{k-1}-1} & \bar{X}^T \\ 0 & X & Y \end{bmatrix} \text{ and } \begin{bmatrix} 0 & \frac{w(A-\epsilon A^T)}{w-1} & 0 \\ \frac{A-\epsilon A^T}{w-1} & \frac{w(\epsilon A^T)-A}{w-1} & 0 \\ 0 & 0 & Y \end{bmatrix}$$

for $k \neq 1$ and $k = 1$, respectively.

Now we prove the lemma. Note that $\epsilon(i_\alpha^q A)^T = i_{-\alpha}^q A$ and $\sigma_A^q(-\phi) = \sigma_{\epsilon A^T}^q(\phi)$, where $-(s_1, \dots, s_r) = (-s_1, \dots, -s_r)$. We may assume that $n_\alpha \geq 0$ and furthermore by rearranging s_i we may assume $s_1 = \dots = s_u = 1$ and $s_{u+1} = \dots = s_r = -1$ where $u \geq r/2$. For, if $n_\alpha < 0$ and the lemma is proved for the case of $n_\alpha \geq 0$,

$$\begin{aligned} \sigma_{i_\alpha^q A}^q(\phi) &= \sigma_{\epsilon(i_{(-\alpha)}^q A)^T}^q(\phi) = \sigma_{i_{-\alpha}^q A}^q(-\phi) \\ &= \begin{cases} \sigma_A^+((-n_\alpha)(-\phi)) & \text{if } \epsilon = 1 \\ \sigma_A^-((-n_\alpha)(-\phi)) + \eta_{-n_\alpha}(-\phi) \text{sign}(A + A^T) & \text{if } \epsilon = -1 \end{cases} \end{aligned}$$

and the conclusion follows.

Suppose $n_\alpha > 0$. By applying repeatedly the first and third congruence relations in the above to the matrix

$$(1) \quad \frac{w(i_\alpha^q A) - (i_\alpha^q A)^T}{w-1} = \begin{bmatrix} A_1 & A & A & \cdots & A \\ \epsilon A^T & A_2 & A & \cdots & A \\ \epsilon A^T & \epsilon A^T & A_3 & \cdots & A \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \epsilon A^T & \epsilon A^T & \epsilon A^T & \cdots & A_r \end{bmatrix}$$

where

$$A_n = \begin{cases} \frac{wA-\epsilon A^T}{w-1} & \text{if } s_n = 1 \\ \frac{w(\epsilon A^T)-\epsilon(\epsilon A^T)^T}{w-1} & \text{if } s_n = -1 \end{cases},$$

we obtain a matrix that is a block sum of

$$(2) \quad \frac{(w^2-1)(A-\epsilon A^T)}{(w-1)(w-1)}, \dots, \frac{(w^u-1)(A-\epsilon A^T)}{(w-1)(w^{u-1}-1)}, \\ \frac{(w^{u-1}-1)(A-\epsilon A^T)}{(w^{-1}-1)(w^u-1)}, \dots, \frac{(w^{u-(r-u)}-1)(A-\epsilon A^T)}{(w^{-1}-1)(w^{u-(r-u)+1}-1)}$$

and

$$\frac{w^{n_\alpha} A - \epsilon A}{w^{n_\alpha} - 1}.$$

If $\epsilon = 1$, the contribution of (2) to the signature is zero since $\text{sign}(i(A - A^T)) = 0$. (Recall that A is a matrix over \mathbf{Q} .) This proves the conclusion for $n_\alpha > 0$ and $\epsilon = 1$.

If $\epsilon = -1$, let $f_n(\phi) = -\sin(n+1)\phi + \sin n\phi + \sin \phi$. Then we have

$$\operatorname{sgn} f_n(\phi) = \operatorname{sgn} \frac{i(w^{n+1} - 1)}{(w - 1)(w^n - 1)} = -\operatorname{sgn} \frac{i(w^n - 1)}{(w^{-1} - 1)(w^{n+1} - 1)}$$

and therefore

$$\begin{aligned} \sigma_{i_A}^q(\phi) &= \sigma_A^q(n_\alpha \phi) + \operatorname{sign}(A + A^T) \left(\sum_{n=1}^{u-1} \operatorname{sgn} f_n(\phi) - \sum_{n=1}^{r-u} \operatorname{sgn} f_{u-n}(\phi) \right) \\ &= \sigma_A^q(n_\alpha \phi) + \operatorname{sign}(A + A^T) \sum_{n=1}^{n_\alpha-1} \operatorname{sgn} f_n(\phi). \end{aligned}$$

By an elementary calculation, it is easily shown that f_n is zero at $\phi = 2\pi k/(n+1)$, $2\pi k/n$, and $f_n(\phi) > 0$ on $(2\pi(k-1)/n, 2\pi k/(n+1))$ and $f_n(\phi) < 0$ on $(2\pi k/(n+1), 2\pi k/n)$. Hence by induction on n_α the sum in the above equation is equal to $\eta_{n_\alpha}(\phi)$. This proves the conclusion for $n_\alpha > 0$ and $\epsilon = -1$.

Suppose $n_\alpha = 0$. In this case, the proof also proceeds in a similar way. We transform the matrix (1) to a block sum of matrices in (2) and

$$(3) \quad \begin{bmatrix} 0 & \frac{w}{w-1}(A - \epsilon A^T) \\ \frac{1}{w-1}(A - \epsilon A^T) & \frac{w(\epsilon A^T) - A}{w-1} \end{bmatrix}$$

by applying all of the four congruence relations shown before. The signature of (3) is zero since $A - \epsilon A^T$ is nonsingular. Hence we have $\sigma_{i_{\alpha A}}^q(\phi) = 0$. On the other hand, $\sigma_A^+(n_\alpha \phi) = \sigma_A^+(0) = 0$ for $\epsilon = 1$, and $\eta_{n_\alpha}(\phi) \operatorname{sign}(A + A^T) + \sigma_A^-(n_\alpha \phi) = 0$ for $\epsilon = -1$. This completes the proof. \square

Now we are ready to prove the main theorem of this section.

Proof of Theorem 2.1. If $n_\alpha = 0$ and $A - \epsilon A^T$ is nonsingular, $\sigma_{i_{\alpha A}}^q$ is the zero function by Lemma 2.2, and so $\delta_{i_{\alpha A}}^q(\theta) = 0 = \operatorname{sgn} n_\alpha \cdot \delta_A^q(n_\alpha \theta)$ for all θ .

Suppose $n_\alpha > 0$. By Lemma 2.1, we can choose $\epsilon_0, \epsilon_1 > 0$ such that $\sigma_\phi^q(A)$ is constant on each of $[\theta - \epsilon_0, \theta)$, $(\theta, \theta + \epsilon_1]$, $[n_\alpha \theta - n_\alpha \epsilon_0, n_\alpha \theta)$ and $(n_\alpha \theta, n_\alpha \theta + n_\alpha \epsilon_1]$. We may assume that $(\theta - \epsilon_0)/\pi$ and $(\theta + \epsilon_1)/\pi$ are irrational. For $\epsilon = 1$, by Lemma 2.2

$$\begin{aligned} \delta_{i_{\alpha A}}^+(\theta) &= \sigma_{i_{\alpha A}}^+(n_\alpha \theta + \epsilon_1) - \sigma_{i_{\alpha A}}^+(n_\alpha \theta - \epsilon_0) \\ &= \sigma_A^+(n_\alpha(\theta + \epsilon_1)) - \sigma_A^+(n_\alpha(\theta - \epsilon_0)) \\ &= \delta_A^+(n_\alpha \theta). \end{aligned}$$

For $\epsilon = -1$, if $\theta = 2\pi k$ then the conclusion is trivial since $\delta_A^-(2\pi k) = 0$ for any A . If not, we have

$$\begin{aligned} \delta_{i_{\alpha A}}^-(\theta) &= (\eta_{n_\alpha}(\theta + \epsilon_1) - \eta_{n_\alpha}(\theta - \epsilon_0)) \operatorname{sign}(A + A^T) \\ &\quad + (\sigma_A^-(n_\alpha \theta + n_\alpha \epsilon_1) - \sigma_A^-(n_\alpha \theta - n_\alpha \epsilon_0)). \end{aligned}$$

by Lemma 2.2. If $\theta \neq 2\pi k/n_\alpha$ for any k , we may assume that η_{n_α} is constant near θ , and the theorem is proved from the above equation. If $\theta = 2\pi k/n_\alpha$ for some k that is not a multiple of n_α , then $\delta_{i_{\alpha A}}^-(\theta) = -2 \operatorname{sign}(A + A^T) + (\sigma_A^-(2\pi k + n_\alpha \epsilon_1) - \sigma_A^-(2\pi k - n_\alpha \epsilon_0)) = 0 = \delta_A^-(n_\alpha \theta)$ since $\sigma_A^-(\pm \phi) = \pm \operatorname{sign}(A + A^T)$ for all sufficiently small $\phi > 0$.

If $n_\alpha < 0$, we have $\delta_{i_\alpha^q}^q(\theta) = \delta_{\epsilon(i_{-\alpha}^q A)^T}^q(\theta) = -\delta_{i_{-\alpha}^q A}^q(-\theta) = -\delta_A^q(n_\alpha \theta)$, as in the proof of Lemma 2.2. \square

We remark that a similar argument shows an analogous result for Alexander polynomials defined by $\Delta_A^q(t) = \det(tA - \epsilon A^T)$: If $A - \epsilon A^T$ is nonsingular, then $\Delta_{i_\alpha^q A}^q(t) = \Delta_A^q(t^{n_\alpha})$ up to units of $\mathbf{Q}[t, t^{-1}]$. This result is mentioned and used in [1].

We finish this section with a remark on the matrix cobordism groups. Consider the set of rational matrices A such that $A - \epsilon A^T$ is nonsingular. Define the block sum operation and cobordisms of such matrices as in [17]. Then arguments of [17] show that the cobordism of such matrices is an equivalence relation and that the set $G_\epsilon^{\mathbf{Q}}$ of cobordism classes becomes an abelian group under the block sum. Then i_α^q induces a group homomorphism on $G_\epsilon^{\mathbf{Q}}$ and both δ_A^q and i_α^q are well-defined on $G_\epsilon^{\mathbf{Q}}$. Thus Lemma 2.2 also makes sense when A is regarded as its cobordism class.

3. SEIFERT SURFACES OF LINKS

In this section, we study links and their Seifert surfaces in rational homology spheres. We start with a review of rational-valued linking numbers. Let x, y be disjoint p, q -cycles respectively in a $(n+2)$ -rational homology sphere Σ where $p+q = n+1$. Since $H_p(\Sigma)$, $H_q(\Sigma)$ are torsion, there are $(p+1), (q+1)$ -chains u, v in Σ such that $\partial u = ax, \partial v = by$ for some nonzero integers a, b . The *linking number* of x and y is defined to be $\text{lk}_\Sigma(x, y) = (1/b)(x \cdot v) = (-1)^{q+1}(1/a)(u \cdot y)$ where \cdot denotes the intersection number in Σ . The linking number $\text{lk}_\Sigma(x, y)$ is determined by x and y and is independent of the choices of a, b, u and v . When the ambient space is obvious, we will denote the linking number by $\text{lk}(x, y)$. We remark that $\text{lk}(-, -)$ is well-defined only for disjoint cycles, and its value modulo \mathbf{Z} is well-defined for homology classes.

The following properties of the linking number are easily checked. First, if $c(x' - x) = \partial w$ for some $c \neq 0$ and $(p+1)$ -chain w disjoint to y , then $\text{lk}(-, y)$ assumes the same value on x, x' . $\text{lk}(x, -)$ has the same property. Second, suppose that Σ is a boundary component of an $(n+3)$ -manifold Δ such that $H_{p+1}(\Delta; \mathbf{Q}) = H_{q+1}(\Delta; \mathbf{Q}) = 0$. If x, y are as above and u, v are relative $(p+1), (q+1)$ -cycles of (Δ, Σ) such that $\partial u = ax, \partial v = by$ for some $a, b \neq 0$, $\text{lk}(x, y) = (1/ab)(u \cdot v)$.

We give a formula for linking numbers in a rational homology 3-sphere described by a Kirby diagram. For an m -component framed link, we define the linking matrix A to be a square matrix of dimension m whose (i, j) -entry is the linking number of the i -th component and the j -th longitude with respect to the framing.

Theorem 3.1. *Let $L = K_1 \cup \cdots \cup K_m$ be a framed link in S^3 such that the result of surgery on S^3 along L is a rational homology sphere Σ . Let $A = (a_{ij})$ be the linking matrix of L . For disjoint 1-cycles a, b in $S^3 - L$, we have*

$$\text{lk}_\Sigma(a, b) = \text{lk}_{S^3}(a, b) - x^T A^{-1} y$$

where $x = (x_i)$, $y = (y_i)$ are column vectors with $x_i = \text{lk}_{S^3}(a, K_i)$, $y_i = \text{lk}_{S^3}(b, K_i)$, respectively.

Proof. Let μ_i be a meridian curve on a tubular neighborhood of K_i . Let u and u_i be 2-chains in S^3 that are transverse to L and bounded by b and a 0-linking longitude of K_i , respectively. Let \bar{u} and \bar{u}_i be chains obtained by removing from u and u_i the intersection

with an open tubular neighborhood of L , respectively. Let c_i be the core of the 2-handle attached along K_i . Let $v = k\bar{u} + \sum_i z_i(\bar{u}_i + c_i)$ be a chain in Σ , where k, z_i are integers to be specified later. Then ∂v is homologous to $kb - \sum_j \sum_i (ky_j + z_i a_{ij})\mu_j$. Thus if $z = (z_1, \dots, z_m)$ satisfies $zA = -ky$, then ∂v is homologous to kb . Since Σ is a \mathbf{Q} -homology sphere, A is invertible over \mathbf{Q} . So we can take as k a nonzero common multiple of denominators of entries of $A^{-1}y$, and $z = -kA^{-1}y$. Now we have

$$\begin{aligned} \text{lk}_\Sigma(a, b) &= \frac{1}{k}(a \cdot v) \\ &= \frac{1}{k} \left(\text{lk}_{S^3}(a, kb) + \sum_i x_i z_i \right) \\ &= \text{lk}_{S^3}(a, b) - x^T A^{-1}y \end{aligned}$$

□

Now we turn our attention to links. The main goal of this section is to study the existence of a generalized Seifert surface of a given type τ for links. Let L be an n -link in a rational homology sphere, and suppose $n > 1$. L can be viewed as a framed manifold since all framings are equivalent. Generally, for a framed submanifold M of codimension 2 in a manifold W , we identify a tubular neighborhood of M with $M \times D^2$ with respect to the framing, and in particular identify $M \times S^1$ with a submanifold in the boundary of the exterior $E_M = W - M \times \text{int}(D^2)$. We say (W, M) is *primitive* if there is a homomorphism of $H_1(E_M) \rightarrow \mathbf{Z}$ whose composition with $H_1(M \times S^1) \rightarrow H_1(E_M)$ is the same as the projection $H_1(M \times S^1) \rightarrow H_1(S^1) = \mathbf{Z}$. When W is obvious, we say M is primitive. By a standard use of obstruction theory, transversality and Thom-Pontryagin construction, L is primitive if and only if L bounds a Seifert surface.

Our main observation is that certain unions of parallel copies of L is primitive and hence bound Seifert surfaces. We use the following notations to denote parallel copies of a framed submanifold M in W . For an r -tuple $\alpha = (s_1, \dots, s_r)$ where $s_i = \pm 1$, let n_α be the sum of s_i as before, and $i_\alpha M$ be the union of r parallel copies of M in W with respect to the framing where the i -th copy is oriented according to the sign of s_i . For a nonzero integer r , let $i_r M = i_\alpha M$ where α is the r -tuple $(\text{sgn } r, \dots, \text{sgn } r)$.

We assert that $i_\alpha L$ is primitive if and only if n_α is a multiple of c , where c is a positive integer given as follows. Let $\varphi: H_1(E_L; \mathbf{Q}) \rightarrow \mathbf{Q}$ be the homomorphism sending each meridian to 1, which is uniquely determined by the Alexander duality. Let P be the quotient group of $H_1(E_L)$ by its torsion subgroup. Since $P \otimes \mathbf{Q} = H_1(E_L; \mathbf{Q})$, we can view P as a subgroup of $H_1(E_L; \mathbf{Q})$. Choose a basis $\{e_i\}$ of P , and let c be the positive least common multiple of denominators of the reduced fractional expression of $\varphi(e_i)$. We call c be the *complexity* of L . c is independent to the choice of the basis. Note that $c\varphi(P) \subset \mathbf{Z} \subset \mathbf{Q}$ and hence a homomorphism ϕ from P into \mathbf{Z} is induced by $c\varphi$.

Suppose that $i_\alpha L$ is primitive for $\alpha = (s_1, \dots, s_r)$. Assuming that $i_\alpha L$ is the product of L and r points in the tubular neighborhood $L \times D^2$, we can view E_L as a subspace of $E_{i_\alpha L}$. There is a homomorphism of $H_1(E_{i_\alpha L})$ into \mathbf{Z} sending each meridian of $i_\alpha L$ to 1, and it induces a homomorphism h of $P = H_1(E_L)/\text{torsion}$ to \mathbf{Z} sending each meridian of L to n_α . The homomorphisms $(1/n_\alpha)(h \otimes 1_{\mathbf{Q}})$ and φ on $P \otimes \mathbf{Q} = H_1(E_L; \mathbf{Q})$ are equal since they assume the same value on meridians. Hence $h(e_i)/n_\alpha = \varphi(e_i)$ and so n_α is a

multiple of the denominator of $\varphi(e_i)$. This shows that n_α is a multiple of c . Conversely, suppose $n_\alpha = kc$ for some integer k . Then the homomorphism $k\phi: H_1(E_L) \rightarrow \mathbf{Z}$ sends each meridian of L to n_α , and it extends to a homomorphism of $H_1(E_{i_\alpha L})$ to \mathbf{Z} sending each meridian of $i_\alpha L$ to 1. This proves the assertion.

For a type $\tau = (\tau_i)$ for L , let L^τ be the union of τ_i parallel copies (negatively oriented if $\tau_i < 0$) of the i -th component of L over all i . We call the complexity of L^τ the *complexity of type τ for L* and denote it by $c^\tau(L)$. A generalized Seifert surface of type τ with complexity r for L is a Seifert surface of $i_r L^\tau$, which exists if and only if $i_r L^\tau$ is primitive. This holds exactly when r is a multiple of $c^\tau(L)$. In particular,

Theorem 3.2. *For $n > 1$, an n -link L admits a generalized Seifert surface of any type τ .*

Theorem 3.3. *Fix a basis of $P = H_1(E_L)/\text{torsion}$, and let B be a matrix whose i -th column represents the i -th meridian of L with respect to the basis. Then $c^\tau(L)$ is equal to the least common multiple of denominators (of a reduced fractional expression) of entries of the row vector $\tau^T B^{-1}$. (We view τ as a column vector.)*

Proof. First of all, B is invertible since meridians are independent. Consider the homomorphism of $H_1(E_{L^\tau}; \mathbf{Q})$ to \mathbf{Q} sending meridians to 1. Viewing E_L as a subspace of E_{L^τ} , a homomorphism of $H_1(E_L; \mathbf{Q})$ to \mathbf{Q} sending the i -th meridian of L to τ_i is induced. Suppose that the i -th basis of P is sent to x_i under the induced homomorphism. By the definition of the complexity, the least common multiple of the denominators of x_i is equal to $c^\tau(L)$. The row vector $x = (x_i)$ satisfies $xB = \tau^T$, and so $x = \tau^T B^{-1}$ as desired. \square

Now we consider 1-links in a rational homology sphere. Our result for 1-links is expressed in terms of a linking matrix of L (for any choice of a framing) as follows. We say a 1-link L *admits type τ* if there is a generalized Seifert surface of type τ for L .

Theorem 3.4. *A 1-link L admits type τ if and only if for a linking matrix $A = (a_{ij})$ with respect to a (or any) framing,*

$$\frac{1}{\tau_i} \sum_j a_{ij} \tau_j$$

is an integer for all i .

First we will investigate when a generalized Seifert surface inducing a given fixed framing exists, and then show Theorem 3.4. Let L be a 1-link, f be a framing on L and $A = (a_{ij})$ be the linking matrix with respect to f .

Lemma 3.1. *There exists a generalized Seifert surface of type $\tau = (\tau_i)$ for L that induces the same framing on its boundary as the framing induced by f if and only if*

$$\sum_j a_{ij} \tau_j = 0$$

for all i .

Proof. Let L^τ be the link obtained by taking τ_i parallel copies of the i -th components of L with respect to the given framing f . L admits a desired generalized Seifert surface of complexity r if and only if $i_r L^\tau$ is primitive. Since $H_1(E_{L^\tau}; \mathbf{Q})$ is generated by meridians of

L^τ , for any multiple r of the complexity of L^τ we obtain a homomorphism ϕ of $H_1(E_{i_r L^\tau})$ to \mathbf{Z} sending each meridian of $i_r L^\tau$ to 1, by the arguments for higher dimensional links. In order to show that $i_r L^\tau$ is primitive, we need to check whether the longitudes of $i_r L^\tau$ is sent to zero by ϕ , or equivalently, the longitudes of L is sent to zero by the unique homomorphism from $H_1(E_L; \mathbf{Q}) \rightarrow \mathbf{Q}$ sending meridians μ_j of L to τ_j . Since the i -th longitude of L is equal to $\sum_j a_{ij} \mu_j$ in $H_1(E_L; \mathbf{Q})$, the conclusion follows. \square

Proof of Theorem 3.4. Let A be a linking matrix of a 1-link L with respect to a fixed framing f . Then the i -th longitude with respect to an arbitrary framing f' is homologous to the sum of the i -th longitude with respect to f and a multiple of the i -th meridian in the complement of L . Hence the linking matrix with respect to f' is of the form $A + D$ where D is a diagonal matrix of integers. Conversely, D determines a framing since framings are in 1-1 correspondence with elements of $H_1(L; \pi_1(SO_2)) \cong \mathbf{Z}^m$.

Therefore, by Lemma 3.1, there is a generalized Seifert surface of type τ if and only if $(A + D)\tau = 0$ for some diagonal matrix D of integers, or equivalently, $\frac{1}{\tau_i} \sum_j a_{ij} \tau_j$ is an integer for all i . \square

We remark that generalized Seifert surfaces of the same type induce the same framing, since in the above proof the numbers d_i are determined by $\tau = (\tau_i)$. From now on, for a 1-link L and a type τ admitted by L , we denote by L^τ the union of τ_i parallel copies of the i -th component of L with respect the framing determined by τ .

Example. 1. If a linking matrix of a link L is zero, then any type is admitted.

2. If L is a link in a homology sphere, each entry of a linking matrix is an integer. Hence type $(1 \cdots 1)$ is admitted.

3. Let C be an m -component unlink in S^3 and L be the union of meridian curves of components of C . A rational homology sphere Σ is obtained by n -surgery on each component of C for $n \geq 2$, and L can be viewed as a link in Σ . By Theorem 3.1, the diagonal matrix with diagonals $-1/n$ is a linking matrix, and by Theorem 3.4, L does not admit any type.

4. For any given type $\tau = (a, b)$, there is a link that admits only type $\pm\tau$. For, suppose that a link L has linking matrix $\begin{bmatrix} -b/3a & 1/3 \\ 1/3 & -a/3b \end{bmatrix}$. If a type (x, y) is admitted by L , $(y/x - b/a)/3$ and $(x/y - a/b)/3$ are integers by Theorem 3.4. This implies that $(x, y) = \pm(a, b)$. By the following lemma that classifies linking matrices, such a link L exists.

Lemma 3.2. *For any symmetric square matrix A with rational entries, there is a framed 1-link in a rational homology sphere whose linking matrix is A .*

Proof. We claim that for any symmetric nonsingular m by m matrix V and m by n matrix $B = (b_{ij})$ with integral entries, there is an n -component framed 1-link in a rational homology sphere with linking matrix $B^T V^{-1} B$. Consider the n -component standard unlink L in S^3 with zero-linking framing. We can construct an m -component framed link L' in S^3 with linking matrix $-V$ such that L and L' are disjoint and the linking number of the i -th component L' and the j -th component of L is b_{ij} . The result of surgery along L' is a rational homology 3-sphere Σ since $-V$ is nonsingular. L can be viewed as a link in Σ , and the linking matrix of L in Σ is $B^T V^{-1} B$, by Theorem 3.1.

Any symmetric matrix A over \mathbf{Q} can be written as $A = P^T \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} P$ where D is a nonsingular diagonal matrix and each 0 represents a zero matrix of suitable size. Choose a nonzero common multiple n of denominators of entries of P and D^{-1} and let $B = n \begin{bmatrix} I & 0 \end{bmatrix} P$, $V = n^2 D^{-1}$. Then B and V are matrices over \mathbf{Z} , and we have $A = B^T V^{-1} B$. By the claim, A is realized as a linking matrix of a 1-link in a rational homology sphere. \square

We finish our discussion on 1-links by showing that the set of admitted types of a link is invariant under link concordance. For two \mathbf{Q} -concordant 1-links, a framing on a concordance between the links induces framings on the links, and in fact this is a 1-1 correspondence between the framings of the two links.

Theorem 3.5. *If 1-links L_0 and L_1 are \mathbf{Q} -concordant, then L_0 admits type τ if and only if so does L_1 . Furthermore, τ determines corresponding framings on L_0 and L_1 .*

Proof. Since linking numbers are \mathbf{Q} -concordance invariants, linking matrices of L_0 and L_1 with respect to corresponding framings are the same. Hence by Theorem 3.4, the conclusion is proved. \square

We remark that our complexity is motivated from a similar notion in [3, 4]. The complexity in [3, 4] is similar to $c^\tau(L)$ for $\tau = (1 \cdots 1)$. In general the former is a multiple of the latter.

From now on, we will consider only odd dimensional links. Let (Σ, L) be a $(2q-1)$ -link and F be a generalized Seifert surface of type τ . Define the *Seifert pairing* $S: H_q(F; \mathbf{Q}) \times H_q(F; \mathbf{Q}) \rightarrow \mathbf{Q}$ by $S([x], [y]) = \text{lk}(x^+, y)$ for two q -cycles x, y in F where x^+ denotes the cycle obtained by pushing x slightly along the positive normal direction of F . It is easily verified that S is well-defined. Choosing a basis of $H_q(F; \mathbf{Q})$, a matrix A over \mathbf{Q} is associated to S . A is called a *Seifert matrix* of F . We remark that for $q = 1$, another convention is found in some literature where Seifert pairings and Seifert matrices are defined on the cokernel of $H_1(\partial F) \rightarrow H_1(F)$. These different conventions give different results in the case of links (e.g. consider a full-twisted annulus with Hopf link boundary in S^3).

We remark that if A is a Seifert matrix of F , the matrix $i_\alpha^q A$ is a Seifert matrix of $i_\alpha F$.

For $q > 1$, we can view a Seifert matrix A as a representative of an element $[A]$ of the matrix cobordism group $G_\epsilon^{\mathbf{Q}}$, since $A - \epsilon A^T$ represents the intersection pairing on $H_q(F; \mathbf{Q})$ that is non-degenerated. The following is an analogous result of the well-known result of [15, 16]. Recall that a \mathbf{Q} -concordance between two links is called primitive if there is a homomorphism of the first homology of its exterior into \mathbf{Z} that sends each meridian and each longitude to 1 and 0, respectively.

Lemma 3.3. *Let A_0 and A_1 be Seifert matrices of Seifert surfaces for two $(2q-1)$ -links (Σ_0, L_0) and (Σ_1, L_1) that are \mathbf{Q} -concordant via a primitive \mathbf{Q} -concordance. If $q > 1$, then $[A_0] = [A_1]$ in $G_\epsilon^{\mathbf{Q}}$.*

In this case, everything is primitive and an argument in [16] can be used to prove the conclusion. Details are as follows.

Proof. Let (\mathbf{S}, \mathbf{L}) be a primitive \mathbf{Q} -concordance between (Σ_0, L_0) and (Σ_1, L_1) , and $E_{\mathbf{L}}$ be the exterior of \mathbf{L} in \mathbf{S} . We may assume $\Sigma_0 \cap E_{\mathbf{L}} = E_{L_0}$ and $\Sigma_1 \cap E_{\mathbf{L}} = E_{L_1}$. For $i = 0, 1$, let F_i be a Seifert surface of L_i where A_i is defined. Let C be a parallel copy of

\mathbf{L} in $\partial E_{\mathbf{L}}$ bounded by $L_1 - L_0$. Let $F = F_0 \cup C \cup -F_1$, a closed oriented submanifold of $\partial E_{\mathbf{L}}$ of codimension 1.

Define a map $h: \partial E_{\mathbf{L}} \rightarrow S^1$ by a Thom-Pontryagin construction for $F \subset \partial E_{\mathbf{L}}$. Since \mathbf{L} is primitive, there is a homomorphism $\phi: H_1(E_{\mathbf{L}}) \rightarrow \mathbf{Z}$ sending each meridian to 1. Let $i_*: H_1(\partial E_{\mathbf{L}}) \rightarrow H_1(E_{\mathbf{L}})$ denote the map induced by the inclusion. Then ϕi_* and h_* assume the same value on each meridian and so $\phi i_* = h_*$ since \mathbf{Z} is torsion free. Therefore h is extended to a map $E_{\mathbf{L}} \rightarrow S^1$. By a transversality argument, we construct a $(2q+1)$ -submanifold R of $E_{\mathbf{L}}$ such that $\partial R = F$.

Since $q > 1$, $H_q(F) \cong H_q(F_0) \oplus H_q(F_1)$. Let S be the bilinear pairing on $H_q(F)$ represented by the block sum of A_0 and A_1 . $H = \text{Ker}(H_q(F; \mathbf{Q}) \rightarrow H_q(R; \mathbf{Q}))$ is a half-dimensional subspace of $H_q(F; \mathbf{Q})$ by a duality argument, and S vanishes on H by the previous remarks on the linking number as in [16]. Therefore A_0 and A_1 are cobordant matrices. \square

We note that the above proof does not work in the case of $q = 1$, since $H_q(F)$ is not isomorphic to $H_q(F_0) \oplus H_q(F_1)$.

Theorem 3.6. *Let A_1, A_2 be Seifert matrices of two \mathbf{Q} -concordant $(2q-1)$ -links with $q > 1$. Then $i_r^q[A_0] = i_r^q[A_1]$ in $G_\epsilon^{\mathbf{Q}}$ for some integer $r > 0$.*

Proof. Let \mathbf{L} be a \mathbf{Q} -concordance between two links whose Seifert matrices are A_1 and A_2 . By the same argument as that for links, there is a nonzero integer r (the “complexity” of \mathbf{L}) such that $i_r \mathbf{L}$ is a primitive. $i_r \mathbf{L}$ is a \mathbf{Q} -concordance of two links with Seifert matrices $i_r^q A_0$ and $i_r^q A_1$. By Lemma 3.3, $[i_r^q A_0] = [i_r^q A_1]$. \square

4. SIGNATURES OF LINKS

In this section we prove the invariance of signatures under link concordance. In the case of $q > 1$, Seifert matrices can be viewed as an element of $G_\epsilon^{\mathbf{Q}}$ and the invariance of signatures may be shown directly using Theorem 3.6. Because this does not work for $q = 1$, we will interpret signatures via finite branched covers following ideas of [9, 10] to show the invariance of signatures in any odd dimensions.

Suppose that d is a positive integer, (W, M) is a primitive pair, and $H_1(W; \mathbf{Q}) = H_2(W; \mathbf{Q}) = 0$. Then a homomorphism $\phi: H_1(E_M) \rightarrow \mathbf{Z}$ sending each meridian to 1 exists uniquely, and the composition of ϕ and the canonical projection $\mathbf{Z} \rightarrow \mathbf{Z}_d$ determines a d -fold cover \tilde{W} of W branched along M with a transformation t on \tilde{W} corresponding to $1 \in \mathbf{Z}_d$ such that $\tilde{W}/t \cong W$. When W is even dimensional, define $\sigma_{k,d}(W, M)$ to be the signature of the restriction of the intersection form of \tilde{W} on the $e^{2k\pi i/d}$ -eigenspace of the induced homomorphism t_* on the middle dimensional homology of \tilde{W} with complex coefficients.

Let L be a primitive $(2q-1)$ -link (that has the framing induced by a Seifert surface if $q = 1$) in a rational homology sphere Σ .

Lemma 4.1. *There is a primitive $(2q+2, 2q)$ -manifold pair (W, M) such that (Σ, L) is a boundary component of (W, M) and $H_i(W; \mathbf{Q}) = H_i(\partial W; \mathbf{Q}) = 0$ for $i = 1, \dots, 2q$.*

Proof. Put $W = \Sigma \times [0, 1]$ and identify Σ with $\Sigma \times 1$. The map $E_L \rightarrow S^1$ given by primitiveness is extended to a map $\Sigma \rightarrow D^2$ in an obvious way, and the union of it and

a constant map $\Sigma \times 0 \rightarrow S^1$ is extended to a map $W \rightarrow D^2$. Then M is obtained by transversality at an interior point of D^2 . \square

Note that the above argument is similar to that of the well-known special case $\Sigma = S^{2q+1}$ where B^{2q+2} can be taken as W . Hence it seems to be natural to take rational homology balls as W in our general case, however it is not known even when Σ is null-cobordant. This is why we allow boundary components of W other than Σ .

The following result shows that if (W, M) satisfies the conditions of Lemma 4.1, $\sigma_{k,d}(W, M)$ is independent of the choice of W and M .

Lemma 4.2. *Let (W_0, M_0) and (W_1, M_1) be manifold pairs as in Lemma 4.1. Then $\sigma_{k,d}(W_0, M_0) = \sigma_{k,d}(W_1, M_1)$.*

Proof. We consider $(W, M) = -(W_0, M_0) \cup_{(\Sigma, L)} (W_1, M_1)$. The idea is that (W, M) bounds a primitive pair (V, N) so that the signature of the branched cover of W vanishes. Then by the additivity of signatures, the signatures of branched covers of W_0 and W_1 are the same and the proof is completed. This is similar to the well-known argument for the case of $\Sigma = S^{2q+1}$ and $W_i = B^{2q+2}$. In that case, we can take $V = B^{2q+3}$ and construct N by the argument of the proof of the previous lemma. For our general case, we take $V = W \times [0, 1]$ and proceed in the same way. Even though V is not closed, all extra boundary components have vanishing middle dimensional rational homology and hence have no contributions to signatures. \square

We define $\sigma_{k,d}(L) = \sigma_{k,d}(W, M)$. By Lemma 4.2, $\sigma_{k,d}(L)$ is well-defined.

Lemma 4.3. *If L is primitive and A is a Seifert matrix of a Seifert surface of L , $\sigma_{k,d}(L) = \sigma_A^q(2\pi k/d)$.*

In [9, Chapter XII], Kauffman shows this lemma when L is a link in S^3 and $\sigma_{k,d}(L)$ is defined from (D^4, M) where M is a properly embedded surface in D^4 bounded by L . In fact the same argument can be applied to show that if $W = \Sigma \times I$ and M is obtained by pushing into W the interior of a Seifert surface where A is defined, then $\sigma_{k,d}(W, M) = \sigma_A^q(2\pi k/d)$. We omit the details.

The following result as well as Lemma 3.3 is based on the idea that most of the techniques of signature invariants for links in spheres also work for links in rational homology spheres provided everything is primitive. In fact, for $q > 1$, this result is a consequence of Lemma 3.3. We use a geometric argument to prove this result in any odd dimension.

Lemma 4.4. *If A_0 and A_1 are Seifert matrices defined on Seifert surfaces of primitive $(2q-1)$ -links L_0 and L_1 that are \mathbf{Q} -concordant via a primitive \mathbf{Q} -concordance, $\delta_{A_0}^q(\theta) = \delta_{A_1}^q(\theta)$ for all θ .*

Proof. Let (\mathbf{S}, \mathbf{L}) be a primitive \mathbf{Q} -concordance between two links L_0 and L_1 . Choose a primitive pair (W_0, M_0) where $\sigma_{k,d}(L_0)$ can be defined. Let $(W, M) = (W_0, M_0) \cup_{(\Sigma_0, L_0)} (\mathbf{S}, \mathbf{L})$. Then M is also primitive. We will compute $\sigma_{k,d}(L_1)$ from (W, M) .

Note that for any sufficiently large prime p , the pair (\mathbf{S}, \mathbf{L}) can be considered as a concordance between links in \mathbf{Z}_p -homology spheres and in particular $H_{q+1}(E_{\mathbf{L}}, E_{L_0}; \mathbf{Z}_p) = 0$. For such a prime p , let $(\tilde{\mathbf{S}}, \tilde{\mathbf{L}})$ and $(\tilde{\Sigma}_0, \tilde{L}_0)$ be the p -fold branched covers of (\mathbf{S}, \mathbf{L}) and (Σ_0, L_0) , respectively.

We need the following consequence of Milnor's exact sequence for infinite cyclic coverings [21]. Suppose that (\tilde{X}, \tilde{A}) be a p -fold cyclic cover of a CW -complex (X, A) induced by a map $H_1(X) \rightarrow \mathbf{Z}_d$ that factors through the projection $\mathbf{Z} \rightarrow \mathbf{Z}_d$. Then if $H_i(X, A; \mathbf{Z}_p)$ is zero, $H_i(\tilde{X}, \tilde{A}; \mathbf{Z}_p)$ is also zero.

In our case, $H_{q+1}(E_{\tilde{\mathbf{L}}}, E_{\tilde{L}_0}; \mathbf{Z}_p)$ is zero and so $H_{q+1}(E_{\tilde{\mathbf{L}}}, E_{\tilde{L}_0}; \mathbf{Z}_p)$ is also zero, for any large prime p . From a Mayer-Vietoris sequence for $(\tilde{\mathbf{S}}, \tilde{\Sigma}_0) = (E_{\tilde{\mathbf{L}}}, E_{\tilde{L}_0}) \cup ((\tilde{\mathbf{L}}, \tilde{L}_0) \times D^2)$, it follows that $H_{q+1}(\tilde{\mathbf{S}}, \tilde{\Sigma}_0; \mathbf{Z}_p)$ is zero. Hence $\tilde{\mathbf{S}}$ contributes nothing to the signature of the branched covering of (W, M) , and $\sigma_{k,p}(W, M) = \sigma_{k,p}(W_0, M_0)$ by additivity of signatures. Therefore $\sigma_{k,p}(L_0) = \sigma_{k,p}(L_1)$ and by Lemma 4.3, $\sigma_{A_0}^q(\theta) = \sigma_{A_1}^q(\theta)$ if $\theta = 2\pi k/p$ for some large prime p and some integer k . Since the set of such θ is dense in the real line, $\delta_{A_0}^q(\theta) = \delta_{A_1}^q(\theta)$ for all θ . \square

Now we are ready to show the invariance of the signature jump function under \mathbf{Q} -concordance. The key to handle the non-primitive case is to combine the geometric result for the primitive case with the reparametrization formula in Section 2.

Proof of Theorem 1.1. We may similarly proceed as in the proof of Theorem 3.6. (Indeed, for the case of $q > 1$, the result is a consequence of Theorem 3.6.) Let L_0 and L_1 be \mathbf{Q} -concordant links admitting the given type τ . Let A_0 and A_1 be Seifert matrices of Seifert surfaces of $i_{c_0}L_0^\tau$ and $i_{c_1}L_1^\tau$, respectively. Let \mathbf{L}^τ be the union of τ_i parallel copies of the i -th component of a \mathbf{Q} -concordance between L_0 and L_1 . If $q = 1$, we need to take parallel copies with respect to the framing determined by τ . As before, we can choose a positive integer r such that $i_r\mathbf{L}^\tau$ is primitive. Then $i_r i_{c_1} i_{c_0} L_0^\tau$ and $i_r i_{c_0} i_{c_1} L_1^\tau$ are \mathbf{Q} -concordant via the primitive \mathbf{Q} -concordance $i_r i_{c_0} i_{c_1} \mathbf{L}^\tau$. Note that the links $i_r i_{c_1} i_{c_0} L_0^\tau$ and $i_r i_{c_0} i_{c_1} L_1^\tau$ are viewed as the boundaries of unions of parallel copies of Seifert surfaces of the links $i_{c_0}L_0^\tau$ and $i_{c_1}L_1^\tau$. In the case of $q = 1$, the framing used to take $i_{c_k}L_k^\tau$ and the framing induced by the Seifert surface of $i_{c_k}L_k^\tau$ must be compatible in order for $i_r i_{c_1} i_{c_0} L_0^\tau$ and $i_r i_{c_0} i_{c_1} L_1^\tau$ to be \mathbf{Q} -concordant. This is where our framing condition in the definition of generalized Seifert surfaces for 1-links is necessary.

The primitive links $i_r i_{c_1} i_{c_0} L_0^\tau$ and $i_r i_{c_0} i_{c_1} L_1^\tau$ have the Seifert matrices $i_r^q i_{c_1}^q A_0$ and $i_r^q i_{c_0}^q A_1$, respectively. By Theorem 2.1 and Lemma 4.4, we have

$$\begin{aligned} \delta_{L_0}^\tau(\theta) &= \text{sgn } c_0 \cdot \delta_{A_0}^q(\theta/c_0) \\ &= \text{sgn } c_0 \text{sgn } c_1 \cdot \delta_{i_r^q i_{c_1}^q A_0}^q(\theta/rc_0 c_1) \\ &= \text{sgn } c_0 \text{sgn } c_1 \cdot \delta_{i_r^q i_{c_0}^q A_1}^q(\theta/rc_0 c_1) \\ &= \text{sgn } c_1 \cdot \delta_{A_1}^q(\theta/c_1) = \delta_{L_1}^\tau(\theta). \end{aligned}$$

This proves that $\delta_L^\tau(\theta)$ is a \mathbf{Q} -concordance invariant independent of the choice of generalized Seifert surfaces. \square

We remark that our framing condition for generalized Seifert surfaces of 1-links seems to be the minimal requirement for signatures to be well-defined \mathbf{Q} -concordance invariants. For example, as an attempt to remove the framing condition, one may fix corresponding framings on L_k for $k = 0, 1$ and consider $i_r L_k^\tau$ taken with respect to the framings, where r is a nonzero multiple of the complexity of concordance. It can be shown that they admit Seifert surfaces, and are \mathbf{Q} -concordant via a primitive concordance. The signature jump

functions of Seifert matrices of $i_r L_k$ are equal by Lemma 4.4. But the above argument can not be applied to prove that the reparametrizations $\theta = \theta/r$ of the signature jump functions are invariants of the original links L_k and in fact they are not in general.

We also remark that it is meaningful to consider types for higher dimensional links as well as 1-links. In fact the signature jump function of type τ for a link L is no more than the signature jump function of type $(1 \cdots 1)$ for the link L^τ obtained by taking parallel copies, however, more information on L is obtained from types other than $(1 \cdots 1)$ even for higher dimensional links in the sphere. For example, suppose K is a knot that is not torsion in the (algebraic) knot concordance group (so that $\delta_K(\theta)$ is nontrivial [17]) and L be a split union of K and $-K$. Then $\delta_L^{(1,1)}$ is trivial, but if $a \neq b$, $\delta_L^{(a,b)}(\theta)$ is nontrivial for some θ .

For $\tau = (1 \cdots 1)$, we denote $\delta^\tau(L)$ and $c^\tau(L)$ by $\delta(L)$ and $c(L)$, respectively.

Now we prove Theorem 1.2.

Proof of Theorem 1.2. 1. If L is slice, δ_L^τ is equal to the signature of the unlink by Theorem 1.1. Since the null matrix is a Seifert matrix for the unlink, $\delta_L^\tau(\theta) = 0$.

2. Choose a positive multiple r of $c^\tau(L)$, and let A be a Seifert matrix for $i_r L^\tau$. Then since $i_\alpha^q A$ is a Seifert matrix of $i_\tau i_\alpha L^\tau$, $\delta_{i_\alpha L}^\tau(\theta) = \delta_{i_\alpha^q A}^q(\theta/r) = \text{sgn } n_\alpha \cdot \delta_A^q(n_\alpha \theta/r) = \text{sgn } n_\alpha \cdot \delta_L^\tau(n_\alpha \theta)$ by Theorem 2.1.

3. Let A be a Seifert matrix of $i_{c^\tau(L)} L^\tau$. Since $\delta_A^q(\theta)$ is of period 2π , $\delta_L^\tau(\theta) = \delta_A^q(\theta/c^\tau(L))$ is of period $2\pi \cdot c^\tau(L)$.

4. For $i = 1, 2$, let B_i be a $(2q+1)$ -ball in the ambient space Σ_i of L_i such that the connected sum (Σ, L) is obtained by gluing $(\Sigma_i - \text{int}(B_i), L_i - \text{int}(B_i))$ along their boundaries. We view $\Sigma_i - \text{int}(B_i)$ as a subspace of Σ . Let c be a positive common multiple of $c^\tau(L_1)$ and $c^\tau(L_2)$. Then there is a generalized Seifert surface F_i of type τ with complexity c for L_i . We may assume that each component of $F_i \cap B_i$ is a $2q$ -disk whose boundary contains exactly one component of $\partial F_i \cap B_i$ so that the intersection of $F_1 - \text{int}(B_1)$ and $F_2 - \text{int}(B_2)$ consists of disjoint disks, and their union is a generalized Seifert surface F of type τ with complexity c for L in Σ .

For $q > 1$, let A_i be a Seifert matrix of F_i . We have $H_q(F) = H_q(F_1) \oplus H_q(F_2)$, and $A_1 \oplus A_2$ is a Seifert matrix of F . Therefore $\delta_L^\tau(\theta) = \delta_{A_1 \oplus A_2}^q(\theta/c) = \delta_{A_1}^q(\theta/c) + \delta_{A_2}^q(\theta/c) = \delta_{L_1}^\tau(\theta) + \delta_{L_2}^\tau(\theta)$.

For $q = 1$, L_1 and L_2 are 1-knots by hypothesis. We need additional arguments since $H_1(F)$ is not decomposed as above. For a manifold E , let denote the cokernel of $H_i(\partial E) \rightarrow H_i(E)$ by $\bar{H}_i(E)$. We assert that for any 1-knot and a generalized Seifert surface E of complexity c , the Seifert pairing on $H_1(E)$ induces a well-defined ‘‘Seifert pairing’’ on $\bar{H}_1(E)$ that induces the signature jump function of the knot. For, if x is a boundary component of E , a parallel copy of E is a 2-chain whose boundary is homologous to cx in the complement of $\text{int}(E)$ and so the linking number of x and any 1-cycle on $\text{int}(E)$ is zero. This proves the assertion.

Assume that F_i is connected for $i = 1, 2$. Let x_0, \dots, x_{c-1} be boundary components of $F_1 - \text{int}(B_1)$. $\{x_i\}$ are ordered so that parallel cycles in $\Sigma - F$ are obtained by pushing x_i and x_{i+1} slightly along the negative and positive normal directions of F , respectively. Let y_1, \dots, y_{c-1} be curves on $\text{int}(F)$ such that the intersection number of y_i and x_j on F is $\delta_{(i-1)j} - \delta_{ij}$, where δ_{ij} is the Kronecker symbol (see Figure 1 for a schematic picture).

Then $\bar{H}_1(F)$ is the direct sum of $\bar{H}_1(F_1)$, $\bar{H}_1(F_2)$ and a free abelian group generated by x_1, \dots, x_{c-1} and y_1, \dots, y_{c-1} . We will show that the generators x_i and y_i have no contribution to the signature.

Similarly to the proof of the previous assertion, it is shown that the linking number of x_i and any 1-cycle on the interior of $F_j - \text{int}(B_j)$ is zero. Furthermore, the value of the Seifert pairing at x_i and y_j can be computed by counting intersections of the 1-cycle obtained by pushing y_i slightly and the 2-chain obtained by attaching $(c-1)$ annuli bounded by $x_i - x_j$ ($j \neq i$) to $F_1 - \text{int}(B_1)$. By this argument we can verify that the Seifert matrix on $\bar{H}_1(F)$ with respect to generators of $\bar{H}_1(F_i)$ and y_1, \dots, y_{c-1} , x_1, \dots, x_{c-1} is given by

$$A = \left[\begin{array}{cc|cccc|cccccc|cccc} A_1 & & * & * & * & \cdots & * & & & & & & & & & \\ & A_2 & * & * & * & \cdots & * & & & & & & & & & \\ \hline * & * & * & * & * & \cdots & * & -\frac{1}{c} & -\frac{1}{c} & -\frac{1}{c} & \cdots & -\frac{1}{c} & & & & \\ * & * & * & * & * & \cdots & * & \frac{c-2}{c} & -\frac{2}{c} & -\frac{2}{c} & \cdots & -\frac{2}{c} & & & & \\ * & * & * & * & * & \cdots & * & \frac{c-3}{c} & \frac{c-3}{c} & -\frac{3}{c} & \cdots & -\frac{3}{c} & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & & & & \\ * & * & * & * & * & \cdots & * & \frac{1}{c} & \frac{1}{c} & \frac{1}{c} & \cdots & -\frac{c-1}{c} & & & & \\ \hline & & \frac{c-1}{c} & \frac{c-2}{c} & \frac{c-3}{c} & \cdots & \frac{1}{c} & & & & & & & & & \\ & & -\frac{1}{c} & \frac{c-2}{c} & \frac{c-3}{c} & \cdots & \frac{1}{c} & & & & & & & & & \\ & & -\frac{1}{c} & -\frac{2}{c} & \frac{c-3}{c} & \cdots & \frac{1}{c} & & & & & & & & & \\ & & \vdots & \vdots & \vdots & \ddots & \vdots & & & & & & & & & \\ & & -\frac{1}{c} & -\frac{2}{c} & -\frac{3}{c} & \cdots & \frac{1}{c} & & & & & & & & & \end{array} \right]$$

where A_i is a Seifert matrix defined on $\bar{H}_1(F_i)$ and empty blocks represent zero matrices. The bottom-center block and middle-right block are nonsingular. In fact, subtracting the i -th row from the $(i+1)$ -th row for each $i = c-2, \dots, 1$, the bottom-center block becomes an upper-triangular nonsingular matrix, and similarly for the right-middle block. Hence the top-center block and middle-left block can be eliminated by row and column operations. This shows that A is congruent to the block sum of A_1 , A_2 and a metabolic matrix. Therefore $[A] = [A_1] + [A_2]$ in the matrix cobordism group $G_{+1}^{\mathbf{Q}}$, and we have $\delta_L(\theta) = \delta_A^+(\theta/c) = \delta_{A_1}^+(\theta/c) + \delta_{A_2}^+(\theta/c) = \delta_{L_1}(\theta) + \delta_{L_2}(\theta)$ as desired. \square

By Theorem 1.2, the signature jump function of a link L in an integral homology sphere always has the period 2π since L^τ is primitive. However the period of the signature jump function of a link with nontrivial complexity in a rational homology sphere is not 2π in general. In the following example, for any nonzero rational number r we construct a knot whose signature jump function has the period $2\pi r$.

Example. Let K be a knot in S^{2q+1} and A be a Seifert matrix defined on a Seifert surface F of K . For $k \neq 0$, a knot K' with Seifert matrix $i_k^q A$ is obtained by fusing the link $i_k K = \partial(i_k F)$ along $|k| - 1$ bands whose interiors are disjoint to $i_k F$. (We may take as K' a $(k, 1)$ -cable knot with companion K for $q = 1$.)

Let S be an embedded $(2q-1)$ -sphere with knot type K' and J be an unknotted $(2q-1)$ -sphere that bounds a ball disjoint to S in S^{2q+1} . Let C be a simple closed curve

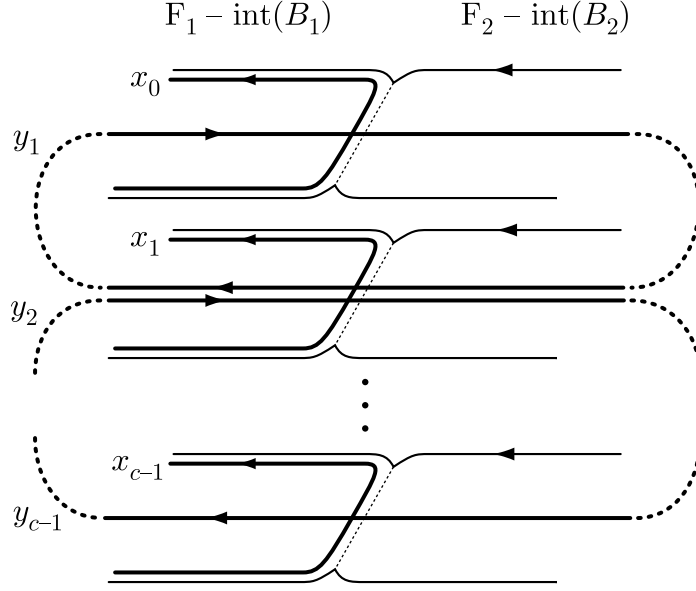


FIGURE 1.

in $S^{2q+1} - S \cup J$ whose linking numbers with S and J are n and -1 , respectively, as shown in Figure 2. For $n \neq 0$, the result of surgery along S and C (with respect to zero-framing if $q = 1$) is a rational homology sphere Σ , and J can be considered as a knot in Σ .

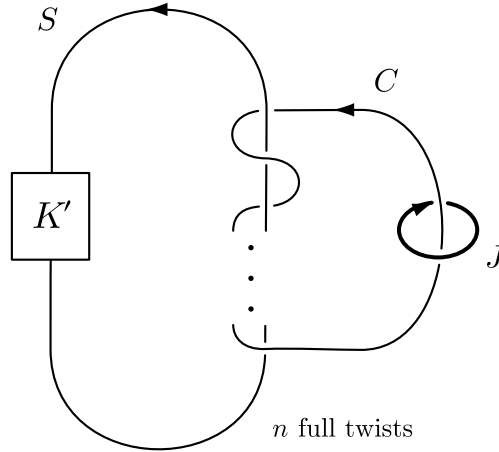


FIGURE 2.

$H_1(\Sigma - J)$ is an infinite cyclic group generated by a meridian u of S . The meridian μ of J is given by $\mu = nu$ in $H_1(\Sigma - J)$. Hence $c(J) = |n|$ by Theorem 3.3. In fact, there is a submanifold in S^{2q+1} bounded by the union of a parallel of S and $i_n J$ as in Figure 3, and a generalized Seifert surface E with complexity n for J is obtained by gluing a $2q$ -disk in Σ to the submanifold along the parallel of S . For $q \neq 1$, $i_k^q A$ is a Seifert matrix of E . For $q = 1$, the block sum of $i_k^q A$ and a zero matrix of dimension $|n| - 1$ is a Seifert matrix of E . In any case, we have $\delta_J(\theta) = \text{sgn } n \cdot \delta_{i_k^q A}^q(\theta/n) = \text{sgn}(k/n) \cdot \delta_K(\theta \cdot k/n)$.

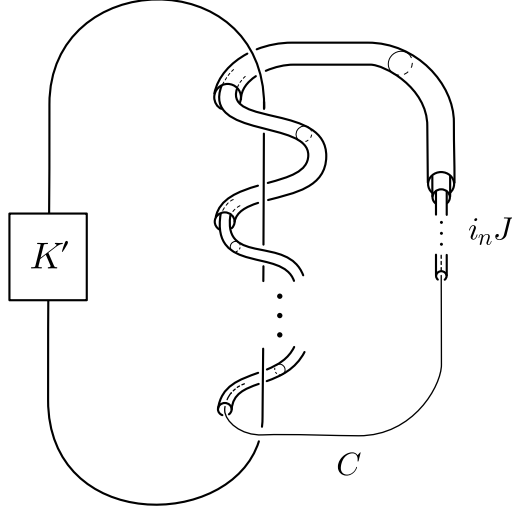


FIGURE 3.

As a special case of this example, there are knots in rational homology spheres whose signature jump functions do not have the period 2π . Such knots are not \mathbf{Q} -concordant to knots in (homology) spheres, by Theorem 1.2. This illustrates that the \mathbf{Q} -concordance theory of links in rational homology spheres is not reduced to that of links in spheres.

Sophisticating the above arguments, we show Theorem 1.3.

Proof of Theorem 1.3. Choose infinitely many distinct primes p_i greater than 7. For odd q , let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and for even q , let

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is easily checked that $\delta_A^q(\theta)$ is nonzero if and only if $\theta = n\pi(k \pm 1/6)$ for some integer k , where $n = 1$ for even q or $n = 2$ for odd q . Since A is a Seifert matrix of a knot in S^{2q+1} by [13, 16], a $(2q-1)$ -knot K_i in a rational homology sphere such that $\delta_{K_i}(\theta) = \delta_A^q(\theta \cdot p_i/7)$ is constructed by the above example. (For $q = 2$, we need to use the block sum of 8 copies of A to satisfy the extra condition [16] forced by a result of Rochlin. Since the signature jump function multiplies by 8, this is irrelevant to our purpose.) In order to show that K_i 's are independent, we consider a connected sum K of $a_i K_i$ over all i , where all but finitely many a_i are zero and $a_j \neq 0$ for some j . We will show that K is not \mathbf{Q} -concordant to a knot in a integral homology sphere. This will complete the proof.

By the additivity of signature jump function, $\delta_K(\theta) = \sum_i a_i \delta_A^q(\theta \cdot p_i/7)$. We claim that $\delta_K(7n\pi/6p_j) \neq 0$ but $\delta_K(7n\pi/6p_j + 2\pi) = 0$. If a summand $a_i \delta_A^q(n\pi p_i/6p_j)$ of $\delta_K(7n\pi/6p_j)$ is nonzero, then $n\pi p_i/6p_j = n\pi(k \pm 1/6)$ for some k , or equivalently $p_i = p_j(6k \pm 1)$. This implies $p_j \mid p_i$ and $i = j$. This shows the first claim. On the other

hand, if a summand $a_i \delta_A^q(n\pi p_i/6p_j + 2\pi p_i/7)$ of $\delta_K(7n\pi/6p_j + 2\pi)$ is nonzero, then we have $p_i(7n + 12p_j) = 7np_j(6k \pm 1)$ for some k , and $7 \mid p_j$ or $7 \mid p_j$. This is a contradiction, and the second claim is proved.

From the claims, $\delta_K(\theta)$ is not of period 2π . By Theorem 1.2, K is not \mathbf{Q} -concordant to knots of period 2π , in particular, to knots in integral homology spheres. \square

Another natural question is whether the homomorphism $C_n \rightarrow C_n^{\mathbf{Q}}$ is injective. More generally, one may ask whether the natural map from the set of concordance classes of links in spheres into that of links in rational homology spheres is injective. We do not address these questions in this paper.

We finish this paper with two applications illustrating that our results are also useful for links in spheres. One is a simple proof of the following result of Litherland [19]: Let K be a satellite knot with companion J , that is, $K = h(J)$ where J is a 1-knot in a standard solid torus V in S^3 and h is a homeomorphism from V onto a tubular neighborhood of a knot K' sending a 0-linking longitude on ∂V to a 0-linking longitude of K' . Let r be the winding number of J in V . Then $\delta_K(\theta) = \delta_J(\theta) + \delta_{K'}(r\theta)$. For, if A and B are Seifert matrices of J and K' , then the block sum $A \oplus i_\alpha^+ B$ is a Seifert matrix of K for some tuple α such that $n_\alpha = r$, and Litherland's formula is proved by Theorem 2.1. We remark that a formula for Alexander polynomials of satellite knots [23] can also be proved using an analogous result of Theorem 2.1 mentioned in Section 2.

Another is a new proof of (a weaker version of) a result of Kawauchi [11]: If a $(2q-1)$ -knot K is not (algebraically if $q = 1$) torsion, then $i_r K$ is a boundary link that is never concordant to any split union for any $r > 0$. For, if $i_r K$ is concordant to a split link, $i_2 K$ is concordant to a split union $K \cup K$, and then by Theorem 1.1 and 1.2, $\delta_K(2\theta) = \delta_{i_2 K}(\theta) = \delta_{K \cup K}(\theta) = 2\delta_K(\theta)$. This implies that $\delta_K(\theta) = 0$ for all θ and K is (algebraically if $q = 1$) torsion by a result of Levine [17].

REFERENCES

- [1] J. Cha and K. Ko, *On equivariant slice knots*, Proc. A. M. S., 127 (1999), 2175–2182.
- [2] J. Cha and K. Ko, *Signatures of covering links*, arXiv:math.GT/0108206.
- [3] T. Cochran and K. Orr, *Not all links are concordant to boundary links*, Bull. A. M. S. 23 (1990), 99–106.
- [4] T. Cochran and K. Orr, *Not all links are concordant to boundary links*, Ann. Math. 138 (1993), 519–554.
- [5] D. Erle, *Die quadratische Form eines Knottens und ein Satz über Knotten-mannigfaltigkeiten*, J. Reine Angw. Math. 236 (1969), 174–218.
- [6] P. Gilmer, *Link cobordism in rational homology 3-spheres*, J. Knot Theory and Its Ramifications 2 (1993), no 3, 285–320.
- [7] P. Gilmer and C. Livingston, *The Casson-Gordon invariant and link concordance*, Topology 31 (1992), 475–492.
- [8] J. Hillman, *Alexander ideals of links*, Springer-Verlag Lecture Notes in Math. 895, Springer, Berlin, 1981.
- [9] L. Kauffman, *On Knots*, Princeton University Press, 1987.
- [10] L. Kauffman and L. Taylor, *Signatures of links*, Trans. A. M. S. 216 (1976), 351–365.
- [11] A. Kawauchi, *On links not cobordant to split links*, Topology 19 (1980), 321–334.
- [12] C. Kearton, *Blanchfield duality and simple knots*, Trans. A. M. S. 202 (1975), 141–160.
- [13] M. Kervaire, *Les noeuds de dimensions supérieures*, Bull. Soc. Math. France 93 (1965), 225–271.
- [14] K. Ko, *Seifert matrices and boundary link cobordisms*, Trans. A. M. S. 299 (1987), 657–681.
- [15] J. Levine, *Unknotting spheres in codimension two*, Topology 4 (1965), 9–16.

- [16] J. Levine, *Knot cobordism group in codimension two*, Comm. Math. Helv. 44 (1969), 229–244.
- [17] J. Levine, *Invariants of knot cobordism*, Invent. Math. 8 (1969), 98–110.
- [18] J. Levine, *Link invariants via the eta invariant*, Comment. Math. Helv. 69 (1994), 82–119.
- [19] R. Litherland, *Signatures of iterated torus knots*, Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977), pp. 71–84, Lecture Notes in Math., 722, Springer, Berlin, 1979.
- [20] T. Matumoto, *On the signature invariants of a non-singular complex sesqui-linear form*, J. Math. Soc. Japan 29 (1977), 67–71.
- [21] J. Milnor, *Infinite cyclic coverings*, Conference on the Topology of Manifolds, Prindle, Weber & Schmidt, Boston, Mass., 1968, pp. 115–133.
- [22] K. Murasugi, *On a numerical invariant of link types*, Trans. A. M. S. 117 (1965), 387–422.
- [23] H. Seifert, *On the homology invariants on knots*, Quart. J. Math. Oxford 2 (1950), 23–32.
- [24] A. G. Tristram, *Some cobordism invariants for links*, Proc. Camb. Phil. Soc. 66 (1969), 252–264.
- [25] H. F. Trotter, *Homology of group systems with applications to knot theory*, Ann. Math. 59 (1962), 464–498.

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