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KNOT SIGNATURE FUNCTIONS ARE INDEPENDENT

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Abstract. A Seifert matrix is a square integral matrix V satisfying

$$\det(V - V^T) = \pm 1.$$

To such a matrix and unit complex number ω there corresponds a signature,

$$\sigma_\omega(V) = \operatorname{sign}((1-\omega)V + (1-\bar{\omega})V^T).$$

Let S denote the set of unit complex numbers with positive imaginary part. We show that $\{\sigma_\omega\}_{\omega\in S}$ is linearly independent, viewed as a set of functions on the set of all Seifert matrices.

If V is metabolic, then $\sigma_\omega(V) = 0$ unless ω is a root of the Alexander polynomial, $\Delta_V(t) = \det(V - tV^T)$. Let A denote the set of all unit roots of all Alexander polynomials with positive imaginary part. We show that $\{\sigma_\omega\}_{\omega\in A}$ is linearly independent when viewed as a set of functions on the set of all metabolic Seifert matrices.

To each knot $K \subset S^3$ one can associate a Seifert matrix V_K, and $\sigma_\omega(V_K)$ induces a knot invariant. Topological applications of our results include a proof that the set of functions $\{\sigma_\omega\}_{\omega\in S}$ is linearly independent on the set of all knots and that the set of two–sided averaged signature functions, $\{\sigma^*_\omega\}_{\omega\in S}$, forms a linearly independent set of homomorphisms on the knot concordance group. Also, if $\nu \in S$ is the root of some Alexander polynomial, then there is a slice knot K whose signature function $\sigma_\omega(K)$ is nontrivial only at $\omega = \nu$ and $\omega = \overline{\nu}$. We demonstrate that the results extend to the higher-dimensional setting.

1. Introduction

Associated to each knot $K \subset S^3$ there is a *Seifert matrix* V_K. The set of such Seifert matrices consists of those square integral matrices V satisfying

$$\det(V - V^T) = \pm 1,$$

where V^T denotes the transpose. For each unit complex number ω the *hermitianized Seifert form*, V_ω, is defined by $V_\omega = (1-\omega)V + (1-\bar{\omega})V^T$; the signature of this matrix is denoted $\sigma_\omega(V)$. It is possible to associate different Seifert matrices to a given knot; however, the value of $\sigma_\omega(V_K)$ is known to depend only on K and not the choice of Seifert matrix [7, 13]; $\sigma_\omega(V_K)$ is usually denoted $\sigma_\omega(K)$.

Let S denote the set of all unit complex numbers with positive imaginary parts. In this paper we study the linear independence of the set of these signature functions, $\{\sigma_\omega\}_{\omega\in S}$, viewed as real-valued functions on the set of all Seifert matrices,

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and consequently as functions on the set of all knots. Our first result is the following:

Theorem 1. *The set of functions* $\{\sigma_\omega\}_{\omega\in S}$ *is linearly independent.*

Previously the only sets D for which it was known that $\{\sigma_\omega\}_{\omega\in D}$ is linearly independent were certain discrete subsets of S, [7, 13]. Applications of this result include the demonstration that a number of results that hold in high-dimensional knot theory fail in dimension 3. This is briefly summarized in Section 6.

A Seifert matrix is necessarily of even dimension, say $2g$. It is called *metabolic* if there is a summand of dimension g of $\mathbf{Z}^{2g}$ on which the associated bilinear form vanishes. It is called *hyperbolic* if $\mathbf{Z}^{2g}$ is the direct sum of two such summands. Over the rational numbers these are identical concepts, but that is not the case over $\mathbf{Z}$. In particular, $\sigma_\omega(V)$ is identically 0 if V is hyperbolic, but can be nonzero for a metabolic form V if ω is a root of the *Alexander polynomial* of V, $\Delta_V(t) = \det(V - tV^T) \in \mathbf{Z}[t, t^{-1}]$. A detailed analysis of the signature functions of metabolic forms was accomplished in [9]. The topological significance of these concepts is that metabolic forms correspond to *slice knots* and hyperbolic forms correspond to *double null–concordant knots* [12]. Renewed interest in double null-concordance, as summarized in Section 6, motivates our study of the signature functions associated to metabolic Seifert forms.

Polynomials that occur as Alexander polynomials are precisely those polynomials that are symmetric, $\Delta(t^{-1}) = \pm t^j\Delta(t)$, for some j, and satisfy $\Delta(-1) = \pm 1$. Let $D \subset S$ denote the set of unit roots of Alexander polynomials with positive imaginary parts.

Theorem 2. *The set of functions* $\{\sigma_\omega\}_{\omega\in D}$ *is linearly independent on the set of all metabolic Seifert matrices.*

The proof of this has the following topological corollary.

Corollary 2.1. *If* ν *is a unit root of some Alexander polynomial, there is a slice knot* K *whose signature function* $\sigma_\omega(K)$ *is nontrivial only at* $\omega = \nu$ *and* $\omega = \overline{\nu}$.

2. Independence of signature functions

Proof of Theorem 1. For a given Seifert matrix V, $\sigma_\omega(V)$ can be viewed as an integer-valued function of $\omega \in S$. Simple arguments show that jumps of this function can occur only at those values of ω that are roots of $\Delta_V(t)$, and if the root is simple the jump is nontrivial. Also, for each V, $\sigma_\omega(V) = 0$ for all ω close to 1. (For proofs, see, for instance, [7].)

We will next show that for any given $\omega \in S$ there are an $\omega' \in S$ arbitrarily close to ω and a Seifert matrix V whose signature function has its only nontrivial jump at ω'. From this the theorem follows, since one easily constructs, for any finite set $\{\sigma_{\omega_i}\}_{1\le i\le N}$ and any chosen $k, 1 \le k \le N$, a Seifert matrix V_k with $\sigma_{\omega_k}(V_k) \ne 0$ and $\sigma_{\omega_i}(V_k) = 0$ if $1 \le i \le N$ and $i \ne k$.

To construct the desired matrix V we construct a polynomial Δ having a unique root in S at a point ω' that can be made as close to ω as desired. Since Δ will be constructed to be integral, to satisfy $\Delta(1) = \pm 1$ and to be symmetric, it is the Alexander polynomial of some Seifert matrix V, [10].

For a given r, $-1 < r < 1$, consider the polynomial

$$F_r(t) = (t-1)^2(t^2 - 2rt + 1) = t^4 + (-2-2r)t^3 + (4r+2)t^2 + (-2-2r)t + 1.$$

It is easily seen that $F_r(t)$ has a pair of (unit) complex roots with real part r. Let $r = \mathrm{Re}(\omega)$, so that F_r has roots at ω and $\bar{\omega}$. For a given ϵ there is a δ such that any perturbation of the coefficients of F_r by less than δ moves the roots less than ϵ. Choose a rational approximation a/b, $b > 0$, to r so that replacing r in the coefficients of F_r by a/b changes the coefficients by less than $\delta/2$. Furthermore, choose b large enough so that $1/b < \delta/2$. Then the roots of

$$G(t) = t^4 + \left(-2 - 2\frac{a}{b}\right)t^3 + \left(4\frac{a}{b} + 2 - \frac{1}{b}\right)t^2 + \left(-2 - 2\frac{a}{b}\right) + 1$$

are within ϵ of those of F_r. Multiplying through by b yields the desired polynomial:

$$\Delta(t) = bt^4 + (-2b - 2a)t^3 + (4a + 2b - 1)t^2 + (-2b - 2a)t + b.$$

Since $\Delta(1) = -1$ and $b > 0$, $\Delta(t)$ has at least two real roots. The remaining roots ω' and $\bar{\omega}'$ are within ϵ of ω and $\bar{\omega}$. Since $\Delta(t)$ is real, symmetric, and it has exactly two nonreal roots, these roots must lie on the unit circle. The result follows. □

3. A remark on concordance

There is an equivalence relation on the set of Seifert forms, called *algebraic concordance*: V_1 and V_2 are *concordant* if $V_1 \oplus -V_2$ is metabolic. The set of equivalence classes forms a group $\mathcal{G}$, the *algebraic concordance group* (see [7]). The signature functions are not well-defined on $\mathcal{G}$, but this difficulty can be overcome by considering the averaged signature function σ^*_ω, obtained as the two-sided average of σ_ω. Our theorem is easily seen to apply to the set $\{\sigma^*_\omega\}_{\omega\in S}$, where this is now a set of *homomorphisms*, not simply functions, on $\mathcal{G}$.

The same result holds for the corresponding topological construction, the *concordance group of knots*, $\mathcal{C}$. Further details will be summarized in the final section.

4. Signature functions of metabolic forms

A simple algebraic calculation shows that the matrix V_ω is singular precisely when the unit ω is a root of $\Delta_V(t)$. If V is metabolic it follows readily that $\sigma_\omega(V) = 0$, except perhaps when ω is a root of $\Delta_V(t)$. In [9], Levine constructed metabolic matrices with nontrivial signature at $\omega = e^{\pi i/3}$, the root of $t^2 - t + 1$.

Our goal is to show that Levine's construction can be expanded to cover any root of any Alexander polynomial. One slightly subtle point in the proof of Theorem 2 is in dealing with Alexander polynomials that have several unit roots; in such cases we must be able to specify at exactly which roots the signature function is nonzero.

Proof of Theorem 2. Let $\Delta(t) = \sum_{i=0}^{2g} d_i t^i$ be an Alexander polynomial. Since multiplication by $\pm t^j$ does not change the unit roots of $\Delta(t)$, we can assume that $d_0 \neq 0$, $d_{2g-i} = d_i$ and that $\Delta(1) = 1$.

Consider the matrix V below, with 0_{2g} a $2g \times 2g$ matrix of zeroes and I_g the $g \times g$ identity. The $g \times g$ and $2g \times 2g$ integer matrices A_g and B_{2g} will be specified in the course of the proof. We will choose B_{2g} to be symmetric, so assume so throughout the discussion. The complex number $\Omega = (1 - \omega)(1 - \overline{\omega})$.

$$V = \begin{pmatrix} 0_{2g} & \begin{pmatrix} I_g & A_g \\ I_g & 0_g \end{pmatrix} \\ \begin{pmatrix} 0 & I_g \\ A_g^T & I_g \end{pmatrix} & \Omega B_{2g} \end{pmatrix}.$$

Since $\det(V - V^T) = 1$, V is the Seifert matrix. The half-dimensional block of zeroes implies that V is metabolic.

Simple work with the matrices and algebraic manipulations yield that $\Delta_V(t)$ is the square of $\det((1-t)^2 A_g + t)$, or, more usefully,

$$\Delta_V(t) = \left((1-t)^{2g}\lambda\left(\frac{t}{(1-t)^2}\right)\right)^2,$$

where $\lambda(x) = \det(A_g + xI_g)$. Writing $\lambda(x) = \sum_{j=0}^{g} a_j x^j$, it follows that $\Delta_V(t) = (P(t))^2$, where

$$P(t) = \sum_{j=0}^{g} (1-t)^{2g-2j} a_j t^j.$$

From this description of P if follows that: 1) $P(t)$ is symmetric, 2) for $k \le g$, the coefficient of t^k in $P(t)$ is a linear function of $\{a_j\}_{j=0\ldots k}$, and 3) in that linear function of $\{a_j\}_{j=0\ldots k}$, a_k appears with coefficient 1.

It follows from these observations that we can choose the a_j so that $P(t) = \Delta(t)$. (Solve first to find $a_0 = d_0$, and then solve recursively for the remaining a_j in order.) We now have $\Delta_V(t) = \Delta(t)^2$. Since $\Delta(1) = 1$, it follows that: 4) $a_g = 1$. Also, since $d_0 \neq 0$ we have: 5) $a_0 \neq 0$. We must now find a matrix A_g having the desired λ; the matrix we use is of the following form, presented here in the case $g = 4$:

$$\begin{pmatrix} 0 & 0 & 0 & a_0 \\ -1 & 0 & 0 & a_1 \\ 0 & -1 & 0 & a_2 \\ 0 & 0 & -1 & a_3 \end{pmatrix}.$$

We now consider the matrix $(1-\omega)V + (1-\bar{\omega})V^T$. Performing appropriate row operations on the top $2g$ rows, and simultaneous conjugate column operations of the first $2g$ columns, quickly yields the following matrix, where $\Omega = (1-\omega)(1-\bar{\omega}) = (1-\omega)+(1-\bar{\omega})$ (notice that Ω is nonzero; it equals 0 only if $\omega = 1$, which is outside our domain):

$$\begin{pmatrix} 0_{2g} & \begin{pmatrix} 0_g & \Omega A_g - I_g \\ I_g & \frac{1}{1-\omega} I_g \end{pmatrix} \\ \begin{pmatrix} 0_g & I_g \\ \Omega A_g^T - I_g & \frac{1}{1-\bar{\omega}} I_g \end{pmatrix} & \Omega B_{2g} \end{pmatrix}.$$

Choose B_{2g} so that all the entries that are not in the lower right $g \times g$ block, denoted B_g, are zero. The $\frac{1}{1-\omega} I_g$ block can be cleared using column operations, and simultaneous row operations will clear the $\frac{1}{1-\bar{\omega}} I_g$ block. It then follows that the signature of V_ω is the signature of the following matrix:

$$\begin{pmatrix} 0 & \Omega A_g - I_g \\ \Omega A_g^T - I_g & \Omega \mathrm{B}_g \end{pmatrix}.$$

Next, simultaneous column operations on the last g columns and row operations on the last g rows can be used to put the $\Omega A - I_g$ block into lower triangular form and the $\Omega A^T - I_g$ block into upper triangular form. If this is done, all entries on the diagonal of the upper right-hand $g \times g$ block are nonzero (actually -1) except the last diagonal element, which becomes $\Omega^g \lambda(-\frac{1}{\Omega})$. If B_g is chosen so that all entries are 0 except its top right and bottom left entries (let us call them b_1) and the bottom right entry, say b_2, then after these row and column operations, the bottom right entry of the entire matrix has become $\Omega_2 b_2 + 2\Omega^2 a_0 b_1$. Hence, the signature of the original matrix V_ω is equal to the signature of the 2×2 matrix

$$\begin{pmatrix} 0 & \lambda(-\frac{1}{\Omega}) \\ \lambda(-\frac{1}{\Omega}) & \Omega b_2 + 2\Omega^2 a_0 b_1 \end{pmatrix}.$$

From the identity $\Delta(t) = (1-t)^{2g}\lambda(\frac{t}{(1-t)^2})$ we see that $\lambda(-\frac{1}{\Omega}) = (1-\omega)^{2g}\Delta(\omega)$. Hence the matrix is nonsingular with 0 signature unless ω is root of Δ. On the other hand, if ω is a root of Δ, then the signature is given by the sign of $\Omega b_2 + 2\Omega^2 a_0 b_1$, which is the same as the sign of $b_2 + 2\Omega a_0 b_1$ since $\Omega = (1-\omega)(1-\bar{\omega})$ is a nonzero norm.

It remains to select b_1 and b_2 appropriately. Suppose that Δ has unit roots with positive imaginary parts $\{\omega_1, \ldots, \omega_k\}$, placed in order of increasing real part, and let p satisfy $1 \leq p \leq k$. Since $\Omega = 2 - 2\mathrm{Re}\,\omega$ is a decreasing function of the real part of ω, it is clear that b_1 and b_2 can be selected so that $b_2 + 2\Omega a_0 b_1$ is positive for ω_j when $j \leq p$ and it is negative for ω_j when $j > p$. The corresponding matrix, say V_1, has signature $\sigma_{\omega_j}(V_1) = 1$ if $k \leq p$ and $\sigma_{\omega_j}(V_1) = -1$ if $k > p$. Similarly, construct V_2 so that $\sigma_{\omega_j}(V_2) = -1$ if $k < p$ and $\sigma_{\omega_j}(V_2) = 1$ if $k \geq p$. The direct sum of these two matrices has the desired property: its only nontrivial signature is $\sigma_{\omega_p}(V_1 \oplus V_2) = 2$. □

Comment. For the matrix constructed we have $\sigma_{\omega_p} = 2$, and all other signatures are 0. Can a similar matrix be constructed, only with $\sigma_{\omega_p} = 1$? If the irreducible polynomial for ω_p has a unique root $\omega \in S$, then the construction gives such an example. If, however, there are multiple roots in S, it can be shown that the parity of the signatures at each of these roots must be the same, as follows. The parity of the signature of V_ω is determined by its rank and nullity of V_ω. Both of these are invariant under the action of the Galois group permuting the roots of $\Delta(t)$.

5. Higher-dimensional knots

The algebraic theory of one-dimensional knots in S^3 extends to a general theory of codimension two knots in S^{2n+1}. According to Levine [7], the case of knots in dimension S^{4n+3} is identical to the classical case. Hence, all the results presented so far apply for knots in these dimensions. In this section we will describe how to modify our previous work to apply to knots in S^{4n+1}. The reference for this is [7].

In the case of knots in S^{4n+1}, a Seifert matrix is a $2g \times 2g$ integral matrix satisfying $\det(V + V^T) = 1$. The signature function of such a Seifert matrix is given by

$$\sigma_\omega(V) = \mathrm{sign}\big[(\omega - \bar{\omega})\left((1-\omega)V - (1-\bar{\omega})V^T\right)\big].$$

The Alexander polynomial is given by $\Delta(t) = \det(tV + V^T)$. An integral polynomial is the Alexander polynomial of some such Seifert matrix if and only if $\Delta(t) = t^{2g}\Delta(t^{-1})$, $\Delta(1) = (-1)^g$, and $\Delta(-1)$ is a square.

Extending Theorem 1. To extend the proof of Theorem 1 to the case of knots in S^{4n+1}, we need to modify the polynomial $\Delta(t)$ constructed in the proof to assure that it satisfies the stricter conditions on Alexander polynomials in these dimensions. To do this, we replace $\Delta(t)$ by the polynomial $D(t) = (ct^2 + (1-2c)t + c)\Delta(t)$, where $c = 2(a+b) > 0$. Then it is straightforward to check that $D(t)$ is an Alexander polynomial of a Seifert matrix V of a knot in S^{4n+1}, using the fact that $\Delta(1) = -1$ and $\Delta(-1) = 8a + 8b - 1$. Even though $D(t)$ has an additional zero ω'' in S, where

$\mathrm{Re}(\omega'') = 1 - 1/2c$, we can control it by choosing a and b carefully, as follows. Since a/b is to be chosen close to $r > -1$, we can assume that $a/b > -1 + \epsilon$ for some fixed positive ϵ. Hence, $c = 2(a+b) > 2b\epsilon$, and we can choose b large enough so that $1 - 1/(2c)$ is sufficiently close 1. Since both σ_1 and $\sigma_{(-1)}$ are zero for any knot in S^{4k+1}, it follows that the signature function $\sigma_\omega(V)$ assumes a constant nonzero value (indeed, ± 2) for $r < \mathrm{Re}(\omega) < 1 - 1/(2c)$, and zero for $\mathrm{Re}(\omega) < r$ or $\mathrm{Re}(\omega) > 1 - 1/(2c)$. Thus it can be used to complete the proof in this setting.

Extending Theorem 2. The extension of the proof of Theorem 2 for knots in S^{4n+1} is more straightforward. We proceed in the same way but start with

$$V = \begin{pmatrix} 0_{2g} & \begin{pmatrix} I_g & A_g \\ I_g & 0_g \end{pmatrix} \\ \begin{pmatrix} 0_g & -I_g \\ -A_g^T & I_g \end{pmatrix} & B_{2g} \end{pmatrix},$$

which is a Seifert matrix of a slice knot in S^{4n+1} for any n by a result of [7], and we assume that the given $\Delta(t)$ satisfies $\Delta(1) = (-1)^g$. Then it can be shown that $\sigma_\omega(V)$ is zero if $\Delta(\omega) \neq 0$, and $\sigma_\omega(V) = b_2 - 2\Omega a_0 b_1$ if $\Delta(\omega) = 0$. Now the arguments of the last paragraph of the proof of Theorem 2 can be applied to construct a desired matrix.

6. Applications to knot concordance

Our detailed examination of signature functions was motivated by problems in studying the concordance group of knots. In this section we give some indication as to the usefulness of the algebraic results of this paper.

The Seifert form V_K associated to a knot K in S^3 is not unique. However, as mentioned in Section 3, by placing an equivalence relation on the set of knots one arrives at the *concordance group* of knots, $\mathcal{C}$. The association $K \to V_K$ induces a surjective homomorphism $\phi : \mathcal{C} \to \mathcal{G}$. These notions were defined and studied by Levine [7, 8].

Levine's work contained two main results. One was topological—in higher dimensions the analogue of ϕ is an isomorphism. The second was algebraic—the group $\mathcal{G}$ is isomorphic to an infinite direct sum, $\mathbf{Z}^\infty \oplus \mathbf{Z}_2^\infty \oplus \mathbf{Z}_4^\infty$. This algebraic result depended in part on the existence of an infinite collection of ω for which the associated signature functions σ_ω are linearly independent.

Several years after Levine's work, Casson and Gordon [1, 2] proved that ϕ is not an isomorphism (for knots in S^3) by developing obstructions to a knot with metabolic Seifert form being trivial in $\mathcal{C}$. The kernel of ϕ is called the group of *algbebraically slice* knots, denoted $\mathcal{A}$. Later, Gilmer [4, 5] demonstrated that these Casson-Gordon obstructions could be interpreted in terms of knot signatures: not those of the original knot, which are necessarily 0 if the knot is algebraically slice, but rather a knot that reflects the knotting in a surface bounded by the original knot.

The most basic applications of Gilmer's approach to Casson-Gordon invariants called on rather simple facts about the signature function. For instance, constructing algebraically slice knots that are nontrivial in the concordance group was reduced to finding knots with nontrivial signature at a single root of unity.

As the subtleties of $\mathcal{A}$ have been explored, deeper facts about signatures have been called on. As one example, Stoltzfus [11] proved that, in higher dimensions, if the Alexander polynomial of a knot K factors into irreducible factors with resultant 1, then the associated knot is concordant to a corresponding connected sum of knots. The second author of this paper, in unpublished work, has shown that this result does not apply in dimension three. The simplest example involves the polynomials $2t^2 - 3t + 2$ and $3t^2 - 5t + 3$. The construction of the counterexample depended, via Gilmer's work, on finding a knot whose signatures at 67–roots of unity satisfy a complicated linear inequality. Finding such a knot could be carried out by ad hoc methods, but Theorem 1 makes the existence of such a knot automatic.

If one takes on more general problems, the ad hoc methods that can be applied to a single example are no longer useful. For instance, it now appears to be the case that for almost any pair of Alexander polynomials with resultant 1 there is a knot with Alexander polynomial the product of those polynomials, and yet the knot is not concordant to a corresponding connected sum. The proof depends on constructing knots whose signature functions at the collection of some (unknown) roots of unity satisfy some (unknown) inequality. Because of the general nature of the problem, little about which roots of unity or what the inequality is can be identified. Yet, with Theorem 1 it is possible to assert that such a knot will exist.

Similar issues arise in a number of related settings. In finding properties of $\mathcal{G}$ that do not apply to $\mathcal{C}$, individual examples can sometimes be constructed using (perhaps very messy) ad hoc constructions, but general results demand complete control over the signature function, something that is offered by Theorem 1.

Applications of Theorem 2 take place in a different realm. Levine's work in [9] offered one proof that the kernel of the natural surjection of $\mathcal{G}^h$ onto $\mathcal{G}$ is nontrivial, where $\mathcal{G}^h$ is the algebraic concordance group built using the equivalence relation based on hyperbolic rather than metabolic forms. In fact, it follows from [9] that the kernel is infinitely generated. This result implied a similar result for the topological setting of double null concordance versus concordance. Levine's work focused on the signature function at a particular root of unity. Recent work of Cochran, Orr, and Teichner [3] has renewed interest in the study of double null concordance; in particular, recently Taehee Kim [6] has made significant progress in studying the case in which all invariants of the knot based on $\mathcal{G}$ and $\mathcal{G}^h$ vanish. This ongoing work points to a need for a further study of the algebra of $\mathcal{G}^h$. The work here demonstrates that a complete analysis of the relationship between $\mathcal{G}$ and $\mathcal{G}^h$ will depend on considerations of all possible unit roots of Alexander polynomials.

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